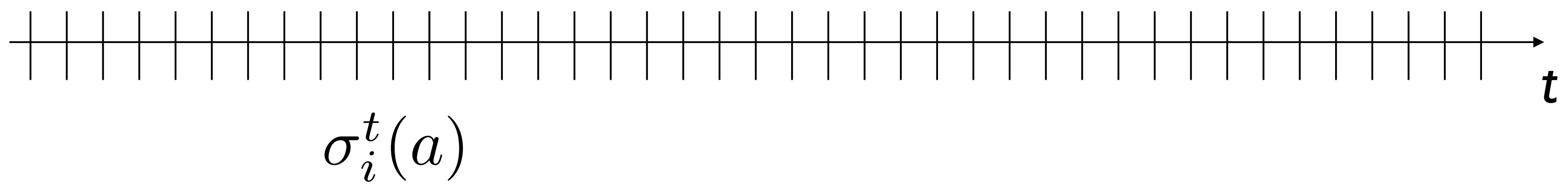
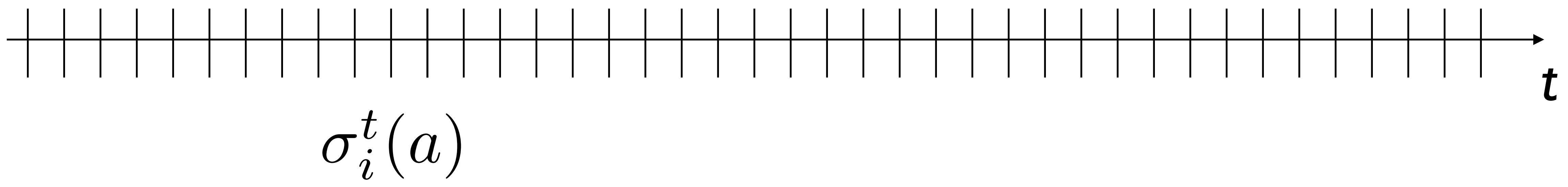
Regret Matching (RM) and Counter Factual Regret (CFR) minimization

POLITECNICO
MILANO 1863

Adaptive strategies

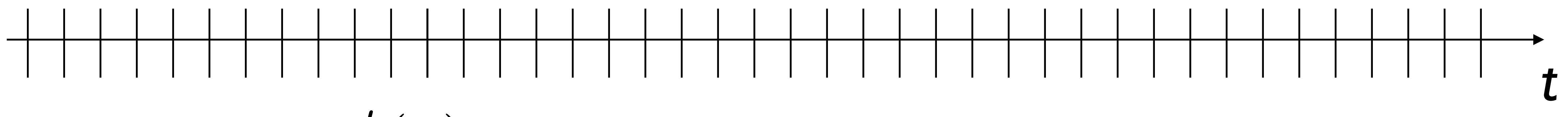


Adaptive strategies



	R	**P**	**S**
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

Adaptive strategies



$\sigma_i^t(a)$

	R	P	S
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

$$\sigma_1^1(a) = \begin{cases} \text{R} & 1/3 \\ \text{P} & 1/3 \\ \text{S} & 1/3 \end{cases} \qquad \sigma_2^1(a) = \begin{cases} \text{R} & 1/3 \\ \text{P} & 1/3 \\ \text{S} & 1/3 \end{cases}$$

Algorithm

- At every iteration
  - Calculate the instantaneous regret
  - Calculate the cumulative regrets
  - Calculate the cumulative regret plus
  - Update the strategies accordingly

Regret

For every action a, the (instantaneous) ***regret*** at time t represents the difference between the expected utility provided by that action and the expected utility of the current strategy

$$r_i^t(a) = \mathbb{E}[U_i(a, \sigma_{-i}^t)] - \mathbb{E}[U_i(\sigma_i^t, \sigma_{-i}^t)]$$

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expected utility provided by action a

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expected utility provided by action a

expected utility provided by the current strategy of player i

Example

	R	P	S
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
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$$\sigma_1^1(a) = \begin{cases} \mathrm{R} & 1/3 \\ \mathrm{P} & 1/3 \\ \mathrm{S} & 1/3 \end{cases} \qquad \sigma_2^1(a) = \begin{cases} \mathrm{R} & 1/3 \\ \mathrm{P} & 1/3 \\ \mathrm{S} & 1/3 \end{cases}$$

Example

$$r_1^1(\mathrm{R}) = \mathbb{E}[U_1(\mathrm{R}, \sigma_2^1)] - \mathbb{E}[U_1(\sigma_1^1, \sigma_2^1)] = 0$$

	R	**P**	**S**
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
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Example

$$r_1^1(\mathrm{R}) = \boxed{\mathbb{E}[U_1(\mathrm{R}, \sigma_2^1)]} - \mathbb{E}[U_1(\sigma_1^1, \sigma_2^1)] = 0$$

$$\left(0 \cdot \tfrac{1}{3} - 1 \cdot \tfrac{1}{3} + 1 \cdot \tfrac{1}{3}\right)$$

	R	**P**	**S**
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

$$\sigma_1^1(a) = \begin{cases} \mathrm{R} & 1/3 \\ \mathrm{P} & 1/3 \\ \mathrm{S} & 1/3 \end{cases} \qquad \sigma_2^1(a) = \begin{cases} \mathrm{R} & 1/3 \\ \mathrm{P} & 1/3 \\ \mathrm{S} & 1/3 \end{cases}$$

Example

$$r_1^1(\mathrm{R}) = \underbrace{\boxed{\mathbb{E}[U_1(\mathrm{R}, \sigma_2^1)]}}_{(0\cdot\frac{1}{3} - 1\cdot\frac{1}{3} + 1\cdot\frac{1}{3})} - \underbrace{\boxed{\mathbb{E}[U_1(\sigma_1^1, \sigma_2^1)]}}_{(0\cdot\frac{1}{3} - \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3})} = 0$$

	R	**P**	**S**
R	0 , 0	-1 , 3	1 , -2
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$$\sigma_1^1(a) = \begin{cases} \mathrm{R} & 1/3 \\ \mathrm{P} & 1/3 \\ \mathrm{S} & 1/3 \end{cases} \qquad \sigma_2^1(a) = \begin{cases} \mathrm{R} & 1/3 \\ \mathrm{P} & 1/3 \\ \mathrm{S} & 1/3 \end{cases}$$

Example

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R	0 , 0	-1 , 3	1 , -2
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$$\sigma_1^1(a) = \begin{cases} \text{R} & 1/3 \\ \text{P} & 1/3 \\ \text{S} & 1/3 \end{cases} \qquad \sigma_2^1(a) = \begin{cases} \text{R} & 1/3 \\ \text{P} & 1/3 \\ \text{S} & 1/3 \end{cases}$$

$$r_1^1(\text{R}) = \boxed{\mathbb{E}[U_1(\text{R}, \sigma_2^1)]} - \boxed{\mathbb{E}[U_1(\sigma_1^1, \sigma_2^1)]} = 0$$

$$\left(0 \cdot \tfrac{1}{3} - 1 \cdot \tfrac{1}{3} + 1 \cdot \tfrac{1}{3}\right) \qquad \left(0 \cdot \tfrac{1}{3} - \tfrac{1}{3} \cdot \tfrac{1}{3} + \tfrac{1}{3} \cdot \tfrac{1}{3}\right)$$

$$r_1^1(\text{P}) = \boxed{\mathbb{E}[U_1(\text{P}, \sigma_2^1)]} - \boxed{\mathbb{E}[U_1(\sigma_1^1, \sigma_2^1)]} = -\frac{1}{3}$$

$$\left(1 \cdot \tfrac{1}{3} + 0 \cdot \tfrac{1}{3} - 2 \cdot \tfrac{1}{3}\right) \qquad \left(0 \cdot \tfrac{1}{3} - \tfrac{1}{3} \cdot \tfrac{1}{3} + \tfrac{1}{3} \cdot \tfrac{1}{3}\right)$$

Example

	R	P	S
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

$$r_1^1(\mathrm{R}) = \boxed{\mathbb{E}[U_1(\mathrm{R}, \sigma_2^1)]} - \boxed{\mathbb{E}[U_1(\sigma_1^1, \sigma_2^1)]} = 0$$

$$(0 \cdot \tfrac{1}{3} - 1 \cdot \tfrac{1}{3} + 1 \cdot \tfrac{1}{3}) \qquad (0 \cdot \tfrac{1}{3} - \tfrac{1}{3} \cdot \tfrac{1}{3} + \tfrac{1}{3} \cdot \tfrac{1}{3})$$

$$r_1^1(\mathrm{P}) = \boxed{\mathbb{E}[U_1(\mathrm{P}, \sigma_2^1)]} - \boxed{\mathbb{E}[U_1(\sigma_1^1, \sigma_2^1)]} = -\frac{1}{3}$$

$$(1 \cdot \tfrac{1}{3} + 0 \cdot \tfrac{1}{3} - 2 \cdot \tfrac{1}{3}) \qquad (0 \cdot \tfrac{1}{3} - \tfrac{1}{3} \cdot \tfrac{1}{3} + \tfrac{1}{3} \cdot \tfrac{1}{3})$$

$$r_1^1(\mathrm{S}) = \boxed{\mathbb{E}[U_1(\mathrm{S}, \sigma_2^1)]} - \boxed{\mathbb{E}[U_1(\sigma_1^1, \sigma_2^1)]} = \frac{1}{3}$$

$$(-2 \cdot \tfrac{1}{3} + 3 \cdot \tfrac{1}{3} + 0 \cdot \tfrac{1}{3}) \qquad (0 \cdot \tfrac{1}{3} - \tfrac{1}{3} \cdot \tfrac{1}{3} + \tfrac{1}{3} \cdot \tfrac{1}{3})$$

Example

	R	P	S
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

$$r_2^1(\mathrm{R}) = \underbrace{\boxed{\mathbb{E}[U_2(\mathrm{R}, \sigma_1^1)]}}_{(0 \cdot \frac{1}{3} - 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3})} - \underbrace{\boxed{\mathbb{E}[U_2(\sigma_2^1, \sigma_1^1)]}}_{(0 \cdot \frac{1}{3} - \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3})} = -\frac{1}{3}$$

$$r_2^1(\mathrm{P}) = \underbrace{\boxed{\mathbb{E}[U_2(\mathrm{P}, \sigma_1^1)]}}_{(1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} - 2 \cdot \frac{1}{3})} - \underbrace{\boxed{\mathbb{E}[U_2(\sigma_2^1, \sigma_1^1)]}}_{(0 \cdot \frac{1}{3} - \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3})} = \frac{2}{3}$$

$$r_2^1(\mathrm{S}) = \underbrace{\boxed{\mathbb{E}[U_2(\mathrm{S}, \sigma_1^1)]}}_{(-2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3})} - \underbrace{\boxed{\mathbb{E}[U_2(\sigma_2^1, \sigma_1^1)]}}_{(0 \cdot \frac{1}{3} - \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3})} = -\frac{1}{3}$$

Cumulative regret

For every action a, the ***cumulative regret*** at time t represents the sum, for every time from 1 to t, of the difference between the expected utility provided by that action and the expected utility of the current strategy

$$R_i^t(a) = \sum_{\tau=1}^{t} r_i^{\tau}(a)$$

Example

	R	P	S
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

$$R_1^1(\mathrm{R}) = \qquad r_1^1(\mathrm{R}) = \qquad 0$$

$$R_1^1(\mathrm{P}) = \qquad r_1^1(\mathrm{P}) = \qquad -\frac{1}{3}$$

$$R_1^1(\mathrm{S}) = \qquad r_1^1(\mathrm{S}) = \qquad \frac{1}{3}$$

$$R_2^1(\mathrm{R}) = \qquad r_2^1(\mathrm{R}) = \qquad -\frac{1}{3}$$

$$R_2^1(\mathrm{P}) = \qquad r_2^1(\mathrm{P}) = \qquad \frac{2}{3}$$

$$R_2^1(\mathrm{S}) = \qquad r_2^1(\mathrm{S}) = \qquad -\frac{1}{3}$$

Cumulative regret plus

We take only positive cumulative regrets

$$R_i^{t,+}(a) = \max\{R_i^t(a), 0\}$$

Example

	R	P	S
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

$$R_1^{1,+}(\mathrm{R}) = \qquad R_1^1(\mathrm{R}) = \qquad 0$$

$$R_1^{1,+}(\mathrm{P}) = \qquad 0$$

$$R_1^{1,+}(\mathrm{S}) = \qquad R_1^1(\mathrm{S}) = \qquad \frac{1}{3}$$

$$R_2^{1,+}(\mathrm{R}) = \qquad 0$$

$$R_2^{1,+}(\mathrm{P}) = \qquad R_2^1(\mathrm{P}) = \qquad \frac{2}{3}$$

$$R_2^{1,+}(\mathrm{S}) = \qquad 0$$

Update rule: Regret Matching (RM)

For every action a, the new strategy is given by the ratio between the cumulative regret plus of that strategy and the sum of the cumulative regret plus of all the actions of the player

$$\sigma_i^{t+1}(a) = \begin{cases} \frac{R_i^{t-1,+}(a)}{\sum_{a'} R_i^{t-1,+}(a')} & \text{if } \sum_{a'} R_i^{t-1,+}(a') > 0 \\ \frac{1}{|A_i|} & \text{if } \sum_{a'} R_i^{t-1,+}(a') = 0 \end{cases}$$

Example

	R	P	S
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

$$\sigma_1^2(\text{R}) = \frac{0}{\frac{1}{3}} = 0$$

$$\sigma_1^2(\text{P}) = \frac{0}{\frac{1}{3}} = 0$$

$$\sigma_1^2(\text{S}) = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$$

$$\sigma_2^2(\text{R}) = \frac{0}{\frac{2}{3}} = 0$$

$$\sigma_2^2(\text{P}) = \frac{\frac{2}{3}}{\frac{2}{3}} = 1$$

$$\sigma_2^2(\text{S}) = \frac{0}{\frac{2}{3}} = 0$$

Example

	R	P	S
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

$$r_1^2(\mathrm{R}) = \boxed{\mathbb{E}[U_1(\mathrm{R}, \sigma_2^2)]} - \boxed{\mathbb{E}[U_1(\sigma_1^2, \sigma_2^2)]} = -4$$

$$(0 \cdot 0 - 1 \cdot 1 + 1 \cdot 0) \qquad 3$$

$$r_1^2(\mathrm{P}) = \boxed{\mathbb{E}[U_1(\mathrm{P}, \sigma_2^2)]} - \boxed{\mathbb{E}[U_1(\sigma_1^2, \sigma_2^2)]} = -3$$

$$(1 \cdot 0 + 0 \cdot 1 - 2 \cdot 0) \qquad 3$$

$$r_1^2(\mathrm{S}) = \boxed{\mathbb{E}[U_1(\mathrm{S}, \sigma_2^2)]} - \boxed{\mathbb{E}[U_1(\sigma_1^2, \sigma_2^2)]} = 0$$

$$(-2 \cdot 0 + 3 \cdot 1 + 0 \cdot 0) \qquad 3$$

Example

	R	P	S
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

$$r_2^2(\mathrm{R}) = \underbrace{\boxed{\mathbb{E}[U_2(\mathrm{R}, \sigma_1^2)]}}_{(0 \cdot 0 - 2 \cdot 0 + 1 \cdot 1)} - \underbrace{\boxed{\mathbb{E}[U_2(\sigma_2^2, \sigma_1^2)]}}_{-1} = 2$$

$$r_2^2(\mathrm{P}) = \underbrace{\boxed{\mathbb{E}[U_2(\mathrm{P}, \sigma_1^2)]}}_{(3 \cdot 0 + 0 \cdot 0 - 1 \cdot 1)} - \underbrace{\boxed{\mathbb{E}[U_2(\sigma_2^2, \sigma_1^2)]}}_{-1} = 0$$

$$r_2^2(\mathrm{S}) = \underbrace{\boxed{\mathbb{E}[U_2(\mathrm{S}, \sigma_1^2)]}}_{(-2 \cdot 0 + 1 \cdot 0 + 0 \cdot 1)} - \underbrace{\boxed{\mathbb{E}[U_2(\sigma_2^2, \sigma_1^2)]}}_{-1} = 1$$

Example

	R	P	S
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

$$R_1^2(\mathrm{R}) = \quad r_1^1(\mathrm{R}) \quad + \quad r_1^2(\mathrm{R}) = \quad -4$$

$$R_1^2(\mathrm{P}) = \quad r_1^1(\mathrm{P}) \quad + \quad r_1^2(\mathrm{P}) = \quad -\frac{10}{3}$$

$$R_1^2(\mathrm{S}) = \quad r_1^1(\mathrm{S}) \quad + \quad r_1^2(\mathrm{S}) = \quad \frac{1}{3}$$

$$R_2^2(\mathrm{R}) = \quad r_2^1(\mathrm{R}) \quad + \quad r_2^2(\mathrm{R}) = \quad \frac{5}{3}$$

$$R_2^2(\mathrm{P}) = \quad r_2^1(\mathrm{P}) \quad + \quad r_2^2(\mathrm{P}) = \quad \frac{2}{3}$$

$$R_2^2(\mathrm{S}) = \quad r_2^1(\mathrm{S}) \quad + \quad r_2^2(\mathrm{S}) = \quad \frac{2}{3}$$

Example

	R	P	S
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

$$R_1^{2,+}(\mathrm{R}) = 0$$
$$R_1^{2,+}(\mathrm{P}) = 0$$
$$R_1^{2,+}(\mathrm{S}) = \frac{1}{3}$$

$$R_2^{2,+}(\mathrm{R}) = \frac{5}{3}$$
$$R_2^{2,+}(\mathrm{P}) = \frac{2}{3}$$
$$R_2^{2,+}(\mathrm{S}) = \frac{2}{3}$$

Example

	R	P	S
R	0 , 0	-1 , 3	1 , -2
P	1 , -2	0 , 0	-2 , 1
S	-2 , 1	3 , -1	0 , 0

$$\sigma_1^3(\mathrm{R}) = \frac{0}{\frac{1}{3}} = 0$$

$$\sigma_1^3(\mathrm{P}) = \frac{0}{\frac{1}{3}} = 0$$

$$\sigma_1^3(\mathrm{S}) = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$$

$$\sigma_2^3(\mathrm{R}) = \frac{\frac{5}{3}}{\frac{9}{3}} = \frac{5}{9}$$

$$\sigma_2^3(\mathrm{P}) = \frac{\frac{2}{3}}{\frac{9}{3}} = \frac{2}{9}$$

$$\sigma_2^3(\mathrm{S}) = \frac{\frac{2}{3}}{\frac{9}{3}} = \frac{2}{9}$$

Convergence

- As t increases, the average strategy from 1 to t returned by the Regret Matching algorithm converges to a Nash equilibrium in 2-player zero-sum games

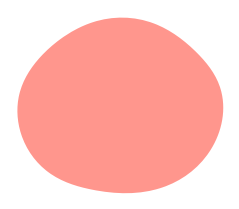
- The cumulative regret decreases as

$$\frac{R_i^t(a)}{t} \leq \frac{m}{\sqrt{t}} \Delta_{\max}$$

- The epsilon of the epsilon-Nash equilibrium decreases as

$$\epsilon \leq \frac{2m}{\sqrt{t}} \Delta_{\max}$$

Approachability



Approachability

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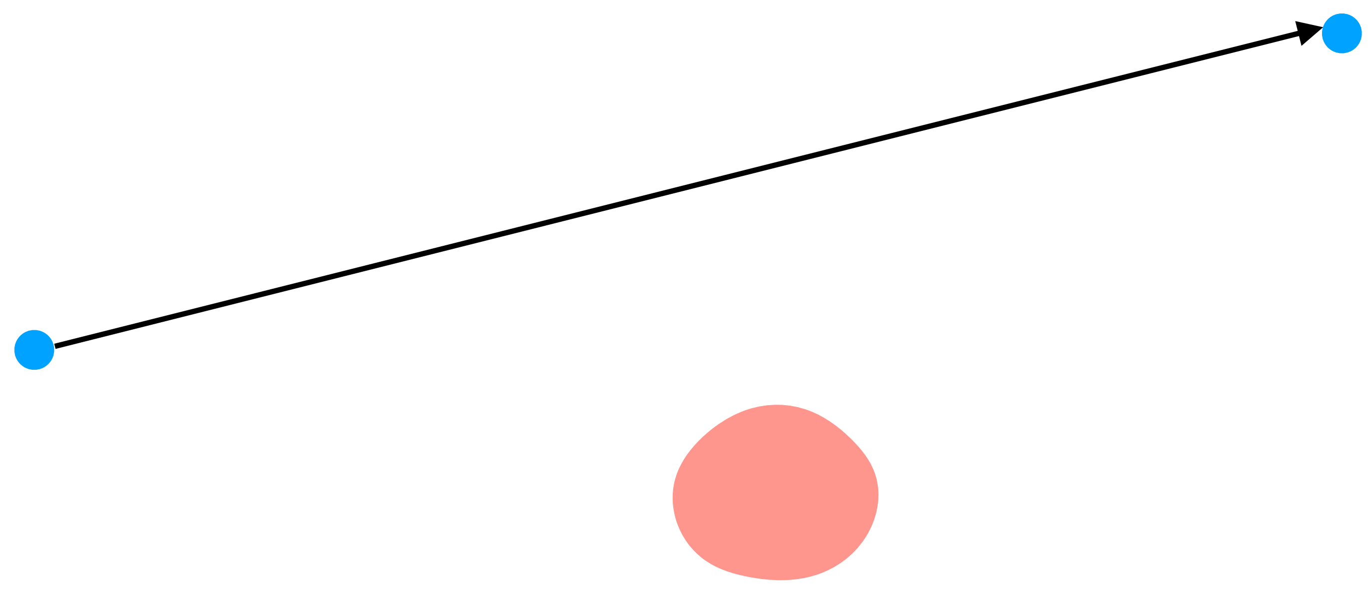
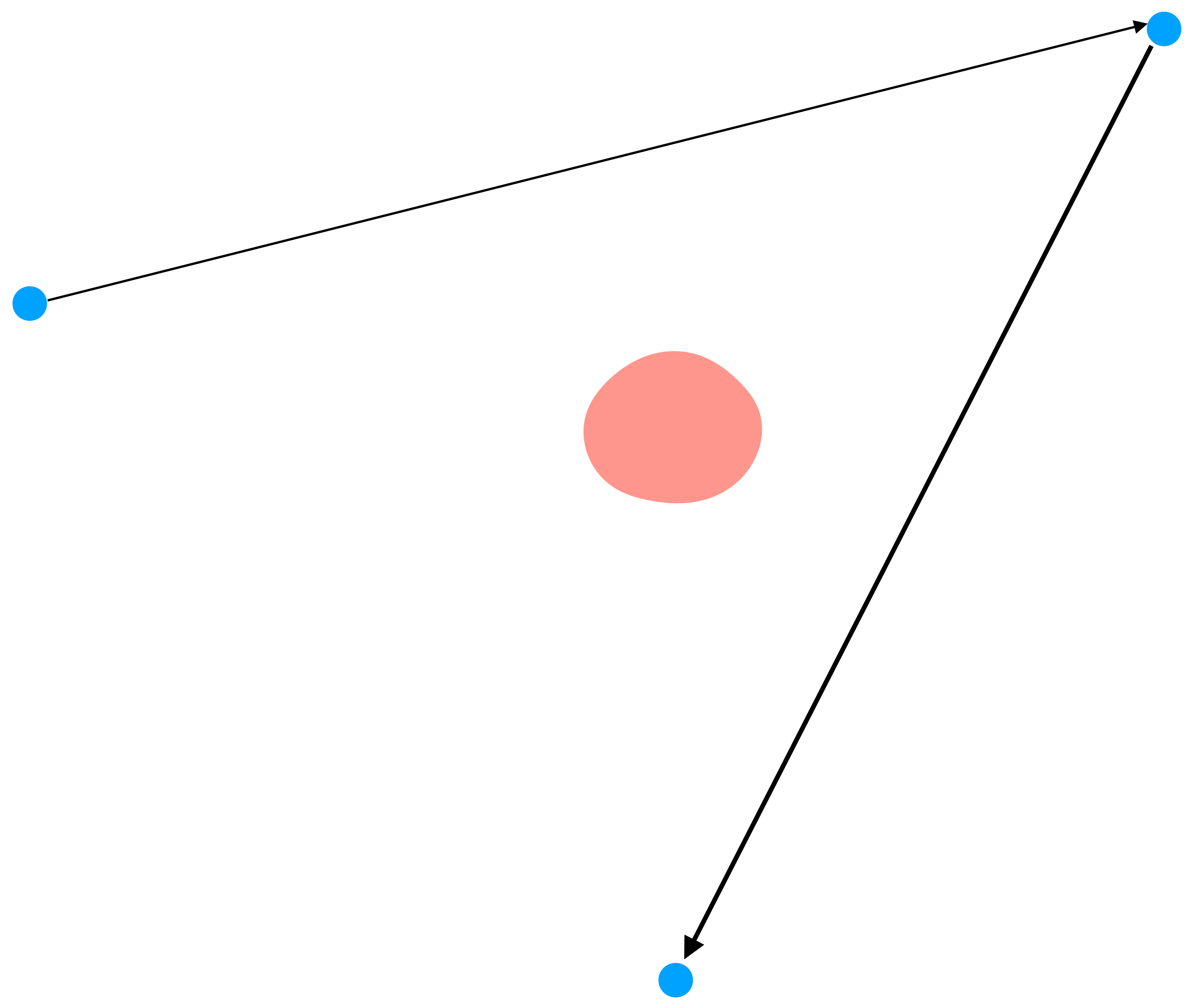


