Data-Driven Synthesis of Optimization-Based Controllers for Regulation of Unknown Linear Systems

Gianluca Bianchin Miguel Vaquero Jorge Cortés Emiliano Dall'Anese

Abstract— **This paper proposes a data-driven framework to solve time-varying optimization problems associated with unknown linear dynamical systems. Making online control decisions to steer a system to the solution trajectory of a time-varying optimization problem is a central goal in many modern engineering applications. Yet, the available methods critically rely on a precise knowledge of the system dynamics, thus requiring ad-hoc system identification and model refinement phases. In this work, we leverage tools from behavioral theory to show that the steady-state transfer function of a system can be computed from control experiments without knowledge or estimation of the system model. Such direct computation allows us to avoid the explicit model identification phase, and is significantly more tractable than the direct model-based computation. We leverage the data-driven representation to design a controller inspired from a gradient-descent method that drives the system to the solution of an unconstrained optimization problem, without any knowledge of time-varying disturbances affecting the model equation. Results are tailored to cost functions that are smooth and satisfy the Polyak-Łojasiewicz inequality. Simulation results illustrate the technical findings.**

I. Introduction

Online optimization problems have attracted significant attention in various disciplines, including machine learning [1], control systems [2], [3], and transportation management [4]. When considering dynamical systems, a basic online optimization setting consists in making online decisions in order to minimize a pre-specified loss function that is time-varying according to an underlying and possibly uncertain dynamical environment [5]–[7]. Such problems correspond to scenarios where the loss function may change over time to reflect dynamic performance objectives or, more generally, to take into account time-varying unknown exogenous disturbances that affect the dynamical system.

The vast majority of works on online optimization of dynamical systems make a strict assumption on the knowledge of the underlying system dynamics (see e.g. [2], [8]–[10]). However, besides their theoretical value, maintaining and refining full system models is often undesirable because: (i) perfect knowledge of the dynamics is rarely available in practice since it requires an explicit system identification phase and periodic re-update, and (ii) identifying a full model of the system is often unnecessary since most optimizing controllers rely on simple model representations that promote convergence. To the best of our knowledge, efficient and numerically reliable online optimization methods that bypass the model identification phase are still lacking.

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In this work, we take a novel approach to design online optimization algorithms that relies on tools from behavioral systems theory [11] to construct a succinct data-driven representations of a dynamical system, which is then used for algorithm synthesis. Precisely, we assume the availability of noise-free historical data, i.e., finite-length trajectories produced by the open-loop dynamics, and we show that the steady-state transfer function of the system can be computed from non-steady input-output data. The noise-free assumption corresponds to scenarios where accurate sensors or sophisticated signal-processing techniques can be utilized to produce noise-free data offline (a similar approach was taken in [12], [13]). Then, we build upon such data-driven representation to propose a controller inspired by online optimization methods that steer the dynamical system towards an equilibrium point that minimizes a given loss function, despite unknown and time-varying noise terms affecting the model equation. Interestingly, our results suggest that a suitable choice of the controller stepsize is sufficient to guarantee asymptotic convergence to the desired optimizers, up to an asymptotic error that is bounded by the time-variability of the exogenous noise terms.

Related Work. The results presented here are tied to the fields of data-driven control and online optimization. The success of data-driven control methods mainly originates from the possibility of synthesizing controllers without first identifying a model for the system. Among these methods, the behavioral framework has recently regained considerable attention [11], [14], [15]. Recent extensions include distributed formulations [16], combinations with model predictive control [12], [17], trajectory tracking [13], and nonlinear systems [18], [19]. In this work, we leverage the behavioral framework to build a data-driven representation of a dynamical system that is the used for optimization purposes. Especially relevant to this work are the recent results [13], [20]–[22] that focus on the presence of noise in the data.

Online optimization approaches aim to optimize loss functions in connection with an underlying and uncertain dynamical system. Linear time-invariant systems are considered in e.g., [2]–[4], [9], stable nonlinear systems in [8], [23], and switched systems in [10]. In contrast with the above line of work, which considers continuous-time dynamics, the focus of this paper is on systems and controllers that operate at discrete time. Although online optimization methods are central in many engineering fields, data-driven techniques to solve online optimization problems without any knowledge of the system dynamics are critically lacking. A relevant exception is the recent work [24], which however considers scenarios where the model operates in the absence of noise, and results are limited to regret analysis.

Contributions. This work features two main contributions. First, we show that the steady-state transfer function of a linear time-invariant dynamical system can be obtained from non steady-state input-output trajectories of the open-loop system, without any knowledge or estimation of the system matrices. Interestingly, our results also suggest that the steady-state transfer function can be computed exactly from input-output data even when the trajectories are affected by constant noise terms. In contrast with the vast majority of the available literature on data-driven control that consider disturbances affecting the system output (see e.g. [20], [22]), our work focuses on disturbances affecting the model equation. Second, we prove input-to-state stability (ISS) of the dynamical system obtained by interconnecting the dynamical environment with a gradient-flow controller. Our results build upon the theory of ISS Lyapunov functions for discrete-time dynamical systems [25], properly modified to guarantee stability with respect to compact sets of optimizers [26].

Organization. The paper is organized as follows. Section II presents basic notions used in our analysis and Section III formalizes the problem of interest. Section IV introduces the proposed online-optimization controller, illustrates our data-driven representation of a dynamical plant, and establishes convergence of the proposed method. We present the proofs of all results in Section V and numerical simulations in Section VI. Section VI gathers our conclusions.

II. Preliminaries

We first outline the notation and recall few basic concepts.

