[481] CONTRIBUTIONS TO THE FOUNDING OF THE THEORY OF TRANSFINITE NUMBERS

(FIRST ARTICLE)

"Hypotheses non fingo."

"Neque enim leges intellectui aut rebus damus ad arbitrium nostrum, sed tanquam scribæ fideles ab ipsius naturæ voce latas et prolatas excipimus et describimus."

"Veniet tempus, quo ista quæ nunc latent, in lucem dies extrahat et longioris ævi diligentia."

§ 1

The Conception of Power or Cardinal Number

BY an "aggregate" (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects m of our intuition or our thought. These objects are called the "elements" of M.

In signs we express this thus:

$$(1) \qquad M = \{m\}.$$

We denote the uniting of many aggregates M, N, P, . . ., which have no common elements, into a single aggregate by

$$(2) \qquad (M, N, P, \ldots).$$

The elements of this aggregate are, therefore, the elements of M, of N, of P, . . ., taken together.

We will call by the name "part" or "partial aggregate" of an aggregate M any other aggregate M_1 whose elements are also elements of M.

If M_2 is a part of M_1 and M_1 is a part of M, then M_2 is a part of M.

Every aggregate M has a definite "power," which we will also call its "cardinal number."

We will call by the name "power" or "cardinal number" of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given.

[482] We denote the result of this double act of abstraction, the cardinal number or power of M, by

$$(3) \qquad \overline{\overline{M}}.$$

Since every single element m, if we abstract from its nature, becomes a "unit," the cardinal number $\overline{\overline{M}}$ is a definite aggregate composed of units, and this number has existence in our mind as an intellectual image or projection of the given aggregate M.

We say that two aggregates M and N are "equivalent," in signs

$$(4) \qquad M \sim N \quad \text{or} \quad N \sim M,$$

if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other. To every part M_1 of M there corresponds, then, a definite equivalent part N_1 of N, and inversely.

If we have such a law of co-ordination of two equivalent aggregates, then, apart from the case when each of them consists only of one element, we can modify this law in many ways. We can, for instance, always take care that to a special element m_0 of M a special element n_0 of N corresponds. For if, according to the original law, the elements m_0 and n_0 do not correspond to one another, but to the element m_0 of M the element n_1 of N corresponds, and to the element n_0 of N the element m_1 of M corresponds, we take the modified law according to which m_0 corresponds to n_0 and m_1 to n_1 and for the other elements the original law remains unaltered. By this means the end is attained.

Every aggregate is equivalent to itself:

(5) $$M \sim M.$$

If two aggregates are equivalent to a third, they are equivalent to one another; that is to say:

(6) from $M \sim P$ and $N \sim P$ follows $M \sim N$.

Of fundamental importance is the theorem that two aggregates M and N have the same cardinal number if, and only if, they are equivalent: thus;

(7) from $M \sim N$ we get $\overline{\overline{M}} = \overline{\overline{N}}$,

and

(8) from $\overline{\overline{M}} = \overline{\overline{N}}$ we get $M \sim N$.

Thus the equivalence of aggregates forms the necessary and sufficient condition for the equality of their cardinal numbers.

[483] In fact, according to the above definition of power, the cardinal number $\overline{\overline{M}}$ remains unaltered if in the place of each of one or many or even all elements m of M other things are substituted. If, now, $M \sim N$, there is a law of co-ordination by means of which M and N are uniquely and reciprocally referred to one another; and by it to the element m of M corresponds the element n of N. Then we can imagine, in the place of every element m of M, the corresponding element n of N substituted, and, in this way, M transforms into N without alteration of cardinal number. Consequently

$$\overline{\overline{M}} = \overline{\overline{N}}.$$

The converse of the theorem results from the remark that between the elements of M and the different units of its cardinal number $\overline{\overline{M}}$ a reciprocally univocal (or bi-univocal) relation of correspondence subsists. For, as we saw, $\overline{\overline{M}}$ grows, so to speak, out of M in such a way that from every element m of M a special unit of $\overline{\overline{M}}$ arises. Thus we can say that

$$M \sim \overline{\overline{M}}. \tag{9}$$

In the same way $N \sim \overline{\overline{N}}$. If then $\overline{\overline{M}} = \overline{\overline{N}}$, we have, by (6), $M \sim N$.

We will mention the following theorem, which results immediately from the conception of equivalence. If M, N, P, . . . are aggregates which have no common elements, M′, N′, P′, . . . are also aggregates with the same property, and if

$$M \sim M', \quad N \sim N', \quad P \sim P', \quad \ldots,$$

then we always have

$$(M, N, P, \ldots) \sim (M', N', P', \ldots).$$

§ 2

"Greater" and "Less" with Powers

If for two aggregates M and N with the cardinal numbers $\mathfrak{a} = \overline{\overline{M}}$ and $\mathfrak{b} = \overline{\overline{N}}$, both the conditions:

(*a*) There is no part of M which is equivalent to N,
(*b*) There is a part N_1 of N, such that $N_1 \sim M$,

are fulfilled, it is obvious that these conditions still hold if in them M and N are replaced by two equivalent aggregates M′ and N′. Thus they express a definite relation of the cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ to one another.

[484] Further, the equivalence of M and N, and thus the equality of $\mathfrak{a}$ and $\mathfrak{b}$, is excluded; for if we had $M \sim N$, we would have, because $N_1 \sim M$, the equivalence $N_1 \sim N$, and then, because $M \sim N$, there would exist a part M_1 of M such that $M_1 \sim M$, and therefore we should have $M_1 \sim N$; and this contradicts the condition (*a*).

Thirdly, the relation of $\mathfrak{a}$ to $\mathfrak{b}$ is such that it makes impossible the same relation of $\mathfrak{b}$ to $\mathfrak{a}$; for if in (*a*) and (*b*) the parts played by M and N are interchanged, two conditions arise which are contradictory to the former ones.

We express the relation of $\mathfrak{a}$ to $\mathfrak{b}$ characterized by (*a*) and (*b*) by saying: $\mathfrak{a}$ is "less" than $\mathfrak{b}$ or $\mathfrak{b}$ is "greater" than $\mathfrak{a}$; in signs

$$(1) \qquad \mathfrak{a}<\mathfrak{b} \quad \text{or} \quad \mathfrak{b}>\mathfrak{a}.$$

We can easily prove that,

(2) if $\mathfrak{a}<\mathfrak{b}$ and $\mathfrak{b}<\mathfrak{c}$, then we always have $\mathfrak{a}<\mathfrak{c}$.

Similarly, from the definition, it follows at once that, if P_1 is part of an aggregate P, from $\mathfrak{a}<\overline{\overline{P}}_1$ follows $\mathfrak{a}<\overline{\overline{P}}$ and from $\overline{\overline{P}}<\mathfrak{b}$ follows $\overline{\overline{P}}_1<\mathfrak{b}$.

We have seen that, of the three relations

$$\mathfrak{a}=\mathfrak{b}, \quad \mathfrak{a}<\mathfrak{b}, \quad \mathfrak{b}<\mathfrak{a},$$

each one excludes the two others. On the other hand, the theorem that, with any two cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$, one of those three relations must necessarily be realized, is by no means self-evident and can hardly be proved at this stage.

Not until later, when we shall have gained a survey over the ascending sequence of the transfinite cardinal numbers and an insight into their connexion, will result the truth of the theorem:

A. If $\mathfrak{a}$ and $\mathfrak{b}$ are any two cardinal numbers, then either $\mathfrak{a}=\mathfrak{b}$ or $\mathfrak{a}<\mathfrak{b}$ or $\mathfrak{a}>\mathfrak{b}$.

From this theorem the following theorems, of which, however, we will here make no use, can be very simply derived:

B. If two aggregates M and N are such that M is equivalent to a part N_1 of N and N to a part M_1 of M, then M and N are equivalent;

C. If M_1 is a part of an aggregate M, M_2 is a part of the aggregate M_1, and if the aggregates M and M_2 are equivalent, then M_1 is equivalent to both M and M_2;

D. If, with two aggregates M and N, N is equivalent neither to M nor to a part of M, there is a part N_1 of N that is equivalent to M;

E. If two aggregates M and N are not equivalent, and there is a part N_1 of N that is equivalent to M, then no part of M is equivalent to N.

[485] § 3

The Addition and Multiplication of Powers

The union of two aggregates M and N which have no common elements was denoted in § 1, (2), by (M, N). We call it the "union-aggregate (*Vereinigungsmenge*) of M and N."

If M′ and N′ are two other aggregates without common elements, and if $M \sim M'$ and $N \sim N'$, we saw that we have

$$(M, N) \sim (M', N').$$

Hence the cardinal number of (M, N) only depends upon the cardinal numbers $\overline{\overline{M}} = \mathfrak{a}$ and $\overline{\overline{N}} = \mathfrak{b}$.

This leads to the definition of the sum of $\mathfrak{a}$ and $\mathfrak{b}$. We put

$$(1) \qquad \mathfrak{a} + \mathfrak{b} = \overline{\overline{(M, N)}}.$$

Since in the conception of power, we abstract from the order of the elements, we conclude at once that

(2) $$\mathfrak{a}+\mathfrak{b}=\mathfrak{b}+\mathfrak{a};$$

and, for any three cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, we have

(3) $$\mathfrak{a}+(\mathfrak{b}+\mathfrak{c})=(\mathfrak{a}+\mathfrak{b})+\mathfrak{c}.$$

We now come to multiplication. Any element m of an aggregate M can be thought to be bound up with any element n of another aggregate N so as to form a new element (m, n); we denote by (M . N) the aggregate of all these bindings (m, n), and call it the "aggregate of bindings (*Verbindungsmenge*) of M and N." Thus

(4) $$(\mathrm{M} \,.\, \mathrm{N})=\{(m, n)\}.$$

We see that the power of (M . N) only depends on the powers $\overline{\overline{\mathrm{M}}}=\mathfrak{a}$ and $\overline{\overline{\mathrm{N}}}=\mathfrak{b}$; for, if we replace the aggregates M and N by the aggregates

$$\mathrm{M}'=\{m'\} \quad \text{and} \quad \mathrm{N}'=\{n'\}$$

respectively equivalent to them, and consider m, m' and n, n' as corresponding elements, then the aggregate

$$(\mathrm{M}' . \mathrm{N}')=\{(m', n')\}$$

is brought into a reciprocal and univocal correspondence with (M . N) by regarding (m, n) and (m', n') as corresponding elements. Thus

(5) $$(\mathrm{M}' . \mathrm{N}') \sim (\mathrm{M} \,.\, \mathrm{N}).$$

We now define the product $\mathfrak{a} . \mathfrak{b}$ by the equation

(6) $$\mathfrak{a} \,.\, \mathfrak{b}=\overline{\overline{(\mathrm{M} \,.\, \mathrm{N})}}.$$

[486] An aggregate with the cardinal number $\mathfrak{a} . \mathfrak{b}$ may also be made up out of two aggregates M and N with the cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ according to the following rule: We start from the aggregate N and replace in it every element n by an aggregate $M_n \sim M$; if, then, we collect the elements of all these aggregates M_n to a whole S, we see that

$$S \sim (M . N), \tag{7}$$

and consequently

$$\overline{\overline{S}} = \mathfrak{a} . \mathfrak{b}.$$

For, if, with any given law of correspondence of the two equivalent aggregates M and M_n, we denote by m the element of M which corresponds to the element m_n of M_n, we have

$$S = \{m_n\}; \tag{8}$$

and thus the aggregates S and (M . N) can be referred reciprocally and univocally to one another by regarding m_n and (m, n) as corresponding elements.

From our definitions result readily the theorems:

$$\mathfrak{a} . \mathfrak{b} = \mathfrak{b} . \mathfrak{a}, \tag{9}$$

$$\mathfrak{a} . (\mathfrak{b} . \mathfrak{c}) = (\mathfrak{a} . \mathfrak{b}) . \mathfrak{c}, \tag{10}$$

$$\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}; \tag{11}$$

because:

$$(M . N) \sim (N . M),$$

$$\big(M . (N . P)\big) \sim \big((M . N) . P\big),$$

$$\big(M . (N, P)\big) \sim \big((M . N), (M . P)\big).$$

Addition and multiplication of powers are subject, therefore, to the commutative, associative, and distributive laws.

§ 4

The Exponentiation of Powers

By a "covering of the aggregate N with elements of the aggregate M," or, more simply, by a "covering of N with M," we understand a law by which with every element n of N a definite element of M is bound up, where one and the same element of M can come repeatedly into application. The element of M bound up with n is, in a way, a one-valued function of n, and may be denoted by $f(n)$; it is called a "covering function of n." The corresponding covering of N will be called $f(\mathrm{N})$.

[487] Two coverings $f_1(\mathrm{N})$ and $f_2(\mathrm{N})$ are said to be equal if, and only if, for all elements n of N the equation

$$(1) \qquad f_1(n)=f_2(n)$$

is fulfilled, so that if this equation does not subsist for even a single element $n=n_0$, $f_1(\mathrm{N})$ and $f_2(\mathrm{N})$ are characterized as different coverings of N. For example, if m_0 is a particular element of M, we may fix that, for all n's

$$f(n)=m_0;$$

this law constitutes a particular covering of N with M. Another kind of covering results if m_0 and m_1 are two different particular elements of M and n_0 a particular element of N, from fixing that

$$f(n_0) = m_0$$
$$f(n) = m_1,$$

for all n's which are different from n_0.

The totality of different coverings of N with M forms a definite aggregate with the elements $f(N)$; we call it the "covering-aggregate (*Belegungsmenge*) of N with M" and denote it by $(N \mid M)$. Thus:

$$(2) \qquad (N \mid M) = \{f(N)\}.$$

If $M \sim M'$ and $N \sim N'$, we easily find that

$$(3) \qquad (N \mid M) \sim (N' \mid M').$$

Thus the cardinal number of $(N \mid M)$ depends only on the cardinal numbers $\overline{\overline{M}} = \mathfrak{a}$ and $\overline{\overline{N}} = \mathfrak{b}$; it serves us for the definition of $\mathfrak{a}^{\mathfrak{b}}$:

$$(4) \qquad \mathfrak{a}^{\mathfrak{b}} = \overline{\overline{(N \mid M)}}.$$

For any three aggregates, M, N, P, we easily prove the theorems:

$$(5) \qquad \big((N \mid M) . (P \mid M)\big) \sim \big((N, P) \mid M\big),$$
$$(6) \qquad \big((P \mid M) . (P \mid N)\big) \sim \big(P \mid (M . N)\big),$$
$$(7) \qquad \big(P \mid (N \mid M)\big) \sim \big((P . N) \mid M\big),$$

from which, if we put $\overline{\overline{P}} = \mathfrak{c}$, we have, by (4) and by paying attention to § 3, the theorems for any three cardinal numbers, $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$:

$$(8) \qquad \mathfrak{a}^{\mathfrak{b}} . \mathfrak{a}^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}+\mathfrak{c}},$$
$$(9) \qquad \mathfrak{a}^{\mathfrak{c}} . \mathfrak{b}^{\mathfrak{c}} = (\mathfrak{a} . \mathfrak{b})^{\mathfrak{c}},$$
$$(10) \qquad (\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b} . \mathfrak{c}}.$$

[488] We see how pregnant and far-reaching these simple formulæ extended to powers are by the following example. If we denote the power of the linear continuum X (that is, the totality X of real numbers x such that $x \geqq$ and $\leqq 1$) by $\mathfrak{o}$, we easily see that it may be represented by, amongst others, the formula:

$$(11) \qquad \mathfrak{o} = 2^{\aleph_0},$$

where § 6 gives the meaning of $\aleph_0$. In fact, by (4), $2^{\aleph_0}$ is the power of all representations

$$(12) \qquad x = \frac{f(1)}{2} + \frac{f(2)}{2^2} + \ldots + \frac{f(\nu)}{2^\nu} + \ldots$$

(where $f(\nu) = 0$ or 1)

of the numbers x in the binary system. If we pay attention to the fact that every number x is only represented once, with the exception of the numbers $x = \frac{2\nu + 1}{2^\mu} < 1$, which are represented twice over, we have, if we denote the "enumerable" totality of the latter by $\{s_\nu\}$,

$$2^{\aleph_0} = \overline{\overline{(\{s_\nu\}, \text{X})}}.$$

If we take away from X any "enumerable" aggregate $\{t_\nu\}$ and denote the remainder by X_1, we have:

$$\text{X} = (\{t_\nu\}, \text{X}_1) = (\{t_{2\nu-1}\}, \{t_{2\nu}\}, \text{X}_1),$$
$$(\{s_\nu\}, \text{X}) = (\{s_\nu\}, \{t_\nu\}, \text{X}_1),$$
$$\{t_{2\nu-1}\} \sim \{s_\nu\}, \quad \{t_{2\nu}\} \sim \{t_\nu\}, \quad \text{X}_1 \sim \text{X}_1;$$

so

$$\text{X} \sim (\{s_\nu\}, \text{X}),$$

and thus (§ 1)

$$2^{\aleph_0} = \overline{\overline{X}} = \mathfrak{c}.$$

From (11) follows by squaring (by § 6, (6))

$$\mathfrak{c} \cdot \mathfrak{c} = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0 + \aleph_0} = 2^{\aleph_0} = \mathfrak{c},$$

and hence, by continued multiplication by $\mathfrak{c}$,

(13) $$\mathfrak{c}^{\nu} = \mathfrak{c},$$

where ν is any finite cardinal number.

If we raise both sides of (11) to the power* $\aleph_0$ we get

$$c^{\aleph_0} \quad \mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0}. \quad = 2^{\aleph_0} = c$$

But since, by § 6, (8), $\aleph_0 \cdot \aleph_0 = \aleph_0$, we have

(14) $$\mathfrak{c}^{\aleph_0} = \mathfrak{c}.$$

The formulæ (13) and (14) mean that both the ν-dimensional and the $\aleph_0$-dimensional continuum have the power of the one-dimensional continuum. Thus the whole contents of my paper in Crelle's *Journal*, vol. lxxxiv, 1878,† are derived purely algebraically with these few strokes of the pen from the fundamental formulæ of the calculation with cardinal numbers.

[489] § 5

The Finite Cardinal Numbers

We will next show how the principles which we have laid down, and on which later on the theory of the actually infinite or transfinite cardinal numbers

* [In English there is an ambiguity.]
† [See Section V of the Introduction.]

will be built, afford also the most natural, shortest, and most rigorous foundation for the theory of finite numbers.

To a single thing e_0, if we subsume it under the concept of an aggregate $E_0 = (e_0)$, corresponds as cardinal number what we call "one" and denote by 1; we have

$$(1) \qquad 1 = \overline{\overline{E}}_0.$$

Let us now unite with E_0 another thing e_1, and call the union-aggregate E_1, so that

$$(2) \qquad E_1 = (E_0, e_1) = (e_0, e_1).$$

The cardinal number of E_1 is called "two" and is denoted by 2:

$$(3) \qquad 2 = \overline{\overline{E}}_1.$$

By addition of new elements we get the series of aggregates

$$E_2 = (E_1, e_2), \quad E_3 = (E_2, e_3), \ldots,$$

which give us successively, in unlimited sequence, the other so-called "finite cardinal numbers" denoted by 3, 4, 5, . . . The use which we here make of these numbers as suffixes is justified by the fact that a number is only used as a suffix when it has been defined as a cardinal number. We have, if by $\nu - 1$ is understood the number immediately preceding ν in the above series,

$$(4) \qquad \nu = \overline{\overline{E}}_{\nu-1},$$

$$(5) \qquad E_\nu = (E_{\nu-1}, e_\nu) = (e_0, e_1, \ldots e_\nu).$$

From the definition of a sum in § 3 follows:

$$(6) \qquad \overline{\overline{E}}_\nu = \overline{\overline{E}}_{\nu-1} + 1;$$

that is to say, every cardinal number, except 1, is the sum of the immediately preceding one and 1.

Now, the following three theorems come into the foreground:

A. The terms of the unlimited series of finite cardinal numbers

$$1, 2, 3, \ldots, \nu, \ldots$$

are all different from one another (that is to say, the condition of equivalence established in § 1 is not fulfilled for the corresponding aggregates).

[490] B. Every one of these numbers ν is greater than the preceding ones and less than the following ones (§ 2).

C. There are no cardinal numbers which, in magnitude, lie between two consecutive numbers ν and $\nu+1$ (§ 2).

We make the proofs of these theorems rest on the two following ones, D and E. We shall, then, in the next place, give the latter theorems rigid proofs.