Approachability



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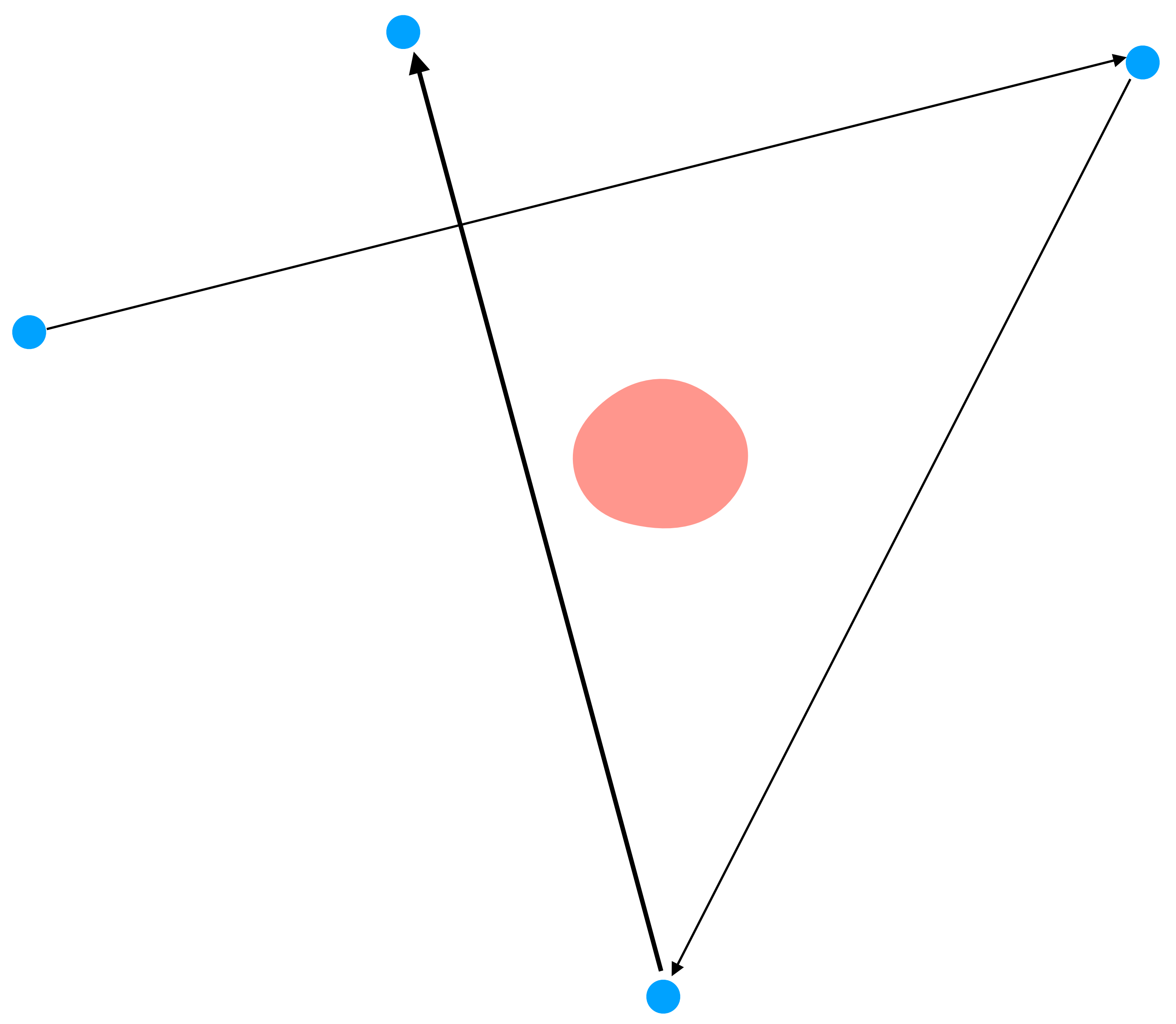

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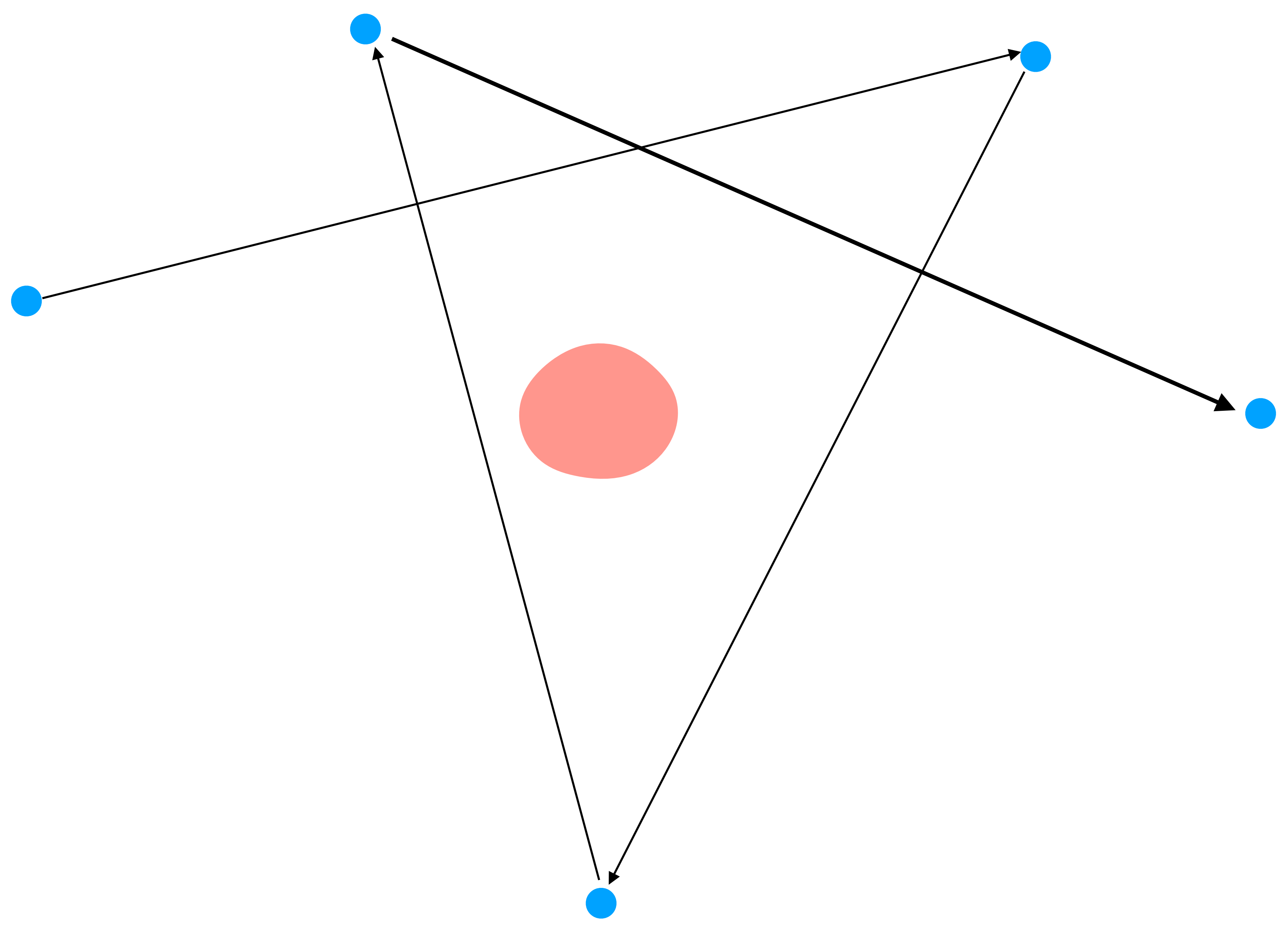
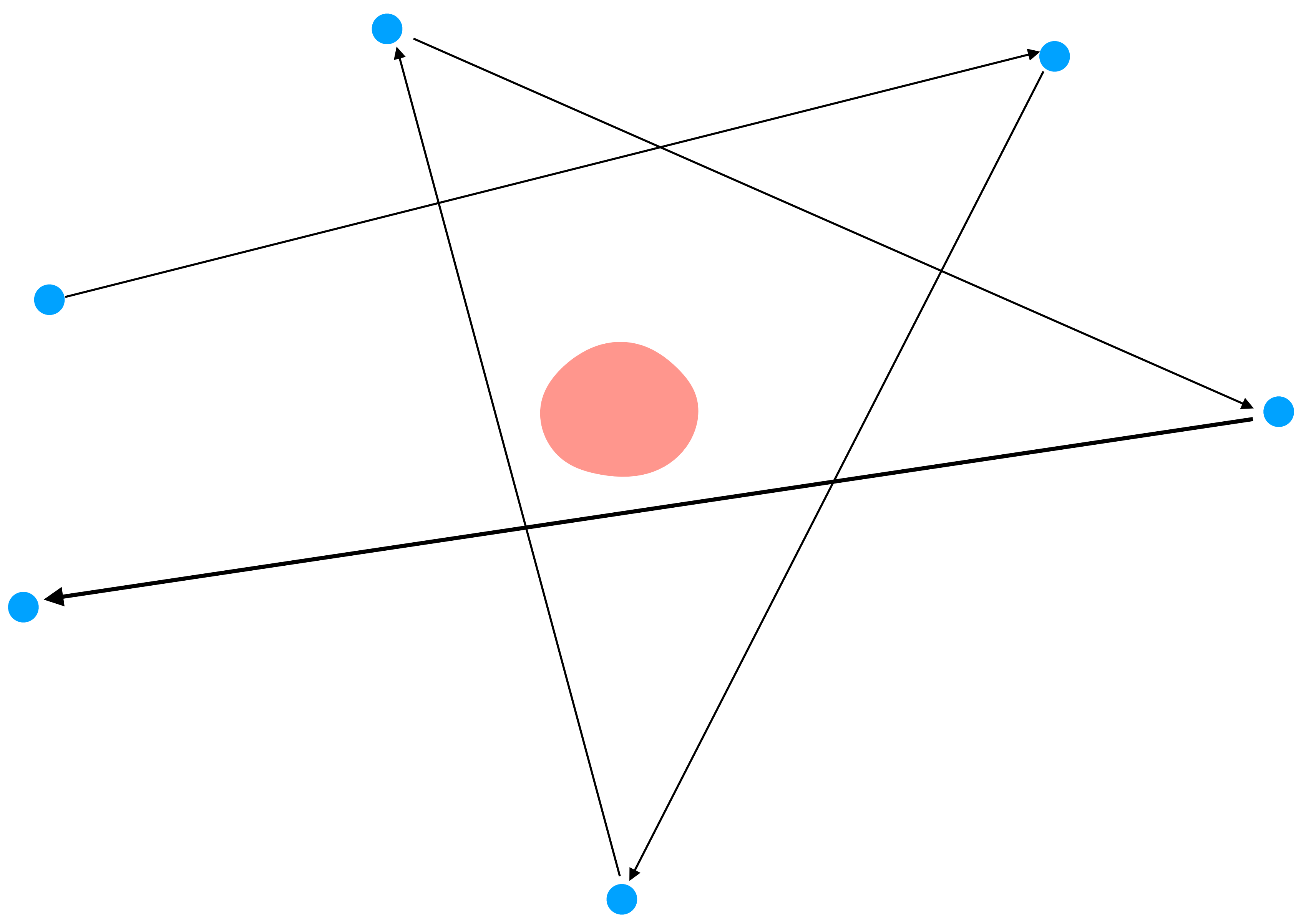


Approachability



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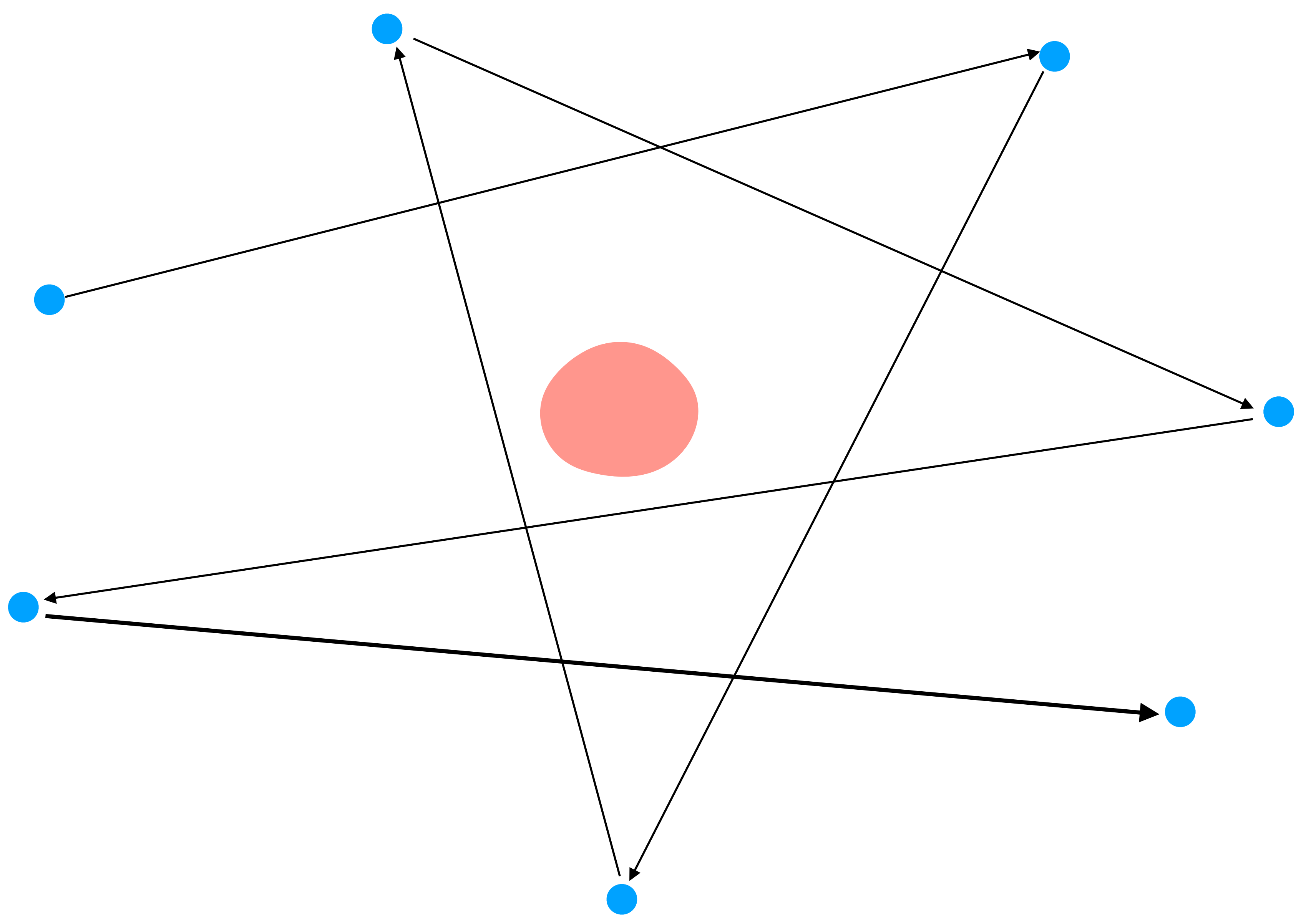

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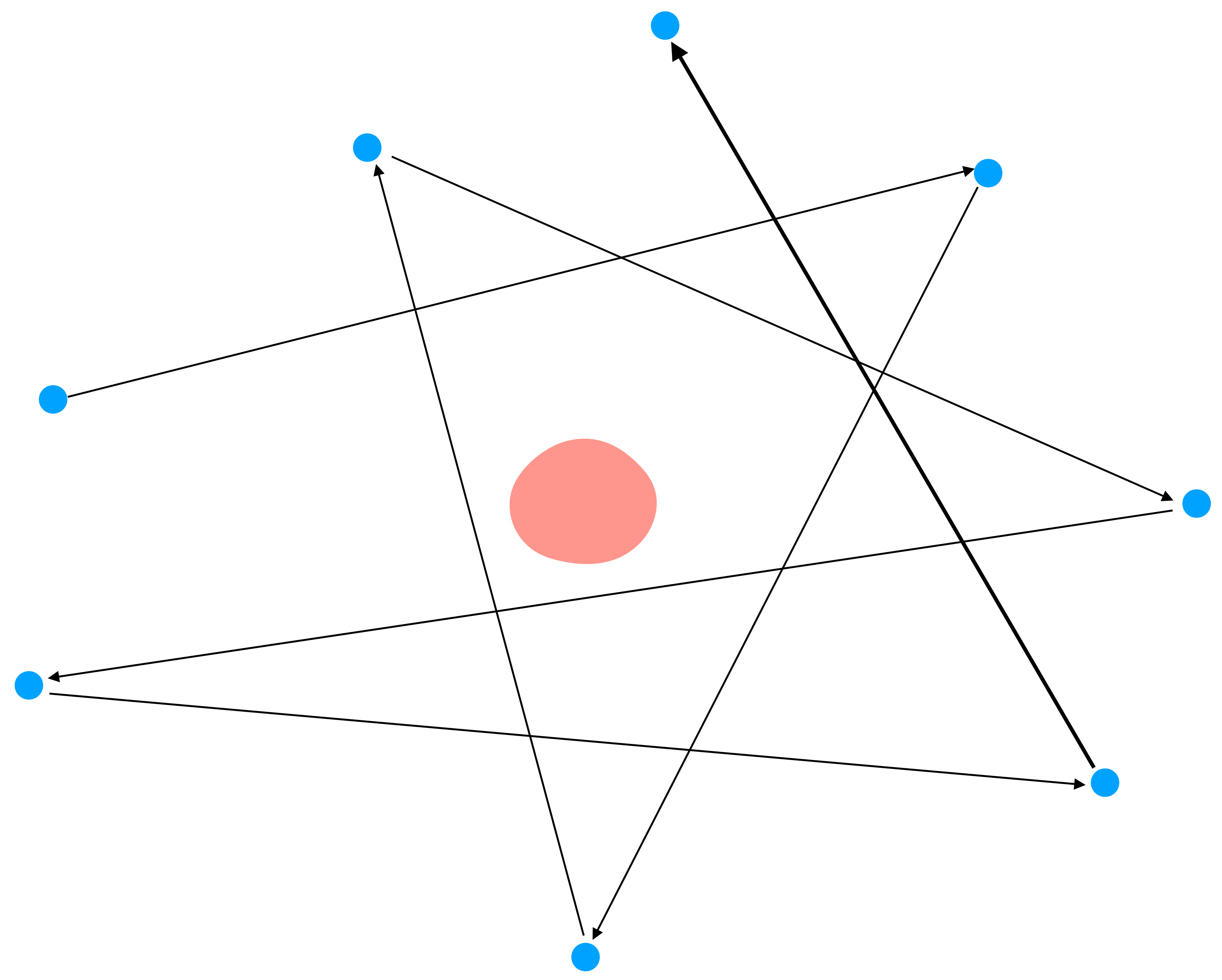

