Microeconomics Exercises

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July 19, 2023

1 Rational Choice

1. Let $X = R$. Consider an individual whose attitude toward the alternatives is "the larger, the better." However, he finds it impossible to determine whether x is greater than y unless the difference is at least 1:

$$x \succeq y \iff x \geq y - 1$$

Are preferences complete?

ANSWER: Completeness means that $\forall x, y \in X$ or $x \succeq y$ or $y \succeq x$ or both. This is plainly obtained by X being equal to R, which is, by definition, an ordered set. Then, it is possible to find two numbers s.t. $x \geq y - 1$, or $y \geq x - 1$, or equal.

Are preferences transitive?

ANSWER: Let's see a counterexample. Take $x = 0.5$, $y = 1$, $z = 2$. Then $x \geq y - 1$, since $0.5 \geq 1 - 1$. $y \geq z$, since $1 \geq 2 - 1$. But $x \ngeq z$, since $0.5 < 2 - 1$. Therefore, the preferences are not transitive. So then, they are not rational.

2. Two friends, L and M, have complete and transitive $\succeq_L$ and $\succeq_M$ preferences on X. Besides, their "group preference" is the following:

$$x \succeq^* y \quad \text{if} \quad x \succeq_L y \quad \text{or} \quad x \succeq_M y$$

Is $\succeq^*$ complete?

ANSWER: Completeness means $x \succeq^* y$ or $y \succeq^* x$ or both, $\forall x, y \in X$. Let's see $x \succeq^* y$. This can be easily detected by the following individual preferences, according to the definition of "group preferences":

- $x \succeq_L y$
- $x \succeq_M y$
- $x \succeq_L y$ and $x \succeq_M y$

Note that by definition, $\succeq_L$ and $\succeq_M$ are complete. So then, the argument is symmetric for $y \succeq^* x$.

Need $\succeq^*$ to be transitive? Provide an example.

ANSWER: To see this point, assume the following individual preferences for L and M.

- $x \succeq_L y$, $y \succeq_L z$ and $x \succeq_L z$
- $z \succeq_M y$, $y \succeq_M x$ and $z \succeq_M x$

Both these preference relations satisfy transitivity. A group preference is the following: $z \succeq^* x \succeq^* y \succeq^* z$. Transitivity is violated, still, the group preferences are consistent with the definition above, and the group reaches a decision.

3. Let $(\mathbf{B}, C(.))$ be a Choice Structure where $\mathbf{B}$ includes all non empty subsets of X, $C(B) \neq \emptyset$ $\forall B \in \mathbf{B}$ and X is nonempty. $C(.)$ is said to be distributive if:

$$\forall B, B' \in \mathbf{B} : C(B) \cap C(B') \neq \emptyset \Rightarrow C(B) \cap C(B') = C(B \cap B')$$

Suppose that $(\mathfrak{B}, C(.))$ satisfies WARP. Can we say that C(.) is distributive?

ANSWER: First, define WARP. Let $B, B' \in \mathbf{B}$, and $x, y \in B$ and B'. If $x \in C(B)$ and $y \in C(B')$, then $x \in C(B')$. To see if $C(.)$ is distributive, the reasoning goes as follows. From WARP, it is apparent that both $C(B)$ and $C(B')$ are not empty. Still, even their intersection is not empty since $C(B)$ and $C(B')$ have at least one element in common, namely x. Furthermore, x belongs to B and B' and, therefore, to the choice set of their intersection, i.e. $C(B \cap B')$. So then, $C(B) \cap C(B') \neq \emptyset \Rightarrow C(B) \cap C(B') = C(B \cap B')$.

Suppose that $C(.)$ is distributive. Can we say that $(\mathbf{B}, C(.))$ satisfies WARP?

ANSWER: This is not necessarily true. To see it, let's define a C(.) that fails to satisfy WARP but it is still distributive according to the above definition. For instance, take $\mathbf{B} = \{\{x, y, z\}\{x, z\}\{z, y\}\{x, y\}\}$ and define the following $C(.)$s

- $C(\{x, y, z\}) = \{y\}$ (B)
- $C(\{x, z\}) = \{z\}$ (B')
- $C(\{z, y\}) = \{y\}$ (B")
- $C(\{x, y\}) = \{x\}$ (B"')

It is apparent that such $C(.)$ does not satisfy WARP, since x is preferred to y when $B''' = \{x, y\}$ but y is preferred to x when $B = \{x, y, z\}$. However, note that C(.) satisfies distributivity as previously defined. Take $C(\{x, y, z\})$ and $C(\{z, y\})$. Then $C(B) \cap C(B'') \neq \emptyset \Rightarrow C(B) \cap C(B'') = C(B \cap B'')$. Indeed:

$$C(\{x, y, z\}) \cap C(\{z, y\}) = C(\{x, y, z\} \cap \{z, y\}) = \{y\} \cap \{y\} = \{y\}$$

4. Consider the following preferences on R^2_+.

$$x \succeq_A y \iff x_1 \geq y_2 \text{ and } x_2 \geq y_2; \qquad x \succeq_B y \iff \exists l : x_l \geq y_l$$

For each preference relation, shows if it is complete or not and transitive or not, providing a counterexample.

ANSWER: Let's start with $\succeq_A$. Take $x = (3, 2)$ and $y = (2, 3)$. It is apparent that completeness it is not satisfied since $x_1 > y_1$ but $x_2 < y_2$. Another way of seeing this is to think of x and y as vectors. One is greater than the other only if all its components are greater than the other. Again, they are equal only if all their components are equal. Otherwise, they simply cannot be compared. Still, transitivity it is easily satisfied since for three vectors x, y, z, if $x \geq y$ and $y \geq z$, by the preference relation defined above, $x \geq z$.

Let's see now $\succeq_B$. Contrary to the above, now, in order to have x be preferred to y, it is sufficient to have one component greater than another (so they are not "really" vectors now). Therefore, it is apparent that the relationship is complete. However, it is not transitive anymore. To see this, it is sufficient to write three "vectors" such that $x_1 \geq y_1$, $y_1 \geq z_1$, but $z_2 \geq x_2$. Say, $(3, 1), (2, 1), (1, 3)$. Now, by the definition above $x \geq y$, since $x_1 \geq y_1$, $y \geq z$, since $y_1 \geq z_1$, but $z \geq x$, because $z_2 \geq x_2$.

Because these preferences are not complete and transitive, they are not rational.

5. We are interested in establishing the existence of a utility function for any rational preference relation of a finite set X. First, let's show that any set $A \subseteq X$ has a minimal element (i.e. a $y \in A$ s.t. $y \preceq x, \quad \forall x \in A$). Note also that if A has only one element, then this element is the minimal one.

a) Take a set A_n with n-elements. Assume that it has a minimal element. Show that also $A_{n+1} = A_n \cup \{x_{n+1}\}$ has a minimal element.

ANSWER: To see this, let's take an $y \in A$, where y is the minimal point of A. Since A_{n+1} is by construction nothing else than A_n "plus" a point x_{n+1}, we face only three options:

- y is still the minimal point of A_{n+1}, since $y \preceq x_{n+1}$;
- x_{n+1} is the new minimal point, since $y \succeq x_{n+1}$;
- the set A_{n+1} has now two minimal points, since $x \sim y$.

In any case, the new set A_{n+1} has at least one minimal point.

b) Explain why you have just shown, by induction, that each finite set $A \subseteq X$ has a minimal element. Second, construct a utility function in the following way. First, since X has some minimal elements, define X_1 their set. If $x \in X_1$, then define $U(x) = 1$. Take $X \setminus X_1$. If it is not empty, define a new set X_2 as the set of minimal elements of $X \setminus X_1$. If $x \in X_2$ then $U(x) = 2$. Repeat until there are no elements left.

ANSWER: The same reasoning as above can be extended to any set $\subset X$, which is, by construction, finite. It is sufficient to assume that A_n, our first example, is a set made up only of one element. Therefore A_{n+1} has two elements, and so on. This is proven by induction.

c) Why do you know that this repetition will stop after a certain number of steps? How many steps can there be? When will you reach the maximum number of steps?

ANSWER: The process described above will stop after n steps, where n is the number of elements of X. However, it could also have less than n steps, whereas for some subset of X there are more than one minimal point. In any case, it cannot have more than n steps.

d) Now take the utility functions and show that:

- If $U(x) > U(y)$ then $x \succ y$
- If $U(x) = U(y)$ then $x \sim y$
- If $x \succ y$, then $U(x) > U(y)$
- If $x \sim y$, then $U(x) = U(y)$

ANSWER: By construction, we know that $U(.) = k$ where k is an index of the set X_k, the set of the minimal points of $X \setminus X_1 \setminus X_2 \cdots \setminus X_k \setminus X_{k+1} \ldots$. That is, to X_1 belong the minimal points of X, to X_2 the minimal points of $X \setminus X_1$, and so on; Define $U(x) = k+1$ and $U(y) = k$. Then, x belongs to the set of minimal points after $k+1$ steps. Since y belongs to the set of minimal points after only k steps, by definition of "minimal point" $x \succ y$ (that is, y is a minimal point of a set which comprises also x). Note that exactly the same reasoning applies to the case of $U(x) = U(y)$, with the difference that now they are both the minimal points of the same set.

Let's see now the last two cases. $x \succ y$ means that y is a minimal element of the set which comprises x and y. Therefore, it is "extracted" from that set before x, and the value of $U(y)$ is lower than the value of $U(x)$. The same reasoning applies to the case of $x \sim y$.

2 Walrasian Demand

1. Assume Walras' Law and WARP hold. Show that demand must then be homogenous of degree zero

ANSWER: First, Let's formally define Walras's Law, WARP, and Homogeneity of Degree Zero.

- Walras' Law (MWG, 2E2): $x(p, w)$ satisfies WL if $\forall p \gg 0$ and $w > 0$, we have $p \cdot x = w \; \forall x \in x(p, w)$
- WARP (MWG, 2F1) : $x(p, w)$ satisfies the WARP if the following property holds: for any two (p, w) and (p', w'): if $p \cdot x(p', w') \leq w$ and $x(p', w') \neq x(p, w) \implies p' \cdot x(p, w) > w'$.

- HDZ (MWG, 2E1): $x(p,w)$ satisfies HDZ if $x(\alpha p, \alpha w) = x(p,w)$ for all p, w and $\alpha > 0$.

In a nutshell, Walras' Law implies that each consumer spends all his income. WARP is a consistency requirement and means that if a certain bundle $x(p', w')$ is disposable when the price is p, and wealth is w but $x(p,w)$ is chosen over it (formally, $x(p', w') \in B_p, w$ but is not on the budget line. That is, $\leq w$), then, the chosen $x(p,w)$ won't be the consumer's choice at the price wealth combination (p', w') if and only if it is not affordable (formally, if $p' \cdot x(p,w) > w'$).

Finally, HDZ simply states that if p and w change of the same amount, say α, then the demand function does not change.

From WL, we know that the choice of a consumer rests on his budget line. So let's show what happens to WARP if demand is not HDZ. A not HDZ demand implies that a change by a positive factor α of p and w determines a change of at least α of the demand. So then, $x(\alpha p, \alpha w) = \alpha x(p,w)$ or more.

