Homework 1 - High Dimensional Probability

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Exercise 1

Prove the inequalities:

$$\left(\frac{n}{m}\right)^m \overset{(1)}{\leq} \binom{n}{m} \overset{(2)}{\leq} \sum_{k=0}^{m} \binom{n}{m} \overset{(3)}{\leq} \left(\frac{en}{m}\right)^m \ \forall m \in [1, n]$$

Solution:

$$(1) \quad \frac{\binom{n}{m}}{\left(\frac{n}{m}\right)^m} = \frac{n!}{m!(n-m)!} \cdot \frac{m^m}{n^m} = \frac{n!}{n^m \cdot (n-m)!} \cdot \frac{m^m}{m!}$$
$$= \prod_{i=1}^{m} \frac{n+1-i}{n} \cdot \prod_{i=1}^{m} \frac{m}{m+1-i} = \star$$

We have to prove $\star \geq 1$. This holds if and only if

$$\prod_{i=1}^{m} \frac{n+1-i}{n} \geq \prod_{i=1}^{m} \frac{m+1-i}{m} \iff \frac{n+1-i}{n} \geq \frac{m+1-i}{m}$$
$$\iff 1 + \frac{1-i}{n} \geq 1 + \frac{1-i}{m} \iff \frac{1-i}{n} \geq \frac{1-i}{m} \iff \frac{1}{m} \geq \frac{1}{n}.$$

$$(2) \quad \sum_{k=0}^{m} \binom{n}{m} = \binom{n}{m} + \sum_{k=0}^{m-1} \binom{n}{m} \geq \binom{n}{m}$$

$$(3) \quad \sum_{k=0}^{m} \binom{n}{k} \left(\frac{m}{n}\right)^m \leq \sum_{k=0}^{m} \binom{n}{k} \left(\frac{m}{n}\right)^k \leq \sum_{k=0}^{m} \frac{n^k}{k!} \left(\frac{m}{n}\right)^k = \sum_{k=0}^{m} \frac{m^k}{k!} \leq e^m$$

$\blacksquare$

Exercise 2

Check that in Corollary 0.0.4[1]

$$\left(C + C\varepsilon^2 N\right)^{\lceil 1/\varepsilon^2 \rceil}$$

suce. Here C is a suitable absolute constant.

Solution:

$$(C + C\mathcal{E}^2 N)^{\left\lceil \frac{1}{\varepsilon^2} \right\rceil} = C^{\left\lceil \frac{1}{\varepsilon^2} \right\rceil} \cdot (1 + \mathcal{E}^2 N)^{\left\lceil \frac{1}{\varepsilon^2} \right\rceil}$$

Following the same reasoning used in the proof of the Corollary and observing that we can choose k elements in a set of N points with repetitions in $\binom{N+k-1}{k}$ ways, we can write:

$$\begin{aligned} |\mathcal{N}| \leq \binom{N+k-1}{k} \leq \left(\frac{e(N+k-1)}{k}\right)^k &= e^k \left(\frac{N+k-1}{k}\right)^k \\ &= e^k \left(\frac{N}{k} + 1 - \frac{1}{k}\right)^k \\ &\leq e^k \left(1 + \frac{N}{k}\right)^k \end{aligned}$$

Setting $C = e$ and $k = \lceil \frac{1}{\mathcal{E}^2} \rceil$:

$$|\mathcal{N}| \leq C^{\left\lceil \frac{1}{\varepsilon^2} \right\rceil} \cdot \left(1 + \frac{N}{\left\lceil \frac{1}{\mathcal{E}^2} \right\rceil}\right)^{\left\lceil \frac{1}{\varepsilon^2} \right\rceil} \leq C^{\left\lceil \frac{1}{\varepsilon^2} \right\rceil} \cdot (1 + \mathcal{E}^2 N)^{\left\lceil \frac{1}{\varepsilon^2} \right\rceil}$$

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Exercise 3

Let X be a random variable and $p \in (0, \infty)$. Show that:

$$\mathbb{E}\left[|X|^p\right] = \int_0^\infty p t^{p-1} \mathbb{P}\{|X| > t\} dt$$

Solution:

$$\begin{aligned} \mathbb{E}\left[|X|^p\right] \overset{(1)}{=} \int_0^\infty \mathbb{P}\{|X|^p > t\} dt = \int_0^\infty \mathbb{P}\{|X| > t^{1/p}\} dt \\ = \int_0^\infty p y^{p-1} \mathbb{P}\{|X| > y\} dy \end{aligned}$$

Where in the last step we set $y = t^{1/p} \implies dt = py^{p-1}dy$ and in (1) we applied the result for which, given a positive random variable, $\mathbb{E}[X] = \int_0^\infty \mathbb{P}\{X > t\} dt$.