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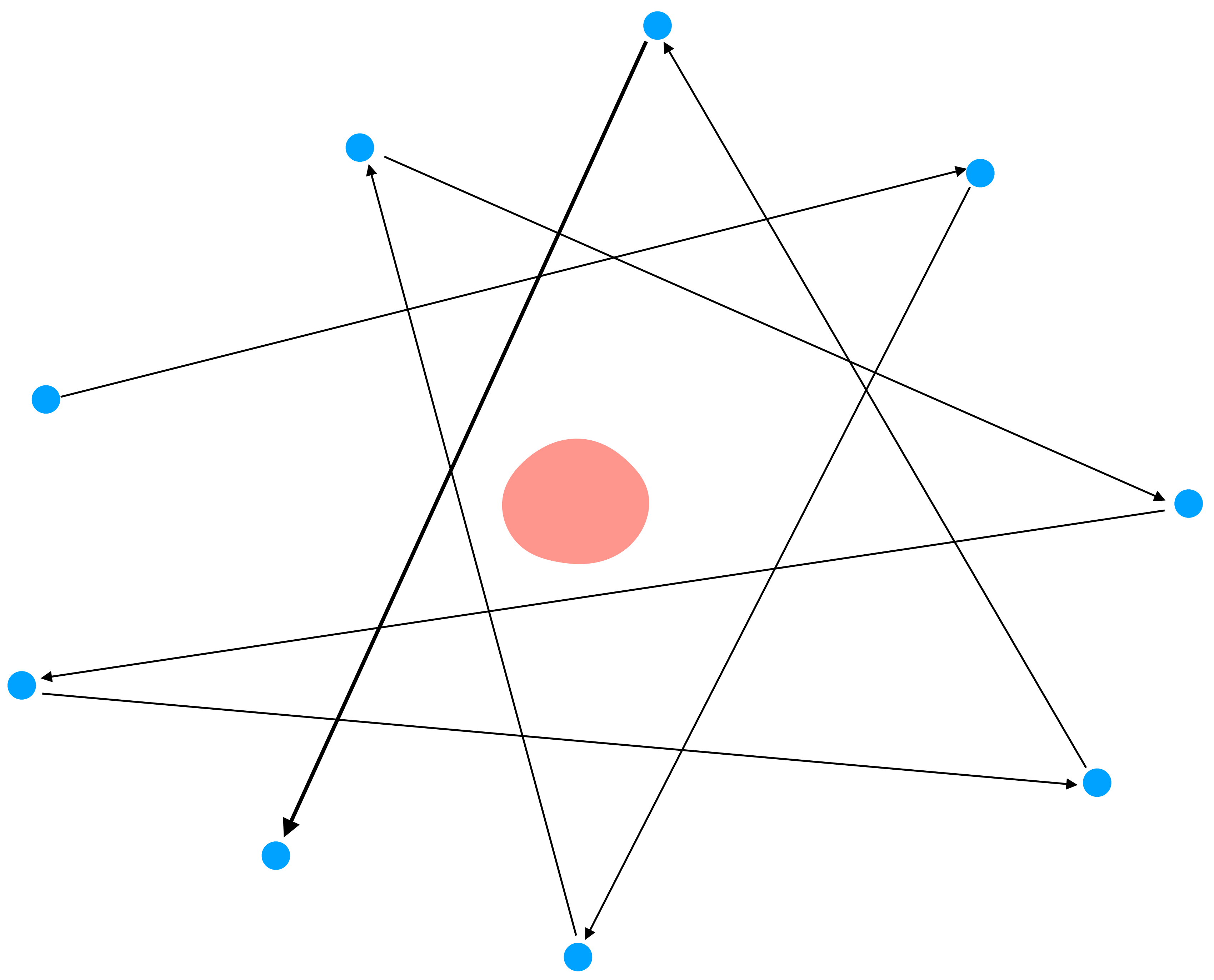

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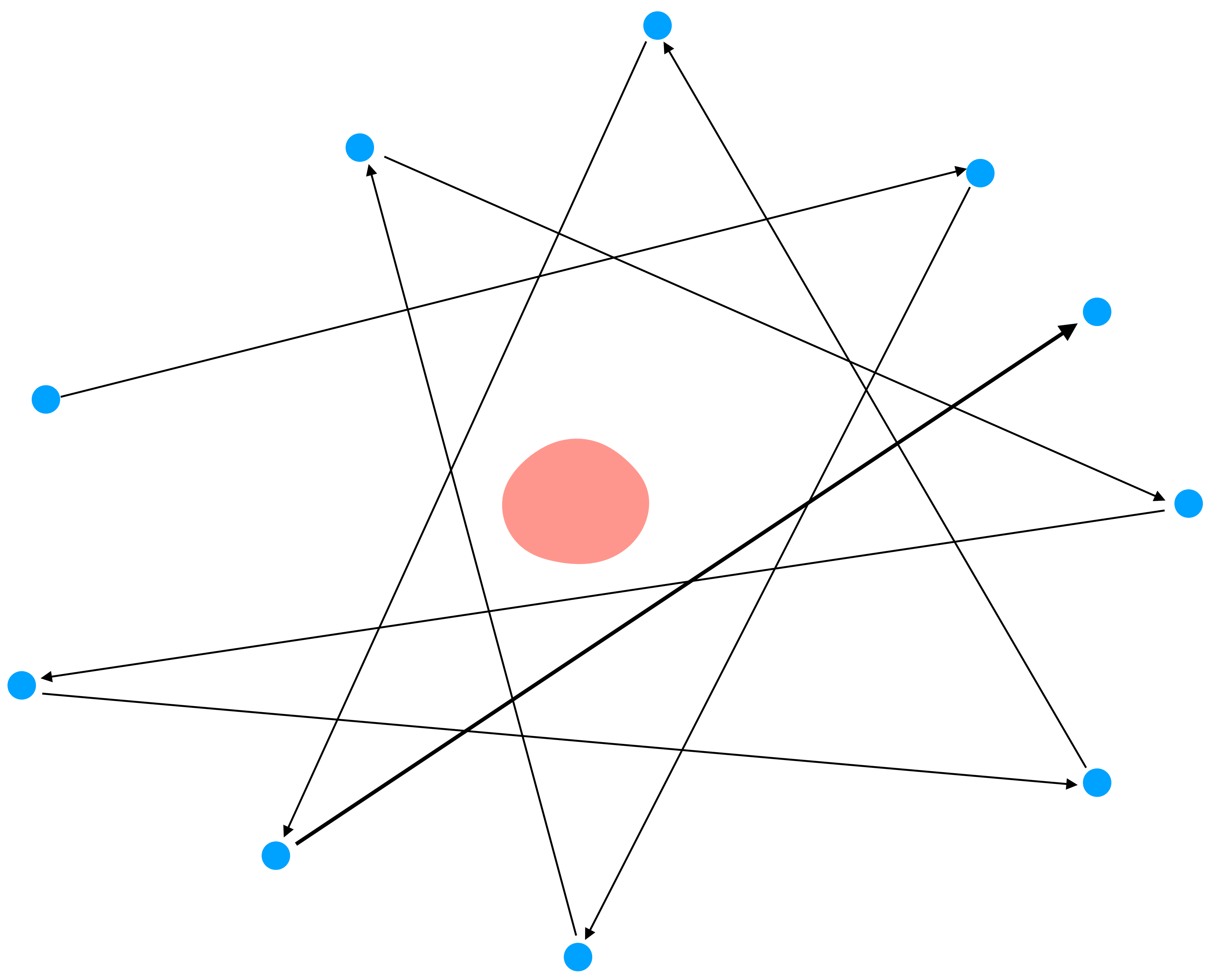
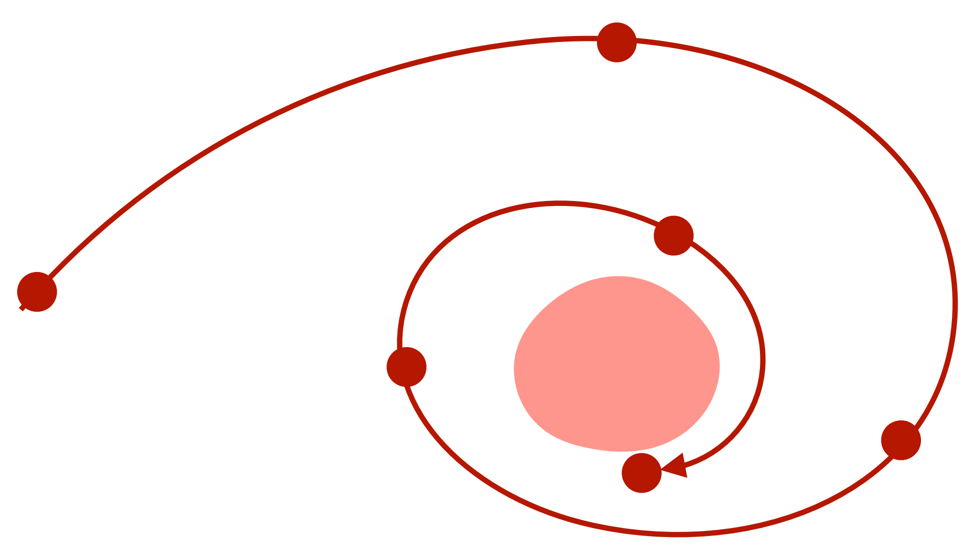


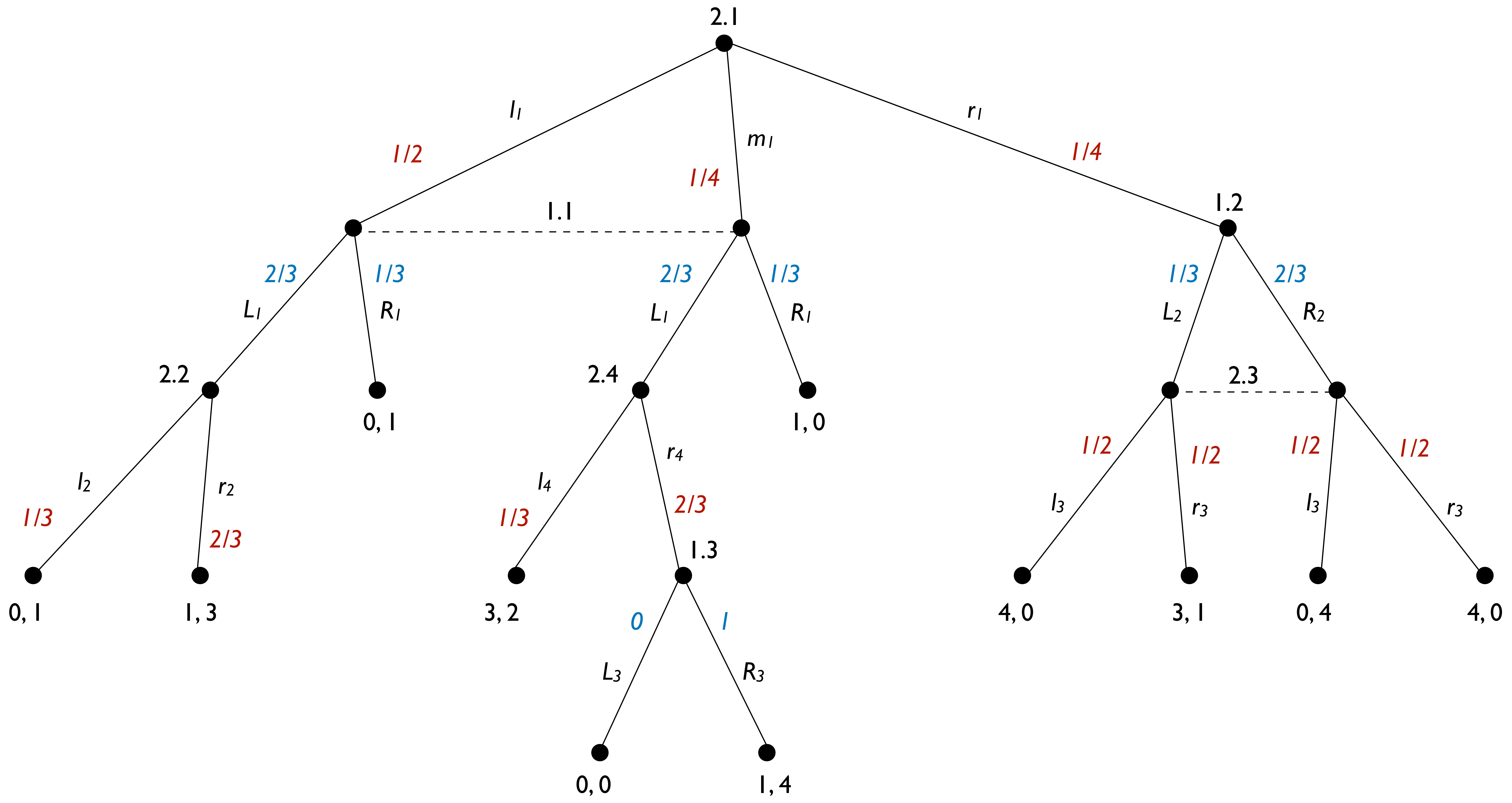
Approachability



Counter Factual Regret minimization

The idea of CFR is the application of RM at every single information set, given that such an information set is reached with positive probability

Example


2.1
l1
1/2
m1
1/4
r1
1/4
1.1
1.2
2/3
L1
1/3
R1
2/3
L1
1/3
R1
1/3
L2
2/3
R2
2.2
0, 1
2.4
1, 0
2.3
l2
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r2
2/3
l4
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r4
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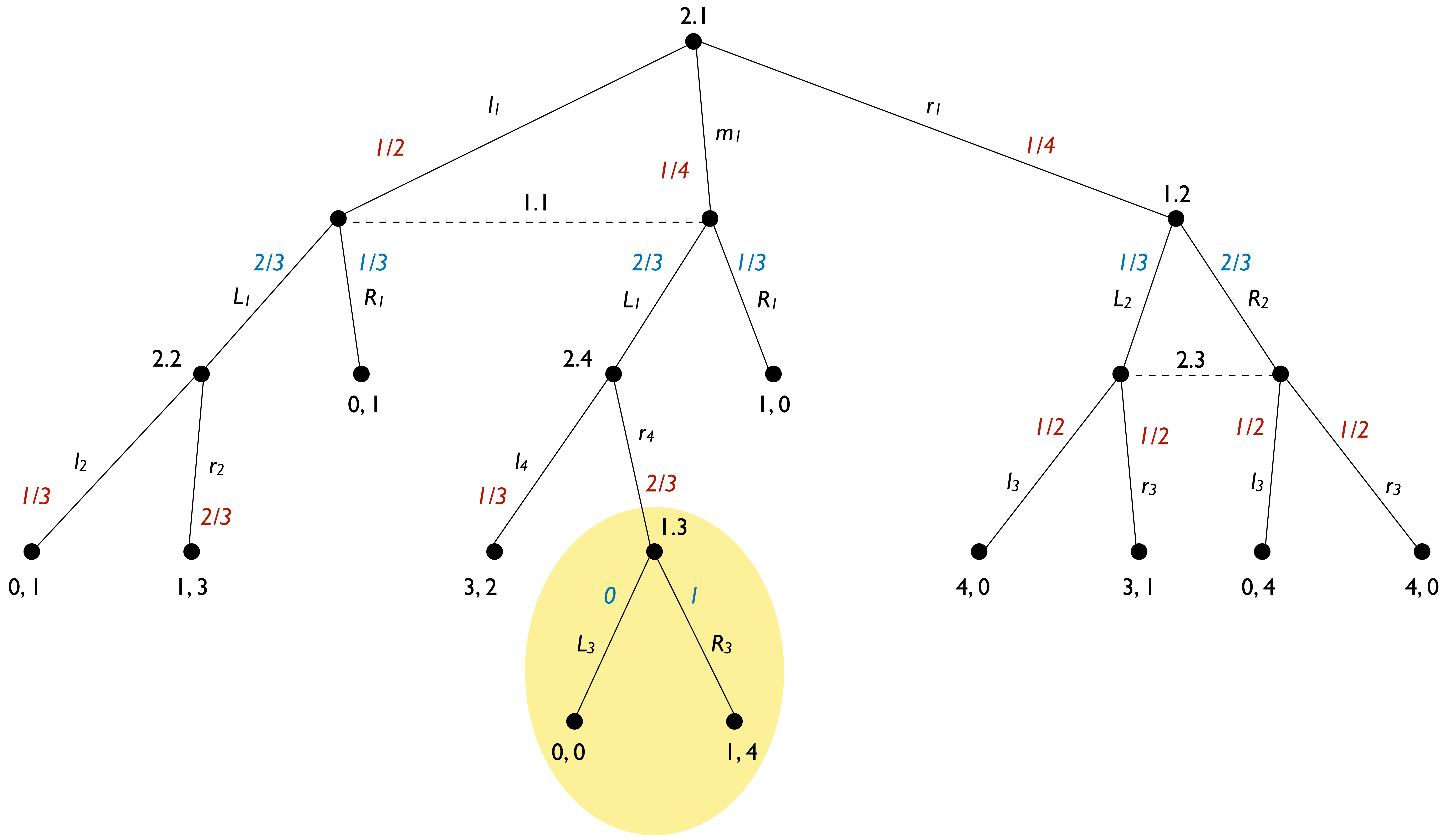
l3
1/2
r3
1/2
l3
1/2
r3
0, 1
1, 3
3, 2
0
L3
1
R3
4, 0
3, 1
0, 4
4, 0
0, 0
1, 4

Example


2.1
l_1
m_1
r_1
1/2
1/4
1/4
1.1
1.2
2/3
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1/3
1/3
2/3
L_1
R_1
L_1
R_1
L_2
R_2
2.2
0, 1
2.4
1, 0
2.3
l_2
r_2
1/3
2/3
l_4
r_4
1/3
2/3
1/2
1/2
1/2
1/2
l_3
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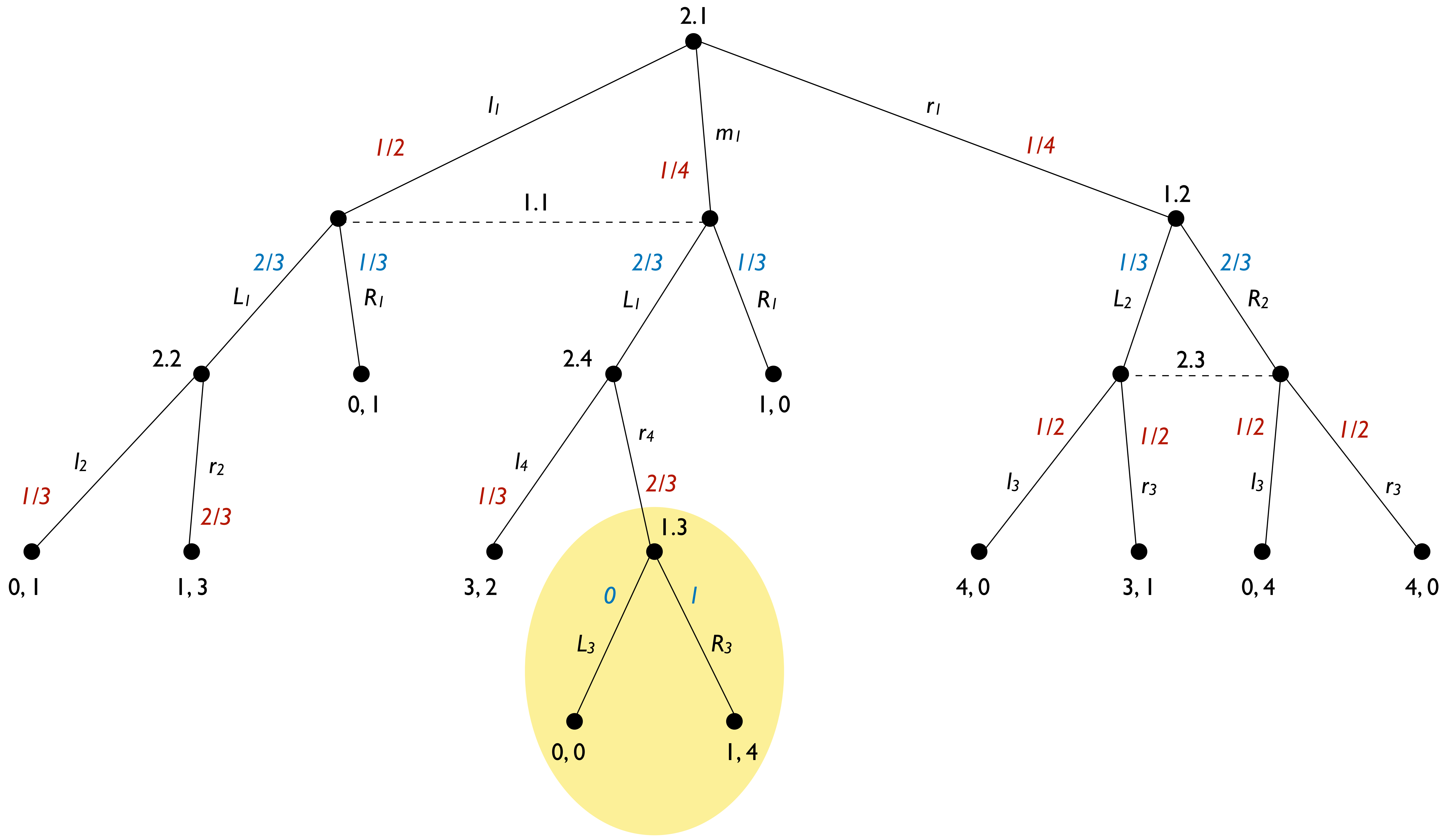
l_3
r_3
1.3
0, 1
1, 3
3, 2
4, 0
3, 1
0, 4
4, 0
0
1
L_3
R_3
0, 0
1, 4

