Feedback Optimization of Dynamical Systems in Time-Varying Environments: An Internal Model Principle Approach

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***Abstract*—Feedback optimization has emerged as a promising approach for regulating dynamical systems to optimal steady states that are implicitly defined by underlying optimization problems. Despite their effectiveness, existing methods face two key limitations: (i) reliable performance is restricted to time-invariant or slowly varying settings, and (ii) convergence rates are limited by the need for the controller to operate orders of magnitude slower than the plant. These limitations can be traced back to the reliance of existing techniques on numerical optimization algorithms. In this paper, we propose a novel perspective on the design of feedback optimization algorithms, by framing these objectives as an output regulation problem. We place particular emphasis on time-varying optimization problems, and show that an algorithm can track time-varying optimizers if and only if it incorporates a model of the temporal variability inherent to the optimization – a requirement that we term the *internal model principle of feedback optimization*. Building on this insight, we introduce a new design methodology that couples output-feedback stabilization with a control component that drives the system toward the critical points of the optimization problem. This framework enables feedback optimization algorithms to overcome the classical limitations of slow tracking and poor adaptability to time variations.**

I. Introduction

Feedback optimization techniques [1]–[6] are concerned with the problem of controlling dynamical systems to an optimal steady-state point, where optimality is implicitly defined by an underlying mathematical optimization problem [3]. This framework has been successfully applied in a variety of domains, including optimal scheduling in communication networks, resource allocation in power systems, transportation system optimization, and the operation of industrial control processes [7]. The foundational design approach underlying these methods begins with an established optimization algorithm [8], and in suitably adapting it to incorporate feedback measurements in place of unknown quantities [3], [7]. This modification results in a feedback loop between the optimization algorithm and the dynamical system. Although numerous feedback optimization methods have been developed, these techniques inherently carry over the limitations of the optimization algorithms from which they originate. Such limitations include fundamental bounds on the achievable convergence rate [9], limited robustness to uncertainty [10], and the inability to exactly track optimizers when in time-varying settings [6], [10].

In this work, we approach the design of feedback optimization algorithms from a novel perspective: by formulating it as an output regulation problem [11]–[15]. Specifically, in this

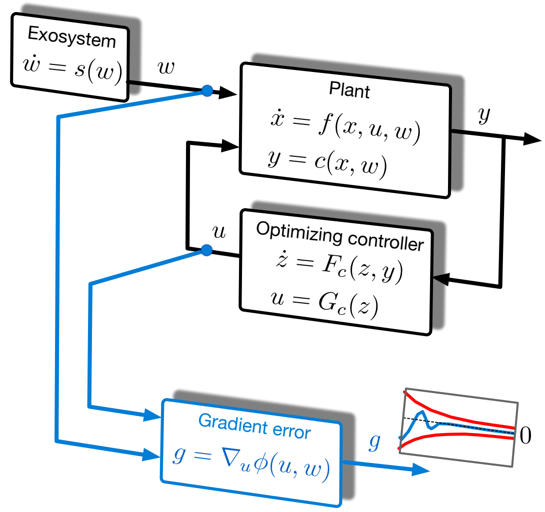

Fig. 1: Architecture of the feedback optimization scheme studied in this work. Given a plant subject to a time-varying disturbance, the goal is to design a control algorithm having access only to measurements $y(t)$ and generating a sequence of control actions $u(t)$ such that the interconnection is stabilized around an equilibrium and the gradient signal $g(t) \to 0$ as $t \to \infty$. Blue indicates quantities that are unmeasurable by the controller; in particular, the controller has no access to the signal $g(t)$ to be regulated to zero. See (9).

setting, the signal to be regulated corresponds to the (unmeasurable) gradient error – see Fig. 1 for an illustration. By establishing this connection, we are able not only to design novel feedback optimization algorithms that are more general than those derived from classical optimization methods, but also to study an entire class of algorithms and establish fundamental limitations for the whole class. This class includes, as special cases, many well-established techniques [1]–[6]. Our approach introduces three key practical innovations compared to the existing literature: (i) under suitable system knowledge, our methods enable exact tracking of optimal steady states despite arbitrary time-varying disturbances – whereas existing techniques can only achieve approximate tracking (see [6] and references therein); (ii) our framework does not require plant pre-stabilization, as it simultaneously addresses both stabilization and setpoint tracking; and (iii) it removes the need for a time-scale separation between the plant and the controller, thereby improving the achievable rate of convergence.

Related work. The field of feedback optimization stems from the seminal works [3], [16], which considered the objective of steering the outputs of a plant to a steady-state that solves a convex optimization problem. While early works focused on static plants, a more recent body of literature [1]–[6], [17] extends feedback optimization to dynamic plants, addressing closed-loop stability and exponential convergence rates [6], [10], including in sampled-data settings [18]. Constrained variants of the problem have also been explored, particularly

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using primal-dual dynamics and projection-based methods [1], [2], [6]. An alternative approach employs barrier function techniques [19], [20]. Nonconvex problems have been considered in [21], [22], gradient-free methods in [23], distributed approaches in [24], [25]. Recent developments have sought to relax the reliance on time-scale separation [9].

A central contribution of this work is the reformulation of the feedback optimization problem as an output regulation problem; this connects our work with the literature on output regulation. This field originated in the 1970s with foundational work on linear systems [11]–[13], and was subsequently extended to nonlinear systems through both local [14] and global [26] analysis techniques. In recent years, output regulation has seen renewed interest through the lens of modern control methods; see, for example, [27] and the recent tutorial [28]. To the best of our knowledge, this is the first work to formally link output regulation theory with the design of feedback optimization algorithms.

Finally, since the problem studied in this work is a time-varying optimization problem, it is naturally connected to the literature on online optimization [29], [30]. Particularly relevant are the recent works [31]–[33], which explore the use of internal models to address such problems.

Contributions. This paper features three main contributions. First, we demonstrate that the feedback optimization problem can be formulated as an output regulation problem (see Section III and Definition 2). This reformulation enables the use of tools from the output regulation literature to address the problem, allowing us to analyze feedback optimization algorithms as a broad class of methods – rather than relying on ad hoc algorithmic constructs from optimization theory. From a methodological perspective, the techniques we develop here extend existing approaches in several key directions: (i) they enable exact asymptotic tracking of a critical point even in the presence of arbitrary time-varying disturbances – whereas prior work has been limited to constant disturbances; (ii) they allow for simultaneous plant stabilization and feedback optimization, while these two objectives have typically been treated separately in the literature; and (iii) they eliminate the need for a time-scale separation between the plant and the controller, which removes inherent limitations on the maximum achievable convergence rate. To the best of our knowledge, all of these features are novel within the literature. Second, we establish fundamental limitations for a broad class of feedback optimization methods (see Section IV and Theorem 4). Notably, we show that exact tracking of a critical point is possible if and only if the algorithm embeds a duplicated representation of the disturbance signal – though this copy may be expressed in a different coordinate system (see Theorem 5). We refer to this concept as an internal model principle, akin to its counterpart in time-varying optimization [32] and controls [12], [34]. Interestingly, since many existing feedback optimization methods in the literature are special cases of the general class of algorithms considered in this work (see Remark 2), our results imply that these methods achieve exact tracking only when the time-varying signals involved in the optimization problem are, in fact, constant. In all other cases, they necessarily yield only inexact tracking (see Remark 6). To the best of our knowledge, this is the first work in the literature that formally proves the necessity of internal models – a requirement that aligns with recent insights presented in [32]. Third, we derive necessary and sufficient conditions for exact asymptotic tracking of a critical trajectory (see Theorem 5) and, by leveraging these, we introduce a novel design procedure for feedback optimization algorithms. Our design technique relies on a separation principle, which combines an observer to estimate unmeasurable dynamical states, and a static feedback control action, which steers the system toward the set of critical points (see Fig. 3, Section VII and Algorithm 1).

Organization. Section II introduces the problem, which is reformulated as an output regulation problem in Section III. Section IV presents necessary conditions for solvability and outlines the controller structure. Static and dynamic feedback optimization are addressed in Sections V and VI, respectively. Section VII details the algorithm design, while Section VIII discusses extensions to constrained problems. Numerical validations are provided in Section IX, and conclusions in Section X. Technical proofs appear in Appendix A, and Appendix B reviews relevant center manifold theory.

Notation. We let $\mathbb{C}_{<} := \{s : \operatorname{Re} s < 0\}$ and $\mathbb{C}_{\geq} := \{s : \operatorname{Re} s \geq 0\}$. We denote the space of $n \times n$ symmetric real matrices by $\mathbb{S}^n$. Given an open set U, we say that $f : U \to \mathbb{R}$ is of differentiability class C^k if it has a k^{th} derivative that is continuous in U. Given $A \in \mathbb{R}^{n\times n}$, denote its eigenvalues by $\lambda_j = a_j + ib_j, a_j, b_j \in \mathbb{R}, j \in \{1, \dots, n\}$ with corresponding (generalized) eigenvectors $w_j = u_j + iv_j, u_j, v_j \in \mathbb{R}^n$. The space $\mathscr{X}_{<}(A) := \operatorname{span}\{u_j, v_j : a_j < 0\}$ is the *stable subspace* of A, and $\mathscr{X}_{\geq}(A) := \operatorname{span}\{u_j, v_j : a_j \geq 0\}$ is the *unstable subspace* of A. Given $B \in \mathbb{R}^{n\times m}$, the *controllable subspace* of (A, B) is $\mathcal{C}(A, B) := \sum_{i=1}^{n} \operatorname{Im}(A^{i-1}B)$. The pair (A, B) is *controllable* if $\mathcal{C}(A, B) = \mathbb{R}^n$ and *stabilizable* if $\mathscr{X}_{\geq}(A) \subseteq \mathcal{C}(A, B)$. Given $C \in \mathbb{R}^{q\times n}$, the *unobservable subspace* of (C, A) is $\mathcal{O}^c(C, A) := \cap_{i=1}^{n} \operatorname{Ker}(CA^{i-1})$. The pair (C, A) is observable if $\mathcal{O}^c(C, A) = \emptyset$ and *detectable* if $\mathcal{O}^c(C, A) \cap \mathscr{X}_{\geq}(A) = \emptyset$.

II. Problem formulation

In this section, we formulate the problem studied and illustrate its applicability through a representative example.

A. Problem statement

We consider plants described by nonlinear systems of differential equations of the form:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), w(t)), \\ y(t) &= c(x(t), w(t)), \end{aligned} \tag{1}$$

where $t \in \mathbb{R}_{\geq 0}$ denotes time, $x(t) \in X \subseteq \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^q$ is the measurable output, and $w(t) \in W \subseteq \mathbb{R}^p$ is a disturbance. We assume that $f : X \times \mathbb{R}^m \times W \to \mathbb{R}^n$ and $c : X \times W \to \mathbb{R}^q$ are C^1.

The signal $w(t)$ models exogenous disturbances that may affect the plant and/or measurements, possibly independently

TABLE I: Comparison between our framework and existing methods in reference tracking [11]–[14], disturbance rejection [15], and feedback optimization [1], [2], [6]. See Remarks 1, 2, and 6 for discussions.

	$w(t) \neq 0$ & time-varying	Plant stabilization	Controller at same timescale	Automatic reference generation	x-dependent objectives
Reference tracking [11]–[14]		✓	✓		
Tracking and disturbance rejection [15]	✓	✓	✓		
Feedback optimization [1], [2], [7]				✓	
Time-varying feedback optimization [6]	✓			✓	
This work	✓	✓	✓	✓	✓

(see Section II-B). Consequently, we treat $w(t)$ as an unknown and unmeasurable time-varying signal in our analysis.

In this work, we study the problem of devising control actions that accomplish two goals: (i) locally exponentially stabilize the plant (1), and (ii) regulate (1) to an optimal equilibrium point. The second objective is formalized by means of the following mathematical optimization problem:

$$\begin{aligned} \underset{u \in \mathbb{R}^m, x \in X}{\text{minimize}} \quad & \phi_0(u, x), \\ \text{subject to:} \quad & 0 = f(x, u, w(t)), \end{aligned} \tag{2}$$

where $\phi_0 : \mathbb{R}^m \times X \to \mathbb{R}$. This optimization formalizes an equilibrium-selection problem, which seeks to select an equilibrium input u and state x for (1) that minimize the loss $\phi_0(u, x)$, which quantifies performance at the equilibrium[1].