A. Notation

Given a symmetric matrix $M \in \mathbb{R}^{n\times n}$, $\underline{\lambda}(M)$ and $\bar{\lambda}(M)$ denote the smallest and largest eigenvalue of M, respectively; $M \succ 0$ indicates that M is positive definite. For vectors $u \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$, $(x, u) \in \mathbb{R}^{n+m}$ denotes their vector concatenation. We denote by $\|u\|$ the Euclidean norm of u; $u^\top$ denotes transposition; given nonempty compact sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^n$, $|u|_{\mathcal{A}} = \inf_{z\in\mathcal{A}} \|z - u\|$ denotes the point-to-set distance, while $\operatorname{dist}(\mathcal{A}, \mathcal{B}) := \max\{\sup_{x\in\mathcal{A}} \inf_{y\in\mathcal{B}} \|x - y\|, \sup_{y\in\mathcal{B}} \inf_{x\in\mathcal{A}} \|x - y\|\}$ denotes the Hausdorff distance.

A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$, and it is of class $\mathcal{K}_\infty$ if in addition $\lim_{r\to\infty} \gamma(r) = \infty$.

B. Persistence of Excitation

We next recall some useful facts on behavioral system theory from [11]. For a signal $k \mapsto z_k \in \mathbb{R}^\sigma$, $k \in \mathbb{Z}_{\geq 0}$, we denote by $z_{[k,k+T]}$, $k \in \mathbb{Z}$, $T \in \mathbb{Z}_{\geq 0}$, the vectorization of z restricted to the interval $[k, k + T]$, namely:

$$z_{[k,k+T]} = (z_k, \ldots, z_{k+T}).$$

Given $z_{[0,T-1]}$, $t \leq T$, and $q \leq T - t + 1$, we let $Z_{t,q}$ denote the Hankel matrix of length t associated with $z_{[0,T-1]}$:

$$Z_{t,q} = \begin{bmatrix} z_0 & z_1 & \cdots & z_{q-1} \\ z_1 & z_2 & \cdots & z_q \\ \vdots & \vdots & \ddots & \vdots \\ z_{t-1} & z_t & \cdots & z_{T-1} \end{bmatrix} \in \mathbb{R}^{\sigma t\times q}.$$

The signal $z_{[0,T-1]}$ is persistently exciting of order t if $Z_{t,q}$ has full row rank; that is, the row rank is σt. Notice that this implicitly requires $q \geq \sigma t$ and $T \geq (\sigma + 1)t - 1$.

The linear dynamical system

$$x_{k+1} = Ax_k + Bu_k, \qquad y_k = Cx_k + Du_k, \tag{1}$$

$x \in \mathbb{R}^n$, is controllable if $\mathcal{C} := [B, AB, A^2B, \ldots, A^{n-1}B]$ satisfies $\operatorname{rank}(\mathcal{C}) = n$. We also recall the following two properties of (1) when its input is persistently exciting.

Lemma 2.1: ***(Fundamental Lemma)*** [11, Corollary 2] Assume (1) is controllable, let $(u_{[0,T-1]}, y_{[0,T-1]})$, $T \in \mathbb{Z}_{>0}$, be an input-output trajectory of (3). If $u_{[0,T-1]}$ is persistently exciting of order $n + L$, then:

$$\operatorname{rank} \begin{bmatrix} U_{L,q} \\ X_{1,q} \end{bmatrix} = Lm + n,$$

where $U_{L,q}$ and $X_{1,q}$ denote the Hankel matrices associated with $u_{[0,T-1]}$ and $x_{[0,T-1]}$, respectively. □

Lemma 2.2: [11, Theorem 1] Assume (1) is controllable, let $(u_{[0,T-1]}, y_{[0,T-1]})$, $T \in \mathbb{Z}_{>0}$, be an input-output trajectory of (3). If $u_{[0,T-1]}$ is persistently exciting of order $n + L$, then any pair of L-long signals $(\tilde{u}_{[0,L-1]}, \tilde{y}_{[0,L-1]})$ is an input-output trajectory of (3) if and only if there exists $\alpha \in \mathbb{R}^q$ such that:

$$\begin{bmatrix} \tilde{u}_{[0,L-1]} \\ \tilde{y}_{[0,L-1]} \end{bmatrix} = \begin{bmatrix} U_{L,q} \\ Y_{L,q} \end{bmatrix} \alpha,$$

where $U_{L,q}$ and $Y_{L,q}$ denote the Hankel matrices associated with $u_{[0,T-1]}$ and $y_{[0,T-1]}$, respectively. □

In words, persistently exciting signals generate output trajectories that can be used to express any other trajectory.

III. Problem Formulation

We consider the problem of steering a dynamical system with unknown dynamics to an equilibrium point that minimizes a pre-specified loss function. The dynamical system is subject to unknown and time-varying disturbances, which make the the objective function inaccessible and time-varying, as described by the system dynamics.

Let $k \mapsto u_k \in \mathbb{R}^m$ be the online decision at time $k \in \mathbb{Z}_{\geq 0}$ and denote by $\phi : \mathbb{R}^m \to \mathbb{R}$, $\psi : \mathbb{R}^p \to \mathbb{R}$ pre-specified loss functions. We are interested in making online control decisions u_k that minimize the loss:

$$(u_k^*, x_k^*, y_k^*) \in \arg\min_{\bar{u},\bar{x},\bar{y}} \quad \phi(\bar{u}) + \psi(\bar{y}), \tag{2a}$$

$$\text{s.t.} \quad \bar{x} = A\bar{x} + B\bar{u} + Ew_k, \tag{2b}$$

$$\bar{y} = C\bar{x}, \tag{2c}$$

where $k \mapsto w_k \in \mathbb{R}^r$ denotes an unknown exogenous input or disturbance, which is assumed to be bounded at all times, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $E \in \mathbb{R}^{n\times r}$, are unknown matrices describing the system dynamics. We stress that the value of the objective functions is unknown and time-varying since it depends on the exogenous disturbance w_k. We make the following assumption on the loss functions.