Counter Factual Regret minimization

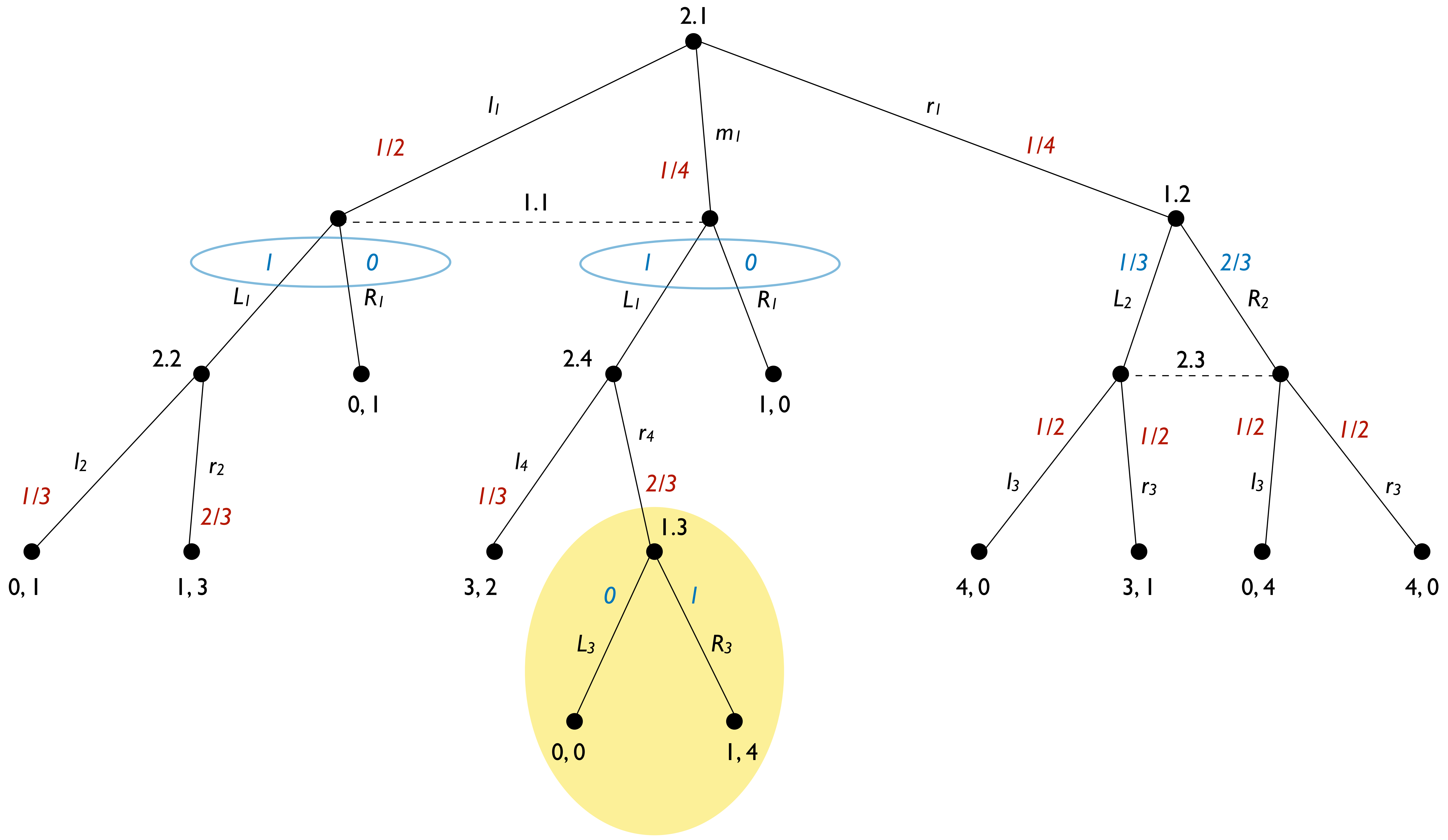
The strategy of player 1 before 1.3 is forced to reach 1.3

Example

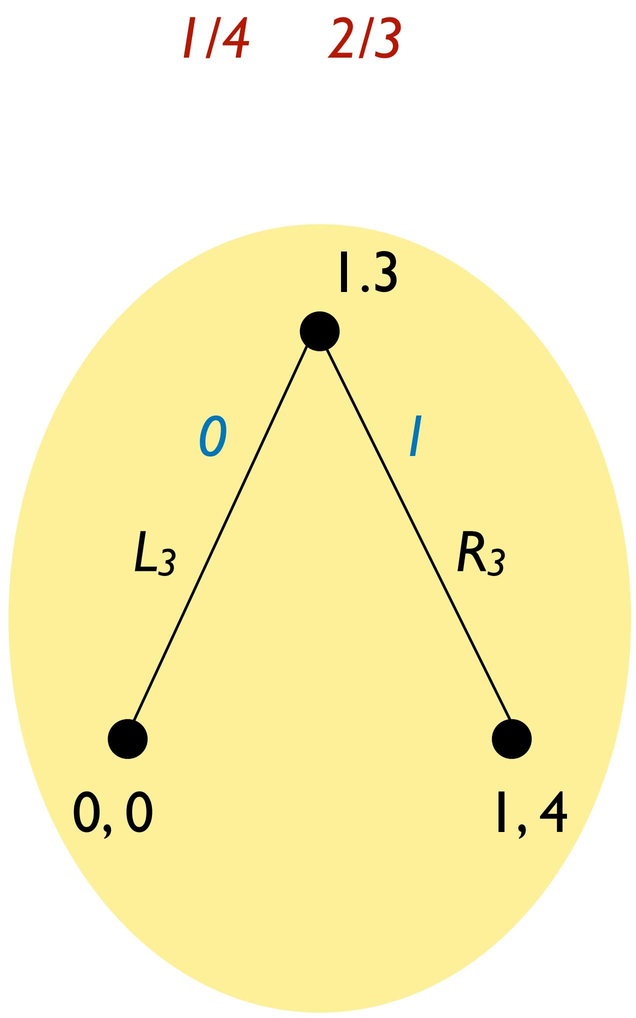


2.1
l_1
1/2
m_1
1/4
r_1
1/4
1.1
1.2
2/3
L_1
1/3
R_1
2/3
L_1
1/3
R_1
1/3
L_2
2/3
R_2
2.2
0, 1
2.4
1, 0
2.3
l_2
1/3
r_2
2/3
l_4
1/3
r_4
2/3
1/2
l_3
1/2
r_3
1/2
l_3
1/2
r_3
0, 1
1, 3
3, 2
1.3
4, 0
3, 1
0, 4
4, 0
0
L_3
1
R_3
0, 0
1, 4

Example


2.1
l_1
1/2
m_1
1/4
r_1
1/4
1.1
1.2
1
0
L_1
R_1
1
0
L_1
R_1
1/3
2/3
L_2
R_2
2.2
0, 1
2.4
1, 0
2.3
l_2
1/3
r_2
2/3
l_4
1/3
r_4
2/3
1/2
1/2
1/2
1/2
l_3
r_3
l_3
r_3
0, 1
1, 3
3, 2
1.3
4, 0
3, 1
0, 4
4, 0
0
1
L_3
R_3
0, 0
1, 4

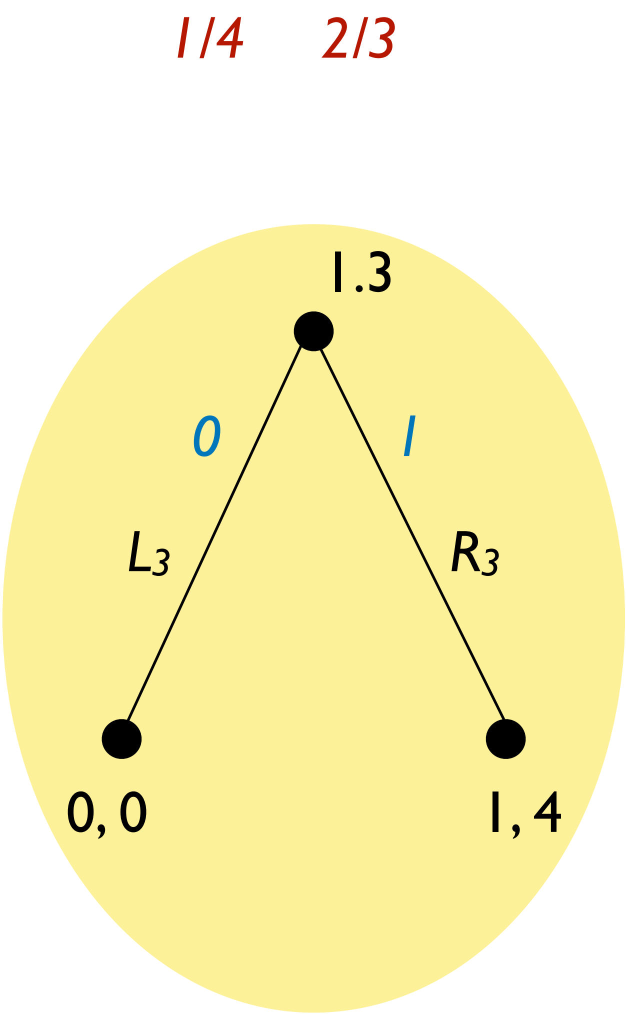
Example



$$r_1^t(L_3) = 1/4 \cdot 2/3 \cdot (0 - 1) = -\frac{1}{6}$$
$$r_1^t(R_3) = 1/4 \cdot 2/3 \cdot (1 - 1) = 0$$

$$\sigma_1^{t+1}(L_3) = \frac{R_1^{t,+}(L_3)}{R_1^{t,+}(L_3) + R_1^{t,+}(R_3)}$$
$$\sigma_1^{t+1}(R_3) = \frac{R_1^{t,+}(R_3)}{R_1^{t,+}(L_3) + R_1^{t,+}(R_3)}$$

Example



$$r_1^t(L_3) = \boxed{1/4 \cdot 2/3} \cdot (0-1) = -\frac{1}{6}$$
$$r_1^t(R_3) = \boxed{1/4 \cdot 2/3} \cdot (1-1) = 0$$

$$\sigma_1^{t+1}(L_3) = \frac{R_1^{t,+}(L_3)}{R_1^{t,+}(L_3) + R_1^{t,+}(R_3)}$$
$$\sigma_1^{t+1}(R_3) = \frac{R_1^{t,+}(R_3)}{R_1^{t,+}(L_3) + R_1^{t,+}(R_3)}$$

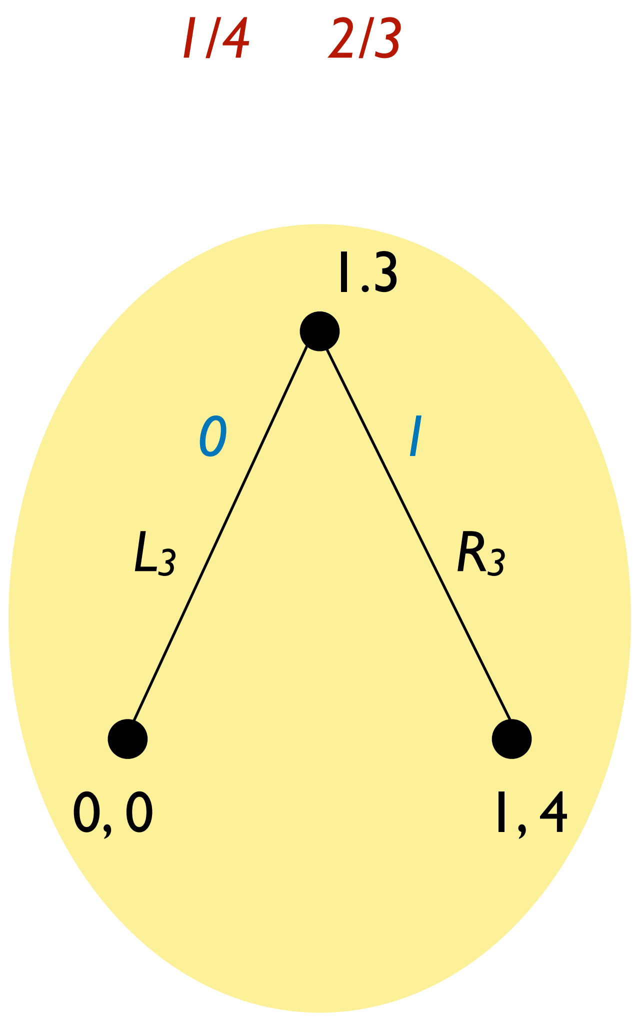
Example

$$r_1^t(L_3) = 1/4 \cdot 2/3 \cdot (0 - 1) = -\frac{1}{6}$$

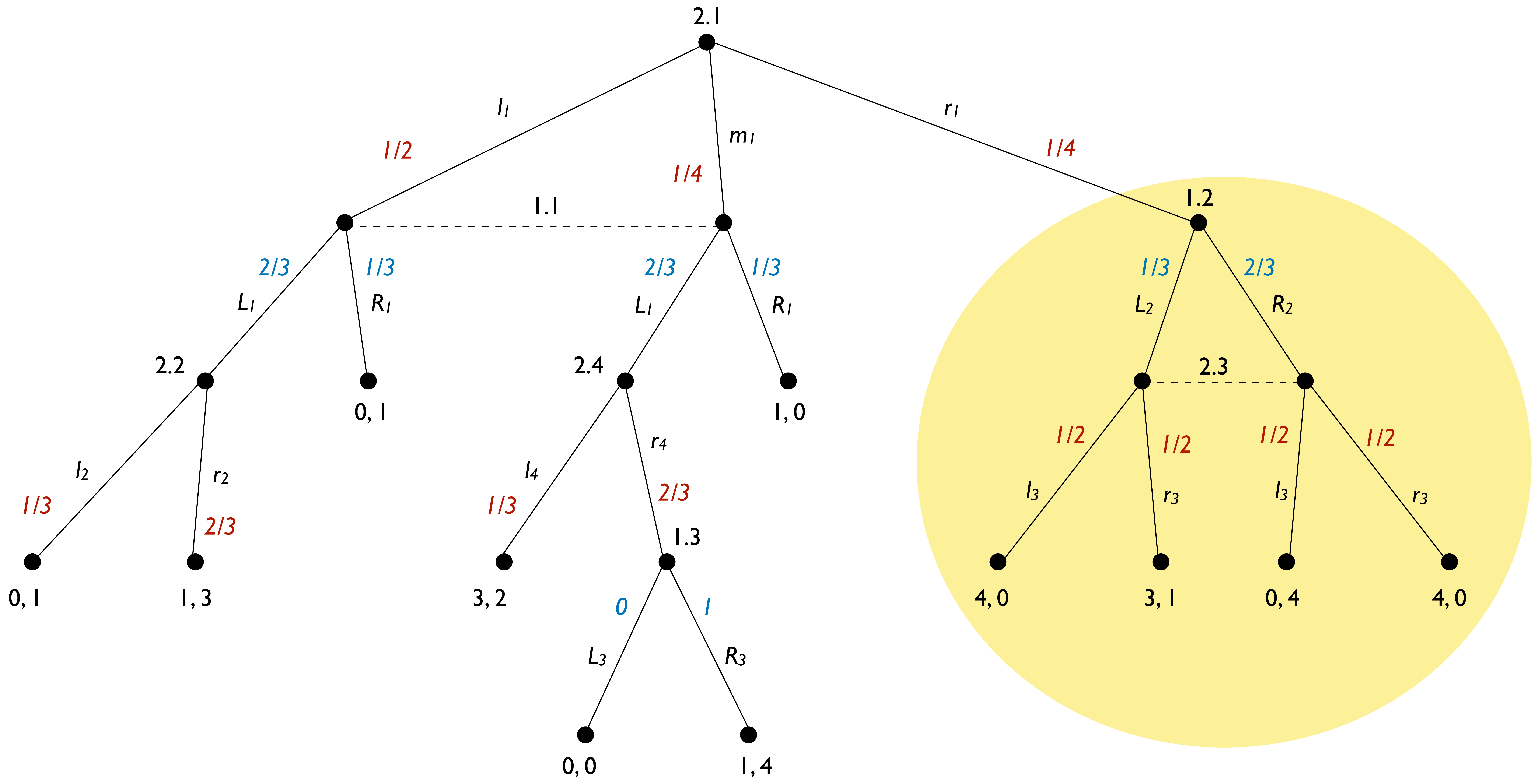
$$r_1^t(R_3) = 1/4 \cdot 2/3 \cdot (1 - 1) = 0$$

$$\sigma_1^{t+1}(L_3) = \frac{R_1^{t,+}(L_3)}{R_1^{t,+}(L_3) + R_1^{t,+}(R_3)}$$

$$\sigma_1^{t+1}(R_3) = \frac{R_1^{t,+}(R_3)}{R_1^{t,+}(L_3) + R_1^{t,+}(R_3)}$$



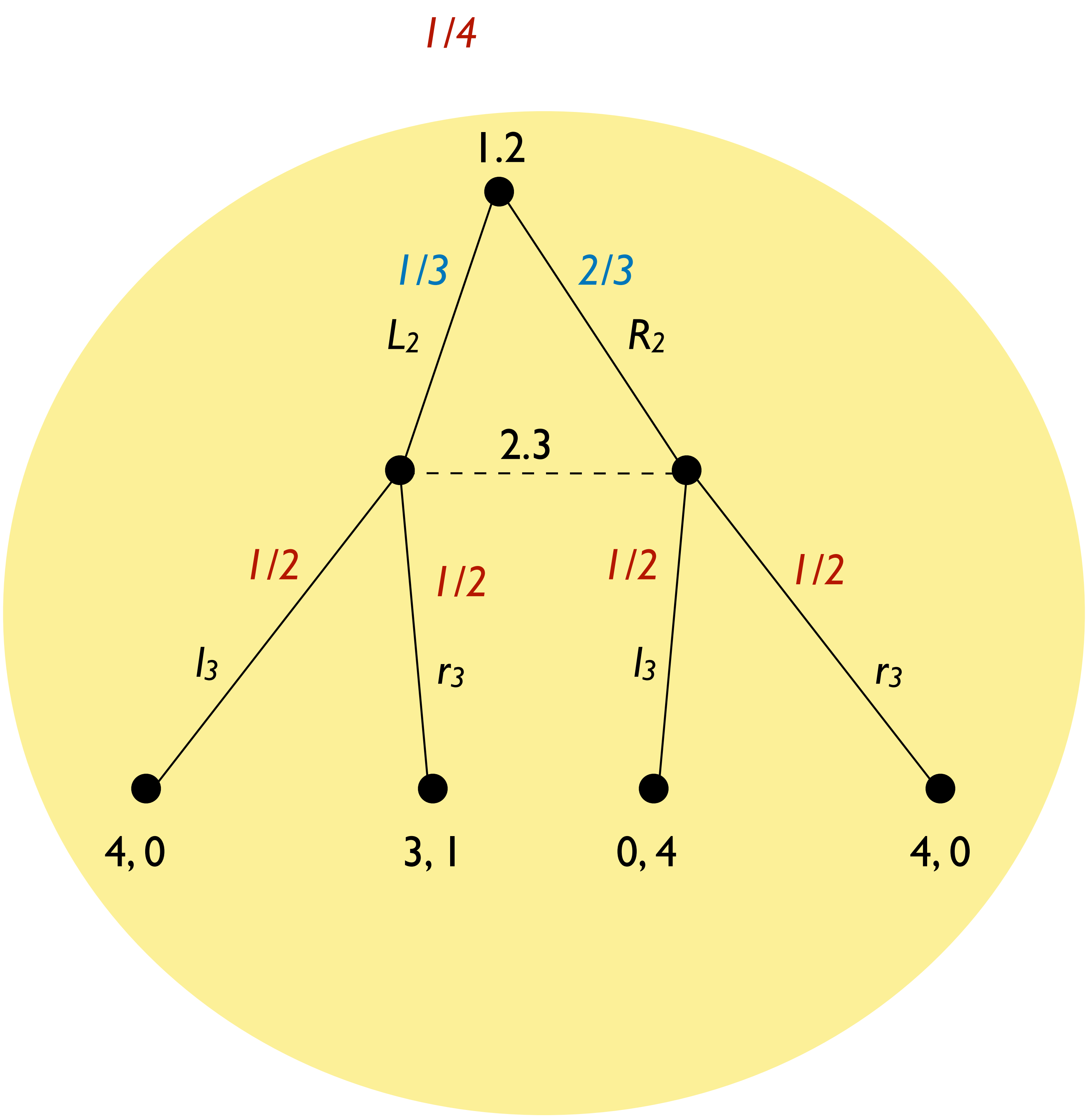
Example



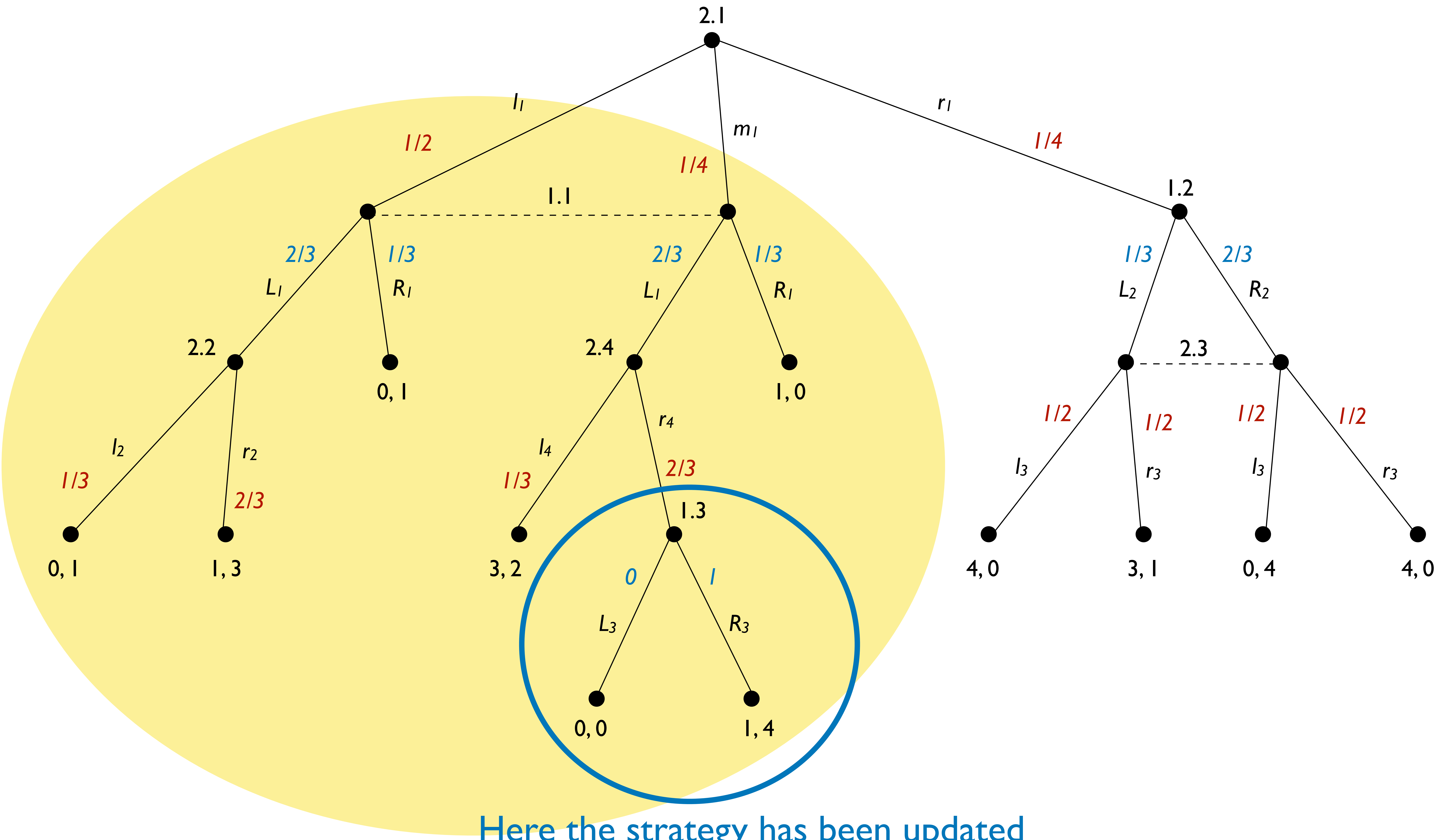
2.1
l1
m1
r1
1/2
1/4
1/4
1.1
1.2
2/3
1/3
2/3
1/3
1/3
2/3
L1
R1
L1
R1
L2
R2
2.2
2.4
2.3
0, 1
1, 0
l2
r2
l4
r4
1/3
2/3
1/3
2/3
1/2
1/2
1/2
1/2
l3
r3
l3
r3
1.3
0, 1
1, 3
3, 2
4, 0
3, 1
0, 4
4, 0
0
1
L3
R3
0, 0
1, 4