From an optimization perspective, (2) is an optimization problem that is *parametrized* by the exogenous signal $w(t)$. This aspect has two important implications: first, because $w(t)$ is unknown and unmeasurable, solutions to (2) cannot be computed explicitly by an optimization solver (since a numerical value for $w(t)$ cannot be substituted into (2) to solve the optimization); second, because $w(t)$ is a time-varying signal, (2) defines a *sequence* of optimization problems (one at each time t), and thus the problem of regulating the plant to solutions of (2) involves also *tracking these solutions over time.* For these reasons, our control objective cannot be accomplished using standard control approaches such as reference generation plus reference tracking [11]–[14]. We informally state the problem of interest as follows.

Problem 1 (***Feedback optimization algorithm design – informal***)**.** Construct, when possible, a control algorithm such that: (i) the equilibrium of (1) with $w(t) \equiv 0$ is locally exponentially stable, and (ii) the states and inputs of (1) converge, with exactly zero asymptotic error, to a solution of (2) for any time-varying signal $w(t)$ in some class. □

We refer to a control algorithm that achieves these goals as a *feedback-optimization algorithm*, in line with the counterparts in [2], [6], [24]. We stress that existing methods are capable of tracking solutions of (2) only *inexactly*, unless $w(t)$ is a constant signal. In contrast, our goal is to design control algorithms capable of achieving exactly zero asymptotic error when $w(t)$ is time-varying.

We conclude this discussion by relating our problem with the existing literature in Remark 1.

[1]Our approach also applies to more general constraints than merely equilibrium selection (such as predictive control), provided that any state that satisfies the constraints can be written as a function of the control input and exogenous noise, as in Assumption 3.

Remark 1 (***Relationship of*** (2) ***with models commonly studied in the literature***)**.** The optimization objective (2) is related to the classical *(reference) tracking problem*, also called the *(output) regulation problem* or *servomechanism problem* [11]–[14], which consists of designing a controller for (1) such that, with $w(t) \equiv 0$, the closed-loop system is stable and $\lim_{t\to\infty} y(t) - r(t) = 0$, where $r(t)$ is a given reference signal. More generally, (2) is related to the *(reference) tracking and disturbance rejection problem* [15], which extends the reference tracking problem by allowing $w(t)$ to be nonzero and time-varying. The optimization objective (2) is also related to the feedback optimization literature [2], [6], where controllers are designed by drawing inspiration from algorithmic optimization. See Table I.

The optimization (2) extends these existing approaches in at least four directions. First, in reference tracking problems, the reference signal to be tracked is prespecified, thus dictating the need for external reference generation mechanisms (e.g., [35]), which often require knowledge of $x(t)$ and $w(t)$, thus making these approaches unfeasible or suboptimal when these signals are unknown. Instead, the formulation (2) allows us to implicitly generate the reference to be tracked as the solution to a mathematical optimization problem. Second, both in feedback optimization and reference tracking, performance objectives or references to be tracked can be specified only in terms of the plant's output $y(t)$ and not in terms of the plant's state $x(t)$ (or, in other words, they can be applied to solve (2) only when the plant's output map is such that $y(t) = x(t)$). By allowing the loss function to depend on the state, (2) provides additional flexibility in applications with partially observed plants (i.e., where $x(t)$ cannot be directly estimated from $y(t)$); this aspect is illustrated in detail Section II-B. Third, feedback optimization techniques require the plant to be prestabilized. Fourth, feedback optimization methods rely on a timescale separation between the (fast) plant and (slow) controller, and this poses limitations on the maximum attainable rate of convergence. Additional connections between our methods and existing techniques are established in Table I. □

B. Illustrative application: automatically optimize an unstable system subject to unknown disturbances

Consider a two-wheeled balancing robot moving on a surface as in Fig. 2(a), and the objective of self-balancing this inverted-pendulum type system. For this task, we consider the kinematic model in Fig. 2(b); letting θ be the rotation angle of the center of mass and r the horizontal displacement of the wheels, we model the robot using the equation of motion:

$$J_e \ddot{\theta}(t) = mg\ell \sin\theta(t) - k\ell^2 \dot{\theta}(t) - m\ell \ddot{r}(t) \cos\theta(t) + w_x(t),$$

where $J_e > 0$ is the moment of inertia of the rod about its end, $m > 0$ the mass, $\ell > 0$ the distance from the center of gravity to the pivot, g the gravitational acceleration, and we assumed the presence of a frictional force with coefficient $k > 0$. The signal $w_x(t)$ is used to describe external disturbances or model discrepancies, such as uneven surface profiles or frictions. Viewing the cart's acceleration $\ddot{r}(t)$ as the control input (i.e., letting $u(t) := \ddot{r}(t)$), the problem of self-balancing the robot can be formulated as the following instance of (2):

$$
\begin{aligned}
&\underset{u,\theta,\dot\theta\in\mathbb{R}}{\text{minimize}} && \frac{1}{2}(\theta^2+\dot\theta^2),\\
&\text{subject to:} && 0=\dot\theta,\\
& && 0 = mg\ell\sin\theta - m\ell u\cos\theta + w_x(t).
\end{aligned} \tag{3}
$$

Suppose the robot is equipped with an integrated Inertial Measurement Unit (IMU) providing noisy measures of the angular rate of change $\dot\theta(t)$:

$$y(t) = \dot\theta(t) + w_y(t), \tag{4}$$

where $w_y(t)$ models sensor noise. Because $y(t)$ does not include displacement information (i.e., it does not explicitly depend on $\theta(t)$), any method that achieves $y(t) \to 0$ as $t \to \infty$ (such as reference tracking or feedback optimization, see Remark 1) will not guarantee that $\theta(t) \to 0$, but only that $\dot\theta(t) \to 0$. This limitation is illustrated in Fig. 2(c), where a reference tracking technique [14] is applied to this problem. In contrast, the techniques proposed in this work are capable of achieving $\theta(t) \to 0$ (see Fig. 2(d)). See Section IX for additional details on this problem and a description of the techniques used.

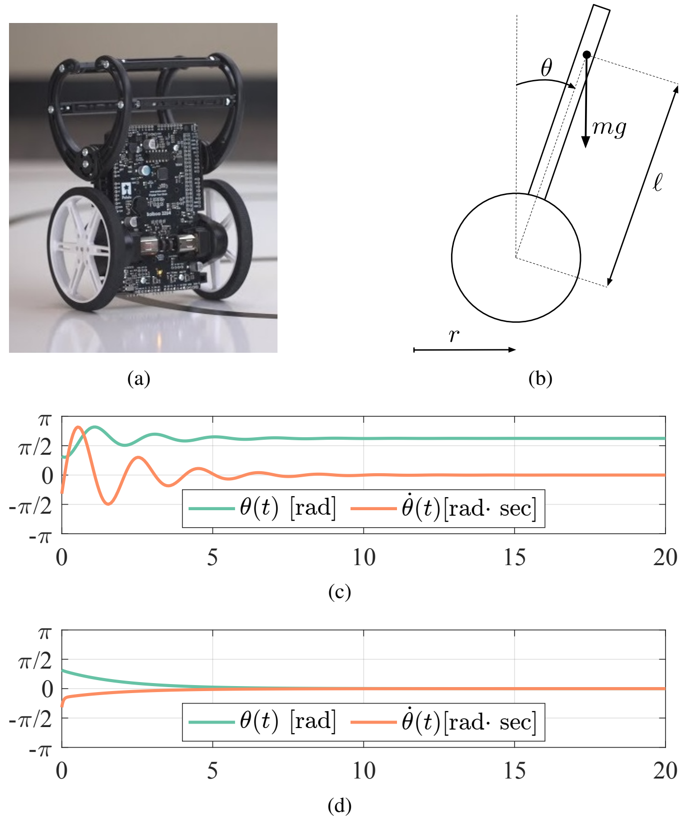

Fig. 2: (a) Illustration of the balancing robot discussed in Section II-B. (b) Free-body diagram of the robot. (c) State evolution under control via a standard output regulation algorithm. (d) State evolution under the control strategy proposed in this work. The simulation demonstrates that traditional output regulation algorithms fail to stabilize the robot in the upright position – i.e., to achieve $\theta(t) \to 0$ – due to the inability to measure $\theta(t)$. Parameters used: $\ell = 0.023$ [m], $m = 0.316$ [Kg], $k = 0.1$ [Kg/s], $g = 9.81$ [m/s^2], $J_e = 0.000444$ [Kg m^2]. For simplicity of the illustration, this simulation uses $w_x(t) \equiv 0$ and $w_y(t) \equiv 0$; see Section IX for additional simulations.

III. Technical reformulation of the problem: from feedback optimization to output regulation

In this section, we present a formal reformulation of Problem 1, connecting it with the output regulation framework.

A. Standing assumptions

We now present the assumptions on which our approach is based. We make the following standard assumption on the loss.

Assumption 1 (***Properties of the objective function***)**.** The map $(u, x) \mapsto \phi_0(u, x)$ is convex and differentiable, and $(u, x) \mapsto \nabla\phi_0(u, x)$ is Lipschitz continuous in $\mathbb{R}^m \times X$.

Convexity and smoothness are standard assumptions in optimization [29], which have been widely used in works on related problems [2], [6], [24].

We assume that the temporal variability of the disturbance $w(t)$ belongs to the following class.

Assumption 2 (***Class of temporal variabilities of the disturbance signal***)**.** There exists a C^1 vector field $s : W \to \mathbb{R}^p$ such that the disturbance signal $w(t)$ satisfies:

$$\dot w(t) = s(w(t)) \qquad \forall t \in \mathbb{R}_{\geq 0}. \tag{5}$$

Moreover, (5) has an equilibrium at $w = 0$ and its trajectories are bounded. □

We do not assume *a priori* that $s(w)$ or $w(t)$ are known; see Sections V–VI, where different assumptions on the knowledge of these two quantities are discussed. Assumption 2 characterizes the class of temporal variabilities for the disturbance taken into account. This assumption is mild, as it only requires that $w(t)$ is deterministic, sufficiently smooth (so that its derivative is some C^1 function), and its trajectories remain bounded. Following established terminology in the literature [11], [14], we refer to the model in (5) as the *exosystem*.

We observe that asymptotically stable modes in $w(t)$ do not affect the optimization (2) when $t \to \infty$ (precisely, because these modes converge to 0 as $t \to \infty$). Hence, without loss of generality, in what follows, we will assume that (5) has no asymptotically stable modes. For simplicity of the presentation, we also assume that W is some neighborhood of the origin of $\mathbb{R}^p$. We put no restrictions on the size of this neighborhood (which is, e.g., allowed to be the entire space $W = \mathbb{R}^p$), and thus on the size of $w(t)$ nor of its temporal variation. Moreover, there is no restriction with asking that W contains the origin because, if $w(t)$ takes values in the neighborhood of any other point, such a point can be shifted to the origin through a time-invariant change of variables without altering the solutions of (2).

We make the following assumption on the plant to control.

Assumption 3 (***Existence of a steady-state map***)**.** There exists a unique C^1 function $h : \mathbb{R}^m \times W \to X$ such that $f(h(u,w),u,w) = 0$, for any $u \in \mathbb{R}^m$ and $w \in W$. $\square$

In words, Assumption 3 guarantees that the plant (1), with constant inputs, admits a steady-state operating point. Existence of such a steady-state map is guaranteed in most practical cases of interest [36]. For example, this condition is ensured when $\nabla_x f(x,u,w)$ is invertible in $\mathbb{R}^m \times W$; alternatively, it can be guaranteed by application of the implicit function theorem [37, Thm. 2], [38, Thm. 6] to the equation $f(x,u,w) = 0$.

Using Assumption 3, the optimization problem (2) can be reformulated as an unconstrained problem:

$$\underset{u \in \mathbb{R}^m}{\text{minimize}} \quad \phi_0(u, h(u, w(t)) = \phi(u, w(t)), \tag{6}$$

where $\phi(u,w) := \phi_0(u, h(u,w))$ for $u \in \mathbb{R}^m$ and $w \in W$. Next, we define a critical trajectory as an input signal that results in a vanishing gradient of the unconstrained problem.

Definition 1 (***Critical trajectory***)**.** The function $u^\star : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is said to be a *critical trajectory* of (6) if it satisfies:

$$0 = \nabla_u \phi(u^\star(t), w(t)), \quad \forall t \in \mathbb{R}_{\geq 0}. \qquad \square$$

Notice that, if $u^\star(t)$ is a critical trajectory of (6), then $(u^\star(t), h(u^\star(t), w(t)) \in \mathbb{R}^m \times X$ is a critical trajectory of (2),

$$0 = \nabla_u \phi_0(u, h(u,w))|_{(u,w)=(u^\star(t),w(t))}, \quad \forall t \in \mathbb{R}_{\geq 0}.$$

We therefore refer to the critical trajectories of (2) and those of (6) interchangeably. For the optimization problem to be well-posed, we make the following assumption.