Assumption 1: The functions $u \mapsto \phi(u)$ and $y \mapsto \psi(y)$ are differentiable and have Lipschitz-continuous gradients with constants ℓ_ϕ, ℓ_ψ, respectively. Moreover, $f(u) = \phi(u) + \psi(Gu + Hw)$ is radially-unbounded, has a nonempty set of minimizers, and satisfies the Polyak-Łojasiewicz (PL) inequality, i.e., there exists $\mu > 0$ such that $\frac{1}{2}\|\nabla f(u)\|^2 \geq \mu(f(u) - f(u^*))$, for all $u \in \mathbb{R}^m$ and all minimizer u^*. □

Lipschitz-continuity assumptions are standard for the analysis of first-order optimization algorithms. The PL inequality guarantees that every critical point of (2) is a global minimizer. We note that the PL inequality is a weaker condition than strong convexity (in particular, it implies invexity).

We assume that the environment is described by a system of linear time-invariant dynamical equations:

$$x_{k+1} = Ax_k + Bu_k + Ew_k, \;\; y_k = Cx_k, \tag{3}$$

where $k \in \mathbb{Z}_{\geq 0}$ is the time index, $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ describes the control decision, and $w_k \in \mathbb{R}^r$ denotes the unknown disturbance. We assume that the equilibrium points of (3) are asymptotically stable, as formalized next.

Assumption 2: The matrix A is Schur stable; i.e, for all $Q \succ 0$ there exists $P \succ 0$ such that $A^\mathsf{T} PA - P = -Q$. Moreover, (3) is controllable and the columns of C are linearly independent. □

Remark 1: The linear-independence requirement concerning the columns of C implies that the state of (3) can be fully determined given the system output. Our results suggest that this assumption is necessary for the claims presented in Section IV. Indeed, when C is not of full column rank, there exists an infinite number of data-driven representations of the system that are compatible with the observed data. We refer to Section V-A for a precise discussion. □

Under Assumption 2, the system (3) admits a well-posed steady-state input-output relationship, which is given by:

$$y_k = \underbrace{C(I-A)^{-1}B}_{:=G}\, u_k + \underbrace{C(I-A)^{-1}E}_{:=H}\, w_k. \tag{4}$$

We seek to synthesize an online algorithm that does not require any prior knowledge of the matrices (A, B, C, E) as well as of the unknown disturbance w_k, and uses only output-feedback from (3) (see Fig. 1 for an illustration):

$$u_{k+1} = F_\text{c}(u_k, y_k), \tag{5}$$

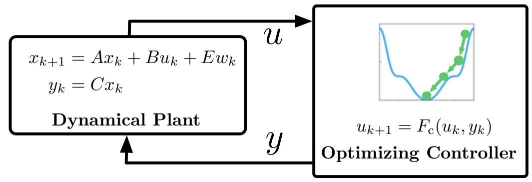

Fig. 1. Online gradient-flow optimizer used as an output feedback controller for unknown LTI systems subject to time-varying disturbances.

to track the minimizers of the time-varying objective function in (2). Precisely, we note that: (i) since the exogenous input w_k is time-varying, the optimizers of (2) are time-varying quantities; i.e., by letting $\mathcal{U}_k^* := \{u_k^* \;:\; 0 = \nabla\phi(u_k^*) + \nabla\psi(Gu_k^* + Hw_k)\}$ denote the set of optimizers of (2) at time k, we have that $\mathcal{U}_{k+1}^* \neq \mathcal{U}_k^*$ in general, and (ii) because we assume no prior knowledge on w_k, any controller of the form (5) can track the solutions of (2) up to an error that depends on the time-variability of $w_{k+1} - w_k$. For these reasons, we aim at guaranteeing that the output of the system (3) satisfies the following tracking bound:

$$\begin{aligned} |\xi_k - \xi_k^*|_{\mathcal{U}_k^*} &\leq \beta(|\xi_{k_0} - \xi_{k_0}^*|_{\mathcal{U}_{k_0}^*}, k - k_0) \\ &+ \gamma_u(\sup_{t\geq k_0} \operatorname{dist}(\mathcal{U}_{t+1}^*, \mathcal{U}_t^*)) + \gamma_w(\sup_{t\geq k_0} \|w_{t+1} - w_t\|) \end{aligned} \tag{6}$$

for all $0 \leq k_0 \leq k$, where β is a class-$\mathcal{KL}$ function, γ_u, γ_w are class-$\mathcal{K}$ functions, $\xi_k := (x_k, u_k)$ denotes the joint system-controller state, $\xi_k^* := ((I - A)^{-1}(Bu_k^* + Ew_k), u_k^*)$ denotes the corresponding optimizer of (2), and $\operatorname{dist}(\mathcal{U}_{t+1}^*, \mathcal{U}_t^*)$ denotes the Hausdorff distance between the compact sets $\mathcal{U}_{t+1}^*$ and $\mathcal{U}_t^*$.

We note that, because the cost function is radially-unbounded (see Assumption 1), the set of optimizers of (2) is compact and thus the Hausdorff distance $\operatorname{dist}(\mathcal{U}_{t+1}^*, \mathcal{U}_t^*)$ is finite for all k. Finally, we note that (6) also guarantees output tracking, as specified by the constraint (2c).