Example

As in a normal-form game, except that the instantaneous regrets are multiplied by the probability (i.e., 1/4) with which player 2 reaches 1.2

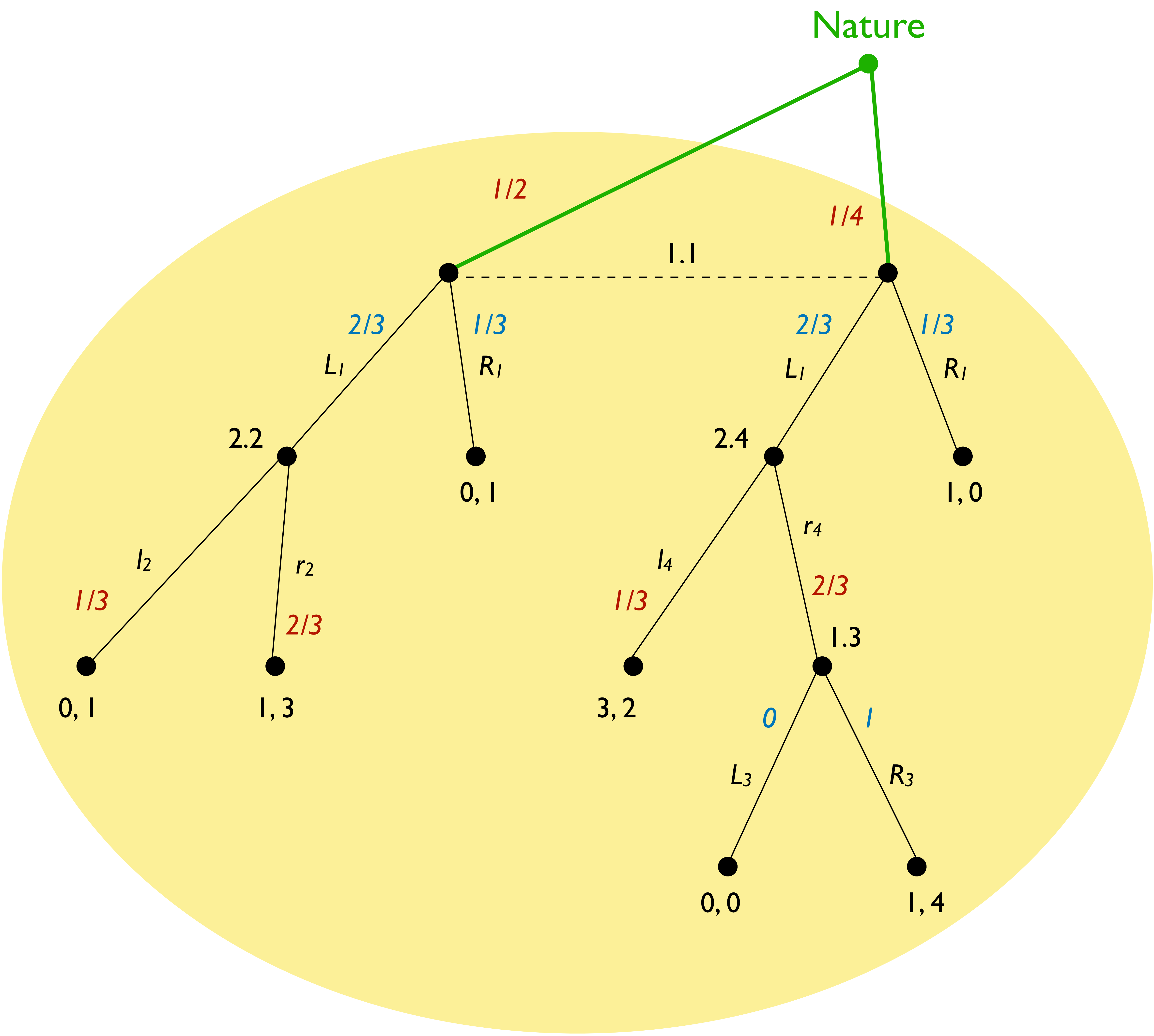


Example



Here the strategy has been updated

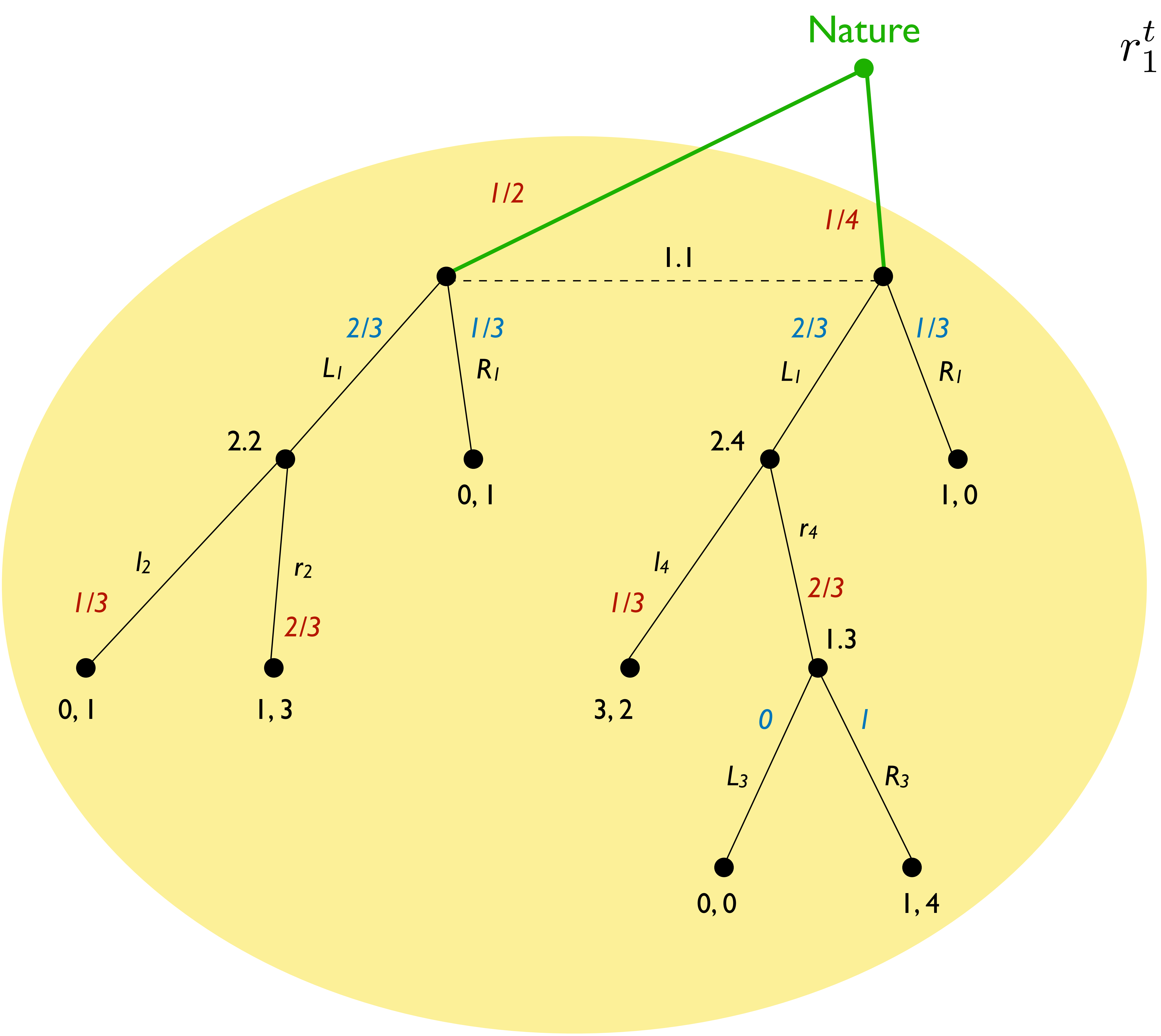
Example



Here the peculiarities are

- the strategy of player 1 at 1.3 is fixed when calculating the regret at information set 1.1
- the nodes of information set 1.1 are reached with different probabilities as the Nature would have played just before the infoset and these probabilities vary in time

Example



$$r_1^t(L_1) = [1/2\,(1/3 \cdot 0 + 2/3 \cdot 1) +$$

Expected utility of L1 from the left node

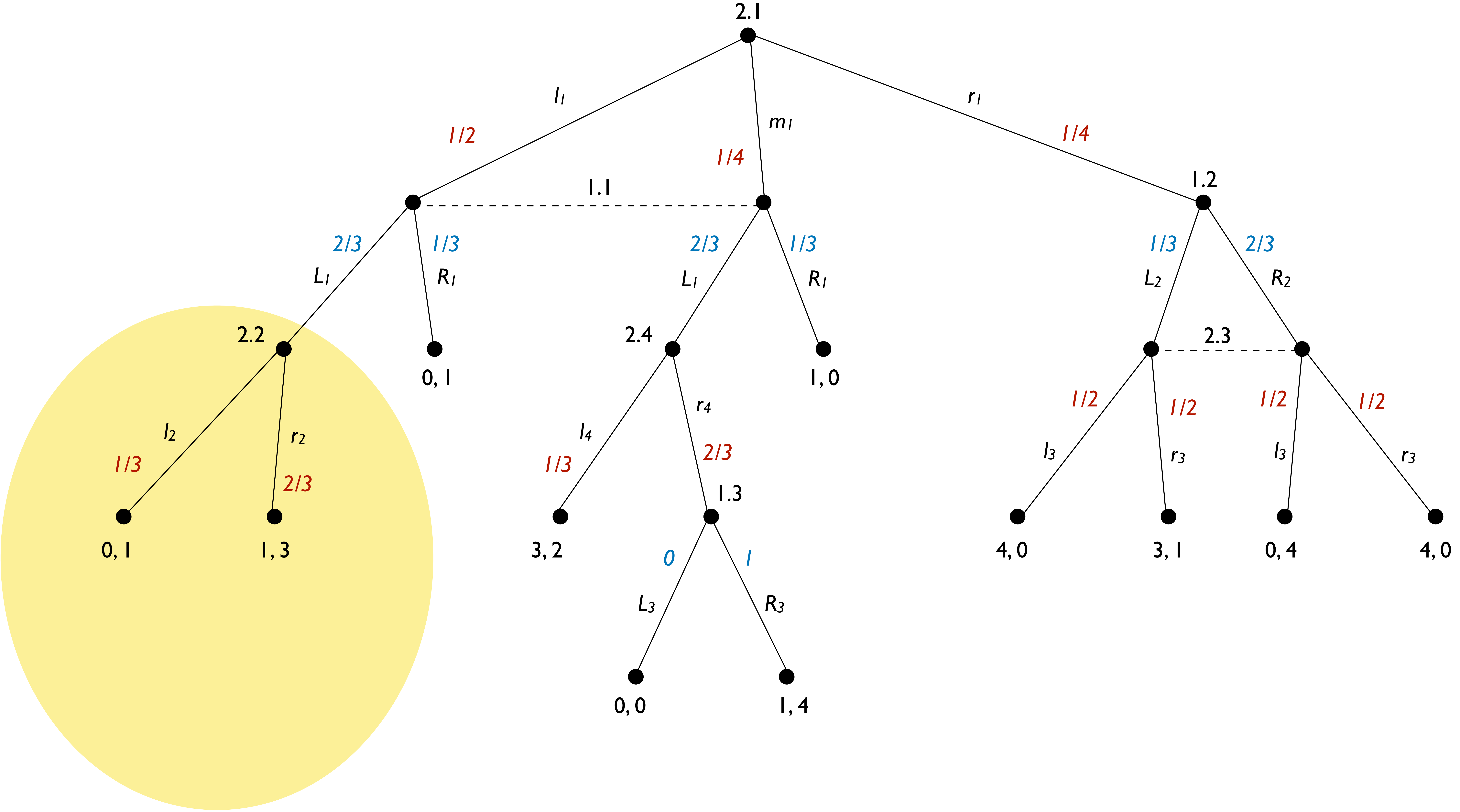
$$1/4(1/3 \cdot 3 + 2/3 \cdot 0 \cdot 0 + 2/3 \cdot 1 \cdot 1)] -$$

Expected utility of L1 from the right node

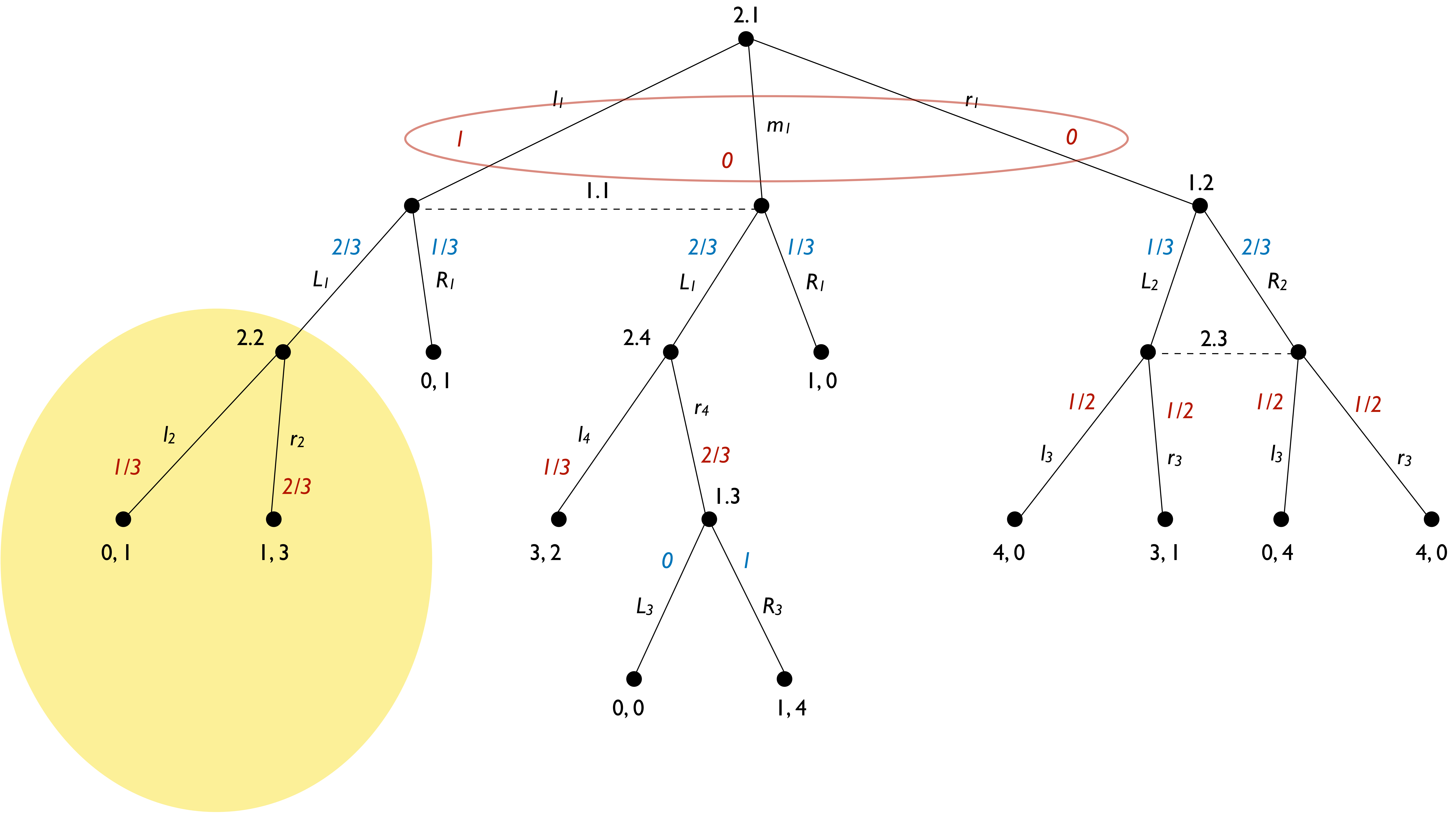
$$[1/2 \cdot 4/3 + 1/4 \cdot (2/3 + 4/9 + 1/3)]$$

Expected utility of the entire strategy

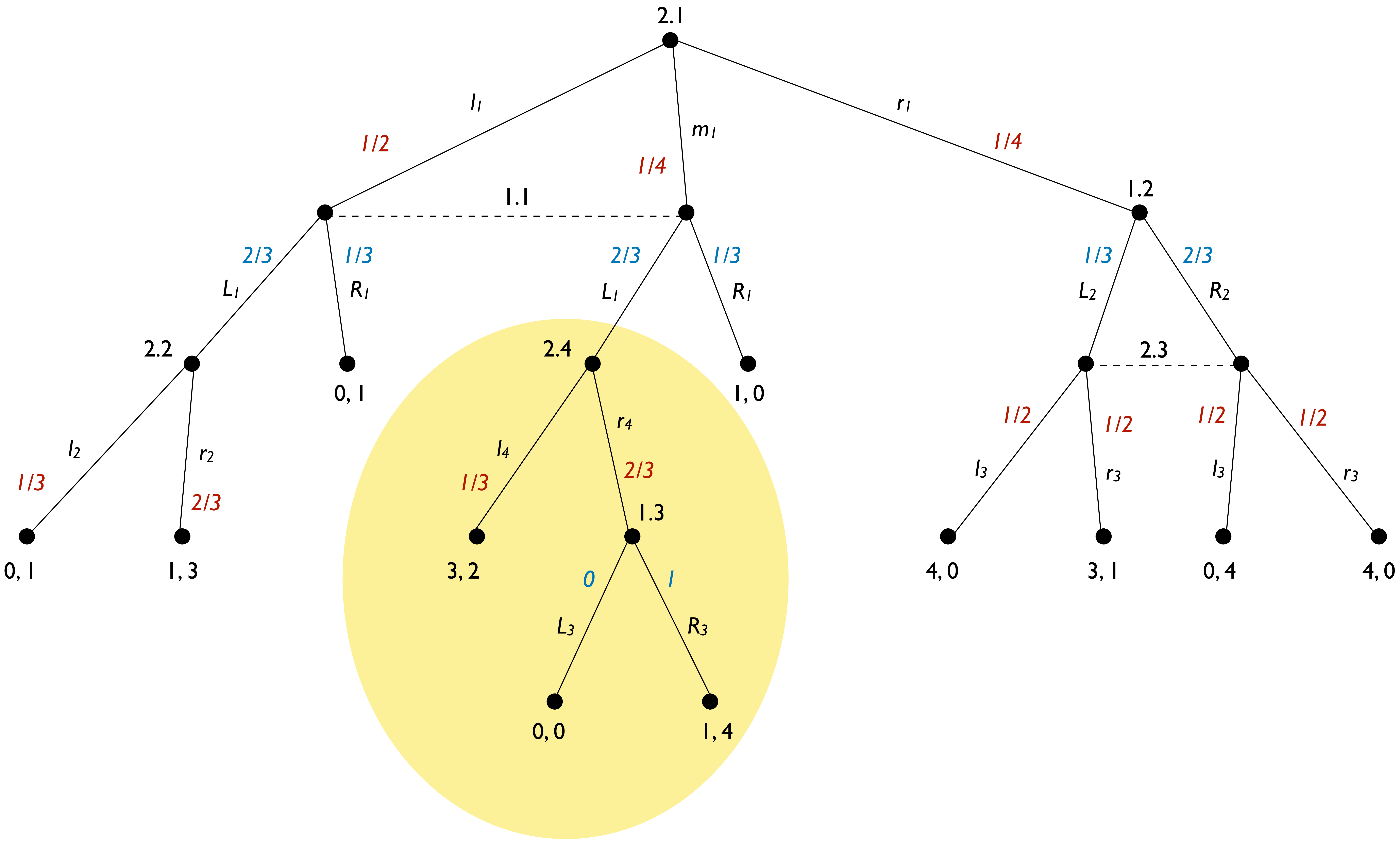
Example


2.1
l1
1/2
m1
1/4
r1
1/4
1.1
1.2
2/3
L1
1/3
R1
2/3
L1
1/3
R1
1/3
L2
2/3
R2
2.2
0, 1
2.4
1, 0
2.3
l2
1/3
r2
2/3
l4
1/3
r4
2/3
1.3
1/2
l3
1/2
r3
1/2
l3
1/2
r3
0, 1
1, 3
3, 2
0
L3
1
R3
0, 0
1, 4
4, 0
3, 1
0, 4
4, 0