Assumption 4 (***Existence and continuity of critical trajectories***)**.** The optimization (6) admits a critical trajectory. Moreover, every critical trajectory is continuous. $\square$

Existence of a critical trajectory for (2) can be guaranteed under mild assumptions; for example, coercivity of the cost function (i.e., $\phi_0(u,x) \to \infty$ when $\|u\| \to \infty$), or by requiring that the search domain can be restricted to a compact set without altering the optimizers (by Weierstrass' theorem [39]). Continuity of the critical trajectories can also be ensured under standard assumptions; for example, by requiring that the loss $\phi_0(u,x)$ is continuous in x (by Berge's theorem [39]).

B. Controller structure

To address Problem 1, we search within a class of candidate control algorithms that do not require measurements of $w(t)$. Instead, we consider algorithms that rely solely on the plant's measurable output $y(t)$, and are described by an internal state $z(t)$, evolving on an open subset $Z \subseteq \mathbb{R}^{n_c}$ for some $n_c \in \mathbb{N}_{\geq 0}$. Based on this information, the algorithm generates a control input $u(t) \in \mathbb{R}^m$ at each time instant. See Fig. 1 for a block diagram illustration. Formally, we consider the following class of control algorithms[2]:

$$\begin{aligned} \dot{z}(t) &= F_c(z(t), y(t)), \\ u(t) &= G_c(z(t)), \end{aligned} \tag{7}$$

where $F_c : Z \times \mathbb{R}^q \to \mathbb{R}^{n_c}$, $G_c : Z \to \mathbb{R}^m$, and the algorithm's state space dimension $n_c \in \mathbb{N}_{\geq 0}$ are to be designed. We note that (7) defines a general class of algorithms within which we will seek our design. We discuss in Remark 2 how this class encompasses several existing methods as special cases.

Problem 1 then involves designing the functions $F_c(z,y)$ and $G_c(z)$, together with n_c, such that the *gradient error* signal:

$$g(t) = \nabla_u \phi(u(t), w(t)), \tag{8}$$

satisfies $g(t) \to 0$ as $t \to \infty$. Note that the gradient error signal cannot be evaluated by the controller, as $w(t)$ is unmeasurable. Instead, the controller must regulate $g(t)$ to zero despite only having access to the plant's output $y(t)$.

The dynamics of the plant (1), combined with the exosystem (5) and controller (7), form a nonlinear autonomous system:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), w(t)), & y(t) &= c(x(t), w(t)), \\ \dot{z}(t) &= F_c(z(t), y(t)), & u(t) &= G_c(z(t)), \\ \dot{w}(t) &= s(w(t)), & g(t) &= \nabla_u \phi(u(t), w(t)). \end{aligned} \tag{9}$$

See Fig. 1 for an illustration of the interconnection.

Remark 2 (***Generality of the controller class*** (7))**.** To solve problem (6), the authors of [2], [7], under the additional assumption that $h(u,w) = \hat{h}(u) + w$, proposed the controller

$$\dot{u}(t) = -\eta[\nabla_1 \phi_0(u(t), y(t)) + J_{\hat{h}}^\mathsf{T}(u(t)) \nabla_2 \phi_0(u(t), y(t))], \tag{10}$$

where $\eta > 0$ is a tunable gain, $J_{\hat{h}}$ denotes the Jacobian matrix of $\hat{h}(u)$, and $\nabla_1 \phi_0$, and $\nabla_2 \phi_0$ are used to denote the gradient of ϕ_0 with respect to the first and second variable, respectively. When the disturbance $w(t)$ is constant and the plant (1) is exponentially pre-stabilized, a sufficiently-small choice of the tunable gain η ensures that condition (D2b) is met [1], [2]. It is immediate to verify that (10) is a special case of our controller class (7) with the choices:

$$\begin{aligned} F_c(z,y) &= -\eta[\nabla_1 \phi_0(z,y) + J_{\hat{h}}^\mathsf{T}(u) \nabla_2 \phi_0(u,y)], \\ G_c(z) &= z. \end{aligned} \tag{11}$$

It follows that the controller structure (7) is sufficiently general to include a number of existing methods as specific instances. Therefore, any conclusions drawn about the class of algorithms in (7) will also hold for these existing methods. $\square$

C. Technical problem statement

As stated previously, our goal is to design a control algorithm that (i) stabilizes the closed-loop system with zero disturbance locally about an equilibrium and (ii) regulates the state and control input to an optimal solution of (2) for all disturbances generated by the exosystem (5). Therefore, we assume existence of an equilibrium point $(x_\circ^\star, z_\circ^\star, u_\circ^\star, y_\circ^\star)$ of the closed-loop system (9) with zero disturbance for which the gradient is zero. That is[3],

$$\begin{aligned} 0 &= f(x_\circ^\star, u_\circ^\star, 0), & 0 &= F_c(z_\circ^\star, y_\circ^\star), & 0 &= \nabla_u \phi(u_\circ^\star, 0), \\ y_\circ^\star &= c(x_\circ^\star, 0), & u_\circ^\star &= G_c(z_\circ^\star). \end{aligned} \tag{12}$$

[2]Although one could consider a more general control algorithm of the form $\dot{z}(t) = F_c(z(t), y(t))$ and $u(t) = G_c(z(t), y(t))$, we will show in Section VI that allowing for G_c to depend on $y(t)$ is unnecessary. Hence, we focus the simpler formulation (7) for the sake of notation.

[3]The equilibrium point can be constructed by first finding $x_\circ^\star$ and $u_\circ^\star$ such that $0 = f(x_\circ^\star, u_\circ^\star, 0)$ and $0 = \nabla_u \phi(u_\circ^\star, 0)$, and then constructing $y_\circ^\star$ and $z_\circ^\star$ along with the controller F_c and G_c to satisfy the remaining conditions.

Moreover, we assume that $u_\circ^\star$ and $z_\circ^\star$ are locally unique. We can now formalize the notion of exact asymptotic tracking.

Definition 2 (***Exact asymptotic tracking***)**.** The controlled plant (9) is said to *exactly asymptotically track a critical trajectory* of (2) when the following conditions hold:

(D2a) The equilibrium $(x_\circ^*, z_\circ^\star)$ of the autonomous system:

$$\begin{aligned}\dot{x}(t) &= f(x(t), G_c(z(t)), 0),\\ \dot{z}(t) &= F_c(z(t), c(x(t), G_c(z(t)))),\end{aligned} \tag{13}$$

is locally exponentially stable.

(D2b) There exists a neighborhood $\Upsilon \subseteq X \times Z \times W$ of $(x_\circ^\star, z_\circ^\star, 0)$ such that, for each initial condition $(x(0), z(0), w(0)) \in \Upsilon$, the solution of (9) satisfies:

$$\lim_{t\to\infty} g(t) = 0. \qquad \square$$

With this background, we are now ready to make the objectives of Problem 1 mathematically rigorous.

Problem 2 (***Feedback optimization algorithm design – formal***)**.** Design, when possible, $F_c(z, y)$, $G_c(z)$, and n_c so that (9) exactly asymptotically tracks a critical trajectory of (2). $\square$

We conclude this section with an illustration of our framework.

Example 1 (***State regulation for linear systems***)**.** Consider an instance of (1) with linear dynamics:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Pw(t),\\ y(t) &= Cx(t) + Qw(t),\end{aligned} \tag{14}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, P \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{q\times n}, Q \in \mathbb{R}^{q\times p}$. Consider an optimal state-regulation problem (2), where the objective is to regulate the state of the plant to an optimal equilibrium that balances operational costs and control effort:

$$\begin{aligned}&\underset{u\in\mathbb{R}^m, x\in\mathbb{R}^n}{\text{minimize}} \quad \tfrac{1}{2}\|x\|^2 + \tfrac{\lambda}{2}\|u\|^2,\\ &\text{subject to:}\;\; 0 = Ax + Bu + Pw(t),\end{aligned} \tag{15}$$

where $\lambda \geq 0$ is a regularization parameter. If A is invertible, a mapping $h(u, w)$ satisfying Assumption 3 exists and is given by $h(u, w) = T_{xu}u + T_{xw}w$, where $T_{xu} = -A^{-1}B$ and $T_{xw} = -A^{-1}P$. The corresponding unconstrained problem (6) is:

$$\underset{u\in\mathbb{R}^m}{\text{minimize}} \quad \tfrac{1}{2}\|T_{xu}u + T_{xw}w(t)\|^2 + \tfrac{\lambda}{2}\|u\|^2.$$

Restricting, for simplicity of the illustration, the class (7) to linear controllers, the objective of this work is thus to design:

$$\begin{aligned}\dot{z}(t) &= A_c z(t) + B_c y(t),\\ u(t) &= C_c z(t),\end{aligned} \tag{16}$$

where $A_c \in \mathbb{R}^{n_c\times n_c}, B_c \in \mathbb{R}^{n_c\times q}, C_c \in \mathbb{R}^{m\times n_c}$, such that the interconnected system is exponentially stable and the gradient $g(t) = Ru(t) + Tw(t)$ converges to zero as $t \to \infty$, where $R := T_{xu}^\mathsf{T} T_{xu} + \lambda I$ and $T := T_{xu}^\mathsf{T} T_{xw}$. $\square$

IV. Necessary conditions for exact tracking and proposed controller structure

In this section, we derive a set of necessary conditions for the existence of an algorithm that solves Problem 2, and we outline the structure of the controller we propose to address this problem.

A. Necessary conditions for exact asymptotic tracking

To state our conditions, we will require the Jacobian matrices $A \in \mathbb{R}^{n\times n}, P \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{q\times n}, Q \in \mathbb{R}^{q\times p}, B \in \mathbb{R}^{n\times m}, S \in \mathbb{R}^{p\times p}$, and $T \in \mathbb{R}^{m\times p}$, defined as follows:

$$\begin{aligned}
A &:= \left[\frac{\partial f}{\partial x}\right]_{(x,u,w)=(x_\circ^\star,u_\circ^\star,0)}, & P &:= \left[\frac{\partial f}{\partial w}\right]_{(x,u,w)=(x_\circ^\star,u_\circ^\star,0)},\\
C &:= \left[\frac{\partial c}{\partial x}\right]_{(x,w)=(x_\circ^\star,0)}, & Q &:= \left[\frac{\partial c}{\partial w}\right]_{(x,w)=(x_\circ^\star,0)},\\
B &:= \left[\frac{\partial f}{\partial u}\right]_{(x,u,w)=(x_\circ^\star,u_\circ^\star,0)}, & S &:= \left[\frac{\partial s}{\partial w}\right]_{w=0},\\
T &:= \left[\frac{\partial \nabla_u \phi}{\partial w}\right]_{(u,w)=(u_\circ^\star,0)}.
\end{aligned} \tag{17}$$

Recall also that we denote by $\mathscr{X}_{\geq}(A)$ the unstable subspace of A and by $\mathcal{O}^c(C, A)$ the unobservable subspace of the pair (C, A) (see Section I-Notation).

