A Network Optimization Approach for the Optimization of Intersection Signaling

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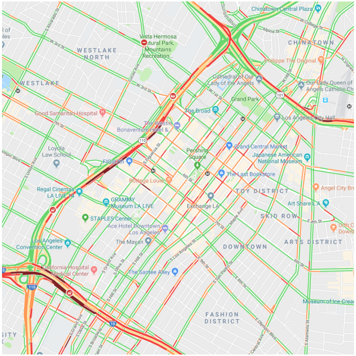
- Transportation: critical infrastructure for development of smart cities
- High complexity
- Efficiency strongly depend on **control of traffic signaling**
- Current control techniques rely on infrastructure sensing
- Intelligent vehicle technologies:
    - **New layer of communication**
    - Enormous potential for control



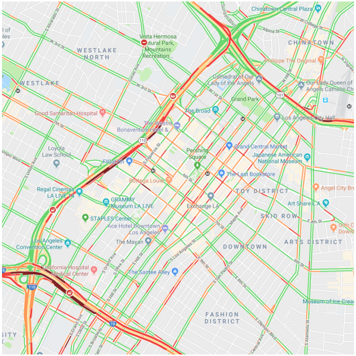
Google live traffic, Downtown LA

Goal

minimize (network congestion)
(lights schedule)

subject to (traffic conditions)
(network interconnection)

- Current methods: distributed
  - Local (infrastructure) sensing
  - Scale well
- Centralized
  - Use V2I and I2V communication
  - Insights for new control variables
  - Higher complexity



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Trade-off between model complexity and performance

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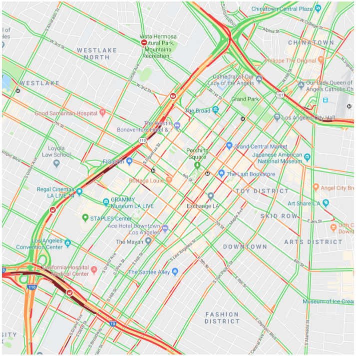

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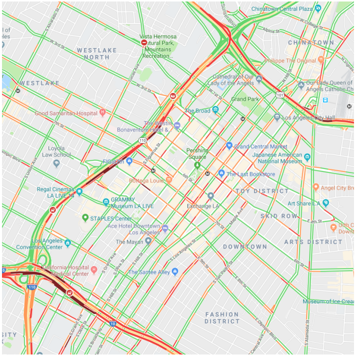
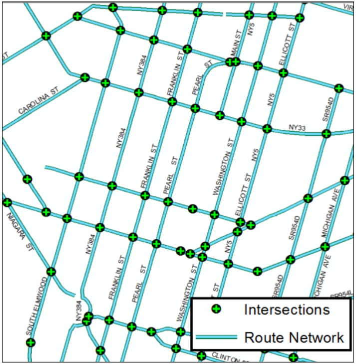


Google live traffic, Downtown LA

Trade-off between model complexity and performance

Model of traffic network

- Traffic network described by $\mathcal{N} = (\mathcal{R}, \mathcal{I})$
  - $\mathcal{R} = \{r_1, \ldots, r_{n_r}\}$ set of one-way roads
  - $\mathcal{I} = \{\mathcal{I}_1, \ldots, \mathcal{I}_{n_\mathcal{I}}\}$ intersections



- Exogenous inflows and outflows
  - Enter at source roads $\mathcal{S} \subseteq \mathcal{R}$
  - Exit at destination roads $\mathcal{D} \subseteq \mathcal{R}$

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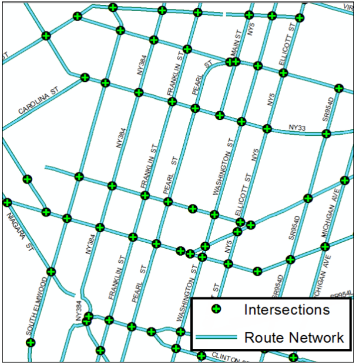

Modeling roads

$$\frac{\partial \rho}{\partial t} + \frac{\partial f}{\partial s} = 0$$

Cell Transmission Model

$$\dot{x}^k = \frac{\gamma}{h}(x^{k-1} - x^k)$$

- x^k density of cell k
- γ average speed of the flow
- h discretization step

- Road dynamics:

$$\begin{bmatrix} \dot{x}_i^1 \\ \dot{x}_i^2 \\ \vdots \\ \dot{x}_i^{\sigma_i} \end{bmatrix} = \frac{\gamma_i}{h} \begin{bmatrix} -1 & & & \\ 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} x_i^1 \\ x_i^2 \\ \vdots \\ x_i^{\sigma_i} \end{bmatrix} + \begin{bmatrix} f_{r_i}^{\text{in}} \\ 0 \\ \vdots \\ -f_{r_i}^{\text{out}} \end{bmatrix}$$