Assume (p,w) and (p', w'), where the second is $(\alpha p, \alpha w)$. Then $x(\alpha p, \alpha w) \neq x(p,w)$. Furthermore, $p \cdot \alpha x(p,w) \leq w$ by the assumption of WARP. But then we can also write $\alpha p \cdot x(p,w) \leq \alpha w$, since α is positive. Then WARP is violated. Indeed, with the price-wealth combination (p,w), to the demand $\alpha x(p,w)$ is preferred $x(p,w)$. But with the price wealth combination $(\alpha p, \alpha w)$, the 'old' demand $x(p,w)$ is disposable (it is inside the budget set), but it is chosen $\alpha x(p,w)$.

2. Filippo consumes only two goods: apples (A) and balloons (B). Filippo's choices in the last two days are depicted in the following table:

	Q_{day1}	P_{day2}	Q_{day2}	P_{day2}
Apples	3	1	5	1
Balloons	1	1	$\frac{1}{2}$	2

a) Are Filippo's choices consistent with the weak axiom of revealed preferences?

ANSWER: seeing this is simple since we have the values associated with Filippo's choices. We have the following price-wealth combinations: $x_1 = (3,1), p_1 = (1,1), x_2 = (5, \frac{1}{2}), p_2 = (1,2)$. Furthermore $w_1 = x_1 \cdot p_1$ and $w_2 = x_2 \cdot p_2$.

First, $x_1 \neq x_2$. To verify WARP, note that a simple way is to check that these two conditions are not satisfied together: $p_1 \cdot x_2 \leq p_1 \cdot x_1$ and $p_2 \cdot x_1 \leq p_2 \cdot x_2$. Since $(1,1) \cdot (5, \frac{1}{2}) > (1,1) \cdot (3,1)$ but $(1,2) \cdot (3,1) \leq (5, \frac{1}{2}) \cdot (1,2)$, we can conclude that WARP holds.

b) Assume Filippo's preferences are rational and his Walrasian demand is a function. Can you argue that Filippo prefers a bundle of 1 apple and 3 balloons to a bundle of 5 apples and $\frac{1}{2}$? Can we argue the opposite, that is, that Filippo prefers a bundle of 5 apples and $\frac{1}{2}$ balloon to a bundle of 1 apple and 3 ballons?

ANSWER: To see this, we know that Filippo's choice on day 1 is $x(3,1)$. On day 2 Filippo chooses $x(5, \frac{1}{2})$. But we know that he prefers $x(5, \frac{1}{2})$ to $x(3,1)$ (see above). Therefore, since his preferences are rational and consequently transitive, he prefers $x(5, \frac{1}{2})$ to $x(1,3)$.

3. Hicks (1956) offered the following example to demonstrate how WARP can fail to result in transitive revealed preferences where there are more than two goods. The consumer choose bundle x^j at prices p^i, $i = 0, 1, 2$, where:

$$p^0 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$x^0 = \begin{pmatrix} 5 \\ 19 \\ 9 \end{pmatrix}$$

$$p^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x^1 = \begin{pmatrix} 12 \\ 12 \\ 12 \end{pmatrix}$$

$$p^2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$x^2 = \begin{pmatrix} 27 \\ 11 \\ 1 \end{pmatrix}$$

a) Show that these data satisfy WARP. Do it by considering all possible pairwise comparisons of the bundles and showing that in each case, one bundle of the pair is revealed preferred to the other.

ANSWER: to show that this data satisfy WARP, let's do the following: first, note that $x^0 \neq x^1$, $x^1 \neq x^2$, $x^0 \neq x^2$. Then, verify that is never the case that:

- $p^0 \cdot x^1 \leq p^0 \cdot x^0$ and $p^1 \cdot x^0 \leq p^1 \cdot x^1$
- $p^1 \cdot x^2 \leq p^1 \cdot x^1$ and $p^2 \cdot x^1 \leq p^2 \cdot x^2$
- $p^0 \cdot x^2 \leq p^0 \cdot x^0$ and $p^2 \cdot x^0 \leq p^2 \cdot x^2$

So:

- $(1, 1, 2) \cdot (12, 12, 12) \leq (1, 1, 2) \cdot (5, 19, 19)$ and $(1, 1, 1) \cdot (5, 19, 9) \leq (1, 1, 1) \cdot (12, 12, 12) = 48 \nleq 42$ and $33 \leq 36$
- $(1, 1, 1) \cdot (27, 11, 1) \leq (1, 1, 1) \cdot (12, 12, 12)$ and $(1, 2, 1) \cdot (12, 12, 12) \leq (1, 2, 1) \cdot (27, 11, 1) = 48 \leq 50$ and $39 \nleq 36$

- $(1,1,2)\cdot(27,11,1) \leq (1,1,2)\cdot(5,19,19)$ and $(1,2,1)\cdot(5,19,9) \leq (1,2,1)\cdot(27,11,1)$ $= 40 \leq 42$ and $52 \nleq 50$.

So, WARP holds.

b) Find the intransitivity in the revealed preferences.

ANSWER: Revealed preferences of x^j mean that x^j is preferred to x^i at p_i and p_j. Then, we can see that x_1 is preferred to x_0, at prices p_1 and p_0. x_2 is preferred to x_1 at prices p_2 and p_1. Finally, x_0 is preferred to x_2 at prices p_0 and p_2. Therefore, the revealed preferences are intransitive.

5. Suppose that a choice function $x(p,w)$ is homogenous of degree zero. Show that the weak axiom of revealed preference is satisfied for all (p,w) if and only if it is satisfied for all $(p,1)$.

ANSWER: This is an if and only if statement. Therefore we need to show that:

- If the WARP is satisfied $\forall(p,w)$, it satisfied also $\forall(p,1)$;

- If the WARP is satisfied $\forall(p,1)$, it is satisfied also $\forall(p,w)$

Starting with the latter, we assume that WARP holds for all $x(p,1)$, where the demand is HDZ. Therefore, we can write $p \cdot x(p',1) \leq 1 \Rightarrow p' \cdot x(p,1) > 1$. This can also be rewritten as $\frac{p}{w} \cdot x(\frac{p'}{w'}, \frac{w'}{w'}) \leq \frac{w}{w} \Rightarrow \frac{p'}{w'} \cdot x(\frac{p}{w}, \frac{w}{w}) > \frac{w'}{w'}$. By HDZ, assuming $\alpha = w'$ and $\alpha' = w$, we have $p \cdot x(p',w') \leq w \Rightarrow p' \cdot x(p,w) > w'$.

The first part of the statement is true since $(p,1) \subset (p,w)$.

3 Classical Demand Theory I

1. Let $(-\infty,+\infty) \times R_+^{L-1}$ denote the consumption set and assume that preferences are strictly convex and quasilinear. Normalize $p_1 = 1$. (We say preferences are quasilinear if 1) $x \succeq y$ implies that $x + \alpha e_1 \succeq y + \alpha e_1$ and if 2) $x + \alpha e_1 \succ x$, for $\alpha > 0$ and $e_1 = (1,0,\ldots,0)$.

a) Show that the Walrasian demand functions for goods $2,\ldots,L$ are independent of wealth. What does this imply about the wealth effect (see Section 2.E.) of demand for good 1?

ANSWER: considering that preferences are quasilinear, we can have $x(p,w) \succ x + (w'-w)e_1$, and $\delta w = w - w'$. $x + (w'-w)e_1$ is equal to $px' = w'$. Therefore, $x' = x(p,w') = x(p,w) + (w'-w)e_1$, which becomes, for good 1, $x_1(p,w) = w + u(p)$.

b) Argue that the indirect utility function can be written in the form $v(p,w) = w + \phi(p)$ for some function $\phi(.)$.

ANSWER: The indirect utility function can be written as $v(p,w) = w + \phi(p)$ since it is linear in one of its components (in this case, the income).

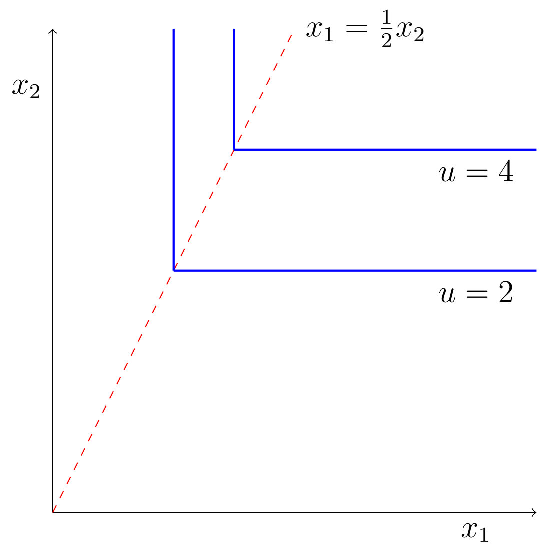
2. David consumes only two goods. His preference can be represented by the utility function $v(x_1,x_2) = a \cdot \min\{b_1x_1, b_2x_2\}$ with $a, b_1, b_2 > 0$.

a) To make your life easier, show first that his preferences can also be represented by $u(x_1, x_2) = \min\{cx_1, x_2\}$ with $c > 0$.

ANSWER: We can assume that c is obtained by b_1 and b_2, i.e. $c = \frac{b_1}{b_2}$. So, since $a > 0$, $u(x_1, x_2)$ is a monotonic positive transformation of $v(x_1, x_2)$.

b) Draw David's indifference curves for the case $a = 2, b_1 = 6, b_2 = 3$

ANSWER:



This is the graph for $u(2)$ and $u(4)$. Note that from the graph it is evident that the two goods are perfect complements.

c) Show that David's preferences are convex

ANSWER: First, let's define convexity (MWG, 3B4): The preference relation $\succeq$ on X is convex if, for every $x \in X$, the upper contour set $\{y \in X : y \succeq x\}$ is convex; that is, if $y \succeq x$ and $z \succeq x$, then $\alpha y + (1-\alpha)z \succeq x$ for any $\alpha \in [0, 1]$.

In other words, if you pick two bundles, x' and x'', and both are weakly preferred to x, then also their convex combination is preferred to x.

To see this point, let's apply the definitions. First, we know that preferences can be represented by utility functions, so then:

- $x' \succeq x = u(x') \geq u(x) = \min\{cx'_1, x'_2\} \geq \min\{cx_1, x_2\}$

To show convexity, we must show that this implies:

$$\alpha x' + (1-\alpha)x'' \succeq x$$

In terms of utility functions, this can be written as:

$$u(\alpha x' + (1-\alpha)x'') \geq u(x)$$

Then, one can show:

$$\min\{c(\alpha x_1' + (1-\alpha)x_1''), \alpha x_2' + (1-\alpha x_2'')\} \geq \min\{cx_1, x_2\} \Rightarrow$$
$$\Rightarrow \min\{c\alpha x_1' + (1-\alpha)cx"_1, \alpha x_2' + (1-\alpha)x_2'')\} \geq \min\{cx_1, x_2\} \Rightarrow$$
$$\Rightarrow \min\{c\alpha x_1', \alpha x_2'\} + \min\{(1-\alpha)x_1'', (1-\alpha)x_2''\} \geq \min\{cx_1, x_2\} \Rightarrow$$
$$\Rightarrow \alpha u(x') + (1-\alpha)u(x'') \geq u(x)$$

Therefore, the convex combination of the utility functions of $x,' x''$ is greater than $u(x)$, as well as the utility of the convex combination of x', x''. So then, preferences are convex.

d) Show that David's preferences are not strictly monotone

ANSWER: Let's define monotonicity and strictly monotonicity (MWG, 3B2): The preference relation $\succeq$ on X is monotone if $x \in X$ and $x \gg y$ implies $x \succ y$. It is strongly monotone if $y \geq x$ and $x \neq y$ imply that $y \succ x$. Note that strict monotonicity rules out an apparent "contradiction" of monotonicity. Namely, in the latter case, a consumer could be indifferent with respect to an increase in the amount of some (but not all) commodities. Still, this is what happens in the case of David's preferences. Indeed take two bundles $x = (1,2)$ and $y = (1,3)$. David attributes the same utility to both bundles since he does not care if the quantity of only one good raises (the goods are perfect complements). Therefore his preferences are not strictly monotone.

e) Show that David's preferences are not strictly convex

ANSWER: Let's define strict convexity (MWG 3B5). The preference relation $\succeq$ on X is strictly convex is for every x, we have that $y \succeq x$ and $z \succeq x$, then $\alpha y + (1-\alpha)z \succ x$ for any $\alpha \in [0,1]$.