Proposition 1 (***Necessary conditions for exact tracking***)**.** System (9) exactly asymptotically track a critical trajectory of (2) only if the following conditions hold:

1) The pair (A, B) is stabilizable.
2) The pair (C, A) is detectable.
3) The following inclusion holds:

$$\mathcal{O}^c(C_L, A_L) \cap \mathscr{X}_{\geq}(A_L) \subseteq \operatorname{Ker}\begin{bmatrix}0_{m\times n} & T\end{bmatrix}.$$

where $A_L := \begin{bmatrix}A & P\\ 0 & S\end{bmatrix}$ and $C_L := \begin{bmatrix}C & Q\end{bmatrix}$. $\square$

Proof. The Jacobian matrix of the dynamics (13) has the form:

$$\begin{bmatrix}A & BC_c\\ B_c C & A_c + B_c Q C_c\end{bmatrix}, \tag{18}$$

where A, B, and C are as in (17), and

$$\begin{aligned}
A_c &:= \left[\frac{\partial F_c}{\partial x}\right]_{(z,y)=(z_\circ^\star,y_\circ^\star)}, & B_c &:= \left[\frac{\partial F_c}{\partial y}\right]_{(z,y)=(z_\circ^\star,y_\circ^\star)},\\
C_c &:= \left[\frac{\partial G_c}{\partial z}\right]_{z=z_\circ^\star}.
\end{aligned} \tag{19}$$

Since $(x_\circ^*, z_\circ^\star)$ is locally exponentially stable (by (D2a)), all eigenvalues of the matrix in (18) must be in $\mathbb{C}_<$. A necessary condition for asymptotic tracking is then, for all $\lambda \in \mathbb{C}_{\geq}$,

$$\operatorname{Ker}\begin{bmatrix}A-\lambda I\\ B_c C\end{bmatrix} = 0 \quad \text{which implies} \quad \operatorname{Ker}\begin{bmatrix}A-\lambda I\\ C\end{bmatrix} = 0.$$

This proves part 2) of the claim. Similarly, Hurwitz stability of the matrix in (18) necessitates that, for all $\lambda \in \mathbb{C}_{\geq}$,

$$\begin{aligned}&\operatorname{Im}\begin{bmatrix}A-\lambda I, & BC_c\end{bmatrix} = \mathbb{R}^n\\ &\qquad\qquad \text{which implies} \quad \operatorname{Im}\begin{bmatrix}A-\lambda I & B\end{bmatrix} = \mathbb{R}^n,\end{aligned}$$

thus proving part 1) of the claim. To prove 3) assume, by contradiction, that there exists a vector $v \in \mathcal{O}^c(C_L, A_L) \cap \mathscr{X}_{\geq}(A_L)$ that does not belong to $\operatorname{Ker}\begin{bmatrix}0 & T\end{bmatrix}$. This implies that there exists an unstable mode of (13) in the direction v for which the gradient signal (8) is nonzero in a neighborhood of the origin of W. But this contradicts (D2b) in Definition 2. ∎

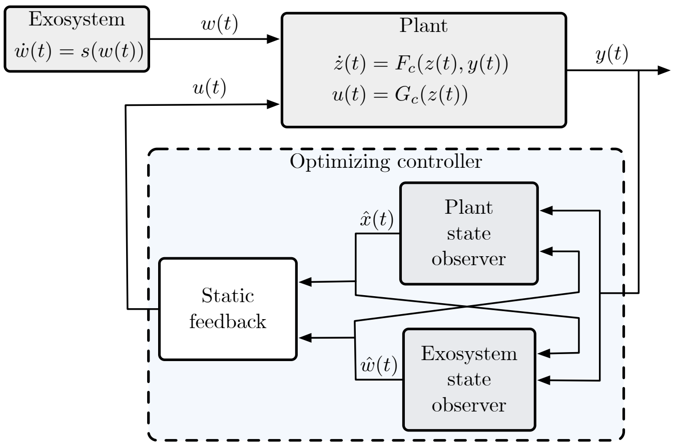

Fig. 3: Block-diagram representation of the optimizing controller developed in this work. The proposed controller is structured according to a separation principle, combining a state estimator – responsible for reconstructing the unmeasurable states $(\hat{x}(t), \hat{w}(t))$ – with a static feedback law that regulates the plant toward a point where the gradient error vanishes. See Section IV-B and the controller components (C1)–(C2).

Condition 1) in the proposition asks that the plant is stabilizable; 2) that it is detectable; and 3) that any undetectable mode of the pair (C_L, A_L) lies in the null space of the matrix $[0, T]$. Intuitively, the requirement 1) is needed to ensure that (1) can be stabilized (see Definition (D2a)) and is a classical requirement in stabilization problems via state feedback; the requirement 2) is needed to ensure that the state of (1) can be estimated from output measurements, and is a standard requirement in the existing literature on stabilization via output feedback; the requirement 3) is specific to our problem, and asks that all unstable modes of (5) are either detectable from $y(t)$ or do not affect the gradient signal $g(t)$. Notice that condition 3) implies condition 2); moreover, condition 3) automatically holds when the pair (C_L, A_L) is detectable. Motivated by these necessary conditions, in the remainder we will impose the following assumption.

Assumption 5 (***Stabilizability and Detectability***)**.** The pair (A, B) is stabilizable and the pair (C_L, A_L) is detectable. □

Assumption 5 guarantees that the conditions of Proposition 1 are met. Moreover, this assumption is in line with the feedback optimization literature; this assumption appears explicitly in [5], [17] and implicitly in [2], [6], [17], where the plant is assumed to be pre-stabilized (so that 1) and 2) are implicitly required), and $w(t)$ is assumed bounded so that, together with the stability of the plant, the requirement 3) is ensured to hold.

B. Structure of the proposed controller

Proposition 1 suggests the existence of a separation principle [40] between the problems of asymptotic stabilization (see Definition (D2a)) and gradient error regulation (see Definition (D2b)). Concretely, this is reflected in the need for two independent properties: stabilizability and detectability. Although a separation is, for now, only necessary, our control design technique relies on showing that such a separation is also sufficient to address Problem 2. Establishing this fact will be the objective of the subsequent sections. Motivated by this observation, we anticipate that the controller we propose in this work has a structure that is inspired from such a separation principle. Specifically, it consists of two main components (C):

(C1) A static feedback algorithm (based on the dynamic states $x(t), w(t)$, or their estimates $\hat{x}(t), \hat{w}(t)$), responsible for stabilizing the closed-loop system (i.e., ensure (D2a)) and drive the plant to operating conditions where the gradient error signal $g(t)$ vanishes (i.e., achieve (D2b)).

(C2) A dynamic observer, responsible for estimating the unmeasurable dynamic states $x(t)$ and $w(t)$ (i.e., generating the estimates $\hat{x}(t)$ and $\hat{w}(t)$).

A block diagram of the proposed controller is illustrated in Fig. 3. Developing the components (C1)–(C2) is the objective of the subsequent sections; precisely, (C1) is developed in Section V, the effectiveness of this controller structure is discussed in Section VI, and (C2) is developed in Section VII.

V. The static-feedback optimization problem

In this section, we develop the controller component (C1). Precisely, we begin by considering an idealization of the problem where the disturbance $w(t)$ and the plant's internal state $x(t)$ can be measured by the control algorithm. In these cases, because the control algorithm has access to all the dynamic states in (9), the controller model (7) can be replaced by a static (i.e., algebraic) map of the form[4]:

$$u(t) = H_c(x(t), w(t)), \tag{20}$$

where $H_c : X \times W \to \mathbb{R}^m$ is a C^0 mapping to be designed. In line with the requirements (12), we will assume that the mapping $H_c(x, w)$ to be designed is such that $H_c(x_\circ^\star, 0) = u_\circ^\star$. In other words, (20) assumes that the state of the plant $x(t)$ and the exogenous disturbance $w(t)$ are measurable and can be used directly for feedback. Because of the explicit dependence on $x(t)$ and $w(t)$, we will refer to (20) as a *static-feedback optimization* algorithm.

the framework developed here will be used in combination with the controller component (C1) in Section VI to tackle Problem 2 in its full generality.

Composing (1), (5), and (20) yields the closed-loop system:

$$\begin{aligned}
&\dot{x}(t) = f(x(t), u(t), w(t)), \quad y(t) = c(x(t), w(t)),\\
&\dot{w}(t) = s(w(t)),\\
&u(t) = H_c(x(t), w(t)), \qquad g(t) = \nabla_u \phi(u(t), w(t)).
\end{aligned} \tag{21}$$

With this framework, Problem 2 is reformulated as follows.

Problem 3 (***Static exact asymptotic tracking***)**.** Find, if possible, $H_c(x, w)$ such that:

(P3a) The equilibrium $x_\circ^*$ of the autonomous system:

$$\dot{x}(t) = f(x(t), H_c(x(t), 0), 0), \tag{22}$$

is locally exponentially stable.

[4]While one could consider a dynamic control algorithm of the form $\dot{z}(t) = F_c(z(t), w(t), x(t))$ and $u(t) = G_c(z(t))$, we will prove in Theorem 2 that such a dynamic structure is unnecessary.

(P3b) There exists a neighborhood $\Upsilon \subseteq X \times W$ of $(x_\circ^\star, 0)$ such that, for each initial condition $(x(0), w(0)) \in \Upsilon$, the solution of (21) satisfies:

$$\lim_{t\to\infty} g(t) = 0. \tag{23}$$

When these conditions hold, (21) is said to *exactly asymptotically track a critical trajectory* of (2). □

The following definition is instrumental to our characterization of this problem.

Definition 3 (***Limit point and limit set***)**.** A point $\bar{w} \in W$ is a *limit point with respect to the initialization* $w_\circ \in W$ if there exists a non-decreasing sequence $\{k_i\}_{i\in\mathbb{N}_{\geq 0}}$, with $k_i \to \infty$ as $i \to \infty$, such that the trajectory of (5) with $w(0) = w_\circ$ satisfies $w(k_i) \to \bar{w}$ as $i \to \infty$. For $w_\circ \in W$, let $\Omega(w_\circ)$ denote the set of all limit points (i.e., for all sequences $\{k_i\}_{i\in\mathbb{N}_{\geq 0}}$) of (5) with respect to the initialization $w_\circ$. Given $W_\circ \subseteq W$, the set $\Omega(W_\circ) := \cup_{w_\circ \in W_\circ} \Omega(w_\circ)$ is called the *limit set with respect to initializations in* $W_\circ$ [41]. □

Intuitively, $\Omega(W_\circ)$ denotes the set of all limit points (equilibria, limit cycles, etc.) that can be reached by the disturbance state when initialized at points in $W_\circ$. By Assumption 2, $\Omega(W_\circ)$ is contained in some neighborhood of the origin of $\mathbb{R}^p$, whose radius depends on the initialization set $W_\circ$.

The next result provides necessary and sufficient conditions for the existence of a static-feedback optimization algorithm.

Theorem 2 (***Solvability of the parameter-feedback tracking problem***)**.** Let Assumptions 1–5 hold. Problem 3 is solvable if and only if there exist C^2 mappings $\pi : W_\circ \to X$ and $\gamma : W_\circ \to \mathbb{R}^m$, where $W_\circ \subset W$ is some neighborhood of the origin of $\mathbb{R}^p$, such that:

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), \gamma(w), w), \tag{24a}$$

$$0 = \nabla_u \phi(\gamma(w), w), \tag{24b}$$

hold at all limit points $w \in \Omega(W_\circ)$. □

The proof of this result builds on Theorem 3 (presented shortly below), and hence is postponed to the appendix.

Theorem 2 asserts that the solvability of the static-feedback optimization problem depends upon the existence of two mappings $x = \pi(w)$ and $u = \gamma(w)$ that make the gradient of the cost identically zero (cf. (24b)) and that relate the plant dynamics $f(x, u, w)$ with the exosystem $s(w)$ in a neighborhood of each limit point of the disturbance (cf. (24a)). Interestingly, the solvability of (24) can be related to the existence of zero dynamics [42] for a composite system that incorporates the plant and the exosystem, as discussed in the following remark.

Remark 3 (***Interpretation of*** (24) ***in terms of zero dynamics***)**.** Conditions (24) state that an algorithm that solves Problem 3 exists if and only if there exists a submanifold of the state space (i.e., $M_s = \{(x, w) : x = \pi(w)\}$) such that:

(i) for some choice of feedback law $u(t) = \gamma(w(t))$, the trajectories of the closed-loop system (21) starting in this manifold remain in this manifold (cf. (24a))
(ii) the corresponding gradient is identically zero (cf. (24b))
(iii) the flow of the zero dynamics on this invariant manifold is a diffeomorphic copy of the disturbance flow (cf. (24a))

It follows from (i)–(ii) that M_s must be contained in the zero dynamics manifold of the composite system that incorporates the plant and the exosystem; in addition, (iii) requires that the flow of the zero dynamics of the composite system on M_s must be a diffeomorphic copy of the flow of the exosystem. □

While Theorem 2 provides a set of conditions for the solvability of the static exact asymptotic tracking problem, the question of how to design such an algorithm remains unanswered. As an intermediate step to address this question, we present the following result, which characterizes the class of all static-feedback algorithms that achieve asymptotic tracking.