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[1]Let P be a polytope in with N vertices and whose diameter is bounded by 1. Then P can be covered by at most $N^{\lceil 1/\varepsilon^2 \rceil}$ Euclidean balls of radii $\varepsilon > 0$.

Exercise 4

Prove the inequality:

$$\cosh(x) \leqslant e^{\left(\frac{x^2}{2}\right)} \quad \forall x \in \mathbb{R}$$

Solution:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{\sum_{k=0}^{+\infty} \frac{x^k}{k!} + \sum_{k=0}^{+\infty} \frac{(-x)^k}{k!}}{2} = \sum_{k=0}^{+\infty} \frac{x^{2k}}{(2k)!}.$$

We now claim that

$$\sum_{k=0}^{+\infty} \frac{x^{2k}}{(2k)!} \leq \sum_{k=0}^{+\infty} \frac{x^{2k}}{2^k k!}.$$

Hence, we need to prove, by induction, that $(2k)! \geq 2^k k! \quad \forall k \in \mathbb{N}$.
Simply, if $k = 1$, $2! \geq 2$.
We can hence assume that the relation holds true for $k = 1, \ldots, n$ and prove it for $k = n + 1$:

$$\begin{aligned}
(2(n+1))! &= (2n+2)(2n+1)(2n)! \\
&\geq (2n+2)(2n+1)2^n n! \\
&= (2n+1)2^{n+1}(n+1)! \\
&\geq 2^{n+1}(n+1)!
\end{aligned}$$

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Exercise 5

Let $X_1, X_2, \ldots, X_N$ be independent $Ber(p_i)$ for $i = 1, 2, \ldots, N$.
Put $S_N = \sum_{i=1}^N X_i$ and $\mathbb{E}[S_N] = \mu$. Show that for $\delta \in (0, 1]$ it holds that

$$\mathbb{P}\left\{|S_N - \mu| \geqslant \delta\mu\right\} \leqslant 2e^{-C\mu\delta^2}$$

where $C > 0$ is a constant.

Solution:
Let us first prove that, for $t < \mu$ it holds:

$$\mathbb{P}\left\{S_N \leq t\right\} \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^t.$$

$$\mathbb{P}\left\{\sum_{i=1}^N X_i \leq t\right\} = \mathbb{P}\left\{e^{-\lambda S_N} \geq e^{-\lambda t}\right\} \leq e^{\lambda t}\mathbb{E}\left[e^{-\lambda S_N}\right] = e^{\lambda t}\prod_{i=1}^N \mathbb{E}\left[e^{-\lambda X_i}\right] \tag{1}$$

where the inequality follows form the Markov's one and the last equality from the independence of the variables. Since X_i are Bernoulli random variables,

$$\mathbb{E}\left[e^{-\lambda X_i}\right] = 1 + (e^{-\lambda} - 1)p_i \leq \exp\left(\left(e^{-\lambda} - 1\right)p_i\right).$$

Plugging this into inequality (1) we get:

$$\mathbb{P}\left\{S_N \leq t\right\} \leq e^{\lambda t}\prod_{i=1}^{N}\exp\left(\left(e^{-\lambda} - 1\right)p_i\right) = e^{\lambda t}\exp\left(\left(e^{-\lambda} - 1\right)\mu\right) = e^{-\mu}\left(\frac{\mu e}{t}\right)^{t},$$

where in the last equality we set $\lambda = \log\left(\frac{\mu}{t}\right)$ which is positive since $t < \mu$.

