k-Dimensional Agreement in Multiagent Systems

Gianluca Bianchin, Miguel Vaquero, Jorge Cortés, and Emiliano Dall'Anese

***Abstract*— We study the problem of k-dimensional agreement, whereby a group of agents is interested in computing k linearly independent weighted means of a global vector whose elements are known only by individual agents. This problem is especially relevant in distributed computing and sensing applications, where the agents would like to evaluate multiple, agent-dependent, functions at a common point by running a single distributed algorithm. We propose the use of a continuous-time linear network protocol for this task, and we show that linear dynamics can agree on quantities that are oblique projections of the global vector. We provide necessary and sufficient conditions that characterize all protocols that can achieve an agreement, and we illustrate a design procedure for selecting the protocol weights, thus engineering the achievement of k-dimensional agreement. Overall, our results show that agreement protocols require higher communication connectivity as compared to consensus algorithms, and we identify Hamiltonian decompositions of graphs as the basic component that allows that graph to sustain an agreement protocol. Theoretical findings are illustrated via numerical simulations on a problem in distributed regression and on a robotic formation control problem.**

I. Introduction

SYNCHRONIZATION algorithms and consensus protocols are central to many network coordination methods, including rendezvous, distributed convex optimization, and distributed computation and sensing. Although an extensive literature has been developed on consensus-based processes – whereby the states of the network nodes asymptotically reach a *common* value that is a weighted mean of the agents' initial conditions [1]–[3] – in many applications it is instead of interest to assign, asymptotically, $k \geq 1$ *independent* weighted means (i.e. whose vectors of weights are linearly independent) of the initial conditions. Relevant examples of this problem include distributed computation [4], [5], where the weighting accounts for different desired outcomes, task allocation algorithms [6], where weights account for heterogeneous computational capabilities, distributed sensing [7], [8] where the weighting is proportional to the accuracy of each sensing device, and robotic formation [9] where one agent would like to achieve a certain configuration relative to another agent. More precisely, given any arbitrary matrix of weights W of rank k, we say that the group of agents reaches a k-dimensional agreement if, asymptotically, the agents' states converge to $Wx(0)$, where $x(0)$ is the vector of initial conditions of the agents. As is commonly done, we model the flow of information among the agents using a directed graph; our goal is to identify what classes of graphs can sustain agreement algorithms and to determine, when possible, a distributed protocol compatible with such graph that allows the agents to compute $Wx(0)$. We note that while this objective may be achieved via classical methods by executing k consensus algorithms [2] in parallel, the communication and computational complexities of such approach do not scale well with the network size (see Fig. 1); instead, our objective is to achieve this goal by running a *single network algorithm*.

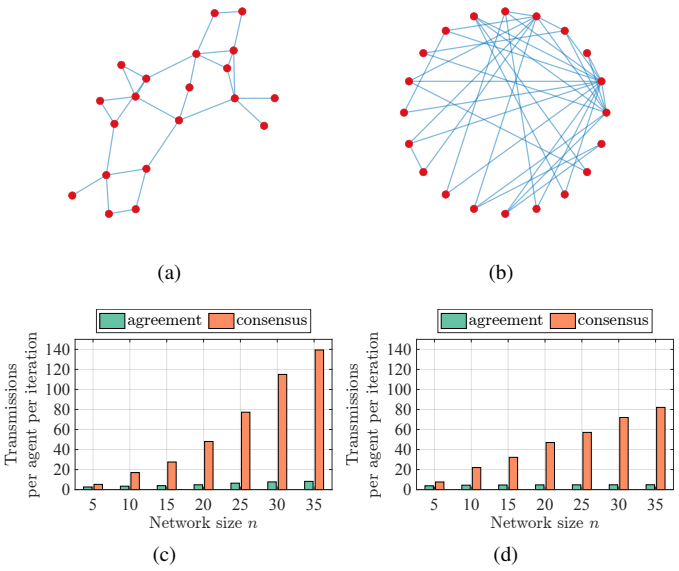

Fig. 1: Communication complexity of running k consensus algorithms in parallel vs one k-dimensional agreement algorithm to compute $k = \lfloor \frac{n}{2} \rfloor$ weighted average means of a global quantity. (a)-(c) Erdős–Rényi network model. (b)-(d) Barabasi-Albert model. Bars denote average number of transmissions per iteration per agent. Further details are provided in Example 3.3.

Related work. The agreement protocol studied in this work stems from the problem of distributed consensus. Because of their centrality, consensus algorithms have been extensively studied in the literature. A necessarily imcomplete list of topics include the following: sufficient and/or necessary conditions to reach consensus are provided in [2], [10]–[14], time delays are accounted in [12], consensus with arbitrary objective maps is studied in [15], convergence rate is considered in [16],

G. Bianchin and E. Dall'Anese are with the Department of Electrical, Computer, and Energy Engineering, University of Colorado Boulder. M. Vaquero is with the School of Human Sciences and Technology, IE University. J. Cortés is with the Department of Mechanical and Aerospace Engineering, University of California San Diego.

[17], and robustness in [18], [19]. Differently from constrained consensus problems [20], [21], where the agents' states are limited to satisfy agent-dependent constraints during transients and the asymptotic value lies in the intersection of the constraint sets, in our setting the values at convergence are constrained instead. In Pareto optimal distributed optimization [22], the group of agents cooperatively seeks to determine the minimizer of a cost function that depends on agent-dependent decision variables; in contrast, here the value at convergence might not be the optimizer of a loss function. Clustering-based consensus [23]–[25] is a closely-related problem where the state of agents in the same cluster of the graph converge to identical values, while inter-cluster states can differ. Differently from this setting, which is obtained by using weakly-connected communication graphs to separate the state of different communities, in this work we focus on cases where the asymptotic state of each agent depends on every other agent in the network. To the best of our knowledge, agreement problems exhibiting this dependence where the agents' states do not converge to identical values have not been studied. A relevant exception is the problem of scaled consensus considered in [26], where agents agree on subspaces of dimension $k = 1$. In this paper, we tackle the more general problem $k \geq 1$; as shown shortly below, the extension to $k \geq 2$ is non-trivial since standard assumptions made for consensus are not sufficient to guarantee agreement (see Example 4.5).

Contributions. The contribution of this work is fourfold. First, we formulate the k-dimensional agreement problem, whereby a group of agents with first-order dynamics is interested in computing, asymptotically, k independent weighted means of the initial conditions. We propose the use of linear protocols to achieve this task and show that they can reach an agreement on oblique projections of the initial conditions. Conversely, we also show that. for any desired oblique projection of the initial conditions, it is always possible to design a corresponding agreement protocol, provided that the communication graph is complete. In this case, we provide an algorithm to design agreement protocols. Second, for sparse graphs, we provide an algebraic characterization of network protocols that achieve agreement to any desired weights. Further, we show how such conditions can be used to design efficient numerical algorithms that yield fast agreement protocols. Third, motivated by this result, we provide graph-theoretic necessary and sufficient conditions to check whether a graph can sustain agreement dynamics. Our necessary conditions illustrate that widely-studied graphs, such as the line and circulant topology, can sustain agreement protocols on subspaces of dimension at most 1. Our sufficient conditions suggest that graphs that admit independent Hamiltonian decompositions can always compute agreement on arbitrary weights. Overall, our findings suggest that agreement protocols require graphs with higher connectivity as compared to consensus algorithms, and that Hamiltonian decompositions are the basic components that enable agreement.

Organization. Section II introduces basic concepts and Section III formalizes the problem. Section IV studies agreement protocols over complete graphs, Section V illustrates our main result pertaining to an algebraic characterization of agreement protocols and illustrates numerical methods to compute agreement algorithms, Section VI provides graph-theoretic conditions for agreement. Section VII extends the approach to tracking problems and Section VIII illustrates the techniques via numerical simulations. We gather our conclusions and ideas for future work in Section IX.

II. Preliminaries

Here, we introduce some basic notions used in the paper.

Notation. We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers, by $\mathbb{C}$ the set of complex numbers, and by $\mathbb{R}$ the set of real numbers. Given $x \in \mathbb{C}$, $\Re(x)$ and $\Im(x)$ denote its real and imaginary parts, respectively. Given vectors $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, we let $(x, u) \in \mathbb{R}^{n+m}$ denote their concatenation. We denote by $\mathbb{1}_n \in \mathbb{R}^n$ the vector of all ones, by $I_n \in \mathbb{R}^{n\times n}$ the identity matrix, by $\mathbb{0}_{n,m} \in \mathbb{R}^{n\times m}$ the matrix of all zeros (subscripts are dropped when dimensions are specified by the context). Given $A \in \mathbb{R}^{n\times n}$, we denote its spectrum by $\sigma(A) = \{\lambda \in \mathbb{C} \,:\, \det(\lambda I - A) = 0\}$, and by $\lambda_{\max}(A) = \max\{\Re(\lambda) \,:\, \lambda \in \sigma(A)\}$ its spectral abscissa. Given $A \in \mathbb{R}^{n\times m}$, $\mathrm{Im}(A)$ and $\ker(A)$ denote its image and null space, respectively.

Graph-theoretic notions. A directed graph (abbreviated digraph), denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, consists of a set of nodes $\mathcal{V} = \{1, \dots, n\}$ and a set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An element $(i, j) \in \mathcal{E}$ denotes a directed edge from node j to i. We will often use the notion of weighted digraphs $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $A \in \mathbb{R}^{n\times n}$ is the graph's adjacency matrix; $A = [a_{ij}]$ satisfies $a_{ij} \neq 0$ only if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ if $(i, j) \notin \mathcal{E}$. More generally, for fixed $\mathcal{G}$ and $A \in \mathbb{R}^{n\times n}$, we say that A is commensurate with $\mathcal{G}$ if A is a feasible adjacency matrix for $\mathcal{G}$. The set of (in)*neighbors* of node $i \in \mathcal{V}$ is $\mathcal{N}_i = \{j \in \mathcal{V} \setminus \{i\} \,:\, (i, j) \in \mathcal{E}\}$. A graph is *complete* if there exists an edge connecting every pair of nodes, and is *sparse* otherwise. A *path* in $\mathcal{G}$ is a sequence of edges $(e_1, e_2, \dots)$ such that the initial node of each edge is the final node of the preceding edge. Notice that a path may contain repeated edges and, also, going along the path one may reach repeated nodes. The *length* of a path is the number of edges contained in the sequence $(e_1, e_2, \dots)$. A graph is *strongly connected* if, for any $i, j \in \mathcal{V}$, there is a directed path from i to j. A *closed path* is a path whose initial and final vertices coincide. A closed path is a *cycle* if, going along the path, one reaches no node, other than the initial-final node, more than once. A cycle of length equal to one is a *self cycle*. We say that $\mathcal{G}_1, \dots, \mathcal{G}_m$ is a *decomposition* of $\mathcal{G}$ if $\mathcal{G}_i$ are pairwise disjoint and the union of the node sets of $\mathcal{G}_i$ is the node set of $\mathcal{G}$. A *Hamiltonian cycle* is a cycle that visits every node of $\mathcal{G}$ exactly once [27]. A *Hamiltonian decomposition* is a decomposition of $\mathcal{G}$ such that each subgraph $\mathcal{G}_i$ is a Hamiltonian cycle. (Notice that not all graphs admit a Hamiltonian decomposition; conversely, certain graphs admit multiple Hamiltonian decompositions.) A *Hamiltonian ℓ-decomposition* of $\mathcal{G}$ is a Hamiltonian decomposition of some subgraph of $\mathcal{G}$ with ℓ nodes, see Fig. 2 for an illustration.

