# Hyperbolic honeycomb

In the following pages, we will take a computer-assisted walkthrough through the Hyperbolic space, and we will admire a Hyperbolic honeycomb.

## Honeycombs

A *honeycomb* is a tight packing of polyhedra that allows for no gaps. It can be considered the generalization of tessellations in three-dimensional spaces (and more generally in spaces with any number of dimensions).

We will consider a specific tessellation of the three-dimensional hyperbolic space. This space has a non-Euclidean geometry: for example, the volume of balls in this space is not proportional to the cube of the radius, but instead, it increases exponentially. In some sense, this space is larger and richer than the familiar Euclidean space, and as we will see, stunning patterns populate it.

Honeycombs are beautiful designs; looking at them sheds light on the space structure. A lovely straightforward example is illustrated in the lithograph *Cubic Space Division (1952)*, by the Dutch graphic artist Maurits Cornelis Escher. On this page, it is possible to see a computer rendering inspired by Escher's drawing (Fig.1). The very nature of the Euclidean space is well represented by the grid of parallel alleys that fade away at infinity.

To make a comparison, let us consider a simple tiling of the Hyperbolic plane (Fig.2). In this tiling, the lines (represented by circle arcs) meet forming right angles (as in the Escher's drawing), but the "alleys" continue to branch and move away incapable of keeping the same direction.

## A long digression

This feature of the Hyperbolic plane (and space) is very appropriate for this presentation: we will start with a (Hyperbolic) digression, and for a while, we will move away from the main subject. The computer programs that we will use to explore the Hyperbolic space have been developed for a precise reason that deserves to be narrated. It is a story that involves two errors: the first very creative and engaging, while the second is embarrassing, but led to the creation of beautiful digital images. The title of this presentation should have been more appropriately:

***Tale of two errors***

*How I lost my way and stumbled upon the Hyperbolic Space*

The protagonist of the story is Denis, a ten-year boy who went to primary school in a small town near Venice. A secondary school math teacher, Sofia Sabatti, visited Denis's class in the context of the *Progetto continuità:* a program that allows the students to get to know and be known by future teachers across educational stages.

The teacher handed the pupils some pieces of *Polydron*, a geometric construction toy. The colourful pieces have the shape of regular polygons (squares, equilateral triangles, and many others); all the sides have the same length. The sides are outfitted with ingenious joints that allow snapping together any pair of pieces. With this toy, it is possible to assemble very many kinds of polyhedra, with different sizes and complexity.

The exercise assigned to the children was simple: they had to build solids using only one type of face. Despite its simplicity, the task is involving and leads to many possible discoveries: the five regular convex polyhedra, the surprisingly rich set of deltahedra, the impossibility to use polygons with more than five edges, the difference between convex and concave, and many others.

Our little hero began working with great determination and absolute contempt for the rules. The teacher saw the first results and was smart enough not to blame or interrupt him. The final result was stunning: he had assembled 6 squares, 32 triangles, and 12 decagons, building a very large polyhedron with the same symmetry of the cube and the octahedron (Fig.3). The model is well built, stable, and robust; it is an enjoyable and satisfactory experience re-building it.

It is worth considering that it is unlikely to construct this kind of model by chance. For example, some triangles are connected to a square, but others are not. One must follow a precise mental map to make the correct choice and avoid dead ends. Building the model is a nontrivial task, even for adults; discovering it is much more difficult, requiring imagination in addition to precision. From a didactical point of view, it is engaging that Denis, having decided to disobey the rules, selected a more difficult task compared to the assigned one instead of an easier one.

# Finding the name

Mathematicians like classifications. While some bad educators may degenerate this attitude into a sterile memorization of names, sorting concepts and inventing groups is nevertheless one of the main challenges of math. Names and definitions can contain precious pieces of information. For example, let us consider the concept of *regular polyhedra.* There may seem to be a never-ending variety of solid objects with identical faces; instead, only nine exist (five are convex, and four are star-shaped), which is a fascinating and surprising fact.

Let us consider a negative example and try to find a polyhedron with a different number of sides for each face. We can readily persuade ourselves that such a polyhedron can not exist. Let us consider the face with the largest number of sides and let us call $m$ this number. This face must have $m$ adjacent faces with the number of sides between three and m. At least two of them must have the same number of sides. So this kind of polyhedron can not exist. Again an intriguing fact.