D. If M is an aggregate such that it is of equal power with none of its parts, then the aggregate (M, e), which arises from M by the addition of a single new element e, has the same property of being of equal power with none of its parts.

E. If N is an aggregate with the finite cardinal number ν, and N_1 is any part of N, the cardinal number of N_1 is equal to one of the preceding numbers 1, 2, 3, . . ., $\nu - 1$.

Proof of D.—Suppose that the aggregate (M, e) is equivalent to one of its parts which we will call N. Then two cases, both of which lead to a contradiction, are to be distinguished:

(*a*) The aggregate N contains e as element; let $N = (M_1, e)$; then M_1 is a part of M because N is a part of (M, e). As we saw in § 1, the law of correspondence of the two equivalent aggregates (M, e) and (M_1, e) can be so modified that the element e of the one corresponds to the same element e of the other; by that, then, M and M_1 are referred reciprocally and univocally to one another. But this contradicts the supposition that M is not equivalent to its part M_1.

(*b*) The part N of (M, e) does not contain e as element, so that N is either M or a part of M. In the law of correspondence between (M, e) and N, which lies at the basis of our supposition, to the element e of the former let the element f of the latter correspond. Let $N = (M_1, f)$; then the aggregate M is put in a reciprocally univocal relation with M_1. But M_1 is a part of N and hence of M. So here too M would be equivalent to one of its parts, and this is contrary to the supposition.

Proof of E.—We will suppose the correctness of the theorem up to a certain ν and then conclude its validity for the number $\nu + 1$ which immediately follows, in the following manner:—We start from the aggregate $E_\nu = (e_0, e_1, \ldots, e_\nu)$ as an aggregate with the cardinal number $\nu+1$. If the theorem is true for this aggregate, its truth for any other aggregate with the same cardinal number $\nu+1$ follows at once by § 1. Let E′ be any part of E_ν; we distinguish the following cases:

(*a*) E′ does not contain e_ν as element, then E is either $E_{\nu-1}$ [491] or a part of $E_{\nu-1}$, and so has as cardinal number either ν or one of the numbers $1, 2, 3, \ldots, \nu-1$, because we supposed our theorem true for the aggregate $E_{\nu-1}$, with the cardinal number ν.

(*b*) E′ consists of the single element e_ν, then $\overline{\overline{E'}}=1$.

(*c*) E′ consists of e_ν and an aggregate E″, so that $E'=(E'', e_\nu)$. E″ is a part of $E_{\nu-1}$ and has therefore by supposition as cardinal number one of the numbers $1, 2, 3, \ldots, \nu-1$. But now $\overline{\overline{E'}}=\overline{\overline{E''}}+1$, and thus the cardinal number of E′ is one of the numbers $2, 3, \ldots, \nu$.

Proof of A.—Every one of the aggregates which we have denoted by E_ν has the property of not being equivalent to any of its parts. For if we suppose that this is so as far as a certain ν, it follows from the theorem D that it is so for the immediately following number $\nu+1$. For $\nu=1$, we recognize at once that the aggregate $E_1=(e_0, e_1)$ is not equivalent to any of its parts, which are here (e_0) and (e_1). Consider, now, any two numbers μ and ν of the series $1, 2, 3, \ldots$; then, if μ is the earlier and ν the later, $E_{\mu-1}$ is a part of $E_{\nu-1}$. Thus $E_{\mu-1}$ and

$E_{\nu-1}$ are not equivalent, and accordingly their cardinal numbers $\mu=\overline{\overline{E}}_{\mu-1}$ and $\nu=\overline{\overline{E}}_{\nu-1}$ are not equal.

Proof of B.—If of the two finite cardinal numbers μ and ν the first is the earlier and the second the later, then $\mu<\nu$. For consider the two aggregates $M=E_{\mu-1}$ and $N=E_{\nu-1}$; for them each of the two conditions in § 2 for $\overline{\overline{M}}<\overline{\overline{N}}$ is fulfilled. The condition (*a*) is fulfilled because, by theorem E, a part of $M=E_{\mu-1}$ can only have one of the cardinal numbers 1, 2, 3, . . ., $\mu-1$, and therefore, by theorem A, cannot be equivalent to the aggregate $N=E_{\nu-1}$. The condition (*b*) is fulfilled because M itself is a part of N.

Proof of C.—Let $\mathfrak{a}$ be a cardinal number which is less than $\nu+1$. Because of the condition (*b*) of § 2, there is a part of E_ν with the cardinal number $\mathfrak{a}$. By theorem E, a part of E_ν can only have one of the cardinal numbers 1, 2, 3, . . ., ν. Thus $\mathfrak{a}$ is equal to one of the cardinal numbers 1, 2, 3, . . ., ν. By theorem B, none of these is greater than ν. Consequently there is no cardinal number $\mathfrak{a}$ which is less than $\nu+1$ and greater than ν.

Of importance for what follows is the following theorem:

F. If K is any aggregate of different finite cardinal numbers, there is one, κ_1, amongst them which is smaller than the rest, and therefore the smallest of all.

[492] *Proof.*—The aggregate K either contains

the number 1, in which case it is the least, $\kappa_1 = 1$, or it does not. In the latter case, let J be the aggregate of all those cardinal numbers of our series, 1, 2, 3, . . ., which are smaller than those occurring in K. If a number ν belongs to J, all numbers less than ν belong to J. But J must have one element ν_1 such that $\nu_1 + 1$, and consequently all greater numbers, do not belong to J, because otherwise J would contain all finite numbers, whereas the numbers belonging to K are not contained in J. Thus J is the segment (*Abschnitt*) $(1, 2, 3, \ldots, \nu_1)$. The number $\nu_1 + 1 = \kappa_1$ is necessarily an element of K and smaller than the rest.

From F we conclude:

G. Every aggregate $K = \{\kappa\}$ of different finite cardinal numbers can be brought into the form of a series

$$K = (\kappa_1, \kappa_2, \kappa_3, \ldots)$$

such that

$$\kappa_1 < \kappa_2 < \kappa_3, \ldots.$$

§ 6

The Smallest Transfinite Cardinal Number Aleph-Zero

Aggregates with finite cardinal numbers are called "finite aggregates," all others we will call "transfinite aggregates" and their cardinal numbers "transfinite cardinal numbers."

The first example of a transfinite aggregate is given by the totality of finite cardinal numbers ν;

we call its cardinal number (§ 1) "Aleph-zero" and denote it by $\aleph_0$; thus we define

(1) $$\aleph_0 = \overline{\overline{\{\nu\}}}.$$

That $\aleph_0$ is a *transfinite* number, that is to say, is not equal to any finite number μ, follows from the simple fact that, if to the aggregate $\{\nu\}$ is added a new element e_0, the union-aggregate $(\{\nu\}, e_0)$ is equivalent to the original aggregate $\{\nu\}$. For we can think of this reciprocally univocal correspondence between them: to the element e_0 of the first corresponds the element 1 of the second, and to the element ν of the first corresponds the element $\nu + 1$ of the other. By § 3 we thus have

(2) $$\aleph_0 + 1 = \aleph_0.$$

But we showed in § 5 that $\mu + 1$ is always different from μ, and therefore $\aleph_0$ is not equal to any finite number μ.

The number $\aleph_0$ is greater than any finite number μ:

(3) $$\aleph_0 > \mu.$$

[493] This follows, if we pay attention to § 3, from the three facts that $\mu = \overline{\overline{(1, 2, 3, \ldots, \mu)}}$, that no part of the aggregate $(1, 2, 3, \ldots, \mu)$ is equivalent to the aggregate $\{\nu\}$, and that $(1, 2, 3, \ldots, \mu)$ is itself a part of $\{\nu\}$.

On the other hand, $\aleph_0$ is the least transfinite cardinal number. If $\mathfrak{a}$ is any transfinite cardinal number different from $\aleph_0$, then

(4) $$\aleph_0 < \mathfrak{a}.$$

This rests on the following theorems:

A. Every transfinite aggregate T has parts with the cardinal number $\aleph_0$.

Proof.—If, by any rule, we have taken away a finite number of elements $t_1, t_2, \ldots, t_{\nu-1}$, there always remains the possibility of taking away a further element t_ν. The aggregate $\{t_\nu\}$, where ν denotes any finite cardinal number, is a part of T with the cardinal number $\aleph_0$, because $\{t_\nu\} \sim \{\nu\}$ (§ 1).

B. If S is a transfinite aggregate with the cardinal number $\aleph_0$, and S_1 is any transfinite part of S, then $\overline{\overline{S}}_1 = \aleph_0$.

Proof.—We have supposed that $S \sim \{\nu\}$. Choose a definite law of correspondence between these two aggregates, and, with this law, denote by s_ν that element of S which corresponds to the element ν of $\{\nu\}$, so that

$$S = \{s_\nu\}.$$

The part S_1 of S consists of certain elements s_κ of S, and the totality of numbers κ forms a transfinite part K of the aggregate $\{\nu\}$. By theorem G of § 5 the aggregate K can be brought into the form of a series

$$K = \{\kappa_\nu\},$$

where

$$\kappa_\nu < \kappa_{\nu+1};$$

consequently we have

$$S_1 = \{s_{\kappa_\nu}\}.$$

Hence follows that $S_1 \sim S$, and therefore $\overline{\overline{S_1}} = \aleph_0$.

From A and B the formula (4) results, if we have regard to § 2.

From (2) we conclude, by adding 1 to both sides,

$$\aleph_0 + 2 = \aleph_0 + 1 = \aleph_0,$$

and, by repeating this

(5) $$\aleph_0 + \nu = \aleph_0.$$

We have also

(6) $$\aleph_0 + \aleph_0 = \aleph_0.$$

[494] For, by (1) of § 3, $\aleph_0 + \aleph_0$ is the cardinal number $\overline{\overline{(\{a_\nu\},\ \{b_\nu\})}}$ because

$$\overline{\overline{\{a_\nu\}}} = \overline{\overline{\{b_\nu\}}} = \aleph_0.$$

Now, obviously

$$\{\nu\} = (\{2\nu - 1\},\ \{2\nu\}),$$
$$(\{2\nu - 1\},\ \{2\nu\}) \sim (\{a_\nu\},\ \{b_\nu\}),$$

and therefore

$$\overline{\overline{(\{a_\nu\},\ \{b_\nu\})}} = \overline{\overline{\{\nu\}}} = \aleph_0.$$

The equation (6) can also be written

$$\aleph_0 \cdot 2 = \aleph_0;$$

and, by adding $\aleph_0$ repeatedly to both sides, we find that

(7) $$\aleph_0 \cdot \nu = \nu \cdot \aleph_0 = \aleph_0.$$

We also have

(8) $$\aleph_0 \cdot \aleph_0 = \aleph_0.$$

Proof.—By (6) of § 3, $\aleph_0 \cdot \aleph_0$ is the cardinal number of the aggregate of bindings

$$\{(\mu, \nu)\},$$

where μ and ν are any finite cardinal numbers which are independent of one another. If also λ represents any finite cardinal number, so that $\{\lambda\}$, $\{\mu\}$, and $\{\nu\}$ are only different notations for the same aggregate of all finite numbers, we have to show that

$$\{(\mu, \nu)\} \sim \{\lambda\}.$$

Let us denote $\mu+\nu$ by ρ; then ρ takes all the numerical values 2, 3, 4, . . ., and there are in all $\rho-1$ elements (μ, ν) for which $\mu+\nu=\rho$, namely:

$$(1, \rho-1), (2, \rho-2), \ldots, (\rho-1, 1).$$

In this sequence imagine first the element (1, 1), for which $\rho=2$, put, then the two elements for which $\rho=3$, then the three elements for which $\rho=4$, and so on. Thus we get all the elements (μ, ν) in a simple series:

$$(1, 1); (1, 2), (2, 1); (1, 3), (2, 2), (3, 1); (1, 4), (2, 3), \ldots,$$

and here, as we easily see, the element (μ, ν) comes at the λth place, where

$$(9) \qquad \lambda = \mu + \frac{(\mu+\nu-1)(\mu+\nu-2)}{2}.$$

The variable λ takes every numerical value 1, 2, 3, . . ., once. Consequently, by means of (9), a reciprocally univocal relation subsists between the aggregates $\{\nu\}$ and $\{(\mu, \nu)\}$.

[495] If both sides of the equation (8) are multiplied by $\aleph_0$, we get $\aleph_0^3 = \aleph_0^2 = \aleph_0$, and, by repeated multiplications by $\aleph_0$, we get the equation, valid for every finite cardinal number ν:

$$(10) \qquad \aleph_0^\nu = \aleph_0.$$

The theorems E and A of § 5 lead to this theorem on finite aggregates:

C. Every finite aggregate E is such that it is equivalent to none of its parts.

This theorem stands sharply opposed to the following one for transfinite aggregates:

D. Every transfinite aggregate T is such that it has parts T_1 which are equivalent to it.

Proof.—By theorem A of this paragraph there is a part $S = \{t_\nu\}$ of T with the cardinal number $\aleph_0$. Let $T = (S, U)$, so that U is composed of those elements of T which are different from the elements t_ν. Let us put $S_1 = \{t_{\nu+1}\}$, $T_1 = (S_1, U)$; then T_1 is a part of T, and, in fact, that part which arises out of T if we leave out the single element t_1. Since $S \sim S_1$, by theorem B of this paragraph, and $U \sim U$, we have, by § 1, $T \sim T_1$.

In these theorems C and D the essential difference between finite and transfinite aggregates, to which I referred in the year 1877, in volume lxxxiv [1878] of Crelle's *Journal*, p. 242, appears in the clearest way.

After we have introduced the least transfinite cardinal number $\aleph_0$ and derived its properties that lie the most readily to hand, the question arises as to the higher cardinal numbers and how they proceed from $\aleph_0$. We shall show that the transfinite cardinal numbers can be arranged according to their magnitude, and, in this order, form, like the finite numbers, a "well-ordered aggregate" in an extended sense of the words. Out of $\aleph_0$ proceeds, by a definite law, the next greater cardinal number $\aleph_1$, out of this by the same law the next greater $\aleph_2$, and so on. But even the unlimited sequence of cardinal numbers

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\nu, \ldots$$

does not exhaust the conception of transfinite cardinal number. We will prove the existence of a cardinal number which we denote by $\aleph_\omega$ and which shows itself to be the next greater to all the numbers $\aleph_\nu$; out of it proceeds in the same way as $\aleph_1$ out of $\aleph$ a next greater $\aleph_{\omega+1}$, and so on, without end.

[496] To every transfinite cardinal number $\mathfrak{a}$ there is a next greater proceeding out of it according to a unitary law, and also to every unlimitedly ascending well-ordered aggregate of transfinite cardinal numbers, $\{\mathfrak{a}\}$, there is a next greater proceeding out of that aggregate in a unitary way.

For the rigorous foundation of this matter, discovered in 1882 and exposed in the pamphlet *Grundlagen einer allgemeinen Mannichfaltigkeitslehre* (Leipzig, 1883) and in volume xxi of the

Mathematische Annalen, we make use of the so-called "ordinal types" whose theory we have to set forth in the following paragraphs.

§ 7

The Ordinal Types of Simply Ordered Aggregates

We call an aggregate M "simply ordered" if a definite "order of precedence" (*Rangordnung*) rules over its elements m, so that, of every two elements m_1 and m_2, one takes the "lower" and the other the "higher" rank, and so that, if of three elements m_1, m_2, and m_3, m_1, say, is of lower rank than m_2, and m_2 is of lower rank than m_3, then m_1 is of lower rank than m_3.

The relation of two elements m_1 and m_2, in which m_1 has the lower rank in the given order of precedence and m_2 the higher, is expressed by the formulæ:

$$(1) \qquad m_1 \prec m_2, \quad m_2 \succ m_1.$$

Thus, for example, every aggregate P of points defined on a straight line is a simply ordered aggregate if, of every two points p_1 and p_2 belonging to it, that one whose co-ordinate (an origin and a positive direction having been fixed upon) is the lesser is given the lower rank.

It is evident that one and the same aggregate can be "simply ordered" according to the most different laws. Thus, for example, with the aggregate R of all positive rational numbers p/q (where p and q are relatively prime integers) which are greater than 0 and less than 1, there is, firstly, their "natural" order according to magnitude; then they can be arranged (and in this order we will denote the aggregate by R_0) so that, of two numbers p_1/q_1 and p_2/q_2 for which the sums p_1+q_1 and p_2+q_2 have different values, that number for which the corresponding sum is less takes the lower rank, and, if $p_1+q_1=p_2+q_2$, then the smaller of the two rational numbers is the lower. [497] In this order of precedence, our aggregate, since to one and the same value of $p+q$ only a finite number of rational numbers p/q belongs, evidently has the form

$$R_0=(r_1, r_2, \ldots, r_\nu, \ldots)=(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{2}{3}, \tfrac{1}{5}, \tfrac{1}{6}, \tfrac{2}{5}, \tfrac{3}{4}, \ldots),$$

where

$$r_\nu \prec r_{\nu+1}.$$

Always, then, when we speak of a "simply ordered" aggregate M, we imagine laid down a definite order or precedence of its elements, in the sense explained above.

There are doubly, triply, ν-ply and α-ply ordered aggregates, but for the present we will not consider them. So in what follows we will use the shorter expression "ordered aggregate" when we mean "simply ordered aggregate."

Every ordered aggregate M has a definite "ordinal type," or more shortly a definite "type," which we will denote by

$$(2) \qquad \overline{M}.$$

By this we understand the general concept which results from M if we only abstract from the nature of the elements m, and retain the order of precedence among them. Thus the ordinal type $\overline{M}$ is itself an ordered aggregate whose elements are units which have the same order of precedence amongst one another as the corresponding elements of M, from which they are derived by abstraction.

We call two ordered aggregates M and N "similar" (*ähnlich*) if they can be put into a bi-univocal correspondence with one another in such a manner that, if m_1 and m_2 are any two elements of M and n_1 and n_2 the corresponding elements of N, then the relation of rank of m_1 to m_2 in M is the same as that of n_1 to n_2 in N. Such a correspondence of similar aggregates we call an "imaging" (*Abbildung*) of these aggregates on one another. In such an imaging, to every part—which obviously also appears as an ordered aggregate—M_1 of M corresponds a similar part N_1 of N.

We express the similarity of two ordered aggregates M and N by the formula:

$$(3) \qquad M \simeq N.$$

Every ordered aggregate is similar to itself.

If two ordered aggregates are similar to a third, they are similar to one another.

[498] A simple consideration shows that two ordered aggregates have the same ordinal type if, and only if, they are similar, so that, of the two formulæ

(4) $$\overline{M} = \overline{N}, \quad M \sim N,$$

one is always a consequence of the other.

If, with an ordinal type $\overline{M}$ we also abstract from the order of precedence of the elements, we get (§ 1) the cardinal number $\overline{\overline{M}}$ of the ordered aggregate M, which is, at the same time, the cardinal number of the ordinal type $\overline{M}$. From $\overline{M} = \overline{N}$ always follows $\overline{\overline{M}} = \overline{\overline{N}}$, that is to say, ordered aggregates of equal types always have the same power or cardinal number; from the similarity of ordered aggregates follows their equivalence. On the other hand, two aggregates may be equivalent without being similar.

We will use the small letters of the Greek alphabet to denote ordinal types. If α is an ordinal type, we understand by

(5) $$\bar{\alpha}$$

its corresponding cardinal number.

The ordinal types of finite ordered aggregates offer no special interest. For we easily convince ourselves that, for one and the same finite cardinal number ν, all simply ordered aggregates are similar to one another, and thus have one and the same type. Thus the finite simple ordinal types are subject to the same laws as the finite cardinal numbers, and it is allowable to use the same signs $1, 2, 3, \ldots, \nu, \ldots$ for them, although they are conceptually different from the cardinal numbers. The case is quite different with the transfinite ordinal types; for to one and the same cardinal number belong innumerably many different types of simply ordered aggregates, which, in their totality, constitute a particular "class of types" (*Typenclasse*). Every one of these classes of types is, therefore, determined by the transfinite cardinal number $\mathfrak{a}$ which is common to all the types belonging to the class. Thus we call it for short the class of types $[\mathfrak{a}]$. That class which naturally presents itself first to us, and whose complete investigation must, accordingly, be the next special aim of the theory of transfinite aggregates, is the class of types $[\aleph_0]$ which embraces all the types with the least transfinite cardinal number $\aleph_0$. From the cardinal number which *determines* the class of types $[\mathfrak{a}]$ we have to distinguish that cardinal number $\mathfrak{a}'$ which for its part **[499]** *is determined by* the class of types $[\mathfrak{a}]$. The latter is the cardinal number which (§ 1) the class $[\mathfrak{a}]$ has, in so far as it represents a well-defined aggregate whose elements are all the types α with the cardinal number $\mathfrak{a}$. We will see that $\mathfrak{a}'$ is different from $\mathfrak{a}$, and indeed always greater than $\mathfrak{a}$.

