Routing Apps May Deteriorate Performance in Traffic Networks: Oscillating Congestions and Robust Information Design

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***Abstract*—A fundamental question in traffic theory concerns how drivers behave in response to sudden fluctuations of traffic congestion, and to what extent navigation apps can benefit the overall traffic system in these cases. In this paper, we study the stability of the equilibrium points of traffic networks under real-time app-informed routing. We propose a dynamical routing model to describe the real-time route selection mechanism that is at the core of app-informed routing, and we leverage the theory of passivity for nonlinear dynamical systems to provide a theoretical framework that explains emerging dynamical behaviors in real-world networks. We demonstrate for the first time the existence of oscillatory trajectories due to the general adoption of routing apps, which demonstrate how drivers continuously switch between highways in the attempt of minimizing their travel time to destination. Further, we propose a family of control policies to stabilize the system around its equilibrium points, which relies on the idea of regulating the rate at which travelers react to traffic congestion. Illustrative numerical simulations combined with empirical data from highway sensors illustrate our findings.**

I. Introduction

Traffic networks are fundamental components of modern societies, making economic activity possible by enabling the transfer of passengers, goods, and services in a timely and reliable fashion. Despite their economical importance, these transportation systems are impaired by the outstanding problem of traffic congestion, which wastes of billions of fuel each year in the United States [1] and worldwide [2]. Accompanied by increasing levels of congestion is a drop in the system performance, a phenomenon that increasingly conditions the behavior of its users, forcing travelers to shift the time of their commutes or to adopt alternative routes that are often undesirable for city planners. These alternative routes are increasingly made available by routing apps (such as Google Maps, Inrix, Waze, etc.), which provide reliable minimum-time routing suggestions to the travelers based on real-time congestion information. Despite the unprecedented widespread use of app-based routing systems, a characterization of the impact of these devices on the traffic dynamics for general, possibly capacitated, networks has remained elusive until now.

In this work, we study the stability properties of a traffic system composed of an interconnection between a dynamical model of traffic network (inspired from the Cell Transmission Model [3]) and a dynamical model of routing (inspired from the Replicator Dynamics [4]). Our models allow us to take into account the fact that traffic conditions can change while travelers are traversing the network, and that navigation apps will instantaneously respond by updating the route of each driver at her next available junction. Differently from standard methods to study congestion-responsive routing, the focus of this work is on the dynamical behavior of the traffic network, rather than on the economic properties of its equilibria. Our results suggest that: (i) when the drivers make decisions based on global congestion information, then the system admits an equilibrium point that maximizes the flow of traffic through the network, but (ii) if travelers can instantaneously respond to congestion by updating their route at the next available junction, then the stability of the system may deteriorate, and unstable behaviors such as temporal oscillations of traffic densities may originate in the highways. As a result, our models suggest that the benefits in the adoption of real-time routing systems may come at the cost of increased system fragility, indicating that adequate information design is necessary to prevent instabilities and guarantee system robustness.

Related Work. In the transportation literature, the dynamics of the traffic highways and the routing behavior of travelers have been studied in an independent fashion under the assumption that the two systems operate at different timescales.

On the one hand, dynamical traffic models with static routing preferences have been widely studied after the popularization of the Cell Transmission Model [3]. In this line of research, the main emphasis has been on the development of precise numerical models to simulate the traffic system in several regimes [5], and on characterizing the properties of its equilibria [6]. Critically, the use of time-invariant or approximate routing models limits the capability of these frameworks to predict the behavior of the system in the presence of sudden fluctuations of congestion.

On the other hand, the description of the drivers' behavior in congested networks is a well-studied topic that has been analyzed in a game-theoretic setting. This framework is commonly known as *the routing game* (see e.g. [7]–[9]), which relies on static flow models whereby traffic flows propagate instantaneously across the network. Recently, the routing game has received renewed interest when tools from Evolutionary Game-Theory [4] have been applied to study not only the equilibria of the game but also the dynamics of the decision-making mechanism [10], [11]. Unfortunately, because most of these frameworks rely on a static flow model, the validity of these results is limited to cases where routing decisions take place at a slower time-scale with respect to that of the traffic system; corresponding, for instance, to cases where routing decisions are not updated in real-time but from day-to-day.

Few recent works have explored the problem of coupling a dynamical model of traffic with a dynamical model of routing. For instance, [12]–[17] propose an effective model for simulation, and study the well-posedness of the corresponding initial

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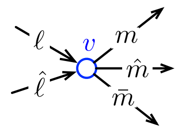
value problem. The work [18] characterizes the robustness of the system when drivers follow a local routing model. The authors in [19] show that if the timescale of the traffic network and that of the routing decisions are sufficiently separated, then the system is guaranteed to converge to a Wardrop equilibrium. The recent work [20] proposes a toll mechanism in order to steer the system to a preferred equilibrium point, while leveraging a singular-perturbation reasoning for the analysis.

Finally, we note that although microscopic traffic instabilities – such as stop-and-go traffic, showckwaves – have been widely studied and characterized, the tools available for the analysis of macroscopic traffic oscillations are to date limited to numerical simulations or empirical interpretations [21].

Contribution. The contribution of this work is fourfold. First, we propose a dynamical equation to describe congestion-dependent routing ratios in dynamical traffic models. According to the proposed setting, the routing decisions take place at the same timescale as the physics of traffic, and relative to the existing literature it does not rely on a timescale separation between the two systems. Our model is inspired from evolutionary models (or learning models) in biology and game theory, and describes a setting in which a routing app uses the observations of other travelers to quickly adjust routing suggestions. Second, we study the properties of the equilibrium points of a dynamical system that couples the dynamical model of routing with a dynamical model of traffic. We establish a connection between the properties of the equilibrium points and the notion of Wardrop equilibrium [22]. Our results show that, when app-informed travelers can update their routing preferences in real-time as they are traversing the network, the system admits an equilibrium point that satisfies the Wardrop First Principle. This observation extends the standard results in the trasportation literature, which were so far limited to scenarios where travelers update their congestion information from day to day.

Third, we characterize the Lyapunov stability of the fixed points of the interconnected system. Our analysis relies on the theory of passivity for nonlinear dynamical systems [23], and it shows that the equilibrium points are marginally stable, but may not be asymptotically stable (see Definition 2 for a formal classification of equilibrium points). Moreover, for a network composed of two parallel highways we show the existence of limit cycles, thus demonstrating that traffic congestion can oscillate over time: a phenomenon that was recently observed in [21], [24]. Fourth, we propose a control technique to guarantee the asymptotic stability of the fixed points. Our method relies on regulating the rates at which travelers react to changes in traffic congestion: a behavior that can be achieved by appropriately designing the frequency at which navigation apps update the routing suggestions.

Organization. This paper is organized as follows. Section II illustrates our traffic network model, our routing decision model, and reviews the Wardrop First Principle. Section III characterizes the properties of the equilibrium points, and presents a set of necessary and sufficient conditions for their existence. Section IV contains the stability analysis of the equilibrium points, and illustrates through an example the existence of oscillatory trajectories. Section V proposes a control technique to ensure the asymptotic stability of the fixed points, and Section VI illustrates our findings through a set of simulations. Finally, Section VII concludes the paper.



(a)

$$
\begin{aligned}
&v^{\text{in}} = \{\ell, \hat{\ell}\} \quad v^{\text{out}} = \{m, \hat{m}, \bar{m}\}\\
&\mathcal{A}_\ell = \mathcal{A}_{\hat{\ell}} = \{m, \hat{m}, \bar{m}\}\\
&\mathcal{A} = \{(\ell, m), (\ell, \hat{m}), (\ell, \bar{m}),\\
&\qquad (\hat{\ell}, m), (\hat{\ell}, \hat{m}), (\hat{\ell}, \bar{m})\}
\end{aligned}
$$

(b)

Fig. 1. Illustration of adopted notation (see (1)).

Notation. A directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{L})$, consists of a set of vertices $\mathcal{V}$ and a set of directed links (or edges) $\mathcal{L} \subseteq \mathcal{V} \times \mathcal{V}$. We use the notation $\ell = (v, w)$ to denote a directed link from node $v \in \mathcal{V}$ to node $w \in \mathcal{V}$. For $\ell = (v, w) \in \mathcal{L}$, we let $\ell^+ = v$ denote its upstream node and $\ell^- = w$ be its downstream node. Moreover, we let $v^{\text{out}} = \{(z, w) \in \mathcal{L} \;:\; z = v\}$ be the set of outgoing links from node v, and $v^{\text{in}} = \{(w, z) \in \mathcal{L} \;:\; z = v\}$ be the set of incoming links to the node (see Fig. 1). A path in $\mathcal{G}$ is a subgraph $p = (\{v_1, \dots, v_k\}, \{\ell_1, \dots, \ell_k\})$, $k \in \mathbb{N}$, such that $v_i \neq v_j$ for all $i \neq j$, and $\ell_i = (v_i, v_{i+1})$ for each $i \in \{1, \dots, k-1\}$. Path p is simple if no link is repeated in p. At times, we denote a path by $p = p_{v_1 \to v_k}$ if it starts at v_1 and ends at v_k (notice that such path may not be unique). A cycle is a path where the first and last vertex are identical, i.e., $v_1 = v_k$. Finally, $\mathcal{G}$ is acyclic if it contains no cycles.

II. Traffic Network and App Routing Models

This section is organized into three parts. First, we discuss a model of traffic network to describes the physics of traffic highways and junctions. Second, we introduce a decision model to describe congestion-dependent routing ratios. Third, we review Wardrop's First Principle.

A. *Traffic Network Model*

We model a network of highways as a directed graph (digraph) $\mathcal{G} = (\mathcal{V}, \mathcal{L})$, where $\mathcal{V} = \{v_1, \dots, v_\nu\}$ models the set of nodes (traffic junctions), and $\mathcal{L} \subseteq \mathcal{V} \times \mathcal{V}$, models the set of links (traffic highways). We use the following enumeration for the links: $\mathcal{L} = \{1, \dots n\}$, $n \in \mathbb{N}$. At every node, an incoming highway $\ell \in \mathcal{L}$ is connected to a set of outgoing highways by means of a traffic ramp. We denote by $\mathcal{A}_\ell \subseteq \mathcal{L}$ the set of highways at downstream of ℓ, and by $\mathcal{A} \subseteq \mathcal{L} \times \mathcal{L}$ (adjacent link pairs) the set of all ramps in the network (see Fig. 1 for an illustration). Formally:

$$
\begin{aligned}
&\mathcal{A} := \{(\ell, m) \;:\; \exists\, v \in \mathcal{V} \text{ s.t. } \ell \in v^{\text{in}} \text{ and } m \in v^{\text{out}}\},\\
&\mathcal{A}_\ell := \{m \in \mathcal{L} \;:\; \exists (\ell, m) \in \mathcal{A}\}.
\end{aligned} \tag{1}
$$

We describe the macroscopic behavior of each link $\ell \in \mathcal{L}$ by means of a dynamical equation that captures the conservation of flows between upstream and downstream:

$$
\dot{x}_\ell = f_\ell^{\text{in}}(x) - f_\ell^{\text{out}}(x_\ell), \tag{2}
$$

where $x_\ell : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the traffic density, $f_\ell^{\text{in}} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the traffic inflow at upstream, and $f_\ell^{\text{out}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

is the traffic outflow at downstream. We make the following assumptions on $\mathcal{G}$ and on the flow functions.

Assumption 1: The di-graph $\mathcal{G}$ admits a unique source $s \in \mathcal{L}$ where flows enter the network and a unique destination $d \in \mathcal{L}$ where flows exit the network. Moreover, for all $\ell \in \mathcal{L}$, there exists a path from s to ℓ and a path from ℓ to d. □

Assumption 2: For all $\ell \in \mathcal{L}$, $x_\ell \mapsto f_\ell^{\text{out}}(x_\ell)$ is locally Lipschitz-continuous, non-decreasing, and $f_\ell^{\text{out}}(x_\ell) = 0$ only if $x_\ell = 0$. Moreover, there exists $C_\ell \in \mathbb{R}_{>0} \cup \{+\infty\}$ such that:

$$\sup_{x_\ell \in \mathbb{R}_{\geq 0}} f_\ell^{\text{out}}(x_\ell) = C_\ell.$$

□

As formalized by Assumption 1, this work focuses on networks with one origin and one destination (often called single-commodity). Although all the results can be extended to multiple origins and destinations, this comes at the cost of considerably-higher notational complexity. For this reason, in what follows we focus our attention on a single-commodity. In the remainder, we adopt the convention $s = 1$ and $d = n$ (conventionally, we also include in the set $\mathcal{V}$ two additional nodes, respectively at the head and tail of s and d). Assumption 2 is standard in the Cell Transmission Model [3], and allows us to model networks with limited flow capacity. The following example illustrates common flow functions used in practice.