The world of polyhedra is so beautiful and exciting that its taxonomy has grown large and complex, and is still growing. A natural question that arises is: what is the name of the Denis' polyhedron? It has three types of faces, so it can not be a regular polyhedron. It has two kinds of vertices (decagon-triangle-square-triangle and decagon-triangle-decagon), so it can not be a uniform polyhedron. Therefore it should be a *Johnson solid*: by definition, they are not-uniform, convex polyhedra with regular polygons as faces. In 1966 Norman W. Johnson enumerated 92 Johnson solids [1], and in 1969 (relatively recently), Victor A. Zalgaller proved that the list is complete.

Surprisingly the Denis' polyhedron is not on the list. How is that possible?

Sadly we can not conclude that Denis discovered a new polyhedron: the Zalgaller's demonstration rules this out. We must assume that the polyhedron does not exist or, more precisely, that the Denis' model does not represent a polyhedron. It turns out that because the plastic pieces are flexible, their faces are imperceptibly bent (they are not flat polygons) and the hinges along the edges can hide small triangular gaps between adjacent faces. The model is indeed not a polyhedron consisting of regular flat polygons, but this fact does not make it less attractive. It has been used as the logo for "*problemi.xyz"* [2], a website that collects math problems for math teachers.

# Near misses

Beside our model, there are very many entities that are close, but not quite members of the Johnson solids club. Johnson himself found "a number of tantalizingly close misses". The *Johnson solids near-misses* have become an appealing topic. The very definition is not trivial: when exactly a solid can be considered a "near-miss"? Some many papers and webpages propose reasonable definitions and compile lists of specimens. Even Wikipedia has a page dedicated to *Near-miss Johnson solids*.

A paper published in 2001 by Craig S. Kaplan and George W. Hart has a section about *Near Misses* and shows three examples. The first one is the Denis model. [3]

# Evidence of absence

Maria Dedò, a math professor, wrote the story of the "never happening polyhedron" [4] and challenged herself to prove that the polyhedron does not exist. Of course, we already have a demonstration: the model is not in the Johnson list, and the list is complete (because of the Zalgaller's demonstration).

This demonstration is correct, but it is not wholly satisfactory for it says nothing about our model. It would be more helpful to understand exactly where the model failed to be a true polyhedron. To demonstrate that something does not exist is an unusual, stimulating problem.

A possible approach is exploiting the symmetry of our shape.

Fig.4 displays the twelve decagons aligned with a cube. The twelve cube edges split each decagon into two equal parts. For symmetry reasons, we can assume that each decagon forms 45° angles with two cube faces. If we assume that edges are one unit long, then the figure is entirely determined by a single parameter $D$: the distance between the origin and the decagon centres.

Adjacent decagons should share two consecutive vertices. We can compute the distances between the two vertex pairs as functions of parameter D. Equating the first distance to zero fixes the model and allows us to compute the parameter D and the distance between the second vertex pair.

The result is small (relative to the length of the edges), but it is not zero: a small gap separates two adjacent faces. To fill the gap, we must adjust the position of the second vertex, making the polygon not-planar or not regular.

Another approach considers two adjacent decagons, hinged by a shared segment, and an equilateral triangle that completes one of the vertices of the Denis model. The triangle fixes the dihedral angle between the two decagons. Then we add two more decagons and two more triangles. Fig. 5 shows the result. According to the Denis model, the chain of decagons and triangles should close, and the four triangles should connect each other, producing the inner square. The figure is entirely determined (as in the previous case). We can compute the position of all vertices, and therefore it is possible to compute the distance between the chain ends. The distance is not nil: it is not even small. The chain does not close, and the gap is significant. This construction (unlike the previous one) concentrates the defect in a single point instead of spreading it among the whole model; therefore, the gap is more substantial.

# The second error

The "never happening polyhedron" story is a good one. It would be nice to "repair" the model in order to make it an existing polyhedron. I had the (very wrong) intuition that the gap between the decagons could change in curved space and could reduce to zero for a given curvature. This idea, as we will see, is charming, yet utterly wrong.

The root of this fake-intuition is a previous edition of this conference. In *Math & Culture, 2009,* I presented a digitally animated representation of another Escher's lithography: *Circle Limit III (1959)*. The presentation aimed to show the relations between this drawing and one of the models of Hyperbolic geometry. [5]

# The Hyperbolic Geometry

The Hyperbolic Geometry is a non-Euclidean geometry.

Two-dimensional Euclidean geometry describes correctly flat surfaces (and surfaces that can be flattened, such as cones and cylinders). Curved surfaces such as spheres or saddles require other geometries. For example, it is easy to see that on the globe, most theorems of Euclidean geometry do not fit: the sum of the angles of a triangle is larger than 180°, and the ratio between the length of a circle and its diameter is less than π. All these geometrical laws are just a good approximation if we consider small parts of the sphere.