If in an ordered aggregate M all the relations of precedence of its elements are inverted, so that "lower" becomes "higher" and "higher" becomes "lower" everywhere, we again get an ordered aggregate, which we will denote by

$$^*M \tag{6}$$

and call the "inverse" of M. We denote the ordinal type of *M, if $\alpha = \overline{M}$, by

$$^*\alpha. \tag{7}$$

It may happen that $^*\alpha = \alpha$, as, for example, in the case of finite types or in that of the type of the aggregate of all rational numbers which are greater than 0 and less than 1 in their natural order of precedence. This type we will investigate under the notation η.

We remark further that two similarly ordered aggregates can be imaged on one another either in one manner or in many manners; in the first case the type in question is similar to itself in only one way, in the second case in many ways. Not only all finite types, but the types of transfinite "well-ordered aggregates," which will occupy us later and which we call transfinite "ordinal numbers," are such that they allow only a single imaging on themselves. On the other hand, the type η is similar to itself in an infinity of ways.

We will make this difference clear by two simple examples. By ω we understand the type of a well-ordered aggregate

$$(e_1, e_2, \ldots, e_\nu, \ldots),$$

in which

$$e_\nu \prec e_{\nu+1},$$

and where ν represents all finite cardinal numbers in turn. Another well-ordered aggregate

$$(f_1, f_2, \ldots, f_\nu, \ldots),$$

with the condition

$$f_\nu \prec f_{\nu+1},$$

of the same type ω can obviously only be imaged on the former in such a way that e_ν and f_ν are corresponding elements. For e_1, the lowest element in rank of the first, must, in the process of imaging, be correlated to the lowest element f_1 of the second, the next after e_1 in rank (e_2) to f_2, the next after f_1, and so on. **[500]** Every other bi-univocal correspondence of the two equivalent aggregates $\{e_\nu\}$ and $\{f_\nu\}$ is not an "imaging" in the sense which we have fixed above for the theory of types.

On the other hand, let us take an ordered aggregate of the form

$$\{e_\nu\},$$

where ν represents all positive and negative finite integers, including 0, and where likewise

$$e_\nu \prec e_{\nu+1}.$$

This aggregate has no lowest and no highest element in rank. Its type is, by the definition of a sum given in § 8,

$$^*\omega + \omega.$$

It is similar to itself in an infinity of ways. For let us consider an aggregate of the same type

$$\{f_{\nu'}\},$$

where

$$f_{\nu'} \prec f_{\nu'+1}.$$

Then the two ordered aggregates can be so imaged on one another that, if we understand by ν_0' a definite one of the numbers ν', to the element $e_{\nu'}$ of the first the element $f_{\nu_0'+\nu'}$ of the second corresponds. Since ν_0' is arbitrary, we have here an infinity of imagings.

The concept of "ordinal type" developed here, when it is transferred in like manner to "multiply ordered aggregates," embraces, in conjunction with the concept of "cardinal number" or "power" introduced in § 1, everything capable of being numbered (*Anzahlmässige*) that is thinkable, and in this sense cannot be further generalized. It contains nothing arbitrary, but is the natural extension of the concept of number. It deserves to be especially emphasized that the criterion of equality (4) follows with absolute necessity from the concept of ordinal type and consequently permits of no alteration. The chief cause of the grave errors in G. Veronese's *Grundzüge der Geometrie* (German by A. Schepp, Leipzig, 1894) is the non-recognition of this point.

On page 30 the "number (*Anzahl oder Zahl*) of an ordered group" is defined in exactly the same way as what we have called the "ordinal type of a simply ordered aggregate" (*Zur Lehre vom Transfiniten*, Halle, 1890, pp. 68–75; reprinted from the *Zeitschr. für Philos. und philos. Kritik* for 1887). **[501]** But Veronese thinks that he must make an addition to the criterion of equality. He says on page 31: "Numbers whose units correspond to one another uniquely and in the same order and of which the one is neither a part of the other nor equal to a part of the other are equal."* This definition of equality contains a circle and thus is meaningless. For what is the meaning of "not equal to a part of the other" in this addition? To answer this question, we must first know when two numbers are equal or unequal. Thus, apart from the arbitrariness of his definition of equality, it presupposes a definition of equality, and this again presupposes a definition of equality, in which we must know again what equal and unequal are, and so on *ad infinitum*. After Veronese has, so to speak, given up of his own free will the indispensable foundation for the comparison of numbers, we ought not to be surprised at the lawlessness with which, later on, he operates with his pseudo-transfinite numbers, and ascribes properties to them which they cannot possess simply because they themselves, in the form imagined by him, have no existence except on paper. Thus, too, the striking similarity of his "numbers" to the very absurd "infinite numbers" in Fontenelle's *Géométrie de l'Infini* (Paris, 1727) becomes comprehensible. Recently, W. Killing has given welcome expression to his doubts concerning the foundation of Veronese's book in the *Index lectionum* of the Münster Academy for 1895–1896.†

* In the original Italian edition (p. 27) this passage runs: "Numeri le unità dei quali si corrispondono univocamente e nel medesimo ordine, e di cui l' uno non è parte o uguale ad una parte dell' altro, sono uguali."

† [Veronese replied to this in *Math. Ann.*, vol. xlvii, 1897, pp. 423–432. *Cf.* Killing, *ibid.*, vol. xlviii, 1897, pp. 425–432.]

§ 8

Addition and Multiplication of Ordinal Types

The union-aggregate (M, N) of two aggregates M and N can, if M and N are ordered, be conceived as an ordered aggregate in which the relations of precedence of the elements of M among themselves as well as the relations of precedence of the elements of N among themselves remain the same as in M or N respectively, and all elements of M have a lower rank than all the elements of N. If M′ and N′ are two other ordered aggregates, $M \simeq M'$ and $N \simeq N'$, [502] then $(M, N) \simeq (M', N')$; so the ordinal type of (M, N) depends only on the ordinal types $\overline{M} = \alpha$ and $\overline{N} = \beta$. Thus, we define:

$$(1) \qquad \alpha + \beta = \overline{(M, N)}.$$

In the sum $\alpha + \beta$ we call α the "augend" and β the "addend."

For any three types we easily prove the associative law:

$$(2) \qquad \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma.$$

On the other hand, the commutative law is not valid, in general, for the addition of types. We see this by the following simple example.

If ω is the type, already mentioned in § 7, of the well-ordered aggregate

$$E = (e_1, e_2, \ldots e_\nu \ldots), \quad e_\nu \prec e_{\nu+1},$$

then $1+\omega$ is not equal to $\omega+1$. For, if f is a new element, we have by (1):

$$1+\omega=\overline{(f, \mathrm{E})},$$
$$\omega+1=\overline{(\mathrm{E}, f)}.$$

But the aggregate

$$(f, \mathrm{E})=(f, e_1, e_2, \ldots, e_\nu, \ldots)$$

is similar to the aggregate E, and consequently

$$1+\omega=\omega.$$

On the contrary, the aggregates E and (E, f) are not similar, because the first has no term which is highest in rank, but the second has the highest term f. Thus $\omega+1$ is different from $\omega=1+\omega$.

Out of two ordered aggregates M and N with the types α and β we can set up an ordered aggregate S by substituting for every element n of N an ordered aggregate M_n which has the same type α as M, so that

$$\text{(3)} \qquad \overline{\mathrm{M}}_n=\alpha;$$

and, for the order of precedence in

$$\text{(4)} \qquad \mathrm{S}=\{\mathrm{M}_n\}$$

we make the two rules:

(1) Every two elements of S which belong to one and the same aggregate M_n are to retain in S the same order of precedence as in M_n;

(2) Every two elements of S which belong to two different aggregates M_{n_1} and M_{n_2} have the same relation of precedence as n_1 and n_2 have in N.

The ordinal type of S depends, as we easily see, only on the types α and β; we define

(5) $$\alpha \cdot \beta = \overline{S}.$$

[503] In this product α is called the "multiplicand" and β the "multiplier."

In any definite imaging of M on M_n let m_n be the element of M_n that corresponds to the element m of M; we can then also write

(6) $$S = \{m_n\}.$$

Consider a third ordered aggregate $P = \{p\}$ with the ordinal type $\overline{P} = \gamma$, then, by (5),

$$\alpha \cdot \beta = \overline{\{m_n\}}, \quad \beta \cdot \gamma = \overline{\{n_p\}}, \quad (\alpha \cdot \beta) \cdot \gamma = \overline{\{(m_n)_p\}},$$
$$\alpha \cdot (\beta \cdot \gamma) = \overline{\{m_{(n_p)}\}}.$$

But the two ordered aggregates $\{(m_n)_p\}$ and $\{m_{(n_p)}\}$ are similar, and are imaged on one another if we regard the elements $(m_n)_p$ and $m_{(n_p)}$ as corresponding. Consequently, for three types α, β, and γ the associative law

(7) $$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

subsists. From (1) and (5) follows easily the distributive law

(8) $$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma;$$

but only in this form, where the factor with two terms is the multiplier.

On the contrary, in the multiplication of types as in their addition, the commutative law is not generally valid. For example, $2 . \omega$ and $\omega . 2$ are different types; for, by (5),

$$2 . \omega = \overline{(e_1, f_1; \ e_2, f_2; \ \ldots; \ e_\nu, f_\nu; \ \ldots)} = \omega;$$

while

$$\omega . 2 = \overline{(e_1, e_2, \ldots, e_\nu, \ldots; \ f_1, f_2, \ldots, f_\nu, \ldots)}$$

is obviously different from ω.

If we compare the definitions of the elementary operations for cardinal numbers, given in § 3, with those established here for ordinal types, we easily see that the cardinal number of the sum of two types is equal to the sum of the cardinal numbers of the single types, and that the cardinal number of the product of two types is equal to the product of the cardinal numbers of the single types. Every equation between ordinal types which proceeds from the two elementary operations remains correct, therefore, if we replace in it all the types by their cardinal numbers.

[504] § 9

The Ordinal Type η of the Aggregate R of all Rational Numbers which are Greater than 0 and Smaller than 1, in their Natural Order of Precedence

By R we understand, as in § 7, the system of all rational numbers p/q (p and q being relatively prime) which >0 and <1, in their natural order of precedence, where the magnitude of a number determines its rank. We denote the ordinal type of R by η:

(1) $$\eta = \overline{R}.$$

But we have put the same aggregate in another order of precedence in which we call it R_0. This order is determined, in the first place, by the magnitude of $p+q$, and in the second place—for rational numbers for which $p+q$ has the same value—by the magnitude of p/q itself. The aggregate R_0 is a well-ordered aggregate of type ω:

(2) $R_0 = (r_1, r_2, \ldots, r_\nu, \ldots)$, where $r_\nu < r_{\nu+1}$,

(3) $\overline{R}_0 = \omega$.

Both R and R_0 have the same cardinal number since they only differ in the order of precedence of their elements, and, since we obviously have $\overline{\overline{R}}_0 = \aleph_0$, we also have

(4) $$\overline{\overline{R}} = \overline{\eta} = \aleph_0.$$

Thus the type η belongs to the class of types $[\aleph_0]$.

Secondly, we remark that in R there is neither an element which is lowest in rank nor one which is highest in rank. Thirdly, R has the property that between every two of its elements others lie. This property we express by the words: R is "everywhere dense" (*überalldicht*).

We will now show that these three properties characterize the type η of R, so that we have the following theorem:

If we have a simply ordered aggregate M such that

(*a*) $\overline{\overline{M}} = \aleph_0$;

(*b*) M has no element which is lowest in rank, and no highest;

(*c*) M is everywhere dense;

then the ordinal type of M is η:

$$\overline{M} = \eta.$$

Proof.—Because of the condition (*a*), M can be brought into the form [505] of a well-ordered aggregate of type ω; having fixed upon such a form, we denote it by M_0 and put

$$(5) \qquad M_0 = (m_1, m_2, \ldots, m_\nu, \ldots).$$

We have now to show that

$$(6) \qquad M \simeq R;$$

that is to say, we must prove that M can be imaged on R in such a way that the relation of precedence of any and every two elements in M is the same as that of the two corresponding elements in R.

Let the element r_1 in R be correlated to the element m_1 in M. The element r_2 has a definite relation of precedence to r_1 in R. Because of the condition (*b*), there are infinitely many elements m_ν of M which have the same relation of precedence in M to m_1 as r_2 to r_1 in R; of them we choose that one which has the smallest index in M_0, let it be m_{ι_2} and correlate it to r_2. The element r_3 has in R definite relations of precedence to r_1 and r_2; because of the conditions (*b*) and (*c*) there is an infinity of elements m_ν of M which have the same relation of precedence to m_1 and m_{ι_2} in M as r_3 to r_1 and r_2 to R; of them we choose that—let it be m_{ι_3}—which has the smallest index in M_0, and correlate it to r_3. According to this law we imagine the process of correlation continued. If to the ν elements

$$r_1, r_2, r_3, \ldots, r_\nu$$

of R are correlated, as images, definite elements

$$m_1, m_{\iota_2}, m_{\iota_3}, \ldots, m_{\iota_\nu},$$

which have the same relations of precedence amongst one another in M as the corresponding elements in R, then to the element $r_{\nu+1}$ of R is to be correlated that element $m_{\iota_{\nu+1}}$ of M which has the smallest index in M_0 of those which have the same relations of precedence to

$$m_1, m_{\iota_2}, m_{\iota_3}, \ldots, m_{\iota_\nu}$$

in M as $r_{\nu+1}$ to $r_1, r_2, \ldots, r_\nu$ in R.

In this manner we have correlated definite elements m_{ι_ν} of M to all the elements r_ν of R, and the elements m_{ι_ν} have in M the same order of precedence as the corresponding elements r_ν in R. But we have still to show that the elements m_{ι_ν} include *all* the elements m_ν of M, or, what is the same thing, that the series

$$1, \iota_2, \iota_3, \ldots, \iota_\nu, \ldots$$

[506] is only a permutation of the series

$$1, 2, 3, \ldots \nu, \ldots$$

We prove this by a complete induction: we will show that, if the elements $m_1, m_2, \ldots, m_\nu$ appear in the imaging, that is also the case with the following element $m_{\nu+1}$.

Let λ be so great that, among the elements

$$m_1, m_{\iota_2}, m_{\iota_3}, \ldots, m_{\iota_\lambda},$$

the elements

$$m_1, m_2, \ldots, m_\nu,$$

which, by supposition, appear in the imaging, are contained. It may be that also $m_{\nu+1}$ is found among them; then $m_{\nu+1}$ appears in the imaging. But if $m_{\nu+1}$ is not among the elements

$$m_1, m_{\iota_2}, m_{\iota_3}, \ldots, m_{\iota_\lambda},$$

then $m_{\nu+1}$ has with respect to these elements a definite ordinal position in M; infinitely many elements in R have the same ordinal position in R with respect to $r_1, r_2, \ldots, r_\lambda$, amongst which let $r_{\lambda+\sigma}$ be that with the least index in R_0. Then $m_{\nu+1}$ has, as we can easily make sure, the same ordinal position with respect to

$$m_1, m_{\iota_2}, m_{\iota_3}, \ldots, m_{\iota_{\lambda+\sigma-1}}$$

in M as $r_{\lambda+\sigma}$ has with respect to

$$r_1, r_2, \ldots, r_{\lambda+\sigma-1}$$

in R. Since $m_1, m_2, \ldots, m_\nu$ have already appeared in the imaging, $m_{\nu+1}$ is that element with the smallest index in M which has this ordinal position with respect to

$$m_1, m_{\iota_2}, \ldots, m_{\iota_{\lambda+\sigma-1}}.$$

Consequently, according to our law of correlation,

$$m_{\iota_{\lambda+\sigma}} = m_{\nu+1}.$$

Thus, in this case too, the element $m_{\nu+1}$ appears in the imaging, and $r_{\lambda+\sigma}$ is the element of R which is correlated to it.

We see, then, that by our manner of correlation, the *whole aggregate* M is imaged on the *whole aggregate* R; M and R are similar aggregates, which was to be proved.

From the theorem which we have just proved result, for example, the following theorems:

[507] The ordinal type of the aggregate of all negative and positive rational numbers, including zero, in their natural order of precedence, is η.

The ordinal type of the aggregate of all rational numbers which are greater than a and less than b, in their natural order of precedence, where a and b are any real numbers, and $a < b$, is η.

The ordinal type of the aggregate of all real algebraic numbers in their natural order of precedence is η.

The ordinal type of the aggregate of all real algebraic numbers which are greater than a and less than b, in their natural order of precedence, where a and b are any real numbers and $a < b$, is η.

For all these ordered aggregates satisfy the three conditions required in our theorem for M (see Crelle's *Journal*, vol. lxxvii, p. 258).*

If we consider, further, aggregates with the types —according to the definitions given in § 8—written

[* *Cf.* Section V of the Introduction.]

$\eta + \eta$, $\eta\eta$, $(1 + \eta)\eta$, $(\eta + 1)\eta$, $(1 + \eta + 1)\eta$, we find that those three conditions are also fulfilled with them. Thus we have the theorems:

(7) $$\eta + \eta = \eta,$$

(8) $$\eta\eta = \eta,$$

(9) $$(1 + \eta)\eta = \eta,$$

(10) $$(\eta + 1)\eta = \eta,$$

(11) $$(1 + \eta + 1)\eta = \eta.$$

The repeated application of (7) and (8) gives for every finite number ν:

(12) $$\eta \cdot \nu = \eta,$$

(13) $$\eta^{\nu} = \eta.$$

On the other hand we easily see that, for $\nu > 1$, the types $1 + \eta$, $\eta + 1$, $\nu . \eta$, $1 + \eta + 1$ are different both from one another and from η. We have

(14) $$\eta + 1 + \eta = \eta,$$

but $\eta + \nu + \eta$, for $\nu > 1$, is different from η.

Finally, it deserves to be emphasized that

(15) $$^{*}\eta = \eta.$$

[508] § 10

The Fundamental Series contained in a Transfinite Ordered Aggregate

Let us consider any simply ordered transfinite aggregate M. Every part of M is itself an ordered aggregate. For the study of the type $\overline{M}$, those parts of M which have the types ω and $^*\omega$ appear to be especially valuable; we call them "fundamental series of the first order contained in M," and the former—of type ω—we call an "ascending" series, the latter—of type $^*\omega$—a "descending" one. Since we limit ourselves to the consideration of fundamental series of the first order (in later investigations fundamental series of higher order will also occupy us), we will here simply call them "fundamental series." Thus an "ascending fundamental series" is of the form

$$(1) \qquad \{a_\nu\}, \quad \text{where} \quad a_\nu \prec a_{\nu+1};$$

a "descending fundamental series" is of the form

$$(2) \qquad \{b_\nu\}, \quad \text{where} \quad b_\nu \succ b_{\nu+1}.$$

The letter ν, as well as κ, λ, and μ, has everywhere in our considerations the signification of an arbitrary finite cardinal number or of a finite type (a finite ordinal number).

We call two ascending fundamental series $\{a_\nu\}$ and $\{a'_\nu\}$ in M "coherent" (*zusammengehörig*), in signs

$$(3) \qquad \{a_\nu\} \parallel \{a'_\nu\},$$

if, for every element a_ν there are elements a'_λ such that

$$a_\nu \prec a'_\lambda,$$

and also for every element a'_ν there are elements a_μ such that

$$a'_\nu \prec a_\mu.$$

Two descending fundamental series $\{b_\nu\}$ and $\{b'_\nu\}$ in M are said to be "coherent," in signs

(4) $$\{b_\nu\} \parallel \{b'_\nu\},$$

if for every element b_ν there are elements b'_λ such that

$$b_\nu > b'_\lambda,$$

and for every element b'_ν there are elements b_μ such that

$$b'_\nu > b_\mu.$$

An ascending fundamental series $\{a_\nu\}$ and a descending one $\{b_\nu\}$ are said to be "coherent," in signs

[509] (5) $$\{a_\nu\} \parallel \{b_\nu\},$$

if (*a*) for all values of ν and μ

$$a_\nu < b_\mu,$$

and (*b*) in M exists at most one (thus either only one or none at all) element m_0 such that, for all ν's,

$$a_\nu < m_0 < b_\nu.$$

Then we have the theorems:

A. If two fundamental series are coherent to a third, they are also coherent to one another.

B. Two fundamental series proceeding in the same direction of which one is part of the other are coherent.

If there exists in M an element m_0 which has such a position with respect to the ascending fundamental series $\{a_\nu\}$ that:

(*a*) for every ν

$$a_\nu \prec m_0,$$

(*b*) for every element m of M that precedes m_0 there exists a certain number ν_0 such that

$$a_\nu \succ m, \quad \text{for} \quad \nu \geqq \nu_0,$$

then we will call m_0 a "limiting element (*Grenzelement*) of $\{a_\nu\}$ in M" and also a "principal element (*Hauptelement*) of M." In the same way we call m_0 a "principal element of M" and also "limiting element of $\{b_\nu\}$ in M" if these conditions are satisfied:

(*a*) for every ν

$$b_\nu \succ m_0,$$

(*b*) for every element m of M that follows m_0 exists a certain number ν_0 such that

$$b_\nu \succ m, \quad \text{for} \quad \nu \geqq \nu_0.$$

A fundamental series can never have more than one limiting element in M; but M has, in general, many principal elements.