Structural analysis and sparse matrices spaces. We rely on the structural approach to systems theory originally proposed in [28], [29]. We are concerned with linear subspaces

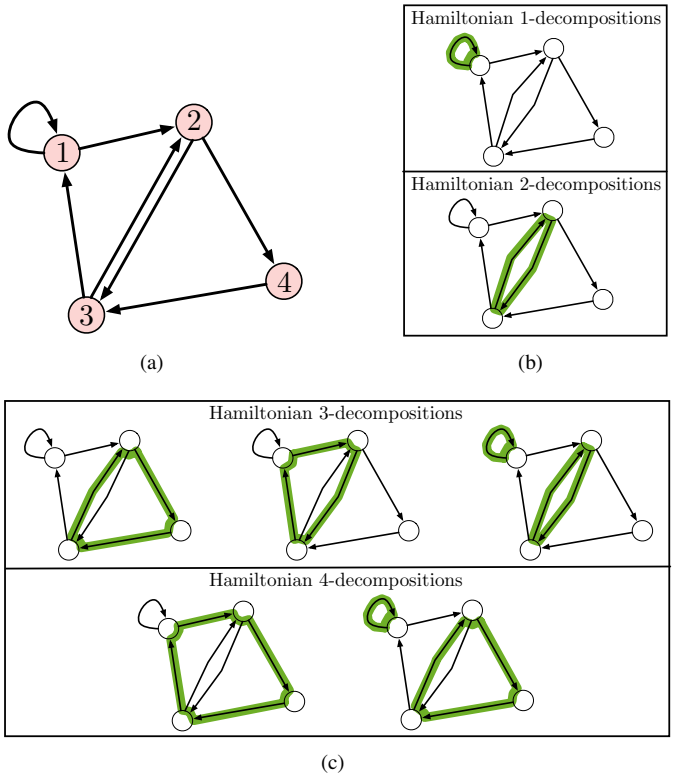

Fig. 2: (a) Illustration of a digraph and (b)-(c) associated Hamiltonian ℓ-decompositions for $\ell \in \{1,\dots,4\}$. Notice that a graph might admit one, multiple, or no Hamiltonian ℓ-decompositions.

obtained by forcing certain entries of the matrices in $\mathbb{R}^{n\times n}$ to be zero. More precisely, given a digraph $\mathcal{G}$, we define $\mathcal{A}_{\mathcal{G}}$ to be the vector space in $\mathbb{R}^{n\times n}$ that contains all matrices that are commensurate with $\mathcal{G}$, and whose nonzero entries are independent parameters. The vector space $\mathcal{A}_{\mathcal{G}}$ can be represented by a $n \times n$ *structured matrix* $\mathbf{A}_{\mathcal{G}}$, whose entries are either algebraically independent parameters in $\mathbb{R}$ (denoted by $\divideontimes$) or fixed zeros (denoted by 0). We define a vector of parameters $a = (\{a_{ij}\}_{(i,j)\in\mathcal{E}})$ such that $\mathbf{A}_{\mathcal{G}}(a)$ defines a numerical realization of the structured matrix $\mathbf{A}_{\mathcal{G}}$, namely, $\mathbf{A}_{\mathcal{G}}(a) \in \mathcal{A}_{\mathcal{G}}$. For instance, for the graph in Fig. 2, we have

$$\mathbf{A}_{\mathcal{G}} = \begin{bmatrix} \divideontimes & 0 & \divideontimes & 0 \\ \divideontimes & 0 & \divideontimes & 0 \\ 0 & \divideontimes & 0 & \divideontimes \\ 0 & \divideontimes & 0 & 0 \end{bmatrix}, \mathbf{A}_{\mathcal{G}}(a) = \begin{bmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & a_{34} \\ 0 & a_{42} & 0 & 0 \end{bmatrix},$$

where $a = (a_{11}, a_{13}, a_{21}, a_{23}, a_{32}, a_{34}, a_{42})$.

Projections and linear subspaces. We next recall some basic notions regarding projections in linear algebra [30]. Two vectors $x, y \in \mathbb{R}^n$ are orthogonal if $x^{\mathsf{T}}y = 0$; the orthogonal complement of $\mathcal{M} \subset \mathbb{R}^n$ is a a linear subspace defined as $M^{\perp} := \{x \in \mathbb{R}^n \ : \ x^{\mathsf{T}}y = 0, \forall y \in \mathcal{M}\}$. Given two subspaces $\mathcal{M}, \mathcal{N} \subseteq \mathbb{R}^n$, the subspace $\mathcal{W} \subseteq \mathbb{R}^n$ is a direct sum of $\mathcal{M}$ and $\mathcal{N}$, denoted $\mathcal{W} = \mathcal{M} \oplus \mathcal{N}$, if $\mathcal{M} \cap \mathcal{N} = \{0\}$, and $\mathcal{M} + \mathcal{N} = \{u + v \ : \ u \in \mathcal{M}, v \in \mathcal{N}\} = \mathcal{W}$. The subspaces $\mathcal{M}, \mathcal{N} \subset \mathbb{R}^n$ are complementary if $\mathcal{M} \oplus \mathcal{N} = \mathbb{R}^n$. A matrix $\Pi \in \mathbb{R}^{n\times n}$ is called a *projection* in $\mathbb{R}^{n\times n}$ if $\Pi^2 = \Pi$. Given complementary subspaces $\mathcal{M}, \mathcal{N} \subset \mathbb{R}^n$, for any $z \in \mathbb{R}^n$ there exists a unique decomposition of the form $z = x + y$, where $x \in \mathcal{M}$, $y \in \mathcal{N}$. The transformation $\Pi_{\mathcal{M},\mathcal{N}}$ defined by $\Pi_{\mathcal{M},\mathcal{N}} z := x$ is called *projection onto $\mathcal{M}$ along $\mathcal{N}$*, and the transformation $\Pi_{\mathcal{N},\mathcal{M}}$ defined by $\Pi_{\mathcal{N},\mathcal{M}} z := y$ is called *projection onto $\mathcal{N}$ along $\mathcal{M}$*. Moreover, the vector x is the projection of z onto $\mathcal{M}$ along $\mathcal{N}$, and the vector y is the projection of z onto $\mathcal{N}$ along $\mathcal{M}$. The projection $\Pi_{\mathcal{M},\mathcal{M}^{\perp}}$ onto $\mathcal{M}$ along $\mathcal{M}^{\perp}$ is called *orthogonal projection onto $\mathcal{M}$*. Because the subspace $\mathcal{M}$ uniquely determines $\mathcal{M}^{\perp}$, in what follows we will denote $\Pi_{\mathcal{M},\mathcal{M}^{\perp}}$ in compact form as $\Pi_{\mathcal{M}}$. General projections that are not orthogonal are called *oblique projections*. We recall the following instrumental results.

*Lemma 2.1: (**[30, Thm. 2.11 and Thm. 2.31]**)* If $\Pi \in \mathbb{R}^{n\times n}$ is a projection with $\operatorname{rank}(\Pi) = k$, then there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that

$$\Pi = T \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} T^{-1}.$$

Moreover, if Π is an orthogonal projection, then T can be chosen to be an orthogonal matrix, i.e., $TT^{\mathsf{T}} = I$. □

*Lemma 2.2: (**[30, Thm. 2.26]**)* Let $\mathcal{M}, \mathcal{N}$ be complementary subspaces and let the columns of $V \in \mathbb{R}^{n\times k}$ and $W \in \mathbb{R}^{n\times k}$ form a basis for $\mathcal{M}$ and $\mathcal{N}^{\perp}$, respectively. Then $\Pi_{\mathcal{M},\mathcal{N}} = V(W^{\mathsf{T}}V)^{-1}W^{\mathsf{T}}$. □

By recalling the properties [30, Thm. 1.60]:

$$\operatorname{Im}(M^{\mathsf{T}}) = \operatorname{Im}(M^{\dagger}) = \operatorname{Im}(M^{\dagger}M) = \operatorname{Im}(M^{\mathsf{T}}M),$$
$$\ker(M) = \operatorname{Im}(M^{\mathsf{T}})^{\perp} = \ker(M^{\dagger}M) = \operatorname{Im}(I - M^{\dagger}M),$$

it follows from Lemma 2.2 that if $M \in \mathbb{R}^{m\times n}$, then $\Pi_{\operatorname{Im}(M)} = MM^{\dagger}$ and $\Pi_{\ker(M)} = I - M^{\dagger}M$, where $M^{\dagger} \in \mathbb{R}^{n\times m}$ is the Moore-Penrose inverse of M.

III. Problem Formulation

Consider a group of n agents whose interaction topology is described by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\mathcal{V} = \{1,\dots,n\}$, and where the state of the i-th agent, $i \in \mathcal{V}$, is described by a variable $x_i \in \mathbb{R}$. We are interested in a model where each agent exchanges its state with its neighbors and updates it by forming linear combinations of the received estimates:

$$\dot{x}_i = a_{ii}x_i + \sum_{j\in\mathcal{N}_i} a_{ij}x_j, \qquad i \in \mathcal{V}, \tag{1}$$

where $a_{ij} \in \mathbb{R}$, $(i,j) \in \mathcal{E}$, is a weighting factor. Setting $A = [a_{ij}]$ with $a_{ij} = 0$ if $(i,j) \notin \mathcal{E}$, and $x = (x_1,\dots,x_n)$, this iteration can be written in vector form as

$$\dot{x} = Ax. \tag{2}$$

We say that the network process (2) reaches a k-dimensional agreement if, asymptotically, each state variable reaches a (agent-dependent) weighted sum of the initial conditions $x(0)$, such that the value at convergence is confined to a subspace of dimension k. This notion is formalized next.

*Definition 3.1: (**k-dimensional agreement**)* For $k \in \mathbb{N}$ with $1 \le k \le n-1$, let $W \in \mathbb{R}^{n\times n}$ with $\operatorname{rank}(W) = k$. The update (2) *reaches k-dimensional agreement on W* if

$$\lim_{t\to\infty} x(t) = Wx(0), \tag{3}$$

for any $x(0) \in \mathbb{R}^n$. □

Reaching a k-dimensional agreement, in general, does not imply that the values of all state variables converge to a common quantity (i.e., it is not guaranteed that $\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0$, $i\neq j$). Conversely, (3) formalizes a notion of agreement whereby, asymptotically, the agents' state is confined to the k-dimensional subspace described by the range space of W.

Remark 3.2: ***(Relationship with consensus problems)*** In the special case $k=1$, the matrix of weights W can be written as $W=vw^{\mathsf{T}}$ for some $v,w\in\mathbb{R}^n$. In this case, when $v=\mathbb{1}$ and w is such that $w^{\mathsf{T}}\mathbb{1}=1$, we recover the well-studied *consensus* problem [2] and when, in addition, $w=\frac{1}{n}\mathbb{1}$ we recover the *average consensus* problem [2, Sec. 2]. Notice that, when $v=\mathbb{1}$, all the state variables converge to the same quantity, namely, $\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0,\forall\, i,j\in\mathcal{V}$. □

In this paper, we are interested in cases where the matrix of weights W is fixed but arbitrary and, accordingly, in these cases we will say that (2) can reach an *agreement on arbitrary weights*. This motivates the following assumption.

Assumption 1: ***(Strong connectivity)*** The communication digraph $\mathcal{G}$ is strongly connected. □

It is easy to see that, when $\mathcal{G}$ is not strongly connected, agreement can only be reached on weight matrices W that are sparse (i.e., at least one of their entries is equal to zero); and thus agreement on arbitrary weights cannot be reached. In fact, by recalling that when $\mathcal{G}$ is not strongly connected at least one of the entries of $\lim_{t\to\infty}e^{At}$ is identically zero for any A that is commensurate with $\mathcal{G}$, it follows that at least one entry of $W=\lim_{t\to\infty}e^{At}$ in (3) must be identically zero. We conclude this section illustrating the applicability of the framework in distributed computation applications.