To show that David's preferences are non-convex, take this counterexample using the bundles $x = (1,2)$, $y = (1,3)$, and $z = (1,1)$. The convex combination of x and y is:

$$\begin{aligned}
&\alpha(1,2) + (1-\alpha)(1,3) = \\
&= (\alpha, 2\alpha) + (1-\alpha, 3-3\alpha) \\
&= (\alpha + 1 - \alpha, 2\alpha + 3 - 3\alpha) \\
&= (1, 3-\alpha)
\end{aligned}$$

We see that it is not strictly preferred to $z = (1,1)$.

f) What is the fastest way of realizing that David's preferences are complete and transitive?

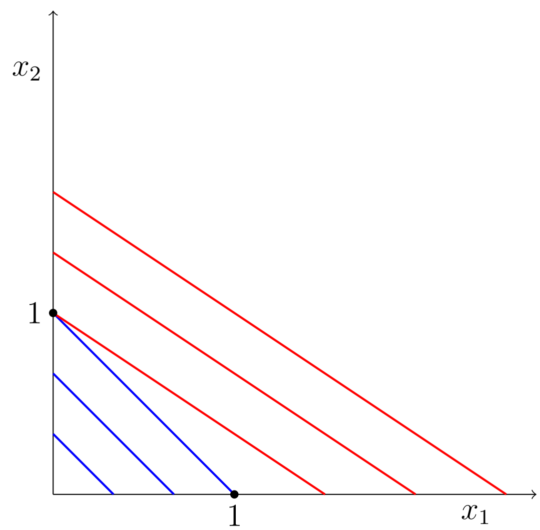
ANSWER: Since, by assumption, we know that David's preferences can be represented by a utility function, and this is possible only if they are rational.

3. Let the utility of a consumer be represented by

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 & \text{if } x_1 + x_2 < 1 \\ x_1 + 4x_2 & \text{if } x_1 + x_2 \geq 1 \end{cases}$$

a) Draw a map of indifference curves for the consumer

ANSWER: See next page. Note that the indifference curves show that the two goods are perfect substitutes, even if the slope differs.



b) Show that these preferences are not continuous

ANSWER: take two bundles x and y, where $x = (1, \frac{1}{n}$ and $y = (-\frac{1}{n}, 1)$. Taking the utility function associated with each of them, we see that $u(x) > u(y)$, since $1 + \frac{4}{n} > 1 - \frac{1}{n}$. Still, if we take the limit of the two sequences, we have the following:

$$\lim_{x \to +\infty} (1, \frac{1}{n}) = (1, 0) \qquad \lim_{n \to +\infty} (-\frac{1}{n}, 1) = (0, 1)$$

Therefore, taking the utility functions, we have $u(x) < u(y)$, since $1 < 4$, and then $y \succ x$. In the limit, it is not preserved preference relation; then these preferences are not continuous.

c) Let $w = 1$. Derive the Walrasian Demand of the consumer

ANSWER: From the utility function, we can derive Walrasian Demand by looking at the ratio between prices and therefore assuming which will be the optimal choice for the consumer in each case. Indeed, since goods are perfect substitutes, the consumer will be interested in spending all his income on buying the less expensive item.

If $\frac{p_1}{p_2} < \frac{1}{4}$, this means that $p_1 < p_2$ and therefore only x_1 will be consumed. So $x(p_1, p_2) = (\frac{w}{p_1}, 0)$.

If $\frac{p_1}{p_2} > 1$, then $p_1 > p_2$, in this case, as above $x(p_1, p_2) = (0, \frac{w}{p_1})$.

For values $\frac{1}{4} < \frac{p_1}{p_2} < 1$, we have that: if $p_1 < 1, x_1(p_1, p_2) = \frac{p_2 - 1}{p_2 - p_1}$ and if $p_2 > 1$, $x_2(p_1, p_2) = \frac{1 - p_1}{p_2 - p_1}$.

d) Let $w = 1$ and $p_2 = 2$. Assume also that you have shown in the previous step that when $p_1 \leq 1$ and $1 \leq p_2 \leq 4p_1$ then $x(p_1, p_2) = \frac{p_2 - 1}{p_2 - p_1}$. Calculate the demand for commodity 1 when $p_1 = \frac{2}{3}$. Do it also when $p_1 = \frac{1}{2}$. What do you find? Be surprised.

ANSWER: Given $w = 1$ and $p_2 = 2$. If $p_1 = \frac{2}{3}$ then $x_1(\frac{2}{3}, 2) = \frac{2-1}{2-\frac{2}{3}} = \frac{3}{4}$. If $p_1 = \frac{1}{2}$, then $x_1(\frac{1}{2}, 2) = \frac{2-1}{2-\frac{1}{2}} = \frac{2}{3}$. That is, p_1 decreases, still also x_1 decreases.

4. A monotone and continuous preference relation $\succeq$ on $X = R^L_+$ is homothetic if all indifference sets are related by proportional expansion along rays; that is if $x \sim y$ then $\alpha x \sim \alpha y$ for all $\alpha \geq 0$.

a) First, show that $x \succeq y \Rightarrow \alpha x \succeq \alpha y$ [Hint: to do so, you might find it useful to have in mind our proof for the existence of a utility function for continuous preferences. There, we showed that there is an element αe in the 45 line such that $\alpha e \sim x$ for any $x \in X$]

ANSWER: Assume $x \succeq y$. This can also be written as $x \sim y + \delta$. Since preferences are homothetic, then we can write $\alpha x \sim \alpha(y + \delta)$. Therefore, $\alpha x \sim \alpha y + \alpha\delta$. And, $\alpha x \succeq \alpha y$ for all $\alpha, \delta \geq 0$.

b) Show that a utility function that is homogenous of degree 1 represents homothetic preferences (recall $u(x)$ is homogenous of degree one if $u(\alpha x) = \alpha u(x)$).

ANSWER: From $u(x) = u(y)$, $x \sim y$. Furthermore, from the definition of HD1, $u(\alpha x) = u(\alpha y) = \alpha u(x) = \alpha u(y)$. Then, $\alpha x \sim \alpha y$.

c) Take a utility function $u(x_1, x_2) = x_1 x_2$. Is this utility homogenous with degree one? Are preferences homothetic? Why?

ANSWER: Let's first check for HD1. $u(\alpha x_1, \alpha_2) = \alpha x_1 \alpha x_2 = \alpha^2 x_1 x_2$ But this is different from $\alpha(x_1, x_2)$. Therefore this utility function does not satisfy HD1. Still, these preferences are homothetic. Indeed, taking two bundles x and y, their utility functions can be written as $x_1 x_2$ and $y_1 y_2$. If $x \sim y$, then $x_1 x_2 = y_1 y_2$, and $\alpha x_1 x_2 = \alpha y_1 y_2$. Therefore $\alpha x \sim \alpha y$.

This is because any monotonic transformation of a utility function is still a utility function. But such transformation not necessarily preserves the homogeneity of a function.

d) Show that homothetic preferences lead to demand functions that are homogenous of degree one in income, that is, a demand which is linear in income: $x(p, \alpha w) = \alpha x(p, w)$.

ANSWER: We have $x(p, w) \succ x$, $\forall x : px \preceq w$. If preferences are homothetic, we have $\alpha x(p, w) \succeq y$, $\forall y : py \preceq \alpha w$. Suppose that the second condition is not true, so $y \succ \alpha x$, then $\frac{y}{\alpha} \succ x$. But $p\frac{y}{\alpha} \preceq w$. But this contradicts $x(p, w)$ is the best bundle under (p, w).

e) In particular, given the point before, show that $x(p, w) = x(p, 1)w$.

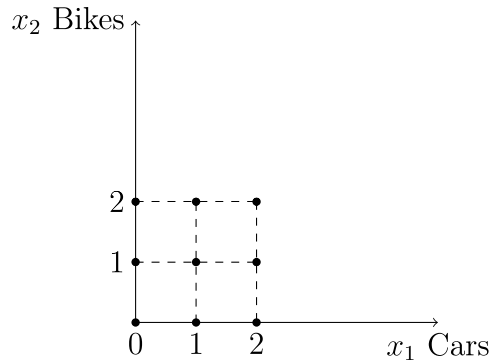
ANSWER: From above, we know that $x(p, \alpha w) = \alpha x(p, w)$. Then $x(p, \frac{1}{w}w) = \frac{1}{w}x(p, w)$. So $x(p, w) = x(p, 1)w$.

5. Millard has preferences over cars (x_1) and bikes (x_2). Both goods can only be consumed in integer non-negative amounts. Let Millard's preferences $\succeq$ on $X = Z_+^2$ be given by:

$$x \succeq y \quad \text{if} \quad \begin{cases} x_1 > y_1 & \text{or} \\ x_1 = y_1 & \text{and } x_2 \geq y_2 \end{cases}$$

a) Let us first restrict X even further. Say that $x_1 \leq 2$ and $x_2 \leq 2$. Present a figure showing X.

ANSWER: See the figure.



b) Consider again the case with X also restricted by $x_1 \leq 2$ and $x_2 \leq 2$. Find a utility function that represents Millard's preferences.

ANSWER: In this case, a utility function must satisfy two features. It assigns less utility to bikes over cars. And all bundles with more cars are strongly preferred to the bundles with fewer cars. In this sense, the preference ordering is the following one. $(0,0) \prec (0,1) \prec (0,2) \prec (0,3) \prec (1,0) \prec (1,1) \prec (1,2) \prec (2,0) \prec (2,1) \prec (2,2)$. Each $f : X \subseteq Z_+^2 \to R$ which preserves such ordering is a utility function for these preferences.

c) Let us now stick to $X = Z_+^2$. Find a utility function that represents Millard's preferences.

ANSWER: a utility function for the preferences can be the following:

$$u(x_1, x_2) = x_1 - \frac{1}{1 + x_2} + 1.$$

For the first part, $x_1 > y_1$, we have $x_1 - \frac{1}{1+x_2} + 1 \geq y_1 - \frac{1}{1+y_2} + 1 \Rightarrow x_1 - y_1 \geq \frac{1}{1+x_2} - \frac{1}{1+y_2}$. Given that $x_1 - y_1 \geq 1$, we can write $1 \geq \frac{1}{1+x_2} - \frac{1}{1+y_2}$, but $\frac{1}{1+x_2}$ is less or equal than 1. Then $1 \geq 1 - \frac{1}{1+y_2}$, and therefore $0 \geq -\frac{1}{1-y_2}$.