We perceive the truth of the following theorems:

C. If a fundamental series has a limiting element in M, all fundamental series coherent to it have the same limiting element in M.

D. If two fundamental series (whether proceeding in the same or in opposite directions) have one and the same limiting element in M, they are coherent.

If M and M′ are two similarly ordered aggregates, so that

$$(6) \qquad \overline{M} = \overline{M}',$$

and we fix upon any imaging of the two aggregates, then we easily see that the following theorems hold:

[510] E. To every fundamental series in M corresponds as image a fundamental series in M′, and inversely; to every ascending series an ascending one, and to every descending series a descending one; to coherent fundamental series in M correspond as images coherent fundamental series in M′, and inversely.

F. If to a fundamental series in M belongs a limiting element in M, then to the corresponding fundamental series in M′ belongs a limiting element in M′, and inversely; and these two limiting elements are images of one another in the imaging.

G. To the principal elements of M correspond as images principal elements of M′, and inversely.

If an aggregate M consists of principal elements, so that every one of its elements is a principal element, we call it an "aggregate which is dense in itself (*insichdichte Menge*)." If to every fundamental series in M there is a limiting element in M, we call M a "closed (*abgeschlossene*) aggregate." An aggregate which is both "dense in itself" and "closed" is called a "perfect aggregate." If an aggregate has one of these three predicates, every similar aggregate has the same predicate; thus these predicates can also be ascribed to the corresponding ordinal types, and so there are "types which are dense in themselves," "closed types," "perfect types," and also "everywhere-dense types" (§ 9).

For example, η is a type which is "dense in itself," and, as we showed in § 9, it is also "everywhere-dense," but it is not "closed." The types ω and $^*\omega$ have no principal elements, but $\omega+\nu$ and $\nu+{}^*\omega$ each have a principal element, and are "closed" types. The type $\omega . 3$ has two principal elements, but is not "closed"; the type $\omega . 3+\nu$ has three principal elements, and is "closed."

§ 11

The Ordinal Type θ of the Linear Continuum X

We turn to the investigation of the ordinal type of the aggregate $X=\{x\}$ of all real numbers x, such that $x \geqq 0$ and $\leqq 1$, in their natural order of precedence, so that, with any two of its elements x and x',

$$x \prec x', \quad \text{if} \quad x < x'.$$

Let the notation for this type be

$$\overline{X}=\theta. \tag{1}$$

[511] From the elements of the theory of rational and irrational numbers we know that every fundamental series $\{x_\nu\}$ in X has a limiting element x_0 in X, and that also, inversely, every element x of X

is a limiting element of coherent fundamental series in X. Consequently X is a "perfect aggregate" and θ is a "perfect type."

But θ is not sufficiently characterized by that; besides that we must fix our attention on the following property of X. The aggregate X contains as part the aggregate R of ordinal type η investigated in § 9, and in such a way that, between any two elements x_0 and x_1 of X, elements of R lie.

We will now show that these properties, taken together, characterize the ordinal type θ of the linear continuum X in an exhaustive manner, so that we have the theorem:

If an ordered aggregate M is such that (*a*) it is "perfect," and (*b*) in it is contained an aggregate S with the cardinal number $\overline{\overline{S}} = \aleph_0$ and which bears such a relation to M that, between any two elements m_0 and m_1 of M elements of S lie, then $\overline{M} = \theta$.

Proof.—If S had a lowest or a highest element, these elements, by (*b*), would bear the same character as elements of M; we could remove them from S without S losing thereby the relation to M expressed in (*b*). Thus, we suppose that S is without lowest or highest element, so that, by § 9, it has the ordinal type η. For since S is a part of M, between any two elements s_0 and s_1 of S other elements of S must, by (*b*), lie. Besides, by (*b*) we have $\overline{\overline{S}} = \aleph_0$. Thus the aggregates S and R are "similar" to one another.

$$(2) \qquad S \simeq R.$$

We fix on any "imaging" of R on S, and assert that it gives a definite "imaging" of X on M in the following manner:

Let all elements of X which, at the same time, belong to the aggregate R correspond as images to those elements of M which are, at the same time, elements of S and, in the supposed imaging of R on S, correspond to the said elements of R. But if x_0 is an element of X which does not belong to R, x_0 may be regarded as a limiting element of a fundamental series $\{x_\nu\}$ contained in X, and this series can be replaced by a coherent fundamental series $\{r_{\kappa_\nu}\}$ contained in R. To this [512] corresponds as image a fundamental series $\{s_{\lambda_\nu}\}$ in S and M, which, because of (a), is limited by an element m_0 of M that does not belong to S (F, § 10). Let this element m_0 of M (which remains the same, by E, C, and D of § 10, if the fundamental series $\{x_\nu\}$ and $\{r_{\kappa_\nu}\}$ are replaced by others limited by the same element x_0 in X) be the image of x_0 in X. Inversely, to every element m_0 of M which does not occur in S belongs a quite definite element x_0 of X which does not belong to R and of which m_0 is the image.

In this manner a bi-univocal correspondence between X and M is set up, and we have now to show that it gives an "imaging" of these aggregates.

This is, of course, the case for those elements of X which belong to R, and for those elements of M which belong to S. Let us compare an element r of R with an element x_0 of X which does not belong to R; let the corresponding elements of M be s and m_0. If $r < x_0$, there is an ascending fundamental series $\{r_{\kappa_\nu}\}$, which is limited by x_0 and, from a certain ν_0 on,

$$r < r_{\kappa_\nu} \quad \text{for} \quad \nu \geqq \nu_0.$$

The image of $\{r_{\kappa_\nu}\}$ in M is an ascending fundamental series $\{s_{\lambda_\nu}\}$, which will be limited by an m_0 of M, and we have (§ 10) $s_{\lambda_\nu} \prec m_0$ for every ν, and $s \prec s_{\lambda_\nu}$ for $\nu \geqq \nu_0$. Thus (§ 7) $s \prec m_0$.

If $r > x_0$, we conclude similarly that $s \succ m_0$.

Let us consider, finally, two elements x_0 and x'_0 not belonging to R and the elements m_0 and m'_0 corresponding to them in M; then we show, by an analogous consideration, that, if $x_0 < x'_0$, then $m_0 \prec m'_0$.

The proof of the similarity of X and M is now finished, and we thus have

$$\overline{\mathrm{M}} = \theta.$$

HALLE, *March* 1895.

[207] CONTRIBUTIONS TO THE FOUNDING OF THE THEORY OF TRANSFINITE NUMBERS

(SECOND ARTICLE)

§ 12

Well-Ordered Aggregates

AMONG simply ordered aggregates "well-ordered aggregates" deserve a special place; their ordinal types, which we call "ordinal numbers," form the natural material for an exact definition of the higher transfinite cardinal numbers or powers,—a definition which is throughout conformable to that which was given us for the least transfinite cardinal number Aleph-zero by the system of all finite numbers ν (§ 6).

We call a simply ordered aggregate F (§ 7) "well-ordered" if its elements f ascend in a definite succession from a lowest f_1 in such a way that:

I. There is in F an element f_1 which is lowest in rank.

II. If F′ is any part of F and if F has one or many elements of higher rank than all elements of F′, then there is an element f' of F which follows immediately after the totality F′, so that no elements in rank between f' and F′ occur in F.*

In particular, to every single element f of F, if it is not the highest, follows in rank as next higher another definite element f'; this results from the condition II if for F′ we put the single element f. Further, if, for example, an infinite series of consecutive elements

$$e' \prec e'' \prec e''' \prec \ldots \prec e^{(\nu)} \prec e^{(\nu+1)} \ldots$$

is contained in F in such a way, however, that there are also in F elements of [208] higher rank than all elements $e^{(\nu)}$, then, by the second condition, putting for F′ the totality $\{e^{(\nu)}\}$, there must exist an element f' such that not only

$$f' \succ e^{(\nu)}$$

for all values of ν, but that also there is no element g in F which satisfies the two conditions

$$g \prec f',$$
$$g \succ e^{(\nu)}$$

for all values of ν.

Thus, for example, the three aggregates

$$(a_1, a_2, \ldots, a_\nu, \ldots),$$
$$(a_1, a_2, \ldots, a_\nu, \ldots, b_1, b_2 \ldots, b_\mu, \ldots),$$
$$(a_1, a_2, \ldots, a_\nu, \ldots b_1, b_2, \ldots, b_\mu, \ldots c_1, c_2, c_3),$$

where

$$a_\nu \prec a_{\nu+1} \prec b_\mu \prec b_{\mu+1} \prec c_1 \prec c_2 \prec c_3,$$

* This definition of "well-ordered aggregates," apart from the wording, is identical with that which was introduced in vol. xxi of the *Math. Ann.*, p. 548 (*Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, p. 4). [See Section VII of the Introduction.]

are well-ordered. The two first have no highest element, the third has the highest element c_3; in the second and third b_1 immediately follows all the elements a_ν, in the third c_1 immediately follows all the elements a_ν and b_μ.

In the following we will extend the use of the signs $\prec$ and $\succ$, explained in § 7, and there used to express the ordinal relation of two elements, to groups of elements, so that the formulæ

$$M \prec N,$$
$$M \succ N$$

are the expression for the fact that in a given order all the elements of the aggregate M have a lower, or higher, respectively, rank than all elements of the aggregate N.

A. Every part F_1 of a well-ordered aggregate F has a lowest element.

Proof.—If the lowest element f_1 of F belongs to F_1, then it is also the lowest element of F_1. In the other case, let F′ be the totality of all elements of F which have a lower rank than all elements F_1, then, for this reason, no element of F lies between F′ and F_1. Thus, if f' follows (II) next after F′, then it belongs necessarily to F_1 and here takes the lowest rank.

B. If a simply ordered aggregate F is such that both F and every one of its parts have a lowest element, then F is a well-ordered aggregate.

[209] *Proof.*—Since F has a lowest element, the condition I is satisfied. Let F′ be a part of F

such that there are in F one or more elements which follow F′; let F_1 be the totality of all these elements and f' the lowest element of F_1, then obviously f' is the element of F which follows next to F′. Consequently, the condition II is also satisfied, and therefore F is a well-ordered aggregate.

C. Every part F′ of a well-ordered aggregate F is also a well-ordered aggregate.

Proof.—By theorem A, the aggregate F′ as well as every part F″ of F′ (since it is also a part of F) has a lowest element; thus by theorem B, the aggregate F′ is well-ordered.

D. Every aggregate G which is similar to a well-ordered aggregate F is also a well-ordered aggregate.

Proof.—If M is an aggregate which has a lowest element, then, as immediately follows from the concept of similarity (§ 7), every aggregate N similar to it has a lowest element. Since, now, we are to have $G \simeq F$, and F has, since it is a well-ordered aggregate, a lowest element, the same holds of G. Thus also every part G′ of G has a lowest element; for in an imaging of G on F, to the aggregate G′ corresponds a part F′ of F as image, so that

$$G' \simeq F'.$$

But, by theorem A, F′ has a lowest element, and therefore also G′ has. Thus, both G and every part of G have lowest elements. By theorem B, consequently, G is a well-ordered aggregate.

E. If in a well-ordered aggregate G, in place of its elements g well-ordered aggregates are substituted in such a way that, if F_g and $F_{g'}$ are the well-ordered aggregates which occupy the places of the elements g and g' and $g \prec g'$, then also $F_g \prec F_{g'}$, then the aggregate H, arising by combination in this manner of the elements of all the aggregates F_g, is well-ordered.

Proof.—Both H and every part H_1 of H have lowest elements, and by theorem B this characterizes H as a well-ordered aggregate. For, if g_1 is the lowest element of G, the lowest element of F_{g_1} is at the same time the lowest element of H. If, further, we have a part H_1 of H, its elements belong to definite aggregates F_g which form, when taken together, a part of the well-ordered aggregate $\{F_g\}$, which consists of the elements F_g and is similar to the aggregate G. If, say, F_{g_0} is the lowest element of this part, then the lowest element of the part of H_1 contained in F_{g_0} is at the same time the lowest element of H_1.

[210] § 13

The Segments of Well-Ordered Aggregates

If f is any element of the well-ordered aggregate F which is different from the initial element f_1, then we will call the aggregate A of all elements of F which precede f a "segment (*Abschnitt*) of F," or, more fully, "the segment of F which is defined by the element f." On the other hand, the aggregate R of all the other elements of F, including f, is a "remainder of F," and, more fully, "the remainder which is determined by the element f." The aggregates A and R are, by theorem C of § 12, well-ordered, and we may, by § 8 and § 12, write:

$$(1) \qquad F=(A, R),$$

$$(2) \qquad R=(f, R'),$$

$$(3) \qquad A \prec R.$$

R′ is the part of R which follows the initial element f and reduces to 0 if R has, besides f, no other element.

For example, in the well-ordered aggregate

$$F=(a_1, a_2, \ldots, a_\nu, \ldots b_1, b_2, \ldots, b_\mu, \ldots c_1, c_2, c_3),$$

the segment

$$(a_1, a_2)$$

and the corresponding remainder

$$(a_3, a_4, \ldots a_{\nu+2}, \ldots b_1, b_2, \ldots b_\mu, \ldots c_1, c_2, c_3)$$

are determined by the element a_3; the segment

$$(a_1, a_2, \ldots, a_\nu, \ldots)$$

and the corresponding remainder

$$(b_1, b_2, \ldots, b_\mu, \ldots c_1, c_2, c_3)$$

are determined by the element b_1; and the segment

$$(a_1, a_2, \ldots, a_\nu, \ldots b_1, b_2, \ldots, b_\mu, \ldots c_1)$$

and the corresponding ~~segment~~ remainder

$$(c_2, c_3)$$

by the element c_2.

If A and A′ are two segments of F, f and f' their determining elements, and

$$(4) \qquad f' < f,$$

then A′ is a segment of A. We call A′ the "less," and A the "greater" segment of F:

$$(5) \qquad A' < A.$$

Correspondingly we may say of every A of F that it is "less" than F itself:

$$A < F.$$

[211] A. If two similar well-ordered aggregates F and G are imaged on one another, then to every segment A of F corresponds a similar segment B of G, and to every segment B of G corresponds a similar segment A of F, and the elements f and g of F and G by which the corresponding segments A and B are determined also correspond to one another in the imaging.

Proof.—If we have two similar simply ordered aggregates M and N imaged on one another, m and n are two corresponding elements, and M′ is the aggregate of all elements of M which precede m and N′ is the aggregate of all elements of N which precede n, then in the imaging M′ and N′ correspond to one another. For, to every element m' of M that precedes m must correspond, by § 7, an element

n' of N that precedes n, and inversely. If we apply this general theorem to the well-ordered aggregates F and G we get what is to be proved.

B. A well-ordered aggregate F is not similar to any of its segments A.

Proof.—Let us suppose that $F \simeq A$, then we will imagine an imaging of F on A set up. By theorem A the segment A' of A corresponds to the segment A of F, so that $A' \simeq A$. Thus also we would have $A' \simeq F$ and $A' < A$. From A' would result, in the same manner, a smaller segment A'' of F, such that $A'' \simeq F$ and $A'' < A'$; and so on. Thus we would obtain an infinite series

$$A > A' > A'' \dots A^{(\nu)} > A^{(\nu+1)} \dots$$

of segments of F, which continually become smaller and all similar to the aggregate F. We will denote by $f, f', f'', \dots, f^{(\nu)}, \dots$ the elements of F which determine these segments; then we would have

$$f \succ f' \succ f'' \succ \dots \succ f^{(\nu)} \succ f^{(\nu+1)} \dots$$

We would therefore have an infinite part

$$(f, f', f'', \dots, f^{(\nu)}, \dots)$$

of F in which no element takes the lowest rank. But by theorem A of § 12 such parts of F are not possible. Thus the supposition of an imaging F on one of its segments leads to a contradiction, and consequently the aggregate F is not similar to any of its segments.

Though by theorem B a well-ordered aggregate F is not similar to any of its segments, yet, if F is infinite, there are always **[212]** other parts of F to which F is similar. Thus, for example, the aggregate

$$F = (a_1, a_2, \ldots, a_\nu, \ldots)$$

is similar to every one of its remainders

$$(a_{\kappa+1}, a_{\kappa+2}, \ldots, a_{\kappa+\nu}, \ldots).$$

Consequently, it is important that we can put by the side of theorem B the following:

C. A well-ordered aggregate F is similar to no part of any one of its segments A.

Proof.—Let us suppose that F′ is a part of a segment A of F and $F' \simeq F$. We imagine an imaging of F on F′; then, by theorem A, to a segment A of the well-ordered aggregate F corresponds as image the segment F″ of F′; let this segment be determined by the element f' of F′. The element f' is also an element of A, and determines a segment A′ of A of which F″ is a part. The supposition of a part F′ of a segment A of F such that $F' \simeq F$ leads us consequently to a part F″ of a segment A′ of A such that $F'' \simeq A$. The same manner of conclusion gives us a part F‴ of a segment A″ of A′ such that $F''' \simeq A'$. Proceeding thus, we get, as in the proof of theorem B, an infinite series of segments of F which continually become smaller:

$$A > A' > A'' \ldots A^{(\nu)} > A^{(\nu+1)} \ldots,$$

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and thus an infinite series of elements determining these segments:

$$f > f' > f'' \ldots f^{(\nu)} > f^{(\nu+1)} \ldots,$$

in which is no lowest element, and this is impossible by theorem A of § 12. Thus there is no part F′ of a segment A of F such that $F' \underline{\backsim} F$.

D. Two different segments A and A′ of a well-ordered aggregate F are not similar to one another.

Proof.—If $A' < A$, then A′ is a segment of the well-ordered aggregate A, and thus, by theorem B, cannot be similar to A.

E. Two similar well-ordered aggregates F and G can be imaged on one another only in a single manner.

Proof.—Let us suppose that there are two different imagings of F on G, and let *f* be an element of F to which in the two imagings different images *g* and *g′* in G correspond. Let A be the segment of F that is determined by *f*, and B and B′ the segments of G that are determined by *g* and *g′*. By theorem A, both $A \underline{\backsim} B$ **[213]** and $A \underline{\backsim} B'$, and consequently $B \underline{\backsim} B'$, contrary to theorem D.

F. If F and G are two well-ordered aggregates, a segment A of F can have at most one segment B in G which is similar to it.

Proof.—If the segment A of F could have two segments B and B′ in G which were similar to it, B and B′ would be similar to one another, which is impossible by theorem D.

G. If A and B are similar segments of two well-ordered aggregates F and G, for every smaller segment $A' < A$ of F there is a similar segment $B' < B$ of G and for every smaller segment $B' < B$ of G a similar segment $A' < A$ of F.

The proof follows from theorem A applied to the similar aggregates A and B.

H. If A and A' are two segments of a well-ordered aggregate F, B and B' are two segments similar to those of a well-ordered aggregate G, and $A' < A$, then $B' < B$.

The proof follows from the theorems F and G.

I. If a segment B of a well-ordered aggregate G is similar to no segment of a well-ordered aggregate F, then both every segment $B' > B$ of D and G itself are similar neither to a segment of F nor F itself.

The proof follows from theorem G.

K. If for any segment A of a well-ordered aggregate F there is a similar segment B of another well-ordered aggregate G, and also inversely, for every segment B of G a similar segment A of F, then $F \simeq G$.