Example 3.3: ***(Motivating example: distributed computation of k weighted means)*** Consider a scenario where each agent is interested in computing, using only local communication, an agent-dependent weighted mean of a certain quantity $x^*=(x_1^*,\dots,x_n^*)$, where $x_i^*\in\mathbb{R}$ is known only by agent i. More formally, let $w_i\in\mathbb{R}^n$, with $w_i^{\mathsf{T}}\mathbb{1}=1$, denote desired weighting coefficients for agent i, with k of these vectors being linearly independent. We are thus interested in guaranteeing that $\lim_{t\to\infty}x_i(t)=w_i^{\mathsf{T}}x^*$ for all $i\in\{1,\dots,n\}$. There are two natural solutions to this problem. The first and more standard solution consists in executing k linear consensus algorithms in parallel [31], where the i-th algorithm is designed to converge asymptotically to $w_i^{\mathsf{T}}x^*$. A second solution, proposed in this work, consists in executing a single linear algorithm of the form (2). We next detail these two approaches and discuss their communication complexities.

(Approach 1) Consider k duplicates of the agents states $x^{(1)},\dots,x^{(k)}\in\mathbb{R}^n$, and let the i-th duplicate implement the protocol $\dot{x}^{(i)}=A^{(i)}x^{(i)}$. A simple choice to achieve $x^{(i)}(t)\to\mathbb{1}w_i^{\mathsf{T}}x^*$ is to adopt a Laplacian-based consensus algorithm described by $x^{(i)}(0)=x^*$ and

$$A^{(i)}:=P^{(i)}-D^{(i)},\quad\text{where}\quad w_i^{\mathsf{T}}(P^{(i)}-\operatorname{diag}(P^{(i)}\mathbb{1}))=0.$$

As shown in [31, Thm. 1], this choice guarantees $\lim_{t\to\infty}x^{(i)}(t)=\mathbb{1}w_i^{\mathsf{T}}x^*$, provided that $\mathcal{G}$ is strongly connected. Unfortunately, it is easy to see that the spatial and communication complexities of this approach do not scale well with the network size. Indeed, each agent must maintain k independent scalar state variables and, at every time step, each agent must transmit these k variables to all its neighbors. It follows that the per-agent spatial complexity is $\mathcal{O}(k)$ (since each agent maintains k copies of a scalar state variable), and the per-agent communication complexity is $\mathcal{O}(k\cdot\deg(\mathcal{G}))$ (and thus is $\mathcal{O}(n\cdot\deg(\mathcal{G}))$ when k is proportional to n). Here, $\deg(\mathcal{G})$ denotes the largest in/out node degree in $\mathcal{G}$.

(Approach 2) Motivated by the undesirable spatial and communication complexities of *(Approach 1)*, we propose the use of a linear agreement protocol of the form (2) that achieves agreement as in (3). Here, A is chosen as described shortly below. In this case, the per-agent spatial complexity is $\mathcal{O}(1)$ since agents maintains a single scalar state variable, and the communication complexity is $\mathcal{O}(\deg(\mathcal{G}))$. Fig. 1 compares the communication volumes of *(Approach 1)* and *(Approach 2)*. □

IV. Characterization of the agreement space and centralized agreement algorithms

In this section, we provide a characterization of the agreement space for the iterates of (2) and illustrate how it is possible to construct the matrix A that defines the agreement protocol. We begin with the following instrumental result.

Lemma 4.1: ***(Spectral properties of agreement protocols)*** The update (2) reaches a k-dimensional agreement if and only if there exists a nonsingular $T\in\mathbb{R}^{n\times n}$ such that A admits the following decomposition:

$$A=T\begin{bmatrix}\mathbb{0}&\mathbb{0}\\ \mathbb{0}&B\end{bmatrix}T^{-1},\tag{4}$$

where $B\in\mathbb{R}^{(n-k)\times(n-k)}$ satisfies $\lambda_{\max}(B)<0$. □

Proof: *(If)* When (4) holds, we have:

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}e^{At}x(0)=\underbrace{T\begin{bmatrix}I_k&\mathbb{0}\\ \mathbb{0}&\mathbb{0}\end{bmatrix}T^{-1}}_{:=W}x(0)=Wx(0).$$

(Only if) Since $\lim_{t\to\infty}e^{At}$ exists, $\lambda_{\max}(A)\leq 0$ and if $\lambda\in\sigma(A)$ and $\Re(\lambda)=0$, then $\lambda=0$ and its algebraic and geometric multiplicities coincide (see, e.g., [32, Lemma 1.7]). It follows that A must satisfy (4). ■

Lemma 4.1 shows that matrices A that define agreement protocols satisfy two properties: (i) their spectrum is $\sigma(A)=\{0\}\cup\{\lambda_1,\dots,\lambda_{n-k}\}$, where $\Re(\lambda_i)<0$, and (ii) the eigenvalue $\lambda=0$ has identical algebraic and geometric multiplicities, which are equal to k.

In the following result, we characterize the class of weight matrices W on which an agreement can be achieved.

Proposition 4.2: ***(Characterization of agreement space)*** Let $x(t)$ denote the solution of (2) with initial condition $x(0)$. If $\lim_{t\to\infty}x(t):=x_\infty$ exists, then there exist complementary subspaces $\mathcal{M},\mathcal{N}\subset\mathbb{R}^n$ such that $x_\infty=\Pi_{\mathcal{M},\mathcal{N}}x(0)$. □

Proof: When (2) reaches an agreement, matrix A reads as in (4), from which we obtain

$$W=\lim_{t\to\infty}e^{At}=T\begin{bmatrix}I_k&\mathbb{0}\\ \mathbb{0}&\mathbb{0}\end{bmatrix}T^{-1}.$$

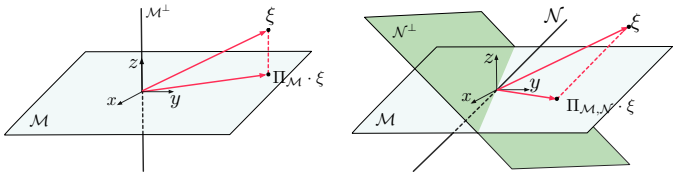

Fig. 3: (a) Illustration of orthogonal projection of a vector $\xi \in \mathbb{R}^3$ onto $\mathcal{M} \subset \mathbb{R}^3$. (b) Illustration of complementary subspaces $\mathcal{M}, \mathcal{N} \in \mathbb{R}^3$ and oblique projection of ξ. Notice that the projection ray belongs to $\text{span}(\mathcal{N})$.

To conclude, notice that W is a projection since $W^2 = W$. ■

Proposition 4.2 provides a characterization of the class of weights on which an agreement can be reached: it shows that linear protocols of the form (2) can agree on matrix weights that are oblique projections. Accordingly, the asymptotic value at agreement can be interpreted geometrically as the corresponding oblique projection of the initial conditions $x(0)$, cf. Fig. 3 for an illustration.

Remark 4.3: ***(Geometric interpretation of agreement space of consensus algorithms)*** In the case of standard consensus ($k = 1$), the group of agents is known to converge to $\mathbb{1}w^\mathsf{T}x(0)$, where w is a left eigenvector of A ($w^\mathsf{T}A = 0$) that satisfies $w^\mathsf{T}\mathbb{1} = 1$ (see Remark 3.2). Proposition 4.2 provides a geometric interpretation of the consensus value: $\mathbb{1}w^\mathsf{T}x(0) = \Pi_{\mathcal{M},\mathcal{N}}$ is the oblique projection of the initial conditions $x(0)$ onto $\mathcal{M} = \text{Im}(\mathbb{1})$ along $\mathcal{N} = \text{Im}(w)^\perp$ (see Lemma 2.2). In the case of average consensus, the value at convergence is $\frac{1}{n}\mathbb{1}\mathbb{1}^\mathsf{T}x(0)$, which is the orthogonal projection of $x(0)$ onto $\mathcal{M} = \text{Im}(\mathbb{1})$. □

It is now natural to consider the following converse question: given any pair of complementary subspaces $\mathcal{M}, \mathcal{N}$, is it possible to construct a protocol A such that $\lim_{t\to\infty} x(t) = \Pi_{\mathcal{M},\mathcal{N}}x(0)$? The following result provides a positive answer to this question for the case where the digraph $\mathcal{G}$ is complete.

Proposition 4.4: ***(Existence of agreement algorithms over complete digraphs)*** Let $\mathcal{M}, \mathcal{N} \subset \mathbb{R}^n$ be complementary subspaces and $\mathcal{G}$ the complete graph with $|\mathcal{V}| = n$. Then, exists $A \in \mathbb{R}^{n\times n}$ such that the iterates (2) satisfy $\lim_{t\to\infty} x(t) = \Pi_{\mathcal{M},\mathcal{N}}x(0)$. □

Proof: For any pair of complementary subspaces $\mathcal{M}, \mathcal{N}$, [30, Thm. 2.26] guarantees existence of an oblique projection matrix $\Pi_{\mathcal{M},\mathcal{N}}$. Moreover, by Lemma 2.1, there exists invertible $\bar{T} \in \mathbb{R}^{n\times n}$ such that $\Pi_{\mathcal{M},\mathcal{N}}$ can be decomposed as

$$\Pi_{\mathcal{M},\mathcal{N}} = T_\Pi \begin{bmatrix} I_k & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix} T_\Pi^{-1},$$

where $k = \dim(\mathcal{M})$. The statement follows by choosing A as in (4) with $T = T_\Pi$ and by noting that, with this choice, $\lim_{t\to\infty} e^{At}x(0) = \Pi_{\mathcal{M},\mathcal{N}}x(0)$. ■

Proposition 4.4 shows that for any pair of complementary subspaces $\mathcal{M}, \mathcal{N}$, it is always possible to construct an agreement protocol A such that the iterates (2) converge to the projection of the initial conditions $\Pi_{\mathcal{M},\mathcal{N}}x(0)$. The proof of this result provides an explicit way to construct A given $\mathcal{M}, \mathcal{N}$. We formalize a technique to construct agreement matrices A over complete graphs in Algorithm 1. We remark that matrices A constructed from Proposition 4.4 or Algorithm 1 are, in general, non-sparse or non-commensurate with a pre-specified digraph $\mathcal{G}$.

Algorithm 1 Construction of agreement matrix A

Require: $V \in \mathbb{R}^{n\times k}$ whose columns form a basis for $\mathcal{M}$
Require: $U \in \mathbb{R}^{n\times k}$ whose columns form a basis for $\mathcal{N}^\perp$
$\Pi_{\mathcal{M},\mathcal{N}} \leftarrow V(U^\mathsf{T}V)^{-1}U^\mathsf{T}$;
Determine T such that $\Pi_{\mathcal{M},\mathcal{N}} = T \begin{bmatrix} I_k & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix} T^{-1}$;
Choose $B \in \mathbb{R}^{(n-k)\times(n-k)}$ such that $\lambda_{\max}(B) < 0$;
return $A = T \begin{bmatrix} \mathbb{0}_k & \mathbb{0} \\ \mathbb{0} & B \end{bmatrix} T^{-1}$;

It is thus natural to consider the question of whether k-dimensional agreement can be achieved by protocols implemented over an arbitrary (non-complete) digraph $\mathcal{G}$, provided that $\mathcal{G}$ satisfies standard assumptions (such as strong connectivity, see Assumption 1). The following example provides a negative answer to this question by showing that when $n = 3$ and $k = 2$, agreement can be achieved only if $\mathcal{G}$ is complete.

Example 4.5: ***(Not every digraph can reach $k \geq 2$ agreement on arbitrary weights)*** Consider a case where a network of $n = 3$ agents is interested in reaching an agreement on a subspace of dimension $k = 2$. By using Lemma 4.1, (2) reaches an agreement if and only if A can be diagonalized as:

$$A = T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix} T^{-1}, \tag{5}$$

for some $\beta < 0$ and some $T \in \mathbb{R}^{3\times3}$ invertible. By denoting by t_i the i-th column of T and by τ_i^T the i-th row of T^{-1}, $i \in \{1, 2, 3\}$, the above identity yields $A = \beta t_3\tau_3^\mathsf{T}$. Moreover, by recalling that the agreement weights are $W = \lim_{t\to\infty} e^{At}$, we conclude that $W = t_3\tau_3^\mathsf{T}$. The above discussion shows that W is non-sparse if and only if $\mathcal{G}$ is the complete graph. It thus follows that agreement on arbitrary weights can be achieved only of $\mathcal{G}$ is complete. □

In the following section, we focus on providing conditions on $\mathcal{G}$ that guarantee the existence of an agreement protocol with communication pattern commensurate with $\mathcal{G}$.