Example 1: ***(Common Link Outflow Functions)*** A common choice of outflow function, originally adopted in the Cell Transmission Model [3], is the linear function with saturation: $f_\ell^{\text{out}}(x_\ell) = \min\{v_\ell x_\ell, C_\ell\}$, where $v_\ell \in \mathbb{R}_{>0}$ models the *free-flow speed* of the link. Linear outflow functions have also been considered in the literature thanks to their simplicity [25]: $f_\ell^{\text{out}}(x_\ell) = v_\ell x_\ell$, where, in this case, $C_\ell = +\infty$. Alternatively, exponential saturation functions have widely been adopted in the recent literature (see e.g. [26]): $f_\ell^{\text{out}}(x_\ell) = C_\ell(1 - \exp(-a_\ell x_\ell))$, where $a_\ell \in \mathbb{R}_{\geq 0}$. □

To every pair of adjacent links $(\ell, m) \in \mathcal{A}$, we associate a time-varying routing ratio $r_{\ell m} : \mathbb{R}_{\geq 0} \to [0, 1]$ describing the fraction of traffic flow entering link m upon exiting ℓ, with $\sum_m r_{\ell m} = 1$. We denote by $\mathcal{R}_\mathcal{G}$ the set of feasible routing ratios for the network defined by $\mathcal{G}$. That is,

$$\mathcal{R}_\mathcal{G} := \{r_{\ell m} \in [0, 1] : (\ell, m) \in \mathcal{A}, \sum_{m \in \mathcal{A}_\ell} r_{\ell m} = 1\}, \quad (3)$$

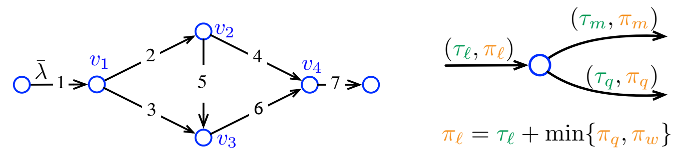
where we recall that $\mathcal{A}$ and $\mathcal{A}_\ell$ are as in (1). At every node, traffic flows are transferred from the incoming links to the outgoing links as described by the routing ratios:

$$f_m^{\text{in}}(x) = \sum_{\ell \in \mathcal{L}} r_{\ell m} f_\ell^{\text{out}}(x_\ell).$$

To describe the dynamics of the overall network, we denote by r the joint vector of routing ratios, and we combine the routing parameters into a matrix-valued map $r \mapsto R(r) = [r_{\ell m}] \in \mathbb{R}^{n \times n}$, where we let $r_{\ell m} = 0$ if $(\ell, m) \notin \mathcal{A}$. Moreover, we denote by $x = [x_1, \dots, x_n]^\mathsf{T}$ the joint vector of traffic densities in the links. The network dynamics read as:

$$\dot{x} = (R(r) - I)^\mathsf{T} f(x) + \lambda, \quad x(0) = x_0 \in \mathbb{R}^n_{\geq 0}, \quad (4)$$

where $I \in \mathbb{R}^{n \times n}$ denotes the identity matrix, $f(x) = [f_1^{\text{out}}(x_1), \dots, f_n^{\text{out}}(x_n)]^\mathsf{T}$ is the vector of link outflows, and



(a) (b)

Fig. 2. (a) Seven-highway network discussed in examples 2 and 3. (b) We associate two variables to each link: the link travel cost τ_ℓ (the travel time to traverse that link) and the link perceived cost π_ℓ (the total travel cost of reaching the network destination from that link).

$\lambda = [\bar{\lambda}, \dots, 0]^\mathsf{T}$, where $\bar{\lambda} \in \mathbb{R}_{>0}$ is the inflow of traffic at the source (see Assumption 1). Finally, we illustrate our traffic model in Example 2, and we discuss our modeling assumptions in Remark 1.

Example 2: ***(Dynamical Traffic Model)*** Consider the network illustrated in Fig. 2. The traffic model (4) reads as:

$$\begin{aligned}
\dot{x}_1 &= -f_1^{\text{out}}(x_1) + \bar{\lambda}, \\
\dot{x}_2 &= -f_2^{\text{out}}(x_2) + r_{12} f_1^{\text{out}}(x_1), \\
\dot{x}_3 &= -f_3^{\text{out}}(x_3) + r_{13} f_1^{\text{out}}(x_1), \\
\dot{x}_4 &= -f_4^{\text{out}}(x_4) + r_{24} f_2^{\text{out}}(x_2), \\
\dot{x}_5 &= -f_5^{\text{out}}(x_5) + r_{25} f_2^{\text{out}}(x_2), \\
\dot{x}_6 &= -f_6^{\text{out}}(x_6) + f_3^{\text{out}}(x_3) + f_4^{\text{out}}(x_4), \\
\dot{x}_7 &= -f_7^{\text{out}}(x_7) + f_5^{\text{out}}(x_5) + f_6^{\text{out}}(x_6).
\end{aligned}$$

The set of feasible routing ratios reads as: $\mathcal{R}_\mathcal{G} = \{r_{12}, r_{13}, r_{24}, r_{25} \in [0, 1] \, : \, r_{12} + r_{13} = 1, r_{24} + r_{25} = 1\}$. □

Remark 1: ***(Capturing Backwards Propagation)*** Our traffic model can be interpreted as a simplified version of the Cell Transmission Model [3] that does not account for backwards propagation through the junctions. More precisely, our model: (i) can capture bounds in the largest flow admissible in the links as described in Assumption 2, but (ii) congestion cannot propagate backwards through the junctions since the links do not include a link supply function (as done in e.g. [3]). □

B. Congestion-Responsive Routing Model

We next present a dynamical model to describe congestion-dependent routing ratios.

Our models rely on the observation that travelers select their routing in order to minimize their travel time to destination (similarly to e.g. [15].) To this aim, we associate to each link $\ell \in \mathcal{L}$ a function that describes the travel time (or travel delay) required to traverse link ℓ:

$$x_\ell \mapsto \tau_\ell(x_\ell) \in [0, +\infty].$$

We denote by $\tau(x) = [\tau_1(x_1), \dots, \tau_n(x_n)]^\mathsf{T}$ the joint vector of travel times, and we make the following assumption.

Assumption 3: For all $\ell \in \mathcal{L}$, $x_\ell \mapsto \tau_\ell(x_\ell)$ is Lipschitz-continuous and non-decreasing. Moreover, there exists $B_\ell \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that:

$$\lim_{x_\ell \to B_\ell} \tau_\ell(x_\ell) = +\infty, \text{ and } f_\ell(B_\ell) = C_\ell.$$

□

Assumption 3 asserts that for all $\ell \in \mathcal{L}$ there exists a critical value B_ℓ such that the travel time on highway ℓ grows unbounded as x_ℓ approaches B_ℓ. Notice that $B_\ell = +\infty$ is a feasible choice. In this case, $\tau_\ell(x_\ell) < \infty$ for any $x_\ell \in \mathbb{R}_{\geq 0}$. The value B_ℓ can be interpreted as a critical density such that when $x_\ell \to B_\ell$, then no additional vehicle will enter link ℓ.

To model networks where drivers minimize travel time to destination, we associate to each $\ell \in \mathcal{L}$ a function $x \mapsto \pi_\ell(x) \in \mathbb{R}_{\geq 0}$ that models the minimum travel time from ℓ to destination (perceived cost to destination).

Formally, let $\mathcal{P}_\ell$ denote the set of all paths from the node upstream of ℓ to destination. We model the perceived cost of ℓ as:

$$\pi_\ell(x) := \min_{p \in \mathcal{P}_\ell} \sum_{m \in p} \tau_m(x_m). \tag{5}$$

The following lemma relates travel times with perceived costs.

Lemma 2.1: ***(Recursive Definition of Perceived Costs)*** Let Assumptions 1 and 3 be satisfied. Recall that $\mathcal{A}_\ell$ denotes the set of highways downstream of $\ell \in \mathcal{L}$, and define

$$\hat{\pi}_\ell(x) := \tau_\ell(x_\ell) + \min_{m \in \mathcal{A}_\ell} \hat{\pi}_m(x), \qquad \text{for all } \ell \in \mathcal{L}. \tag{6}$$

Then, $\hat{\pi}_\ell(x) = \pi_\ell(x)$ for all $\ell \in \mathcal{L}$.

Proof: Fix $\ell \in \mathcal{L}$, let $\pi_\ell(x)$ be as in (5), and let $\hat{\pi}_\ell(x)$ be as in (6). Moreover, let $(i_1, i_2, \dots, i_h)$, with $i_1 = \ell$ and $i_h = d$ denote the sequence of links that defines $\hat{\pi}_\ell(x)$, according to (6). Then, we can express $\pi_\ell(x)$ as (we removed all function arguments for clarity):

$$\begin{aligned}\pi_\ell &= (\pi_{i_1} - \pi_{i_2}) + (\pi_{i_2} - \pi_{i_3}) + \dots + (\pi_{i_{h-1}} - \pi_{i_h}) + \pi_{i_h} \\ &= \tau_{i_1} + \tau_{i_2} + \dots + \underbrace{\tau_{i_{h-1}} + \hat{\pi}_{i_h}}_{=\hat{\pi}_{i_h - 1}}, \\ &= \tau_{i_1} + \hat{\pi}_{i_2} = \hat{\pi}_\ell,\end{aligned}$$

where the first identity follows by adding and subtracting the quantities $\pi_{i_2}, \pi_{i_3}, \dots, \pi_{i_h}$, the second identity follows from $\pi_{i_j} - \pi_{i_{j+1}} = \tau_{i_j}$ for all $j \in \{1, \dots, h\}$ and by noting that $\pi_{i_h} = \hat{\pi}_{i_h}$ since i_h is the destination, and the last identities follow from (6). Thus, we conclude $\pi_\ell(x) = \hat{\pi}_\ell(x)$. Finally, the claim follows by iterating the reasoning for all $\ell \in \mathcal{L}$. ■

Lemma 2.1 shows that the perceived costs can be computed in a recursive fashion given the travel costs. In words, the lemma shows that the minimum travel time from any link to destination can be obtained by minimizing the perceived cost at each junction. Hence, if a traveler updates her/his route at every encountered junction by selecting the outgoing link with the smallest perceived cost, then she/he will traverse a shortest path to destination. This observation suggests than an accurate decision model for the drivers seeks to minimize the perceived costs at every junction, as we do shortly below.

Remark 2: ***(Choices of Perceived Costs)*** A choice of perceived costs that generalizes (5) is the weighted combination:

$$\pi_\ell(x) = \alpha_{\ell,1} \tau_\ell(x_\ell) + \alpha_{\ell,2} \min_{m \in \mathcal{A}_\ell} \pi_m(x), \text{ for all } \ell \in \mathcal{L}, \tag{7}$$

where $\alpha_{\ell,1} \in \mathbb{R}_{\geq 0}$ and $\alpha_{\ell,2} \in \mathbb{R}_{\geq 0}$, describe the drivers' confidence on local information and global information, respectively. For instance, when $\alpha_{\ell,1} = \alpha_{\ell,2} = 1$, we recover the full-information model (5), while $\alpha_{\ell,1} = 1$, $\alpha_{\ell,2} = 0$ corresponds to a situation in which routing decisions rely only on one-link ahead congestion information. Although all the stability results presented in this paper hold for the general model (7), in what follows we focus on (5) due to its connection to the Wardrop conditions (see Section III-C). □

Remark 3: ***(Distinction Between Perceived and Actual Travel Time)*** It is worth noting that the travel time $\tau_\ell(x_\ell)$ as well as the perceived cost $\pi_\ell(x)$ are instantaneous quantities that *influence the decisions of the travelers*, but may significantly differ from the *actual* travel time experienced by a traveler that traverses the link (see [13] for a precise discussion). Discrepancies between the *perceived* travel delays and the *actual* travel delays arise because traffic densities are time-varying quantities that vary while travelers are traversing the network. We stress that the main goal of (5) is to describe the quantities that govern the decisions, rather than on providing a precise model of the actual travel delays. □

We describe the aggregate behavior of the travelers at the junctions by modeling the time-varying routing ratios according to a selection mechanism that seeks to minimize the perceived travel costs at all times:

$$\dot{r}_{\ell m} = r_{\ell m} \underbrace{\Big(\sum_{q \in \mathcal{A}_\ell} r_{\ell q} \pi_q(x) - \pi_m(x)\Big)}_{:= a_{\ell m}(x)}, \text{ for all } \ell \in \mathcal{L}. \tag{8}$$

The above equation states that the rate of growth of the routing parameter $r_{\ell m}$ is governed by the difference between the average travel cost from ℓ to destination and the perceived cost of m. The model is commonly known as *replicator equation* [4]. Recall that r denotes the joint vector of routing ratios. We rewrite (8) in compact form as:

$$\dot{r} = \varrho(r, \pi(x)), \tag{9}$$

where the map $(r, \pi) \mapsto \varrho(r, \pi)$ is obtained by stacking (8).