Proof.—We can image F and G on one another according to the following law: Let the lowest element f_1 of F correspond to the lowest element g_1 of G. If $f \succ f_1$ is any other element of F, it determines a segment A of F. To this segment belongs by supposition a definite similar segment B of G, and let the element g of G which determines the segment B be the image of F. And if g is any element of G that follows g_1, it determines a segment B of G, to which by supposition a similar segment A of F belongs. Let the element f which determines this segment A be the image of g. It easily follows that the bi-univocal correspondence of F and G defined in this manner is an imaging in the sense of § 7. For if f and f' are any two elements of F, g and g' **[214]** the corresponding elements of G, A and A′ the segments determined by f and f', B and B′ those determined by g and g', and if, say,

$$f' \prec f,$$

then

$$A' < A.$$

By theorem H, then, we have

$$B' < B,$$

and consequently

$$g' \prec g.$$

L. If for every segment A of a well-ordered aggregate F there is a similar segment B of another well-ordered aggregate G, but if, on the other hand, there is at least one segment of G for which there is no similar segment of F, then there exists a definite segment B_1 of G such that $B_1 \simeq F$.

Proof.—Consider the totality of segments of G for which there are no similar segments in F. Amongst them there must be a least segment which we will call B_1. This follows from the fact that, by theorem A of § 12, the aggregate of all the elements determining these segments has a lowest element; the segment B_1 of G determined by that element is the least of that totality. By theorem I, every segment of G which is greater than B_1 is such that no segment similar to it is present in F. Thus the segments B of G which correspond to similar segments of F must all be less than B_1, and to every segment $B < B_1$ belongs a similar segment A of F, because B_1 is the least segment of G among those to which no similar segments in F correspond. Thus, for every segment A of F there is a similar segment B of B_1, and for every segment B of B_1 there is a similar segment A of F. By theorem K, we thus have

$$F \simeq B_1.$$

M. If the well-ordered aggregate G has at least one segment for which there is no similar segment in the well-ordered aggregate F, then every segment A of F must have a segment B similar to it in G.

Proof.—Let B_1 be the least of all those segments of G for which there are no similar segments in F.* If there were segments in F for which there were no corresponding segments in G, amongst these, one, which we will call A_1, would be the least. For every segment of A_1 would then exist a similar segment of B_1, and also for every segment of B_1 a similar segment of A_1. Thus, by theorem K, we would have

$$B_1 \simeq A_1.$$

[215] But this contradicts the datum that for B_1 there is no similar segment of F. Consequently, there cannot be in F a segment to which a similar segment in G does not correspond.

* See the above proof of L.

N. If F and G are any two well-ordered aggregates, then either :

(*a*) F and G are similar to one another, or

(*b*) there is a definite segment B_1 of G to which F is similar, or

(*c*) there is a definite segment A_1 of F to which G is similar ;

and each of these three cases excludes the two others.

Proof.—The relation of F to G can be any one of the three :

(*a*) To every segment A of F there belongs a similar segment B of G, and inversely, to every segment B of G belongs a similar one A of F ;

(*b*) To every segment A of F belongs a similar segment B of G, but there is at least one segment of G to which no similar segment in F corresponds;

(*c*) To every segment B of G belongs a similar segment A of F, but there is at least one segment of F to which no similar segment in G corresponds.

The case that there is both a segment of F to which no similar segment in G corresponds and a segment of G to which no similar segment in F corresponds is not possible ; it is excluded by theorem M.

By theorem K, in the first case we have

$$F \simeq G.$$

In the second case there is, by theorem L, a definite segment B_1 of B such that

$$B_1 \simeq F ;$$

and in the third case there is a definite segment A_1 of F such that

$$A_1 \simeq G.$$

We cannot have $F \simeq G$ and $F \simeq B_1$ simultaneously, for then we would have $G \simeq B_1$, contrary to theorem B; and, for the same reason, we cannot have both $F \simeq G$ and $G \simeq A_1$. Also it is impossible that both $F \simeq B_1$ and $G \simeq A_1$, for, by theorem A, from $F \simeq B_1$ would follow the existence of a segment B'_1 of B_1 such that $A_1 \simeq B'_1$. Thus we would have $G \simeq B'_1$, contrary to theorem B.

O. If a part F' of a well-ordered aggregate F is not similar to any segment of F, it is similar to F itself.

Proof.—By theorem C of § 12, F' is a well-ordered aggregate. If F' were similar neither to a segment of F nor to F itself, there would be, by theorem N, a segment F'_1 of F' which is similar to F. But F'_1 is a part of that segment A of F which **[216]** is determined by the same element as the segment F'_1 of F'. Thus the aggregate F would have to be similar to a part of one of its segments, and this contradicts the theorem C.

§ 14

The Ordinal Numbers of Well-Ordered Aggregates

By § 7, every simply ordered aggregate M has a definite ordinal type $\overline{M}$; this type is the general concept which results from M if we abstract from the nature of its elements while retaining their order of precedence, so that out of them proceed units (*Einsen*) which stand in a definite relation of precedence to one another. All aggregates which are similar to one another, and only such, have one and the same ordinal type. We call the ordinal type of a well-ordered aggregate F its "ordinal number."

If α and β are any two ordinal numbers, one can stand to the other in one of three possible relations. For if F and G are two well-ordered aggregates such that

$$\overline{F} = \alpha, \quad \overline{G} = \beta,$$

then, by theorem N of § 13, three mutually exclusive cases are possible:

(*a*) $$F \simeq G;$$

(*b*) There is a definite segment B_1 of G such that

$$F \simeq B_1;$$

(*c*) There is a definite segment A_1 of F such that

$$G \simeq A_1.$$

As we easily see, each of these cases still subsists if F and G are replaced by aggregates respectively similar to them. Accordingly, we have to do with three mutually exclusive relations of the types α and β to one another. In the first case $\alpha = \beta$; in the second we say that $\alpha < \beta$; in the third we say that $\alpha > \beta$. Thus we have the theorem:

A. If α and β are any two ordinal numbers, we have either $\alpha=\beta$ or $\alpha<\beta$ or $\alpha>\beta$.

From the definition of minority and majority follows easily :

B. If we have three ordinal numbers α, β, γ, and if $\alpha<\beta$ and $\beta<\gamma$, then $\alpha<\gamma$.

Thus the ordinal numbers form, when arranged in order of magnitude, a simply ordered aggregate ; it will appear later that it is a well-ordered aggregate.

[217] The operations of addition and multiplication of the ordinal types of any simply ordered aggregates, defined in § 8, are, of course, applicable to the ordinal numbers. If $\alpha=\overline{F}$ and $\beta=\overline{G}$, where F and G are two well-ordered aggregates, then

$$(1) \qquad \alpha+\beta=\overline{(F, G)}.$$

The aggregate of union (F, G) is obviously a well-ordered aggregate too ; thus we have the theorem :

C. The sum of two ordinal numbers is also an ordinal number.

In the sum $\alpha+\beta$, α is called the "augend" and β the "addend."

Since F is a segment of (F, G), we have always

$$(2) \qquad \alpha<\alpha+\beta.$$

On the other hand, G is not a segment but a remainder of (F, G), and may thus, as we saw in § 13, be similar to the aggregate (F, G). If this is not the case, G is, by theorem O of § 13, similar to a segment of (F, G). Thus

(3) $$\beta \leqq \alpha + \beta.$$

Consequently we have :

D. The sum of the two ordinal numbers is always greater than the augend, but greater than or equal to the addend. If we have $\alpha+\beta=\alpha+\gamma$, we always have $\beta=\gamma$.

In general $\alpha+\beta$ and $\beta+\alpha$ are not equal. On the other hand, we have, if γ is a third ordinal number,

(4) $$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma).$$

That is to say :

E. In the addition of ordinal numbers the associative law always holds.

If we substitute for every element g of the aggregate G of type β an aggregate F_g of type α, we get, by theorem E of § 12, a well-ordered aggregate H whose type is completely determined by the types α and β and will be called the product $\alpha.\beta$:

(5) $$\overline{F}_g=\alpha,$$

(6) $$\alpha.\beta=\overline{H}.$$

F. The product of two ordinal numbers is also an ordinal number.

In the product $\alpha.\beta$, α is called the "multiplicand" and β the "multiplier."

In general $\alpha.\beta$ and $\beta.\alpha$ are not equal. But we have (§ 8)

(7) $$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

That is to say:

[218] G. In the multiplication of ordinal numbers the associative law holds.

The distributive law is valid, in general (§ 8), only in the following form:

(8) $$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma.$$

With reference to the magnitude of the product, the following theorem, as we easily see, holds:

H. If the multiplier is greater than 1, the product of two ordinal numbers is always greater than the multiplicand, but greater than or equal to the multiplier. If we have $\alpha . \beta = \alpha . \gamma$, then it always follows that $\beta = \gamma$.

On the other hand, we evidently have

(9) $$\alpha \cdot 1 = 1 \cdot \alpha = \alpha.$$

We have now to consider the operation of subtraction. If α and β are two ordinal numbers, and α is less than β, there always exists a definite ordinal number which we will call $\beta - \alpha$, which satisfies the equation

(10) $$\alpha + (\beta - \alpha) = \beta.$$

For if $\overline{G} = \beta$, G has a segment B such that $\overline{B} = \alpha$; we call the corresponding remainder S, and have

$$G = (B, S),$$

$$\beta = \alpha + \overline{S};$$

and therefore

$$(11) \qquad \beta - \alpha = \overline{\overline{S}}.$$

The determinateness of $\beta - \alpha$ appears clearly from the fact that the segment B of G is a completely definite one (theorem D of § 13), and consequently also S is uniquely given.

We emphasize the following formulæ, which follow from (4), (8), and (10):

$$(12) \qquad (\gamma + \beta) - (\gamma + \alpha) = \beta - \alpha,$$

$$(13) \qquad \gamma(\beta - \alpha) = \gamma\beta - \gamma\alpha.$$

It is important to reflect that an infinity of ordinal numbers can be summed so that their sum is a definite ordinal number which depends on the sequence of the summands. If

$$\beta_1, \beta_2, \ldots, \beta_\nu, \ldots$$

is any simply infinite series of ordinal numbers, and we have

$$(14) \qquad \beta_\nu = \overline{G}_\nu,$$

[219] then, by theorem E of § 12,

$$(15) \qquad G = (G_1, G_2, \ldots, G_\nu, \ldots)$$

is also a well-ordered aggregate whose ordinal number represents the sum of the numbers β_ν. We have, then,

$$(16) \qquad \beta_1 + \beta_2 + \ldots + \beta_\nu + \ldots = \overline{G} = \beta,$$

and, as we easily see from the definition of a product, we always have

(17) $$\gamma \cdot (\beta_1 + \beta_2 + \ldots + \beta_\nu + \ldots) = \gamma \cdot \beta_1 + \gamma \cdot \beta_2 + \ldots + \gamma \cdot \beta_\nu + \ldots$$

If we put

(18) $$\alpha_\nu = \beta_1 + \beta_2 + \ldots + \beta_\nu,$$

then

(19) $$\alpha_\nu = \overline{(G_1, G_2, \ldots G_\nu)}.$$

We have

(20) $$\alpha_{\nu+1} > \alpha_\nu,$$

and, by (10), we can express the numbers β_ν by the numbers α_ν as follows:

(21) $$\beta_1 = \alpha_1; \quad \beta_{\nu+1} = \alpha_{\nu+1} - \alpha_\nu.$$

The series

$$\alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots$$

thus represents *any* infinite series of ordinal numbers which satisfy the condition (20); we will call it a "fundamental series" of ordinal numbers (§ 10). Between it and β subsists a relation which can be expressed in the following manner:

(*a*) The number β is greater than α_ν for every ν, because the aggregate $(G_1, G_2, \ldots, G_\nu)$, whose ordinal number is α_ν, is a segment of the aggregate G which has the ordinal number β;

(*b*) If β' is any ordinal number less than β, then, from a certain ν onwards, we always have

$$\alpha_\nu > \beta'.$$

For, since $\beta' < \beta$, there is a segment B′ of the aggregate G which is of type β'. The element of G which determines this segment must belong to one of the parts G_ν; we will call this part G_{ν_0}. But then B′ is also a segment of $(G_1, G_2, \ldots, G_{\nu_0})$, and consequently $\beta' < \alpha_{\nu_0}$. Thus

$$\alpha_\nu > \beta'$$

for $\nu \geqq \nu_0$.

Thus β is the ordinal number which follows next in order of magnitude after all the numbers α_ν; accordingly we will call it the "limit" (*Grenze*) of the numbers α_ν for increasing ν and denote it by $\operatorname*{Lim}_\nu \alpha_\nu$, so that, by (16) and (21):

$$(22)\quad \operatorname*{Lim}_\nu \alpha_\nu = \alpha_1 + (\alpha_2 - \alpha_1) + \ldots + (\alpha_{\nu+1} - \alpha_\nu) + \ldots$$

[220] We may express what precedes in the following theorem:

I. To every fundamental series $\{\alpha_\nu\}$ of ordinal numbers belongs an ordinal number $\operatorname*{Lim}_\nu \alpha_\nu$ which follows next, in order of magnitude, after all the numbers α_ν; it is represented by the formula (22).

If by γ we understand any constant ordinal number, we easily prove, by the aid of the formulæ (12), (13), and (17), the theorems contained in the formulæ:

$$(23)\qquad \operatorname*{Lim}_\nu (\gamma + \alpha_\nu) = \gamma + \operatorname*{Lim}_\nu \alpha_\nu;$$

$$(24)\qquad \operatorname*{Lim}_\nu \gamma \cdot \alpha_\nu = \gamma \cdot \operatorname*{Lim}_\nu \alpha_\nu.$$

We have already mentioned in § 7 that all simply ordered aggregates of given finite cardinal number ν have one and the same ordinal type. This may be proved here as follows. Every simply ordered aggregate of finite cardinal number is a well-ordered aggregate; for it, and every one of its parts, must have a lowest element,—and this, by theorem B of § 12, characterizes it as a well-ordered aggregate. The types of finite simply ordered aggregates are thus none other than finite ordinal numbers. But two different ordinal numbers α and β cannot belong to the same finite cardinal number ν. For if, say, $\alpha < \beta$ and $\overline{G} = \beta$, then, as we know, there exists a segment B of G such that $\overline{B} = \alpha$. Thus the aggregate G and its part B would have the same finite cardinal number ν. But this, by theorem C of § 6, is impossible. Thus the finite ordinal numbers coincide in their properties with the finite cardinal numbers.

The case is quite different with the transfinite ordinal numbers; to one and the same transfinite cardinal number $\mathfrak{a}$ belong an infinity of ordinal numbers which form a unitary and connected system. We will call this system the "number-class $Z(\mathfrak{a})$," and it is a part of the class of types $[\mathfrak{a}]$ of § 7. The next object of our consideration is the number-class $Z(\aleph_0)$, which we will call "the second number-class." For in this connexion we understand by "the first number-class" the totality $\{\nu\}$ of finite ordinal numbers.

[221] § 15

The Numbers of the Second Number-Class $Z(\aleph_0)$

The second number-class $Z(\aleph_0)$ is the totality $\{\alpha\}$ of ordinal types α of well-ordered aggregates of the cardinal number $\aleph_0$ (§ 6).

A. The second number-class has a least number $\omega = \operatorname*{Lim}_{\nu} \nu$.

Proof.—By ω we understand the type of the well-ordered aggregate

$$(1) \qquad F_0 = (f_1, f_2, \ldots, f_\nu, \ldots),$$

where ν runs through all finite ordinal numbers and

$$(2) \qquad f_\nu \prec f_{\nu+1}.$$

Therefore (§ 7)

$$(3) \qquad \omega = \overline{F}_0,$$

and (§ 6)

$$(4) \qquad \overline{\overline{\omega}} = \aleph_0.$$

Thus ω is a number of the second number-class, and indeed the least. For if γ is any ordinal number less than ω, it must (§ 14) be the type of a segment of F_0. But F_0 has only segments

$$A = (f_1, f_2, \ldots, f_\nu),$$

with *finite* ordinal number ν. Thus $\gamma = \nu$. Therefore there are no transfinite ordinal numbers which are less than ω, and thus ω is the least of them. By the definition of $\operatorname*{Lim}_{\nu} \alpha_\nu$ given in § 14, we obviously have $\omega = \operatorname*{Lim}_{\nu} \nu$.

B. If α is any number of the second number-class, the number $\alpha+1$ follows it as the next greater number of the same number-class.

Proof.—Let F be a well-ordered aggregate of the type α and of the cardinal number $\aleph_0$:

$$\overline{F}=\alpha, \tag{5}$$

$$\overline{\alpha}=\aleph_0. \tag{6}$$

We have, where by g is understood a new element,

$$\alpha+1=\overline{(F, g)}. \tag{7}$$

Since F is a segment of (F, g), we have

$$\alpha+1>\alpha. \tag{8}$$

We also have

$$\overline{\alpha+1}=\overline{\alpha}+1=\aleph_0+1=\aleph_0 \quad (\S\ 6).$$

Therefore the number $\alpha+1$ belongs to the second number-class. Between α and $\alpha+1$ there are no ordinal numbers; for every number γ **[222]** which is less than $\alpha+1$ corresponds, as type, to a segment of (F, g), and such a segment can only be either F or a segment of F. Therefore γ is either equal to or less than α.

C. If $\alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots$ is any fundamental series of numbers of the first or second number-class, then the number $\operatorname*{Lim}_\nu \alpha_\nu$ (§ 14) following them next in order of magnitude belongs to the second number-class.

Proof.—By § 14 there results from the fundamental series $\{\alpha_\nu\}$ the number $\operatorname*{Lim}_\nu \alpha_\nu$ if we set up another series $\beta_1, \beta_2, \ldots, \beta_\nu, \ldots$, where

$$\beta_1 = \alpha_1,\ \beta_2 = \alpha_2 - \alpha_1,\ \ldots,\ \beta_{\nu+1} = \alpha_{\nu+1} - \alpha_\nu,\ \ldots$$

If, then, $G_1, G_2, \ldots, G_\nu, \ldots$ are well-ordered aggregates such that

$$\overline{G}_\nu = \beta_\nu,$$

then also

$$G = (G_1, G_2, \ldots, G_\nu, \ldots)$$

is a well-ordered aggregate and

$$\operatorname*{Lim}_\nu \alpha_\nu = \overline{G}.$$

It only remains to prove that

$$\overline{\overline{G}} = \aleph_0.$$

Since the numbers $\beta_1, \beta_2, \ldots, \beta_\nu, \ldots$ belong to the first or second number-class, we have

$$\overline{\overline{G}}_\nu \leqq \aleph_0,$$

and thus

$$\overline{\overline{G}} \leqq \aleph_0 \cdot \aleph_0 = \aleph_0.$$

But, in any case, G is a transfinite aggregate, and so the case $\overline{\overline{G}} < \aleph_0$ is excluded.

We will call two fundamental series $\{\alpha_\nu\}$ and $\{\alpha'_\nu\}$ of numbers of the first or second number-class (§ 10) "coherent," in signs :

$$(9) \qquad \{\alpha_\nu\} \parallel \{\alpha'_\nu\},$$

if for every ν there are finite numbers λ_0 and μ_0 such that

$$(10) \qquad \alpha'_\lambda > \alpha_\nu, \quad \lambda \geqq \lambda_0,$$

and

(11) $$\alpha_\mu > \alpha'_\nu, \quad \mu \geqq \mu_0.$$

[223] D. The limiting numbers $\operatorname*{Lim}_\nu \alpha_\nu$ and $\operatorname*{Lim}_\nu \alpha'_\nu$ belonging respectively to two fundamental series $\{\alpha_\nu\}$ and $\{\alpha'_\nu\}$ are equal when, and only when, $\{\alpha_\nu\} \parallel \{\alpha'_\nu\}$.

Proof. — For the sake of shortness we put $\operatorname*{Lim}_\nu \alpha_\nu = \beta$, $\operatorname*{Lim}_\nu \alpha'_\nu = \gamma$. We will first suppose that $\{\alpha_\nu\} \parallel \{\alpha'_\nu\}$; then we assert that $\beta = \gamma$. For if β were not equal to γ, one of these two numbers would have to be the smaller. Suppose that $\beta < \gamma$. From a certain ν onwards we would have $\alpha'_\nu > \beta$ (§ 14), and consequently, by (11), from a certain μ onwards we would have $\alpha_\mu > \beta$. But this is impossible because $\beta = \operatorname*{Lim}_\nu \alpha_\nu$. Thus for all μ's we have $\alpha_\mu < \beta$.

If, inversely, we suppose that $\beta = \gamma$, then, because $\alpha_\nu < \gamma$, we must conclude that, from a certain λ onwards, $\alpha'_\lambda > \alpha_\nu$, and, because $\alpha'_\nu < \beta$, we must conclude that, from a certain μ onwards, $\alpha_\mu > \alpha'_\nu$. That is to say, $\{\alpha_\nu\} \parallel \{\alpha'_\nu\}$.