Remark 4.6: ***(Graphs that guarantee consensus on arbitrary weights)*** We recall that in the case of consensus ($k = 1$) the property of strong connectivity of the underlying digraph is sufficient to guarantee that consensus can be achieved on any arbitrary weighted average [31]. More formally, $\forall\ W : \text{rank}(W) = 1, \exists A : \lim_{t\to\infty} x(t) = Wx(0)$. It follows from the discussion in Example 4.5 that agreement on subspaces of dimension $k \geq 2$ mandates the use of digraphs with higher connectivity as compared to the case of consensus. □

V. Agreement algorithms over sparse digraphs

Here, we restrict our attention to agreement protocols that are commensurate with a given graph $\mathcal{G}$. We adopt the notation $\mathbf{A}_\mathcal{G}(a)$ to denote matrices associated with (2) that are commensurate with $\mathcal{G}$ and whose entries are parametrized by

the vector $a \in \mathbb{R}^{|\mathcal{E}|}$ (see Section II). Moreover, motivated by Proposition 4.2, we restrict ourselves to cases where W is a projection matrix.

Assumption 2: ***(Matrix of weights is a projection)*** The matrix of weights W is a projection onto a k-dimensional space, namely, $W^2 = W$ and $\operatorname{rank}(W) = k$, with $k \in \mathbb{N}$, $1 \leq k \leq n-1$. □

Building upon Assumption 2, in what follows we make use of the following decomposition for W (see Lemma 2.1):

$$W = T_W \begin{bmatrix} I_k & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{bmatrix} T_W^{-1}. \tag{6}$$

Moreover, we often use the following notation:

$$T_W = \begin{bmatrix} t_1 & \cdots & t_n \end{bmatrix}, \quad (T_W^{-1})^{\mathsf{T}} = \begin{bmatrix} \tau_1 & \cdots & \tau_n \end{bmatrix}, \tag{7}$$

where $t_i \in \mathbb{R}^n$, $i \in \{1, \dots, n\}$, denotes the i-th column of T_W, and $\tau_i \in \mathbb{R}^n$ denote the i-th column of T_W and the i-th row of T_W^{-1}, respectively, $i \in \{1, \dots, n\}$.

A. Algebraic conditions for agreement

We next provide an algebraic characterization of agreement protocols over sparse digraphs. Our characterization builds upon a graph-theoretic characterization of characteristic polynomials of matrices associated with sparse digraphs, which we present next. Recall that a *Hamiltonian ℓ-decomposition* is a decomposition of $\mathcal{G}$ that contains ℓ nodes and such that each subgraph is a Hamiltonian cycle (see Section II for a more formal definition). In what follows, we denote by $\mathcal{C}_\ell(\mathcal{G})$ the set of all Hamiltonian ℓ-decomposition of the graph $\mathcal{G}$. The following characterization, proved in [33], is instrumental for the subsequent analysis[1].

Lemma 5.1: ***([33, Thm. 1])*** Let $\mathcal{G}$ be a digraph, let $A \in \mathcal{A}_\mathcal{G}$, and $\det(\lambda I - A) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n$ denote its characteristic polynomial. Each coefficient $p_\ell, \ell \in \{1, \dots, n\}$, can be written as:

$$p_\ell = \sum_{\xi \in \mathcal{C}_\ell(\mathcal{G})} (-1)^{d(\xi)} \prod_{(i,j)\in\xi} a_{ij},$$

where $d(c)$ denotes the number of Hamiltonian cycles in ξ. □

Lemma 5.1 provides a graph-theoretic interpretation of the characteristic polynomial: it shows that the ℓ-th coefficient of $\det(\lambda I - A)$ can be written as a sum of terms such that each summand is the product of edges in a Hamiltonian ℓ-decomposition of $\mathcal{G}$. We illustrate this claim next.

Example 5.2: Consider the digraph in Fig. 2(a). We have:

$$A = \begin{bmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & 0 \\ 0 & a_{32} & 0 & a_{34} \\ 0 & a_{42} & 0 & 0 \end{bmatrix},$$

and we refer to Fig. 2(b)-(c) for an illustration of all Hamiltonian ℓ-decompositions for this graph. Lemma 5.1 yields:

$$p_1 = -a_{11}, \qquad p_3 = -a_{13}a_{21}a_{32} + a_{11}a_{23}a_{32} - a_{23}a_{42}a_{34},$$
$$p_2 = -a_{23}a_{32} \quad p_4 = -a_{13}a_{21}a_{42}a_{34} + a_{11}a_{23}a_{34}a_{42}.$$

Notice that each summand in p_ℓ is the product of weights in a Hamiltonian decomposition of corresponding size. □

Finally, we say that a polynomial is *stable* if all its roots have negative real part. We are now ready to state the following necessary and sufficient conditions for agreement.

Theorem 5.3: ***(Algebraic characterization of sparse agreement matrices)*** Let $\mathcal{G}$ be a fixed digraph as in Assumption 1, let W satisfy Assumption 2. The linear update $\dot{x} = \mathbf{A}_\mathcal{G}(a)x$ reaches a k-dimensional agreement on W if and only if the following conditions hold simultaneously:

(i) $\mathbf{A}_\mathcal{G}(a) \cdot t_i = 0, \quad \tau_i^{\mathsf{T}} \mathbf{A}_\mathcal{G}(a) = 0, \qquad \forall i \in \{1, \dots, k\};$

(ii) The polynomial $\lambda^{n-k-1} + p_1\lambda^{n-k-2} + \cdots + p_{n-k-1}$, whose coefficients are defined as

$$p_\ell = \sum_{\xi \in \mathcal{C}_\ell(\mathcal{G})} (-1)^{d(\xi)} \prod_{(i,j)\in\xi} a_{ij}, \quad \ell \in \{1, \dots, n-k-1\},$$

is stable. □

Proof: *(If)* Let A be any matrix that satisfies *(i)-(ii)*. If A is diagonalizable, then, by letting T be the matrix of its eigenvectors, we conclude that A satisfies (4) and thus the linear update reaches an agreement. If A is not diagonalizable, let T be a similarity transformation such that $T^{-1}AT$ is in Jordan normal form:

$$T^{-1}AT = \begin{bmatrix} J_{\lambda_1} & & & \\ & J_{\lambda_2} & & \\ & & \ddots & \\ & & & J_{\lambda_n} \end{bmatrix}, J_{\lambda_i} = \begin{bmatrix} \lambda_1 & 1 & & \\ & \ddots & \ddots & \\ & & & \lambda_1 \end{bmatrix},$$

From *(i)* we conclude that $\lambda = 0$ is an eigenvalue with algebraic multiplicity k, moreover, since the vectors t_i are linearly independent (see (6)), we conclude that its geometric multiplicity is also equal to k, and thus all Jordan blocks associated with $\lambda = 0$ have dimension 1. Namely, $J_{\lambda_1} = \cdots = J_{\lambda_k} = 0$. By combining this with *(ii)*, we conclude that the characteristic polynomial of A is

$$\det(\lambda I - A) = \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-k-1}\lambda^{k-1},$$

and, since by assumption such polynomial is stable, we conclude that all remaining eigenvalues $\{\lambda_{k+1}, \dots, \lambda_n\}$ of A satisfy $\Re(\lambda_i) < 0$. Since all Jordan blocs associated to $\lambda = 0$ have dimension 1 and all the remaining eigenvalues of A are stable, we conclude that A admits the representation (4) and thus the linear update reaches an agreement.

(Only if) We will prove this claim by showing that (4) implies *(i)-(ii)*. To prove that *(i)* holds, we rewrite (4) as

$$T^{-1}AT = \begin{bmatrix} \mathbb{0} & \mathbb{0} \\ \mathbb{0} & B \end{bmatrix},$$

and, by taking the first k columns of the above identity we conclude $At_i = 0$, $i \in \{1, \dots, k\}$, thus showing that *(i)* holds. To prove that *(ii)* holds, notice that (4) implies that

[1]The claim [33, Thm. 1] is stated in terms of *cycles* and *cycle families* instead than Hamiltonian cycles and Hamiltonian decompositions. In this work, we used the latter wording, which is more standard and better aligned with the recent literature [27].

the characteristic polynomial of A is a stable polynomial with k roots at zero. Namely,

$$\begin{aligned}\det(\lambda I - A) &= \lambda^k(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_{n-k})\\ &= \lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-k-1}\lambda^{k-1},\end{aligned}$$

where $\Re(\lambda_i) < 0, i \in \{1,\dots,n-k\}$, and $p_j, j \in \{1,\dots,n-k-1\}$, are nonzero real coefficients. The statement *(ii)* thus follows by by applying the graph-theoretic interpretation of the coefficients of the characteristic polynomial in Lemma 5.1. ■

Theorem 5.3 provides an algebraic characterization of linear protocols that reach an agreement on arbitrary weights W. Given a graph $\mathcal{G}$ and a matrix of weights W, this result can be used to construct agreement protocols on W that are commensurate with $\mathcal{G}$ as follows: by interpreting the vector of weights a as an unknown, conditions *(i)-(ii)* define a system of nk linear equations and $n-k$ multilinear polynomial equations with $|\mathcal{E}|$ unknowns described by the vector a and $n-k$ arbitrary but fixed real numbers $p_1,\dots p_{n-k}$. Since the set of equations *(i)-(ii)* is nonlinear in the parameter a, a solution might not exist in general (unless the digraph is complete, see Proposition 4.4). Its solvability can be assessed via numerical techniques for systems of polynomial equations. We discuss one of these techniques next.

Remark 5.4: ***(Determining numerical solutions to polynomial equations)*** A powerful and general technique for solving systems of polynomial equations uses the tool of Gröbner bases, as applied using Buchberger's algorithm. The technique relies on transforming a system of polynomial equations into a canonical form, expressed in terms of a Gröbner basis, for which it then easier to determine a solution. We refer to [34], [35] for a complete discussion. We also remark that Hilbert's Nullstellensatz theorem provides a powerful tool to assess existence of solutions to conditions *(i)-(ii)*. More precisely, the result states that a system of polynomial equations has no solution if its Gröbner basis is $\{1\}$. In this sense, the Gröbner basis method provides an easy way to check solvability of *(i)-(ii)*. Finally, we should note that in general the computational complexity of solving a system of polynomial equations via Gröbner bases is exponential [35]. □

B. Fast distributed agreement algorithms

Theorem 5.3 motivates the study of the following problem: given a digraph $\mathcal{G}$ and a matrix of weights W, determine an agreement protocol with sparsity pattern $A \in \mathcal{A}_\mathcal{G}$ determined by $\mathcal{G}$ such that (2) reaches an agreement on W and such that its rate of convergence is maximal. This question can be posed as the following optimization problem:

$$\begin{aligned}\min_{A}\quad & r(A)\\ \text{s.t.}\quad & A \in \mathcal{A}_\mathcal{G},\quad \lim_{t\to\infty} e^{At} = W.\end{aligned} \tag{8}$$

In (8), $r : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a function that measures the rate of convergence of e^{At} over time. The problem of determining the fastest distributed agreement algorithm (8) is closely related to the problem of fastest average consensus studied in [17]; the main difference is that while the average consensus problem is guaranteed to be feasible when $\mathcal{G}$ is strongly connected, there is no simple way to check feasibility of (8) for general graphs. When $\mathcal{G}$ is the complete graph Proposition 4.4 guarantees that (8) is feasible.