We illustrate our routing model in Example 3, we provide an interpretation of the replicator equations in Remark 4, and we discuss in Remark 5 why the replicator equation is amenable to model networks in the presence of routing apps.

Example 3: ***(Dynamical Routing Model)*** Consider the seven-link network illustrated in Fig. 2(a) and discussed in Example 2. The perceived costs read as (see (6)):

$$\begin{aligned}&\pi_1 = \tau_1 + \min\{\pi_2, \pi_3\}, \quad \pi_3 = \tau_3 + \pi_6, \quad \pi_5 = \tau_5 + \pi_6, \\ &\pi_2 = \tau_2 + \min\{\pi_4, \pi_5\}, \quad \pi_4 = \tau_4 + \pi_7, \quad \pi_6 = \tau_6 + \pi_7,\end{aligned}$$

and $\pi_7 = \tau_7$. The routing model (9) reads as:

$$\begin{aligned}\dot{r}_{12} &= r_{12}((r_{12}\pi_2 + r_{13}\pi_3) - \pi_2), \\ \dot{r}_{13} &= r_{13}((r_{12}\pi_2 + r_{13}\pi_3) - \pi_3), \\ \dot{r}_{24} &= r_{24}((r_{24}\pi_4 + r_{25}\pi_5) - \pi_4), \\ \dot{r}_{25} &= r_{25}((r_{24}\pi_4 + r_{25}\pi_5) - \pi_5).\end{aligned}$$

□

Remark 4: ***(Interpretation of the Dynamics** (8)**)*** By rearranging (8) (and when $r_{\ell m} \neq 0$), we obtain $\frac{\dot{r}_{\ell m}}{r_{\ell m}} = \sum_{q \in \mathcal{A}_\ell} r_{\ell q} \pi_q(x) - \pi_m(x)$, which suggests that: (i) if $\pi_m(x)$ is larger than the average cost to destination (i.e. $\sum_q r_{\ell q} \pi_q(x)$)

then the fraction of travelers choosing link m will decrease over time (i.e. $\dot{r}_{\ell m} < 0$), (ii) if $\pi_m(x)$ is smaller than the average cost to destination, then the fraction of travelers choosing link m will decrease over time (i.e. $\dot{r}_{\ell m} > 0$). As a result, we interpret $a_{\ell m}$ in (8) as the appeal of link m with respect to alternative links. Thus, the model (8) describes a continuous route-selection mechanism where the rate of change of $\dot{r}_{\ell m}/r_{\ell m}$ depends on the appeal of link m with respect to the alternatives. □

Remark 5: ***(Modeling Routing Apps Via Time-Varying Routing Ratios)*** The classical Routing Game [7] critically relies on the assumption that the path-selection process operates at a slower timescale as compared to that of the network dynamics. This setting, which can be interpreted as a scenario in which drivers maintain their routing constant while they are traversing the network and update their path preferences from day to day, lacks to capture the fact that app-informed travelers can respond in real-time to changes in the traffic state. As opposed to this classical approach, equation (8) describes a mechanism in which routing decisions take place at the same timescale as the traffic dynamics. We note that, although the steady-state operating point of our system coincides with that of the classical routing game (see Section III-C), the dynamical behavior of the system during the transients can significantly different (see Section IV). □

Recall that $\mathcal{R}_\mathcal{G}$ denotes the set of feasible routing ratios for graph $\mathcal{G}$ (see (3)). The following lemma guarantees that the trajectories of (9) exist and remain in the set $\mathcal{R}_\mathcal{G}$ at all times.

Lemma 2.2: ***(Conservation of Flows)*** Let Assumption 3 be satisfied, and let $r_0 \in \mathcal{R}_\mathcal{G}$. Then, there exists a unique solution $r(t)$ to (9) that satisfies $r(t) \in \mathcal{R}_\mathcal{G}$ for all $t \in \mathbb{R}_{>0}$.

Proof: We begin by observing that, under Assumption 3, the vector field in (9) is Lipschitz continuous in r and locally Lipschitz in t, and thus standard results guarantee existence of a unique solution to the initial value problem (9). To prove that $r(t) \in \mathcal{R}_\mathcal{G}$ for all $t \in \mathbb{R}_{>0}$, we first show that $\sum_m r_{\ell m} = 1$ at all times or, equivalently, $\sum_m \dot{r}_{\ell m} = 0$. By substituting (8):

$$\begin{aligned}\sum_m \dot{r}_{\ell m} &= \sum_m r_{\ell m}(\sum_q r_{\ell q}\pi_q - \pi_m)\\ &= \underbrace{\sum_m r_{\ell m}}_{=1} \sum_q r_{\ell q}\pi_q - \sum_m r_{\ell m}\pi_m\\ &= \sum_q r_{\ell q}\pi_q - \sum_m r_{\ell m}\pi_m = 0.\end{aligned}$$

Second, we show that for all $(\ell, m) \in \mathcal{A}$, $r_{\ell m} \in [0,1]$ at all times. To show $r_{\ell m} \geq 0$:

$$r_{\ell m} = 0 \Rightarrow \dot{r}_{\ell m} = r_{\ell m} a_{\ell m}(x) = 0.$$

To show $r_{\ell m} \leq 1$, let t_0 be the first time instant such that $r_{\ell m}(t_0) = 1$ for some $(\ell, m) \in \mathcal{A}$. By using $\sum_q r_{\ell q}(t_0) = 1$, we obtain $r_{\ell q}(t_0) = 0$ for all $q \neq m$, which implies:

$$a_{\ell m}(x(t_0)) = \sum_q r_{\ell q}\pi_q - \pi_m = r_{\ell m}\pi_m - \pi_m = 0.$$

Hence, we obtained the following implication:

$$r_{\ell m}(t_0) = 1 \Rightarrow \dot{r}_{\ell m}(t_0) = r_{\ell m}(t_0) a_{\ell m}(x(t_0)) = 0,$$

which shows the second claim and concludes the proof. ■

C. The Wardrop First Principle

The goal of this section is to establish a connection between the classical game-theoretic setting and our framework. The routing game [22] consists of a static (time-invariant) traffic model combined with a path-selection model. A Wardop Equilibrium corresponds to a situation where the costs on all used origin-destination paths in a graph are identical, and there are no unused paths with strictly smaller cost. Because in a static traffic model traffic flows can propagate instantaneously across the network, this framework corresponds to a scenario where the routing decisions take place on slower timescale as compared to that of the traffic system, and thus describes a situation where travelers do not update their path *as they are traversing the network* but rather *from day to day*.

Next, we recall the notion Wardrop Equilibrium. To comply with the static nature of the routing game, we will assume that the dynamical system (4) is at an equilibrium point. Let x^* be an equilibrium of (4), and let $f_\ell^* := f_\ell^{\text{out}}(x_\ell^*)$, for all $\ell \in \mathcal{L}$, be the set of equilibrium flows on the links. In vector form, $f^* := [f_1^* \dots, f_n^*]^\mathsf{T}$. Moreover, let $\mathcal{P} = \{p_1, \dots, p_\zeta\}$, $\zeta \in \mathbb{N}$, be the set of simple paths between origin and destination, and let $f_p^* := [f_{p_1}^*, \dots, f_{p_\zeta}^*]^\mathsf{T}$ be the set of flows on the paths. The flows on the origin-destination paths are related to the flows on the links by means of the following relationship:

$$f_\ell^* = \sum_{p \in \mathcal{P}: \ell \in p} f_p^*,$$

which establishes that the flow on each link is the superposition of all the flows in the paths passing through that link. By inverting the above set of equations, the vector of path flows can be computed from the vector of link flows as follows

$$f_p^* = E^\dagger f^*, \tag{10}$$

where $E \in \mathbb{R}^{n \times \nu}$ is the edge-path incidence matrix:

$$E_{\ell p} = \begin{cases} 1, & \text{if } \ell \in p,\\ 0, & \text{otherwise,}\end{cases}$$

and $E^\dagger$ denotes the pseudoinverse of E. Finally, we extend the definition of travel costs to the paths by letting the cost of a path be the sum of the costs of all the links in that path:

$$\tau_p^* := E^\mathsf{T} \tau(x^*).$$

The Wardrop First Principle states that all paths with nonzero flow have identical travel cost, and is formalized next.

Definition 1: (***Wardrop First Principle***) Let x^* be an equilibrium of (4). The vector x^* is a *Wardrop Equilibrium* if the following condition is satisfied for all origin-destination paths $p \in \mathcal{P}$:

$$f_p^*(\tau_p^* - \tau_{\bar{p}}^*) \leq 0, \text{ for all } \bar{p} \in \mathcal{P}.$$

We refer to [7] for a discussion on Wardrop's First Principle, and we conclude this section by showing that, when $\mathcal{G}$ is acyclic, the path flows are unique for any choice of link flows.

Lemma 2.3: ***(Uniqueness of the Path Flow Vectors)*** Let Assumptions 1-2, and assume $\mathcal{G}$ is acyclic. For any $f^* \in \mathbb{R}^n_{\geq 0}$ there exists a unique $f_p^* \in \mathbb{R}^\zeta_{\geq 0}$ that solves (10).

Proof: We equivalently show that the linear map $f^* = Ef_p^*$ is injective, that is, $\operatorname{Ker}(E) = \emptyset$. By noting that E is a $n \times \zeta$ matrix, we will show that the columns of E are linearly independent, that is, $\operatorname{Rank}(E) = \zeta$. We organize the proof into two parts. First, we show that $\zeta \leq n$. Second, we prove that E contains ζ linearly independent columns.

To show that $\zeta \leq n$, we let $\mathcal{S} \subseteq \mathcal{V}$ and we partition $\mathcal{V}$ into two subsets, $\mathcal{S}$ and $\bar{\mathcal{S}} = \mathcal{V} - \mathcal{S}$, such that $s \in \mathcal{S}$ and $d \in \bar{\mathcal{S}}$. Moreover, we let $\mathcal{S}^{\text{out}} = \{(v_i, v_j) \in \mathcal{L} \;:\; v_j \in \mathcal{S} \text{ and } v_i \in \bar{\mathcal{S}}\}$ be the set of all links from $\mathcal{S}$ to $\bar{\mathcal{S}}$. By application of the Max-Flow Theorem [27], the number of simple paths ζ satisfies

$$\zeta = \min_{\mathcal{S} \subseteq \mathcal{V}} |\mathcal{S}^{\text{out}}|,$$

where $|\mathcal{S}^{\text{out}}|$ denotes the cardinality of the set $\mathcal{S}^{\text{out}}$. Since $\mathcal{S}^{\text{out}} \subseteq \mathcal{L}$, we have $|\mathcal{S}^{\text{out}}| \leq n$ and thus $\zeta \leq n$.

To show that the columns of E are linearly independent, we denote by E_i the i-th column of E, $i \in \{1, \dots, \zeta\}$, and we observe E_i can be written as a linear combination:

$$E_i = \sum_{j=1}^{n} b_{ij} e_j,$$

where e_j denotes the j-th canonical vector of dimension n, and $b_{ij} \in \{0, 1\}$, with $b_{ij} = 1$ only if link i belongs to path p_j. To conclude, we note that because $\mathcal{G}$ is acyclic, each pair of paths differs by at least one edge, and thus each pair of columns E_i, E_k are generated by independent choices of coefficients $(b_{i1}, \dots, b_{in})$ and $(b_{k1}, \dots, b_{kn})$, which shows that E contains ζ linearly-independent columns. ■

III. Existence and Properties of the Equilibria

In this section, we study the fixed points of a traffic system composed of the interconnection between the traffic dynamics (4) and the routing dynamics (9), which reads as:

$$\begin{aligned} \dot{x} &= (R(r) - I)^\mathsf{T} f(x) + \lambda, \\ \dot{r} &= \varrho(r, \pi(x)). \end{aligned} \tag{11}$$

Fig. 3 graphically illustrates the interactions between the two systems and depicts the quantities that establish the coupling. Existence and uniqueness of the solutions of (11) are formalized in the following lemma.

Lemma 3.1: ***(Existence and Uniqueness of Solutions)*** Let $x_0 \in \mathbb{R}^n_{\geq 0}$, $r_0 \in \mathcal{R}_\mathcal{G}$. Then, there exists a unique solution $(x(t), r(t))$ to (11) with $x(0) = x_0$ and $r(0) = r_0$.