E. If a is any number of the second number-class and ν_0 any finite ordinal number, we have $\nu_0 + a = a$, and consequently also $a - \nu_0 = a$.

Proof.—We will first of all convince ourselves of the correctness of the theorem when $a = \omega$. We have

$$\omega = (\overline{f_1, f_2, \ldots, f_\nu, \ldots}),$$

$$\nu_0 = (\overline{g_1, g_2, \ldots g_{\nu_0}}),$$

and consequently

$$\nu_0+\omega=\overline{(g_1, g_2, \ldots g_{\nu_0}, f_1, f_2, \ldots, f_\nu, \ldots)}=\omega.$$

But if $\alpha>\omega$, we have

$$\alpha=\omega+(\alpha-\omega),$$

$$\nu_0+\alpha=(\nu_0+\omega)+(\alpha-\omega)=\omega+(\alpha-\omega)=\alpha.$$

F. If ν_0 is any finite ordinal number, we have $\nu_0 \cdot \omega=\omega$.

Proof.—In order to obtain an aggregate of the type $\nu_0 \cdot \omega$ we have to substitute for the single elements f_ν of the aggregate $(f_1, f_2, \ldots, f_\nu, \ldots)$ aggregates $(g_{\nu,1}, g_{\nu,2}, \ldots, g_{\nu,\nu_0})$ of the type ν_0. We thus obtain the aggregate

$$(g_{1,1}, g_{1,2}, \ldots, g_{1,\nu_0}, g_{2,1}, \ldots, g_{2,\nu_0}, \ldots, g_{\nu,1}, g_{\nu,2}, \ldots, g_{\nu,\nu_0}, \ldots),$$

which is obviously similar to the aggregate $\{f_\nu\}$. Consequently

$$\nu_0 \cdot \omega=\omega.$$

The same result is obtained more shortly as follows. By (24) of § 14 we have, since $\omega=\operatorname*{Lim}_\nu \nu$,

$$\nu_0\,\omega=\operatorname*{Lim}_\nu \nu_0\,\nu.$$

On the other hand,

$$\{\nu_0\,\nu\} \parallel \{\nu\},$$

and consequently

$$\operatorname*{Lim}_\nu \nu_0\,\nu=\operatorname*{Lim}_\nu \nu=\omega\,;$$

so that

$$\nu_0\,\omega=\omega.$$

[224] G. We have always

$$(\alpha+\nu_0)\omega=\alpha\omega,$$

where α is a number of the second number-class and ν_0 a number of the first number-class.

Proof.—We have

$$\operatorname*{Lim}_{\nu}\nu=\omega.$$

By (24) of § 14 we have, consequently,

$$(\alpha+\nu_0)\omega=\operatorname*{Lim}_{\nu}(\alpha+\nu_0)\nu.$$

But

$$\begin{aligned}(\alpha+\nu_0)\nu &= \overbrace{(\alpha+\nu_0)}^{1}+\overbrace{(\alpha+\nu_0)}^{2}+\ldots+\overbrace{(\alpha+\nu_0)}^{\nu}\\ &=\alpha+\overbrace{(\nu_0+\alpha)}^{1}+\overbrace{(\nu_0+\alpha)}^{2}\ldots\overbrace{(\nu_0+\alpha)}^{\nu-1}+\nu_0\\ &=\overbrace{\alpha}^{1}+\overbrace{\alpha}^{2}+\ldots+\overbrace{\alpha}^{\nu}+\nu_0\\ &=\alpha\nu+\nu_0.\end{aligned}$$

Now we have, as is easy to see,

$$\{\alpha\nu+\nu_0\}\parallel\{\alpha\nu\},$$

and consequently

$$\operatorname*{Lim}_{\nu}(\alpha+\nu_0)\nu=\operatorname*{Lim}_{\nu}(\alpha\nu+\nu_0)=\operatorname*{Lim}_{\nu}\alpha\nu=\alpha\omega.$$

H. If α is any number of the second number-class, then the totality $\{\alpha'\}$ of numbers α' of the first and second number-classes which are less than α form, in their order of magnitude, a well-ordered aggregate of type α.

Proof.—Let F be a well-ordered aggregate such that $\overline{\mathrm{F}}=\alpha$, and let f_1 be the lowest element of F. If α' is any ordinal number which is less than α, then, by § 14, there is a definite segment A′ of F such that

$$\overline{\mathrm{A}'}=\alpha',$$

and inversely every segment A′ determines by its type $\overline{\mathrm{A}'}=\alpha'$ a number $\alpha'<\alpha$ of the first or second number-class. For, since $\overline{\overline{\mathrm{F}}}=\aleph_0$, $\overline{\overline{\mathrm{A}'}}$ can only be either a finite cardinal number or $\aleph_0$. The segment A′ is determined by an element $f' \succ f_1$ of F, and inversely every element $f' \succ f_1$ of F determines a segment A′ of F. If f' and f'' are two elements of F which follow f_1 in rank, A′ and A″ are the segments of F determined by them, α' and α'' are their ordinal types, and, say $f' \prec f''$, then, by § 13, $\mathrm{A}'<\mathrm{A}''$ and consequently $\alpha'<\alpha''$. **[225]** If, then, we put $\mathrm{F}=(f_1, \mathrm{F}')$, we obtain, when we make the element f' of F′ correspond to the element α' of $\{\alpha'\}$, an imaging of these two aggregates. Thus we have

$$\overline{\{\alpha'\}}=\overline{\mathrm{F}'}.$$

But $\overline{\mathrm{F}'}=\alpha-1$, and, by theorem E, $\alpha-1=\alpha$. Consequently

$$\overline{\{\alpha'\}}=\alpha.$$

Since $\bar{\bar{\alpha}}=\aleph_0$, we also have $\overline{\overline{\{\alpha'\}}}=\aleph_0$; thus we have the theorems:

I. The aggregate $\{\alpha'\}$ of numbers α' of the first and second number-classes which are smaller than a number α of the second number-class has the cardinal number $\aleph_0$.

K. Every number α of the second number-class is either such that (a) it arises out of the next smaller number α_{-1} by the addition of 1:

$$\alpha = \alpha_{-1} + 1,$$

or (b) there is a fundamental series $\{\alpha_\nu\}$ of numbers of the first or second number-class such that

$$\alpha = \operatorname*{Lim}_\nu \alpha_\nu.$$

Proof.—Let $\alpha = \overline{\overline{F}}$. If F has an element g which is highest in rank, we have $F = (A, g)$, where A is the segment of F which is determined by g. We have then the first case, namely,

$$\alpha = \overline{\overline{A}} + 1 = \alpha_{-1} + 1.$$

There exists, therefore, a next smaller number which is that called α_{-1}.

But if F has no highest element, consider the totality $\{\alpha'\}$ of numbers of the first and second number-classes which are smaller than α. By theorem H, the aggregate $\{\alpha'\}$, arranged in order of magnitude, is similar to the aggregate F; among the numbers α', consequently, none is greatest. By theorem I, the aggregate $\{\alpha'\}$ can be brought into the form $\{\alpha'_\nu\}$ of a simply infinite series. If we set out from α'_1, the next following elements $\alpha'_2, \alpha'_3, \ldots$ in this order, which is different from the order of magnitude, will, in general, be smaller than α'_1; but in every case, in the further course of the process, terms will occur which are greater than α'_1; for α'_1 cannot be greater than all other terms, because among the numbers $\{\alpha'_\nu\}$ there is no greatest. Let that number α'_ν which has the least index of those greater than α'_1 be α'_{ρ_2}. Similarly, let α'_{ρ_3} be that number of the series $\{\alpha'_\nu\}$ which has the least index of those which are greater than α'_{ρ_2}. By proceeding in this way, we get an infinite series of increasing numbers, a fundamental series in fact,

$$\alpha'_1, \alpha'_{\rho_2}, \alpha'_{\rho_3}, \ldots, \alpha'_{\rho_\nu}, \ldots$$

[226] We have

$$1 < \rho_2 < \rho_3 < \ldots < \rho_\nu < \rho_{\nu+1} \ldots,$$

$$\alpha'_1 < \alpha'_{\rho_2} < \alpha'_{\rho_3} < \ldots < \alpha'_{\rho_\nu} < \alpha'_{\rho_{\nu+1}} \ldots,$$

$$\alpha'_\mu < \alpha'_{\rho_\nu} \quad \text{always if} \quad \mu < \rho'_\nu;$$

and since obviously $\nu \overline{\overline{<}} \rho_\nu$, we always have

$$\alpha'_\nu \leqq \alpha'_{\rho_\nu}.$$

Hence we see that every number α'_ν, and consequently every number $\alpha' < \alpha$, is exceeded by numbers α'_{ρ_ν} for sufficiently great values of ν. But α is the number which, in respect of magnitude, immediately follows all the numbers α', and consequently is also the next greater number with respect to all α'_{ρ_ν}. If, therefore, we put $\alpha'_1 = \alpha_1$, $\alpha_{\rho_{\nu+1}} = \alpha_{\nu+1}$, we have

$$\alpha = \operatorname*{Lim}_\nu \alpha_\nu.$$

From the theorems B, C, . . ., K it is evident that the numbers of the second number-class result from smaller numbers in two ways. Some numbers, which we call "numbers of the first kind (*Art*)," are got from a next smaller number α_{-1} by addition of 1 according to the formula

$$\alpha = \alpha_{-1} + 1\,;$$

The other numbers, which we call "numbers of the second kind," are such that for any one of them there is not a next smaller number α_{-1}, but they arise from fundamental series $\{\alpha_\nu\}$ as limiting numbers according to the formula

$$\alpha = \operatorname*{Lim}_{\nu} \alpha_\nu.$$

Here α is the number which follows next in order of magnitude to all the numbers α_ν.

We call these two ways in which greater numbers proceed out of smaller ones "the first and the second principle of generation of numbers of the second number-class." *

§ 16

The Power of the Second Number-Class is equal to the Second Greatest Transfinite Cardinal Number Aleph-One

Before we turn to the more detailed considerations in the following paragraphs of the numbers of the second number-class and of the laws which rule them, we will answer the question as to the

* [*Cf.* Section VII of the Introduction.]

cardinal number which is possessed by the aggregate $Z(\aleph_0) = \{\alpha\}$ of all these numbers.

[227] A. The totality $\{\alpha\}$ of all numbers α of the second number-class forms, when arranged in order of magnitude, a well-ordered aggregate.

Proof.—If we denote by A_α the totality of numbers of the second number-class which are smaller than a given number α, arranged in order of magnitude, then A_α is a well-ordered aggregate of type $\alpha - \omega$. This results from theorem H of § 14. The aggregate of numbers α' of the first and second number-class which was there denoted by $\{\alpha'\}$, is compounded out of $\{\nu\}$ and A_α, so that

$$\{\alpha'\} = (\{\nu\}, A_\alpha).$$

Thus

$$\overline{\{\alpha'\}} = \overline{\{\nu\}} + \overline{A}_\alpha\,;$$

and since

$$\overline{\{\alpha'\}} = \alpha, \quad \overline{\{\nu\}} = \omega,$$

we have

$$\overline{A}_\alpha = \alpha - \omega.$$

Let J be any part of $\{\alpha\}$ such that there are numbers in $\{\alpha\}$ which are greater than all the numbers of J. Let, say, α be one of these numbers. Then J is also a part of A_{α_0+1}, and indeed such a part that at least the number α_0 of A_{α_0+1} is greater than all the numbers of J. Since A_{α_0+1} is a well-ordered aggregate, by § 12 a number α' of A_{α_0+1}, and therefore also of $\{\alpha\}$, must follow next to all the numbers of J. Thus the condition II of § 12 is fulfilled in the case of $\{\alpha\}$; the condition I of § 12 is also fulfilled because $\{\alpha\}$ has the least number ω.

Now, if we apply to the well-ordered aggregate $\{\alpha\}$ the theorems A and C of § 12, we get the following theorems:

B. Every totality of different numbers of the first and second number-classes has a least number.

C. Every totality of different numbers of the first and second number-classes arranged in their order of magnitude forms a well-ordered aggregate.

We will now show that the power of the second number-class is different from that of the first, which is $\aleph_0$.

D. The power of the totality $\{\alpha\}$ of all numbers α of the second number-class is not equal to $\aleph_0$.

Proof.—If $\overline{\overline{\{\alpha\}}}$ were equal to $\aleph_0$, we could bring the totality $\{\alpha\}$ into the form of a simply infinite series

$$\gamma_1, \gamma_2, \ldots, \gamma_\nu, \ldots$$

such that $\{\gamma_\nu\}$ would represent the totality of numbers of the second [228] number-class in an order which is different from the order of magnitude, and $\{\gamma_\nu\}$ would contain, like $\{\alpha\}$, no greatest number.

Starting from γ_1, let γ_{ρ_2} be the term of the series which has the least index of those greater than γ_1, γ_{ρ_3} the term which has the least index of those greater than γ_{ρ_2}, and so on. We get an infinite series of increasing numbers,

$$\gamma_1, \gamma_{\rho_2}, \ldots, \gamma_{\rho_\nu}, \ldots,$$

such that

$$1 < \rho_2 < \rho_3 \ldots < \rho_\nu < \rho_{\nu+1} < \ldots,$$

$$\gamma_1 < \gamma_{\rho_2} < \gamma_{\rho_3} \ldots < \gamma_{\rho_\nu} < \gamma_{\rho_{\nu+1}} < \ldots,$$

$$\gamma_\nu \leqq \gamma_{\rho_\nu}.$$

By theorem C of § 15, there would be a definite number δ of the second number-class, namely,

$$\delta = \operatorname*{Lim}_\nu \gamma_{\rho_\nu},$$

which is greater than all numbers γ_{ρ_ν}. Consequently we would have

$$\delta > \gamma_\nu$$

for every ν. But $\{\gamma_\nu\}$ contains *all* numbers of the second number-class, and consequently also the number δ; thus we would have, for a definite ν_0,

$$\delta = \gamma_{\nu_0},$$

which equation is inconsistent with the relation $\delta > \gamma_{\nu_0}$. The supposition $\overline{\{\alpha\}} = \aleph_0$ consequently leads to a contradiction.

E. Any totality $\{\beta\}$ of different numbers β of the second number-class has, if it is infinite, either the cardinal number $\aleph_0$ or the cardinal number $\overline{\{\alpha\}}$ of the second number-class.

Proof.—The aggregate $\{\beta\}$, when arranged in its order of magnitude, is, since it is a part of the well-ordered aggregate $\{\alpha\}$, by theorem O of § 13, similar either to a segment A_{α_0}, which is the totality of all numbers of the same number-class which are less than α_0, arranged in their order of magnitude, or to the totality $\{\alpha\}$ itself. As was shown in the proof of theorem A, we have

$$\overline{A}_{\alpha_0} = \alpha_0 - \omega.$$

Thus we have either $\overline{\{\beta\}} = \alpha_0 - \omega$ or $\overline{\{\beta\}} = \overline{\{\alpha\}}$, and consequently $\overline{\overline{\{\beta\}}}$ is either equal to $\overline{\overline{\alpha_0 - \omega}}$ or $\overline{\overline{\{\alpha\}}}$. But $\overline{\overline{\alpha_0 - \omega}}$ is either a finite cardinal number or is equal to $\aleph_0$ (theorem I of § 15). The first case is here excluded because $\{\beta\}$ is supposed to be an infinite aggregate. Thus the cardinal number $\overline{\overline{\{\beta\}}}$ is either equal to $\aleph_0$ or $\overline{\overline{\{\alpha\}}}$.

F. The power of the second number-class $\{\alpha\}$ is the second greatest transfinite cardinal number Aleph-one.

[229] *Proof.*—There is no cardinal number $\mathfrak{a}$ which is greater than $\aleph_0$ and less than $\overline{\overline{\{\alpha\}}}$. For if not, there would have to be, by § 2, an infinite part $\{\beta\}$ of $\{\alpha\}$ such that $\overline{\overline{\{\beta\}}} = \mathfrak{a}$. But by the theorem E just proved, the part $\{\beta\}$ has either the cardinal number $\aleph_0$ or the cardinal number $\overline{\overline{\{\alpha\}}}$. Thus the cardinal number $\overline{\overline{\{\alpha\}}}$ is necessarily the cardinal number which immediately follows $\aleph_0$ in magnitude; we call this new cardinal number $\aleph_1$.

In the second number-class $Z(\aleph_0)$ we possess, consequently, the natural representative for the second greatest transfinite cardinal number Aleph-one.

§ 17

The Numbers of the Form $\omega^\mu \nu_0 + \omega^{\mu-1}\nu_1 + \ldots + \nu_\mu$.

It is convenient to make ourselves familiar, in the first place, with those numbers of $Z(\aleph_0)$ which are whole algebraic functions of finite degree of ω. Every such number can be brought—and brought in only one way—into the form

$$(1) \qquad \phi = \omega^\mu \nu_0 + \omega^{\mu-1}\nu_1 + \ldots + \nu_\mu,$$

where μ, ν_0 are finite and different from zero, but $\nu_1, \nu_2, \ldots, \nu_\mu$ may be zero. This rests on the fact that

$$(2) \qquad \omega^{\mu'}\nu' + \omega^\mu \nu = \omega^\mu \nu,$$

if $\mu' < \mu$ and $\nu > 0$, $\nu' > 0$. For, by (8) of § 14, we have

$$\omega^{\mu'}\nu' + \omega^\mu \nu = \omega^{\mu'}(\nu' + \omega^{\mu-\mu'}\nu),$$

and, by theorem E of § 15,

$$\nu' + \omega^{\mu-\mu'}\nu = \omega^{\mu-\mu'}\nu.$$

Thus, in an aggregate of the form

$$\ldots + \omega^{\mu'}\nu' + \omega^\mu \nu + \ldots,$$

all those terms which are followed towards the right by terms of higher degree in ω may be omitted. This method may be continued until the form given in (1) is reached. We will also emphasize that

$$(3) \qquad \omega^\mu \nu + \omega^\mu \nu' = \omega^\mu(\nu + \nu').$$

Compare, now, the number ϕ with a number ψ of the same kind:

$$(4) \qquad \psi = \omega^{\lambda}\rho_0 + \omega^{\lambda-1}\rho_1 + \ldots + \rho_{\lambda}.$$

If μ and λ are different and, say, $\mu < \lambda$, we have by (2) $\phi + \psi = \psi$, and therefore $\phi < \psi$.

[230] If $\mu = \lambda$, ν_0, and ρ_0 are different, and, say, $\nu_0 < \rho_0$, we have by (2)

$$\phi + \left(\omega^{\lambda}(\rho_0 - \nu_0) + \omega^{\lambda-1}\rho_1 + \ldots + \rho_{\mu}\right) = \psi,$$

and therefore

$$\phi < \psi.$$

If, finally,

$$\mu = \lambda, \quad \nu_0 = \rho_0, \; \nu_1 = \rho_1, \ldots \nu_{\sigma-1} = \rho_{\sigma-1}, \quad \sigma \overline{\overline{<}} \mu,$$

but ν_σ is different from ρ_σ and, say, $\nu_\sigma < \rho_\sigma$, we have by (2)

$$\phi + \left(\omega^{\lambda-\sigma}(\rho_\sigma - \nu_\sigma) + \omega^{\lambda-\sigma-1}\rho_{\sigma+1} + \ldots + \rho_{\mu}\right) = \psi,$$

and therefore again

$$\phi < \psi.$$

Thus, we see that only in the case of complete identity of the expressions ϕ and ψ can the numbers represented by them be equal.