Remark 2: We notice that although $\operatorname{dist}(\mathcal{U}_{t+1}^*, \mathcal{U}_t^*)$ may be reconducted to the time-variability of w_k, (6) reflects more accurately the role of individual error terms, since a precise relationship between $w_{k+1} - w_k$ and $\operatorname{dist}(\mathcal{U}_{t+1}^*, \mathcal{U}_t^*)$ may not be known in general. Furthermore, in cases where the loss functions $\phi(u)$ and $\psi(y)$ are time-varying, the former term can be used to model time-variability of the optimizers. □

IV. Data-Driven Method for Online Optimization

To track the optimizers of (2), we propose the following controller inspired from an online gradient method:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Ew_k, \quad y_k = Cx_k, \\ u_{k+1} &= u_k - \eta(\nabla\phi(u_k) + G^\mathsf{T}\nabla\psi(y_k)), \end{aligned} \tag{7}$$

where $\eta \in \mathbb{R}_{>0}$ is the step size, which in the remainder is interpreted as a tunable controller parameter. Fig. 1 illustrates the interconnection (7). We note that the controller (7) does not rely on any knowledge of the system matrices (A, B, C, E), instead, it requires an exact expression for the map G. In order to implement (7), we propose a two-phase control method, where input-output data from the system

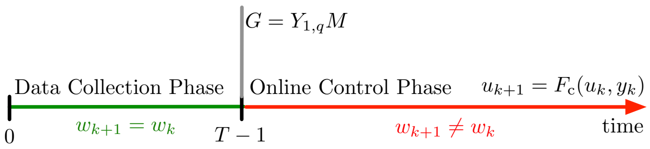

Fig. 2. Two-phase control method where noise-free historical data is used to compute the map G, and then a dynamical controller is used to track the optimizers of (2) despite unknown and time-varying disturbances.

is used to determine the map G, and then the feedback-loop (7) is used to track the time-varying optimizers of (2). Figure 2 illustrates the two phases. We note that a similar two-phase approach leveraging offline data was proposed in [13] for output tracking as well as in [12] for model predictive control.

A. Data-Driven Characterization of the Transfer Function

The following result shows that, when $w_k = 0$ at all times, the steady-state transfer function G can be computed from a non-steady-state input-output trajectory of the system.

Theorem 4.1: ***(Data-Driven Characterization of Steady-State Transfer Function in the Absence of Noise)*** Let Assumption 2 be satisfied, let $u_{[0,T-1]}$ be persistently exciting of order $n+1$, let $q = T-1$, and assume $W_{1,q} = 0$. Then, there exists $M \in \mathbb{R}^{q\times m}$ such that:

$$Y_{1,q}^{\text{diff}} M = 0, \qquad U_{1,q}M = I, \tag{8}$$

where $Y_{1,q}^{\text{diff}} = [y_1 - y_0,\ y_2 - y_1,\ \dots\ y_q - y_{q-1}]$. Moreover, for any M that satisfies (8), the steady-state transfer function of (3) equals $G = Y_{1,q}M$. □

We postpone the proof of Theorem 4.1 to Section V. Theorem 4.1 asserts that the map G can be computed from (non steady-state) input-output data originated from (3), by solving the set of linear equations given by (8). It is worth noting that, in general, the matrix M defined by the set of equations (8) depends on the specific choice of $u_{[0,T-1]}$. Moreover, given an input sequence $u_{[0,T-1]}$, in general, there exists an infinite number of choices of M that satisfy (8). Despite M not being unique, Theorem 4.1 shows that $Y_{1,q}M$ is unique and independent of the choice of $u_{[0,T-1]}$ adopted to generate the data, and coincides with the steady-state transfer function of (3).

Theorem 4.1 assumes the availability of noise-free historical data, i.e., finite-length trajectories produced by the open-loop system (3) in the absence of exogenous disturbance w_k. When w_k is non-zero but constant at all times, Theorem 4.1 can still be used to determine the input-to-output map of (3), as described next. For all $k \in \mathbb{Z}_{\geq 0}$, define

$$d_k := x_{k+1} - x_k, r_k := y_{k+1} - y_k, v_k := u_{k+1} - u_k. \tag{9}$$

By substituting into (3), the new variables follow the dynamical update:

$$d_{k+1} = Ad_k + Bv_k, \qquad r_k = Cd_k. \tag{10}$$

By leveraging the representation (10), Theorem 4.1 can be used determine the map G when the available data is affected by constant disturbance, as formalized next.

Corollary 4.2: ***(Data-Driven Characterization of Steady-State Transfer Function With Constant Noise)*** Let Assumption 2 be satisfied, assume $u_{[0,T-1]}$ is persistently exciting of order $n+1$, and let $w_k = w \in \mathbb{R}^r$ for all $k \in \mathbb{Z}_{\geq 0}$. Then, the steady-state transfer function of (3) equals $G = R_{1,q}M$, $q = T-1$, where

$$R_{1,q}^{\text{diff}} M = 0, \qquad V_{1,q}M = I, \tag{11}$$

and $R_{1,q}^{\text{diff}} = [r_1 - r_0,\ r_2 - r_1,\ \dots\ r_q - r_{q-1}]$. □

Two comments are in order. First, we note that when $u_{[0,T]}$ is persistently exciting of order $n+1$, then $v_{[0,T-1]}$ is persistently exciting of the same order (notice that $u_{[0,T]}$ contains one additional sample as compared to $v_{[0,T-1]}$). Indeed, the columns of $V_{1,q}$ are obtained by subtracting disjoint pairs of columns of $U_{1,q+1}$, which are linearly independent. Second, the Hankel matrices $R_{1,q}, V_{1,q}$, and $R_{1,q}^{\text{diff}}$ can be computed directly from an input-output trajectory of the open-loop system (3) by pre-processing the data as described by (9).