Theorem 3 (***Characterization of static-feedback optimization algorithms***)**.** Let Assumptions 1–5 hold, and assume that $H_c(x, w)$ is such that condition (P3a) is met. Then, (P3b) holds if and only if there exists a C^2 mapping $\pi : W_\circ \to X$, with $W_\circ \subset W$ some neighborhood of the origin of $\mathbb{R}^p$, such that:

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), H_c(\pi(w), w), w), \tag{25a}$$

$$0 = \nabla_u \phi(H_c(\pi(w), w), w), \tag{25b}$$

hold at all limit points $w \in \Omega(W_\circ)$. □

Proof. (Only if) Suppose $g(t) \to 0$ as $t \to \infty$; we will show that (25) holds. The closed-loop system (9) has the form:

$$\begin{aligned} \dot{x} &= (A + BK)x + (P + BL)w + \phi(x, w), \\ \dot{w} &= Sw + \psi(w), \end{aligned} \tag{26}$$

where A, B, P, S are defined in (17), $K := \left[\frac{\partial H_c}{\partial x}\right]_{(x,w)=(x_\circ^\star,0)}$, $L := \left[\frac{\partial H_c}{\partial w}\right]_{(x,w)=(x_\circ^\star,0)}$, and $\phi(x, w)$ and $\psi(w)$ are functions that vanish at $(x, w) = (x_\circ^\star, 0)$, together with their first-order derivatives. By assumption, the eigenvalues of $(A+BK)$ are in $\mathbb{C}_<$, and those of S are on the imaginary axis. By Theorem 7, the system admits a center manifold at $(x_\circ^\star, 0)$, which can be represented as the graph of a continuous mapping $x = \pi(w)$, with $\pi(w)$ satisfying (25a). This establishes (25a).

To establish (25b), note that, under Assumption 2, there exists a neighborhood $W_\circ$ of the origin such that every trajectory of (5) initialized in $W_\circ$ remains bounded. Consequently, each such trajectory admits a subsequence that converges to a limit point $\bar{w} \in \Omega(W_\circ)$. Furthermore, we have:

$$\begin{aligned} \lim_{i\to\infty} g(k_i) &= \lim_{i\to\infty} \nabla_u \phi\left(H_c\left(\pi(w(k_i)), w(k_i)\right), w(k_i)\right) \\ &= \nabla_u \phi\left(H_c\left(\pi(\bar{w}), \bar{w}\right), \bar{w}\right). \end{aligned}$$

When $\lim_{i\to\infty} g(k_i) = 0$, the left-hand side is zero, which implies that (25b) holds at $\bar{w}$. Since this must hold at every limit point $\bar{w}$ and by continuity of $H_c(\cdot, \cdot)$, (25b) must hold everywhere in a neighborhood of each point of $\Omega(W_\circ)$.

(Only if) We now prove that (25) implies $\lim_{t\to\infty} g(t) = 0$. By Theorem 8, the center manifold $x = \pi(w)$ is locally attractive; namely, $x(t) \to \pi(w(t))$ as $t \to \infty$. Then, the fulfillment of (25b) guarantees that $g(t) \to 0$. ■

Theorem 3 provides a complete characterization of the class of static-feedback optimization algorithms that achieve exact asymptotic tracking. In words, a control algorithm (20) satisfying (P3a) asymptotically tracks a critical trajectory if and only if, on a neighborhood of the limit set of the exosystem $\Omega(W_\circ)$, the composite function $H_c(\cdot, w) \circ \pi(\cdot)$ zeros the gradient (cf. (25b)) and the plant $f(x, u, w)$ is algebraically related to the disturbance $s(w)$ as described by (25a). Finally, it is worth commenting on the assumption that $H_c(x, w)$ satisfies (P3a) in the statement of Theorem 3: conditions for the existence of such mappings are given in Theorem 2 and a technique to design such $H_c(x, w)$ is given shortly below (see (27)). In this context, the value of Theorem 3 lies in the conditions (25), which will be used below to design $H_c(x, w)$ in place of (P3b).

The proofs of Theorems 2–3 are constructive, as they provide a method to design control algorithms that solve the asymptotic tracking problem (Problem 3). Explicitly, given mappings $x = \pi(w)$ and $u = \gamma(w)$ satisfying (24), a parameter feedback algorithm that solves Problem 3 is given by:

$$H_c(x, w) = \gamma(w) + K(x - \pi(w)), \tag{27}$$

where K is any matrix such that the eigenvalues of $A + BK$ are in $\mathbb{C}_<$. It is also worth noting that the control action in (27) is the superposition of two terms: a state-error action $K(x - \pi(w))$, responsible for stabilizing the plant around the manifold $\pi(w)$, and a control action $\gamma(w)$, responsible for zeroing the gradient. We illustrate the design procedure next.

Example 2 (***Illustration of the design procedure for static-feedback optimization algorithms*****).** Consider the control problem discussed in Example 1. By application of Theorem 2, there exists a state-feedback optimization algorithm achieving exact asymptotic tracking if and only if there exist matrices $\Pi \in \mathbb{R}^{n\times p}, \Gamma \in \mathbb{R}^{m\times p}$ such that the following identities hold:

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma + P, \\ 0 &= R\Gamma + T. \end{aligned} \tag{28}$$

Note that, in this case, the dependence on w can be dropped, since these identities must hold anywhere in a neighborhood of the origin of $\mathbb{R}^p$. When these two conditions hold, an algorithm solving Problem 3 can be computed from (27), yielding

$$u(t) = \Gamma w(t) + K(x(t) - \Pi w(t)),$$

where K is any matrix such that $A + BK$ is Hurwitz. □

Remark 4 (***Knowledge of the limit set*****).** In applications, the limit set $\Omega(W_\circ)$ may be unknown. When this is the case, to design a static-feedback optimization algorithm, it is possible to seek mappings $\gamma(w)$ and $\pi(w)$ that verify (25) on some set that includes $\Omega(W_\circ)$, which can be more easily determined (e.g., when $w(t)$ is periodic or uniformly bounded). This approach overcomes the need for knowing the limit set precisely. □

VI. The dynamic feedback-optimization problem: existence and conditions for asymptotic tracking

In this section, we formalize the effectiveness of a controller architecture based on the two independent components (C1)–(C2). Precisely, we now show how the conclusions drawn in Section V extend when the dynamic controller (7) is used in place of the static one (20). We begin with the following solvability result.

Theorem 4 (***Solvability of the dynamic feedback-optimization problem*****).** Let Assumptions 1–5 hold. Problem 2 is solvable if and only if there exist C^2 mappings $\pi : W_\circ \to X$ and $\gamma : W_\circ \to \mathbb{R}^m$, where $W_\circ \subset W$ is some neighborhood of the origin of $\mathbb{R}^p$, such that:

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), \gamma(w), w), \tag{29a}$$

$$0 = \nabla_u \phi(\gamma(w), w), \tag{29b}$$

hold at all limit points $w \in \Omega(W_\circ)$. □

The proof of this result builds on Theorem 5 (presented shortly below); hence, we postpone it to the appendix.

Interestingly, the conditions for the solvability of the static problem (Problem 3) derived in Theorem 2 and those for the solvability of the dynamic problem (Problem 2) derived in Theorem 4 are identical. This shall not be surprising since, on the one hand, the static optimization algorithm (20) has access to measurements of the signals $x(t)$ and $w(t)$ while, on the other hand, the dynamic optimization algorithm (7) has additional flexibility thanks to the availability of a dynamic state $z(t)$. We will reinterpret this property under the lens of additional results shortly below (see (33)).

The following result provides a characterization of the class of all dynamic-feedback optimization algorithms that achieve exact asymptotic tracking.

Theorem 5 (***Characterization of dynamic feedback-optimization algorithms*****).** Let Assumptions 1–5 hold, and assume that the controller (7) is such that condition (D2a) is met. Then, (D2b) holds if and only if there exist C^2 mappings $\pi : W_\circ \to X$ and $\sigma : W_\circ \to Z$, with $W_\circ \subset W$ some neighborhood of the origin of $\mathbb{R}^p$, such that:

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), G_c(\sigma(w)), w), \tag{30a}$$

$$\frac{\partial \sigma}{\partial w} s(w) = F_c(\sigma(w), c(\pi(w), w)), \tag{30b}$$

$$0 = \nabla_u \phi(G_c(\sigma(w)), w), \tag{30c}$$

hold at all limit points $w \in \Omega(W_\circ)$. □

Proof. (Only if) Suppose that conditions (D2a)–(D2b) are satisfied. We will now show that (30) holds. The closed-loop system (21) has the form:

$$\begin{aligned} \dot{x} &= Ax + BC_c z + Pw + \phi(x, z, w), \\ \dot{z} &= B_c C x + A_c z + B_c Q w + \chi(x, z, w), \\ \dot{w} &= Sw + \psi(w), \end{aligned} \tag{31}$$

where all matrices involved are defined in (17), (19), and $\phi(x, z, w)$, $\chi(x, z, w)$, and $\psi(w)$ are functions that vanish at $(x, z, w) = (x_\circ^\star, z_\circ^\star, 0)$, together with their first-order derivatives. By (D2a), the eigenvalues of the matrix

$$\begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}$$

are in $\mathbb{C}_<$, and those of S are on the imaginary axis. By Theorem 7, the system admits a center manifold at $(x_\circ^\star, z_\circ^\star, 0)$, which can be expressed as the graph of two continuous mappings $x = \pi(w)$ and $z = \sigma(w)$ that satisfy (30a) and (30b).

To establish (30c), note that, under Assumption 2, there exists a neighborhood $W_\circ$ of the origin such that every trajectory of (5) starting in $W_\circ$ is bounded, and therefore it has a subsequence that converges to some limit point $\bar{w} \in \Omega(W_\circ)$. Thus, we have:

$$\begin{aligned}\lim_{i\to\infty} g(k_i) &= \lim_{i\to\infty} \nabla_u \phi\left(G_c\left(\sigma(w(k_i))\right), w(k_i)\right),\\ &= \nabla_u \phi\left(G_c\left(\pi(\bar{w})\right), \bar{w}\right).\end{aligned}$$

When $\lim_{i\to\infty} g(k_i) = 0$, the left-hand side is zero, which implies that (30c) holds at $\bar{w}$. Since this must hold at every limit point $\bar{w}$ and by continuity of $G_c(\cdot)$, (30c) must hold everywhere in a neighborhood of each point of $\Omega(W_\circ)$. This establishes (30c).

(If) We now show that, if condition (D2a) is satisfied and (30) holds, then (D2b) holds. By Theorem 8, the center manifold $x = \pi(w)$ and $z = \sigma(w)$ are locally attractive; namely, $x(t) \to \pi(w)$ and $z(t) \to \sigma(w)$ as $t \to \infty$. Then, fulfillment of (30b) guarantees that $g(t) \to 0$, thus establishing (D2b). ■

The conditions in (30) fully characterize the class of dynamic-feedback algorithms that achieve asymptotic tracking. In other words, an algorithm from the class (9) satisfying (D2a) achieves exact tracking if and only if, on a neighborhood of the limit set of the exosystem $\Omega(W_\circ)$, the composite function $G_c \circ \sigma$ zeros the gradient (cf. (30c)) and both the plant and the controller are algebraically related to the exosystem as given by, respectively, (30a) and (30b). It is also worth relating Theorem 5 with Theorem 3: by comparison, it is immediate to see that the design of a dynamic algorithm must follow the same criteria of the design of a static algorithm (compare (30a), (30c) with (25)), now with the additional requirement that the controller is algebraically related to the exosystem, as given by (30c). This property establishes the correctness of the controller architecture proposed in (C1)–(C2).

It is worth commenting on the requirement that (7) satisfies (D2a) in the statement: conditions for the existence of an algorithm satisfying (D2a) are given in Theorem 4 and a technique to design a stabilizing controller is given shortly below (see Algorithm 1). In analogy with Theorem 3, the value of Theorem 5 lies in the conditions (30), which will be used in Algorithm 1 to design $F_c(z, y)$ and $G_c(z)$ in place of (D2b). We illustrate these findings in the following example.

Example 3 (***Illustration of the conditions for dynamic feedback-optimization algorithms***)**.** Consider the problem discussed in Examples 1–2. By application of Theorem 4, Problem 2 is solvable if and only there exist matrices $\Pi \in \mathbb{R}^{n\times p}$ and $\Gamma \in \mathbb{R}^{m\times p}$ such that (28) hold. Notice that, as observed immediately after Theorem 4, the conditions for the solvability of the dynamic problem (Problem 2) coincide with the conditions for the solvability of the static problem (Problem 3).