Considering $\delta \in (0, 1]$, we now want to apply the Chernoff's bound to the following:

$$\begin{aligned}\mathbb{P}\left\{|S_N - \mu| \geq \delta\mu\right\} &= \mathbb{P}\left\{S_N \leq (1-\delta)\mu\right\} + \mathbb{P}\left\{S_N \geq (1+\delta)\mu\right\}\\ &\overset{\star}{\leq} e^{-\mu}\left(\frac{e\mu}{(1-\delta)\mu}\right)^{(1-\delta)\mu} + e^{-\mu}\left(\frac{e\mu}{(1+\delta)\mu}\right)^{(1+\delta)\mu},\end{aligned}$$

where in $\star$ we applied the Chernoff's bound on both the addends.

Given what we have obtained so far, we need to prove that the last quantity is smaller than $2e^{-C\mu\delta^2}$ for some constant C. Indeed,

$$\begin{aligned}e^{-\mu}\left(\frac{e\mu}{(1-\delta)\mu}\right)^{(1-\delta)\mu} &+ e^{-\mu}\left(\frac{e\mu}{(1+\delta)\mu}\right)^{(1+\delta)\mu} =\\ &= e^{-\delta\mu-(1-\delta)\mu\log(1-\delta)} + e^{\delta\mu-(1+\delta)\mu\log(1+\delta)}\\ &\overset{\star}{\leq} e^{-\delta\mu+(1-\delta)\mu\left(\delta+\frac{\delta^2}{2}\right)} + e^{\delta\mu-(1+\delta)\mu\left(\delta-\frac{\delta^2}{2}+\frac{\delta^3}{3}\right)}\\ &= e^{-\frac{\delta^2\mu}{2}}\underbrace{\left(e^{\frac{-\delta^3\mu}{2}} + e^{\frac{(\delta^3-2\delta^4)\mu}{6}}\right)}_{\text{A}},\end{aligned}$$

where $\star$ holds since $\log(1-\delta) \geq -\delta - \frac{\delta^2}{2}$ and $\log(1+\delta) > \delta - \frac{\delta^2}{2} + \frac{\delta^3}{3}$ for any $\delta \geq 0$. Finally, $A \leq 2$ for any $\delta \in (0, 1]$ and hence the exercise is solved. In fact, putting all together we proved:

$$\mathbb{P}\{|S_N - \mu| \leq \delta\mu\} \leq e^{-\frac{\delta^2\mu}{2}}\underbrace{\left(e^{\frac{-\delta^3\mu}{2}} + e^{\frac{(\delta^3-2\delta^4)\mu}{6}}\right)}_{\text{A}} \leq 2e^{-C\mu\delta^2},$$

where we set $C = \frac{1}{2}$.

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Exercise 6

Show that for each $p \geq 1$, the random variable $X \sim N(0,1)$ satisfies

$$\|X\|_{L^p} = \left(\mathbb{E}\left[|X|^p\right]\right)^{1/p} = \sqrt{2}\left[\frac{\Gamma((1+p)/2)}{\Gamma(1/2)}\right]^{1/p}.$$

Moreover, deduce that $\|X\|_{L^p} = O(\sqrt{p})$ as $p \longrightarrow +\infty$.

Solution:
Remembering that

$$\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t)\, dt$$

we can write:

$$\begin{aligned}
\|X\|_{L^p}^p &= \int_{-\infty}^{+\infty} |t|^p \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt = \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} t^p \exp\left(-\frac{t^2}{2}\right) dt = \dots
\end{aligned}$$

We now make the following change of variable:

$$y = \frac{t^2}{2}; \quad t = \sqrt{2y}; \quad dt = \frac{1}{\sqrt{2y}} dy.$$

$$\begin{aligned}
\dots &= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} (2y)^{\frac{p}{2}} \exp(-y) \frac{1}{\sqrt{2y}} dy = \\
&= \sqrt{\frac{2}{\pi}} \int_0^{+\infty} 2^{\frac{p}{2}-\frac{1}{2}} y^{\frac{p}{2}-\frac{1}{2}} \exp(-y)\, dy = \\
&= \frac{2^{\frac{p}{2}}}{\sqrt{\pi}} \int_0^{+\infty} y^{\frac{1+p}{2}-1} \exp(-y)\, dy = 2^{\frac{p}{2}} \frac{\Gamma\left(\frac{1+p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}
\end{aligned}$$