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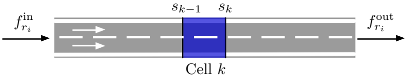

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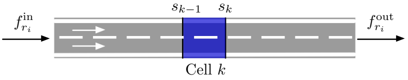

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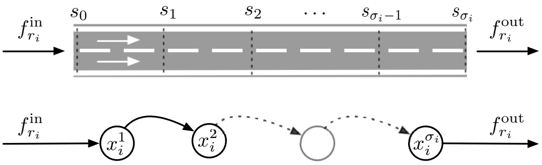
Modeling roads

Analogy between road and the associated network model



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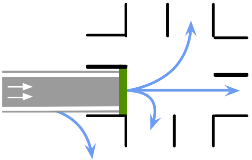
Model of interconnection flows

Intersections control road outflows



$$f_{r_i}^{\text{out}} = \sum_{r_k} s(r_k, r_i, t)\, f(r_k, r_i)$$

- Green split $s : \mathcal{R} \times \mathcal{R} \times \mathbb{R}_{\geq 0} \to \{0, 1\}$
- Transmission rate $f : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_{\geq 0}$

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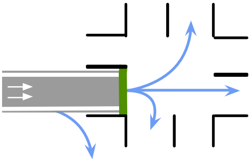
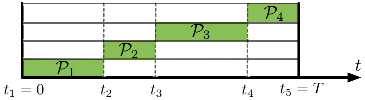


Alternate the right of way over time



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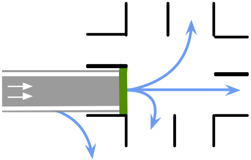

$$f(r_i, r_k) = \min\{c(r_i, r_k) x_k^{\sigma_k}, \bar{f}\}$$

- Green split $s : \mathcal{R} \times \mathcal{R} \times \mathbb{R}_{\geq 0} \to \{0, 1\}$
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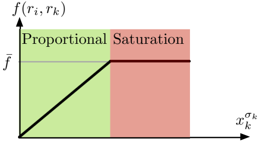
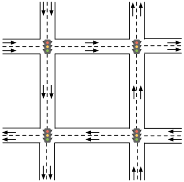


$$f(r_i, r_k) \approx c(r_i, r_k) x_k^{\sigma_k}$$

Network model



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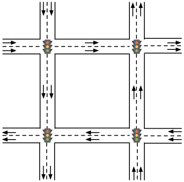

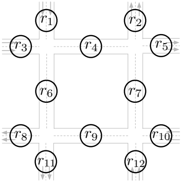

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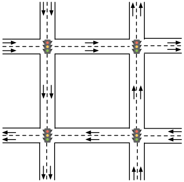
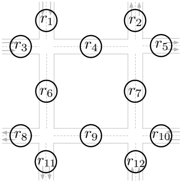
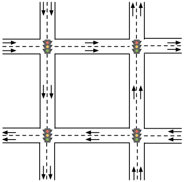
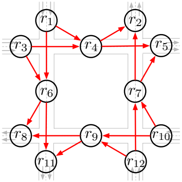




$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n_\mathsf{r}} \end{bmatrix} = \begin{bmatrix} A_{11} & & & \\ & A_{22} & & \\ & & \ddots & \\ & & & A_{n_\mathsf{r} n_\mathsf{r}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_\mathsf{r}} \end{bmatrix}$$

Network model





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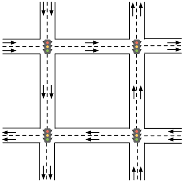
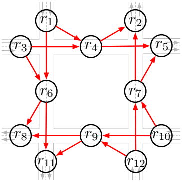
- Nodes: roads with dynamics
- Edges: intersections, time-varying

Network model





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- Outflows: proportional to roads occupancy

- Nodes: roads with dynamics
- Edges: intersections, time-varying

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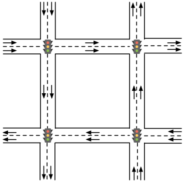

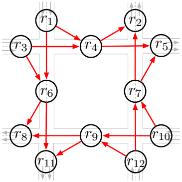

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- Outflows: proportional to roads occupancy

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n_r} \end{bmatrix} = \begin{bmatrix} e_{\sigma_1}^\mathsf{T} & \cdots & 0 \\ \vdots & \ddots & \\ 0 & \cdots & e_{\sigma_{n_r}}^\mathsf{T} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_r} \end{bmatrix}$$