For the second point: $x_1 - \frac{1}{1+x_2} + 1 \geq x_1 - \frac{1}{1+y_2} + 1$. Taking $x_1 = y_1$, then $-\frac{1}{1+x_2} \geq -\frac{1}{1+x_2} \Rightarrow \frac{1}{1+x_2} \leq \frac{1}{1+y_2} \Rightarrow 1 + x_2 \geq 1 + y_2 \Rightarrow x_2 \geq y_2$.

4 Classical Demand Theory II

1. Show that if $\succeq$ is quasilinear with respect to good 1, the Hicksian demand functions for goods $2, ...L$ do not depend on u (you can assume $p = 1$). What is the form of the expenditure function in this case?

ANSWER: The Hicksian Demand can be defined as:

$$h(p,u) = \arg\min \sum_{i=1}^{n} p \cdot x \quad s.t. \;\; u(x) \geq x$$

If $h_l(p,u)$ does not depend of u, then $h_l(p,u) = h_l(p,u')$, for u, u' and $l = 2, \ldots L$. To show that $h(p,u) = h(p,u')$, pick a u' so defined: $x + \alpha e_1$. Then $u(h(p,u) + \alpha e_1) = u'$. Assume $h(p,u) \neq h(p,u') + \alpha e_1$. We can find an x' such that $u(x') = u'$ and $p \cdot x' < p \cdot h(p,u) + \alpha e_1)$. If $h(p,u) + \alpha e_1 \sim x'$ (from the definition of quasi-linear preferences), so:

$$h(p,u) \sim x' - \alpha e_1$$

and

$$p(x' - \alpha e_1) < p \cdot h(p,u)$$

But this contradicts the definition of Hicksian Demand. So $h_l(p,u)$ does not change with u for all $l \neq 1$.

Note now that quasilinear preferences with respect to x_1 by definition admit a utility function of the following form:

$$u(x_1, \ldots x_L) = x_1 + \Psi(x_2, \ldots x_L)$$

x_1 is equal to $u(x_1)$ which is equal to $u(h_1(p,u))$, and therefore u. Furthermore $h(p,u') = h(p,u) + \alpha e_1$. Then, we can write $e(p,u)$ as follows:

$$e(p,u) = p_1 \cdot h_1(p,u) + \sum_{l=2}^{L} p_l \cdot h_l(p,u)$$

2. Ellsworth's utility function in $U(x,y) = \min\{x,y\}$. Ellsworth has 150 dollars, and the price of y are both 1. Ellsworth's boss is thinking of sending him to another town where the price x is 1, and the price of y is 2. The boss offers no raise in pay. Ellsworth, who understands compensating and equivalent variation perfectly, complains bitterly. He says that although he doesn't mind moving for its own sake and the new town is just as pleasant as the old, having to move is as bad as a cut in pay of A dollars. He also says he wouldn't mind moving if, when he moved, he got a raise of B dollars. What are A and B equal to?

a) First, solve this exercise without doing any calculations.

1. Graph Ellsworth's indifference curves for levels $u = 75$ and $u = 50$.

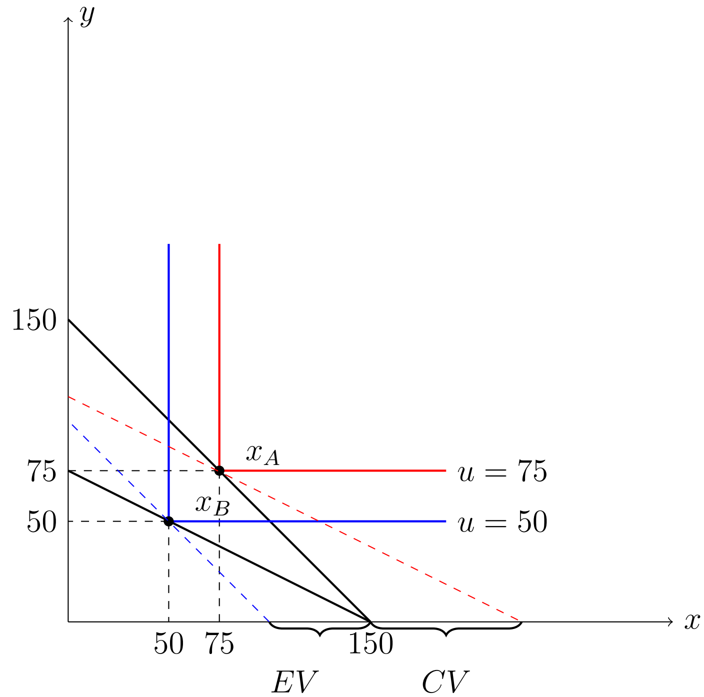


Figure 1: Ellsworth's Indifference curves, the Budget Constrains and the Compensating and Equivalent Variation

2. Graph Ellsworth's budget constraint before and after moving

3. Claim that Ellsworth would choose bundle $(75, 75)$ in the old town

4. Claim that Ellsworth would choose bundle $(50, 50)$ in the new town

5. Show graphically the compensating and equivalent variation. That should be enough to calculate A and B.

ANSWER: See the Figure 1. Note that since his preferences are Leontief, the consumer will equally share his wealth between x and y, so the optimal bundle is always the kink. This also makes clear why the dashed lines pass through x_B and x_A.

b) Now, let's do the same exercise but in a different way. Find Ellsworth's Hicksian Demands and Expenditure Function. Calculate CV and EV from the following formulae:

$$CV(p^0, p^1, w) = e(p^1, v(p^1, w)) - e(p^1, v(p^0, w))$$

$$EV(p^0, p^1, w) = e(p^0, v(p^1, w)) - e(p^0, v(p^0, w))$$

ANSWER: We know from the utility function that x and y are perfect complements. Then, a rational consumer will always demand the same amount of the two goods, otherwise, he will spend more on something which does not generate any utility to her. Therefore, we can write the Expenditure function $e(p,u)$ as $p_x h_x(p,u) + p_y h_y(p,u) = (p_x + p_y)u$, since $h_x(p,u) = h_y(p,u) = u$.

So:

$$\begin{aligned}
CV(p^0, p^1, w) &= e(p^1, v(p^1, w)) - e(p^1, v(p^0, w)) = \\
&= w - (p_x^1 + p_y^1)u = \\
&= 150 - (1+2)75 \\
&= -75
\end{aligned}$$

Similar reasoning holds for the case of EV:

$$\begin{aligned}
EV(p^0, p^1, w) &= e(p^0, v(p^1, w)) - e(p^0, v(p^0, w)) = \\
&= (p_x^0 + p_y^1)u - w = \\
&= (1+1)50 - 150 \\
&= -50
\end{aligned}$$

c) Finally, using the Hicksian demands you found in the last point, calculate CV and EV from the following formulae:

$$CV(p^0, p^1, w) = \int_{p_y^0}^{p_y^1} h_y(p_y, \bar{p_x}, u^0) dp_y$$

$$EV(p^0, p^1, w) = \int_{p_y^0}^{p_y^1} h_y(p_y, \bar{p_x}, u^1) dp_y$$

ANSWER: We know that the Hicksian demand corresponds to the kink of the indifference curve since the goods are perfect complements. Therefore:

$$\begin{aligned}
CV(p^0, p^1, w) &= \int_{p_y^0}^{p_y^1} h_y(p_y, \bar{p_x}, u^0) dp_y = \\
&= \int_1^2 75 dp_y = \\
&= 75 p_y = \\
&= 75(1) - 75(2) = -75
\end{aligned}$$

$$\begin{aligned}
EV(p^0, p^1, w) &= \int_{p_y^0}^{p_y^1} h_y(p_y, \bar{p_x}, u^0) dp_y = \\
&= \int_1^2 50 dp_y = \\
&= 50 p_y = \\
&= 50(1) - 50(2) = -50
\end{aligned}$$

3. Marty's utility funtion has the form: $u(x_1, x_2) = f(x_1) + x_2$. Good 1 is discrete and takes only one of two possible values, either $x_1 = 0$ or $x_1 = 1$.

a) Find the demands $x_1(p_1, p_2, w)$ and $x_2(p_1, p_2, w)$ as functions of $f(0)$ and $f(1)$.

ANSWER: Since x_1 can assume only two values, then we can compare the utility Marty receives from two bundles, one where $x_1 = 1$ and one where $x_1 = 0$. Then, in the first case, we can also write x_2 as $\frac{w-p_1}{p_2}$ because he spends all his wealth, with the exception of p_1, in buying x_2. Equally, when $x_1 = 0$, $x_2 = \frac{w}{p}$. Marty buys x_1 as long as:

$$u(1, \frac{w-p_1}{p_2}) \geq u(0, \frac{w}{p_2})$$

What above can also be written in terms of $f(1)$ and $f(0)$ as follows:

$$f(1) + \frac{w-p_1}{p_2} \geq f(0) + \frac{w}{p_2}$$

Which becomes: $f(1) - f(0) \geq \frac{p_1}{p_2}$.

Therefore, we can write the demand as follows:

$$x(p_1, p_2, w) = \begin{cases} (1, \frac{w-p_1}{p_2}) & f(1) - f(0) \geq \frac{p_1}{p_2} \\ (0, \frac{w}{p_2}) & \text{if } f(1) - f(0) < \frac{p_1}{p_2} \end{cases}$$

b) Find the Indirect utility function as a function of $f(0)$ and $f(1)$.

ANSWER: The indirect utility function can be written as:

$$v(p_1, p_2, w) = \begin{cases} f(1) + \frac{w-p_1}{p_2} & f(1) - f(0) \geq \frac{p_1}{p_2} \\ f(0) + \frac{w}{p_2} & \text{if } f(1) - f(0) < \frac{p_1}{p_2} \end{cases}$$

4. An Individual has preferences over cars (x) and food (y) that can be represented with the following utility function:

$$u(x, y) = x - y^-1$$

Normalize the price of a car to 1 and let the price of food be p. Cars are not divisible, and the individual cannot buy more than one car. Food is perfectly divisible but cannot be consumed in negative quantities. Consequently, $x \in [0, 1]$ whereas $y \geq 0$. Let $w > 1$, so the problem is interesting.

a) Find the demand functions

ANSWER: In a way similar to the above, we know that cars are not divisible, and one individual cannot buy more than one car. So we can write the utility related to the two different bundles, one with cars and one without, as follows: $u(1, \frac{w-1}{p})$ and $u(0, \frac{w}{p})$. The individuals will buy cars only if:

$$u(1, \frac{w-1}{p}) \geq u(0, \frac{w}{p})$$

Plotting the utility functions in the inequality above, we have:

$$
\begin{aligned}
1-(\frac{w-1}{p})^{-}1 \geq -(\frac{w}{p})^{-}1 \\
1-\frac{p}{w-1} &\geq -\frac{p}{w} \\
\frac{w-1-p}{w-1} &\geq -pw \\
w\cdot(w-1-p) &\geq -p\cdot w+p \\
w^2-w-p\cdot w &\geq -p\cdot w+p \\
w^2-w &\geq p
\end{aligned}
$$

Then, we can write the demand functions:

$$
x(1,p_2,w)=\begin{cases}(1,\frac{w-1}{p}) & \text{if } w^2-w\geq p\\ (0,\frac{w}{p}) & \text{if } p>w^2-w\end{cases}
$$

b) Let $w=3$. Calculate demands for $p=5$ and $p=7$. Mention if you see anything surprising.