B. Convergence to First-Order Optimizers

We now turn our attention to the online control phase, where the map G computed according to Theorem 4.1 is used in the gradient-based controller in (7). The following result guarantees global convergence of to the set of optimizers of (2) under a suitable choice of the stepsize η.

Theorem 4.3: Let Assumptions 1-2 hold, let $\ell = \ell_\phi + \|G\|^2\ell_\psi$, and assume that Q satisfies $\underline{\lambda}(Q) > \frac{a}{\epsilon(1-\epsilon)}$, where $a = \frac{1}{2}\ell_\psi^2\|C\|^2\|G\|^2$ and $\epsilon \in (0,1)$ is a given parameter. Moreover, let

$$\eta^* := \frac{1-\epsilon}{\ell/2 + b}, \qquad b := \frac{2\|A^\mathsf{T} P\bar{G}\|^2}{\epsilon\underline{\lambda}(Q)} + \|\bar{G}^\mathsf{T} P\bar{G}\|,$$

where $\bar{G} := (I-A)^{-1}B$. Then, for every $\eta < \eta^*$ the solutions of (7) satisfy (6). □

We postpone the proof of Theorem 4.3 to Section V.

Some observations are in order. First, Theorem 4.3 asserts that, when the dynamics of the controller are sufficiently slower than those of the system (that is, when the stepsize η is sufficiently small), (7) tracks the desired optimal solutions up to an asymptotic error that depends on the time-variability of the optimal set and on the time-variability of the unknown exogenous signal w_k. Such tracking bound has a similar structure as the ones in [10] for gradient flows and hybrid accelerated methods (see also [8], [9] for the case of static optimization problem and disturbances). Second, although an exact computation of η^* requires the knowledge of the system matrices (A, C), Theorem 4.3 provides an existence claim, namely, it guarantees that a sufficiently small choice of η ensures convergence to the desired optimizers. Third, we note that the requirement $\underline{\lambda}(Q) > \frac{a}{\epsilon(1-\epsilon)}$ is non-restrictive, since the choice of Q is arbitrary in Assumption 2.

Remark 3: We recall that, when the plant is replaced by a static map (i.e. the plant dynamics in (7) are replaced by the algebraic map $y_k = Gu_k + Hw_k$), standard results guarantee the convergence of gradient-flow dynamics for all $\eta < \eta_{\text{static}} := 2/\ell$ (see [27]). Thus, by noting that $\eta^* <$

η_{static}, Theorem 4.3 suggests that a strictly smaller stepsize is necessary to guarantee converge of dynamical systems. □

V. Technical Analysis

In this section, we provide the proofs of the main results.

A. Proof of Theorem 4.1

To prove existence of M, we note that by application of Lemma 2.1, all the rows of $D := [(Y_{1,q}^{\text{diff}})^\mathsf{T}, U_{1,q}^\mathsf{T}]^\mathsf{T}$, are linearly independent. Hence, D admits a right inverse, and the existence of M follows by partitioning $D^\dagger = [R,\ M]$.

Next, we show $G = Y_{1,q}M$. Let $(\tilde{u}, \tilde{x}, \tilde{y})$ denote an equilibrium point of (3) with $w(k) = 0$. By application of Lemma 2.2, there exists $\alpha \in \mathbb{R}^q$ such that:

$$\begin{bmatrix}\tilde{x}\\ \tilde{u}\end{bmatrix} = \begin{bmatrix}X_{1,q}\\ U_{1,q}\end{bmatrix}\alpha. \tag{12}$$

By combining the above expression with $\tilde{x} = A\tilde{x} + B\tilde{u}$ and $\tilde{y} = C\tilde{x}$, we have:

$$0 = C\begin{bmatrix}A - I & B\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{u}\end{bmatrix} = C\begin{bmatrix}A - I & B\end{bmatrix}\begin{bmatrix}X_{1,q}\\ U_{1,q}\end{bmatrix}\alpha,$$

and by noting that $Y_{1,q}^{\text{diff}} = C(A - I)X_{1,q} + CBU_{1,q}$ we conclude $0 = Y_{1,q}^{\text{diff}}\alpha$. By combining the relationships $\tilde{u} = U_{1,q}\alpha$ and $0 = Y_{1,q}^{\text{diff}}\alpha$ we have:

$$\alpha = D^\dagger\begin{bmatrix}0\\ \tilde{u}\end{bmatrix} + \nu, \text{ where } D := \begin{bmatrix}Y_{1,q}^{\text{diff}}\\ U_{1,q}\end{bmatrix}, \tag{13}$$

where $D^\dagger$ is a right inverse of D and $\nu \in \ker(D)$. Hence, by using (12) and by recalling the partitioning $D^\dagger = [R, M]$, we conclude $\tilde{y} = CX_{1,q}\alpha = Y_{1,q}M\tilde{u} + Y_{1,q}\nu$.