Suppose now that (28) holds, consider the controller (16), and assume that A_c and B_c are such that the origin of:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BC_c z(t),\\ \dot{z}(t) &= A_c z(t) + B_c C x(t),\end{aligned}$$

is exponentially stable. By application of Theorem 5, exact asymptotic tracking is achieved if and only if there exist matrices $\Pi \in \mathbb{R}^{n\times p}$ and $\Sigma \in \mathbb{R}^{n_c\times p}$ such that:

$$\begin{aligned}\Pi S &= A\Pi + BC_c\Sigma + P,\\ \Sigma S &= A_c\Sigma + B_c(C\Pi + Q),\\ 0 &= RC_c + T\Sigma.\end{aligned} \tag{32}$$

Given A_c, B_c, C_c, (32) is a set of linear equations in the unknowns Π and Σ, which can thus be checked efficiently. □

We conclude this section by discussing an important implication stemming from Theorem 5. By (30b), the state of the control algorithm z and that of the exosystem w must be related, everywhere in the limit set $\Omega(W_\circ)$, by the relationship:

$$z = \sigma(w). \tag{33}$$

Condition (33) can be interpreted as the existence of a coordinate transformation between the state of the exosystem and that of the control algorithm. This, in turn, implies that any controller capable of achieving exact tracking must use an internal state $z(t)$ that is a reduplicated copy of the disturbance signal $w(t)$, albeit potentially expressed in a different coordinate system. We discuss further this property in Remarks 5–6, and illustrate some of its properties in Example 4.

Remark 5 (***The internal model principle of feedback optimization***)**.** We interpret condition (30) as the *internal model principle of feedback optimization,* akin to its counterpart in control systems [11]–[14] and time-varying optimization [32]. In fact, (30) expresses the requirement that any feedback optimization algorithm that achieves asymptotic tracking must include an internal model of the exosystem. The use of a copy of the temporal variability of the optimization problem is explicit in existing feedback optimization algorithms [2], [6], where an internal model for an integrator-type exosystem is used to reject constant disturbances. In this sense, our characterization encompasses all these existing approaches as special cases. □

Remark 6 (***Internal-model based interpretation of existing feedback optimization algorithms***)**.** Recall that the basic feedback-optimization algorithm (10) can be viewed as an instance of (7) with $F_c(z, y)$ and $G_c(z)$ as in (11) (see Remark 2). By direct substitution into (30), it is immediate to see that this algorithm satisfies (30) with $s(w) = 0$ and $\sigma(w)$ arbitrary. Since $s(w) = 0$ is the internal model of a constant signal, it follows from Theorem 5 that these algorithms can achieve exact asymptotic tracking only when $w(t)$ is a constant signal. This observation is in line with the inexact convergence results given for these algorithms when $w(t)$ is time-varying [6], [43], and motivates the need for developing new algorithms when $w(t)$ is time-varying. □

Example 4 (***The mapping $\sigma(w)$ may not be injective***)**.** Consider the scalar plant: $\dot{x} = x + u$ with output $y = -2x + w$ subject to a constant disturbance $\dot{w} = 0$. Denoting $w := w(t)$ for compactness, consider the following instance of (6),

$$\underset{u\in\mathbb{R}}{\text{minimize}} \quad \tfrac{1}{2}|u|^2 + \tfrac{1}{2}|w|^2,$$

which results in the gradient signal: $g(t) = u(t) + w$. In this case, a possible choice of algorithm that achieves asymptotic tracking is $u(t) = y(t)$. Indeed, with this choice, the controlled system dynamics are $\dot{x}(t) = -x(t) + w$, yielding: $x(t) \to w$, $y(t) \to -w$, $u(t) \to -w$, and $g(t) \to 0$, which ensures asymptotic tracking. This is a case where $\sigma(w)$ is not injective; in fact, this corresponds to a situation where $\sigma : W_\circ \to \emptyset$, since this controller has no internal state. Notice that this is not the only controller choice possible, as other solutions (for example, using a dynamic state) are possible. □

VII. Feedback optimization algorithm design

Building on the results derived in the previous section, we now turn our attention to designing algorithms that solve Problem 2. Our controller construction follows the separation principle defined by components (C1)–(C2). Notably, component (C2) is further structured into two parts: (i) an error-feedback mechanism that stabilizes the plant around the center manifold, and (ii) a control action that drives the gradient to zero asymptotically (see (27)).

Formally, our construction is as follows: given functions $\gamma(w)$ and $\pi(w)$ that satisfy (29), we construct a controller with state $z = (z_1, z_2)$, $z_1 \in \mathbb{R}^n, z_2 \in \mathbb{R}^p$, (i.e., the controller state space dimension is $n_c = n + p$) and dynamics:

$$\begin{aligned} \dot{z}_1(t) &= f(z_1(t), u(t), z_2(t)) - L_1 e_y(t), \\ \dot{z}_2(t) &= s(z_2(t)) - L_2 e_y(t), \end{aligned} \tag{34}$$

where

$$\begin{aligned} u(t) &= \gamma(z_2(t)) + K(z_1(t) - \pi(z_2(t))), \\ e_y(t) &= c(z_1(t), z_2(t)) - y(t). \end{aligned}$$

Intuitively, the state variable $z_1(t)$ in (34) operates as a dynamic observer for the plant's state $x(t)$, driven by the output-based estimation error $e_y(t)$ (which describes the error between the observer's output and true output $y(t)$) with gain L_1. The state $z_2(t)$ acts as a dynamic observer for the exosystem state $w(t)$, driven by the output-based estimation error $e_y(t)$ with gain L_2. The control input $u(t)$ is designed to be a static-feedback optimization algorithm (cf. (27)), driven by the estimated states $z_1(t)$ and $z_2(t)$ (in place of the true states $x(t)$ and $w(t)$ as in (27)).

In (34), the matrices $K \in \mathbb{R}^{m\times n}, L_1 \in \mathbb{R}^{n\times q}$, and $L_2 \in \mathbb{R}^{p\times q}$ are designed such that the closed-loop matrices:

$$A + BK \qquad \text{and} \qquad A_L - LC_L, \tag{35}$$

where $L = [L_1^\mathsf{T}, L_2^\mathsf{T}]^\mathsf{T}$, have eigenvalues[5] in $\mathbb{C}_<$. We note that the Hurwitz stability of $A_L - LC_L$ guarantees asymptotic stability of the dynamic observer, while the Hurwitz stability of $A + BK$ ensures that the state-feedback controller $u(t)$ renders the closed-loop system asymptotically stable.

We summarize our algorithm design procedure in Algorithm 1, and we prove the effectiveness of this method in Theorem 6.

Theorem 6 (***Correctness of Algorithm 1***)**.** Let Assumptions 1–5 hold, and let the control algorithm (7) be designed following Algorithm 1. Then, the controlled system (9) exactly asymptotically tracks a critically trajectory of (2). □

[5]Notice that existence of K and L is guaranteed by Assumption 5.

Algorithm 1: Dynamic-feedback optimization algorithm design

Data: Mappings $f(x, u, w)$, $c(x, w)$, $s(w)$; matrices A, B as in (17), A_L, C_L as in Proposition 1; mappings $\pi(w), \gamma(w)$ as in Theorem 4

Let K be any matrix such that $A + BK$ is Hurwitz;
Let L be any matrix such that $A_L - LC_L$ is Hurwitz;
Decompose $z = (z_1, z_2), z_1 \in \mathbb{R}^n, z_2 \in \mathbb{R}^p$;
Decompose $L = [L_1^\mathsf{T}, L_2^\mathsf{T}]^\mathsf{T}, L_1 \in \mathbb{R}^{n\times q}, L_2 \in \mathbb{R}^{p\times q}$;
$G_c(z) \leftarrow \gamma(z_2) + K(z_1 - \pi(z_2))$;
$F_c(z, y) \leftarrow \begin{bmatrix} f(z_1, G_c(z), z_2) - L_1(c(z_1, z_2) - y) \\ s(z_2) - L_2(c(z_1, z_2) - y) \end{bmatrix}$;

Result: $F_c(z, y)$, $G_c(z)$, that solve Problem 2

Proof. Because K and L are designed so that $A + BK$ and $A_L - LC_L$ have eigenvalues in $\mathbb{C}_<$, the matrices:

$$A + BK \qquad \text{and} \qquad \begin{bmatrix} A - L_1 C & P - L_1 Q \\ -L_2 C & S - L_2 Q \end{bmatrix}$$

have eigenvalues in $\mathbb{C}_<$. A standard calculation (see the proof of Theorem 4) shows that, for any Γ and Π, the matrix

$$\begin{bmatrix} A & BK & B(\Gamma - K\Pi) \\ L_1 C & A + BK - L_1 C & P + B(\Gamma - K\Pi) - L_1 Q \\ L_2 C & -L_2 C & S - L_2 Q \end{bmatrix}$$

also has eigenvalues in $\mathbb{C}_<$. It immediate to see that this matrix is the Jacobian of the dynamics

$$\begin{aligned} \dot{x}(t) &= f(x(t), G_c(z(t)), 0), \\ \dot{z}(t) &= F_c(z(t), c(x(t), G_c(z(t)))), \end{aligned}$$

when $F_c(z, y)$ and $G_c(z, y)$ are constructed according to Algorithm 1, when letting

$$\Gamma := \left[\frac{\partial \gamma}{\partial w}\right]_{w=0} \qquad \text{and} \qquad \Pi := \left[\frac{\partial \pi}{\partial w}\right]_{w=0}. \tag{36}$$

This proves that (D2a) holds. To show that (D2b) also holds, observe that the construction of $G_c(z)$ and $F_c(z, y)$ in Algorithm 1 implies the the fulfillment of (30) with

$$\sigma(w) = (\pi(w), w).$$

Hence, (D2b) follows by application of Theorem 5. ■

We conclude this section by illustrating the design procedure on a quadratic optimization problem with a linear plant.

Example 5 (***Illustration of the design procedure for dynamic-feedback optimization algorithms***)**.** Consider the problem discussed in Examples 1–3. Let $\Pi \in \mathbb{R}^{n\times p}, \Gamma \in \mathbb{R}^{m\times p}$ be matrices that satisfy the matrix identities (28). For this problem, the design procedure of Algorithm 1 reads as follows:

1) Let K be any matrix such that $A + BK$ is Hurwitz;
2) Let L be any matrix such that $A_L - LC_L$ is Hurwitz;
3) Decompose $z = (z_1, z_2), z_1 \in \mathbb{R}^n, z_2 \in \mathbb{R}^p$;
4) Decompose $L = [L_1^\mathsf{T}, L_2^\mathsf{T}]^\mathsf{T}, L_1 \in \mathbb{R}^{n\times q}, L_2 \in \mathbb{R}^{p\times q}$;

5) Design the map $G_c(z)$ as follows:

$$\begin{aligned} G_c(z) &= \Gamma z_2 + K(z_1 - \Pi z_2) \\ &= C_c z, \end{aligned}$$

with $C_c = \begin{bmatrix} K & \Gamma - K\Pi \end{bmatrix}$;

6) Design the map $F_c(z, y)$ as follows:

$$\begin{aligned} F_c(z,y) &= \begin{bmatrix} Az_1 + BC_c z + Pz_2 - L_1(Cz_1 + Qz_2 - y) \\ Sz_2 - L_2(Cz_1 + Qz_2 - y) \end{bmatrix} \\ &= A_c z + B_c y, \end{aligned}$$

with

$$\begin{aligned} A_c &= \begin{bmatrix} A + BK - L_1 C & P + B(\Gamma - K\Pi) - L_1 Q \\ -L_2 C & S - L_2 Q \end{bmatrix}, \\ B_c &= \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}. \end{aligned}$$

□

VIII. Extensions to constrained problems

The optimization problem (2) contains only an equilibrium constraint, which we assume can be eliminated using the steady-state map $x = h(u, w)$ (cf. Assumption 3) to obtain the unconstrained problem (6). We now discuss extensions to optimization problems with more general constraints. To that end, consider the equality-constrained problem

$$\begin{aligned} &\text{minimize} \quad \phi(u, w(t)) \\ &\text{subject to} \quad \psi_i(u, w(t)) = 0, \quad i = 1, \dots, r, \end{aligned}$$

where the constraint functions $\psi_i(u, w(t))$ may depend on the control input and disturbance. The associated Lagrangian is

$$L(u, \lambda, w(t)) = \phi(u, w(t)) + \sum_{i=1}^{r} \lambda_i \, \psi_i(u, w(t)),$$

where λ_i is the Lagrange multiplier associated with the i^{th} equality constraint. A pair $(u, \lambda) \in \mathbb{R}^m \times \mathbb{R}^r$ is said to be a saddle-point of the Lagrangian if, for all $(\bar{u}, \bar{\lambda}) \in \mathbb{R}^m \times \mathbb{R}^r$,

$$L(u, \bar{\lambda}, w(t)) \leq L(u, \lambda, w(t)) \leq L(\bar{u}, \lambda, w(t)).$$

For any such saddle-point, if strong duality holds, u is primal optimal, λ is dual optimal, and the optimal duality gap is zero. Moreover, the gradient of the Lagrangian (assuming it exists) is zero at any saddle-point. It follows from the derivations in the previous sections that the gradient-feedback and parameter-feedback algorithms can be directly applied to seek a stationary point of the Lagrangian function by replacing the variable u with the extended decision variable $\tilde{u} = (u, \lambda)$ and by considering an augmented loss function in (6) defined as $\phi(\tilde{u}, w) = L(u, \lambda, w)$. Notice that, if the critical point computed by the controller is also a saddle-point and strong duality holds, then it is also a solution to the equality-constrained problem. When strong duality does not hold, however, such a saddle-point may not correspond to an optimizer [44, Ch. 5].