ANSWER: Let $w=3$ and $p=5$.

$$
x(p,w)=\begin{cases}(1,\frac{2}{5}) & \text{if } 9-3\geq 5\\ (0,\frac{3}{5}) & \text{if } 5>9-3)\end{cases}
$$

It is apparent that with $(p_1,p_2,w)=(1,5,3)$, the demand is $(1,\frac{2}{5})$. Let now fix $w=3$ and $p=7$.

$$
x(p',w)=\begin{cases}(1,\frac{2}{7}) & \text{if } 9-3\geq 7\\ (0,\frac{3}{7}) & \text{if } 7>9-3)\end{cases}
$$

In this case the demand at $(p_1,p_2',w)=(1,7,3)$ is $(0,\frac{3}{7})$. That is, the price of food has increased, and the demand for food has increased too. In other words, food is a Giffen Good.

c) Graph the demand for cars as a function of the price of food

ANSWER: See Figure 2. The red line is the demand for cars, which is discrete and can be equal only to 0 and 1. The blue line is the demand for food. Note that when the price is 7, the demand for food is higher than the demand for food when the price is 5.

5. A consumer with wealth w faces first a price vector p and then a price vector p'. Two fairies may compensate her for the price change, the Slutsky fair or the Hicksian fairy. The Slutsky fairy provides extra income so that bundle $x(p,w)$ is available at prices p'. Instead, the Hicksian fairy provides income so that she can attain utility $u(x(p,w))$ at prices p'.

a) Show that the Slutsky fairy never gives compensation which makes our consumers worse off than with a Hicksian fairy.

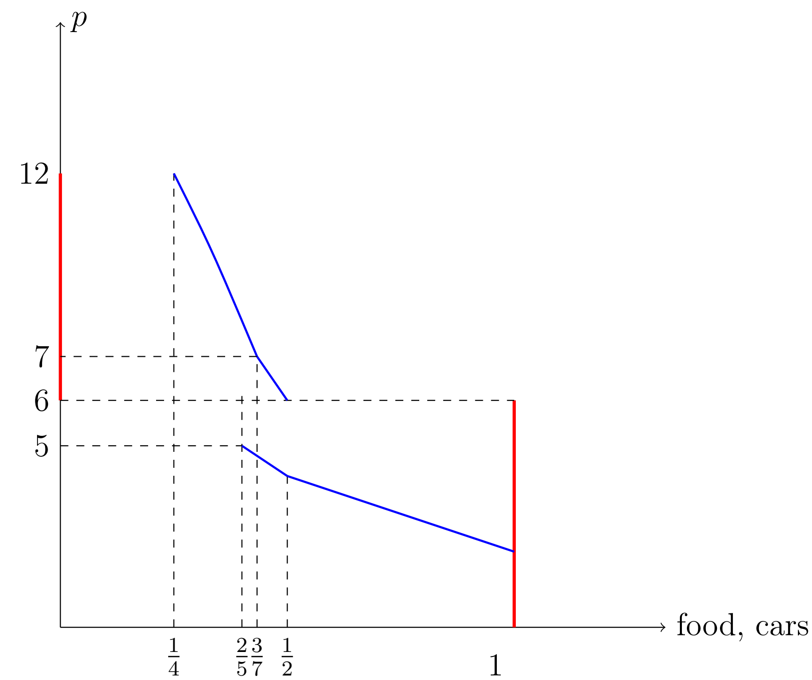


Figure 2: The demand for cars as a function of the price of food

ANSWER: By definition, we know that the Hicksian wealth compensation is the amount: $\Delta w_{Hicks} = e(p'u) - w$, so that wealth after Hicks compensation is $w_{Hicks} = e(p', v(p, w))$. A Slutsky compensation instead is the amount for which: $p' \cdot x(p, w) = w_{Slutsky}$. Since we are in the expenditure minimization world, the utility associated with prices p' and wealth w_{Hicks} must be equal to the utility associated with (p, w). But by definition, $p' \cdot x(p, w) \leq w_{Slutsky}$. Since consumer maximizes utility, $v(p', w_{Slutsky}) \geq v(p, w)$.

b) Give an example of preferences where the compensation is the same

ANSWER: The simplest example that comes to my mind is that of Leontief Preferences, $u(x, y) = \min\{x, y\}$. The demand associated to these preferences is:

$$x(p, w) = \begin{cases} \frac{w}{p_1+p_2} \\ \frac{w}{p_1+p_2} \end{cases}$$

Since $v(p, w) = \frac{w}{p_1+p_2}$ and $e(p, u) = (p_1 + p_2)u$:

$$w_{Hicks} = e(p', v(p, w)) = (p'_1 + p'_2)\frac{w}{p_1 + p_2} = p' \cdot x(p.w) = w_{Slutsky}$$

5 Aggregate Demand and Production Theory

1. There are three worlds, *Ariel, Belinda* and *Caliban*. The utility functions of aliens in each of these worlds are as follows:

$$u_i^A(x_i, y_i) = x_i^{\frac{1}{3}} y_i^{\frac{1}{3}}$$

$$u_i^B(x_i, y_i) = x_i^{\frac{1}{2}} y_i^{\frac{1}{2}}$$

$$u_i^C(x_i, y_i) = x_i y_i$$

a) Compute the demands for aliens in each world
ANSWER: The preferences are Cobb-Douglas. So we can write:

$$x_i(p, w_i) = \frac{1}{2}\frac{w}{p_x}$$

and

$$y_i(p, w_i) = \frac{1}{2}\frac{w}{p_y}$$

This holds for sure in B. Let's see A. Equating $\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$ we have:

$$\frac{\frac{1}{3}x^{-\frac{2}{3}}y^{\frac{1}{3}}}{\frac{1}{3}x^{\frac{1}{3}}y^{-\frac{2}{3}}} = \frac{p_x}{p_y}$$

Simplifying, we obtain:

$$\frac{y}{x} = \frac{p_x}{p_y}$$
$$\frac{x}{y} = \frac{p_y}{p_x}$$
$$x = \frac{yp_y}{p_x}$$

Substituting in: $x \cdot p_x + y \cdot p_y = w$, we have:

$$p_x\frac{yp_y}{p_x} + yp_y = w$$
$$2yp_y = w$$
$$y = \frac{w}{2p_y}$$

Substituting again in $x \cdot p_x + y \cdot p_y = w$ we found x:

$$xp_x + p_y\frac{w}{2p_y} = w$$
$$xp_x + \frac{w}{2} = w$$
$$x = \frac{w}{2}$$
$$x = \frac{w}{2p_x}$$

Since the coefficients are equal, this holds also for C. So each consumer split her income between the two goods.

b) Compute the indirect utility function for aliens in each world

ANSWER: The indirect utility function is simply $v(u(x^*))$, where x^* is the demand. So, we write:

$$v_i^A(p_x, p_y, w_i) = \left(\frac{1}{2}\frac{w}{p_x}\right)^{\frac{1}{3}}\left(\frac{1}{2}\frac{w}{p_y}\right)^{\frac{1}{3}}$$

$$v_i^B(p_x, p_y, w_i) = \left(\frac{1}{2}\frac{w}{p_x}\right)^{\frac{1}{2}}\left(\frac{1}{2}\frac{w}{p_y}\right)^{\frac{1}{2}}$$

$$v_i^C(p_x, p_y, w_i) = \left(\frac{1}{2}\frac{w}{p_x}\right)\left(\frac{1}{2}\frac{w}{p_y}\right)$$

c) Prices are $p_x = 1$ and $p_y = 1$. There are 10 aliens in each world. A social planner would like to maximize the utilitarian welfare function $W(u_1, u_2, \ldots, u_10) = \sum_{i=1}^{1} 0u_i$. The social planner can distribute w units of wealth in each world. How will he do so in each case?

ANSWER: The social planner solves a problem of utility maximization, where wealth is now represented by the sum of individual wealth, and the prices are normalized.

We can rewrite A's indirect utility function as follows:

$$\sum_{i=10}^{10}\left(\frac{w_i}{2}\right)^{\frac{2}{3}}$$

So A maximizes the function above. However, it is apparent that the function is concave, so there is an interior solution.

For B, the problem is:

$$\max_{w_1,\ldots,w_{10}}\sum_{i=10}^{10}\left(\frac{w_i}{2}\right)$$

Since the function is linear, any wealth distribution solves the problem.

Finally, the problem for C is:

$$\max_{w_1,\ldots,w_{10}}\sum_{i=10}^{10}\left(\frac{w_i^2}{4}\right)$$

This function is convex, so it is optimal to give all wealth to just one consumer.

2. Let's focus on the market of cars now. Consumers obtain utility from cars x and the rest of their purchases m. Each consumer can buy either zero or one cars. All consumers share the following utility function:

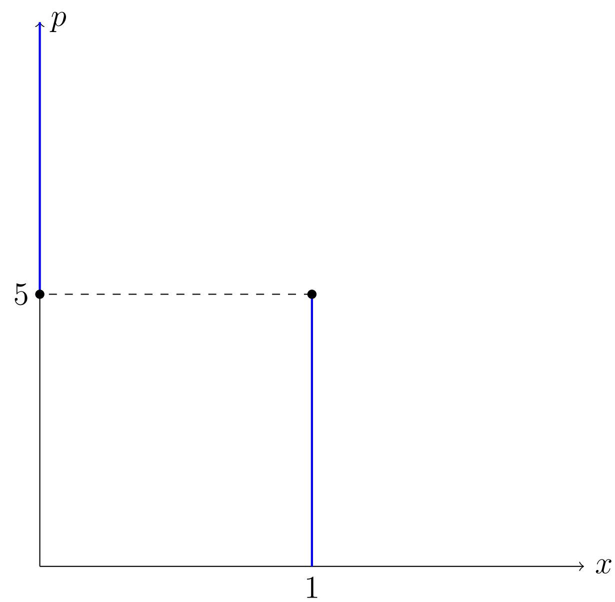


Figure 3: Demand for cars with $w_1 = 10$

$$u(x,m) = \begin{cases} \log(m) & \text{if } x = 0 \\ v + \log(m) & x = 1 \end{cases}$$

Cars are sold at the price of p. Normalize the price of m to 1.

a) Set $v = \log(2)$ for all consumers. Let $x_i(p_i, w_i)$ denote the demand for cars for individual consumers i with wealth level w_i. Recall that cars are indivisible and find $x_i(p, w_i)$.

ANSWER: Cars are bought if $u(1, w_i - p) \geq u(0, w)$, that is, if the utility associated to a car and wealth minus the price of the car is greater than the utility of no car and full wealth. Substituting $\log(2)$ above, we have $\log(2) + \log(w_i - p) \geq \log(w)$. From the properties of logs we can write $2(w_i - p) \geq w_i$ and therefore $\frac{w_i}{2} \geq p$.

Then:

$$x_i(p,w) = \begin{cases} 1 & \text{if } p \leq \frac{w_i}{2} \\ 0 & \text{if } p > \frac{w_i}{2} \end{cases}$$

b) Assume that consumer i buys a car if and only if $p \leq \frac{w_i}{2}$. Draw the demand for cars for consumers $1, 2, 3$ and 4 with $w_1 = 10$,$w_2 = 8$, $w_3 = 6$ and $w_4 = 4$. Are individual demand functions continuous?

ANSWER:

From these graphs, it is apparent that demand is not continuous.

c) Let $x(p, w_1, \ldots, w_I) = \sum_{i=1}^{I} x_i(p, w)$. Draw the aggregate demand for consumers 1 to 4 with wealth as specified above. Is aggregate demand continuous?