Lastly, we show $Y_{1,q}\nu = 0$. By recalling that ν satisfies $Y_{1,q}^{\text{diff}}\nu = 0$ and by substituting $Y_{1,q}^{\text{diff}} = C(A - I)X_{1,q} + CBU_{1,q}$ we obtain:

$$\begin{aligned}0 = Y_{1,q}^{\text{diff}}\nu &= C(A - I)X_{1,q}\nu + CBU_{1,q}\nu\\ &= C(A - I)X_{1,q}\nu.\end{aligned} \tag{14}$$

where we used $U_{1,q}\nu = 0$. By noting that $\ker(C) = \ker(A - I) = \emptyset$ (see Assumption 2), the above identity implies $X_{1,q}\nu = 0$. Finally, $Y_{1,q}\nu = 0$ follows by observing that each column of $Y_{1,q}$ is obtained by multiplying the corresponding column in $X_{1,q}$ by matrix C. ■

Remark 4: It follows from (14) that when the columns of C are not linearly independent, the vector ν does not satisfy $Y_{1,q}\nu = 0$ in general, and thus the stedy-state transfer function G is described by $G = Y_{1,q}M + Y_{1,q}\nu$. In other words, when the state is not fully observable from the output, there exists an infinite number of maps G that are compatible with the observed data, as described by the vector ν. □

B. Proof of Theorem 4.3

The proof of Theorem 4.3 relies on the following result, which can be interpreted as an extension of the characterization of input-to-state stability for equilibrium points studied in [25] to the case of compact forward invariant sets [26].

Lemma 5.1: Consider the system $z_{k+1} = f(z_k, v_k)$ where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz and v_k is bounded. Let $\mathcal{A} \subset \mathbb{R}^n$ be a nonempty and compact set that is forward invariant for the unforced system. Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuous function such that:

$$\alpha_1(|z|_{\mathcal{U}_k^*}) \leq V(z) \leq \alpha_2(|z|_{\mathcal{U}_k^*}), \tag{15a}$$

$$V(f(z, u)) - V(z) \leq -\alpha_3(|z|_{\mathcal{U}_k^*}) + \sigma(|v|_{\mathcal{U}_k^*}), \tag{15b}$$

holds for all $z \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$, where $\alpha_1, \alpha_2, \alpha_3$ are class $\mathcal{K}_\infty$ functions and σ is of class $\mathcal{K}$. Then, there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that:

$$|z_k|_{\mathcal{U}_k^*} \leq \beta(|z_{k_0}|_{\mathcal{U}_k^*}, k - k_0) + \gamma(\sup_{t \geq k_0}\|v_t\|),$$

holds for all $0 \leq k_0 \leq k$, and for any $z_{k_0} \in \mathbb{R}^n$. □

Proof: The claim follows by iterating the proof of [25, Lemma 3.5]. Precisely, we note that, when $\mathcal{A}$ is nonempty and compact, the quantity $|z|_{\mathcal{U}_k^*}$ is well-defined and bounded for any $z \in \mathbb{R}^n$, and thus all Euclidean norms in [25, Lemma 3.5] can be replaced with $|\cdot|_{\mathcal{U}_k^*}$. ■

To prove Theorem 4.3, we begin by performing a change of variables for (7). Let $\mathcal{M}(u, w) := Gu + Hw$, and define the new state variables of the plant as $\tilde{x}_k := x_k - \mathcal{M}(u_k, w_k)$. In the new variables:

$$\begin{aligned}\tilde{x}_{k+1} &= A\tilde{y}_k - \bar{G}(u_{k+1} - u_k) - \bar{H}(w_{k+1} - w_k),\\ u_{k+1} &= u_k - \eta(\nabla\phi(u_k) + G^\mathsf{T}\nabla\psi(C\tilde{x}_k + Gu_k + Hw_k)),\end{aligned} \tag{16}$$

where $\bar{G} := (I - A)^{-1}B$ and $\bar{H} := (I - A)^{-1}E$.

Next, we let $f(u) := \phi(u) + \psi(Gu + Hw_k)$, and we denote by u^* any global minimizer of $f(u)$. We will show that the following Lyapunov function satisfies the assumptions of Lemma 5.1:

$$U(x, u) := V(u) + W(x), \tag{17}$$

where $V(u) = \frac{1}{\eta}(f(u) - f(u^*))$, $W(x) = x^\mathsf{T}Px$, and P is as in Assumption 2.

First, we observe that (15a) follows by noting that the function $U(x, u)$ is continuous, positive definite, and radially unbounded. Next, we prove that (15b) holds. By denoting in compact form $F_{\text{c}}(x, u) := -\nabla\phi(u) - G^\mathsf{T}\nabla\psi(Cx + Gu + Hw_k)$, and by noting that $\nabla f(u) = -F_{\text{c}}(0, u)$, the Lyapunov function V satisfies:

$$\begin{aligned}V(u_{k+1}) - V(u_k) &= \frac{f(u_{k+1}) - f(u_k)}{\eta} - \frac{f(u_{k+1}^*) + f(u_k^*)}{\eta}\\ &\leq \langle\nabla f(u_k), \frac{u_{k+1} - u_k}{\eta}\rangle + \frac{\ell}{2\eta}|u_{k+1} - u_k|_{\mathcal{U}_k^*}^2\\ &\quad + \underbrace{\langle\nabla f(u_k^*), \frac{u_{k+1}^* - u_k^*}{\eta}\rangle}_{=0} + \frac{\ell}{2\eta}\|u_{k+1}^* - u_k^*\|^2\\ &\leq -(1 - \frac{\ell\eta}{2})|F_{\text{c}}(\tilde{x}_k, u_k)|_{\mathcal{U}_k^*}^2 + \ell_\psi\|C\|\|G\||\tilde{x}|_{\mathcal{U}_k^*}|F_{\text{c}}(\tilde{x}_k, u_k)|_{\mathcal{U}_k^*}\\ &\quad + \frac{\ell}{2\eta}\|u_{k+1}^* - u_k^*\|^2\end{aligned} \tag{18}$$

where the first inequality follows from Lipschitz-smoothness of $f(u_k)$, and the last inequality follows from:

$$
\begin{aligned}
&-\langle F_c(0,u_k), F_c(\tilde{x}_k,u_k)\rangle \\
&\quad = -\langle F_c(0,u_k) + F_c(\tilde{x}_k,u_k) - F_c(\tilde{x}_k,u_k), F_c(\tilde{x}_k,u_k)\rangle \\
&\quad \leq -|F_c(\tilde{x}_k,u_k)|^2_{\mathcal{U}_k^*} + \ell_\psi \|C\|\|G\|\|\tilde{x}_k\||F_c(\tilde{x}_k,u_k)|_{\mathcal{U}_k^*}.
\end{aligned}
$$