since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
It remains to be solve the second part of the exercise:

$$\begin{aligned}
\lim_{p\to\infty} \frac{\|X\|_{L^p}}{\sqrt{p}} &= \lim_{p\to\infty} \frac{\sqrt{2}\left(\frac{\Gamma\left(\frac{1+p}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{\frac{1}{p}}}{\sqrt{p}} = \\
&= \lim_{p\to\infty} \frac{\sqrt{2}\left(\frac{\Gamma\left(\frac{p-1}{2}+1\right)}{\Gamma\left(\frac{1}{2}\right)}\right)^{\frac{1}{p}}}{\sqrt{p}}.
\end{aligned}$$

Now, since we know that $\Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{e}\right)^z$ for $z \to +\infty$ (Stirling's approximation), we can write:

$$
\begin{aligned}
&= \lim_{p\to\infty} \frac{\sqrt{2}\left(\frac{\sqrt{2\pi}\sqrt{\frac{p-1}{2}}\left(\frac{p-1}{2e}\right)^{\frac{p-1}{2}}}{\sqrt{\pi}}\right)^{\frac{1}{p}}}{\sqrt{p}} = \\
&= \lim_{p\to\infty} \frac{\sqrt{2}\left(\sqrt{p-1}\left(\sqrt{\frac{p-1}{2e}}\right)^{p-1}\right)^{\frac{1}{p}}}{\sqrt{p}} = \\
&= \lim_{p\to\infty} \frac{\sqrt{2}\left(\sqrt{p-1}\right)^{\frac{1}{p}}\left(\sqrt{\frac{p-1}{2e}}\right)^{1-\frac{1}{p}}}{\sqrt{p}} = \\
&= \lim_{p\to\infty} \sqrt{2}\cdot\frac{1}{(\sqrt{2e})^{1-\frac{1}{p}}}\cdot\frac{\sqrt{p-1}}{\sqrt{p}} = \frac{1}{\sqrt{e}}
\end{aligned}
$$

Therefore we can conclude that:

$$\|X\|_{L^p} = O(\sqrt{p}) \quad as \quad p \to +\infty.$$

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Exercise 7

Compute the sub-Gaussian norm of $X \sim N(0,\sigma^2)$.

Solution:

$$
\begin{aligned}
\mathbb{E}\left[\exp\left(\frac{X^2}{t^2}\right)\right] &= \int_{-\infty}^{+\infty}\exp\left(\frac{x^2}{t^2}\right)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2}{2\sigma^2}\right)dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty}\exp\left(-\frac{x^2}{2}\left(\frac{1}{\sigma^2}-\frac{2}{t^2}\right)\right)dx \\
&\overset{(2)}{=} \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty}\frac{t\sigma}{\sqrt{t^2-2\sigma^2}}\exp\left(-\frac{y^2}{2}\right)dy \\
&= \frac{t}{\sqrt{t^2-2\sigma^2}},
\end{aligned}
$$

where in (2) we used:

$$y = \frac{x\sqrt{t^2-2\sigma^2}}{t\sigma}, \text{with } t > \sqrt{2}\sigma \Longrightarrow dx = \frac{t\sigma}{\sqrt{t^2-2\sigma^2}}dy. \tag{2}$$

If we now study $\mathbb{E}\left[\exp\left(\frac{X^2}{t^2}\right)\right] \leq 2$ we get:

$$\frac{t}{\sqrt{t^2 - 2\sigma^2}} \leq 2 \iff t^2 \geq \frac{8}{3}\sigma^2 \implies t < -2\sqrt{\frac{2}{3}}\sigma \vee t > 2\sqrt{\frac{2}{3}}\sigma.$$

Since, by the definition of the sub-Gaussian norm, we are considering the infimum among $t \geq 0$ we obtain

$$||X||_{\psi_2} = 2\sqrt{\frac{2}{3}}\sigma.$$

Exercise 8

Compute the sub-Gaussian norm of $X \sim SymBer$.