ANSWER: See Figure 5. Aggregate demand is not continuous.

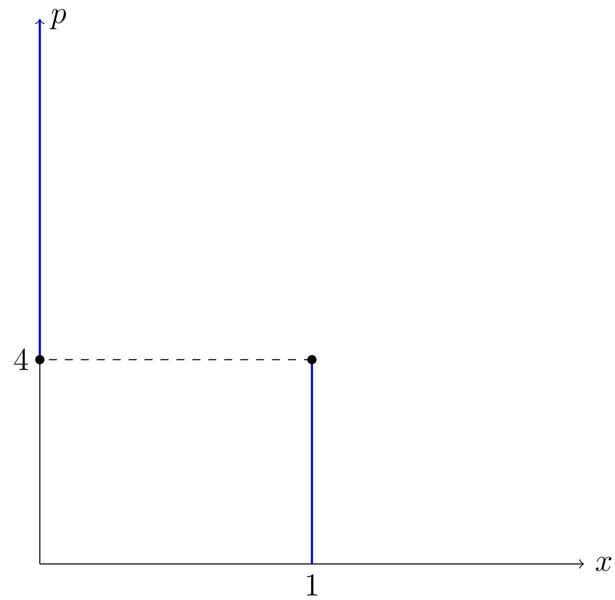


Figure 4: Demand for cars with $w_2 = 8$

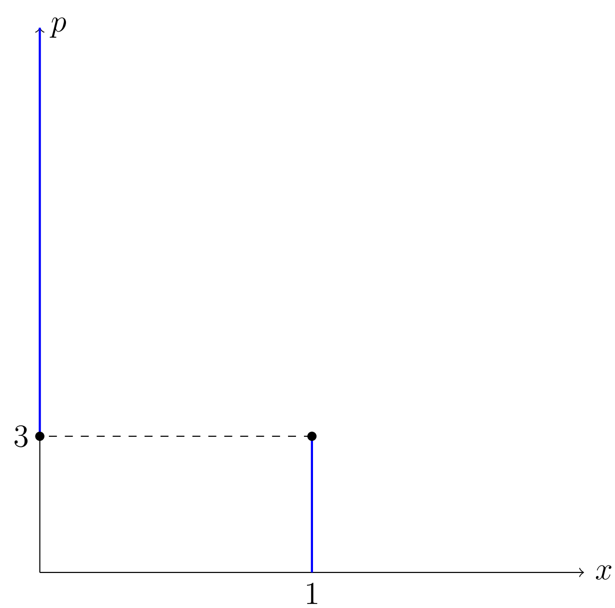


Figure 5: Demand for cars with $w_3 = 6$

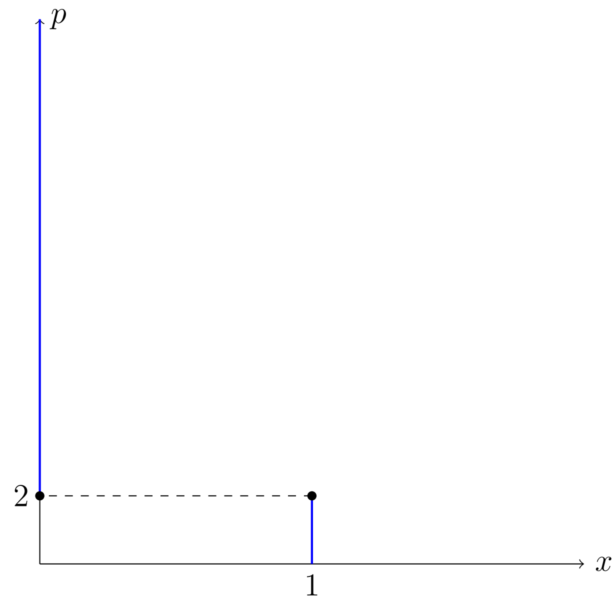


Figure 6: Demand for cars with $w_4 = 4$

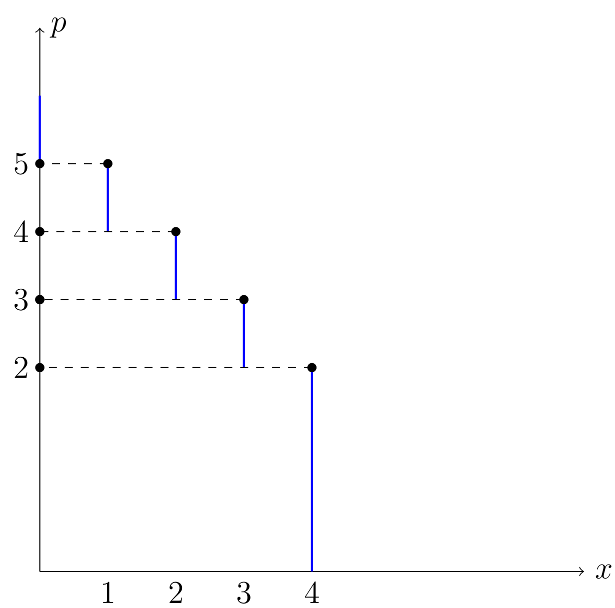


Figure 7: Aggregate Demand for consumers $1, 2, 3$ and 4

d)Assume that there is a continuum of individuals with wealth uniformly distributed in $[0, 10]$. i) Define a notion of aggregate demand appropriate for this setup

ANSWER: Since the consumers are now a continuum of real numbers, we can write the aggregate demand using the integral:

$$\int_{w_i} x_i(p, w_i) f(w_i) dw_i$$

ii) Compute aggregate demand

ANSWER: Assuming that $x_i(p, w) = 1$, and wealth is uniformly distributed in $[0, 10]$ we can write:

$$\begin{aligned} x(p, w) = & \int_{w_i} x_i(p, w_i) \frac{1}{10} dw_i = \\ & \int_{2p}^{10} \frac{1}{10} dw_i = \\ & \frac{1}{10} \int_{2p}^{10} dw_i = \\ & \frac{1}{10} w \Big|_{2p}^{10} = \\ & 1 - \frac{p}{5} \end{aligned}$$

iii) Draw aggregate demand. Is it continuous?

ANSWER:

As apparent from Figure 6, the demand is continuous.

3)Consider a firm that has a distinct set of inputs and outputs. The firm produces M outputs: let $q = (q_1 \ldots q_M)$ denote a vector of its output levels. Holding factor prices fixed, $C(q_1, \ldots q_M)$ is the firm's cost function. We say that $C(.)$ is subadditive if, for all $(q_1 \ldots q_M)$, there is no way to break up the production of amounts $(q_1 \ldots q_M)$ among several firms, each with cost function $C(.)$ and lower cost of production. That is, there is no set of, say, J firms and collections of production vectors $\{q_j = (q_j 1, \ldots q_M j\}_{j=1}^{j}$ such that $\sum_{j=1}^{J} q_j = q$ and $\sum_{j=1}^{j} C(q_j) < C(q)$. When $C(.)$ is subadditive, it is usual to say that the industry is a *natural monopoly* because production is cheapest when it is done by only one firm.

a) Consider the single output case, $M = 1$. Show that if $C(.)$ exhibits decreasing average costs, then $C(.)$ is subadditive.

ANSWER: In the single output case $C(.)$ exhibits decreasing average costs if:

$$\frac{C(kq)}{kq} < \frac{C(q)}{q} \quad \text{for all } k > 1$$

If $q \in R$ we can split in in n-parts, so: $q = \sum_{i=1}^{n} q_i$. For each i we can write an $\alpha > 1$ as follows: $\alpha_n = \frac{q}{q_i}$. Then:

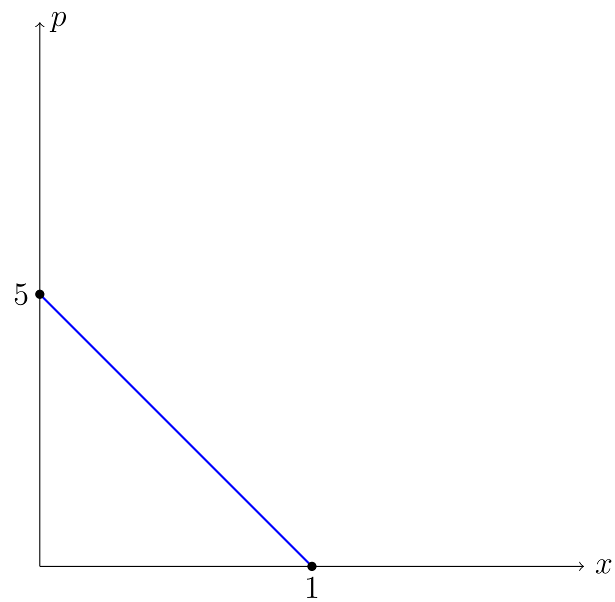


Figure 8: Aggregate Demand for a continuum of consumers

$$\frac{C(q)}{q} = \frac{C(\alpha_i q_i)}{\alpha_i q_i} < \frac{C(q_i)}{q_i}$$

Let's see subadditivity:

$$\sum_{i=1}^{n} C(q)q_i < \sum_{i=1}^{n} C(q_i)q$$

$$C(q)\sum_{i=1}^{n} q_i < q\sum_{i=1}^{n} C(q_i)$$

$$C(q)q < q\sum_{j=1}^{n} C(q_j)$$

$$C(q) < \sum_{i=1}^{n} C(q_i)$$

b) Now consider the multi-output case, $M > 1$. Show by example that the following multiple-output extension of the decreasing average cost assumption is not sufficient for $C(.)$ to be subadditive: $C(.)$ exhibits *decreasing ray average* cost if for any $q \in R_+^M$

$$C(q) > \frac{C(kq)}{k} \text{ for all } k > 1$$

ANSWER: To see this point, one has to show that a cost function that satisfied *Decreasing Ray Average Cost* property can fail to satisfy sub-additivity. Let's first find a cost function that exhibits DRAC. Assume $M = 2$ and define $C(q) = (q_1 + q_2)$. We can write $C(kq_1, kq_w) = \sqrt{k}q_1 + \sqrt{k}q_2 = \sqrt{k}(q_1 + q_2) = \sqrt{k}C(q_1, q_2)$. Then:

$$\frac{\sqrt{k}C(q_1, q_2)}{kq} = \frac{k^{-\frac{1}{2}}C(q_1, q_2)}{q} < \frac{C(q_1, q_2)}{q}$$

However, taking a trivial example as: $q_1 = \alpha$ and $q_2 = -\alpha$, it is apparent that the function above is not sub-additive.

4. Suppose there are J single-output plants. Plant j's average cost is $AC(q_j) = \alpha + \beta_j q_j$ for $q_j \geq 0$. Note that the coefficient α is the same for all plants but that the coefficient β_j may differ from plant to plant. Consider the problem of determining the cost-minimizing aggregate production plan for producing a total output of q, where $q < \frac{\alpha}{\max_j |\beta_j|}$.

From the Average Cost Function, we can define both the cost function and the marginal costs. For the cost function, simply multiply by q.

$$C(q_j) = \alpha q + \beta_j q_j^2$$

For marginal costs, instead:

$$MC(q_i) = \frac{dC}{dq} = \alpha + 2\beta q_j$$

a) If $\beta_j > 0$ for all j, how should output be allocated among the J plants?