Queue lengths are cell densities at the downstream

A linear switching system

Linear switching system, where the switching signals are the green split functions $s(r_i, r_k, t)$

$$\dot{x} = A_{s(r_i, r_k, t)} x$$
$$y = Cx$$

Problem formulation

Design problem

- Assume the network has a certain initial density x_0
- Find the green split functions that minimize the queue lengths

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- Assume the network has a certain initial density x_0
- Find the green split functions that minimize the queue lengths

$$
\begin{aligned}
\min_{s(r_i, r_k, t)} \quad & \int \|y\|_2^2 \, dt \\
\text{s.t.} \quad & \dot{x} = A_{s(r_i, r_k, t)} x \\
& y = Cx \\
& x(0) = x_0 \\
& s(r_i, r_k, t) \text{ is a feasible green split}
\end{aligned}
$$

Approximating traffic switching system

$$\dot{x} = A_{s(r_i,r_k,t)}x$$
$$s(r_i, r_k, t) = \text{piecewise constant}$$

$$\dot{x}_{\text{av}} = A_{\text{av}}x_{\text{av}}$$
$$A_{\text{av}} = \frac{1}{T}(A_1 d_1 + \cdots + A_m d_m)$$

Define $\{d_1, \ldots, d_m\}$ durations where $s(r_i, r_k, t) = \text{constant}$

"Average" network dynamics

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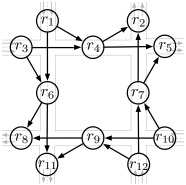

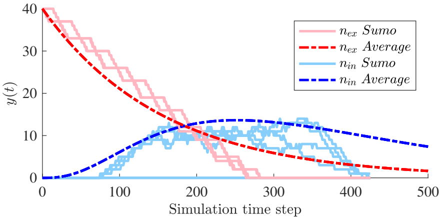

Mode durations $\{d_1, \ldots, d_m\}$ are the new design parameters

A network optimization problem

$$
\begin{aligned}
\min_{d_1,\dots,d_m} \quad & \int_0^\infty \|y_{\text{av}}\|_2^2 \, dt \\
\text{subject to} \quad & \dot{x}_{\text{av}} = A_{\text{av}} x_{\text{av}} \\
& y_{\text{av}} = C_{\text{av}} x_{\text{av}} \\
& A_{\text{av}} = \frac{1}{T} \left(d_1 A_1 + \cdots + d_m A_m \right) \\
& x_{\text{av}}(0) = x_0 \\
& T = d_1 + \cdots + d_m \\
& d_i \geq 0 \quad i \in \{1, \dots, m\}
\end{aligned}
$$

- Measurements will enter the optimization, updating x_0
- We optimize over $[0, \infty]$ and follow a “receding horizon” approach

Cost function

$$\min_{d_1,\ldots,d_m} \int_0^\infty \|y_{\mathsf{av}}\|_2^2 \; dt \quad = \quad \min_{d_1,\ldots,d_m} \int_0^\infty x_0^{\mathsf{T}} e^{A_{\mathsf{av}}^{\mathsf{T}} t} C_{\mathsf{av}}^{\mathsf{T}} C_{\mathsf{av}} e^{A_{\mathsf{av}} t} x_0 \; dt$$

- Finite if exist $\{d_1, \ldots, d_m\}$ that lead to A_{av} Hurwitz

(Theorem) Network stability = Graph-theoretic property

If there exists a path in $\mathcal{N}$ between any source $s \in \mathcal{S}$ and some destination $d \in \mathcal{D}$, then there exists $\{d_1, \ldots, d_m\}$:

$$\alpha(A_{\mathsf{av}}) < 0$$

Spectral abscissa of A_{av}

$$\alpha(A_{\mathsf{av}}) := \sup\{\Re(s) : s \in \mathbb{C}, \det(sI - A_{\mathsf{av}}) = 0\}$$

Cost function

$$\min_{d_1,\dots,d_m} \int_0^\infty \|y_{\mathsf{av}}\|_2^2 \; dt \quad = \quad \min_{d_1,\dots,d_m} \int_0^\infty x_0^\mathsf{T} e^{A_{\mathsf{av}}^\mathsf{T} t} C_{\mathsf{av}}^\mathsf{T} C_{\mathsf{av}} e^{A_{\mathsf{av}} t} x_0 \; dt$$

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Controllability metrics of traffic networks

- Define the Gramian matrix

$$\mathcal{W}(A_{\mathsf{av}}, x_0) = \int_0^\infty e^{A_{\mathsf{av}} t} x_0 x_0^\mathsf{T} e^{A_{\mathsf{av}}^\mathsf{T} t} \;\; dt$$