When the optimization problem (8) is feasible, it is natural to consider two possible choices for the cost function $r(\cdot)$, motivated by the size of $\|e^{At}\|$ as a function of time. The first limiting case is $t \to \infty$. In this case, we consider the following asymptotic measure of convergence motivated by [36, Ch. 14]:

$$r_\infty(A) := \lim_{t\to\infty} t^{-1}\log\|e^{At}\| = \lambda_{\max}(A), \tag{9}$$

where we recall that $\lambda_{\max}(A)$ denotes the spectral abscissa of A (i.e., the largest real part of the eigenvalues of A, see Section II). The second limiting case is $t \to 0$. In this case, we consider the following measure of the initial growth rate of $\|e^{At}\|$:

$$\begin{aligned}r_0(A) := \frac{d}{dt}\|e^{At}\|\Big|_{t=0} &= \lim_{t\downarrow 0} t^{-1}\log\|e^{At}\|\\ &= \lambda_{\max}\left(\frac{A+A^\mathsf{T}}{2}\right),\end{aligned} \tag{10}$$

where $\lambda_{\max}\left(\frac{A+A^\mathsf{T}}{2}\right)$ is the numerical abscissa of A (see [36]). Motivated by these measures, we have the following two characterizations.

Proposition 5.5: ***(Fast agreement via spectral abscissa minimization)*** Assume that the optimization problem (8) is feasible, and let the performance measure be $r(\cdot) = r_\infty(\cdot)$. Any solution of the following optimization problem:

$$\begin{aligned}\min_{a\in\mathbb{R}^{|\mathcal{E}|}}\quad & \lambda_{\max}(\mathbf{A}_\mathcal{G}(a))\\ \text{s.t.}\quad & \mathbf{A}_\mathcal{G}(a)t_i = 0,\quad \tau_i^\mathsf{T}\mathbf{A}_\mathcal{G}(a) = 0,\qquad i\in\{1,\dots,k\},\end{aligned} \tag{11}$$

where t_i, τ_i are as in (7) is also a solution of (8). □

Proof: Since the optimization problem (8) is feasible, condition *(i)* of Theorem 5.3 guarantees that (11) is also feasible. Let a^* denote a solution of (11), and let $A = \mathbf{A}_\mathcal{G}(a^*)$. By construction, we have $A \in \mathcal{A}_\mathcal{G}$, while the two constraints in (11) guarantee that $\lim_{t\to\infty} e^{At} = W$, which shows that a^* is a feasible point for (8). The claim thus follows by noting that the cost functions of (8) and that of (11) coincide. ■

Proposition 5.5 allows us to recast the optimization problem (8) as a finite-dimensional search over the vector of parameters $a \in \mathbb{R}^{|\mathcal{E}|}$. Unfortunately, even though the constraints of (11) are linear equalities, finding a solution may be computationally burdensome because the objective function (i.e., the spectral abscissa) is not a convex function (in fact, it is not even Lipschitz, as shown in [37]). On the other hand, we have the following.

Proposition 5.6: ***(Fast agreement via numerical abscissa minimization)*** Assume that the optimization problem (8) is feasible, and let the performance measure be $r(\cdot) = r_0(\cdot)$. Any solution of the following convex optimization problem:

$$\begin{aligned}\min_{a\in\mathbb{R}^{|\mathcal{E}|}}\quad & \lambda_{\max}\left(\frac{\mathbf{A}_\mathcal{G}(a)+\mathbf{A}_\mathcal{G}(a)^\mathsf{T}}{2}\right)\\ \text{s.t.}\quad & \mathbf{A}_\mathcal{G}(a)t_i = 0,\quad \tau_i^\mathsf{T}\mathbf{A}_\mathcal{G}(a) = 0,\qquad i\in\{1,\dots,k\},\end{aligned} \tag{12}$$

where t_i, τ_i are as in (7), is also a solution of (8). □

Proof: The proof repeats the same steps as those in the proof of Proposition 5.5, by replacing the cost with (10). ■

In contrast with the spectral abscissa formulation (9), the cost function in (12) is always convex in the parameter a, and thus a global solution of (12) can be computed efficiently.

VI. Graph-theoretic conditions for agreement

Although Theorem 5.3 provides a way to construct agreement protocols that are commensurate with a given graph, the system of algebraic equations in its statement might not admit a solution for certain graph topologies (as illustrated, e.g., by Example 4.5). Motivated by this observation, in this section we provide necessary and sufficient conditions on the graph $\mathcal{G}$ that guarantee solvability of this set of equations. We begin with a necessary condition.

Proposition 6.1: ***(Graph-theoretic necessary conditions)*** Let $\mathcal{G}$ satisfy Assumption 1. There exists $a \in \mathbb{R}^{|\mathcal{E}|}$ such that the linear update $\dot{x} = \mathbf{A}_{\mathcal{G}}(a)x$ reaches a k-dimensional agreement on arbitrary weights only if

$$|\mathcal{E}| \geq k(n-1) + n. \tag{13}$$

□

Proof: It follows from the algebraic characterization in Theorem 5.3 that $\dot{x} = \mathbf{A}_{\mathcal{G}}(a)$ reaches an agreement if and only if the following set of algebraic equations hold

$$0 = \mathbf{A}_{\mathcal{G}}(a)t_i, \qquad i \in \{1, \dots, k\}, \tag{14a}$$

$$p_\ell = \sum_{\xi \in \mathcal{C}_\ell(\mathcal{G})} (-1)^{d(c)} \prod_{(i,j)\in\xi} a_{ij}, \quad \ell \in \{1, \dots, n-k\}. \tag{14b}$$

By letting $p = (p_1, \dots, p_{n-k})$, equation (14b) relates p and a by meas of a mapping

$$p = g(a), \tag{15}$$

where $g : \mathbb{R}^{|\mathcal{E}|} \to \mathcal{T}$ is a smooth mapping and $\mathcal{T}$ is a smooth manifold in $\mathbb{R}^{n-k}$. Since $g(\cdot)$ is a multilinear polynomial, it is immediate to verify that $g(\tilde{a}) = (\partial g/\partial a)|_{a=\tilde{a}} \cdot \tilde{a}$. By denoting in compact form $G_{\tilde{a}} = (\partial g/\partial a)|_{a=\tilde{a}} \in \mathbb{R}^{n-k\times|\mathcal{E}|}$, the system of equations (14) can thus be rewritten as

$$\begin{aligned} 0 &= \mathbf{A}_{\mathcal{G}}(a)t_i, \qquad i \in \{1, \dots, k\}, \\ p &= G_a a. \end{aligned} \tag{16}$$

The system of equations (16) to be solved consists of $nk+n-k$ equations with $|\mathcal{E}|$ unknowns and $n-k$ arbitrarily chosen real numbers $p_1, \dots, p_{n-k}$. It follows from the implicit function theorem [38, Thm. 9.28] that its generic solvability requires the following necessary condition: $|\mathcal{E}| \geq nk + n - k$, from which the statement follows. ■

Condition (13) can be interpreted as a lower bound on the minimal graph connectivity required to achieve agreement. The condition shows that the number of edges in $\mathcal{G}$ must grow at least linearly with k (when n is kept constant), or at least linearly with n (when k is kept constant). The necessary condition (13) has some important implications for agreement on two widely-studied topologies, namely, line and circulant digraphs; we discuss these cases in the following remarks.

Remark 6.2: ***(Agreement over circulant digraphs)*** Consider the directed graph with circulant topology illustrated in Fig. 4(a). In this case, $|\mathcal{E}| = 2n$, and thus (16) yields $k \leq \left\lfloor \frac{n}{n-1} \right\rfloor = 1$, provided that $3 \leq n < \infty$. This implies that, protocols of the form (2) with circulant communication digraphs can reach an agreement on arbitrary subspaces of dimension at most 1. More generally, circulant digraphs with in/out degree $\alpha \geq 1$ (illustrated in Fig. 4(b)) are characterized by $|\mathcal{E}| = (\alpha+1)n$, and thus can achieve agreements on subspaces of dimension up to

$$k \leq \left\lfloor \frac{\alpha n}{n-1} \right\rfloor = \alpha,$$

where the last inequality follows by using $3 \leq n < \infty$.

Conversely, by fixing k, condition (13) can also be used to determine the minimal in/out degree that is necessary to achieve an agreement. In fact, (13) with $|\mathcal{E}| = (\alpha+1)n$ gives the following necessary condition:

$$\alpha \geq \left\lfloor \frac{k(n-1)}{n} \right\rfloor = k - 1, \tag{17}$$

which shows that a necessary condition for agreement in that the in/out degree α must grow at least linearly with k. □

Remark 6.3: ***(Agreement over line digraphs)*** Consider the directed digraph with line topology illustrated in Fig. 4(c). In this case, $|\mathcal{E}| = n+2(n-1)$, and thus (16) yields $k \leq 2$. More generally, line digraphs with in/out degree α (see Fig. 4(d) for an illustration) have $|\mathcal{E}| = n + \alpha n - \frac{\alpha}{2}(\frac{\alpha}{2}+1)$, and thus can achieve agreement on subspaces of dimension up to

$$k \leq \left\lfloor \frac{\alpha n - \frac{\alpha}{2}(\frac{\alpha}{2}+1)}{n-1} \right\rfloor \leq \alpha,$$

where the last inequality follows by using $3 \leq n < \infty$. Conversely, to determine the smallest degree α that is necessary to agree on a subspace of dimension k (fixed), notice that (13) with $|\mathcal{E}| = n+\alpha n-\frac{\alpha}{2}(\frac{\alpha}{2}+1)$ gives the second order inequality

$$\alpha^2 + 2\alpha(1-2n) + 4k(n-1) \leq 0.$$

By noting that the discriminant satisfies $\Delta \leq 4(2n-k)^2$, a necessary condition for agreement is

$$\alpha \geq \frac{4n - 2 - 2(2n-k)}{2} = k - 1.$$

Hence, similarly to the circulant graph (see (17)), a necessary condition for agreement is that the in/out degree α must grow at least linearly with k. □

The following result provides a graph-theoretic characterization of graphs that can achieve agreement on arbitrary weights.

Proposition 6.4: ***(Graph-theoretic sufficient conditions)*** Let $\mathcal{G}$ be a graph that satisfies Assumption 1 and the necessary condition (13). If there exists a partitioning of the edge parameters $a = (\{a_{ij}\}_{(i,j)\in\mathcal{E}})$ into two disjoint sets $a_v = \{a_1, \dots a_{n-k}\}$ and $a_c = \{a_{n+1}, \dots a_{|\mathcal{E}|}\}$ such that:

(i) For all $\ell \in \{1, \dots, n-k\}$, there exists a Hamiltonian ℓ-decomposition, denoted by $\mathcal{C}^*_\ell$, such that $a_\ell \in \mathcal{C}^*_\ell$;

(ii) Any edge in $\mathcal{C}^*_\ell$ other than a_ℓ belongs to a_c,

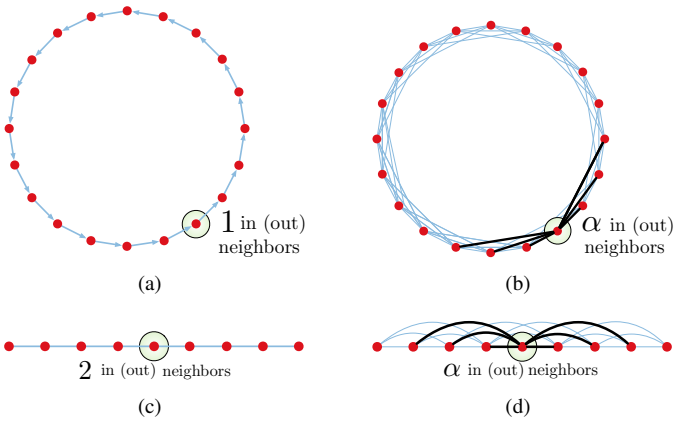

Fig. 4: (a) and (c) Circulant and line graphs, respectively. (b) Circulant graph with α in/out neighbors (when $\alpha = 1$, we recover the graph in (a)). (d) Line graph with α in/out neighbors (when $\alpha = 2$, we recover the graph in (c)). In all plots, edges with no arrows indicate bi-directional communication, and all nodes have self-cycles, which are omitted here for illustration purposes.