Proof: The proof of this claim follows immediately from Lipschitz continuity of the vector fields in (11). ■

In what follows, we denote by $(x(t), r(t))$ a solution trajectory of (11) and by (x^*, r^*) an equilibrium point of (11). We recall the notion of equilibrium point (or fixed point) of a system with (constant) inputs in the following definition.

Definition 2: ***(Fixed Points and Lyapunov Stability)***

- A triplet (x^*, r^*, λ^*), $x^* \in \mathbb{R}^n_{\geq 0}$, $r^* \in [0,1]^{|\mathcal{A}|}$, $\lambda^* \in \mathbb{R}^n_{\geq 0}$ is a *fixed point* (or equilibrium point) of (11) if:

$$(R(r^*) - I)^\mathsf{T} f(x^*) + \lambda^* = 0, \text{ and } \varrho(r^*, \pi(x^*)) = 0.$$

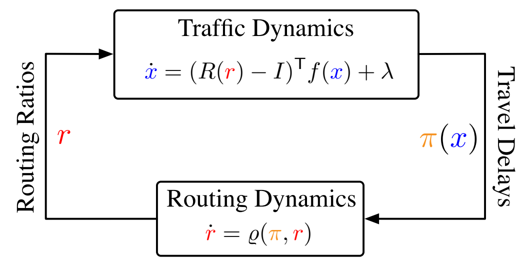


Fig. 3. Coupled interconnection between traffic and routing dynamics.

- A fixed point (x^*, r^*, λ^*) is *marginally stable* if, for every $\epsilon_x > 0, \epsilon_r > 0$, there exists $\delta_x > 0, \delta_r > 0$, such that

$$\begin{aligned} &\|x(0) - x^*\| < \delta_x \text{ and } \|r(0) - r^*\| < \delta_r \\ &\qquad \Rightarrow \|x(t) - x^*\| < \epsilon_x \text{ and } \|r(t) - r^*\| < \epsilon_r, \end{aligned}$$

  for all $t \geq 0$.
- A fixed point is *asymptotically stable* if it is stable and $\delta_x > 0, \delta_r > 0$ can be chosen such that:

$$\begin{aligned} &\|x(0) - x^*\| < \delta_x \text{ and } \|r(0) - r^*\| < \delta_r \\ &\qquad \Rightarrow \lim_{t \to +\infty} x(t) = x^* \text{ and } \lim_{t \to +\infty} r(t) = r^*. \end{aligned}$$

- A fixed point is *unstable* if it is not stable. □

A. Existence of Equilibria

We begin by characterizing the existence of fixed points of (11). To this aim, we first recall the graph-theoretic notion of min-cut capacity [27]. Let the set of nodes $\mathcal{V}$ be partitioned into two subsets $\mathcal{S} \subseteq \mathcal{V}$ and $\bar{\mathcal{S}} = \mathcal{V} - \mathcal{S}$, such that the network source $s \in \mathcal{S}$ and the network destination $d \in \bar{\mathcal{S}}$. Let $\mathcal{S}^{\text{out}} = \{(v, u) \in \mathcal{L} \;:\; v \in \mathcal{S} \text{ and } u \in \bar{\mathcal{S}}\}$ be a cut, namely, the set of all links from $\mathcal{S}$ to $\bar{\mathcal{S}}$, and let $C_\mathcal{S} = \sum_{\ell \in \mathcal{S}^{\text{out}}} C_\ell$ be the capacity of the cut. The min-cut capacity is defined as

$$C_{\text{m-cut}} = \min_{\mathcal{S}} C_\mathcal{S}.$$

The following result relates the existence of fixed points to the magnitude of the exogenous inflow to the network. We note that a similar conclusion was obtained in [19] under the more-stringent assumption that travel costs are continuous functions of the traffic flows.

Theorem 3.2: ***(Existence of Equilibria)*** Let Assumptions 1-3 hold. The interconnected system (11) admits an equilibrium if and only if $\bar{\lambda} < C_{\text{m-cut}}$.

Proof: (If) The proof of this claim consists in showing that the equilibrium points of (16) satisfy a variational inequality (V.I.), and on proving that such V.I. admits a solution under the stated assumptions. More precisely, we first show that if a pair of vectors $x^* \in \mathbb{R}^n$, $r^* \in \mathcal{R}_\mathcal{G}$, satisfies:

$$0 = (R(r^*) - I)^\mathsf{T} f(x^*) + \lambda, \quad (r^* - \hat{r})^\mathsf{T} \pi(x^*) \geq 0, \tag{12}$$

for all $\hat{r} \in \mathcal{R}_\mathcal{G}$, then (x^*, r^*) is an equilibrium of the interconnected system that satisfies (16). Indeed, fix an $\ell \in \mathcal{L}$. Under (12), there exists a constant $c \in \mathbb{R}_{\geq 0}$ such that $\pi_m(x^*) = c$ for all $m \in \mathcal{A}_\ell$ such that $r_{\ell m} > 0$. It follows that the weighted mean reads as $\sum_q r^*_{\ell q} \pi_q(x^*)$ = c, which implies that (x^*, r^*) is an equilibrium that satisfies (16).

Second, we show that there exists a solution to (12). Namely, there exists $r^* \in \mathcal{R}_\mathcal{G}$ that satisfies:

$$(r^* - \hat{r})^\mathsf{T} \pi(x^*) \geq 0, \forall \hat{r} \in \mathcal{R}_\mathcal{G}, \text{ and } \forall x^* \in \mathcal{X}_{r^*}, \tag{13}$$

where $\mathcal{X}_r := \{x \in \mathbb{R}^n_{\geq 0} \ : \ (R(r) - I)^\mathsf{T} f(x) + \lambda = 0\}$. We note that, because x^* is a function of r^* through the constraint $x^* \in \mathcal{X}_{r^*}$, (13) is of the form of a V.I. [28, Def. 1.1]. Next, we leverage the standard result [29, Thm 12] on existence of solutions of V.I.s to prove that (13) admits a solution. First, we note that the set $\{(x, r) \ : \ r \in \mathcal{R}_\mathcal{G}, x \in \mathcal{X}_r\}$ is nonempty. Indeed, by application of the max-flow min-cut theorem [27], there exists a feasible assignment of flows to the links of the graph $\mathcal{G}$, that is, a set of scalars $\{\varphi_1, \dots, \varphi_n\}$ such that $\varphi_1 = \bar{\lambda}$, and:

$$\begin{aligned} &0 \leq \varphi_\ell < C_\ell, && \text{for all } \ell \in \mathcal{L}, \\ &\sum_{\ell \in v^{\text{in}}} \varphi_\ell = \sum_{\ell \in v^{\text{out}}} \varphi_\ell, && \text{for all } v \in \mathcal{V}. \end{aligned}$$

By choosing $r^*_{\ell m} := \frac{\varphi_m}{\sum_{m \in \mathcal{A}_\ell} \varphi_m}$ for all $(\ell, m) \in \mathcal{A}$, we have $(R^{*\mathsf{T}} - I)\varphi + \lambda = 0$, where $\varphi = [\varphi_1 \dots, \varphi_n]^\mathsf{T}$, which shows that the domain set is nonempty. Second, we note that Assumption 3 guarantees that the mapping $\pi(x)$ is monotone, complete, and continuous. Third, the domain set of $\pi(x)$ is $\mathbb{R}^n_{\geq 0}$, which is open and convex. Fourth, the set $\mathcal{R}_\mathcal{G}$ is convex and closed. Under these assumptions, [29, Thm 12] guarantees the existence of a solution to the V.I. (13). Since every solution of (13) is an equilibrium point of (11), and (13) admits at least one solution, the implication follows.

(Only if) The proof of this statement follows by adopting a contradiction reasoning. To this aim, assume (x^*, r^*) is an equilibrium point and that $\bar{\lambda} > C_{\text{m-cut}}$. The latter assumption, combined with the Maximum Flow Theorem, implies that for any assignment of flows to the links of the graph $\mathcal{G}$ there exists $\ell \in \mathcal{L}$ such that $\varphi_\ell > C_\ell$. In other words, link ℓ is required to transfer a traffic flow $f^{\text{in}}_\ell(x^*) = \varphi_\ell$, and thus:

$$\begin{aligned} \dot{x}_\ell &= f^{\text{in}}_\ell(x^*) - f^{\text{out}}_\ell(x^*_\ell) \\ &= \varphi_\ell - f^{\text{out}}_\ell(x^*_\ell) \\ &\geq \varphi_\ell - C_\ell > 0, \end{aligned}$$

which shows shows that x_ℓ grows unbounded, and hence contradicts the assumption that (x^*, r^*) is an equilibrium. ■

This result relates the existence of equilibrium points of a dynamical system with the notion of minimum-cut capacity, which is a property of the static graph. Two important implications follow from Theorem 3.2. First, by recalling that the minimum-cut capacity coincides with the maximum flow that can be transferred through a static graph (see Maximum-Flow Theorem [27]), the result shows that the dynamical system (11) admits an equilibrium point that transfers such flow. Hence, our models suggest that routing apps not only minimize the traveler's travel time to destination, but could also benefit the overall system by ensuring the existence of equilibrium points that transfer a maximum flow. Second, it shows that when the traffic demand is too large ($\bar{\lambda} \geq C_{\text{m-cut}}$), then the network does not admit any equilibrium point, in fact, it operates at a condition in which some densities grow unbounded.

B. Restricted Set of Equilibria

Let (x^*, r^*) denote a fixed point of (11). It follows from the expressions of the routing model (8) that every link $(\ell, m) \in \mathcal{A}$ satisfies one of the following conditions equilibrium:

$$a_{\ell m}(x^*) = 0, \text{ or } r^*_{\ell m} = 0.$$

We next show that a subset of these points are unstable.

Lemma 3.3: ***(Unstable Equilibria)*** Let Assumption 3 hold, let (x^*, r^*) be a fixed point of (11) and assume there exists $(\ell, m) \in \mathcal{A}$ such that:

$$r^*_{\ell m} = 0, \text{ and } a_{\ell m}(x^*) > 0.$$

Then, (x^*, r^*) is unstable.

Proof: To prove this lemma, we adopt a perturbation reasoning and show that there exists an arbitrarily-small perturbation from the equilibrium that implies $\dot{r}_{\ell m} > 0$. The proof is organized into two main parts.

First, we show that at equilibrium all links alternative to m have identical perceived cost. To this aim, we note that $r^*_{\ell m} = 0$ combined with $r^* \in \mathcal{R}_\mathcal{G}$ (i.e. $\sum_q r^*_{\ell q} = 1$) implies that there exists (at least) one alternative link w such that $r^*_{\ell w} > 0$. In general, let $\mathcal{W} = \{w_1, \dots, w_\xi\}$, $\xi \in \mathbb{N}$, denote the set of all such links. Since $r^*_{\ell w_i} > 0$ and x^* is an equilibrium, we necessarily have $a_{\ell w_i}(x^*) = 0$ or, equivalently,

$$0 = a_{\ell w_i}(x^*) = \sum_q r^*_{\ell q} \pi^*_q - \pi^*_{w_i},$$

for all $i \in \{1, \dots, \xi\}$. The above system of equations admits the explicit solution $\pi^*_{w_i} = \sum_q r^*_{\ell q} \pi^*_q$ for all $i \in \{1, \dots, \xi\}$, which implies

$$\pi^*_{w_i} = \pi^*_{w_j}, \text{ for all } i, j \in \{1, \dots, \xi\}, \tag{14}$$

and proves the first claim.

Second, we show that at equilibrium all links alternative to m (i.e. links $w \in \mathcal{W}$), have strictly suboptimal travel cost: $\pi^*_w > \pi^*_m$. To this aim, we use the assumption $a_{\ell m}(x^*) > 0$ to obtain

$$\begin{aligned} 0 < a_{\ell m}(x^*) &= \sum_q r^*_{\ell q} \pi^*_q - \pi^*_m = \pi^*_w \sum_q r^*_{\ell q} - \pi^*_m \\ &= \pi^*_w - \pi^*_m, \end{aligned} \tag{15}$$

where we substituted (8) to obtain the first identity, and (14) to obtain the second identity, which proves the second claim.