An essential difference between the geometry of the flat plane and the geometry of the sphere is the nature of parallel lines (lines that never meet). In the Euclidean Geometry, given a line and a point not on it, there is one and only one line parallel to the other. This statement is called the *Playfair's axiom* and can be considered equivalent to the *fifth Euclid's postulate*. The statement is not valid on the surface of a sphere: two *lines* (the great circles of the sphere) always cross; therefore, parallel lines do not exist. To describe such a surface we must use a non-Euclidean geometry: the Spherical geometry or the Elliptical geometry (the last one considers a set of antipodal points as a single geometry point; with this smart trick we can say that two lines intersect in a single point, precisely as the non-parallel line of the Euclidean geometry).

The Hyperbolic geometry goes in the opposite direction: we postulate that there are infinite lines parallel to any given line. Even in this case, the consequences are notable: in the Hyperbolic geometry, the sum of angles in triangles is smaller than 180°, while the ratio between the length of a circle and its diameter is larger than π.

There are many models of Hyperbolic geometry that play the role of the globe for Elliptical geometry, but they are less evident. This fact makes the Hyperbolic geometry slightly less intuitive and more exotic.The model that struck Escher's imagination is the *Poincaré disk model*.

The ***points*** of the model are inside the unit disk. The ***lines*** are circle arcs orthogonal to the boundary of the disk or diameters of the disk.

It is possible to demonstrate that two lines meet at most in a single point, and it is clear that for a point not on a line, there are infinite other lines that do not cross the given line.

Angles are measured ordinarily: the Hyperbolic angle between two intersecting lines is equal to the Euclidean angle between the two correspondent curves in the model.

Distance measurement is not straightforward: the scale factor of the model shortens without limit on approaching the disk boundary. If something moves toward the boundary, it appears to shrink more and more. We must consider this effect a sort of perspective deformation: shape and size remain the same despite the alterations of the visual appearance. The Hyperbolic plane, likewise the surface of the sphere, is not flat, but curved: to represent it on a flat surface, it is necessary to accept some deformation. The limiting circle of the Poincaré disk is infinitely far away, and the disk embraces an infinite area.

In Hyperbolic geometry, the first four Euclid's postulates are valid; all the theorems and propositions which depend only on them are also correct. Other theorems are entirely different as we have already pointed out.

For instance, in the Hyperbolic plane do not exist similar figures: if two figures have the same shape, they must also have the same size and therefore, they must be congruent. The similarity is an exclusive feature of the Euclidean world. Indeed the mere existence of a pair of similar, non-congruent triangles is equivalent to Euclid's fifth postulate [6].

Two Hyperbolic squares (polygons with four sides with the same length and with equal angles) with different sizes must have different shapes. The interior angles of the two squares must be different (both less than 90°): the biggest square must have the smallest interior angle.

The fact that interior angles depend on the polygon size is the key feature that makes the Hyperbolic plane so interesting for Escher, the "self-made pattern man", as he defines himself [7]. In Euclidean geometry, interior angles of regular polygons depend only on side number. For instance, each interior angle of a hexagon is always 120°, regardless of the polygon size. Thus it is always possible to put exactly three hexagons around a common vertex so that edges match up. This fact constrains the possible tessellations of the plane. We can use only three kinds of regular polygons to tessellate the whole plane leaving no gaps: hexagons, squares and triangles.

Conversely, the Hyperbolic geometry honours its name by offering a hyperbolical amount of choices. We can adjust the interior angle making it small as we like, just by changing the polygon size. Therefore we can make tessellation with any polygon.

Figure 7 shows a tessellation made of octagons in the Poincaré disk and a digital rendering of the Escher's masterpiece *Circle Limit III*, built on top of this tessellation. All the octagons (and all the Escher's fishes) are congruent, but the model distortion changes their Euclidean visual appearance in the model.

The computer can animate the rendering translating the whole tessellation in the Hyperbolic plane. Through the animation, small octagons near the boundary grow larger while they approach the centre and then shrink again, moving toward the opposite side.

# A possible fix for the Denis' polyhedron?

Figure 8 shows two triples of octagons around a vertex. The Hyperbolic size of the first triplet is too small: the interior angles of the octagons are too large, and the octagons overlap. The computer animation allows to increase the octagons size slowly until the angle is precisely 120° and the three tiles fit nicely together.

There is a tempting analogy with Fig. 5: the assemblage of four pairs of decagons and triangles. Here too, the angles are wrong, and the two decagons at the ends of the chain overlap.