(iii) Any Hamiltonian ℓ-decomposition other than $\mathcal{C}_\ell^*$ that contains edges in a_v also contains at least one edge in a_c that does not appear in C_ℓ^*,

then, for any W as in Assumption 2, there exists $a \in \mathbb{R}^{|\mathcal{E}|}$ such that the linear update $\dot{x} = \mathbf{A}_\mathcal{G}(a)x$ reaches a k-dimensional agreement on W. □

Proof: We recall from the proof of Proposition 6.1 that $\dot{x} = \mathbf{A}_\mathcal{G}(a)$ reaches an agreement when the system of equations (16) is solvable, where $p_1, \dots, p_{n-k}$ are arbitrarily chosen real numbers. Thus, we seek solutions in a neighborhood of $p = 0$ by confining ourselves to the case $|\mathcal{E}| \geq nk + n - k$. By using the decomposition $a = a_c \cup a_v$, it is clear that solvability of (16) is equivalent to the existence of a particular point $a_c^* \in \mathbb{R}^{|\mathcal{E}|-n+k}$ for which the equations

$$\begin{aligned} 0 &= \mathbf{A}_\mathcal{G}(a_c^*, a_v) \cdot t_i, \qquad i \in \{1, \dots, k\}, \\ 0 &= p = g(a_c^*, a_v), \end{aligned} \tag{18}$$

(in the variable a_v) have a solution. Thus, existence of solutions is guaranteed provided that there exists a certain a_v^* such that (18) are satisfied and $(\partial g/\partial a_v)|_{(a_c,a_v)=(a_c^*,a_v^*)}$ is invertible.

Notice that $a_v = 0$ is a solution of (18), and thus when $a_c^* \neq 0$, the point $(a_c^*, 0)$ is a nontrivial solution of (16). Thus, we are left to show that there exists a particular choice of a_c^* such that

$$G(a_c^*, a_v) := \left.\frac{\partial g}{\partial a}\right|_{(a_c,a_v)=(a_c^*,a_v)} \in \mathbb{R}^{n-k \times n-k},$$

is invertible when $a_v = 0$. Precisely, we will show that there a choice of a_c^* such that $G(a_c^*, 0)$ is diagonally dominant. Thus, in what follows we provide a way to construct a_c^* by induction.

Let $a_c^{(1)} \in \mathbb{R}^{|\mathcal{E}|-n+k} \setminus \{0\}$ be fixed. Notice that condition *(i)* in the statement guarantees that there exists a nonzero product in entry $(1,1)$ of $G(a_c^{(1)}, a_v)$, and condition *(ii)* guarantees that such product is independent of a_v. Thus, by adopting the notation $G^{(1)}(a_c, a_v) = G(a_c, a_v)$, we have the partitioning:

$$G^{(1)}(a_c^{(1)}, a_v) = \begin{bmatrix} G_{11}^{(1)}(a_c^{(1)}) & G_{12}^{(1)}(a_c^{(1)}, a_v) \\ G_{21}^{(1)}(a_c^{(1)}, a_v) & G_{22}^{(1)}(a_c^{(1)}, a_v) \end{bmatrix},$$

where $G_{11}^{(1)} \in \mathbb{R}$, $G_{12}^{(1)} \in \mathbb{R}^{1 \times n-k-1}$, $G_{21}^{(1)} \in \mathbb{R}^{n-k-1 \times 1}$, $G_{22}^{(1)} \in \mathbb{R}^{n-k-1 \times n-k-1}$. By condition *(ii)* and provided that all entries of $a_c^{(1)}$ are nonzero, we have $G_{11}^{(1)}(a_c^{(1)}) \neq 0$. Moreover, either no element of a_v appears in any Hamiltonian 1- decomposition, in which case we have $G_{12}^{(1)}(a_c^{(1)}, a_v) = 0$ or, otherwise, by condition *(iii)* the corresponding product term contains one element in $a_c^{(1)}$ that does not appear in $G_{11}^{(1)}(a_c^{(1)})$. By choosing such element sufficiently small, there thus always exists a vector $\tilde{a} \in \mathbb{R}^{|\mathcal{E}|-n+k}$ such that $|G_{11}^{(1)}(\tilde{a})| > \|G_{12}^{(1)}(\tilde{a}, a_v)\|_1$, which guarantees diagonal dominance of the first row of $G^{(1)}(\tilde{a}, a_v)$.

For the inductive step, we notice that $G^{(i)}(a_c^{(i)}, a_v)$ is diagonally dominant if $G_{22}^{(i)}(a_c^{(i)}, a_v)$ is diagonally dominant. Thus, by defining $G^{(i+1)}(\cdot,\cdot) = G_{22}^{(i)}(\cdot,\cdot)$, $i \in \{1, \dots, n-1\}$ and by letting $a_c^{(i+1)} = \min\{\tilde{a}, a_c^{(i)}\}$ (entrywise minimum) at each iteration i, and by iterating the above argument, we conclude that $G(a_c^{(n-k)}, a_v)$ is diagonally dominant for any a_v. The claim thus follows by letting $a_c^* = a_c^{(n-k)}$ and by application of the inverse function theorem [38]. ■

Proposition 6.4 provides a set of graph-theoretic properties that are sufficient to guarantee that a certain graph can sustain agreement protocols on arbitrary weights W. The result identifies Hamiltonian decompositions as the fundamental component that guarantees existence of agreement protocols. Indeed, as shown in the proof, the existence of $n-k$ independent Hamiltonian decompositions in $\mathcal{G}$ guarantees that a can be chosen so that $n-k$ modes of $\mathbf{A}(a)$ are stable. Finally, we note that determining the Hamiltonian decompositions of a graph can be done efficiently in polynomial time $\mathcal{O}(n^2)$ by using, e.g., Palmer's Algorithm [39].

The usefulness of Proposition 6.4 depends largely on determining a partitioning of a into two disjoint sets of variables a_v and a_c. An algorithm to determine whether such partitioning exists can be constructed by using ideas similar to [40], where a_v and a_c are derived from a directed spanning tree of $\mathcal{G}$.

Remark 6.5: ***(Minimal graphs for agreement)*** It is worth noting that if $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ admits the set of Hamiltonian ℓ-decompositions $\mathcal{C}_\ell(\mathcal{G})$, then any graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ such that $\mathcal{V}' = \mathcal{V}$ and $\mathcal{E} \subset \mathcal{E}'$ has a set of Hamiltonian ℓ-decompositions $\mathcal{C}'_\ell(\mathcal{G})$ that satisfies $\mathcal{C}_\ell(\mathcal{G}) \subseteq \mathcal{C}'_\ell(\mathcal{G})$. In words, any graph obtained by adding edges to $\mathcal{G}$ admits a set of Hamiltonian ℓ-decompositions that includes $\mathcal{C}_\ell(\mathcal{G})$. It follows that if $\mathcal{G}$ admits agreement protocols on arbitrary weights, then any digraph obtained by adding edges to $\mathcal{G}$ will also admit agreement protocols on arbitrary weights. □

We conclude this section by demonstrating the applicability of Proposition. 6.4 through an example.

Example 6.6: ***(Illustration of Hamiltonian decomposition condition)*** Consider the communication graph illustrated in Fig. 5(a). The corresponding agreement protocol is given by:

$$\mathbf{A}_\mathcal{G}(a) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & a_{15} \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & 0 \\ a_{51} & 0 & 0 & a_{54} & a_{55} \end{bmatrix}.$$

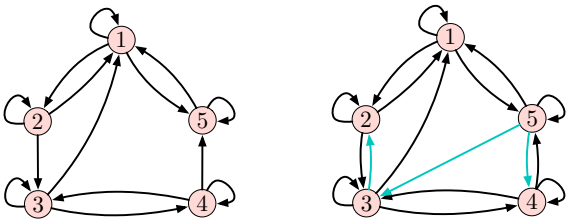

Fig. 5: (a) Example of graph that can sustain a 2-dimensional agreement. (b) Graph obtained by adding green edges to (a); this graph can sustain a 3-dimensional agreement. See Example 6.6.

By Proposition 6.1, a necessary condition for agreement is

$$k \leq \left\lfloor \frac{|\mathcal{E}| - n}{n-1} \right\rfloor = \left\lfloor \frac{9}{4} \right\rfloor = 2.$$

Thus, in what follows we fix $k = 2$. To illustrate the conditions of Proposition 6.4, for simplicity, we let $a_{22} = a_{33} = a_{44} = a_{55} = 0$ (according to Remark 6.5, if the graph without self-cycles has an independent set of Hamiltonian decompositions, then the graph obtained by adding these self-cycles will retain the same set of decompositions). With this choice, the set of all Hamiltonian ℓ-decompositions, $\ell \in \{1, \dots, n-k\}$, is:

$$\begin{aligned} \mathcal{C}_1 &= \{\{a_{11}\}\}, \\ \mathcal{C}_2 &= \{\{a_{12}, a_{21}\}, \{a_{34}, a_{43}\}, \{a_{15}, a_{51}\}\}, \\ \mathcal{C}_3 &= \{\{a_{11}, a_{34}, a_{43}\}, \{a_{13}, a_{21}, a_{32}\}\}. \end{aligned} \tag{19}$$

By selecting a_v and a_c as follows

$$\begin{aligned} a_v &= \{a_{11}, a_{12}, a_{13}\}, \\ a_c &= \{a_{51}, a_{54}, a_{21}, a_{32}, a_{34}, a_{43}, a_{15}\}, \end{aligned}$$

it is immediate to see that a set of Hamiltonian ℓ-decompositions that satisfies the conditions in Proposition 6.4 is:

$$\mathcal{C}_1^* = \{a_{11}\}, \quad \mathcal{C}_2^* = \{a_{12}, a_{21}\}, \quad \mathcal{C}_3^* = \{a_{13}, a_{21}, a_{32}\}.$$

Indeed, with this choice, the set of equations (14b) reads as:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -a_{21} & 0 \\ a_{34}a_{43} & 0 & -a_{21}a_{32} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} - \begin{bmatrix} 0 \\ \gamma \\ 0 \end{bmatrix},$$

where $\gamma = a_{34}a_{43} + a_{15}a_{51}$, which is generically solvable for any $(p_1, p_2, p_3) \in \mathbb{R}^3$.

To achieve agreements on subspaces of dimension $k = 3$, consider the graph in Fig. 5(b) obtained by adding edges to the graph in Fig. 5(a). The necessary condition (13) yields

$$k \leq \left\lfloor \frac{|\mathcal{E}| - n}{n-1} \right\rfloor = \frac{12}{4} = 3,$$

which is satisfied. The set of relevant Hamiltonian decompositions (19) modifies to:

$$\begin{aligned} \mathcal{C}_1 &= \{\{a_{11}\}\}, \\ \mathcal{C}_2 &= \{\{a_{12}, a_{21}\}, \{a_{34}, a_{43}\}, \{a_{15}, a_{51}\}\}, \{a_{23}, a_{32}\}\}, \end{aligned}$$

By selecting a_v and a_c as follows

$$\begin{aligned} a_v &= \{a_{11}, a_{12}\}, \\ a_c &= \{a_{13}, a_{23}, a_{45}, a_{35}, a_{51}, a_{54}, a_{21}, a_{32}, a_{34}, a_{43}, a_{15}\}, \end{aligned}$$

a set of Hamiltonian ℓ-decompositions that satisfies Proposition 6.4 is:

$$\mathcal{C}_1^* = \{a_{11}\}, \qquad \mathcal{C}_2^* = \{a_{12}, a_{21}\},$$

thus showing that the sufficient conditions also holds. □

VII. Tracking dynamics for agreement

In analogy with classical consensus processes [41], agreement protocols can be modified to track the oblique projection of a time-varying forcing signal $u(t)$. Specifically, given a graph $\mathcal{G}$, consider the network process

$$\dot{x} = \mathbf{A}_{\mathcal{G}}(a)x + \dot{u}, \qquad x(0) = u(0), \tag{20}$$

where a is chosen so that $\mathbf{A}_{\mathcal{G}}(a)$ is an agreement matrix (as in Theorem 5.3) and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a continuously-differentiable function. In this framework, the i-th entry of $\dot{u}$ is known only by agent i, and the objective is to guarantee that $x(t)$ tracks a k-dimensional projection $Wu(t)$ of $u(t)$ asymptotically. The protocol (20) can be interpreted as a generalization of the dynamic average consensus algorithm [41], where the communication matrix is an agreement matrix instead than a Laplacian. The following result characterizes the transient behavior of (20).