Finally, by completing the squares in (18), we obtain

$$
\begin{aligned}
V(u_{k+1}) - V(u_k) &\leq -(1 - \frac{\epsilon}{2} - \frac{\ell\eta}{2})|F_c(\tilde{x}_k,u)|^2_{\mathcal{U}_k^*} \qquad (19)\\
&+ \underbrace{\frac{\ell_\psi^2\|C\|^2\|G\|^2}{2\epsilon}}_{:=a_1}|\tilde{x}_k|^2_{\mathcal{U}_k^*} + \underbrace{\frac{\ell}{2\eta}}_{:=a_2}\|u^*_{k+1} - u^*_k\|^2.
\end{aligned}
$$

By denoting in compact form $\Delta w_k := w_{k+1} - w_k$ and by recalling that $u_{k+1} - u_k = \eta F_c(\tilde{x}_k,u_k)$, the Lyapunov function W satisfies:

$$
\begin{aligned}
&W(\tilde{x}_{k+1}) - W(\tilde{x}_k) = \tilde{x}_k^\mathsf{T}(A^\mathsf{T}PA - P)\tilde{x}_k \\
&\quad + \eta^2 F_c^\mathsf{T}(\tilde{x}_k,u_k)\bar{G}^\mathsf{T}P\bar{G}\eta F_c(\tilde{x}_k,u_k) + \Delta w_k^\mathsf{T}\bar{H}^\mathsf{T}P\bar{H}\Delta w_k \\
&\quad - 2\eta\tilde{x}_k^\mathsf{T}A^\mathsf{T}P\bar{G}F_c(\tilde{x}_k,u_k) - 2\tilde{x}_k^\mathsf{T}A^\mathsf{T}P\bar{H}\Delta w_k \\
&\quad + 2\eta F_c^\mathsf{T}(\tilde{x}_k,u_k)\bar{G}^\mathsf{T}P\bar{H}\Delta w_k \\
&\leq -\underline{\lambda}(Q)|x_k|^2_{\mathcal{U}_k^*} + 2\eta\|A^\mathsf{T}P\bar{G}\||\tilde{x}_k|_{\mathcal{U}_k^*}|F_c(\tilde{x}_k,u_k)|_{\mathcal{U}_k^*} \\
&\quad + \eta^2\|\bar{G}^\mathsf{T}P\bar{G}\||F_c(\tilde{x}_k,u_k)|^2_{\mathcal{U}_k^*} + 2\|A^\mathsf{T}P\bar{H}\||\tilde{x}_k|_{\mathcal{U}_k^*}\|\Delta w_k\| \\
&\quad + \|\bar{H}^\mathsf{T}P\bar{H}\|\|\Delta w_k\|^2 + 2\eta\|\bar{G}^\mathsf{T}P\bar{H}\||F_c(\tilde{x}_k,u_k)|_{\mathcal{U}_k^*}\|\Delta w_k\|
\end{aligned}
$$

By completing the squares:

$$
\begin{aligned}
W(\tilde{x}_{k+1}) &- W(\tilde{x}_k) \leq -(1-\epsilon)\underline{\lambda}(Q)|\tilde{x}_k|^2_{\mathcal{U}_k^*} \\
&+ \eta\underbrace{\left(\frac{2\|A^\mathsf{T}P\bar{G}\|^2}{\epsilon\underline{\lambda}(Q)} + \|\bar{G}^\mathsf{T}P\bar{G}\|\right)}_{:=b_1}|F_c(\tilde{x}_k,u_k)|^2_{\mathcal{U}_k^*} \\
&+ \underbrace{\left(\frac{2\|A^\mathsf{T}P\bar{H}\|^2}{\epsilon\underline{\lambda}(Q)} + \|\bar{H}^\mathsf{T}P\bar{H}\|\right)}_{:=b_2}\|\Delta w_k\|^2 \\
&+ 2\eta\|\bar{G}^\mathsf{T}P\bar{H}\||F_c(\tilde{x}_k,u_k)|^2_{\mathcal{U}_k^*}\|\Delta w_k\|^2 \qquad (20)
\end{aligned}
$$

By combining (19)-(20) and by completing the squares, we obtain the following bound:

$$
\begin{aligned}
&U(x_{k+1},u_{k+1}) - U(x_k,u_k) \leq -(1-\epsilon-\frac{\ell\eta}{2})|F_c(\tilde{x}_k,u)|^2_{\mathcal{U}_k^*} \\
&\quad + a_1|\tilde{x}_k|^2_{\mathcal{U}_k^*} + a_2\|u^*_{k+1} - u^*_k\|^2 - (1-\epsilon)\underline{\lambda}(Q)|\tilde{x}_k|^2_{\mathcal{U}_k^*} \\
&\quad + \eta b_1|F_c(\tilde{x}_k,u_k)|^2_{\mathcal{U}_k^*} + (b_2+b_3)\|\Delta w_k\|^2,
\end{aligned}
$$

where $b_3 = \frac{2\eta^2\|\bar{G}^\mathsf{T}P\bar{H}\|}{\epsilon}$. Thus, by letting $z := (|F_c(\tilde{x}_k,u)|_{\mathcal{U}_k^*}, |\tilde{x}_k|_{\mathcal{U}_k^*})$, $v_1 := \|u^*_{k+1} - u^*_k\|$, and $v_2 := \|\Delta w_k\|$, $U(x_k,u_k)$ satisfies (15b) with:

$$
\begin{aligned}
&\alpha_3(z) = \min\{(1-\epsilon) - \frac{\eta\ell}{2} - \eta b_1, (1-\epsilon)\underline{\lambda}(Q) - a_1\}z^2, \\
&\sigma(v_1,v_2) = a_2 v_1^2 + (b_2+b_3)v_2^2,
\end{aligned}
$$

when the following conditions are satisfied:

$$
\eta < \frac{1-\epsilon}{\ell/2 + b_1}, \qquad \underline{\lambda}(Q) > \frac{a_1}{1-\epsilon}.
$$

Finally, the claim follows by taking the supremum among all optimizers u_k^* and by replacing the Hausdorff distance.