We could consider assembling the model in a curved, Hyperbolic space. Maybe the angles could be different, yielding to a better result. Then we could adjust the angles by changing the model size and trying to find a perfect match.

Sadly, as we will see, the idea is entirely wrong. Nevertheless, it is a very seductive idea that challenges to explore the Hyperbolic three-dimensional space.

# The Hyperbolic space

In the next paragraphs, we will consider a three-dimensional space obeying the Hyperbolic geometry. That is a very natural extension of the Hyperbolic plane although it is a somewhat more difficult concept to digest: in the bidimensional case, we can easily imagine curved surfaces (a sphere for Elliptic geometry and a saddle for Hyperbolic geometry) that adequately explain why we need non-Euclidean geometries. This idea also applies to the three-dimensional case, even if it is rather difficult to visualize curved spaces. We could instead measure angles and lengths and verify if the Euclidean theorems are true or false.

If we make the measurements with reasonable precision, we must admit that the space around us appears Euclidean. However, in September 2015, we had stunning evidence that our space can warp, and that therefore it is not flat. The LIGO (Laser Interferometer Gravitational-Wave Observatory) made the first direct observation of gravitational waves: tiny ripples of the spacetime that altered the length of the detector [8]. The deformation was infinitesimal (it is an extraordinary challenge being able to measure it), but its meaning is enormous.

Gravitational waves are minuscule and temporary deformations of the space geometry, but we know that if we were able to measure the sum of the angles of a vast triangle, the result would be noticeably less than 180°. Indeed the Sun gravity bends the light; we know that light goes straight (it follows the geodesic line of the space) and therefore we must acknowledge that the geometry of our space is authentically not Euclidean.

Of course, the mathematicians like to study even concepts that have no immediate connection with our physical reality, but it is interesting to know that the Hyperbolic spaces (both three- and four-dimensional) are helpful mathematical tools to investigate our universe.

# The Poincaré ball

In order to explore the Hyperbolic space, we will use a model that it is analogous to the Poincaré disk: the Poincaré ball. Hyperbolic **points** are represented by the points inside the unit ball. Ball diameters and specific circle arcs represent the Hyperbolic **lines**. The circle arcs must be contained in the ball, and must be perpendicular to the sphere that bounds the ball. Similarly to the disk model, angles are preserved (Hyperbolic angles are equal to angles in the ball), while distances must be computed with a given formula and appear distorted: Hyperbolic objects closer to the bounding sphere appear smaller when represented in the model, and if a point approaches the sphere, then its distance from the ball centre approaches infinity: from the point of view of the Hyperbolic geometry the sphere is infinitely far.

Our model represents planes as regular planes passing through the origin or spherical surfaces orthogonal to the ball boundary (we consider only the part of the plane or spherical surface inside the ball). We can populate the ball with points, lines, segments, planes and any other geometrical entities we want.

# Hyperbolic transformations

To build and manipulate models in the Poincaré ball, we need some mathematical tools. Moreover, if we want to create a digital representation of the ball and the geometric objects inside it, we must also take care of an efficient implementation of those tools.

Our primary needs are the transformations that move and rotate objects. These belong to the group of transformations that preserve distances: the so-called *isometries* (from two Greek words that mean "equal measure"). Rotation around the ball centre is straightforward; rotation around any other point requires special rules. A possible analogous of Euclidean translation, however, is almost entirely different from the Euclidean case: as we have already noticed, it will change the visual appearance of translated figures; moreover, there is only one translation axis (the line left unchanged by the translation). For comparison, a Euclidean translation leaves a whole family of parallel lines unchanged.

To build the needed transformations, we may start from another isometry, which has a remarkably simple definition in our model: the reflection in a plane Combining an even number of reflections, we can build direct isometries, that is the analogous of euclidean rotations and translations. In the Poincaré ball model the reflection in a Hyperbolic plane is represented by the *geometric inversion* in the sphere that contains the plane (if a Hyperbolic plane passes through the origin then it looks like a regular Euclidean plane, and the reflection is straightforward). The inversion is a well known geometric transformation that fixes points of a sphere and swaps inside and outside points.

# Implementation

In computer graphics, it is natural to represent three-dimensional points by homogeneous coordinates: a vector of four numbers represents a point, and two different vectors that are multiples of each other represent the same point.

With this convention, 4x4 matrices can represent many kinds of Euclidean transformations: translations, rotations, perspective projections and many others. The computers graphics boards have dedicated hardware specialized in multiplying 4x4 matrices and 4-vectors very efficiently.