Proposition 7.1: ***(Convergence of dynamic agreement protocol)*** Consider the update (20) and let $\mathbf{A}_{\mathcal{G}}(a)$ be an agreement protocol on W as in Theorem 5.3. Then, for all $t \geq 0$:

$$\|x(t) - Wu(t)\| \leq e^{-\hat{\lambda}t}\|x(0) - Wu(0)\| + \frac{1}{\hat{\lambda}} \sup_{0 \leq \tau \leq t} \|\dot{u}(\tau)\|, \tag{21}$$

where $\hat{\lambda} = \lambda_{\max}\left(\frac{\mathbf{A}_{\mathcal{G}}(a) + \mathbf{A}_{\mathcal{G}}(a)^{\mathsf{T}}}{2}\right)$. □

Proof: The proof is inspired from [41, Thm. 2] and extends the result to non Laplacian-based protocols and non weight-balanced digraphs. Let W be decomposed as in (6), and consider the following decompositions for T_W and T_W^{-1}:

$$T_W = \begin{bmatrix} T_1 & T_2 \end{bmatrix}, \qquad (T_W^{-1})^{\mathsf{T}} = \begin{bmatrix} U_1 & U_2 \end{bmatrix}, \tag{22}$$

where $T_1, U_1 \in \mathbb{R}^{n \times k}$ and $T_2, U_2 \in \mathbb{R}^{n \times n-k}$. Let $e = x - Wu$ denote the tracking error, and consider the change of variables $\bar{e} = T_W^{-1}e$. In the new variables:

$$\begin{aligned} \dot{\bar{e}} &= T_W^{-1}(\dot{x} - W\dot{u}) \\ &= T_W^{-1}AT_W\bar{e} + T_W^{-1}AWu + T_W^{-1}\dot{u} - T_W^{-1}W\dot{u}, \\ &= T_W^{-1}AT_W\bar{e} + T_W^{-1}\dot{u} - T_W^{-1}W\dot{u}, \end{aligned}$$

where the last identity follows by using (6), which implies $AW = 0$. By substituting (22) and by noting that $T^{-1}W = [U_1 \ 0]^{\mathsf{T}}$:

$$\begin{aligned} \dot{\bar{e}} &= \begin{bmatrix} U_1^{\mathsf{T}}AT_1 & U_1^{\mathsf{T}}AT_2 \\ U_2^{\mathsf{T}}AT_1 & U_2^{\mathsf{T}}AT_2 \end{bmatrix} \bar{e} + \begin{bmatrix} 0 \\ U_2^{\mathsf{T}} \end{bmatrix} \dot{u} \\ &= \begin{bmatrix} 0 \\ U_2^{\mathsf{T}}AT_2 \end{bmatrix} \bar{e} + \begin{bmatrix} 0 \\ U_2^{\mathsf{T}} \end{bmatrix} \dot{u}, \end{aligned} \tag{23}$$

where the last inequality follows by noting that $0 = U_1^{\mathsf{T}}AT_1 = U_1^{\mathsf{T}} = AT_1$ according to condition *(i)* in Theorem 5.3.

Next, decompose $e = (e_1, e_2)$ and $\bar{e} = (\bar{e}_1, \bar{e}_2)$, where $e_1, \bar{e}_1 \in \mathbb{R}^k$ and $e_2, \bar{e}_2 \in \mathbb{R}^{n-k}$, and notice that the following identities hold:

$$\bar{e}_2 = U_2^\mathsf{T} e, \qquad e = T_2 \bar{e}_2. \tag{24}$$

The first identity follows immediately from (22), while the second follows from (22) and $\bar{e}_1(t) = 0$ at all times. To see that $\bar{e}_1(t) = 0\ \forall t \geq 0$, notice that $\bar{e}_1(0) = U_1^\mathsf{T}(x(0) - u(0)) = 0$ thanks to the initialization (20), and $\dot{\bar{e}}_1 = 0$ according to (23). By using (24), we conclude that $\dot{e} = Ae + \dot{u}$, from which (21) follows by noting that

$$e(t) = \exp(At) \cdot e(0) + \int_0^t \exp(A(t-\tau)) B \dot{u}(\tau) d\tau,$$

and by using $\| \exp(At) \| \leq \exp\left(-\lambda_{\max}\left(\frac{\mathbf{A}_\mathcal{G}(a) + \mathbf{A}_\mathcal{G}(a)^\mathsf{T}}{2} \right) t \right)$. ∎

VIII. Applications and Numerical Validation

In this section, we illustrate our theoretical findings via numerical simulations. We consider two application scenarios.

Applications to Linear Regression. We consider a problem in distributed computation characterized by a regression model of the form $y = H\theta$, where $H \in \mathbb{R}^{n \times k}, n > k$ and θ is an unknown parameter. We assume that each agent i can sense the i-the entry of vector y, denoted by y_i, and is interested in computing the point $\hat{y}_i$ that is the closest to y_i according to the regression model. To this end, we consider the following regression problem:

$$\theta_{\text{ls}} := \arg\min_\theta \| H\theta - y \|. \tag{25}$$

It is well-known that θ_{ls} can be obtained by setting the gradient of $\|H\theta - y\|^2$ equal to zero, which yields $0 = \nabla_\theta \|H\theta - y\|^2 = 2H^\mathsf{T}H\theta - 2H^\mathsf{T}y$, and thus when $H^\mathsf{T}H$ is invertible we have $\theta_{\text{ls}} = (H^\mathsf{T}H)^{-1}H^\mathsf{T}y$. Hence, the desired point $\hat{y} = H\theta_{\text{ls}}$ to be computed by the agents is the orthogonal projection of y onto $\text{Im}(H)$. For illustration purposes, in our simulations we consider the case $n = 50$ and $k = 2$ and we computed an agreement protocol by using the optimization problem (12) with weights matrix $W = H(H^\mathsf{T}H)^{-1}H^\mathsf{T}$ implemented on the circulant graph in Fig. 4(d) with $\alpha = k = 4$ in/out neighbors. Fig. 6(b) shows the time-evolution of the agents states and Fig. 6(a) shows the sampling points y and asymptotic estimates $\hat{y}$ in comparison with the true regression model. As expected, the distributed algorithm (1) allows the agents to cooperatively converge to n points corresponding to the best approximation of the collected data samples. Fig. 6(b) shows the trajectories of the agents states. As expected, at convergence the states of the agents do not coincide, instead, the agreement state is a $n = 50$-dimensional vector constrained on a 2-dimensional space.

Applications to robotic formation control. We next illustrate how agreement protocols can be applied to solve formation control problems [9] in multi-agent robotic networks. Consider a team of $n = 8$ single-integrator robots initially arranged at equal intervals around a unit circle (grey lines in Fig. 7(a)-(c)). By using $x-$ and $y-$coordinates to describe the robots positions, we use $x_0 = (\cos(0), \sin(0), \cos(\frac{\pi}{4}), \sin(\frac{\pi}{4}), \ldots, \cos(\frac{7\pi}{4}), \sin(\frac{7\pi}{4})) \in \mathbb{R}^{16}$. Fig. 7 illustrates the trajectories of the robots obtained by using the 2D agreement protocol

$$\dot{x} = (A \otimes I_2)x, \qquad x(0) = x_0,$$

using the circulant communication graph illustrated in Fig. 7(b) with $\alpha = 4$. For comparison, in Fig. 7(a)-(d) we illustrate the trajectories obtained by a consensus algorithm described by weights $W = \frac{1}{n}\mathbb{1}\mathbb{1}^\mathsf{T}$. As expected, the robots meet at the point $(0, 0)$ thus achieving rendezvous [9]. In Fig.7(b)-(e), we illustrate the trajectories resulting from running an agreement protocol (computed by solving (11)) with weights $W = \Pi_\mathcal{M}$, where $\Pi_\mathcal{M}$ is the orthogonal projection onto $\mathcal{M} = \ker(M_1) \subset \mathbb{R}^8$ with

$$M_1 = \begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

Matrix M_1 encodes attraction and repulsion forces between certain robots at convergence. Indeed, by recalling that the agreement value is $x(\infty) = \Pi_\mathcal{M} x(0)$, it follows that at steady state the agents positions satisfy $M_1 x(\infty) = 0$. Hence, the rows of M_1 are interpreted as algebraic constraints on the asymptotic value of agreement. From Fig.7(b), which reports the corresponding time-evolution of the $x-$ and $y-$coordinates of the robots, we observe that the robots asymptotically achieve a formation that is characterized by a 2-dimensional subspace. Finally, Fig.s 7(d)-(f) illustrate the trajectories of the robots generated by an agreement protocol (computed by solving (11)) where the weights are described by an oblique projection $W = \Pi_{\mathcal{M},\mathcal{N}}$, where $\mathcal{M} = \ker(M_1) \subset \mathbb{R}^8$ and $\mathcal{N} = \text{Im}(N_1) \subset \mathbb{R}^8$ with

$$N_1^\mathsf{T} = \begin{bmatrix} -1 & 5 & 5 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 5 & 5 & -1 \end{bmatrix}.$$

The use of an oblique projection can be interpreted as an non-homogeneous weighting for the vector that defines the final configuration. Indeed, as shown by the figure, in this case the robots no longer meet "half way", instead, robots 2 and 3 [respectively, 6 and 7] travel a longer distance as opposed to robots 1 and 4 [respectively, 5 and 8]).

IX. Conclusions

We studied the problem of k-dimensional agreement in multi-agent systems, whereby the agents are interested in agreeing on a quantity that is confined to a certain (k-dimensional) subspace without necessarily agreeing on common quantities. We showed that, in general, agreement protocols require communication graphs that are more connected that those needed to achieve consensus, which is recovered as a subcase with $k = 1$. To this end, we provided both algebraic and graph-theoretic conditions to identify graphs that can sustain agreement protocols. Although our approach provides a graph-theoretic condition that is sufficient to conclude that a certain graph can sustain an agreement protocol, we conjecture this class is much larger in practice. Thus, this work opens the opportunity for several directions for future works, including the derivation of less-stringent graph-theoretic conditions, the use and study of nonlinear agreement protocols, as well the

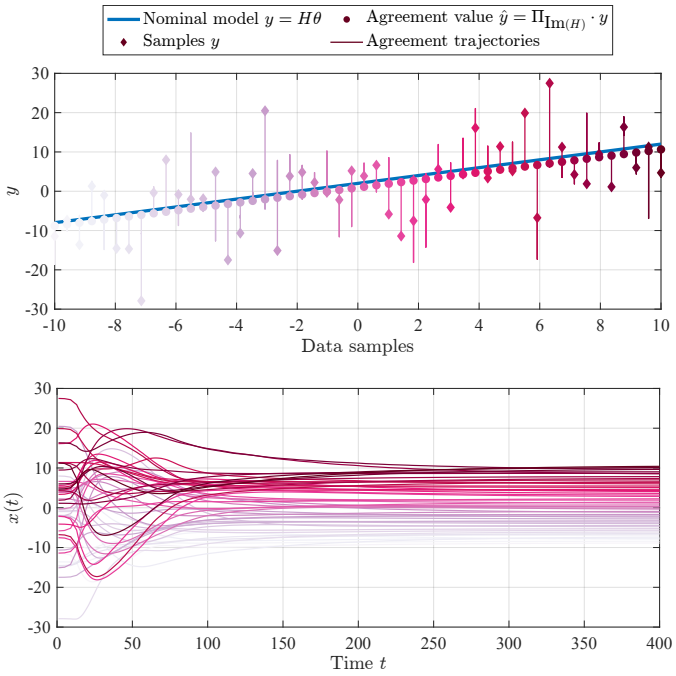

Fig. 6: Application of agreement protocols to solve regression problems. Each agent can measure a sample y_i (represented by diamond markers) and cooperatively computes the projection of $\hat{y}_i$ onto the range of the regression matrix H (represented by circle markers), see (25). In (a), continuous line illustrate the time- evolution of the states of (1). (b) Time-evolution of the trajectories of (1).

development of efficient algorithms to construct agreement protocols in both centralized and distributed fashions.