Example

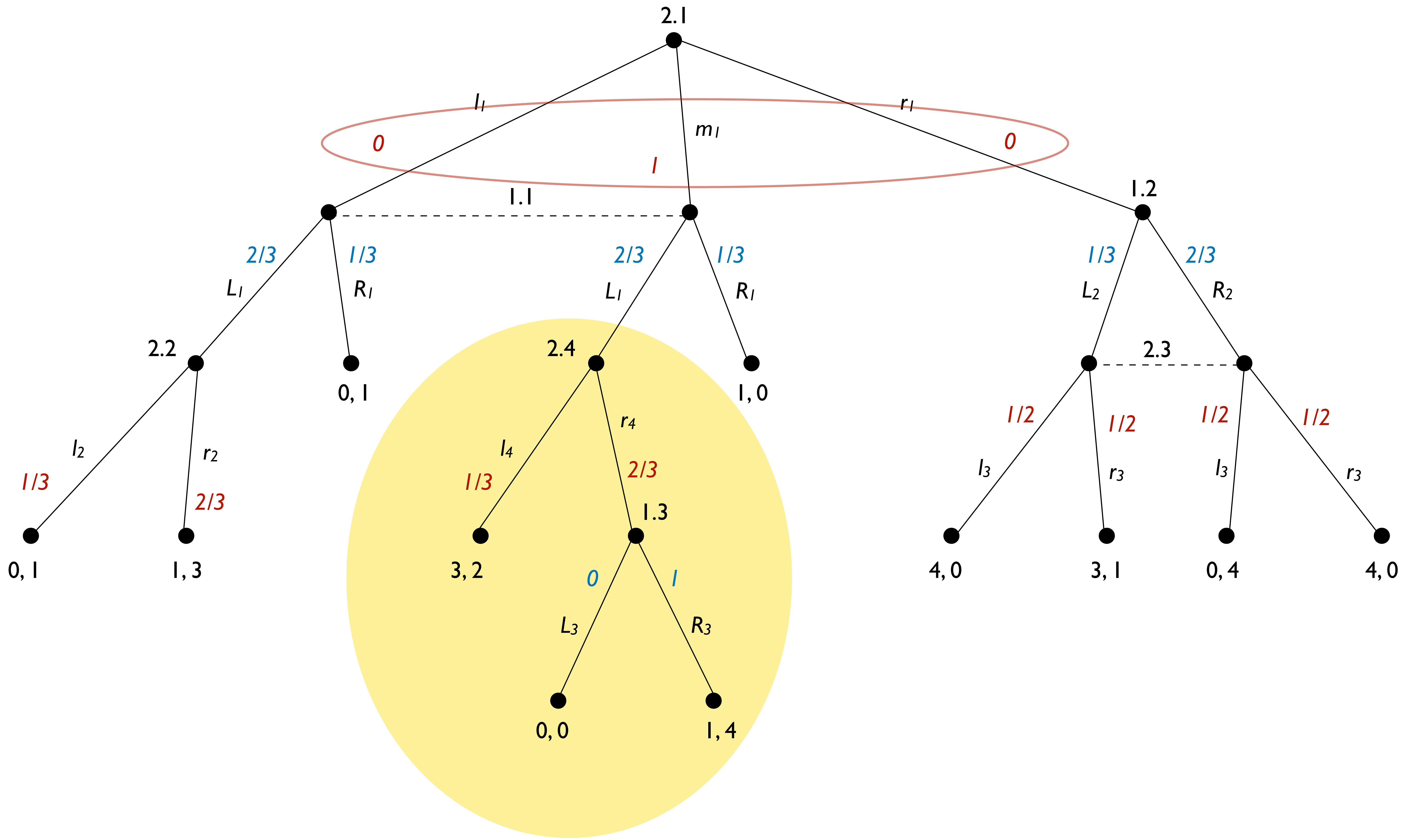

2.1
l1
m1
r1
1
0
0
1.1
1.2
2/3
1/3
2/3
1/3
1/3
2/3
L1
R1
L1
R1
L2
R2
2.2
0, 1
2.4
1, 0
2.3
l2
r2
1/3
2/3
l4
r4
1/3
2/3
1.3
1/2
1/2
1/2
1/2
l3
r3
l3
r3
0, 1
1, 3
3, 2
4, 0
3, 1
0, 4
4, 0
0
1
L3
R3
0, 0
1, 4

Example



2.1
l_1
1/2
m_1
1/4
r_1
1/4
1.1
1.2
2/3
L_1
1/3
R_1
2/3
L_1
1/3
R_1
1/3
L_2
2/3
R_2
2.2
0, 1
2.4
1, 0
2.3
l_2
1/3
r_2
2/3
l_4
1/3
r_4
2/3
1.3
1/2
l_3
1/2
r_3
1/2
l_3
1/2
r_3
0, 1
1, 3
3, 2
4, 0
3, 1
0, 4
4, 0
0
L_3
1
R_3
0, 0
1, 4

Example



2.1
l_1
m_1
r_1
0
1
0
1.1
1.2
2/3
1/3
L_1
R_1
2/3
1/3
L_1
R_1
1/3
2/3
L_2
R_2
2.2
0, 1
2.4
1, 0
2.3
l_2
1/3
r_2
2/3
0, 1
1, 3
l_4
1/3
r_4
2/3
1.3
3, 2
0
1
L_3
R_3
0, 0
1, 4
1/2
1/2
1/2
1/2
l_3
r_3
l_3
r_3
4, 0
3, 1
0, 4
4, 0

Example

	Player 1								Player 2									
	0	L1	R1	L2	R2	L3	R3		0	l1	m1	r1	l2	r2	l3	r3	l4	r4
R								1										
strategy								1										
R								2										
strategy								2										

Algorithm complexity

- The game tree is traversed forward to compute the probability of every terminal and non-terminal sequence
- The game tree is traversed backward to update che regrets and compute the strategies

Regret Matching Plus (RM+)

RM+ distinguishes from RM for

1. The cumulative regret plus is redefined as

$$R_i^{+,t+1}(a) = \max\left\{R_i^{+,t}(a) + r_i^t(a), 0\right\}$$

2. The calculation of the regrets and the updating of the strategies are performed in alternating fashion

3. The strategy returned by RM+ is obtained by linear weighted averaging

Properties

- RM+ has the same worst-case theoretical guarantees of RM
- RM+ empirically converges much faster than RM

Comparison (RM vs. RM+)

	R	P	S
R	2 , -2	1 , -1	0 , 0
P	2 , -2	0 , 0	3 , -3
S	-1 , 1	3 , -3	-3 , 3

	Player 1				Player 2		
					Average strategy		
	R	P	S		R	P	S
R				1			
strategy				1			
R				2			
strategy				2			
R				3			
strategy				3			

Comparison (RM+)

	R	P	S
R	2 , -2	1 , -1	0 , 0
P	2 , -2	0 , 0	3 , -3
S	-1 , 1	3 , -3	-3 , 3

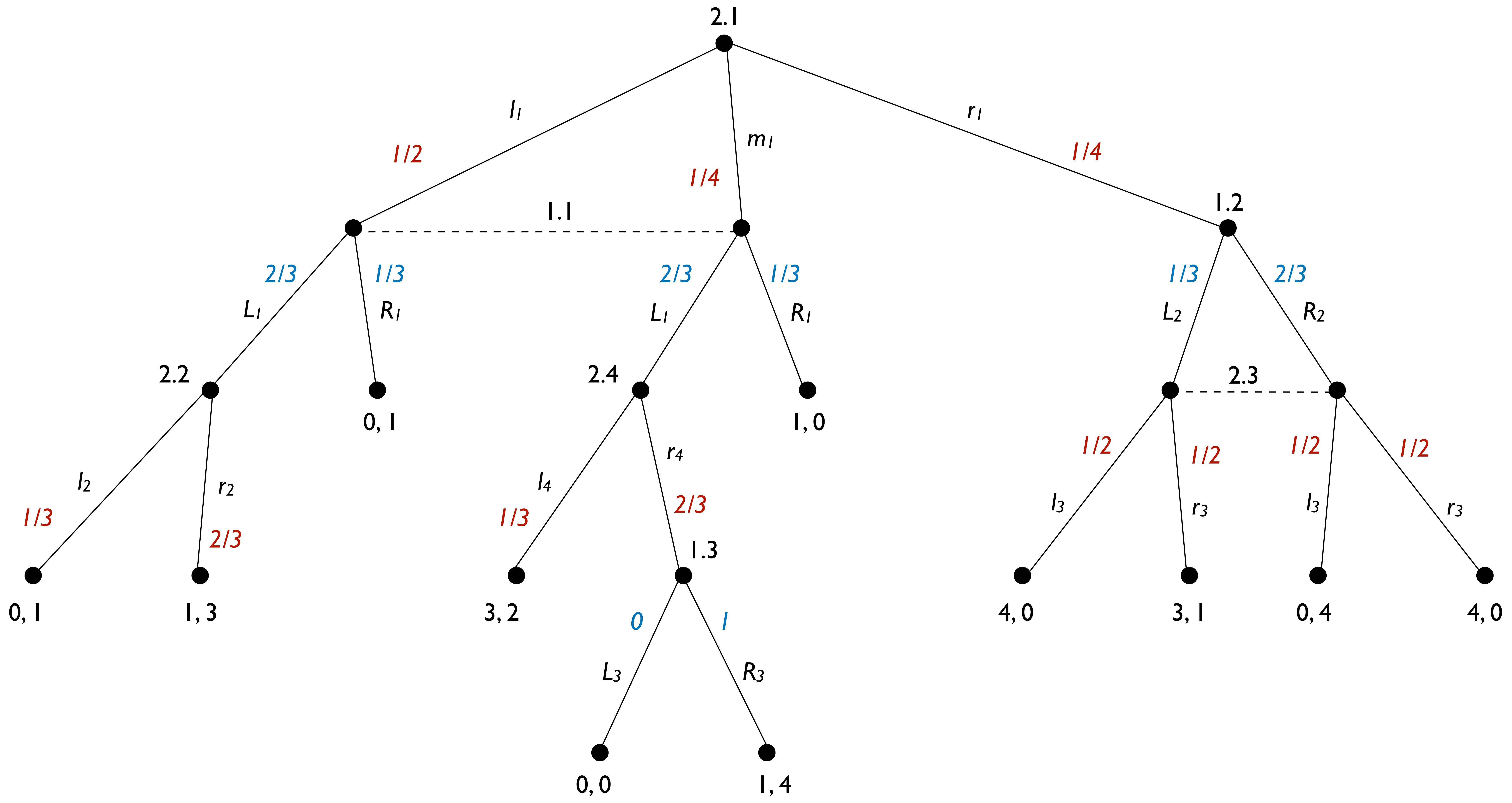
	Player 1				Player 2		
					Average strategy		
	R	P	S		R	P	S
R				1			
strategy				1			
R				2			
strategy				2			
R				3			
strategy				3			

Monte Carlo CFR/CFR+ (external sampling)

When calculating the regret and the strategy of player *i*:

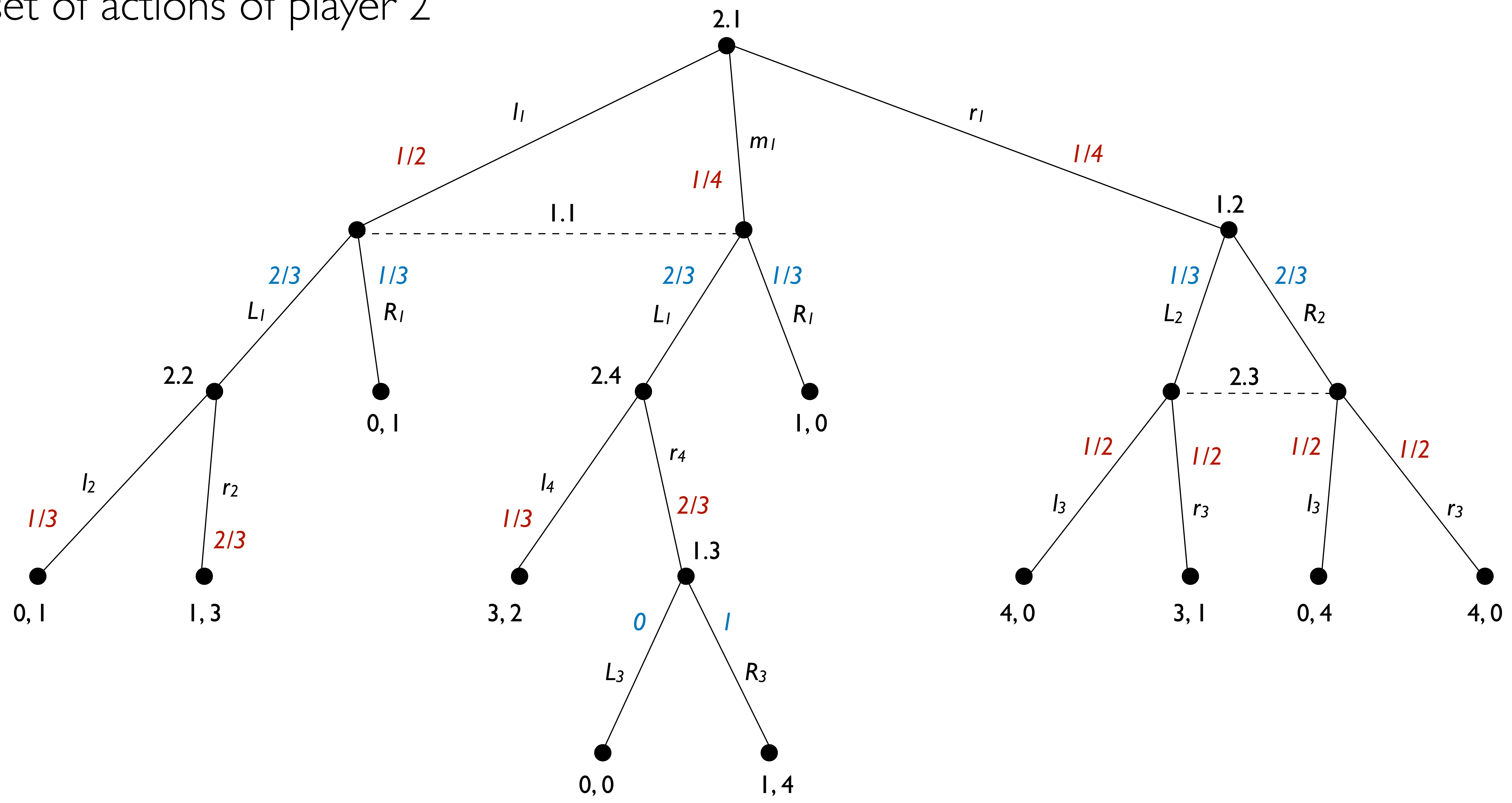
1. Sample a subset of actions of the opponent and of the change
2. Multiply the payoffs by the inverse of the sampling probability of the corresponding terminal sequence of opponent and chance
3. Calculate the instantaneous regrets at every information set reached with strictly positive probability and update the cumulative regrets
4. Update the strategy of player *i* accordingly

Example (1)

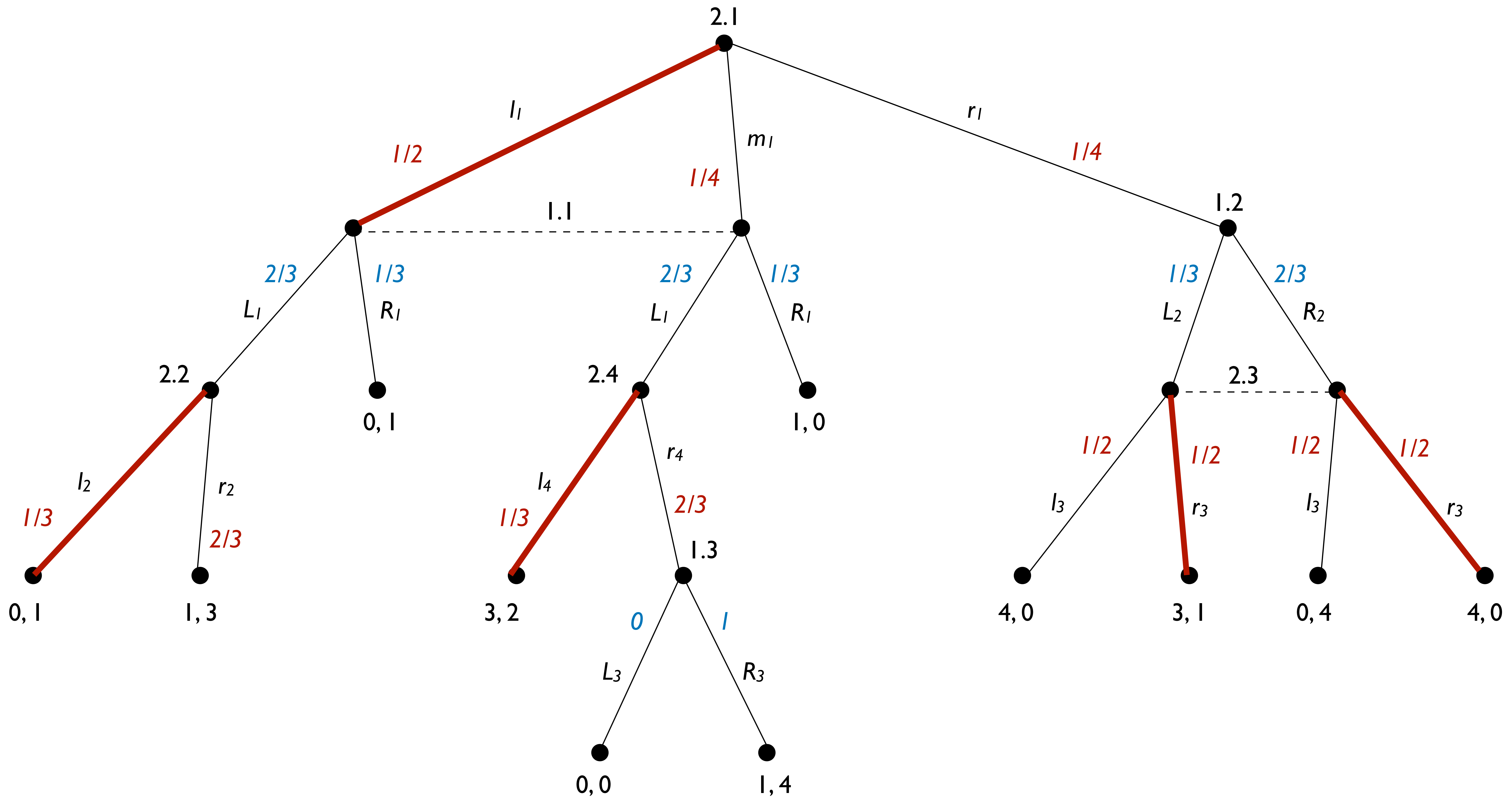

2.1
l1
1/2
m1
1/4
r1
1/4
1.1
1.2
2/3
L1
1/3
R1
2/3
L1
1/3
R1
1/3
L2
2/3
R2
2.2
0, 1
2.4
1, 0
2.3
l2
1/3
r2
2/3
l4
1/3
r4
2/3
1.3
1/2
l3
1/2
r3
1/2
l3
1/2
r3
0, 1
1, 3
3, 2
4, 0
3, 1
0, 4
4, 0
0
L3
1
R3
0, 0
1, 4

Example (1)

Sample a subset of actions of player 2



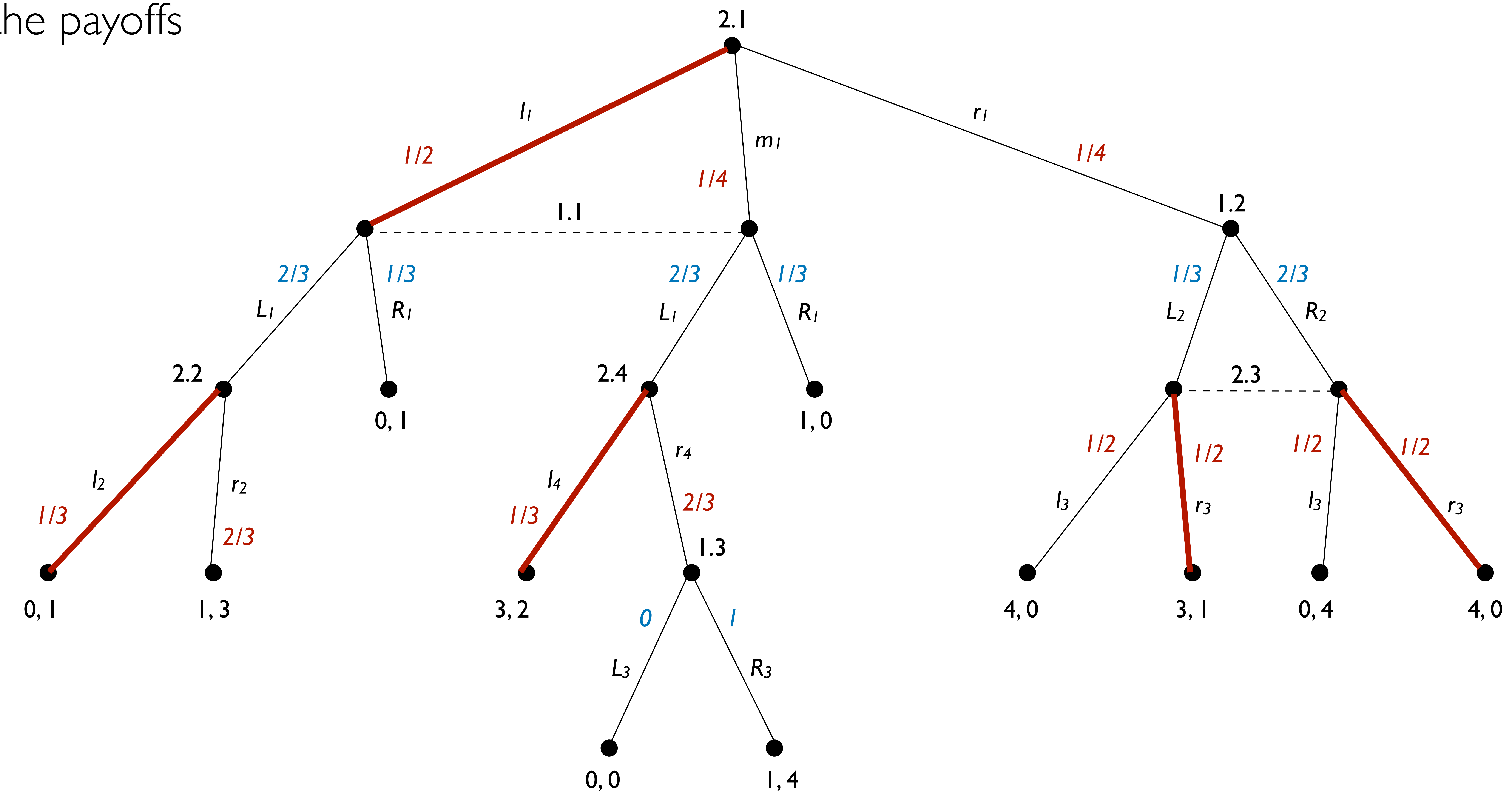
Example (1)