(Lemma) Network performance and controllability

$$\begin{aligned} \min_{d_1,\dots,d_m} \quad & \mathrm{Trace}\left(C_{\mathsf{av}}\, \mathcal{W}(A_{\mathsf{av}}, x_0)\, C_{\mathsf{av}}^\mathsf{T}\right) \\ \text{subject to} \quad & A_{\mathsf{av}} = \frac{1}{T}(d_1 A_1 + \cdots + d_m A_m) \\ & T = d_1 + \cdots + d_m \\ & d_i \geq 0, \quad i \in \{1, \dots, m\} \end{aligned}$$

The optimal split durations minimize
a controllability metric

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(Lemma) Network performance and controllability

$$\begin{aligned} \min_{d_1,\dots,d_m} \quad & \text{Trace}\left(C_{\mathsf{av}} \; \mathcal{W}(A_{\mathsf{av}}, x_0) \; C_{\mathsf{av}}^{\mathsf{T}} \right) \\ \text{subject to} \quad & A_{\mathsf{av}} = \frac{1}{T} \left(d_1 A_1 + \cdots + d_m A_m \right) \\ & T = d_1 + \cdots + d_m \\ & d_i \geq 0, \quad i \in \{1, \dots, m\} \end{aligned}$$

The optimal split durations minimize
a controllability metric

Optimizing network controllability

$$\min_{d_1,\ldots,d_m} \quad \mathrm{Trace}\left(C_{\mathsf{av}}\, \mathcal{W}(A_{\mathsf{av}}, x_0)\, C_{\mathsf{av}}^{\mathsf{T}}\right)$$

Difficulties:

- A_{av} and $\mathcal{W}(A_{\mathsf{av}}, x_0)$ are related by the (**nonlinear**) relation

$$A_{\mathsf{av}}\, \mathcal{W} + \mathcal{W}\, A_{\mathsf{av}}^{\mathsf{T}} = -(x_0 x_0^{\mathsf{T}})(x_0 x_0^{\mathsf{T}})$$

- Similar problems: consider stability $\alpha(A_{\mathsf{av}})$
  - Captures steady state rates (not transient overshoots)
  - Nonconvex in A_{av} and "very hard to optimize"

J. Vanbiervliet, B. Vandereycken, W. Michiels, S. Vandewalle, and M. Diehl, "The smoothed spectral abscissa for robust stability optimization," in *SIAM Journal on Optimization*, vol. 20, no. 1, 2009.

Optimizing network controllability

$$\min_{d_1,\ldots,d_m} \quad \mathrm{Trace}\left(C_{\mathsf{av}}\ \mathcal{W}(A_{\mathsf{av}}, x_0)\ C_{\mathsf{av}}^{\mathsf{T}}\right)$$

Difficulties:

- A_{av} and $\mathcal{W}(A_{\mathsf{av}}, x_0)$ are related by the (**nonlinear**) relation

$$A_{\mathsf{av}}\ \mathcal{W} + \mathcal{W}\ A_{\mathsf{av}}^{\mathsf{T}} = -(x_0 x_0^{\mathsf{T}})(x_0 x_0^{\mathsf{T}})$$

- Similar problems: consider stability $\alpha(A_{\mathsf{av}})$
  - Captures steady state rates (not transient overshoots)
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Optimizing network controllability

- For a certain A_{av} the associated network performance is

$$\mathrm{Trace}\left(C_{\mathsf{av}}\ \mathcal{W}(A_{\mathsf{av}}, x_0)\ C_{\mathsf{av}}^{\mathsf{T}}\right) = 1/\epsilon$$

- Now assume we desire a better performance $\bar{\epsilon} > \epsilon$
- We can make the system "faster" $A_{\mathsf{av}} \to A_{\mathsf{av}} - sI$, $s \in \mathbb{R}$ variable

$$\mathrm{Trace}\left(C_{\mathsf{av}}\ \mathcal{W}(A_{\mathsf{av}} - sI, x_0)\ C_{\mathsf{av}}^{\mathsf{T}}\right) = 1/\bar{\epsilon}$$

- Solution $s := \alpha_\epsilon(A_{\mathsf{av}})$: "smoothed spectral abscissa"
  - Unique
  - Differentiable in A_{av} (and $\{d_1, \dots d_m\}$)

Questions

- Can we design A_{av} so that $s = \alpha_{\bar{\epsilon}}(A_{\mathsf{av}}) = 0$?
- If yes, A_{av} will have performance $1/\bar{\epsilon}$