References

[1] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis, "Convergence in multiagent coordination, consensus, and flocking," in *IEEE Conf. on Decision and Control*, 2005, pp. 2996–3000.

[2] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[3] W. Ren, R. W. Beard, and E. M. Atkins, "A survey of consensus problems in multi-agent coordination," in *American Control Conference*, Portland, OR, Jun. 2005, pp. 1859–1864.

[4] G. Malewicz, M. H. Austern, A. J.-C. Bik, J. C. Dehnert, I. Horn, N. Leiser, and G. Czajkowski, "Pregel: a system for large-scale graph processing," in *International Conference on Management of data*, 2010, pp. 135–146.

[5] D. Peleg, *Distributed computing: a locality-sensitive approach*. SIAM, 2000.

[6] H.-L. Choi, L. Brunet, and J. P. How, "Consensus-based decentralized auctions for robust task allocation," *IEEE Transactions on Robotics*, vol. 25, no. 4, pp. 912–926, 2009.

[7] L. Zhao and A. Abur, "Multi area state estimation using synchronized phasor measurements," *IEEE Transactions on Power Systems*, vol. 20, no. 2, pp. 611–617, 2005.

[8] F. Pasqualetti, R. Carli, A. Bicchi, and F. Bullo, "Distributed estimation and detection under local information," in *IFAC Workshop on Distributed Estimation and Control in Networked Systems*, Annecy, France, Sep. 2010, pp. 263–268.

[9] K.-K. Oh, M.-C. Park, and H.-S. Ahn, "A survey of multi-agent formation control," *Automatica*, vol. 53, pp. 424–440, 2015.

[10] J. N. Tsitsiklis, "Problems in decentralized decision making and computation," Ph.D. dissertation, Massachusetts Institute of Technology, Nov. 1984, available at `http://web.mit.edu/jnt/www/Papers/PhD-84-jnt.pdf`.

[11] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, 2003.

[12] M. Cao, A. S. Morse, and B. D. O. Anderson, "Reaching a consensus in a dynamically changing environment - convergence rates, measurement delays and asynchronous events," *SIAM Journal on Control and Optimization*, vol. 47, no. 2, pp. 601–623, 2008.

[13] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.

[14] J. M. Hendrickx and J. N. Tsitsiklis, "Convergence of type-symmetric and cut-balanced consensus seeking systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 1, pp. 214–218, 2013.

[15] X. Chen, M.-A. Belabbas, and T. Başar, "Consensus with linear objective maps," in *IEEE Conf. on Decision and Control*, 2015, pp. 2847–2852.

[16] A. Olshevsky and J. N. Tsitsiklis, "Convergence speed in distributed consensus and averaging," *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 33–55, 2009.

[17] L. Xiao and S. Boyd, "Fast linear iterations for distributed averaging," *Systems & Control Letters*, vol. 53, pp. 65–78, 2004.

[18] D. Dolev, N. A. Lynch, S. Pinter, E. W. Stark, and W. E. Weihl, "Reaching approximate agreement in the presence of faults," *Journal of the ACM*, vol. 33, no. 3, pp. 499–516, 1986.

[19] A. Khanafer, B. Touri, and T. Başar, "Robust distributed averaging on networks with adversarial intervention," in *IEEE Conf. on Decision and Control*, 2013, pp. 7131–7136.

[20] A. Nedić, A. Ozdaglar, and P. A. Parrilo, "Constrained consensus and optimization in multi-agent networks," *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 922–938, 2010.

[21] F. Morbidi, "Subspace projectors for state-constrained multi-robot consensus," in *IEEE Int. Conf. on Robotics and Automation*, 2020, pp. 7705–7711.

[22] J. Chen and A. H. Sayed, "Distributed pareto optimization via diffusion strategies," *IEEE Journal of Selected Topics in Signal Processing*, vol. 7, no. 2, pp. 205–220, 2013.

[23] S. Ahmadizadeh, I. Shames, S. Martin, and D. Nešić, "On eigenvalues of laplacian matrix for a class of directed signed graphs," *Linear Algebra and its Applications*, vol. 523, pp. 281–306, 2017.

[24] W. Li and H. Dai, "Cluster-based distributed consensus," *IEEE Transactions on Wireless Communications*, vol. 8, no. 1, pp. 28–31, 2009.

[25] G. Bianchin, A. Cenedese, M. Luvisotto, and G. Michieletto, "Distributed fault detection in sensor networks via clustering and consensus," in *IEEE Conf. on Decision and Control*, Osaka, Japan, Dec. 2015, pp. 3828–3833.

[26] S. Roy, "Scaled consensus," *Automatica*, vol. 51, pp. 259–262, 2015.

[27] R. Diestel, *Graph Theory*, 2nd ed., ser. Graduate Texts in Mathematics. Springer, 2000, vol. 173.

[28] C. T. Lin, "Structural controllability," *IEEE Transactions on Automatic Control*, vol. 19, no. 3, pp. 201–208, 1974.

[29] K. Röbenack and K. J. Reinschke, "Digraph based determination of jordan block size structure of singular matrix pencils," *Linear Algebra and its Applications*, vol. 275-276, pp. 495 – 507, 1998.

[30] A. Galántai, *Projectors and projection methods*. Springer Science & Business Media, 2003, vol. 6.

[31] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, Jan 2007.

[32] F. Bullo, J. Cortés, and S. Martínez, *Distributed control of robotic networks: a mathematical approach to motion coordination algorithms*. Princeton University Press, 2009.

[33] K. J. Reinschke, "Graph-theoretic approach to symbolic analysis of linear descriptor systems," *Linear Algebra and its Applications*, vol. 197, pp. 217–244, 1994.

[34] D. Cox, J. Little, and D. OShea, *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. Springer Science & Business Media, 2013.

[35] M. Kreuzer and L. Robbiano, *Computational commutative algebra*. Springer, 2000, vol. 1.

[36] L. N. Trefethen and M. Embree, "Spectra and pseudospectra: The behavior of nonnormal matrices and operators," in *Spectra and Pseudospectra*. Princeton university press, 2005.

[37] M. L. Overton and R. S. Womersley, "On minimizing the special radius of a nonsymmetric matrix function: Optimality conditions and duality theory," *SIAM Journal on Matrix Analysis and Applications*, vol. 9, no. 4, pp. 473–498, 1988.

[38] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., ser. International Series in Pure and Applied Mathematics. McGraw-Hill, 1976.

[39] E. M. Palmer, "The hidden algorithm of Ore's theorem on Hamiltonian cycles," *Computers & Mathematics with Applications*, vol. 34, no. 11, pp. 113–119, 1997.

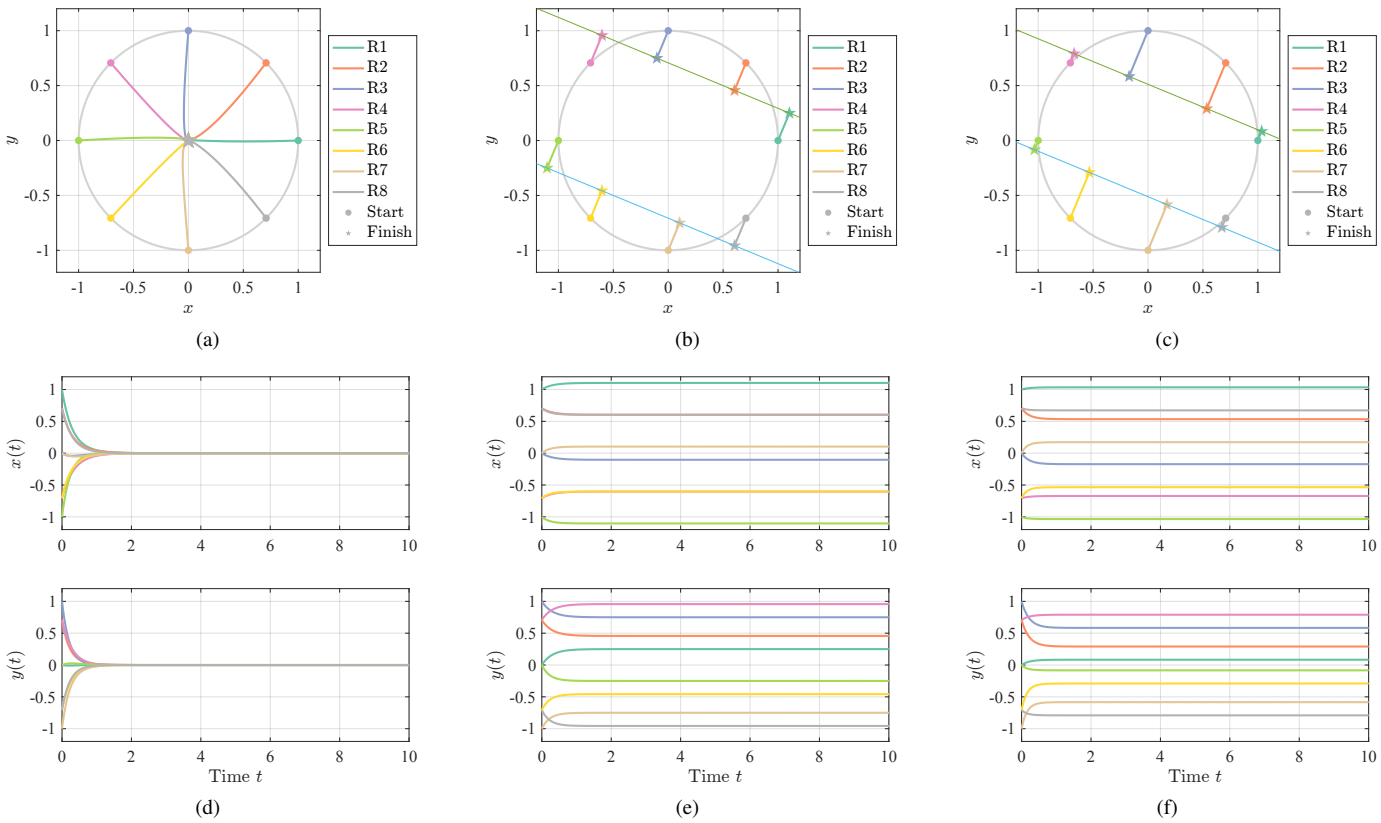

Fig. 7: (a)-(c) Time evolution of the positions of the 8 robots and (d)-(f) trajectories of the $x-$ and $y-$coordinates. (a) and (d) Consensus protocol, which allows the robots to achieve rendezvous. (b) and (e) Agreement protocol on an orthogonal projection onto $\ker(M_1)$. (c) and (f) Agreement on an oblique projection on $\ker(M_1)$ along $\mathrm{Im}(N_1)$.

[40] A. Sefik and M. E. Sezer, "Pole assignment problem: a structural investigation," *International Journal of Control*, vol. 54, no. 4, pp. 973–997, 1991.

[41] S. S. Kia, B. Van Scoy, J. Cortés, R. A. Freeman, K. M. Lynch, and S. Martínez, "Tutorial on dynamic average consensus: The problem, its applications, and the algorithms," *IEEE Control Systems Magazine*, vol. 39, no. 3, pp. 40–72, 2019.