Finally, let $\epsilon \in \mathbb{R}_{>0}$ be a scalar perturbation. By perturbing (8) from the equilibrium point, $r^*_{\ell m} \mapsto r^*_{\ell m} + \epsilon$, we have

$$\begin{aligned} \dot{r}_{\ell m} &= \epsilon(\sum_{q \neq m, w} r^*_{\ell q} \pi^*_q + (r^*_{\ell m} + \epsilon) \pi^*_m + (r^*_{\ell w} - \epsilon) \pi^*_w - \pi^*_m) \\ &= \epsilon(\sum_q r^*_{\ell q} \pi^*_q + \epsilon \pi^*_m - \epsilon \pi^*_w - \pi^*_m) \\ &= \epsilon(\pi^*_w \underbrace{\sum_q r^*_{\ell q}}_{=1} + \epsilon \pi^*_m - \epsilon \pi^*_w - \pi^*_m) \\ &= \epsilon(\pi^*_w + \epsilon \pi^*_m - \epsilon \pi^*_w - \pi^*_m) \\ &= \epsilon(\pi^*_w - \pi^*_m)(1 - \epsilon) > 0, \end{aligned}$$

where we used (14) to obtain the third identity, and the final inequality follows from (15) and from $\epsilon > 0$. The conclusion follows by observing that for any arbitrarily small ϵ, $\dot{r}_{\ell m} > 0$, showing that system trajectories depart from equilibria. ■

Unstable equilibrium points of the form characterized in Lemma 2.2 originate because the replicator equations are a model of imitation, namely they describe a setting where the routing preferences of favourable travelers are imitated by other travelers. The unstable equilibrium points characterized in Lemma 2.2 correspond to a situation where at a certain junction one of the outgoing links has a preferable travel time to destination (i.e. $a_{\ell m} > 0$), but no driver is currently traversing that highway (i.e. $r_{\ell m} = 0$), and thus no imitation is possible. In order to disregard the unstable equilibria from the discussion, in the remainder we focus on the equilibria (x^*, r^*) such that every link $(\ell, m) \in \mathcal{A}$ satisfies one of the following conditions at equilibrium:

$$\begin{aligned} &(i) \quad a_{\ell m}(x^*) = 0, \text{ or} \\ &(ii) \quad r^*_{\ell m} = 0 \text{ and } a_{\ell m}(x^*) < 0. \end{aligned} \tag{16}$$

Remark 6: ***(Relationship to Game-Theoretic Setting)*** In the Game-Theory literature, the rest points (16) are often called *restricted* equilibria [10], and correspond to the set of Nash equilibria of the underlying game. We observe that Lemma 2.2 extends classical results in this literature (see the Folk Theorem of evolutionary game theory [4] and the specific conclusions drawn for the routing game in [10]) by showing that rest points that are not Nash equilibria are unstable also when the payoffs are not algebraic functions of the strategy, but instead evolve according to a dynamical model that is parametrized by the strategy. □

C. Relationship to Wardrop Equilibrium

The following result relates the fixed points of the dynamical system (11) with the established notion of Wardrop equilibria.

Theorem 3.4: ***(Relationship Between Fixed Points and Wardrop Equilibria)*** Let Assumptions 1-3 be satisfied. The following statements are equivalent for the dynamics (11):

(i) $x^* \in \mathbb{R}^n_{\geq 0}$ satisfies the Wardrop First Principle;
(ii) The pair (x^*, r^*) is a fixed point of (11) for some $r^* \in \mathcal{R}_{\mathcal{G}}$. Moreover, (x^*, r^*) satisfies (16).

Proof: (i) ⇒ (ii) We begin by observing that, by assumption, a Wardrop equilibrium is also an equilibrium of the traffic dynamics (4). Thus, we next prove that given a vector x^* that satisfies the Wardrop conditions, there exists a vector $r^* \in \mathcal{R}_{\mathcal{G}}$ such that (16) is satisfied. We note that, although a feasible vector of routing ratios can be recovered given an assignment of flows, to prove $\rho(r^*, \pi(x^*))$, it is necessary to characterize the set of perceived costs, namely, $\pi_\ell(x)$ for all $\ell \in \mathcal{L}$.

Consider the graph $\mathcal{G}$ with link costs $\tau_\ell(x_\ell)$ for all $\ell \in \mathcal{L}$. Since, by Assumption 3, the graph does not admit cycles of negative cost, $\mathcal{G}$ admits a shortest path spanning tree [27], that is, a directed tree rooted at the source with the property that the unique path from the source to any node is a shortest path to that node. Notice that, since in general the Wardrop First Principle allows the existence of multiple paths with optimal travel costs, the shortest-path spanning tree may not be unique. Next, we recall that $\mathcal{P}$ is the set of all origin-destination paths, and we distinguish among three cases.

(Case 1) For all $p, \bar{p} \in \mathcal{P}$, $\tau^*_p - \tau^*_{\bar{p}} = 0$, namely, all paths have identical travel cost. This implies that for every node $v \in \mathcal{V}$, all its outgoing links belong to a shortest-path spanning tree. Since the perceived costs coincide with the shorthest travel cost to destination, the above observation implies:

$$\pi^*_q = \pi^*_m, \text{ for all pairs } m, q \in v^{\text{out}}.$$

As a result,

$$a_{\ell m}(x^*) = \sum_q r^*_{\ell q}\pi^*_q - \pi^*_m = \pi^*_m(\sum_q r^*_{\ell q} - 1) = 0.$$

By iterating the above equation for all $(\ell, m) \in \mathcal{A}$ we proved that the first condition in (16) is satisfied.

(Case 2) There exists a unique $p \in \mathcal{P}$ such that $f^*_p = 0$ and for all $\bar{p} \neq p$, $\tau^*_{\bar{p}} - \tau^*_p \leq 0$. Namely, path p has suboptimal travel cost, while all other paths have optimal travel cost to destination. This assumption implies that there exists a node $v \in \mathcal{V}$ and a pair of links $m, q \in v^{\text{out}}$ such that $m \in p$ and $q \in \bar{p}$. Namely, m belongs to the suboptimal path p, while q belongs to a shortest path spanning tree. This observation, combined with the fact that the perceived costs are equal to the shortest travel cost to destination, implies

$$\pi^*_m > \pi^*_q.$$

Moreover, since m belongs to an origin-destination path with zero flow, there exists $\ell \in v^{\text{in}}$ such that $r^*_{\ell m} = 0$, and thus

$$\begin{aligned} a_{\ell m}(x^*) &= \sum_q r^*_{\ell q}\pi^*_q - \pi^*_m = \sum_{q \neq m} r^*_{\ell q}\pi^*_q + \underbrace{r^*_{\ell m}}_{=0}\pi^*_m - \pi^*_m \\ &= \sum_{q \neq m} r^*_{\ell q}\pi^*_q - \pi^*_m = \pi^*_q \sum_{q \neq m} r^*_{\ell q} - \pi^*_m \\ &= \pi^*_q - \pi^*_m < 0, \end{aligned} \tag{17}$$

which proves that $a_{\ell m}(x^*) < 0$, and shows that the second condition in (16) is satisfied for the pair $(\ell, m) \in \mathcal{A}$.

(Case 3) There exists multiple $p \in \mathcal{P}$ such that $f^*_p = 0$ and for some $\bar{p} \neq p$, $\tau^*_{\bar{p}} - \tau^*_p \leq 0$, namely, there exists multiple origin-destination paths with suboptimal travel cost. Under this assumption, we note that the bound derived in (17) can be iterated for all links m such that $r^*_{\ell m} = 0$, which shows that the second condition in (16) is satisfied for all these pairs, and concludes the proof of the implication.

(ii) ⇒ (i)

To prove this implication we show that if the condition (16) is satisfied at all junctions, then the travel costs on the origin-destination paths satisfy Definition 1. Notice that, in this case, we are proving a condition on the origin-destination paths given a set of conditions at all junctions. We distinguish among three cases.

(Case 1) For all $(\ell, m) \in \mathcal{A}$, $a_{\ell m}(x^*) = 0$, namely, all links have identically zero appeal. Under this assumption, for every $\ell \in \mathcal{L}$, all the perceived travel costs satisfy

$$0 = a_{\ell m}(x^*) = \sum_q r^*_{\ell q}\pi^*_q - \pi^*_m, \text{ for all } m \in \mathcal{A}_\ell, \tag{18}$$

which implies that $\pi^*_m = \pi^*_{\bar{m}}$ for all $m, \bar{m} \in \mathcal{A}_\ell$ are identical (i.e. $\pi^*_m = \pi^*_{\bar{m}} = \sum_q r^*_{\ell q}\pi^*_q$). This observation, combined with the fact that the perceived costs are equal to the shortest travel cost to destination, implies that every link in the network belongs to a shortest path to destination. Hence, all origin-destination paths have identical travel cost, i.e. $\tau^*_p - \tau^*_{\bar{p}} = 0$, which shows that x^* satisfies the Wardrop First Principle.

(Case 2) There exists a unique $(\ell, m) \in \mathcal{A}$ such that $a_{\ell m}(x^*) < 0$ and $r_{\ell m} = 0$. Under this assumption, we first prove that there exists a path $p \in \mathcal{P}$ containing link m such that $f^*_p = 0$. Since the flow on any origin-destination path can be written as the network inflow multiplied by the product of the routing ratios belonging to that path:

$$f^*_p = \bar{\lambda} \prod_{q,w \in p} r^*_{qw},$$

we immediately obtain $f^*_p = 0$.

Second, we prove that for all $\bar{p} \in \mathcal{P}$, $\bar{p} \neq p$, the following inequality holds: $\tau^*_{\bar{p}} - \tau^*_p \leq 0$. By using the assumption $a_{\ell m}(x^*) < 0$, together with $\pi^*_w = \sum_q r^*_{\ell q}\pi^*_q$, which holds for all $w \in \mathcal{A}_\ell$, $w \neq m$, (see (18)), we have

$$\begin{aligned} 0 > a_{\ell m}(x^*) &= \sum_q r^*_{\ell q}\pi^*_q - \pi^*_m \\ &= \pi^*_w \sum_q r^*_{\ell q} - \pi^*_m \\ &= \pi^*_w - \pi^*_m. \end{aligned} \tag{19}$$

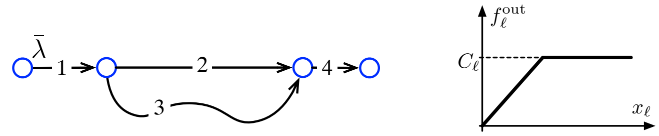
Since path p contains link w and the minimum travel cost from w to destination is suboptimal ($\pi^*_w > \pi^*_m$), we have that any path $\bar{p} \in \mathcal{P}$ containing link m satisfies

$$\tau^*_{\bar{p}} < \tau^*_p,$$

which shows that x^* satisfies the Wardrop First Principle.

(Case 3) There exists multiple ramps $(\ell, m) \in \mathcal{A}$ such that $a_{\ell m}(x^*) < 0$ and $r^*_{\ell m} = 0$. Under this assumption, we note that equation (19) still applies because $r^*_{\ell m} = 0$. Hence, the reasoning adopted for *(Case 2)* can be iterated for all (ℓ, m) such that $a_{\ell m}(x^*) < 0$ and $r^*_{\ell m} = 0$. ∎

Three important implications follow from the above theorem. First, the result shows that a Wardrop equilibrium is also an equilibrium of the dynamical model (11), thus proving that if a dynamical network starts at a Wardrop equilibrium it will remain at that equilibrium at all times (when the network inflow is constant). This result is in accordance with Wardrop's framework, where the network is assumed to be at equilibrium at all times and routing decisions are updated from day to day. Second, the result shows that in traffic systems where the travelers update their routing preferences at every junction by minimizing the perceived travel costs admit equilibrium points that satisfy the Wardrop principle. This observation demonstrates that the perceived costs are representative quantities to describe the driver's decision- making mechanism. Third, by combining Theorem 3.4 with (16), it follows that a Wardrop Equilibrium is perceived by the travelers when all the network links have a nonpositive appeal function. This condition corresponds to a situation where at every junction no link is more appealing that others.



(a) (b)

Fig. 4. Two-link network (a), and piecewise affine outflow function (b).

IV. Stability Analysis

In this section, we characterize the stability of the fixed points of the interconnection (11). Our main findings are summarized in the following theorem.

Theorem 4.1: ***(Stability of Interconnected Traffic Dynamics)*** Let Assumptions 1-3 hold, and let (x^*, r^*) be a fixed point of (11) that satisfies the conditions (16). Then, (x^*, r^*) is marginally stable.

The proof of this result is postponed to Section IV-B.

Marginal stability of the fixed points implies that the state trajectories are not guaranteed to converge asymptotically to an equilibrium point, and can result in nontrivial behaviors, such as oscillations, as illustrated in the following example.

Example 4: ***(Existence of Oscillations in Two Parallel Highways)*** Consider the network illustrated in Fig. 4(a), representing two parallel roads subject to a constant inflow $\bar{\lambda} \in \mathbb{R}_{>0}$. We assume that the travel costs are linear

$$\tau_\ell(x_\ell) = x_\ell, \qquad \ell \in \mathcal{L} = \{1, \dots, 4\},$$

and that all outflows are identical and piecewise-affine:

$$f_\ell^{\text{out}}(x_\ell) = \min\{v x_\ell, C\}, \qquad \ell \in \mathcal{L} = \{1, \dots, 4\},$$

where $v \in \mathbb{R}_{>0}$. Notice that this choice satisfies Assumptions 2-3 and that an equilibrium point exists whenever $\bar{\lambda} < 2C$.