The *addition* of ϕ and ψ leads to the following result:

(*a*) If $\mu < \lambda$, then, as we have remarked above,

$$\phi + \psi = \psi;$$

(*b*) If $\mu = \lambda$, then we have

$$\phi + \psi = \omega^{\lambda}(\nu_0 + \rho_0) + \omega^{\lambda-1}\rho_1 + \ldots + \rho_{\lambda};$$

(*c*) If $\mu > \lambda$, we have

$$\phi+\psi=\omega^{\mu}\nu_0+\omega^{\mu-1}\nu_1+\ldots+\omega^{\lambda+1}\nu_{\mu-\lambda-1}+\omega^{\lambda}(\nu_{\mu-\lambda}+\rho_0)$$
$$+\omega^{\lambda-1}\rho_1+\ldots+\rho_{\lambda}.$$

In order to carry out the *multiplication* of ϕ and ψ, we remark that, if ρ is a finite number which is different from zero, we have the formula :

$$(5)\qquad \phi\rho=\omega^{\mu}\nu_0\rho+\omega^{\mu-1}\nu_1+\ldots+\nu_{\mu}.$$

It easily results from the carrying out of the sum consisting of ρ terms $\phi+\phi+\ldots+\phi$. By means of the repeated application of the theorem G of § 15 we get, further, remembering the theorem F of § 15,

$$(6)\qquad \phi\omega=\omega^{\mu+1},$$

and consequently also

$$(7)\qquad \phi\omega^{\lambda}=\omega^{\mu+\lambda}.$$

By the distributive law, numbered (8) of § 14, we have

$$\phi\psi=\phi\omega^{\lambda}\rho_0+\phi\omega^{\lambda-1}\rho_1+\ldots+\phi\omega\rho_{\lambda-1}+\phi\rho_{\lambda}.$$

Thus the formulæ (4), (5), and (7) give the following result :

(*a*) If $\rho_{\lambda}=0$, we have

$$\phi\psi=\omega^{\mu+\lambda}\rho_0+\omega^{\mu+\lambda-1}\rho_1+\ldots+\omega^{\mu+1}\rho_{\lambda-1}=\omega^{\mu}\psi;$$

(*b*) If ρ_{λ} is not equal to zero, we have

$$\phi\psi=\omega^{\mu+\lambda}\rho_0+\omega^{\mu+\lambda-1}\rho_1+\ldots+\omega^{\mu+1}\rho_{\lambda-1}+\omega^{\mu}\nu_0\rho_{\lambda}$$
$$+\omega^{\mu-1}\nu_1+\ldots+\nu_{\mu}.$$

[231] We arrive at a remarkable resolution of the numbers ϕ in the following manner. Let

(8) $$\phi = \omega^{\mu}\kappa_0 + \omega^{\mu_1}\kappa_1 + \ldots + \omega^{\mu_\tau}\kappa_\tau,$$

where

$$\mu > \mu_1 > \mu_2 > \ldots > \mu_\tau \geqq 0$$

and $\kappa_0, \kappa_1, \ldots, \kappa_\tau$ are finite numbers which are different from zero. Then we have

$$\phi = (\omega^{\mu_1}\kappa_1 + \omega^{\mu_2}\kappa_2 + \ldots + \omega^{\mu_\tau}\kappa_\tau)(\omega^{\mu - \mu_1}\kappa_0 + 1).$$

By the repeated application of this formula we get

$$\phi = \omega^{\mu_\tau}\kappa_\tau(\omega^{\mu_{\tau-1} - \mu_\tau}\kappa_{\tau-1} + 1)(\omega^{\mu_{\tau-2} - \mu_{\tau-1}}\kappa_{\tau-2} + 1) \ldots (\omega^{\mu - \mu_1}\kappa_0 + 1).$$

But, now,

$$\omega^{\lambda}\kappa + 1 = (\omega^{\lambda} + 1)\kappa,$$

if κ is a finite number which is different from zero; so that:

(9) $$\phi = \omega^{\mu_\tau}\kappa_\tau(\omega^{\mu_{\tau-1} - \mu_\tau} + 1)\kappa_{\tau-1}(\omega^{\mu_{\tau-2} - \mu_{\tau-1}} + 1)\kappa_{\tau-2} \ldots \ldots (\omega^{\mu - \mu_1} + 1)\kappa_0.$$

The factors $\omega^{\lambda} + 1$ which occur here are all irresoluble, and a number ϕ can be represented in this product-form in only one way. If $\mu_\tau = 0$, then ϕ is of the first kind, in all other cases it is of the second kind.

The apparent deviation of the formulæ of this paragraph from those which were given in *Math. Ann.*, vol. xxi, p. 585 (or *Grundlagen*, p. 41), is merely a consequence of the different writing of the product of two numbers: we now put the multiplicand on the left and the multiplicator on the right, but then we put them in the contrary way.

§ 18

The Power * γ^α in the Domain of the Second Number-Class

Let ξ be a variable whose domain consists of the numbers of the first and second number-classes including zero. Let γ and δ be two constants belonging to the same domain, and let

$$\delta > 0, \quad \gamma > 1.$$

We can then assert the following theorem :

A. There is one wholly determined one-valued function $f(\xi)$ of the variable ξ such that :

(*a*) $$f(0) = \delta.$$

(*b*) If ξ' and ξ'' are any two values of ξ, and if

$$\xi' < \xi'',$$

then

$$f(\xi') < f(\xi'').$$

[232] (*c*) For every value of ξ we have

$$f(\xi + 1) = f(\xi)\gamma.$$

(*d*) If $\{\xi_\nu\}$ is any fundamental series, then $\{f(\xi_\nu)\}$ is one also, and if we have

$$\xi = \operatorname*{Lim}_\nu \xi_\nu,$$

then

$$f(\xi) = \operatorname*{Lim}_\nu f(\xi_\nu).$$

* [Here obviously it is *Potenz* and not *Mächtigkeit.*]

Proof.—By (a) and (c), we have

$$f(1)=\delta\gamma,\quad f(2)=\delta\gamma\gamma,\quad f(3)=\delta\gamma\gamma\gamma,\quad \ldots,$$

and, because $\delta>0$ and $\gamma>1$, we have

$$f(1)<f(2)<f(3)<\ldots<f(\nu)<f(\nu+1)<\ldots$$

Thus the function $f(\xi)$ is wholly determined for the domain $\xi<\omega$. Let us now suppose that the theorem is valid for all values of ξ which are less than α, where α is any number of the second number-class, then it is also valid for $\xi\leqq\alpha$. For if α is of the first kind, we have from (c):

$$f(\alpha)=f(\alpha_{-1})\gamma>f(\alpha_{-1});$$

so that the conditions (b), (c), and (d) are satisfied for $\xi\leqq\alpha$. But if α is of the second kind and $\{\alpha_\nu\}$ is a fundamental series such that $\operatorname*{Lim}_\nu \alpha_\nu=\alpha$, then it follows from ($b$) that also $\{f(\alpha_\nu)\}$ is a fundamental series, and from (d) that $f(\alpha)=\operatorname*{Lim}_\nu f(\alpha_\nu)$. If we take another fundamental series $\{\alpha'_\nu\}$ such that $\operatorname*{Lim}_\nu \alpha'_\nu=\alpha$, then, because of ($b$), the two fundamental series $\{f(\alpha_\nu)\}$ and $\{f(\alpha'_\nu)\}$ are coherent, and thus also $f(\alpha)=\operatorname*{Lim}_\nu f(\alpha'_\nu)$. The value of $f(\alpha)$ is, consequently, uniquely determined in this case also.

If α' is any number less than α, we easily convince ourselves that $f(\alpha')<f(\alpha)$. The conditions (b), (c), and (d) are also satisfied for $\xi\overline{\gtrless}\alpha$. Hence follows the validity of the theorem *for all values* of ξ. For if there were exceptional values of ξ for which it did not hold, then, by theorem B of § 16, one of

them, which we will call α, would have to be the least. Then the theorem would be valid for $\xi < \alpha$, but not for $\xi \leqq \alpha$, and this would be in contradiction with what we have proved. Thus there is for the whole domain of ξ one and only one function $f(\xi)$ which satisfies the conditions (a) to (d).

[233] If we attribute to the constant δ the value 1 and then denote the function $f(\xi)$ by

$$\gamma^{\xi},$$

we can formulate the following theorem:

B. If γ is any constant greater than 1 which belongs to the first or second number-class, there is a wholly definite function γ^{ξ} of ξ such that:

(a) $\gamma^0 = 1$;

(b) If $\xi' < \xi''$ then $\gamma^{\xi'} < \gamma^{\xi''}$;

(c) For every value of ξ we have $\gamma^{\xi+1} = \gamma^{\xi}\gamma$;

(d) If $\{\xi_\nu\}$ is a fundamental series, then $\{\gamma^{\xi_\nu}\}$ is such a series, and we have, if $\xi = \operatorname*{Lim}_\nu \xi_\nu$, the equation

$$\gamma^{\xi} = \operatorname*{Lim}_\nu \gamma^{\xi_\nu}.$$

We can also assert the theorem:

C. If $f(\xi)$ is the function of ξ which is characterized in theorem A, we have

$$f(\xi) = \delta\gamma^{\xi}.$$

Proof.—If we pay attention to (24) of § 14, we easily convince ourselves that the function $\delta\gamma^{\xi}$ satisfies, not only the conditions (a), (b), and (c) of theorem A, but also the condition (d) of this theorem. On account of the uniqueness of the function $f(\xi)$, it must therefore be identical with $\delta\gamma^{\xi}$.

D. If α and β are any two numbers of the first or second number-class, including zero, we have

$$\gamma^{\alpha+\beta}=\gamma^{\alpha}\gamma^{\beta}.$$

Proof.—We consider the function $\phi(\xi)=\gamma^{\alpha+\xi}$. Paying attention to the fact that, by formula (23) of § 14,

$$\operatorname*{Lim}_{\nu}(\alpha+\xi_\nu)=\alpha+\operatorname*{Lim}_{\nu}\xi_\nu,$$

we recognize that $\phi(\xi)$ satisfies the following four conditions:

(*a*) $\phi(0)=\gamma^{\alpha}$;

(*b*) If $\xi'<\xi''$, then $\phi(\xi')<\phi(\xi'')$;

(*c*) For every value of ξ we have $\phi(\xi+1)=\phi(\xi)\gamma$;

(*d*) If $\{\xi_\nu\}$ is a fundamental series such that $\operatorname*{Lim}_{\nu}\xi_\nu=\xi$, we have

$$\phi(\xi)=\operatorname*{Lim}_{\nu}\phi(\xi_\nu).$$

By theorem C we have, when we put $\delta=\gamma^{\alpha}$,

$$\phi(\xi)=\gamma^{\alpha}\gamma^{\xi}.$$

If we put $\xi=\beta$ in this, we have

$$\gamma^{\alpha+\beta}=\gamma^{\alpha}\gamma^{\beta}.$$

E. If α and β are any two numbers of the first or second number-class, including zero, we have

$$\gamma^{\alpha\beta}=(\gamma^{\alpha})^{\beta}.$$

[234] *Proof.*—Let us consider the function $\psi(\xi)=\gamma^{\alpha\xi}$ and remark that, by (24) of § 14, we

always have $\operatorname*{Lim}_\nu \alpha\xi_\nu = \alpha \operatorname*{Lim}_\nu \xi_\nu$, then we can, by theorem D, assert the following:

(*a*) $\psi(0) = 1$;

(*b*) If $\xi' < \xi''$, then $\psi(\xi') < \psi(\xi'')$;

(*c*) For every value of ξ we have $\psi(\xi + 1) = \psi(\xi)\gamma^\alpha$;

(*d*) If $\{\xi_\nu\}$ is a fundamental series, then $\{\psi(\xi_\nu)\}$ is also such a series, and we have, if $\xi = \operatorname*{Lim}_\nu \xi_\nu$, the equation $\psi(\xi) = \operatorname*{Lim}_\nu \psi(\xi_\nu)$.

Thus, by theorem C, if we substitute in it 1 for δ and γ^α for γ,

$$\psi(\xi) = (\gamma^\alpha)^\xi.$$

On the *magnitude* of γ^ξ in comparison with ξ we can assert the following theorem:

F. If $\gamma > 1$, we have, for every value of ξ,

$$\gamma^\xi \geqq \xi.$$

Proof.—In the cases $\xi = 0$ and $\xi = 1$ the theorem is immediately evident. We now show that, if it holds for all values of ξ which are smaller than a given number $\alpha > 1$, it also holds for $\xi = \alpha$.

If α is of the first kind, we have, by supposition,

$$\alpha_{-1} \leqq \gamma^{\alpha-1},$$

and consequently

$$\alpha_{-1}\gamma \mathrel{\overline{\overline{<}}} \gamma^{\alpha-1}\gamma = \gamma^\alpha.$$

Hence

$$\gamma^\alpha \geqq \alpha_{-1} + \alpha_{-1}(\gamma - 1).$$

Since both α_{-1} and $\gamma - 1$ are at least equal to 1, and $\alpha_{-1} + 1 = \alpha$, we have

$$\gamma^\alpha \geqq \alpha.$$

If, on the other hand, α is of the second kind and

$$\alpha = \operatorname*{Lim}_{\nu} \alpha_\nu,$$

then, because $\alpha_\nu < \alpha$, we have by supposition

$$\alpha_\nu \leqq \gamma^{\alpha_\nu}.$$

Consequently

$$\operatorname*{Lim}_{\nu} \alpha_\nu \leqq \operatorname*{Lim}_{\nu} \gamma^{\alpha_\nu},$$

that is to say,

$$\alpha \leqq \gamma^{\alpha}.$$

If, now, there were values of ξ for which

$$\xi > \gamma^{\xi},$$

one of them, by theorem B of § 16, would have to be the least. If this number is denoted by α, we would have, for $\xi < \alpha$,

[235] $$\xi \leqq \gamma^{\xi};$$

but

$$\alpha > \gamma^{\alpha},$$

which contradicts what we have proved above. Thus we have for all values of ξ

$$\gamma^{\xi} \geqq \xi.$$

§ 19

The Normal Form of the Numbers of the Second Number-Class

Let α be any number of the second number-class. The power ω^{ξ} will be, for sufficiently great values of ξ, greater than α. By theorem F of § 18, this is always the case for $\xi > \alpha$; but in general it will also happen for smaller values of ξ.

By theorem B of § 16, there must be, among the values of ξ for which

$$\omega^{\xi} > \alpha,$$

one which is the least. We will denote it by β, and we easily convince ourselves that it cannot be a number of the second kind. If, indeed, we had

$$\beta = \operatorname*{Lim}_{\nu} \beta_\nu,$$

we would have, since $\beta_\nu < \beta$,

$$\omega^{\beta_\nu} \overline{\overline{<}}\ \alpha,$$

and consequently

$$\operatorname*{Lim}_{\nu} \omega^{\beta_\nu} \overline{\overline{<}}\ \alpha.$$

Thus we would have

$$\omega^{\beta} \overline{\overline{<}}\ \alpha,$$

whereas we have

$$\omega^{\beta} > \alpha.$$

Therefore β is of the first kind. We denote β_{-1} by α_0, so that $\beta = \alpha_0 + 1$, and consequently can assert that there is a wholly determined number α_0 of the first or second number-class which satisfies the two conditions:

$$(1) \qquad \omega^{\alpha_0} \leqq \alpha, \quad \omega^{\alpha_0}\omega > \alpha.$$

From the second condition we conclude that

$$\omega^{\alpha_0}\nu \leqq \alpha$$

does not hold for all finite values of ν, for if it did we would have

$$\operatorname*{Lim}_{\nu} \omega^{\alpha_0}\nu = \omega^{\alpha_0}\omega \leqq \alpha.$$

The least finite number ν for which

$$\omega^{\alpha_0}\nu > \alpha$$

will be denoted by $\kappa_0 + 1$. Because of (1), we have $\kappa_0 > 0$.

[236] There is, therefore, a wholly determined number κ_0 of the first number-class such that

$$(2) \qquad \omega^{\alpha_0}\kappa_0 \leqq \alpha, \quad \omega^{\alpha_0}(\kappa_0 + 1) > \alpha.$$

If we put $\alpha - \omega^{\alpha_0}\kappa_0 = \alpha'$, we have

$$(3) \qquad \alpha = \omega^{\alpha_0}\kappa_0 + \alpha'$$

and

$$(4) \qquad 0 \leqq \alpha' < \omega^{\alpha_0}, \quad 0 < \kappa_0 < \omega.$$

But α can be represented in the form (3) under the conditions (4) in only a single way. For from (3) and (4) follow inversely the conditions (2) and thence the conditions (1). But only the number $\alpha_0 = \beta_{-1}$ satisfies the conditions (1), and by the conditions (2) the finite number κ_0 is uniquely determined. From (1) and (4) follows, by paying attention to theorem F of § 18, that

$$\alpha' < \alpha, \quad \alpha_0 \leqq \alpha.$$

Thus we can assert the following theorem :

A. Every number α of the second number-class can be brought, and brought in only one way, into the form

$$\alpha = \omega^{\alpha_0}\kappa_0 + \alpha',$$

where

$$0 \leqq \alpha' < \omega^{\alpha_0}, \quad 0 < \kappa_0 < \omega,$$

and α' is always smaller than α, but α_0 is smaller than or equal to α.

If α' is a number of the second number-class, we can apply theorem A to it, and we have

$$(5) \qquad \alpha' = \omega^{\alpha_1}\kappa_1 + \alpha'',$$

where

$$0 \leqq \alpha'' < \omega^{\alpha_1}, \quad 0 < \kappa_1 < \omega,$$

and

$$\alpha_1 < \alpha_0, \quad \alpha'' < \alpha'.$$

In general we get a further sequence of analogous equations:

$$(6) \qquad \alpha'' = \omega^{\alpha_2}\kappa_2 + \alpha''',$$

$$(7) \qquad \alpha''' = \omega^{\alpha_3}\kappa_3 + \alpha^{\text{iv}}.$$

. . . . . .

But this sequence cannot be infinite, but must necessarily break off. For the numbers $\alpha, \alpha', \alpha'', \ldots$ decrease in magnitude:

$$\alpha > \alpha' > \alpha'' > \alpha''' \ldots$$

If a series of decreasing transfinite numbers were infinite, then no term would be the least; and this is impossible by theorem B of § 16. Consequently we must have, for a certain finite numerical value τ,

$$\alpha^{(\tau+1)} = 0.$$

[237] If we now connect the equations (3), (5), (6), and (7) with one another, we get the theorem:

B. Every number α of the second number-class can be represented, and represented in only one way, in the form

$$\alpha = \omega^{\alpha_0}\kappa_0 + \omega^{\alpha_1}\kappa_1 + \ldots + \omega^{\alpha_\tau}\kappa_\tau,$$

where $\alpha_0, \alpha_1, \ldots \alpha_\tau$ are numbers of the first or second number-class, such that:

$$\alpha_0 > \alpha_1 > \alpha_2 > \ldots > \alpha_\tau \geqq 0,$$

while $\kappa_0, \kappa_1, \ldots \kappa_\tau, \tau+1$ are numbers of the first number-class which are different from zero.

The form of numbers of the second number-class which is here shown will be called their "normal form"; α_0 is called the "degree" and α_τ the "exponent" of α. For $\tau=0$, degree and exponent are equal to one another.

According as the exponent α_τ is equal to or greater than zero, α is a number of the first or second kind.

Let us take another number β in the normal form:

$$(8) \qquad \beta = \omega^{\beta_0}\lambda_0 + \omega^{\beta_1}\lambda_1 + \ldots + \omega^{\beta_\sigma}\lambda_\sigma.$$

The formulæ:

$$(9) \qquad \omega^{\alpha'}\kappa' + \omega^{\alpha'}\kappa = \omega^{\alpha'}(\kappa'+\kappa),$$

$$(10) \qquad \omega^{\alpha'}\kappa' + \omega^{\alpha''}\kappa'' = \omega^{\alpha''}\kappa'', \quad \alpha' < \alpha'',$$

where $\kappa, \kappa', \kappa''$ here denote finite numbers, serve both for the comparison of α with β and for the carrying out of their sum and difference. These are generalizations of the formulæ (2) and (3) of § 17.

For the formation of the product $\alpha\beta$, the following formulæ come into consideration:

(11) $\alpha\lambda = \omega^{\alpha_0}\kappa_0\lambda + \omega^{\alpha_1}\kappa_1 + \ldots + \omega^{\alpha_\tau}\kappa_\tau, \quad 0 < \lambda < \omega;$

(12) $\alpha\omega = \omega^{\alpha_0 + 1};$

(13) $\alpha\omega^{\beta'} = \omega^{\alpha_0 + \beta'}, \quad \beta' > 0.$

The exponentiation α^β can be easily carried out on the basis of the following formulæ:

(14) $$\alpha^\lambda = \omega^{\alpha_0\lambda}\kappa_0 + \ldots, \quad 0 < \lambda < \omega.$$

The terms not written on the right have a lower degree than the first. Hence follows readily that the fundamental series $\{\alpha^\lambda\}$ and $\{\omega^{\alpha_0\lambda}\}$ are coherent, so that

(15) $$\alpha^\omega = \omega^{\alpha_0\omega}, \quad \alpha_0 > 0.$$

Thus, in consequence of theorem E of § 18, we have:

(16) $$\alpha^{\omega\beta'} = \omega^{\alpha_0\omega\beta'}, \quad \alpha_0 > 0, \quad \beta' > 0.$$

By the help of these formulæ we can prove the following theorems:

[238] C. If the first terms $\omega^{\alpha_0}\kappa_0$, $\omega^{\beta_0}\lambda_0$ of the normal forms of the two numbers α and β are not equal, then α is less or greater than β according as $\omega^{\alpha_0}\kappa_0$ is less or greater than $\omega^{\beta_0}\lambda_0$. But if we have

$$\omega^{\alpha_0}\kappa_0 = \omega^{\beta_0}\lambda_0, \; \omega^{\alpha_1}\kappa_1 = \omega^{\beta_1}\lambda_1, \; \ldots, \; \omega^{\alpha_\rho}\kappa_\rho = \omega^{\beta_\rho}\lambda_\rho,$$

and if $\omega^{\alpha_{\rho+1}}\kappa_{\rho+1}$ is less or greater than $\omega^{\beta_{\rho+1}}\lambda_{\rho+1}$, then α is correspondingly less or greater than β.