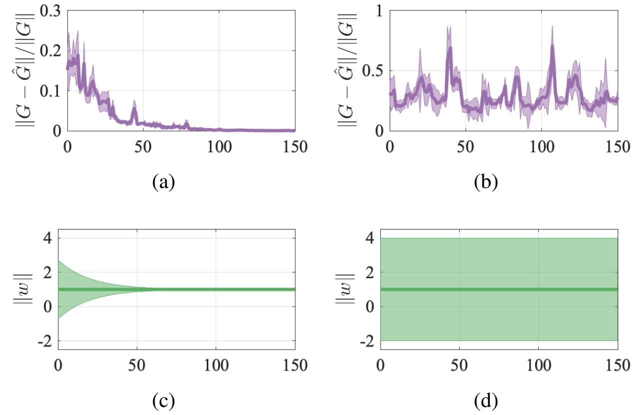

Fig. 3. Montecarlo simulations illustrating precision of data-driven transfer function computed as in Theorem 4.1. Shaded areas illustrate 3-σ confidence intervals. (a)-(c) Variance of disturbance decays to zero asymptotically. (b)-(d) Disturbance is time-varying with constant variance over time.

VI. Simulation Results

To illustrate the conclusions drawn in Theorem 4.1, we consider a linear system with $n = 20$, $m = r = 10$, $p = n$, and matrices (A, B, C, E) with random entries that satisfy Assumption 2. Fig. 3 illustrates the error in the computed transfer function $\|G - \hat{G}\|$, where G denotes true steady-state transfer function (computed by using full knowldge of matrices (A, B, C)), and $\hat{G}$ denotes the steady-state transfer function computed according to Corollary 4.2. The data-driven function $\hat{G}$ has been computed by using a rolling-horizon window that discards old input-output samples. The illustration has been produced by using the output of a Montecarlo simulation where the realization of w was varied over $10,000$ samples of a IID Gaussian process. Continuous lines illustrate the mean of the trajectory, while shaded areas illustrates the 3-standard deviation confidence intervals.

As illustrated by Fig. 3(a)-(c), as the variance of w_k vanishes over time (i.e., w_k becomes constant), the techniques presented in Section III allow us to exactly compute the map G. As illustrated qualitatively by the figure, a precise estimate can also be obtained when w varies slowly over time. In contrast, when the variance of w does not vanish over time (i.e., w_k is a time-varying signal), the method produces a biased estimate of G, as shown in Fig. 3(b)-(d).

Fig. 4(a) illustrates the tracking error for a simulation of the dynamics (7), subject to the time-varying disturbance w illustrated in Fig. 4(b). For simplicity, we consider the case where $\phi(u)$ and $\psi(y)$ are quadratic. As illustrated by the figure, and formally characterized in Theorem 4.3, the tracking error is governed by two main terms: (i) an error component associated with the initial conditions that decays to zero asymptotically, and (ii) an error component associated with the time-variability of w_k, which vanishes only if $w_{k+1} - w_k = 0$ for all $k \geq k_0$, $k_0 \in \mathbb{Z}_{\geq 0}$. The numerical simulations also suggest that values of η larger than η^* prevent the convergence of the method.

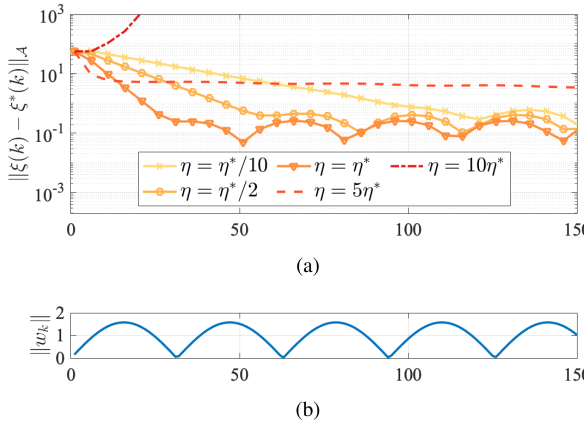

Fig. 4. (a) Tracking error of (7). (b) Time-varying disturbance. The simulation suggests that values of η larger than η^* prevent convergence.

VII. Conclusions

We proposed a data-driven method to steer a dynamical system to the solution trajectory of a time-varying optimization problem. The technique does not rely on any prior knowledge or estimation of the system matrices or of the exogenous disturbances affecting the model equation. Instead, we showed that noise-free input-output data originated by the open-loop system can be used to compute the steady-state transfer function of the dynamical environment. Moreover, we showed that convergence of the proposed algorithm to the time-varying optimizers is guaranteed when the dynamics of the controller are sufficiently slower than those of the dynamical system. This work sets out several opportunities for future works, including extensions to scenarios where historical data is affected by non-constant noise terms, and the development of data-driven methods for the convergence analysis of the interconnected system.

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