ANSWER: If $\beta_j > 0$ for all j, then all plants have decreasing returns to scale. Indeed the average cost is an increasing function, and it is not optimal to concentrate all production in a single plan. Then we can equate marginal costs for all plants.

$$MC(q_i) = \frac{dC}{dq} = \alpha + 2\beta q_j$$

Then we can write:

$$q_j = \frac{K - \alpha}{2\beta_j}$$

and

$$K = \alpha + 2q\frac{1}{\frac{1}{\beta_j}}$$

Finally:

$$q_j = \frac{q}{\beta_j}\frac{1}{\frac{1}{\beta_j}}$$

b) If $\beta_j < 0$ for all j, how should output be allocated among the J plants?

ANSWER: When $\beta_j < 0$, the cost function is decreasing (i.e. there are decreasing returns to scale). Then, the best choice is to concentrate all the production in the plant where the average costs are the lowest.

We can write the total cost as follows:

$$C(q)) = \alpha q + \min_j |\beta_j| q^2 = \alpha q - \max_j |\beta_j| q^2$$

c) What if $\beta_j > 0$ for some plants and $\beta_j < 0$ for others?

ANSWER: In this case, the best choice is to employ only those plants where $\beta_j < 0$.

5) Chocolate q is produces using only one input, labor (z). The price of the chocolate is p, and the price of the input is w. The production function is given by:

$$f(z) = \begin{cases} 0 \text{ if } z \leq 4 \\ \sqrt{z-4} \;\; z > 4 \end{cases}$$

a) Verify that $c(w, q) = 4w + wq^2$. From now on, normalize the price of input to $w = 1$.

ANSWER: To see this, note that $C(w, q)$ solves the following problem:

$$\min_{z \geq 0} w \cdot z \quad \text{s.t. } \sqrt{z-4} > q$$

We can solve the constraint so that $z > q^2 + 4$. Substituting in $w \cdot z$, we have $c(w, q) = w(4 + q^2) = 4w + 4wq^2$.

b) Draw the marginal and average cost functions.

ANSWER: We can write the Average Cost function as follows:

$$AC = \frac{4 + 4q^2}{q} = \frac{4}{q} + q$$

The Marginal Costs are:

$$MC = 2q$$

These are represented in Figure 7.

c) Find the supply function $q(p)$ and the profit function $\pi(p)$. From now on, the price of chocolate is $p = 20$

ANSWER: To find the supply function $q(p)$, one must solve a profit maximization problem:

$$\max_q pq - (4 + q^2)$$

The First Order Condition is $p - 2q = 0$, so then $q = \frac{p}{2}$. To calculate profits, one plugs q in $pq - c(w, q)$. Then:

$$p\frac{p}{2} - 4 - \left(\frac{p}{2}\right)^2$$

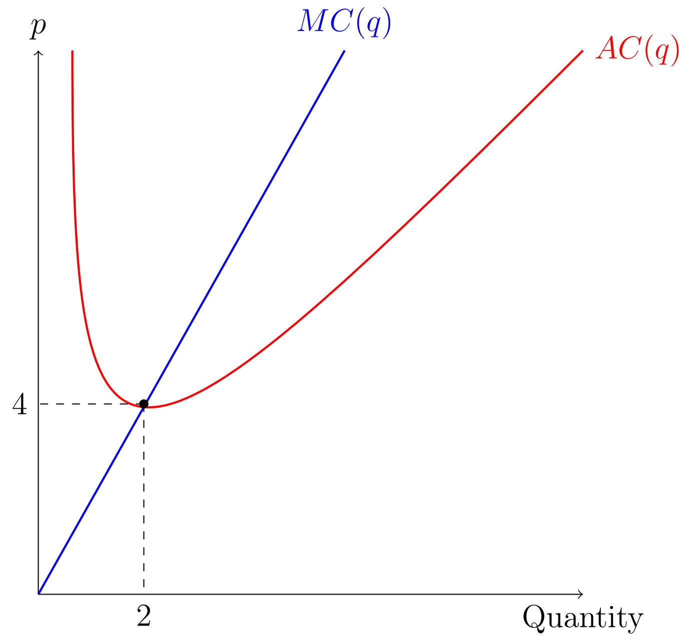


Figure 9: Marginal Cost and Average Cost

So, production lasts until $p > 4$. The supply function is:

$$q(p) = \begin{cases} \frac{p}{2} & \text{if } p \geq 4 \\ 0 & \text{if } p < 4 \end{cases}$$

The profit function is:

$$\pi(q) = \begin{cases} \frac{p^2}{4} - 4 & \text{if } p \geq 4 \\ 0 & \text{if } p < 4 \end{cases}$$

d) A firm that already owns one plant is considering opening a second one. The opening does not affect the market price of chocolate. What is the maximum cost of investment such that the firm would actually open the second plant?

ANSWER: When $p = 20$, then $\pi(20) = \frac{20^2}{4} - 4 = 96$. Then a firm will not spend more than 96 to build a second plant.

e) Assume now that total production (adding up production from all plants) cannot exceed 10. What is the maximum cost of investment such that the firm would actually open the second plant?

ANSWER: The firm's revenue is $p \times q$, so $20 \times 10 = 200$. If are produced 10 units, the cost is $c(10) = 4 + 10^2 = 104$. If 5 units are produced in each plant, then $c(5) = 29$. In two plants, the total costs are $29 + 29 = 58$. The firm saves 46 by opening a second plant.

f) Still, the firm cannot sell more than 10 units of chocolate. Now opening new plants is for free. How many plants should the firm open?

ANSWER: Revenues are still fixed. Assuming n plants, we can write:

$$\begin{aligned} c(q) =& n\Big(4 + \Big(\frac{q}{n}^2\Big)\Big) \\ &= 4n + \frac{q^2}{n} \end{aligned}$$

If $q = 10$:

$$\begin{aligned} \min_n \ & 4n + \frac{100}{n} = \\ & 4n + 100n^{-1} = \\ & 4 - 100n^{-2} = \end{aligned}$$

Taking the FOC and solving:

$$\begin{aligned} & 4 - 100n^{-2} = 0 \\ & -\frac{100}{n^2} = -4 \\ & \frac{100}{n^2} = 4 \\ & \frac{n^2}{100} = \frac{1}{4} = \\ & n^2 = \frac{100}{4} = \\ & n^2 = 25 \Rightarrow n = 5 \end{aligned}$$

So the firm should open 5 plants.

6 General Equilibrium Theory

Competitive Equilibrium in a 2×2 pure exchange economy

Assume a 2×2 economy, with 2 goods, $l = 1, 2$ and 2 agents, $i = 1, 2$. The preferences of the consumers are represented by the following Cobb-Douglas utility function:

$$u_i(x_1, x_2) = x_{1i}^{0.4} x_{2i}^{0.6}$$

The endowments of the two agents are:

- $\omega_1 = (30, 70)$
- $\omega_2 = (70, 30)$

The agents are price-takers (by assumption). Then there is a price vector $p = (p_1, p_2)$ associated to each endowment. Let's find the competitive equilibrium price vector $p^* = (p_1^*, p_2^*)$ and demand x_1^* and x_2^*.

ANSWER: Since the preferences are represented by a Cobb-Douglas Utility Function, we can write each player's demand as follows:

$$x_i = \left(\frac{\alpha w_1}{p_1}, \frac{(1-\alpha)w_2}{p_2}\right)$$

Substituting in the utility function above (and since $w_i = \omega_1 \cdot p_1 + \omega_2 \cdot p_2$)

$$x_1 = \left(\frac{0.4[30p_1 + 70p_2]}{p_1}, \frac{0.6[30p_1 + 70p_2]}{p_2}\right)$$

and

$$x_2 = \left(\frac{0.4[70p_1 + 30p_2]}{p_1}, \frac{0.6[70p_1 + 30p_2]}{p_2}\right)$$

Letting $p_1 = 1$, since what matters is the relative ratio of prices, we have:

$$x_1 = \left(12 + 28p_2, \frac{18}{p_2} + 42\right)$$

and

$$x_2 = \left(28 + 12p_2, \frac{42}{p_2} + 18\right)$$

Notice that:

$$x_{11} + x_{12} \leq 100$$

and

$$x_{21} + x_{22} \leq 100$$

Then let's apply the "Market Clearing Condition" for goods 1 and 2:

$$\begin{aligned}
&12 + 28p_2 + 28 + 12p_2 \leq 100 \\
&40p_2 \leq 100 - 28 - 12 \\
&40p_2 \leq 60 \\
&p_2 = \frac{60}{40} \\
&p_2 = \frac{3}{2}
\end{aligned} \tag{1}$$

And:

$$\begin{aligned}
&\frac{18}{p_2} + 42 + \frac{42}{p_2} + 18 \leq 100 \\
&\frac{60}{p_2} + 60 \leq 100 \\
&\frac{60}{p_2} \leq 40 \\
&p_2 = \frac{3}{2}
\end{aligned} \tag{2}$$

Then, the equilibrium price vector is given by $p^* = (1, \frac{3}{2})$ and the equilibrium allocations (obtained by plugging p_1 and p_2 in the demand functions x_1 and x_2) are:

$$x_1^* = (54, 54)$$

And:

$$x_2^* = (46, 46)$$

Pure exchange economy I

Consider a pure exchange economy with 3 consumers $i \in \{a, b, c\}$ and 2 commodities x_l with $l \in 1, 2$ with prices p_1 and p_2. Let $u_a = x_1^{0.5} x_2^{0.5}$, $u_b = x_1^{0.2} x_2^{0.8}$ and $u_c = x_1^{0.8} x_2^{0.2}$. Initial endowments $\omega_i = (\omega_{1i}, \omega_{2i})$ are given by $\omega_a = (4, 2)$ and $\omega_b = (1, 3)$ and $\omega_c = (2, 1)$.

Define the Walrasian equilibrium of such an economy

ANSWER:

Since each consumer's utility function is a Cobb-Douglas, we can write their demand as follows:

$$x_a = \left(\frac{0.5[4p_1 + 2p_2]}{p_1}, \frac{0.5[4p_1 + 2p_2]}{p_2} \right)$$

$$x_b = \left(\frac{0.2[p_1 + 3p_2]}{p_1}, \frac{0.8[p_1 + 3p_2]}{p_2} \right)$$

$$x_c = \left(\frac{0.8[2p_1 + p_2]}{p_1}, \frac{0.2[2p_1 + p_2]}{p_2} \right)$$

There are two goods. The total endowment of good 1 is $\omega_{1a} + \omega_{1b} + \omega_{1c} = 4 + 1 + 2 = 7$. Then we have to solve: $x_{1a}^* + x_{1b}^* + x_{1c}^* = 7$ (equal and not less than equal because of the non-wastefulness condition). Furthermore, we assume $p_1 = 1$ (since only the ratio of prices matters).

$$
\begin{aligned}
&2+\frac{p_1}{p_2}+\frac{1}{5}+\frac{3}{5}\frac{p_2}{p_1}+\frac{8}{5}+\frac{4}{5}\frac{p_2}{p_1}=7\\
&\frac{19}{5}+\frac{12}{5}\frac{p_2}{p_1}=7\\
&\frac{12}{5}p_2=\frac{16}{5}\\
&p_2=\frac{4}{3}
\end{aligned}
$$

Therefore the equilibrium prices are $(p_1^*, p_2^*) = (1, \frac{4}{3})$. Finally, plug the equilibrium prices and each consumer's demand x_a^*, x_b^*, x_c^* to find their value.

$$
\begin{aligned}
x_a^* &= \left(0.5\left[4+\frac{8}{3}\right], \frac{0.5[4+\frac{8}{3}]}{\frac{4}{3}}\right) = \left(\frac{10}{3}, \frac{5}{2}\right)\\
x_b^* &= \left(0.2\left[1+\frac{16}{3}\right], \frac{0.8[1+\frac{16}{3}]}{\frac{4}{3}}\right) = (1,3)\\
x_c^* &= \left(0.8\left[2+\frac{4}{3}\right], \frac{0.2[2+\frac{4}{3}]}{\frac{4}{3}}\right) = \left(\frac{8}{3}, \frac{1}{2}\right)
\end{aligned}
$$

Pareto Allocations

You must distribute 100 units of a perfectly divisible good among n consumers. An allocation looks like: $x = (x_1, \ldots, x_n)$ with $\sum_i x_i = 100$. Define the set of Pareto Efficient Allocations in these situations:

- $u_1(x) = x_1^2$, $u_2(x) = \log x_2$

ANSWER: First, let's recall the definition of Pareto Efficiency (or Pareto Optimality). An allocation x is Pareto Optimal if there is no other feasible allocation x' such that $x_i' \succeq_i x_i$ for all i and $x_i' \succ_i x_i$ for at least one i. In words, if there are no other allocations that make someone better, all the others are at least as well as before.