D. If the degree α_0 of α is less than the degree β_0 of β, we have

$$\alpha+\beta=\beta.$$

If $\alpha_0=\beta_0$, then

$$\alpha+\beta=\omega^{\beta_0}(\kappa_0+\lambda_0)+\omega^{\beta_1}\lambda_1+\ldots+\omega^{\beta_\sigma}\lambda_\sigma.$$

But if

$$\alpha_0>\beta_0,\ \alpha_1>\beta_0,\ \ldots,\ \alpha_\rho\geqq\beta_0,\ \alpha_{\rho+1}<\beta_0,$$

then

$$\alpha+\beta=\omega^{\alpha_0}\kappa_0+\ldots+\omega^{\alpha_\rho}\kappa_\rho+\omega^{\beta_0}\lambda_0+\omega^{\beta_1}\lambda_1+\ldots+\omega^{\beta_\sigma}\lambda_\sigma.$$

E. If β is of the second kind ($\beta_\sigma>0$), then

$$\alpha\beta=\omega^{\alpha_0+\beta_0}\lambda_0+\omega^{\alpha_0+\beta_1}\lambda_1+\ldots+\omega^{\alpha_0+\beta_\sigma}\lambda_\sigma=\omega^{\alpha_0}\beta;$$

But if β is of the first kind ($\beta_\sigma=0$), then

$$\alpha\beta=\omega^{\alpha_0+\beta_0}\lambda_0+\omega^{\alpha_0+\beta_1}\lambda_1+\ldots+\omega^{\alpha_0+\beta_{\sigma-1}}\lambda_{\sigma-1}+\omega^{\alpha_0}\kappa_0\lambda_\sigma$$
$$+\omega^{\alpha_1}\kappa_1+\ldots+\omega^{\alpha_\tau}\kappa_\tau.$$

F. If β is of the second kind ($\beta_\sigma>0$), then

$$\alpha^\beta=\omega^{\alpha_0\beta}.$$

But if β is of the first kind ($\beta_\sigma=0$), and indeed $\beta=\beta'+\lambda_\sigma$, where β' is of the second kind, we have:

$$\alpha^\beta=\omega^{\alpha_0\beta'}\alpha^{\lambda_\sigma}.$$

G. Every number α of the second number-class can be represented, in only one way, in the product-form:

$$\alpha=\omega^{\gamma_0}\kappa_\tau(\omega^{\gamma_1}+1)\kappa_{\tau-1}(\omega^{\gamma_2}+1)\kappa_{\tau-2}\ldots(\omega^{\gamma_\tau}+1)\kappa_0,$$

and we have

$$\gamma_0=\alpha_\tau,\ \gamma_1=\alpha_{\tau-1}-\alpha_\tau,\ \gamma_2=\alpha_{\tau-2}-\alpha_{\tau-1},\ \ldots,\ \gamma_\tau=\alpha_0-\alpha_1,$$

whilst $\kappa_0, \kappa_1, \ldots \kappa_\tau$ have the same denotation as in the normal form. The factors $\omega^\gamma+1$ are all irresoluble.

H. Every number α of the second kind which belongs to the second number-class can be represented, and represented in only one way, in the form

$$\alpha=\omega^{\gamma_0}\alpha',$$

where $\gamma_0>0$ and α' is a number of the first kind which belongs to the first or second number-class.

[239] I. In order that two numbers α and β of the second number-class should satisfy the relation

$$\alpha+\beta=\beta+\alpha,$$

it is necessary and sufficient that they should have the form

$$\alpha=\gamma\mu, \quad \beta=\gamma\nu,$$

where μ and ν are numbers of the first number-class.

K. In order that two numbers α and β of the second number-class, which are both of the first kind, should satisfy the relation

$$\alpha\beta=\beta\alpha,$$

it is necessary and sufficient that they should have the form

$$\alpha=\gamma^\mu, \quad \beta=\gamma^\nu,$$

where μ and ν are numbers of the first number-class.

In order to exemplify the extent of the *normal form* dealt with and the *product-form* immediately connected with it, of the numbers of the second number-class, the proofs, which are founded on them, of the two last theorems, I and K, may here follow.

From the supposition

$$\alpha+\beta=\beta+\alpha$$

we first conclude that the degree α_0 of α must be equal to the degree β_0 of β. For if, say, $\alpha_0<\beta_0$, we would have, by theorem D,

$$\alpha+\beta=\beta,$$

and consequently

$$\beta+\alpha=\beta,$$

which is not possible, since, by (2) of § 14,

$$\beta+\alpha>\beta.$$

Thus we may put

$$\alpha=\omega^{\alpha_0}\mu+\alpha', \quad \beta=\omega^{\alpha_0}\nu+\beta',$$

where the degrees of the numbers α' and β' are less than α_0, and μ and ν are finite numbers which are different from zero. Now, by theorem D we have

$$\alpha+\beta=\omega^{\alpha_0}(\mu+\nu)+\beta', \quad \beta+\alpha=\omega^{\alpha_0}(\mu+\nu)+\alpha',$$

and consequently

$$\omega^{\alpha_0}(\mu+\nu)+\beta'=\omega^{\alpha_0}(\mu+\nu)+\alpha'.$$

By theorem D of § 14 we have consequently

$$\beta'=\alpha'.$$

Thus we have

$$\alpha=\omega^{\alpha_0}\mu+\alpha', \quad \beta=\omega^{\alpha_0}\nu+\alpha',$$

[240] and if we put

$$\omega^{\alpha_0}+\alpha'=\gamma$$

we have, by (11):

$$\alpha=\gamma\mu, \quad \beta=\gamma\nu.$$

Let us suppose, on the other hand, that α and β are two numbers which belong to the second number-class, are of the first kind, and satisfy the condition

$$\alpha\beta=\beta\alpha,$$

and we suppose that

$$\alpha>\beta.$$

We will imagine both numbers, by theorem G, in their product-form, and let

$$\alpha=\delta\alpha', \quad \beta=\delta\beta',$$

where α' and β' are without a common factor (besides 1) at the left end. We have then

$$\alpha'>\beta',$$

and

$$\alpha'\delta\beta'=\beta'\delta\alpha'.$$

All the numbers which occur here and farther on are of the first kind, because this was supposed of α and β.

The last equation, when we refer to theorem G, shows that α' and β' cannot be both transfinite, because, in this case, there would be a common factor at the left end. Neither can they be both finite; for then δ would be transfinite, and, if κ is the finite factor at the left end of δ, we would have

$$\alpha'\kappa=\beta'\kappa,$$

and thus

$$\alpha' = \beta'.$$

Thus there remains only the possibility that

$$\alpha' > \omega, \quad \beta' < \omega.$$

But the finite number β' must be 1:

$$\beta' = 1,$$

because otherwise it would be contained as part in the finite factor at the left end of α'.

We arrive at the result that $\beta = \delta$, consequently

$$\alpha = \beta\alpha',$$

where α' is a number belonging to the second number-class, which is of the first kind, and must be less than α:

$$\alpha' < \alpha.$$

Between α' and β the relation

$$\alpha'\beta = \beta\alpha'$$

subsists.

[241] Consequently if also $\alpha' > \beta$, we conclude in the same way the existence of a transfinite number of the first kind α'' which is less than α' and such that

$$\alpha' = \beta\alpha'', \quad \alpha''\beta = \beta\alpha''.$$

If also α'' is greater than β, there is such a number α''' less than α'', such that

$$\alpha'' = \beta\alpha''', \quad \alpha'''\beta = \beta\alpha''',$$

and so on. The series of decreasing numbers, α, α', α'', α''', . . ., must, by theorem B of § 16, break off. Thus, for a definite finite index ρ_0, we must have

$$\alpha^{(\rho_0} \leqq \beta.$$

If

$$\alpha^{(\rho_0)} = \beta,$$

we have

$$\alpha = \beta^{\rho_0+1}, \quad \beta = \beta\,;$$

the theorem K would then be proved, and we would have

$$\gamma = \beta, \quad \mu = \rho_0 + 1, \quad \nu = 1.$$

But if

$$\alpha^{(\rho_0)} < \beta,$$

then we put

$$\alpha^{(\rho_0)} = \beta_1,$$

and have

$$\alpha = \beta^{\rho_0}\beta_1, \quad \beta\beta_1 = \beta_1\beta, \quad \beta_1 < \beta.$$

Thus there is also a finite number ρ_1 such that

$$\beta = \beta_1{}^{\rho_1}\beta_2, \quad \beta_1\beta_2 = \beta_2\beta_1, \quad \beta_2 < \beta_1.$$

In general, we have analogously:

$$\beta_1 = \beta_2{}^{\rho_2}\beta_3, \quad \beta_2\beta_3 = \beta_3\beta_2, \quad \beta_3 < \beta_2,$$

and so on. The series of decreasing numbers β_1, β_2, β_3, . . . also must, by theorem B of § 16, break off. Thus there exists a finite number κ such that

$$\beta_{\kappa-1} = \beta_\kappa{}^{\rho_\kappa}.$$

If we put

$$\beta_\kappa = \gamma,$$

then

$$\alpha = \gamma^\mu, \quad \beta = \gamma^\nu,$$

where μ and ν are numerator and denominator of the continued fraction:

$$\frac{\mu}{\nu}=\rho_0+\cfrac{1}{\rho_1+\cdots+\cfrac{1}{\rho_\kappa}}.$$

[242] § 20

The ϵ-Numbers of the Second Number-Class

The degree α_0 of a number α is, as is immediately evident from the normal form :

$$(1) \quad \alpha=\omega^{\alpha_0}\kappa_0+\omega^{\alpha_1}\kappa_1+\ldots, \quad \alpha_0>\alpha_1>\ldots, \quad 0<\kappa_\nu<\omega,$$

when we pay attention to theorem F of § 18, never greater than α; but it is a question whether there are not numbers for which $\alpha_0=\alpha$. In such a case the normal form of α would reduce to the first term, and this term would be equal to ω^α, that is to say, α would be a root of the equation

$$(2) \qquad \omega^\xi=\xi.$$

On the other hand, every root α of this equation would have the normal form ω^α; its degree would be equal to itself.

The numbers of the second number-class which are equal to their degree coincide, therefore, with the roots of the equation (2). It is our problem to determine these roots in their totality. To distinguish them from all other numbers we will call them the "ϵ-numbers of the second number-class."

That there *are* such ϵ-numbers results from the following theorem :

A. If γ is any number of the first or second number-class which does not satisfy the equation (2), it determines a fundamental series $\{\gamma\}$ by means of the equations

$$\gamma_1 = \omega^{\gamma}, \quad \gamma_2 = \omega^{\gamma_1}, \quad \ldots, \quad \gamma_\nu = \omega^{\gamma_{\nu-1}}, \quad \ldots$$

The limit $\operatorname*{Lim}_\nu \gamma_\nu = E(\gamma)$ of this fundamental series is always an ϵ-number.

Proof.—Since γ is not an ϵ-number, we have $\omega^\gamma > \gamma$, that is to say, $\gamma_1 > \gamma$. Thus, by theorem B of § 18, we have also $\omega^{\gamma_1} > \omega^{\gamma}$, that is to say, $\gamma_2 > \gamma_1$; and in the same way follows that $\gamma_3 > \gamma_2$, and so on. The series $\{\gamma_\nu\}$ is thus a fundamental series. We denote its limit, which is a function of γ, by $E(\gamma)$ and have :

$$\omega^{E(\gamma)} = \operatorname*{Lim}_\nu \omega^{\gamma_\nu} = \operatorname*{Lim}_\nu \gamma_{\nu+1} = E(\gamma).$$

Consequently $E(\gamma)$ is an ϵ-number.

B. The number $\epsilon_0 = E(1) = \operatorname*{Lim}_\nu \omega_\nu$, where

$$\omega_1 = \omega, \quad \omega_2 = \omega^{\omega_1}, \quad \omega_3 = \omega^{\omega_2}, \quad \ldots, \quad \omega_\nu = \omega^{\omega_{\nu-1}}, \quad \ldots,$$

is the least of all the ϵ-numbers.

[243] *Proof.*—Let ϵ' be any ϵ-number, so that

$$\omega^{\epsilon'} = \epsilon'.$$

Since $\epsilon' > \omega$, we have $\omega^{\epsilon'} > \omega^{\omega}$, that is to say, $\epsilon' > \omega_1$. Similarly $\omega^{\epsilon'} > \omega^{\omega_1}$, that is to say, $\epsilon' > \omega_2$, and so on. We have in general

$$\epsilon' > \omega_\nu,$$

and consequently

$$\epsilon' \geqq \operatorname*{Lim}_{\nu} \omega_\nu,$$

that is to say,

$$\epsilon' \geqq \epsilon_0.$$

Thus $\epsilon_0 = \mathrm{E}(1)$ is the least of all ϵ-numbers.

C. If ϵ' is any ϵ-number, ϵ'' is the next greater ϵ-number, and γ is any number which lies between them:

$$\epsilon' < \gamma < \epsilon'',$$

then $\mathrm{E}(\gamma) = \epsilon''$.

Proof.—From

$$\epsilon' < \gamma < \epsilon''$$

follows

$$\omega^{\epsilon'} < \omega^\gamma < \omega^{\epsilon''},$$

that is to say,

$$\epsilon' < \gamma_1 < \epsilon''.$$

Similarly we conclude

$$\epsilon' < \gamma_2 < \epsilon'',$$

and so on. We have, in general,

$$\epsilon' < \gamma_\nu < \epsilon'',$$

and thus

$$\epsilon' < \mathrm{E}(\gamma) \leqq \epsilon''.$$

By theorem A, $\mathrm{E}(\gamma)$ is an ϵ-number. Since ϵ'' is the ϵ-number which follows ϵ' next in order of magnitude, $\mathrm{E}(\gamma)$ cannot be less than ϵ'', and thus we must have

$$\mathrm{E}(\gamma) = \epsilon''.$$

Since $\epsilon' + 1$ is not an ϵ-number, simply because all ϵ-numbers, as follows from the equation of definition

$\xi = \omega^\xi$, are of the second kind, $\epsilon' + 1$ is certainly less than ϵ'', and thus we have the following theorem :

D. If ϵ' is any ϵ-number, then $E(\epsilon' + 1)$ is the next greater ϵ-number.

To the least ϵ-number, ϵ_0, follows, then, the next greater one:

$$\epsilon_1 = E(\epsilon_0 + 1),$$

[244] to this the next greater number :

$$\epsilon_2 = E(\epsilon_1 + 1),$$

and so on. Quite generally, we have for the $(\nu + 1)$th ϵ-number in order of magnitude the formula of recursion

$$(3) \qquad \epsilon_\nu = E(\epsilon_{\nu-1} + 1).$$

But that the infinite series

$$\epsilon_0, \epsilon_1, \ldots \epsilon_\nu, \ldots$$

by no means embraces the totality of ϵ-numbers results from the following theorem :

E. If $\epsilon, \epsilon', \epsilon'', \ldots$ is any infinite series of ϵ-numbers such that

$$\epsilon < \epsilon' < \epsilon'' \ldots \epsilon^{(\nu)} < \epsilon^{(\nu+1)} < \ldots,$$

then $\operatorname{Lim}_\nu \epsilon^{(\nu)}$ is an ϵ-number, and, in fact, the ϵ-number which follows next in order of magnitude to all the numbers $\epsilon^{(\nu)}$.

Proof.—

$$\omega^{\operatorname{Lim}_\nu \epsilon^{(\nu)}} = \operatorname{Lim}_\nu \omega^{\epsilon^{(\nu)}} = \operatorname{Lim}_\nu \epsilon^{(\nu)}.$$

That $\operatorname{Lim}_{\nu} \epsilon^{(\nu)}$ is the ϵ-number which follows next in order of magnitude to all the numbers $\epsilon^{(\nu)}$ results from the fact that $\operatorname{Lim}_{\nu} \epsilon^{(\nu)}$ is the number of the second number-class which follows next in order of magnitude to all the numbers $\epsilon^{(\nu)}$.

F. The totality of ϵ-numbers of the second number-class forms, when arranged in order of magnitude, a well-ordered aggregate of the type Ω of the second number-class in its order of magnitude, and has thus the power Aleph-one.

Proof.—The totality of ϵ-numbers of the second number-class, when arranged in their order of magnitude, forms, by theorem C of § 16, a well-ordered aggregate :

$$(4) \qquad \epsilon_0, \epsilon_1, \ldots, \epsilon_\nu, \ldots \epsilon_{\omega+1}, \ldots \epsilon_{\alpha'} \ldots,$$

whose law of formation is expressed in the theorems D and E. Now, if the index α' did not successively take all the numerical values of the second number-class, there would be a least number α which it did not reach. But this would contradict the theorem D, if α were of the first kind, and theorem E, if α were of the second kind. Thus α' takes all numerical values of the second number-class.

If we denote the type of the second number-class by Ω, the type of (4) is

$$\omega+\Omega=\omega+\omega^2+(\Omega-\omega^2).$$

[245] But since $\omega+\omega^2=\omega^2$, we have

$$\omega+\Omega=\Omega\,;$$

and consequently

$$\overline{\omega+\Omega}=\overline{\Omega}=\aleph_1.$$

G. If ϵ is any ϵ-number and α is any number of the first or second number-class which is less than ϵ:

$$\alpha<\epsilon,$$

then ϵ satisfies the three equations:

$$\alpha+\epsilon=\epsilon, \quad \alpha\epsilon=\epsilon, \quad \alpha^{\epsilon}=\epsilon.$$

Proof.—If α_0 is the degree of α, we have $\alpha_0 \leqq \alpha$, and consequently, because of $\alpha<\epsilon$, we also have $\alpha_0<\epsilon$. But the degree of $\epsilon=\omega^{\epsilon}$ is ϵ; thus α has a less degree than ϵ. Consequently, by theorem D of § 19,

$$\alpha+\epsilon=\epsilon,$$

and thus

$$\alpha_0+\epsilon=\epsilon.$$

On the other hand, we have, by formula (13) of § 19,

$$\alpha\epsilon=\alpha\omega^{\epsilon}=\omega^{\alpha_0+\epsilon}=\omega^{\epsilon}=\epsilon,$$

and thus

$$\alpha_0\epsilon=\epsilon.$$

Finally, paying attention to the formula (16) of § 19,

$$\alpha^{\epsilon}=\alpha\omega^{\epsilon}=\omega^{\alpha_0}\omega^{\epsilon}=\omega^{\alpha_0\epsilon}=\omega^{\epsilon}=\epsilon.$$

H. If α is any number of the second number-class, the equation

$$\alpha^{\xi}=\xi$$

has no other roots than the ϵ-numbers which are greater than α.

Proof.—Let β be a root of the equation

$$\alpha^{\xi}=\xi,$$

so that

$$\alpha^{\beta}=\beta.$$

Then, in the first place, from this formula follows that

$$\beta>\alpha.$$

On the other hand, β must be of the second kind, since, if not, we would have

$$\alpha^{\beta}>\beta.$$

Thus we have, by theorem F of § 19,

$$\alpha^{\beta}={}^{\alpha_0\beta}.$$

and consequently

$$\omega^{\alpha_0\beta}=\beta.$$

[246] By theorem F of § 19, we have

$$\omega^{\alpha_0\beta}\geqq\alpha_0\beta,$$

and thus

$$\beta\geqq\alpha_0\beta.$$

But β cannot be greater than $\alpha_0\beta$; consequently

$$\alpha_0\beta=\beta,$$

and thus

$$\omega^{\beta}=\beta.$$

Therefore β is an ϵ-number which is greater than α.

HALLE, *March* 1897.