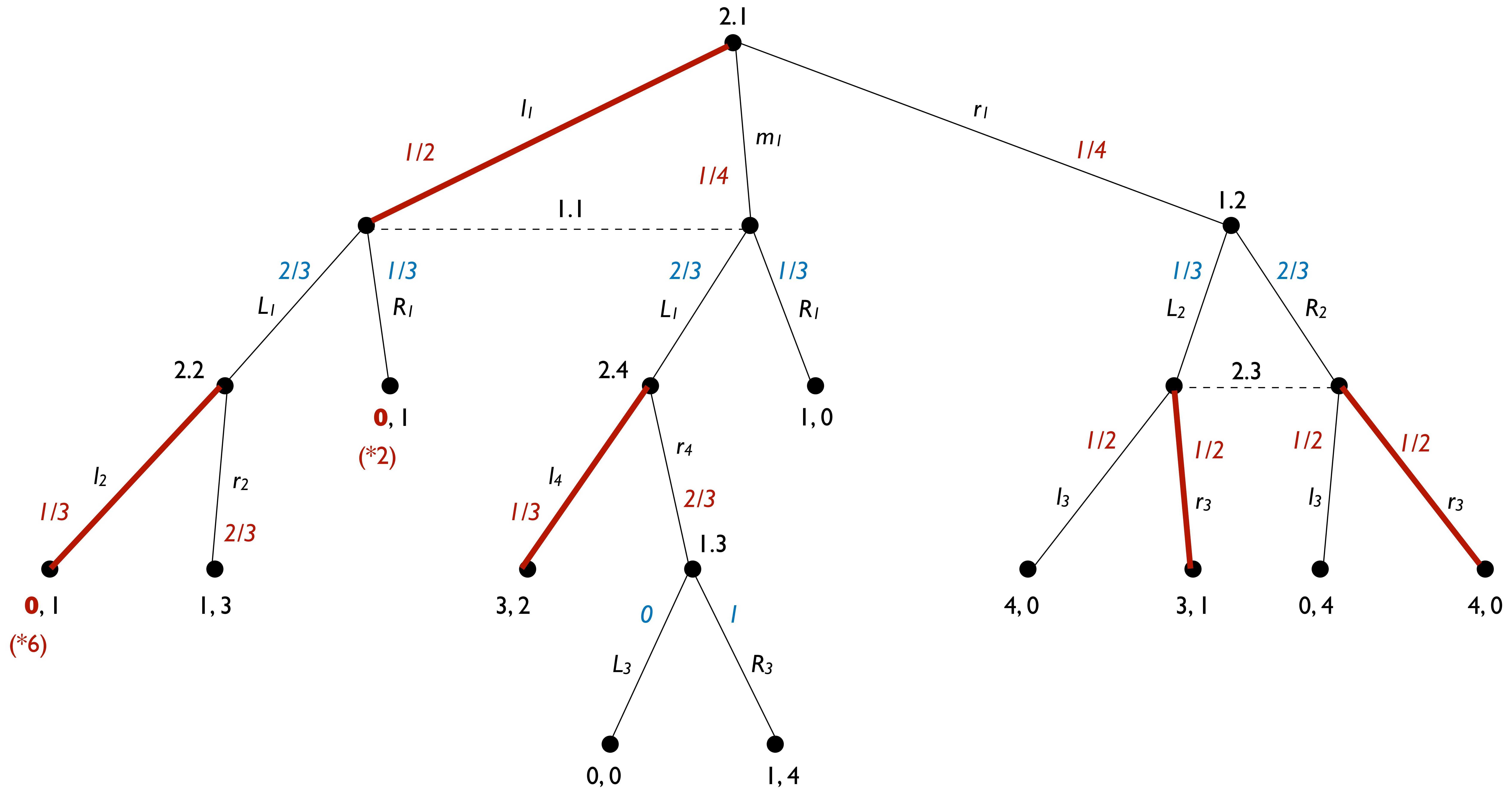
2.1
l_1
1/2
m_1
1/4
r_1
1/4
1.1
1.2
2/3
L_1
1/3
R_1
2/3
L_1
1/3
R_1
1/3
L_2
2/3
R_2
2.2
0, 1
2.4
1, 0
2.3
l_2
1/3
r_2
2/3
l_4
1/3
r_4
2/3
1.3
1/2
l_3
1/2
r_3
1/2
l_3
1/2
r_3
0, 1
1, 3
3, 2
4, 0
3, 1
0, 4
4, 0
0
L_3
1
R_3
0, 0
1, 4

Example (1)

Normalize the payoffs

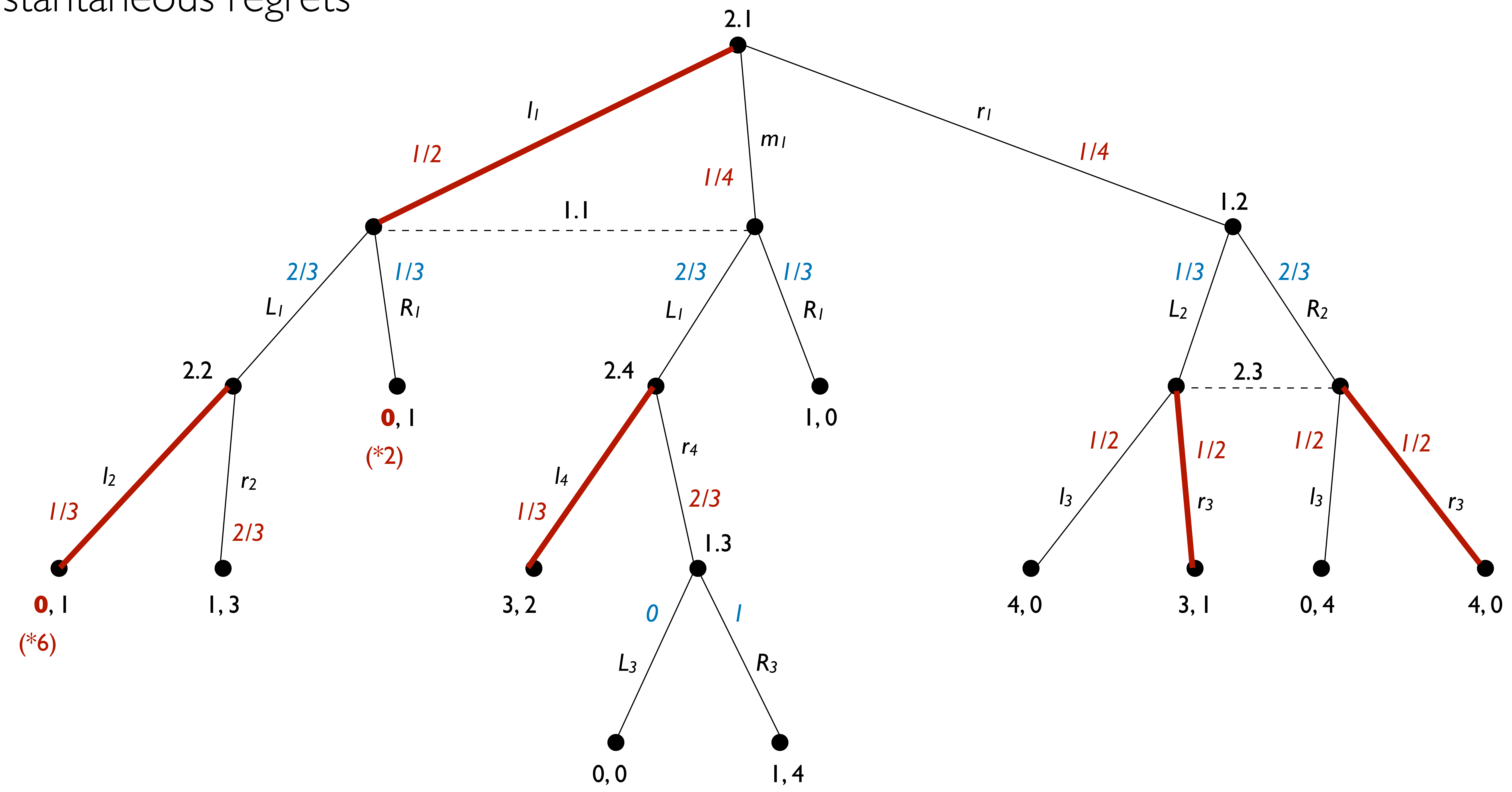


Example (1)


2.1
l1
1/2
m1
1/4
r1
1/4
1.1
1.2
2/3
L1
1/3
R1
2/3
L1
1/3
R1
1/3
L2
2/3
R2
2.2
0, 1
(*2)
2.4
1, 0
2.3
l2
1/3
r2
2/3
l4
1/3
r4
2/3
1/2
l3
1/2
r3
1/2
l3
1/2
r3
0, 1
(*6)
1, 3
3, 2
1.3
4, 0
3, 1
0, 4
4, 0
0
L3
1
R3
0, 0
1, 4

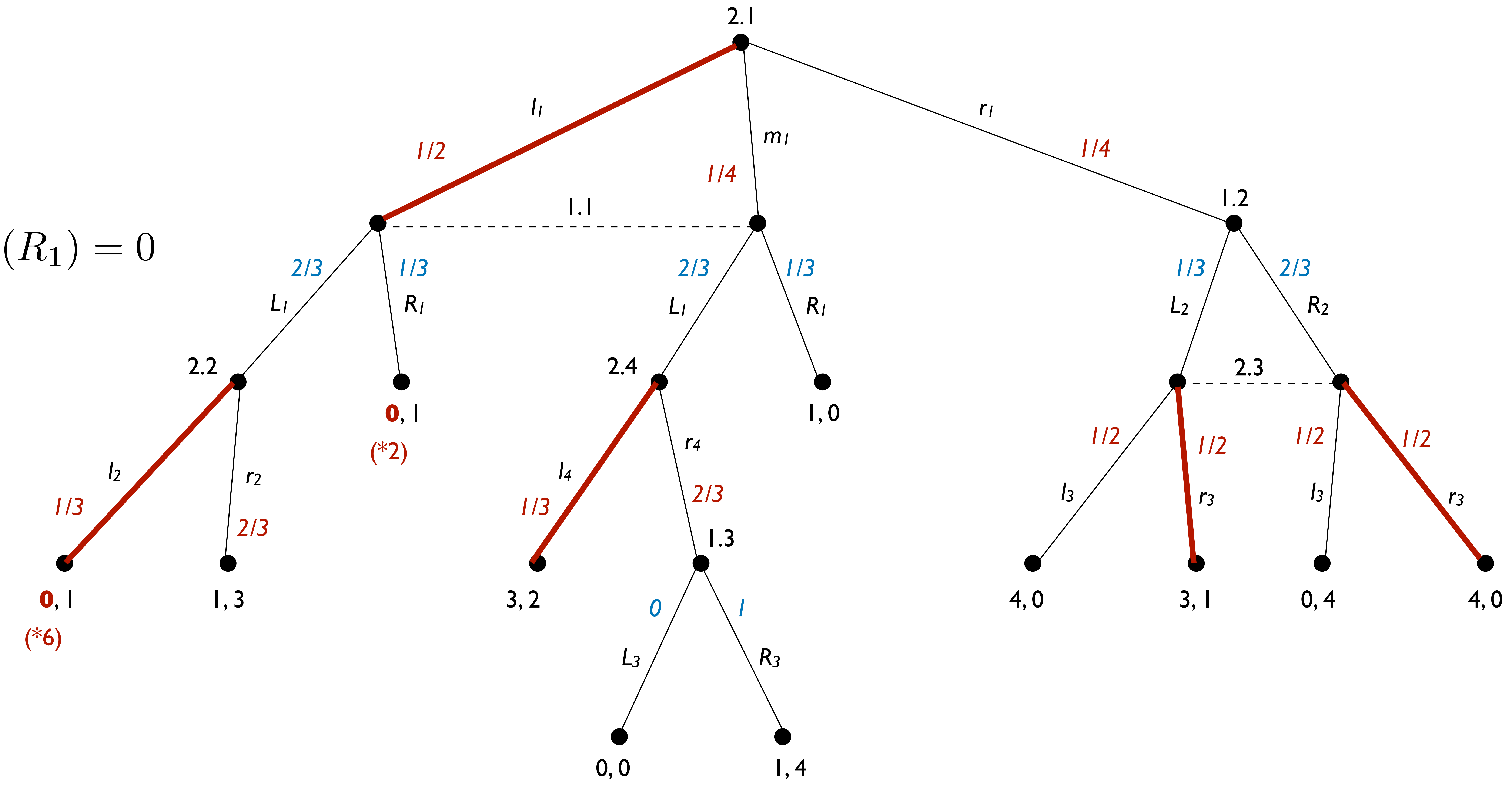
Example (1)

Calculate the instantaneous regrets

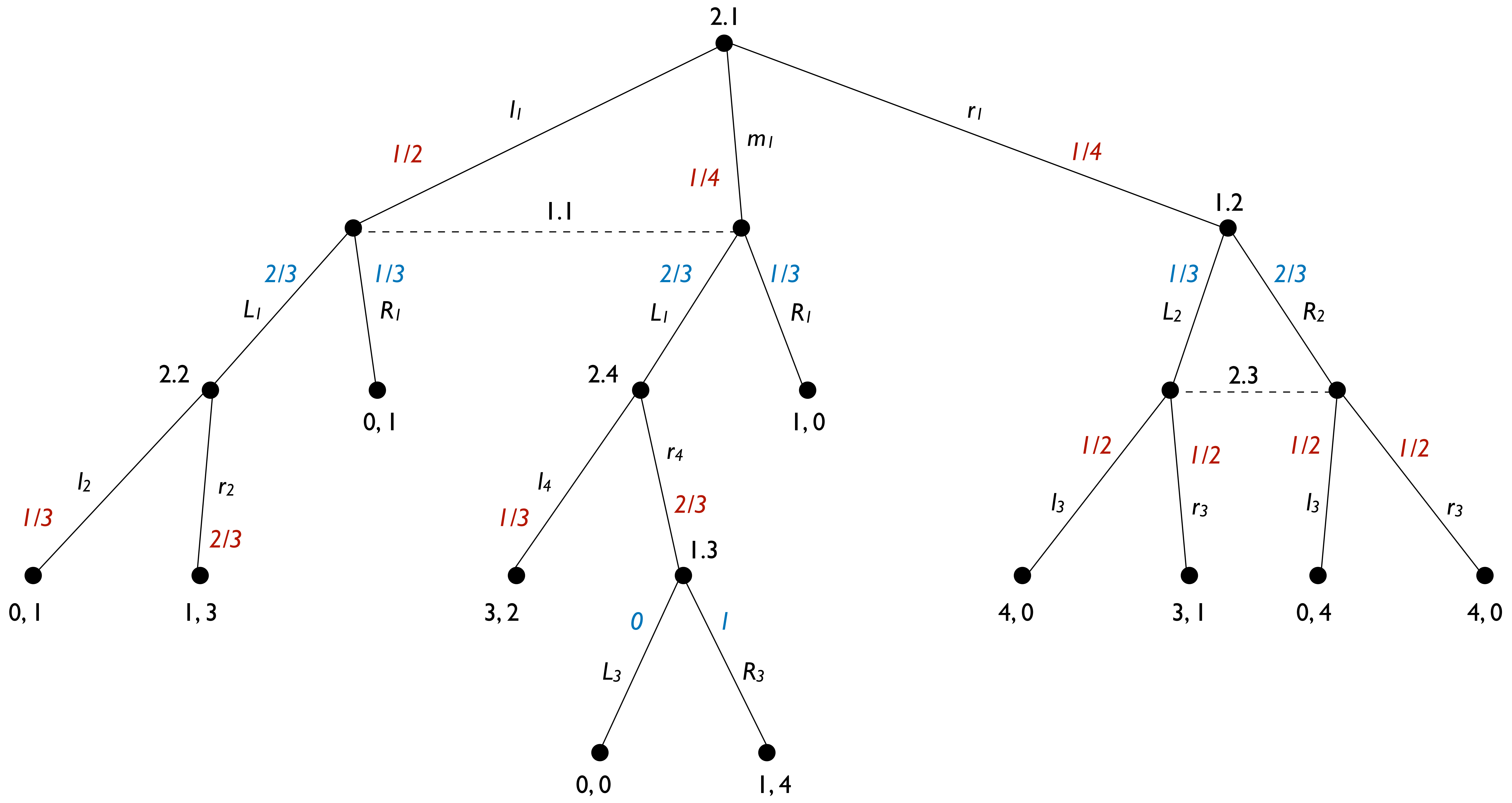


Example (1)

$$r_1^t(L_1) = r_1^t(R_1) = 0$$

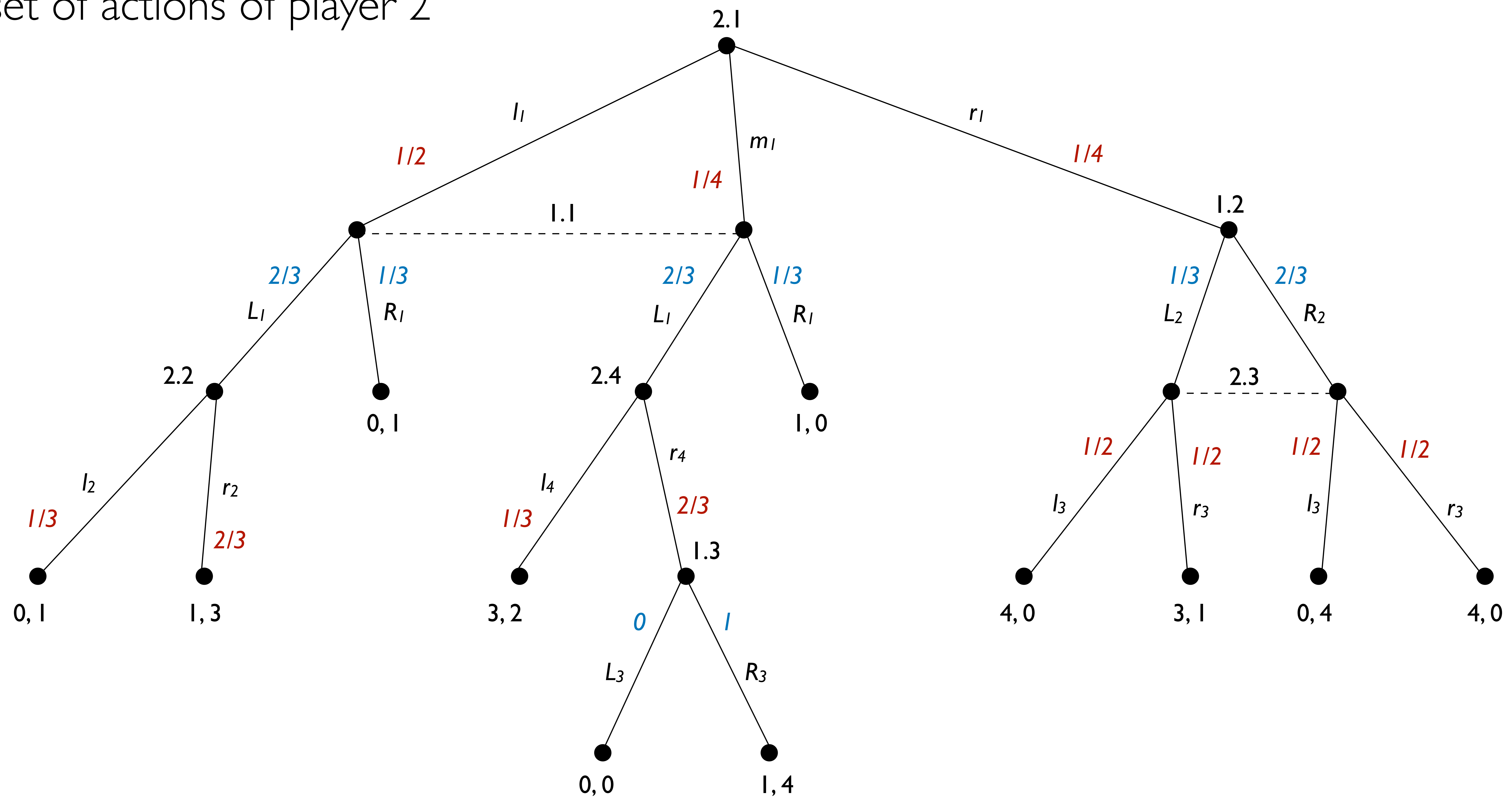


Example (2)

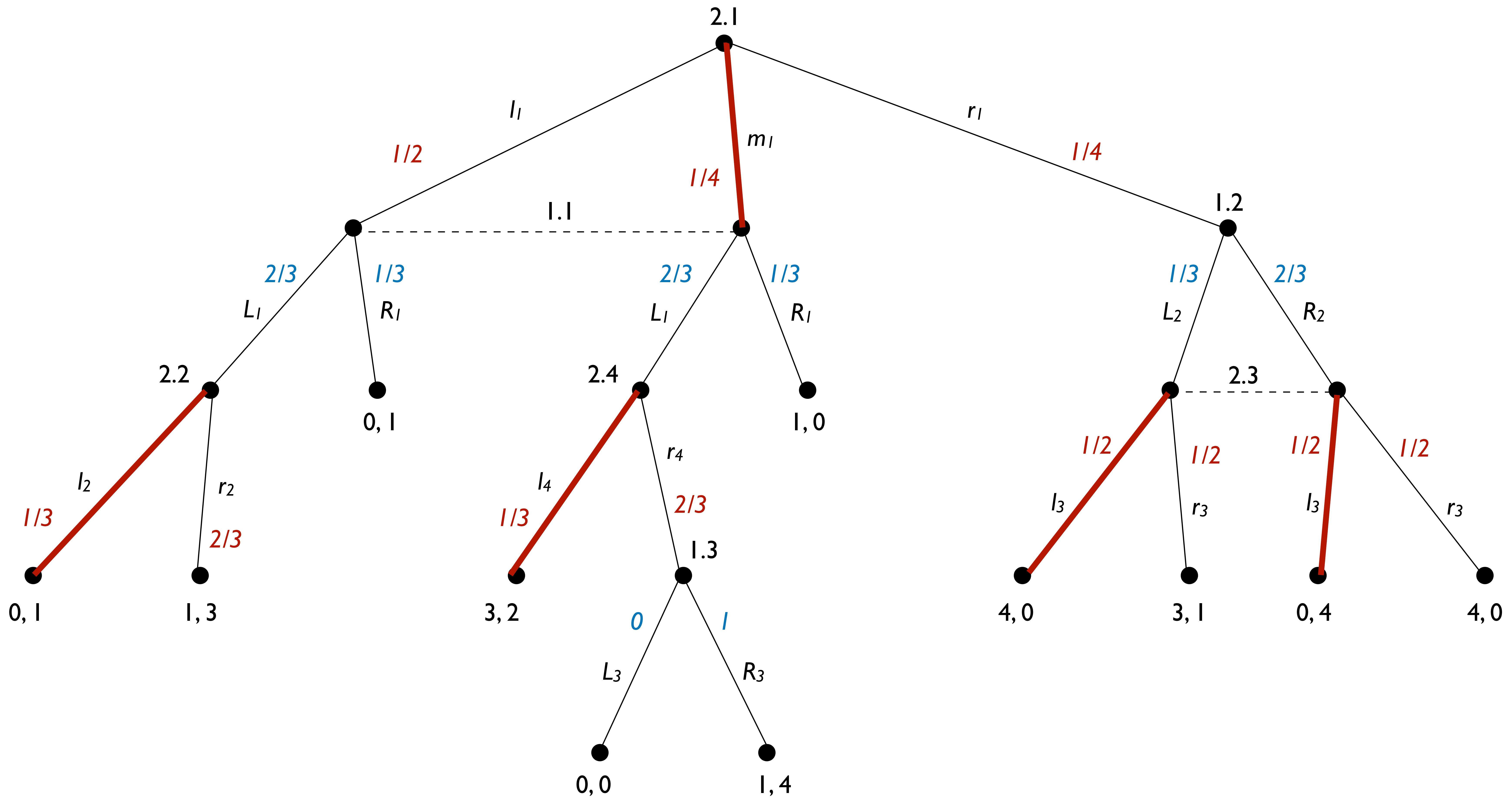

2.1
l_1
1/2
m_1
1/4
r_1
1/4
1.1
1.2
2/3
L_1
1/3
R_1
2/3
L_1
1/3
R_1
1/3
L_2
2/3
R_2
2.2
0, 1
2.4
1, 0
2.3
l_2
1/3
r_2
2/3
l_4
1/3
r_4
2/3
1.3
1/2
l_3
1/2
r_3
1/2
l_3
1/2
r_3
0, 1
1, 3
3, 2
4, 0
3, 1
0, 4
4, 0
0
L_3
1
R_3
0, 0
1, 4