In the case of the utility functions above, it is easy to see that any feasible allocation is Pareto Efficient. Take, for instance, $x = (100, 0)$, that is, all the units to the first agent and zero to the second. This is PE since another allocation, say $x' = (99, 1)$, would make the first agent worst off. At the same time, $x' = (99, 1)$ is Pareto Efficient since an allocation $x'' = (98, 2)$ makes the first agent worst off again. But even $x = (100, 0)$ makes the second agent worst off. So we can write the set of all Pareto Efficient Allocations as:

$$
PS = \Big\{x_1, x_2 : x_1 \in [0, 100], x_2 \in [100 - x_1]\Big\}
$$

- $u_1(x) = 3\sqrt{x_1}$, $u_2(x) = -x_2$

ANSWER: take the following (extreme) allocation $x = (100, 0)$, then $u_1(x) = 3\sqrt{x_1} = 30$ and $u_2(x) = 0$. This is Pareto Efficient. Any less extreme allocation, like $x' = (99, 1)$ makes all the agents worst off.

- $u_1(x) = \min\{x_1, 60\}$, $u_2(x) = x_2^{1000}$

ANSWER: In this case, we can see that any allocation that gives to the first agent more than 60 units of x_1 is not Pareto Efficient.

$$PS = \Big\{x_1, x_2 : x_1 \in [0, 60], x_2 \in [100 - x_1]\Big\}$$

- $u_1(x) = x_1 + x_2$, $u_2(x) = x_2$

ANSWER: in this case we can see that the only PE allocation is $x = (0, 100)$. Indeed then $u_1(x) = 100$ and $u_2(x) = 100$. Any other allocation leaves the first agent utility not affected (it always sums to 100) but makes the second agent worst off.

- $u_i(x) = x_1 x_2 x_3$ for any $i \in \{1, 2, 3\}$

Production Economy

Consider a one producer-one consumer economy. The firm has a production function $f(z) = \sqrt{z}$, z is the input (hours of work), wage is w and the price of the output is p. The consumer has utility $u = x_1^{0.5} x_2^{0.5}$, where x_1 is leisure (with $\bar{L} = 24$) and x_2 is the good produced by the firm.

Find the wage w^* such that the price vector $(p, w) = (8, w^*)$ supports the Walrasian equilibrium. Find the consumer's Walrasian demand in equilibrium and check that both markets clear.

ANSWER: Let's write the profit function as follows:

$$\pi(p, w) = p \cdot \sqrt{z} - w \cdot z$$

And find the z^* that maximizes it:

$$\max_z \quad \pi(p, w)$$

Taking the F.O.Cs and solve:

$$\frac{p}{2\sqrt{z}} - w = 0$$
$$\frac{8}{\sqrt{z}} - w = 0$$
$$\frac{8}{\sqrt{z}} = w$$
$$\frac{\sqrt{z}}{4} = \frac{1}{w}$$
$$\sqrt{z} = \frac{4}{w}$$
$$z^* = \frac{16}{w^2}$$

z^* is the input that maximizes profits. Substituting it in $\pi(p,w)$ we obtain:

$$\pi(p,w) = 8 \cdot \sqrt{\frac{16}{w^2}} - w \cdot \frac{16}{w^2}$$
$$\pi(p,w) = \frac{32}{w} - \frac{16}{w}$$
$$\pi(p,w) = \frac{16}{w}$$

The consumer is also a utility maximizer. So its problem is:

$$\max_{x_1,x_2} \quad x_1^{0.5}x_2^{0.5} \quad \text{subj. to} \quad p \cdot x_2 = w \cdot (\bar{L} - x_1) + \pi(p,w)$$

Recall that in the generic form of the UMP, the constraint is offered by the budget line: $p_1 \cdot x_1 + p_2 \cdot x_2 = w$. So we can rewrite the constraint as follows:

$$p \cdot x_2 + w \cdot x_1 = w \cdot \bar{L} + \pi(p,w)$$

The utility function is a Cobb-Douglas. So we can write the demand for x_1 and x_2 as follows:

$$x^* = \left(\frac{0.5 \cdot [24w - \frac{16}{w}]}{w}, \frac{0.5 \cdot [24w - \frac{16}{w}]}{8} \right)$$

Where:

$$x_1^* = 12 + \frac{8}{w^2}$$
$$x_2^* = \frac{3}{2} + \frac{1}{w}$$

Let's check that both markets clear. We know that x_1 is leisure, then we can write $x_1 = \bar{L} - z^*$, and x_2 is the good produced by the firm, so $x_2 = f(z^*)$.

$$
\begin{aligned}
&x_1^* = \bar{L} - z^* \Rightarrow \\
&\Rightarrow 12 + \frac{8}{w^2} = 24 - \frac{16}{w^2} \\
&12 = \frac{24}{w^2} \\
&\frac{w^2}{24} = \frac{1}{12} \\
&w^2 = 2 \\
&w^* = \sqrt{2}
\end{aligned}
$$

And:

$$
\begin{aligned}
x_2 &= \sqrt{\frac{16}{w^2}} \\
x_2 &= \sqrt{\frac{16}{2}} \\
x_2 &= \sqrt{8}
\end{aligned}
$$

Aggregate excess demand function

Consider a pure exchange economy with three agents $i \in \{1, 2, 3\}$. There are two commodities a and b. Initial endowments are $\omega_1 = (10, 2)$, $\omega_2 = (6, 14)$ and $\omega_3 = (8, 8)$. All agents have the same utility function: $u_i = \max\{a, b\}$.

- Compute the aggregate excess demand and show that this economy has no competitive equilibria.

ANSWER: Let's recall first the formal definition of aggregate excess demand:

$$
z(p) = \sum_i z_i(p) = \sum_i x_i(p, p \cdot \omega) - \sum_i \omega_i \quad \text{for all } i
$$

Then, we need to compute the Walrasian demands for each agent. Since all have the same utility functions, the demand is the same. In particular, we have three cases.

First, if $p_a > p_b$, then, all the income will be spent on buying commodity b. Then, the demand is:

$$
x_i = \left(0, \frac{w_i}{p_b}\right)
$$

Recall that: $w_i = p_a \cdot \omega_{ai} + p_b \cdot \omega_{bi}$. So, we can write:

$$
x_1 = \left(0, \frac{10p_a + 2p_b}{p_b}\right)
$$

$$
x_2 = \left(0, \frac{6p_a + 14p_b}{p_b}\right)
$$

$$x_3 = \left(0, \frac{8p_a + 8p_b}{p_b}\right)$$

Then, the aggregate excess of demand can be written as follows:

$$z(p) = \left(0, \frac{10p_a + 2p_b}{p_b}\right) + \left(0, \frac{6p_a + 14p_b}{p_b}\right) + \left(0, \frac{8p_a + 8p_b}{p_b}\right) - [(10, 2) + (6, 14) + (8, 8)]$$

Then:

$$z(p_a) = \frac{10p_a + 2p_b}{p_b} + \frac{6p_a + 14p_b}{p_b} + \frac{8p_a + 8p_b}{p_b} - 24$$

Letting p_b equal to 1, we have:

$$\begin{aligned} z(p_a) =& 10p_a + 2 + 6p_a + 14 + 8p_a + 8 - 24 = 0 \\ & 24p_a + 24 - 24 = 0 \\ & 24p_a = 0 \\ & p_a = 0 \end{aligned}$$

But this contradicts both the assumption of $p_a > p_b$, and especially that of prices being always positive. So markets do not clear for commodity a.

Even the aggregate excess of demand for commodity b is different from zero. Indeed:

$$\begin{aligned} z(p_b) =& 0 - 24 = 0 \\ & - 24 = 0 \end{aligned}$$

Which is clearly impossible.

The same reasoning applies in the case of $p_a < p_b$. Just let's notice that in this case, the demand for each agent becomes:

$$x_i = \left(\frac{w_i}{p_a}, 0\right)$$

Still, the reasoning does not change.

A third possibility is that of prices being equal, $p_a = p_b$. In this case, the consumer will be indifferent between spending all his income on buying a or in buying b. Still, however, in any case, markets do not clear.

- Is this a counterexample of the existence of a general equilibrium? Does the theorem about the existence of an equilibrium apply? If not why?

ANSWER: I think this is not a counterexample of the existence of a general equilibrium, since the theorem does not apply. Indeed one requirement, for each consumer's utility function is that preferences are continuous, strictly convex, and strongly monotone. But $u = \max\{a, b\}$ is not strongly monotone. To see this, take two bundles $(2, 2)$ and $(2, 1)$. Then $u(2, 2) = 2 = u(2, 1)$ but $(2, 2) > (2, 1)$.

Besides, these preferences are even not strictly convex.

The core of a pure exchange economy I

In a 2×2 economy the two consumers have identical utility function $u_i(x_{1i}, x_{2i}) = (x_{1i} + 1)^{0.5}(x_{2i} + 1)^{0.5}$ for $i \in \{1, 2\}$ but different initial endowments: $\omega_1 = (99, 0)$ and $\omega_2 = (0, 99)$. Describe the core of this economy.

ANSWER:

The core of a pure exchange economy II

All consumers have the same utility function: $u_i = x_{1i}x_{2i} + x_{3i}x_{4i}$. Initial endowments are: $\omega_1 = (1, 2, 3, 4)$, $\omega_2 = (4, 3, 2, 2)$, $\omega_3 = (1, 3, 2, 4)$ and $\omega_4 = (4, 2, 3, 1)$.

Does the allocation x such that $x_{li} = \frac{5}{2}$ for any l and any i belong to the core of this economy? Explain.

ANSWER: The core corresponds to the set of all unbeaten allocations. An allocation can be blocked, or beaten, by a coalition of players if this coalition is able to improve upon the proposed allocation.

Given the utility functions and each agent's initial endowment, the core allocations must satisfy the following conditions:

$$\begin{cases} u_1 \geq 16 \\ u_2 \geq 14 \\ u_3 \geq 11 \\ u_4 \geq 11 \\ x_1, x_2, x_3, x_4 > 0 \end{cases}$$

From this, it is apparent that the proposed allocation does not satisfy the conditions for players 1 and 2, since they would receive less utility than they can reach with their initial endowments.