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Optimizing network controllability

$$\begin{aligned} \alpha_{\bar{\epsilon}}^* = \min_{d_1,\dots,d_m} \quad & |\alpha_{\bar{\epsilon}}(A_{\mathsf{av}})| \\ \text{subject to} \quad & A_{\mathsf{av}} = \frac{1}{T}\left(d_1 A_1 + \cdots + d_m A_m\right) \\ & T = d_1 + \cdots + d_m \\ & d_i \geq 0, \quad i \in \{1, \dots, m\} \end{aligned}$$

Gradient descent can numerically solve this problem

$$\frac{\partial \alpha_\epsilon(A_{\mathsf{av}})}{\partial d} = \operatorname{vec}\left(\frac{QP}{\operatorname{Trace}(QP)}\right)\frac{\partial a_{\mathsf{av}}}{\partial d}$$

$$(A_{\mathsf{av}} - \alpha_\epsilon(A_{\mathsf{av}})I)P + P(A_{\mathsf{av}} - \alpha_\epsilon(A_{\mathsf{av}})I)^{\mathsf{T}} + x_0 x_0^{\mathsf{T}} = 0$$

$$(A_{\mathsf{av}} - \alpha_\epsilon(A_{\mathsf{av}})I)^{\mathsf{T}} + Q(A_{\mathsf{av}} - \alpha_\epsilon(A_{\mathsf{av}})I) + C_{\mathsf{av}} C_{\mathsf{av}}^{\mathsf{T}} = 0$$

Optimizing network controllability

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Optimizing network controllability: gradient-descent algorithm

$$\min_{d_1,\dots,d_m} \quad |\alpha_{\bar\epsilon}(A_{\mathsf{av}})|$$

$$\text{s.t.} \quad A_{\mathsf{av}} = \frac{1}{T}(d_1 A_1 + \dots + d_m A_m)$$

$$T = d_1 + \dots + d_m$$

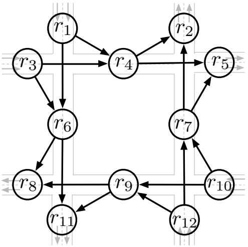
$$d_i \geq 0, \quad i \in \{1, \dots, m\}$$

Input: Matrix C_{av}, vector x_0, scalars μ, ν_{min}, $\bar\epsilon$
Output: $\{d_1^*, \dots d_m^*\}$
Initialize $d^{(0)}$, $k = 1$
repeat
- Compute the current value of $\alpha_\epsilon(A_{\mathsf{av}}^{(k)})$;
- Compute $\frac{\partial \alpha_\epsilon(A_{\mathsf{av}}^{(k)})}{\partial d}$;
- Compute $\frac{\partial \alpha_\epsilon(A_{\mathsf{av}}^{(k)})^2}{\partial d^{(k)}}$;
- Update: $\delta^{(k)} \leftarrow d^{(k-1)} + \mu \frac{\partial \alpha_\epsilon(A_{\mathsf{av}}^{(k)})^2}{\partial d^{(k)}}$;
- Projection: $d^{(k)} \leftarrow \underset{d \in \Delta}{\operatorname{argmin}} \|\delta^{(k)} - d\|$;
- Update $A_{\mathsf{av}}^{(k)}$;
- $k \leftarrow k + 1$;

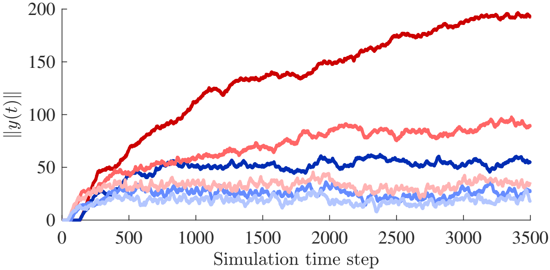
until $|\alpha_\epsilon(A_{\mathsf{av}}^{(k)})^2 - \alpha_\epsilon(A_{\mathsf{av}}^{(k-1)})^2| < \nu_{\mathsf{min}}$;
return $d^{(k)}$;

Preliminary simulation experiments

Additional algorithm complexity is justified by increased performance



Distributed vs Centralized



Dark colors: "slower" network

Light colors: "faster" network

Summary and ongoing effort

Motivation: incorporate new traffic data and model interconnection

Approximate model: tradeoff between complexity and accuracy

- Benefits: centralized techniques give better insight
- Allow network design
- Allow design of new control parameters
- Better performance

Ongoing effort:

- Validation of the technique over existing traffic networks
- Computational:
  - Distributed implementation of gradient descent
- Security analysis

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