Solution:

$$\|X\|_{\psi_2} = \inf\left\{ t > 0 \quad | \quad \mathbb{E}\left[\exp\left(\frac{X^2}{t^2}\right)\right] \leq 2 \right\}$$

$$\mathbb{E}\left[\exp\left(\frac{X^2}{t^2}\right)\right] = \exp\left(\frac{1}{t^2}\right) \leq 2$$

$$\implies \frac{1}{t^2} \leq \log 2$$

$$t \leq -\frac{1}{\sqrt{\log 2}} \vee t \geq \frac{1}{\sqrt{\log 2}}$$

So $\|X\|_{\psi_2} = \frac{1}{\sqrt{\log 2}}$.

Exercise 9

Let X be a bounded r.v.. Show that

$$\|X\|_{\psi_2} \leqslant \frac{\|X\|_\infty}{\sqrt{\log 2}}.$$

Solution:
We recall that

$$\|X\|_{\psi_2} = \inf\left\{ t > 0 \quad | \quad \mathbb{E}\left[\exp\left(\frac{X^2}{t^2}\right) \leq 2\right] \right\}$$

and

$$\|X\|_\infty = \inf\left\{ a \in \mathbb{R} \quad | \quad \mathbb{P}\left(X^{-1}(a, +\infty)\right) = 0 \right\}.$$

Since X is bounded we have that $\exists a \in \mathbb{R} : X(\omega) \leq a \quad \forall \omega$ and therefore $\|X\|_\infty = a$. Now we assume, ab absurdo, that

$$\|X\|_{\psi_2} > \frac{a}{\sqrt{\log(2)}}.$$

Then:

$$\begin{aligned}\mathbb{E}\left[\exp\left(\frac{X^2}{\|X\|_\psi^2}\right)\right] &< \mathbb{E}\left[\exp\left(\frac{X^2\log(2)}{a^2}\right)\right] \\ &\leq \mathbb{E}\left[\exp(\log 2)\right] = 2\end{aligned}$$

Therefore

$$\exists \bar{t} = \frac{a}{\sqrt{\log 2}} \quad : \quad \bar{t} < \|X\|_{\psi_2} \text{ and } \mathbb{E}\left[\exp\left(\frac{X^2}{\bar{t}^2}\right)\right] \leq 2,$$

which is a contradiction by the definition of $\|\cdot\|_{\psi_2}$.

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Exercise 10

Show that $||X||_{\psi_2}$ is a norm.

Solution:
We want to prove that

$$||X||_{\psi_2} = \inf\left\{t > 0 \quad | \quad \mathbb{E}\left[\exp\left(\frac{X^2}{t^2}\right)\right] \leq 2\right\}$$

is a norm, hence we need to demonstrate:

(i) $X \equiv 0 \Longleftrightarrow ||X||_{\psi_2} = 0$;

(ii) $||\lambda X||_{\psi_2} = |\lambda|\, ||X||_{\psi_2} \quad \forall \lambda \in \mathbb{R}$;

(iii) $||X+Y||_{\psi_2} \leq ||X||_{\psi_2} + ||Y||_{\psi_2}$.

(i) Assuming $X \equiv 0$ we have:

$$\begin{aligned}||X||_{\psi_2} &= \inf\left\{t > 0 \quad | \quad \mathbb{E}\left[\exp\left(\frac{0}{t^2}\right)\right] \leq 2\right\} = \\ &= \inf\left\{t > 0 \quad | \quad \mathbb{E}\left[1\right] \leq 2\right\} = \inf\left\{t > 0\right\} = 0.\end{aligned}$$

On the other hand, let $||X||_{\psi_2} = 0$ (we consider as null random variables all those having at most a zero measure set of non-null values) and suppose for a contradiction that there exists a measurable set $\Omega \in \mathbb{R}$ such that $\mathbb{P}\{X \in \Omega\} > 0$. Notice that, by hypothesis, we know that

$$\mathbb{E}\left[\exp\left(\frac{X^2}{t^2}\right)\right] = \int_\Omega p(x)\exp\left(\frac{x^2}{t^2}\right)dx \leq 2,$$

for any arbitrarily small $t > 0$. Choosing any $0 < t_\star < t$, we have that:

$$p(x)\exp\left(\frac{x^2}{t_\star^2}\right) > p(x)\exp\left(\frac{x^2}{t^2}\right) \quad \forall x \in \Omega,$$

which implies:

$$\int_\Omega p(x)\exp\left(\frac{x^2}{t_\star^2}\right)dx > \int_\Omega p(x)\exp\left(\frac{x^2}{t^2}\right)dx \quad \forall x \in \Omega.$$