Luckily 4x4 matrices can also represent Hyperbolic transformations. Therefore it is possible to visualize Hyperbolic worlds taking advantage of all the power of modern computers.[9]

# Building shapes

Fig. 9 visualizes the Hyperbolic ball with some geometric entities inside it. The computer can animate the image in real-time while performing Hyperbolic transformations of the visualized objects.

To create a Hyperbolic regular polyhedron, we can exploit the unique role of the origin (the ball centre). We put a small Euclidean polyhedron at the origin and create Hyperbolic points at the polyhedron vertices.

Then we connect these points with Hyperbolic segments and create Hyperbolic faces limited by these segments. The result is a Hyperbolic polyhedron. For symmetry reasons (in the model, Hyperbolic rotations around the origin are represented by Euclidean rotations) we can assume that also the Hyperbolic polyhedron is regular.

The faces appear concave, and the measure of the dihedral angle between two adjacent faces is less than the Euclidean counterpart.

If we change the polyhedron size (changing the size of the seed Euclidean polyhedron), the dihedral angle changes accordingly, as we expected: the biggest the size, the smaller the angle. (Fig. 10).

# The Denis' polyhedron

Unfortunately, the dream of being able to correct the Denis' polyhedron in the Hyperbolic space stops here. Let us consider the twelve decagons only and recall our previous reasonings: we assumed, for symmetry reasons, that they must be perpendicular to the lines connecting their centres with the centre of the polyhedron. Moreover, we know that such a configuration can not exist in the Euclidean space: pairs of adjacent decagons can share only a vertex each; a tiny triangular fissure separates the facing edges.

Let suppose that Denis' polyhedron can exist in Hyperbolic space, for a carefully selected edge size. Then we should have a configuration of twelve Hyperbolic decagons nicely connected, with no gaps. Each pair of adjacent decagons should share two vertices.

If the centre of the Hyperbolic polyhedron was is in the ball centre, then (for symmetry reasons) the vertices of any given decagon would lie on the same Euclidean plane and would form a Euclidean regular decagon.

Therefore we should have twelve Euclidean regular decagons, with every pair of adjacent decagons sharing an edge, with no gaps.

We know that that is not possible, so we have to admit that even the Hyperbolic version of Denis' polyhedron does not exist.

# The honeycomb

To compensate for the unfortunate discovery, we are going to conclude our quick exploration of Hyperbolic space by building a beautiful model.

We had seen that if we change the size of a regular Hyperbolic polyhedron, then the dihedral angles change as well, as we expected. That allows us to create the three-dimensional analogues of the beautiful tessellations on the Poincaré disk. We will build only one of these tessellations, but their number is infinite (as in the bidimensional case).

We start with a regular Hyperbolic dodecahedron and carefully adjust its size so that the dihedral angle is precisely 90°. In the Poincaré ball, we translate the model so that one vertex lies in the origin. The three edges that share that vertex appear to be straight segments in the model; these segments are all perpendicular to each other, and the whole polyhedron fits nicely into an octant.

Using the reflection transformation, we can create seven replicas of the dodecahedron, filling the whole space around the origin with no gaps.

The replication process can continue forever: the space occupied by the polyhedra is bounded by pentagonal faces; each one of these pentagonal faces belongs to just one dodecahedron. We select one of these faces and the related dodecahedron; then we create a copy of that dodecahedron by reflecting it on the Hyperbolic plane passing through the face. The symmetry of the model ensures that there are no overlaps or fissures between adjacent dodecahedra.

Figure 11 shows a step of the process. We translated the model again so that a dodecahedron sits in the origin. With such large polyhedra, the visual shrinking effect is substantial: the dodecahedra in adjacent shells are smaller and smaller; already in the third shell they are minuscule. On the other side, their number is immense. The whole set resembles dandelion fluff.

# Floating in the Hyperbolic space

We can change the point of view and look at the pattern from the origin of the ball. We will see an immense web of curved lines. At each vertex three lines meet, forming right angles. It is very natural to highlight this feature by visualizing vertices as large cubes and edges as square-section bars. The resulting visual representation recalls Escher's *Cubic Space Division,* the image from which we started. (Fig. 12)

In the computer-generated image, as in the Escher's drawing, details fade away into the distance. This visual effect is also a trick to limit the complexity of the drawing. Even using this trick, the amount of objects that the computer has to draw is immense.

Luckily (as we have already noticed) the graphical board helps us and therefore the last computer animation allows us to float in the Hyperbolic space, following one of the lines while the others diverge and branch out to infinity.

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