IX. Application robotic balancing control

In this section, we demonstrate the effectiveness of the methods in controlling an unstable system. We present two sets of simulations: the first considers a quadratic cost function, while the second explores a more general cost formulation.

A. Quadratic cost

We begin by illustrating our approach on the balancing robot presented in Section II-B. Letting $x_1 = \theta$ and $x_2 = \dot{\theta}$, a state-space model (1) for this robot reads as:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \alpha \sin x_1(t) - \beta x_2(t) - \gamma u(t) \cos x_1(t) + \eta w_x(t), \\ y(t) &= x_2(t) + w_y(t), \end{aligned} \tag{37}$$

where $\alpha = \frac{mg\ell}{J_e}, \beta = \frac{k\ell^2}{J_e}, \gamma = \frac{m\ell}{J_e}$, and $\eta = \frac{1}{Je}$. For our simulations, we used sinusoidal signals: $w_x(t) = \rho_x \cos(\bar{\omega}_x t)$, $\rho_x = 1$ [rad/s^2], $\bar{\omega}_x = 1$ [rad/s] and $w_y(t) = \rho_y \cos(\bar{\omega}_y t)$, $\rho_y = 0.5$ [rad/s^2], $\bar{\omega}_y = 10$ [rad/s]. These signals are generated by an exosystem (5) with state $w = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$, vector field $s(w) = Sw$ with

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\bar{\omega}_x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\bar{\omega}_y^2 & 0 \end{bmatrix}, \tag{38}$$

initial condition $w(0) = (\rho_x, 0, \rho_y, 0)$. This exosystem model generates the desired signals $w_x(t)$ and $w_y(t)$ when letting[6] $w_x(t) = w_1(t)$ and $w_y(t) = w_3(t)$. It can be verified that a mapping $h(u, w)$ satisfying Assumption 3 is given by:

$$h(u, w) = \begin{bmatrix} -2 \tan^{-1} \left(\frac{\alpha - \sqrt{\alpha^2 - \eta^2 w_1^2 + \gamma^2 u^2}}{\eta w_1 + \gamma u} \right) \\ 0 \end{bmatrix},$$

when $\eta w_1 + \gamma u \neq 0$ and $h(u, w) = [\pi, 0]^\mathsf{T}$ otherwise. At first, we consider the following instance of (2):

$$\begin{aligned} &\underset{u \in \mathbb{R},\, x \in \mathbb{R}^2}{\text{minimize}} \quad \tfrac{1}{2}\|x\|^2, \\ &\text{subject to:} \quad 0 = x_2, \\ &\qquad 0 = \alpha \sin x_1(t) - \beta x_2(t) \\ &\qquad\qquad - \gamma u(t) \cos x_1(t) + \eta w_x(t), \end{aligned} \tag{39}$$

which formalizes the objective of balancing the robot at the vertical position ($x = 0$) while rejecting disturbances. To determine mappings $\pi(w)$ and $\gamma(w)$ that solve (29), we sought an approximate solution using polynomial representations:

$$\pi(w) = \sum_{\ell=1}^{d_\pi} \langle \psi_\ell^\pi, \Theta_\ell(w) \rangle, \quad \gamma(w) = \sum_{\ell=1}^{d_\gamma} \langle \psi_\ell^\gamma, \Theta_\ell(w) \rangle, \tag{40}$$

of order d_π and d_γ, respectively. Here, ψ_ℓ^π and ψ_ℓ^γ, are $\binom{\ell p}{\ell}$-dimensional vectors of coefficients and $\Theta_\ell(w)$, $\ell \in \mathbb{N}_{>0}$, is an ℓ-th order polynomial basis of functions; formally:

$$\begin{aligned} \Theta_\ell(w) = [&w_1^\ell, w_1^{\ell-1} w_2, \dots, w_1^{\ell-1} w_p, \\ &w_1^{\ell-2} w_2^2, w_1^{\ell-2} w_2 w_3, \cdots, w_1^{\ell-2} w_2 w_p, \cdots, w_p^\ell]^\mathsf{T}. \end{aligned}$$

The parameter vectors ψ_ℓ^π and ψ_ℓ^γ have been fitted numerically so that (29) holds in a neighborhood of the origin of $\mathbb{R}^4$. In our simulations, we used $d_\pi = d_\gamma = 4$ and (29) were

[6] Recall that the solutions to the dynamical system $\dot{w} = Sw$ with $w = (w_1, w_2) \in \mathbb{R}^2$ and $S = \left[\begin{smallmatrix} 0 & 1 \\ -\omega^2 & 0 \end{smallmatrix}\right]$ are: $w_1(t) = \cos(\omega t) w_1(0) + \sin(\omega t) \frac{w_2(0)}{\omega}$ and $w_2(t) = -\omega \sin(\omega t) w_1(0) + \cos(\omega t) w_2(0)$.

satisfied up to a relative error of $10^{-7.}$ It is worth stressing that, to solve the internal-model conditions (29), knowledge of the magnitude of the disturbance signals ρ_x and ρ_y is not needed (since these quantities do not appear in the exosystem model (38)); it is only required to know their frequencies ω_x and ω_y (which are the only parameters in S). We applied Algorithm 1, selecting the controller gain K and the observer gain L so that the eigenvalues of $A_L - LC_L$ are uniformly distributed in the intervals $[-1, -2]$ and $[-2, -3]$, respectively. Simulation results for this problem are illustrated in Fig. 4. The simulation illustrates that the controller is effective in regulating the gradient signal $g(t)$ (cf. (8)) to zero, up to a numerical error of order 10^{-6} (see Fig. 4(c)); the optimal robot configuration corresponds to a situation where the pendulum is precisely in the vertical position described by $(x_1, x_2) = (0, 0)$ (see Fig. 4(c)). Notice also that the control input that achieves balancing oscillates periodically (see Fig. 4(b)) to cancel out the effect of the oscillatory disturbances (see Fig. 4(a)).

B. Logistic regression

To illustrate the controller performance on a non-quadratic problem, we next replace the loss function in (39) by:

$$\phi_0(u, x) = \tfrac{1}{2}\|x\|^2 + \tfrac{\kappa}{2}(\log[1 + e^{\mu u}] + \log[1 + e^{-\mu u}]), \quad (41)$$

where $\kappa, \mu > 0$ (for our experiments, we choose $\mu = 0.5$ and $\kappa = 1$). In other words, (41) defines a logistic regression problem with a time-varying regularization term; intuitively, an optimizer of (41) is an equilibrium state for the pendulum model such that the robot is as close as possible to being vertically balanced ($x = 0$), while large values (in modulus) of the control input u are penalized by the logistic term $\log[1+e^{\mu u}]$. The mappings $\pi(w)$ and $\gamma(w)$ have been obtained by fitting (40) numerically; in this case, in our simulations, (29) were satisfied up to a relative error of $10^{-4.}$ Simulation results for this problem are illustrated in Fig. 5. By comparing Fig. 2(b) and Fig. 5(a), we observe that in the latter case the controller favors control inputs that are smaller in magnitude, in line with the above interpretation for (41); as a result of avoiding control inputs of large magnitude, the state is not regulated to zero exactly, but the robot swings around the vertical position (see Fig. 5(c)); notice that this configuration does correspond to optimality for this optimization problem (see Fig. 5(b)), in the sense that the gradient of (41) approaches zero, up to a numerical error. We reconduct the discrepancy between Fig. 2(c) (where $\nabla_u \phi(u(t), w(t)) \approx 0$ up to a numerical error of order 10^{-6}) and Fig. 5(b) (where $\nabla_u \phi(u(t), w(t)) \approx 0$ up to a numerical error of order 10^{-4}) to numerical error in the satisfaction of (29) (relative error of 10^{-7} in the former case and 10^{-4} in the latter). We conjecture that the numerical error in achieving $\nabla_u \phi(u(t), w(t)) \approx 0$ can be further reduced by using higher-order polynomials ($d_\pi, d_\gamma > 4$) to fit (29).

X. Conclusions

We have shown that the equilibrium-selection problem studied in feedback optimization can be recast as an output regulation problem. This allows us to develop methods for exact setpoint tracking even when the optimization problem is time-varying. Fundamentally, we show that this requires knowledge of an internal model of the temporal variability as well as a particular control design architecture — a fundamental limitation that is proven here for the first time in the literature. Our algorithm design is novel in the literature, and it combines an output-feedback control action that stabilizes the plant with an additional control action that drives the system toward the set of critical points of the optimization. This work opens the opportunity for several directions of future work, including an investigation of methods that use inexact internal model knowledge or learn the internal model online, the relaxation of the convexity and smoothness assumptions, and an investigation of discrete-time algorithms.

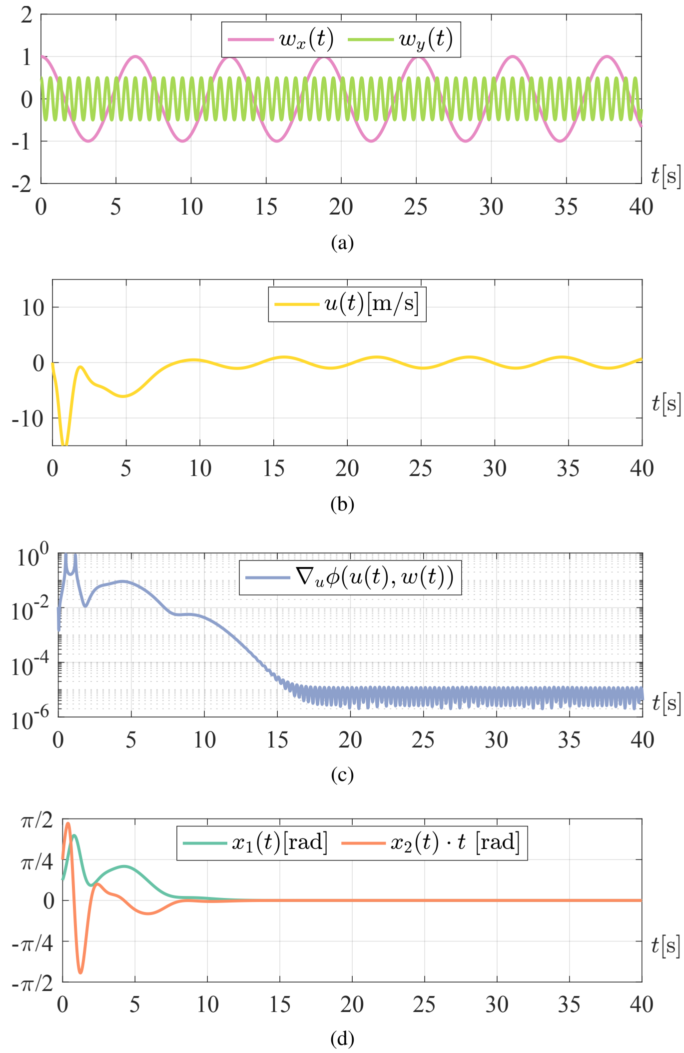

Fig. 4: Performance of the proposed control design applied to solve the equilibrium-selection problem (39) to the balancing robot of Fig 2(a) (see model equations (37)). Despite the presence of unmeasurable time-varying disturbances acting on both the state and output (Fig. 4(a)), the controller successfully regulates the gradient error signal to zero – up to a numerical tolerance of order 10^{-6} (Fig. 4(c)). This corresponds to stabilizing the inverted pendulum in the upright position, characterized by zero vertical angle and zero velocity (Fig. 4(a)). It is worth noting that the control input required to achieve this regulation is not constant (Fig. 4(b)). See Fig. 2 for the parameter values used.