We distinguish between two cases: (a) the network is operating in congested regimes, that is, $x_2(t) > C/v$ and $x_3(t) > C/v$ at all times, and (b) the network is operating in regimes of free-flow, that is, $x_2(t) \leq C/v$ and $x_3(t) \leq C/v$ at all times. Fig. 5 (a) and (b) show the phase portrait of the system trajectories in case (a) and case (b), respectively. As illustrated by the plots: in case (a) the trajectories of the system are oscillating periodic orbits; in contrast, in case (b) the trajectories converge asymptotically to an equilibrium point. The presence of periodic orbits implies that the equilibrium points are marginally stable, but not asymptotically stable, thus supporting the conclusions drawn in Theorem 4.1.

The existence of periodic orbits in case (a) can be further formalized. To this aim, we recall the dynamical equations governing the system in this regime:

$$\begin{aligned} \dot{x}_2 &= -C + r_{12}\bar{\lambda}, \\ \dot{x}_3 &= -C + r_{13}\bar{\lambda}, \\ \dot{r}_{12} &= r_{12}(1 - r_{12})(x_3 - x_2), \end{aligned}$$

where we used the fact that $f_1^{\text{out}} = \bar{\lambda}$ after an initial transient. This system admits an equilibrium point described by $r_{12} = 0.5$ and $x_3 = x_2$. We adopt the change of variables $z :=$

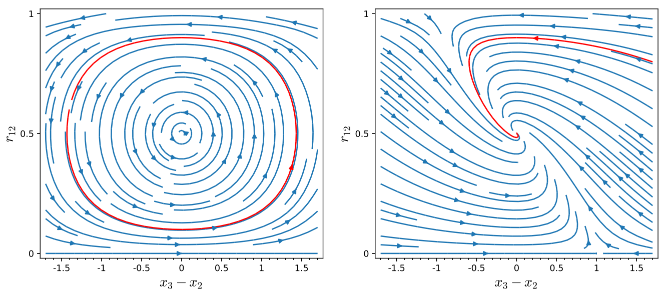


Fig. 5. Phase portrait: (a) oscillatory trajectories, (b) stable trajectories. The red curve illustrates an example of trajectories passing through the conditions $x_3 = x_2$ and $r_{12} = 0.9$.

$x_3 - x_2$, and rewrite the dynamical equations describing the new state $[z,\ r_{12}]^\mathsf{T}$:

$$\dot{z} = (1 - 2r_{12})\bar{\lambda},$$
$$\dot{r}_{12} = r_{12}(1 - r_{12})z.$$

Next, we show that the following quantity:

$$U(z, r_{12}) := \frac{1}{2}z^2 - \bar{\lambda}\left(\ln r_{12} - \ln(1 - r_{12})\right),$$

is conserved along the trajectories of the system. To this aim, we compute its time derivative to obtain

$$\begin{aligned}\frac{d}{dt}U(z(t), r_{12}(t)) &= z\dot{z} - \bar{\lambda}\left(\frac{1}{r_{12}} + \frac{1}{1 - r_{12}}\right)\dot{r}_{12}\\ &= z(1 - 2r_{12})\bar{\lambda} - \bar{\lambda}((1 - r_{12})z - r_{12}z))\\ &= z(1 - 2r_{12})\bar{\lambda} - z(1 - 2r_{12})\bar{\lambda} = 0,\end{aligned}$$

which shows that the quantity $U(z, r_{12})$ is a constant of motion, and proves the existence of periodic orbits. □

A. Basic Notions on Passivity of Dynamical Systems

The proof of Theorem 4.1 leverages passivity tools for nonlinear dynamical systems (we refer to [23], [30] for a comprehensive discussion on this topic). In what follows, we recall some fundamental notions that are used for our analysis.

*Definition 3: **(Passive System** **[23])*** Consider the system $\dot{x} = f(x, u)$, $y = g(x, u)$, $x \in \mathcal{X} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, $y \in \mathcal{Y} \subseteq \mathbb{R}^m$, where $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, $g : \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$, are locally Lipschitz in x and u, and satisfy $f(0, 0) = 0, g(0, 0) = 0$.

- The system is *passive with respect to the input-output pair* (u, y) if there exists a differentiable function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$, $V(0) = 0$, such that:
$$V(x(t)) - V(x_0) \leq \int_0^t u(\sigma)^\mathsf{T} y(\sigma) d\sigma, \qquad (20)$$
holds for all $t \geq 0$, for every $x(0) \in \mathcal{X}$, and for any input u that satisfies $u(\sigma) \in \mathcal{U}$ for all $\sigma \geq 0$.
- A dynamical system is *input strictly passive* if (20) is replaced by:
$$V(x(t)) - V(x_0) \leq \int_0^t u(\sigma)^\mathsf{T} y(\sigma) - u(\sigma)^\mathsf{T}\varphi(u(\sigma))\ d\sigma,$$
where $\varphi : \mathcal{U} \to \mathbb{R}^m$ satisfies $u^\mathsf{T}\varphi(u) > 0$ for every $u \neq 0$.
- A dynamical system is *output strictly passive* if (20) is replaced by:
$$V(x(t)) - V(x_0) \leq \int_0^t u(\sigma)^\mathsf{T} y(\sigma) - y(\sigma)^\mathsf{T}\rho(y(\sigma))\ d\sigma,$$
where $\rho : \mathcal{Y} \to \mathbb{R}^p$ satisfies $y^\mathsf{T}\rho(y) > 0$ for every $y \neq 0$.

□

Loosely speaking, a system is passive if the increase in the storage function during the time interval $[0, t]$ (left hand side of (20)) is no larger than the energy supplied to the system during that interval (right hand side of (20)).

Passivity is a useful tool to assess stability of the equilibrium points, as formalized by the Passivity Theorem [23, Proposition 4.3.1] [30, Theorem 2.30], stated below. In order to present the Passivity Theorem, we recall that the dynamical system in Definition 3 is zero-state observable if $g(x(t), 0) = 0$ at all times implies $x(t) = 0$ at all times [23].

*Theorem 4.2: **(Passivity Theorem)*** Consider the systems:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, u_1), & \dot{x}_2 &= f_2(x_2, u_2),\\ y_1 &= g_1(x_1, u_1), & y_2 &= g_2(x_2, u_2),\end{aligned} \qquad (21)$$

each satisfying the assumptions of Definition 3. Moreover, assume the systems are interconnected by $u_2 = y_1$, $u_1 = -y_2$, and that each system is zero-state observable. Then,

(i) if each system is passive, then $(x_1^*, x_2^*) = (0, 0)$ is a marginally stable fixed point of (21).

(ii) if there exists differentiable functions $V_1 : \mathcal{X}_1 \to \mathbb{R}_{\geq 0}$, $V_2 : \mathcal{X}_2 \to \mathbb{R}_{\geq 0}$, such that for $i \in \{1, 2\}$:
$$\begin{aligned}V_i(x(t)) - V_i(x_0) \leq \int_0^t & u_i(\sigma)^\mathsf{T} y_i(\sigma) -\\ & u_i(\sigma)^\mathsf{T}\varphi_i(u_i(\sigma)) - y_i(\sigma)^\mathsf{T}\rho_i(y_i(\sigma))\ d\sigma,\end{aligned}$$
holds for all $t \geq 0$, for every $x_1(0) \in \mathcal{X}_1$, $x_2(0) \in \mathcal{X}_2$, where $\varphi_1 : \mathcal{U}_1 \to \mathbb{R}^m_{\geq 0}$, $\varphi_2 : \mathcal{U}_2 \to \mathbb{R}^m_{\geq 0}$, $\rho_1 : \mathcal{Y}_1 \to \mathbb{R}^m_{\geq 0}$, $\rho_2 : \mathcal{Y}_2 \to \mathbb{R}^m_{\geq 0}$, satisfy:
$$v^\mathsf{T}\varphi_1(v) + v^\mathsf{T}\rho_2(v) > 0, \text{ and } v^\mathsf{T}\varphi_2(v) + v^\mathsf{T}\rho_1(v) > 0, \qquad (22)$$
for any $v \in \mathbb{R}^m \setminus \{0\}$, then $(x_1^*, x_2^*) = (0, 0)$ is an asymptotically-stable fixed point of (21).

B. Proof of Theorem 4.1

We now present the proof of Theorem 4.1. In short, marginal stability of the fixed points follows from interpreting (11) as a negative feedback interconnection between the traffic dynamics and the routing dynamics (see Fig. 3), and from showing that the open-loop components are passive systems. We first show that the routing dynamics are passive. We begin by fixing a link $\ell \in \mathcal{L}$, and by showing that the following subset of equations define a passive system:

$$\dot{r}_{\ell m} = r_{\ell m}(\sum_q r_{\ell q}\pi_q - \pi_m), \text{ for all } m \in \mathcal{A}_\ell. \qquad (23)$$

By recalling that $\mathcal{A}_\ell$ denotes the set of links downstream of ℓ, we stress that (23) is the set of routing equations associated to a single link ℓ (we will extend the reasoning to all links in the network shortly below). Let $\mathcal{A}_\ell := \{m_1^\ell, \dots, m_\alpha^\ell\}$ where $\alpha = |\mathcal{A}_\ell|$, we interpret the set of equations (23) as a dynamical system with state $r_\ell := [r_{\ell,m_1^\ell}, \dots, r_{\ell,m_\alpha^\ell}]^\top$, and with input and output, respectively:

$$u_\ell := [\pi_{m_1^\ell} \ \dots \ \pi_{m_\alpha^\ell}]^\top, \quad y_\ell := [r_{\ell,m_1^\ell} \ \dots \ r_{\ell,m_\alpha^\ell}]^\top. \tag{24}$$

The following result formalizes passivity of (23).

Lemma 4.3: ***(Passivity of Single-Link Routing System)*** Let Assumption 3 hold. Let $\ell \in \mathcal{L}$, and for any $u_\ell^* = [\pi^*_{m_1^\ell} \ \dots \ \pi^*_{m_\alpha^\ell}]^\top \in \mathbb{R}^\alpha_{\geq 0}$, let $r_\ell^* := [r^*_{\ell,m_1^\ell}, \dots, r^*_{\ell m_\alpha^\ell}]^\top$ denote the equilibrium point of (23) with $u_\ell = u_\ell^*$. Then, there exists a storage function $r_\ell \mapsto V_\ell(r_\ell) \in \mathbb{R}_{\geq 0}$ such that along the trajectories of (23):

$$\frac{d}{dt} V_\ell(r_\ell) \leq -(u_\ell - u_\ell^*)^\top (y_\ell - y_\ell^*).$$

Proof: Consider the storage function:

$$V_\ell(r_\ell) = \sum_{m \in \mathcal{A}_\ell} r^*_{\ell m} \ln\left(\frac{r^*_{\ell m}}{r_{\ell m}}\right). \tag{25}$$

Notice that $r_\ell \mapsto V_\ell(r_\ell)$ is differentiable because it is a combination of natural logarithm functions, $V_\ell(r_\ell^*) = 0$, and by using the log-sum inequality:

$$\begin{aligned} V_\ell(r_\ell) &= \sum_m r^*_{\ell m} \ln\left(\frac{r^*_{\ell m}}{r_{\ell m}}\right) \\ &\geq (\sum_m r^*_{\ell m}) \ln\left(\frac{\sum_m r^*_{\ell m}}{\sum_m r_{\ell m}}\right) = 1 \cdot \ln(1) = 0, \end{aligned}$$

where we used $\sum_m r^*_{\ell m} = \sum_m r_{\ell m} = 1$. The time-derivative of the storage function reds as:

$$\begin{aligned} \tfrac{d}{dt} V_\ell(r_\ell) &= -\sum_m r^*_{\ell m} \frac{\dot r_{\ell m}}{r_{\ell m}} = \sum_m r^*_{\ell m}(\pi_m - \sum_q r_{\ell q}\pi_q) \\ &= \sum_m r^*_{\ell m}\pi_m + \underbrace{\sum_m r^*_{\ell m}}_{=1} \sum_q r_{\ell q}\pi_q \\ &= \sum_m (r^*_{\ell m} - r_{\ell m})\pi_m \\ &= \sum_m (r^*_{\ell m} - r_{\ell m})(\pi_m - \pi_m^*) + \sum_m (r^*_{\ell m} - r_{\ell m})\pi_m^* \\ &\leq \sum_m (r^*_{\ell m} - r_{\ell m})(\pi_m - \pi_m^*) = -(u_\ell - u_\ell^*)^\top (y_\ell - y_\ell^*). \end{aligned}$$

The last inequality follows from the following bound:

$$\begin{aligned} \sum_m (r^*_{\ell m} - r_{\ell m})\pi_m^* &= \pi_i^* - \sum_m r_{\ell m}\pi_m^* \\ &= \pi_i^*(1 - r_{\ell i}) - \sum_{m \neq i} r_{\ell m}\pi_m^* \\ &= \sum_{m \neq i} r_{\ell m}(\pi_i^* - \pi_m^*) \leq 0, \end{aligned}$$

where we used $\sum_m r^*_{\ell m} = \pi_i^*$ with $i := \arg\max_{m \in \mathcal{A}_\ell}\{\pi_m^* : r^*_{\ell m} > 0\}$, $(1 - r_{\ell i}) = \sum_{m \neq i} r_{\ell m}$, and $(\pi_i^* - \pi_m^*) \leq 0$. The above inequalities show the claim and conclude the proof. ∎

Lemma 4.3 shows that, for a fixed $\ell \in \mathcal{L}$, the set of routing equations (23) is a passive dynamical system. We note that, because the equilibrium points of (23) are parametrized by the choice of input u_ℓ, Lemma 4.3 shows passivity with respect to *any* equilibrium point. The latter corresponds to the notion of equilibrium-independent passivity studied in [31]. Next, we use the above lemma to show that the set of routing equations (8) for all $\ell \in \mathcal{L}$ define a passive system. To this aim, define the following input and output vectors for (8), respectively:

$$u_r := [u_1, \dots, u_n]^\top, \qquad y_r := [y_1, \dots, y_n]^\top, \tag{26}$$

where u_ℓ and y_ℓ are as defined in (24) for all $\ell \in \{1, \dots, n\}$.