Example (2)

Sample a subset of actions of player 2



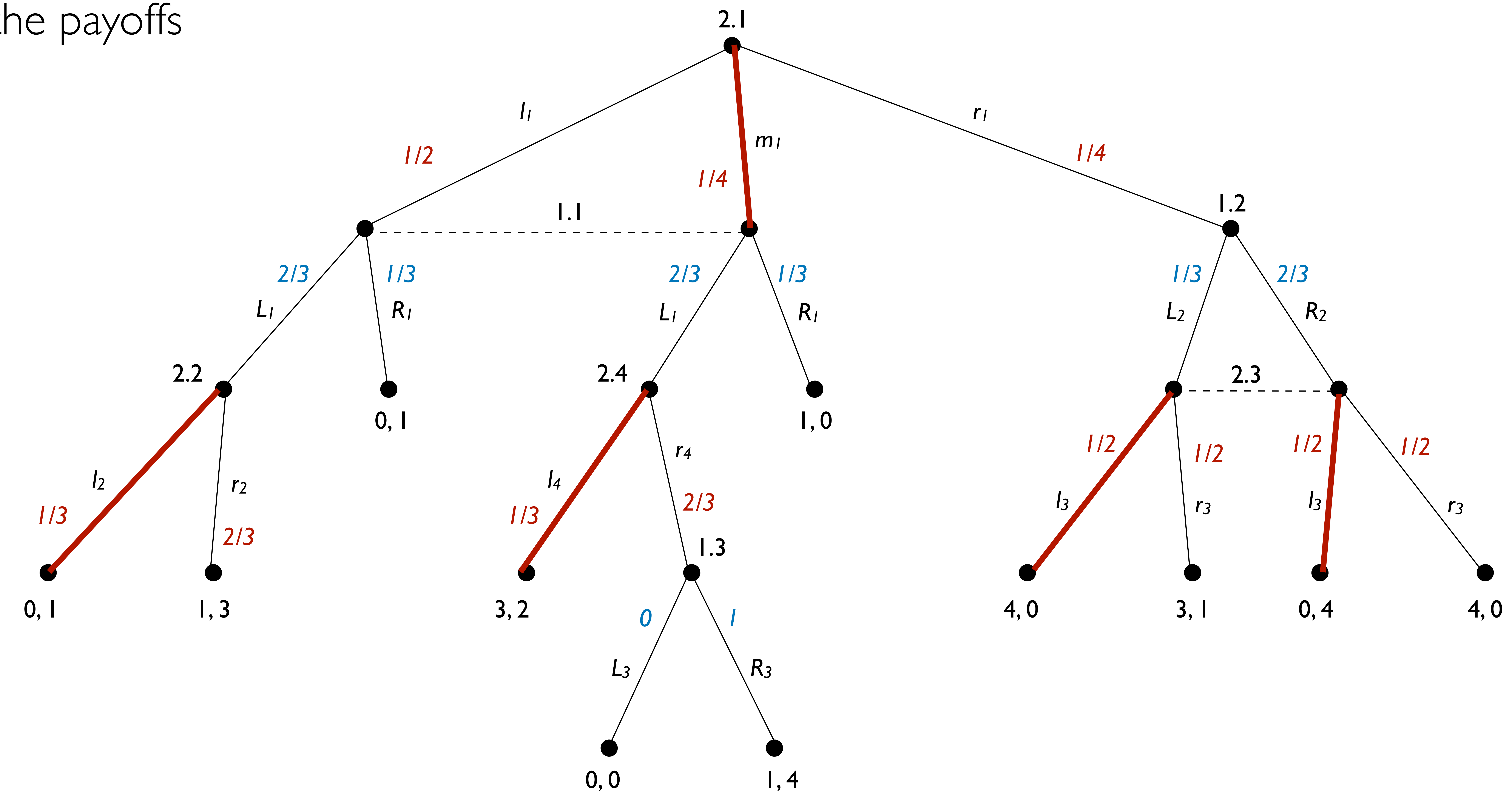
Example (2)



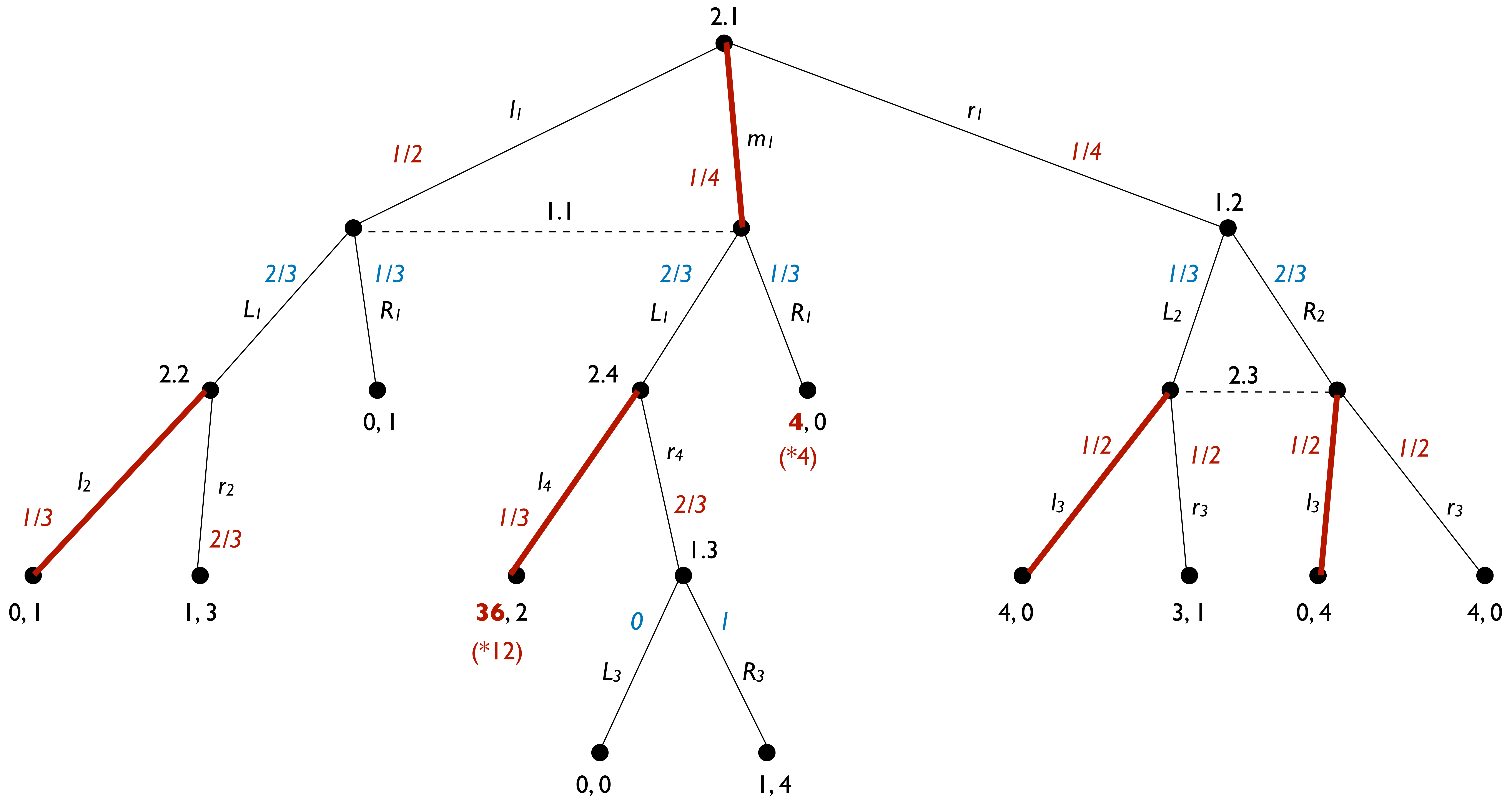
2.1
l1
1/2
m1
1/4
r1
1/4
1.1
1.2
2/3
L1
1/3
R1
2/3
L1
1/3
R1
1/3
L2
2/3
R2
2.2
0, 1
2.4
1, 0
2.3
l2
1/3
r2
2/3
l4
1/3
r4
2/3
1/2
l3
1/2
r3
1/2
l3
1/2
r3
1.3
0, 1
1, 3
3, 2
4, 0
3, 1
0, 4
4, 0
0
L3
1
R3
0, 0
1, 4

Example (2)

Normalize the payoffs



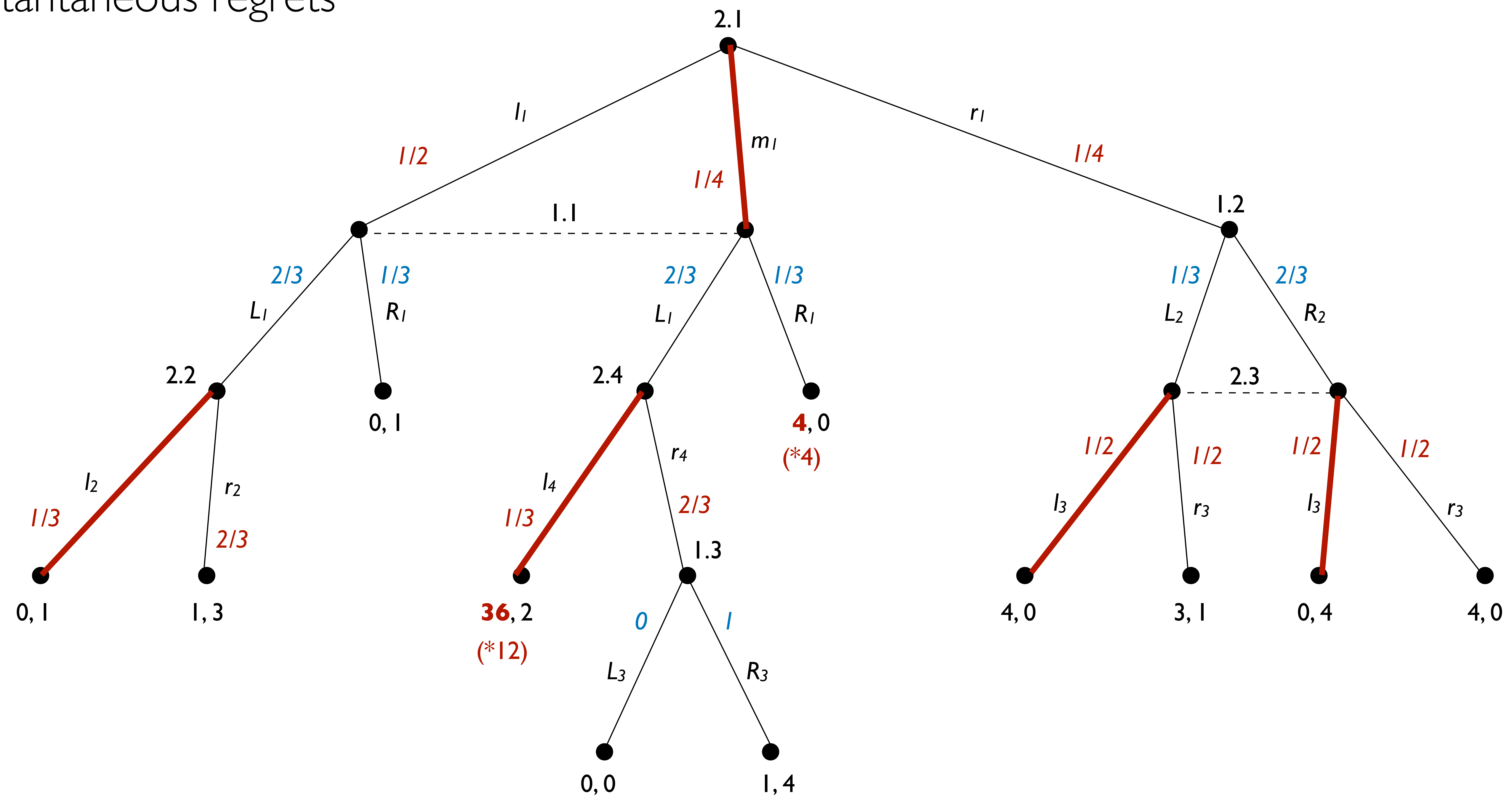
Example (2)



2.1
l1
1/2
m1
1/4
r1
1/4
1.1
1.2
2/3
L1
1/3
R1
2/3
L1
1/3
R1
1/3
L2
2/3
R2
2.2
0, 1
2.4
4, 0
(*4)
2.3
l2
1/3
r2
2/3
l4
1/3
r4
2/3
1/2
l3
1/2
r3
1/2
l3
1/2
r3
1.3
0, 1
1, 3
36, 2
(*12)
4, 0
3, 1
0, 4
4, 0
0
L3
1
R3
0, 0
1, 4

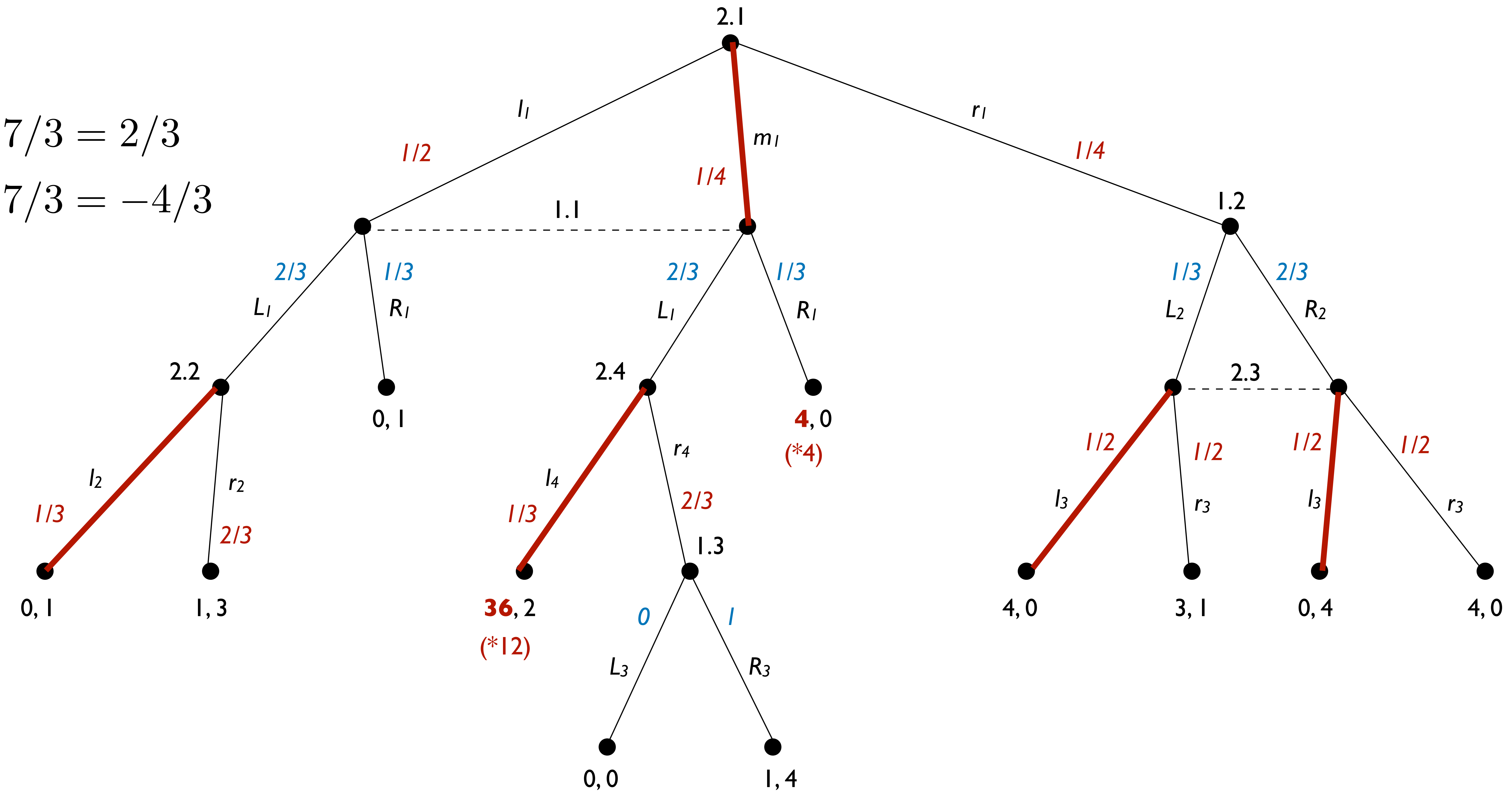
Example (2)

Calculate the instantaneous regrets



Example (2)

$$r_1^t(L_1) = 3 - 7/3 = 2/3$$
$$r_1^t(R_1) = 1 - 7/3 = -4/3$$

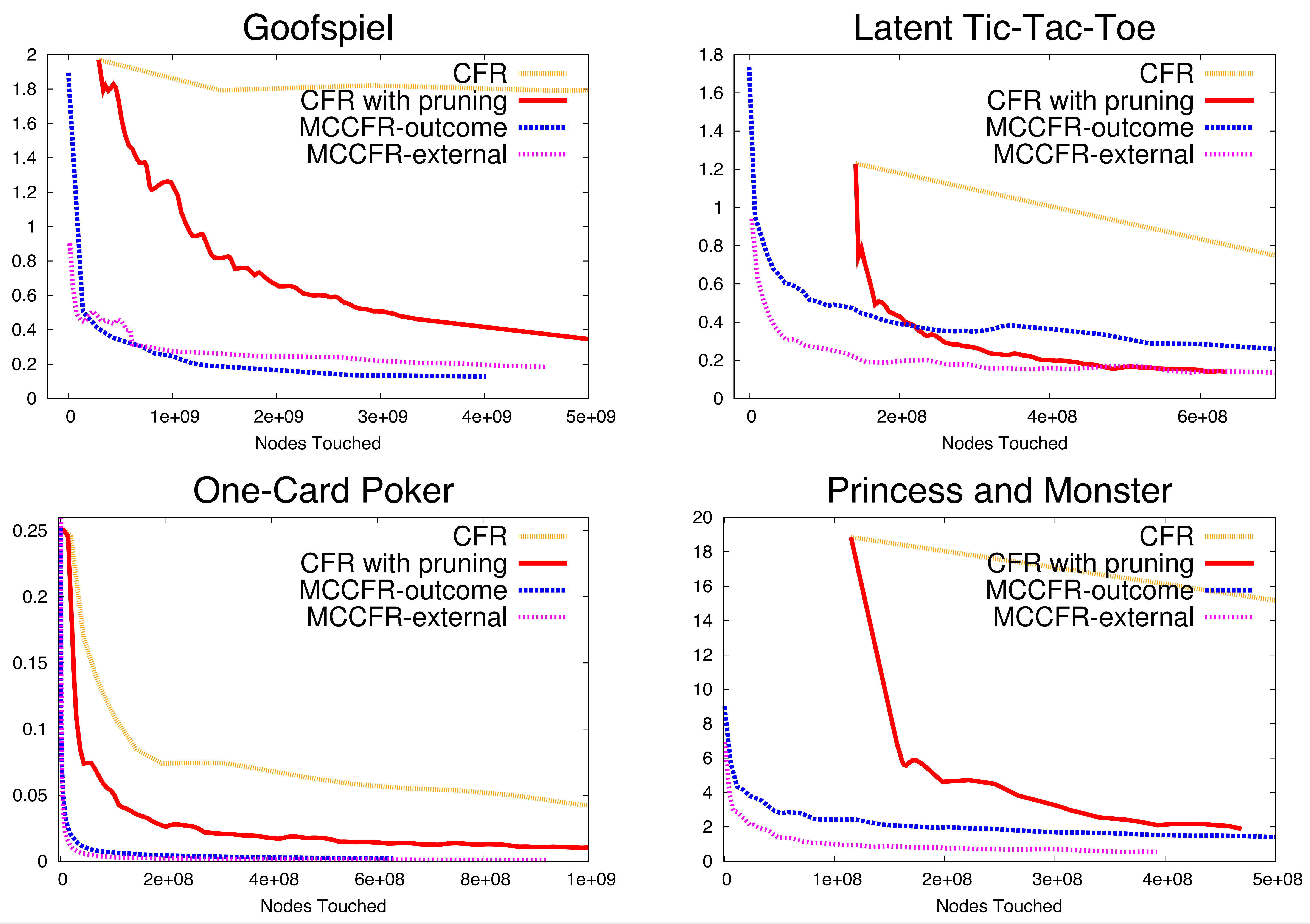


Properties

Monte Carlo CFR converges with high probability

- On the one hand, using Monte Carlo CFR, more iterations are needed due to the randomness of sampling
- On the other hand, using Monte Carlo CFR, every iteration is much less demanding in terms of computational effort
- In many cases, Monte Carlo CFR converges much faster than CFR

Performance comparison



Monte Carlo CFR/CFR+ (other sampling strategy)

- Other sampling strategies also sample actions from the strategy of player i
- This makes a bit more involved the update of the regret
- There is no evidence that these other sampling strategies work better