Therefore, if we define $\{t_n\}_{n\in\mathbb{N}} = \frac{t}{n}$, since $\lim_{n\to\infty} exp(\frac{x^2}{t_n}) = \lim_{n\to\infty} exp(\frac{nx^2}{t}) = +\infty$, it will always be possible to find an $n_\star$ such that:

$$\mathbb{E}\left[\exp\left(\frac{X^2}{t_{n_\star}^2}\right)\right] = \int_\Omega p(x)\exp\left(\frac{x^2}{t_{n_\star}^2}\right)dx = \int_\Omega p(x)\exp\left(\frac{n_\star x^2}{t^2}\right)dx > 2$$

which is a contradiction.

(ii) Fixing $\lambda \in \mathbb{R}$ we have:

$$\begin{aligned}||\lambda X||_{\psi_2} &= \inf\left\{t > 0 \;\mid\; \mathbb{E}\left[\exp\left(\frac{\lambda^2 X^2}{t^2}\right)\right] \leq 2\right\} = \\ &= \inf\left\{|\lambda|\frac{t}{|\lambda|} > 0 \;\mid\; \mathbb{E}\left[\exp\left(\frac{X^2}{\left(\frac{t}{\lambda}\right)^2}\right)\right] \leq 2\right\} = \\ &= |\lambda|\inf\left\{y > 0 \;\mid\; \mathbb{E}\left[\exp\left(\frac{X^2}{y^2}\right)\right] \leq 2\right\} = |\lambda|\,||X||_{\psi_2}.\end{aligned}$$

(iii) Let us consider $f(x) = \exp(x^2)$, which is a convex and increasing function. Now, considering $a, b > 0$ we have that $f\left(\frac{|X+Y|}{a+b}\right) \leq f\left(\frac{|X|+|Y|}{a+b}\right)$. Moreover, for the convexity we can write:

$$f\left(\frac{|X|+|Y|}{a+b}\right) = f\left(\frac{a}{a+b}\frac{|X|}{a} + \frac{b}{a+b}\frac{|Y|}{b}\right) \leq \frac{a}{a+b}f\left(\frac{|X|}{a}\right) + \frac{b}{a+b}f\left(\frac{|Y|}{b}\right).$$

As a consequence, picking $a = \|X\|_{\psi_2}^2$ and $b = \|Y\|_{\psi_2}^2$, substituting f and taking the expectations we get:

$$\mathbb{E}\left[\exp\left(\frac{X+Y}{||X||_{\psi_2}+||Y||_{\psi_2}}\right)^2\right] \leq$$

$$\leq \frac{||X||_{\psi_2}{}^2}{||X||_{\psi_2}{}^2+||Y||_{\psi_2}{}^2}\mathbb{E}\left[\exp\left(\frac{X}{||X||_{\psi_2}}\right)^2\right] + \frac{||Y||_{\psi_2}{}^2}{||X||_{\psi_2}{}^2+||Y||_{\psi_2}{}^2}\mathbb{E}\left[\exp\left(\frac{Y}{||Y||_{\psi_2}}\right)^2\right] \leq$$

$$\leq \frac{||X||_{\psi_2}{}^2}{||X||_{\psi_2}{}^2+||Y||_{\psi_2}{}^2}\cdot 2 + \frac{||Y||_{\psi_2}{}^2}{||X||_{\psi_2}{}^2+||Y||_{\psi_2}{}^2}\cdot 2 = 2.$$

Hence we showed that

$$\mathbb{E}\left[\exp\left(\frac{X+Y}{||X||_{\psi_2}+||Y||_{\psi_2}}\right)^2\right] \leq 2$$

from which it follows that

$$||X||_{\psi_2}{}^2+||Y||_{\psi_2}{}^2 \in \{t>0 \quad | \quad \mathbb{E}\left[\exp\left(\frac{X+Y}{t}\right)^2\right] \leq 2\}$$

$$\Downarrow$$

$$\|X+Y\|_{\psi_2} \leq \|X\|_{\psi_2} + \|Y\|_{\psi_2}.$$

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