Lemma 4.4: ***(Passivity of Joint Routing Dynamics)*** Let Assumption 3 hold and let the perceived costs satisfy (6). For any $u_r^* \in \mathbb{R}^\alpha_{\geq 0}$, let r^* denote an equilibrium point of (9) with $u_\ell = u_\ell^*$. Then, there exists a storage function $r \mapsto V_r(r) \in \mathbb{R}_{\geq 0}$ such that along the trajectories of (8):

$$\frac{d}{dt} V_r(r) \leq -(u_r - u_r^*)^\top (y_r - y_r^*).$$

Proof: We consider the storage function:

$$V_r(r) = \sum_{\ell \in \mathcal{L}} V_\ell(r_\ell), \tag{27}$$

where $V_\ell(r_\ell)$ is as defined in (25) for all $\ell \in \mathcal{L}$. By recalling the definition $r_\ell := [r_{\ell,m_1^\ell}, \dots, r_{\ell m_\alpha^\ell}]^\top$, we note that for every pair of links $\ell, w \in \mathcal{L}$, the vectors r_ℓ and r_w contain no common variables, and thus each term in the summation (27) can be analyzed independently. By computing the time derivative of the storage function:

$$\frac{d}{dt} V_r(r) \leq -\sum_{\ell \in \mathcal{L}} (u_\ell - u_\ell^*)^\top (y_\ell - y_\ell^*) = (u_r - u_r^*)^\top (y_r - y_r^*),$$

where the inequality follows from the proof of Lemma 4.3. ∎

Next, we show passivity of the traffic (4). We interpret (4) as a dynamical system with input and output, respectively,

$$\begin{aligned} u_x &:= [r_{11}, r_{12}, \dots, r_{1n}, r_{21}, \dots, r_{nn}]^\top, \\ y_x &:= [\ \pi_1,\ \pi_2, \dots,\ \pi_n,\ \pi_1, \dots,\ \pi_n]^\top. \end{aligned} \tag{28}$$

Notice that to the scalar input $r_{\ell m}$ we associated the scalar output π_m, for all $(\ell, m) \in \mathcal{A}$.

Lemma 4.5: ***(Passivity of the Traffic Dynamics)*** Let Assumptions 1-2 hold. The traffic system (4) is passive with respect to the input-output pair (u_x, y_x).

Moreover, if for all ℓ there exists $\rho_\ell \in \mathbb{R}_{>0}$ such that

$$f_\ell^{\text{out}}(x_\ell) \geq \rho_\ell \pi_\ell(x), \tag{29}$$

then (4) is output strictly passive with respect to (u_x, y_x).

Proof: We show that the following function

$$V_x(x) := \frac{1}{h} \sum_{\ell \in \mathcal{L}} \int_0^{x_\ell} \tau_\ell(\sigma)\ d\sigma, \tag{30}$$

is a storage function for (4), where $h = \max_{\ell \in \mathcal{L}}\ C_\ell$. We note that V_x is non-negative and it is differentiable, because it is the combination of integral functions, and thus it is an appropriate choice of storage function. By taking the time derivative:

$$\begin{aligned}\frac{d}{dt}V_x(x) &= \frac{1}{h}\sum_{\ell\in\mathcal{L}}\tau_\ell(x_\ell)\left(-f_\ell^{\text{out}}(x_\ell)+\sum_{m\in\mathcal{A}_\ell} r_{m\ell}f_m^{\text{out}}(x_m)\right)\\ &\leq \frac{1}{h}\sum_{\ell\in\mathcal{L}}\tau_\ell(x_\ell)\sum_{m\in\mathcal{A}_\ell} r_{m\ell}f_m^{\text{out}}(x_m)\\ &\leq \sum_{\ell\in\mathcal{L}}\sum_{m\in\mathcal{A}_\ell}\tau_\ell(x_\ell)r_{m\ell}\\ &\leq \sum_{\ell\in\mathcal{L}}\sum_{m\in\mathcal{A}_\ell}\pi_\ell(x)r_{m\ell} = u_x^\mathsf{T} y_x, \end{aligned} \tag{31}$$

where for every $\ell \in \mathcal{L}$ and for any $x_\ell \in \mathbb{R}_{\geq 0}$ we used $\tau_\ell(x_\ell)f_\ell^{\text{out}}(x_\ell) \geq 0$ for the first inequality, $f_\ell^{\text{out}}(x_\ell) \leq h$ for the second inequality, and $\tau_\ell(x_\ell) \leq \pi_\ell(x)$ for the last inequality.

To show output-strict passivity, we substitute (29) to obtain:

$$\begin{aligned}\tfrac{d}{dt}V_x(x) &= \frac{1}{h}\sum_{\ell\in\mathcal{L}}\tau_\ell(x_\ell)\left(-f_\ell^{\text{out}}(x_\ell)+\sum_{m\in\mathcal{A}_\ell} r_{m\ell}f_m^{\text{out}}(x_m)\right)\\ &\leq -\frac{1}{h}\sum_{\ell\in\mathcal{L}}\tau_\ell(x_\ell)f_\ell^{\text{out}}(x_\ell)+\sum_{\ell\in\mathcal{L}}\sum_{m\in\mathcal{A}_\ell}\pi_\ell(x)r_{m\ell}\\ &\leq -\sum_{\ell\in\mathcal{L}}\frac{\rho_\ell}{h}\tau_\ell(x_\ell)\pi_\ell(x)+\sum_{\ell\in\mathcal{L}}\sum_{m\in\mathcal{A}_\ell}\pi_\ell(x)r_{m\ell}\\ &:= -y_x^\mathsf{T}\rho(y_x)+u_x^\mathsf{T} y_x,\end{aligned}$$

where the first inequality follows from (31), and the second inequality follows from (29). Finally, the claim follows by observing that $y_x^\mathsf{T}\rho(y_x) > 0$ for all $y_x \neq 0$, which shows output-strict passivity and concludes the proof. ■

The condition (29) shows that in order to guarantee output-strict passivity, an increase in the perceived cost $\pi_\ell(x)$ must imply an increase in the outflow from the link $f^{\text{out}}(x_\ell)$.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1: To prove the claim, we interpret (11) as an interconnection between the traffic system (4) and the routing system (8) and we leverage the Passivity Theorem [23].

We begin by observing that lemmas 4.4 and 4.5 ensure passivity of the open loop systems. Next, we show that the equilibrium points are minima of the storage functions.First, $V_r(r^*) = 0$ and $V_r(r) \geq 0$ for any $r \in \mathcal{R}_\mathcal{G}$ follow directly from the proof of Lemma 4.4.

Second, we show that $V_x(x)$ attains a minimum at the equilibrium points. To this aim, we first let $\bar\lambda = 0$ and we study the equilibrium points of (4). Every equilibrium point x^* satisfies the following identity

$$0 = (R(r) - I)^\mathsf{T} f(x^*).$$

By observing that $(R^\mathsf{T} - I)$ is invertible (see e.g. [25, Theorem 1]), and that $f(x^*) = 0$ only if $x^* = 0$ (see Assumption 2), the above equation implies that the unique equilibrium point of the system satisfies $x^* = 0$. The choice of $V_x(x)$ in (30) implies that $V_x(x)$ is non-negative and that $V_x(x^*) = 0$, which shows that x^* is a local minima of the storage function. Lastly, we observe that any nonzero $\bar\lambda$ has the effect of shifting the equilibrium point, and thus it does not change the properties of the storage function. Finally, marginal stability of the equilibrium points follows from (i) in Theorem 4.2. ■

V. Robust Information Design

In this section, we propose a control technique to guarantee asymptotic stability of the equilibrium points. The method relies on regulating the rate at which travelers react to changes in traffic congestion, as described next. We consider the following modification of the replicator equations (8):

$$\dot r_{\ell m} = \delta_\ell(\pi(x))\ r_{\ell m}\big(\sum_{q\in\mathcal{A}_\ell} r_{\ell q}\pi_q(x) - \pi_m(x)\big), \quad (\ell, m) \in \mathcal{A}, \tag{32}$$

where for every $\ell \in \mathcal{L}$, $\pi(x) \mapsto \delta_\ell(\pi(x)) \in \mathbb{R}_{\geq 0}$ is a control parameter that modifies the rate of change of the replicator equations (hereafter called congestion-aware reaction rates). We focus our attention on a particular class of reaction rates that satisfies the following condition for all $\ell \in \mathcal{L}$:

$$(\pi_m - \pi_m^*)\delta_\ell(\pi_m) \geq 0, \forall m \in \mathcal{A}_\ell, \pi_m \in \mathbb{R}_{\geq 0}, \tag{33}$$

where $\pi_m^* = \pi_m(x^*)$, denotes the perceived cost of link m at an equilibrium point (x^*, r^*) of (11).

Remark 7: ***(Local Congestion-Aware Reaction Rates)*** By noting that, for each $\ell \in \mathcal{L}$, (33) imposes a condition that depends only on the perceived costs of the links downstream of ℓ (i.e. π_m for all $m \in \mathcal{A}_\ell$), the condition (33) defines a family of reaction rates that are local, namely that can be computed at each link by using only knowledge regarding the perceived costs of the links downstream of ℓ. □

Example 5: ***(Choice of Congestion-Aware reaction Rates)*** Consider the seven-link network illustrated in Fig. 2(a) and discussed in Examples 2-3. A choice of reaction rates for this network that satisfies (33) is:

$$\begin{aligned}\delta_1(\pi_2, \pi_3) &= \max\{\pi_2 - \pi_2^*, 0\} + \max\{\pi_3 - \pi_3^*, 0\},\\ \delta_2(\pi_4, \pi_5) &= \max\{\pi_4 - \pi_4^*, 0\} + \max\{\pi_5 - \pi_5^*, 0\}.\end{aligned}$$

Notice that these functions are Lipschitz continuous and guarantee existence and uniqueness of solutions to (34). □

In vector form, we denote by $\delta(\pi(x))$ the vector of reaction rates, and we rewrite (32) in compact form as $\dot r = \varrho(r, \pi(x), \delta(\pi(x)))$, where the map $(r, \pi, \delta) \mapsto \varrho(r, \pi, \delta)$ is obtained by stacking (32). In what follows, we study the stability of the equilibrium points of the coupled system:

$$\begin{aligned}\dot x &= (R(r) - I)^\mathsf{T} f(x) + \lambda,\\ \dot r &= \varrho(r, \pi(x), \delta(\pi(x))).\end{aligned} \tag{34}$$

We note that (11) and (34) share the same equilibrium points, because the reaction rates are positive multiplicative quantities in the routing equation (32).

Theorem 5.1: ***(Asymptotic Stability Under Congestion-Aware Reaction Rates)*** Let Assumptions 1-3 hold, assume that for all $\ell \in \mathcal{L}$ there exists $\rho_\ell > 0$ such that $f_\ell(x_\ell) \geq \rho_\ell\pi_\ell(x)$, and let the reaction rates satisfy (33). Then, every equilibrium (x^*, r^*) of (34) that satisfies (16) is asymptotically stable.

The above result shows that asymptotic stability of the equilibrium points of the interconnected traffic-routing systems can be achieved by modifying the rate at which travelers respond to congestion, and precisely by adopting a class of congestion-aware reaction rates that satisfies (32).