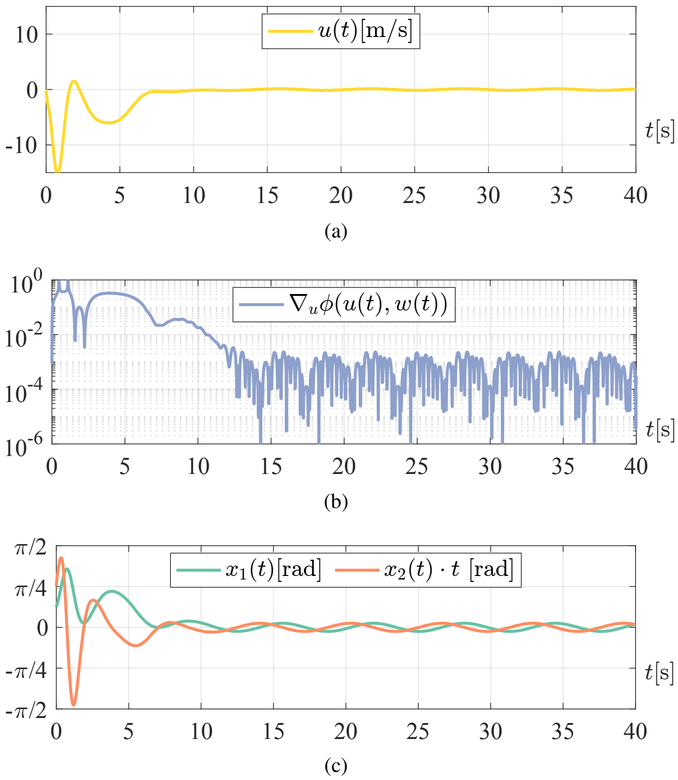

Fig. 5: Performance of the proposed control design applied to solve an optimization problem with logistic cost (41). For simplicity of the illustration, we used constant disturbance signals $\omega_x = 1$ [rad/s], $\omega_y = 10$ [rad/s]. Despite the presence of unknown disturbances acting on both the state and output, the controller regulates the gradient error signal to zero – up to a numerical tolerance of order 10^{-4} (Fig. 5(b)), which corresponds to striking a balance between stabilizing the inverted pendulum in the upright position, characterized by zero vertical angle and zero velocity (Fig. 5(c)) and minimizing the control effort (Fig. 5(a)). See Fig. 2 for the parameter values.

Appendix A
Complementary proofs

A. Proof of Theorem 2

(Only if). Suppose Problem 3 is solvable, meaning that there exists $H_c(x, w)$ that satisfies (P3a) and (P3b). By Theorem 3, there exists $\pi(w)$ such that the conditions in (25) are satisfied. Consequently, (24) is also satisfied by $\gamma(w) = H_c(\pi(w), w)$.

(If). We aim to show that, under condition (24), there exists a function $H_c(x, w)$ such that both (P3a) and (P3b) are satisfied. First, notice that, by Assumption 5, there exists a matrix K such that $(A + BK)$ has eigenvalues in $\mathbb{C}_<$. Moreover, let $\gamma(w)$ and $\pi(w)$ be two functions such that (24) hold, and set:

$$H_c(x, w) = \gamma(w) + K(x - \pi(w)).$$

This choice ensures that condition (P3a) is satisfied: indeed, the Jacobian of $f(x(t), H_c(x(t), 0), 0)$ is $A + BK$ (see (26) for notation), which has eigenvalues in $\mathbb{C}_<$. Moreover, by construction, $H_c(\pi(w), w) = \gamma(w)$, so condition (24a) reduces to (25a), and similarly, (24b) reduces to (25b). Thus, by Theorem 3, condition (P3b) is also satisfied.

B. Proof of Theorem 4

(Only if). Suppose Problem 2 is solvable. Then, by Theorem 5, there exists functions $\pi(w)$, $\sigma(w)$, and $G_c(z)$ such that the conditions in (30) are satisfied. Consequently, the conditions in (29) are also satisfied by letting $\gamma(w) = G_c(\sigma(w))$.

(If). We aim to show that, under condition (29), there exists functions $F_c(z, y)$ and $G_c(z)$ such that both conditions (D2a)-(D2b) are satisfied. First of all, notice that, by Assumption 5, there exist matrices K and L such that (letting $L = [L_1^\mathsf{T}, L_2^\mathsf{T}]^\mathsf{T}, L_1 \in \mathbb{R}^{n\times q}, L_2 \in \mathbb{R}^{p\times q}$):

$$A + BK \quad \text{and} \quad \begin{bmatrix} A - L_1 C & P - L_1 Q \\ -L_2 C & S - L_2 Q \end{bmatrix} \tag{42}$$

have eigenvalues in $\mathbb{C}_<$. This implies that, for any pair of matrices Γ and Π, the matrix

$$\begin{bmatrix} A & BK & B(\Gamma - K\Pi) \\ L_1 C & A + BK - L_1 C & P + B(\Gamma - K\Pi) - L_1 Q \\ L_2 C & -L_2 C & S - L_2 Q \end{bmatrix} \tag{43}$$

has eigenvalues in $\mathbb{C}_<$. To see this, observe that applying the similarity transformation $M \mapsto T^{-1} M T$ with

$$T = \begin{bmatrix} I_n & 0 & 0 \\ I_n & I_n & 0 \\ 0 & 0 & I_p \end{bmatrix}$$

to the matrix in (43) gives the block upper-triangular matrix:

$$\begin{bmatrix} A + BK & BK & B(\Gamma - K\Pi) \\ 0 & A - L_1 C & P - L_1 Q \\ 0 & -L_2 C & S - L_2 Q \end{bmatrix},$$

whose eigenvalues are determined by (42).

Next, let $\pi(w)$ and $\gamma(w)$ be functions that satisfy the conditions (29). With the controller defined by $F_c(z, y)$ and $G_c(z)$ from Algorithm 1, it is immediate to verify that (43), with Γ and Π the Jacobian matrices defined in (36), is the Jacobian matrix of the closed-loop dynamics with zero disturbance (13). This establishes (D2a). To establish (D2b), observe that the construction of $G_c(z)$ and $F_c(z, y)$ in Algorithm 1 ensures the fulfillment of (30) with $\sigma(w) = (\pi(w), w)$. Therefore, condition (D2b) follows directly from Theorem 5.

Appendix B
Basic concepts from center manifold theory

We now summarize relevant facts on center manifold theory from [45]; see also [46]. Consider the nonlinear system:

$$\dot{x} = f(x) \tag{44}$$

where f is a C^k vector field defined on an open subset U of $\mathbb{R}^n$. Consider an equilibrium point for f, which we take without loss of generality to be zero, i.e., $f(0) = 0$. Let $F = \left[\frac{\partial f}{\partial x}\right]_{x=0}$, denote the Jacobian matrix of f at $x = 0$. Suppose the matrix F has n° eigenvalues with zero real part, n^- eigenvalues with negative real part, and n^+ eigenvalues with positive real part. Let E^-, E°, and E^+ be the (generalized) real eigenspaces of F associated with eigenvalues of F lying on the open left half plane, the imaginary axis, and the open right half plane, respectively. Note that E°, E^-, E^+ have dimension n°, n^-, n^+, respectively and that each of these spaces is invariant under the flow of $\dot{x} = Fx$. If the linear mapping F is viewed as a representation of the differential (at $x = 0$) of the nonlinear mapping f, its domain is the tangent space T_0U to U at $x = 0$, and the three subspaces in question can be viewed as subspaces of T_0U satisfying $T_0U = E^\circ \oplus E^- \oplus E^+$. We refer to [47, Sec. A.II] for a precise definition of C^k manifolds; loosely speaking, a set $S \subset U$ is a C^k manifold it can be locally represented as the graph of a C^k function.

Definition 4 (***Locally invariant manifold***)**.** A C^k manifold S of U is locally invariant for (44) if, for each $x_\circ \in S$, there exists $t_1 < 0 < t_2$ such that the integral curve $x(t)$ of (44) satisfying $x(0) = x_\circ$ satisfies $x(t) \in S$ for all $t \in (t_1, t_2)$. $\square$

Intuitively, by letting $x = (y, \theta)$ and expressing (44) as:

$$\dot{y} = f_y(\theta, y), \qquad \dot{\theta} = f_\theta(\theta, y), \tag{45}$$

a curve $y = \pi(\theta)$ is an invariant manifold for (45) if the solution of (45) with $\theta(0) = \theta_\circ$ and $y(0) = \pi(\theta_\circ)$ lies on the curve $y = \pi(\theta)$ for t in a neighborhood of 0. The notion of invariant manifold is useful as, under certain assumptions, it allows us to reduce the analysis of (44) to the study of a reduced system in the variable θ only. The remainder of this section is devoted to formalizing this fact.

Definition 5 (***Center manifold***)**.** Let $x = 0$ be an equilibrium of (44). A manifold S, passing through $x = 0$, is said to be a center manifold for (44) at $x = 0$ if it is locally invariant and the tangent space to S at 0 is exactly E°. $\square$

Intuitively, the invariant manifold $y = \pi(\theta)$ is a center manifold for (45) when all orbits of y decay to zero and those of θ neither decay nor grow exponentially.

In what follows, we will assume that all eigenvalues of F have nonpositive real part, i.e., $n^+ = 0$. When this holds, it is always possible to choose coordinates in U such that (44) is

$$\dot{y} = Ay + g(y, \theta), \qquad \dot{\theta} = B\theta + h(y, \theta), \tag{46}$$

where A is an $n^- \times n^-$ matrix having all eigenvalues with negative real part, B is an $n^\circ \times n^\circ$ matrix having all eigenvalues with zero real part, and the functions g and h are C^k functions vanishing at $(y, \theta) = (0, 0)$, together with all their first-order derivatives. Because of their equivalence, any conclusion drawn for (46) will apply also to (44). The following result ensures the existence of a center manifold.

Theorem 7 (***Center manifold existence theorem***)**.** Assume that $n^+ = 0$. There exists a neighborhood $V \subset \mathbb{R}^{n^0}$ of 0 and a class C^{k-1} mapping $\pi : V \to \mathbb{R}^{n^-}$ such that the set $S = \{(y, \theta) \in \mathbb{R}^{n^-} \times V : y = \pi(\theta)\}$, is a center manifold for (46). $\square$

By definition, a center manifold for (46) passes through $(0, 0)$ and is tangent to the subset of points with $y = 0$. Namely,

$$\pi(0) = 0 \qquad \text{and} \qquad \frac{\partial \pi}{\partial \theta}(0) = 0. \tag{47}$$

Moreover, this manifold is locally invariant for (46): this imposes on the mapping π the constraint:

$$\frac{\partial \pi}{\partial \theta}(B\theta + h(\pi(\theta), \theta)) = A\pi(\theta) + g(\pi(\theta), \theta), \tag{48}$$

as deduced by differentiating with respect to time any solution $(y(t), \theta(t))$ of (46) on the manifold $y(t) = \pi(\theta(t))$. In other words, any center manifold for (46) can equivalently be described as the graph of a mapping $y = \pi(\theta)$ satisfying the partial differential equation (48), with the constraints (47).

Remark 7. Theorem 7 shows existence but not the uniqueness of a center manifold. Moreover, (i) if g and h are $C^k, k \in \mathbb{N}_{>0}$, then (46) admits a C^{k-1} center manifold; (ii) if g and h are C^∞ functions, then (46) has a C^k center manifold for any finite k, but not necessarily a C^∞ center manifold. $\square$

The next result shows that any y-trajectory of (46), starting sufficiently close to the origin converges, as time tends to infinity, to a trajectory that belongs to the center manifold.

Theorem 8 (***Center manifold is locally attractive***)**.** Assume that $n^+ = 0$ and suppose $y = \pi(\theta)$ is a center manifold for (46) at $(0, 0)$. Let $(y(t), \theta(t))$ be a solution of (46). There exists a neighborhood U° of $(0, 0)$ and real numbers $M > 0$ and $K > 0$ such that, if $(y(0), \theta(0)) \in U^\circ$, then for all $t \geq 0$,

$$\|y(t) - \pi(\theta(t))\| \leq Me^{-Kt}\|y(0) - \pi(\theta(0))\|. \qquad \square$$

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