In what follows, we present the proof of Theorem 5.1. We begin by showing strict passivity of the routing dynamics.

Lemma 5.2: ***(Input Strict Passivity of Routing Dynamics)*** Let Assumption 3 hold, let the perceived costs satisfy (6), and assume the reaction rates satisfy (33). Moreover, for any $u_r^* \in \mathbb{R}_{\geq 0}^{\alpha}$, let r^* denote an equilibrium point of (32) with $u_\ell = u_\ell^*$. Then, there exists a storage function $r \mapsto V_r(r) \in \mathbb{R}_{\geq 0}$ such that along the trajectories of (32):

$$\frac{d}{dt} V_r(r) \leq -(u_r - u_r^*)^\mathsf{T}(y_r - y_r^*) - (y_r - y_r^*)^\mathsf{T} \rho(y_r),$$

for some function $\rho(\cdot)$ that satisfies $(y_r - y_r^*)^\mathsf{T} \rho(y_r) \geq 0$ and where u_r, y_r are as defined in (26).

Proof: The proof of this claim uses a similar argument as in Section IV-B. First, we fix a link $\ell \in \mathcal{L}$ and we show that the group of routing equations associated to ℓ are input strictly passive. Consider the Storage function (25), by taking the time-derivative along the trajectories of (34):

$$\begin{aligned}
\tfrac{d}{dt} V_\ell(r_\ell) &= -\sum_m r_{\ell m}^* \frac{\dot{r}_{\ell m}}{r_{\ell m}} = \sum_m r_{\ell m}^* \delta_\ell(\pi)(\pi_m - \sum_q r_{\ell q}\pi_q) \\
&\leq \sum_m (r_{\ell m}^* - r_{\ell m})(\pi_m - \pi_m^*) + \delta_\ell(\pi) \sum_m r_{\ell m}^*(\pi_m - \pi_m^*) \\
&\leq \sum_m (r_{\ell m}^* - r_{\ell m})(\pi_m - \pi_m^*) + \sum_m \delta_\ell(\pi)(\pi_m - \pi_m^*) \\
&:= -(u_\ell - u_\ell^*)^\mathsf{T}(y_\ell - y_\ell^*) - (y_\ell - y_\ell^*)^\mathsf{T} \delta(y_\ell),
\end{aligned}$$

where the first inequality follows by iterating the bound in the proof of Lemma 4.3, and the second inequality follows from $r_{\ell m}^* \geq 0$ for all $(\ell, m) \in \mathcal{A}$. The above bound shows that the single-junction routing (23) is input strictly passive.

Finally, input strict passivity of the joint routing dynamics follows by using the storage $V_r(r) = \sum_{\ell \in \mathcal{L}} V_\ell(r)$ and by adopting a reasoning similar to proof of Lemma 4.4. ■

We are now ready to formally prove Theorem 5.1.

Proof of Theorem 5.1: To prove the claim, we interpret (34) as an interconnection between the traffic system (4) and the routing dynamics (34), and we leverage (ii) in Theorem 4.2.

First, by iterating the reasoning in the proof of Theorem 4.1 we conclude that the equilibrium points of (34) are local minima for the storage functions.

Second, we note that each open loop systems is zero-state observable. Indeed, zero-state observability of the routing dynamics immediately follows by observing that the state of the system coincides with its output. Zero-state observability of the traffic dynamics immediately follows by observing that if the output of the system is identically zero then its state is also identically zero.

Third, we show that the inequalities (22) are satisfied. We begin by observing that input strict passivity of the routing dynamics, proved in Lemma 5.2, ensures the existence of a function $\varphi_{\text{routing}} : \mathcal{T} \to \mathbb{R}_{>0}^n$, such that

$$v^\mathsf{T} \varphi_{\text{routing}}(v) > 0, \text{ for all } v \neq 0.$$

Moreover, output strict passivity of the traffic dynamics, proved in Lemma 4.5, ensures the existence of a function $\rho_{\text{traffic}} : \mathcal{T} \to \mathbb{R}_{>0}^n$, such that

$$v^\mathsf{T} \rho_{\text{traffic}}(v) > 0, \text{ for all } v \neq 0.$$

Finally, the claim follows by combining the above observations, and by application of (ii) in Theorem 4.2. ■

VI. Simulation Results

This section presents two sets of numerical simulations that illustrate our findings.

A. Data From SR60-W and I10-W in Southern California

Consider the traffic network in Fig. 6(a), which schematizes the west bounds of the freeways SR60-W and I10-W in Southern California. Let x_{60} and x_{10} be the average traffic density in the examined sections of SR60-W (absolute miles $13.1 - 22.4$) and in the section of I10-W (absolute miles $24.4 - 36.02$), respectively. Moreover, let r_{60} (resp. $r_{10} = 1 - r_{60}$) be the fraction of travelers choosing freeway SR60-W over I10-W (resp. choosing freeway I10-W over SR60-W) for their commute. Fig. 6(a) illustrates the time-evolution of the recorded traffic densities in the two highways on Friday, March 6, 2020, reconstructed using data from the *Caltrans Freeway Performance Measurement System (PeMS)*. Fig. 6(a)-(b) illustrate the time-evolution of the state of the interconnected model (11). The parameters of the traffic system (4) were derived from the nominal highway characteristics provided by the PeMS. In the routing model (8), the instantaneous link travel costs are computed by integrating traffic speed data, and the perceived costs are computed from the link travel costs by using Lemma 2.1. Fig. 6(a) graphically illustrate that our model of routing is consistent with traffic data observed in practice. Fig. 6(b) graphically illustrates the oscillatory behaviors discussed and characterized in Example 4.

B. Oscillating Trajectories in Seven-Link Network

Consider the network illustrated in Fig. 7(a) and discussed in Examples 2-3 and 5. Assume that the outflows are linear:

$$f_\ell(x_\ell) = x_\ell, \text{ for all } \ell \in \{1, \ldots, 7\},$$

and that the travel costs are affine:

$$\tau_\ell(x_\ell) = a_\ell x_\ell + b_\ell, \text{ for all } i \in \{1, \ldots, 7\},$$

where the parameters a_ℓ and b_ℓ are summarized in Table I. Notice that these choices of flow functions and travel costs satisfy Assumption 2-3. Since the flow capacities of the links are unbounded, $C_{\text{m-cut}} = +\infty$ and thus Theorem 3.2 guarantees the existence of an equilibrium point (x^*, r^*). It can be verified that an equilibrium point that satisfies (16) is:

$$\begin{gathered}
x_1^* = 6,\ x_2^* = 4,\ x_3^* = 2,\ x_4^* = 2,\ x_5^* = 2,\ x_6^* = 4,\ x_7^* = 6, \\
r_{12} = 2/3, \quad r_{13} = 1/3, \quad r_{24} = 1/2, \quad r_{25} = 1/2.
\end{gathered}$$

Fig. 7 illustrates a sample trajectory of (11), demonstrating that the system admits a periodic orbit, which prevents the state

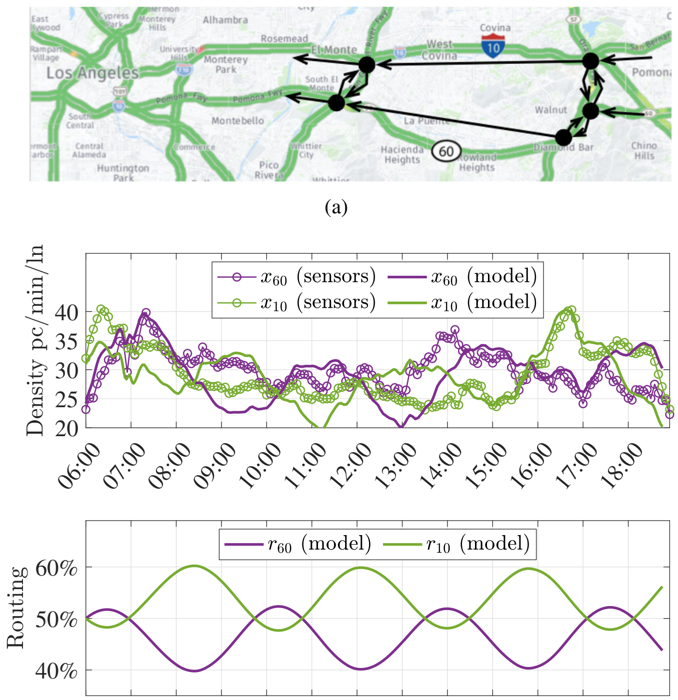


Fig. 6. Time series data for SR60-W and I10-W on March 6, 2020. (a) schematic of traffic network. (b) Sensory data (continuous lines with circles) and trajectories predicted by our models (continuous lines). (c) Routing predicted by our models. Simulation uses constant inflow $\bar{\lambda} = 3340$ veh/hr/ln.

TABLE I
CHOICE OF AFFINE TRAVEL COSTS

ℓ	1	2	3	4	5	6	7
a_ℓ	1	10	1	1	1	10	1
b_ℓ	0	0	50	10	50	0	0

from converging to the equilibrium points asymptotically. Notice that, although the flow functions satisfy the assumptions of Theorem 5.1, constant reaction rates violate condition (33) and thus Theorem 5.1 is not applicable in this case.

In contrast, Fig. 8 illustrates a sample trajectory of (11) when the reaction rates follow the congestion-responsive model described in Example 5. As illustrated by the simulation, this class of control policies guarantees the asymptotic stability of the equilibrium points.

VII. CONCLUSION

This paper proposed a dynamical routing model to understand the impact of app-informed travelers in traffic networks. We studied the stability of the routing model coupled with a dynamical traffic model, and our models and results suggest that the general adoption of routing apps (i) maximizes the throughput of flow across the traffic system, but (ii) could deteriorate the stability of the equilibrium points. To ensure asymptotic stability, we propose a control technique that relies on regulating the rate at which routing apps react to changes in traffic congestion. Our results give rise to several opportunities for future work. By coupling these models with common

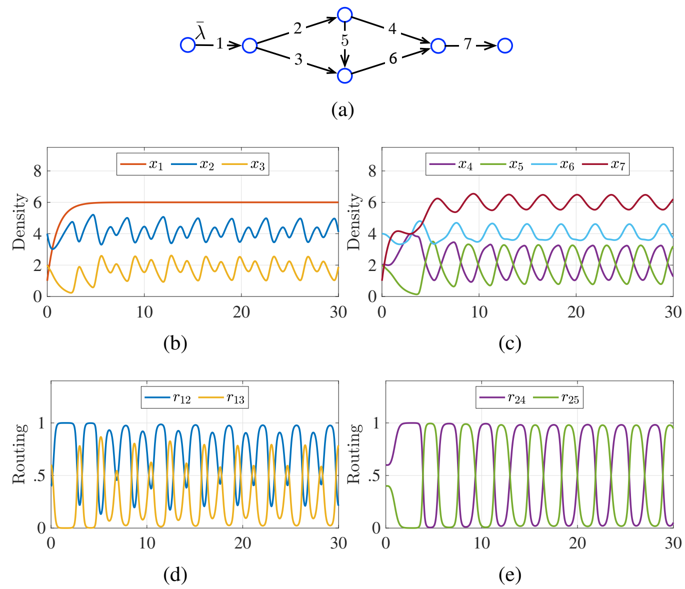


Fig. 7. (a) Seven-link network. (b)-(c) Oscillating traffic state. (d)-(e) Oscillating routing state.

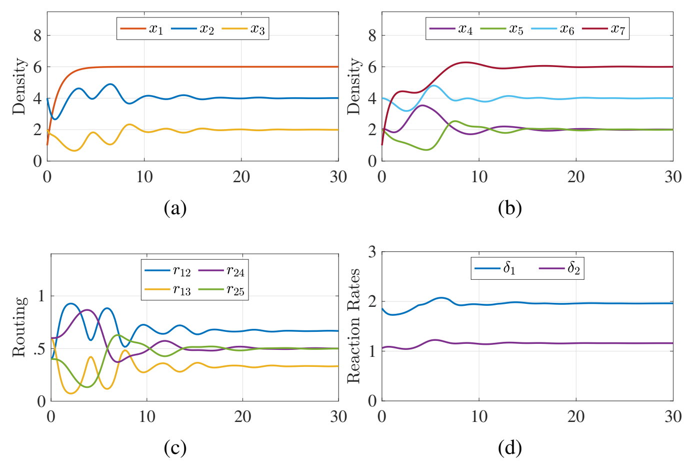


Fig. 8. Asymptotic stability under congestion-aware reaction rates.

infrastructure-control models (such as variable speed limits and freeway metering), these results may play an important role in designing dynamical controllers for congested infrastructures. Furthermore, our models and stability analysis represent a fundamental framework for future studies on robustness and security analysis.

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