# Resolution Analysis in Seismic Imaging Using the Kronecker-Factored Hessian

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SIAM: Mathematical & Computational Issues in the Geosciences, Houston

University of Alberta | Signal Analysis & Imaging Group | NSERC CRSNG | Alberta Innovates

# Outline

- Motivation: Resolution and uncertainty analysis in FWI
- Kronecker-factored Hessian
- Numerical examples
  - I. Local resolution analysis
  - II. Linearized Bayesian inversion
- Conclusion

# Full waveform inversion

Regularized least-squares waveform inversion

$$\underset{\mathbf{m}}{\arg\min}\; J(\mathbf{m}) = \frac{1}{2}\sum_{i=1}^{n} \|\mathscr{P}\mathbf{u}_i(\mathbf{m}) - \mathbf{d}_i\|^2 + R(\mathbf{m})$$

Subject to the PDE constraint

$$\mathbf{L}(\mathbf{m})\mathbf{u}_i = \mathbf{s}_i$$

Wave equation operator

External source

# Full waveform inversion

Regularized least-squares waveform inversion

$$\underset{\mathbf{m}}{\arg\min}\ J(\mathbf{m}) = \frac{1}{2}\sum_{i=1}^{n} \|\mathscr{P}\mathbf{u}_i(\mathbf{m}) - \mathbf{d}_i\|^2 + R(\mathbf{m})$$

Updates performed via gradient-based optimization

$$\mathbf{m}^{k+1} = \mathbf{m}^k + \nu^k \delta\mathbf{m}^k$$

Gradients computed using adjoint-state method (Plessix, 2006)

# Estimating uncertainties

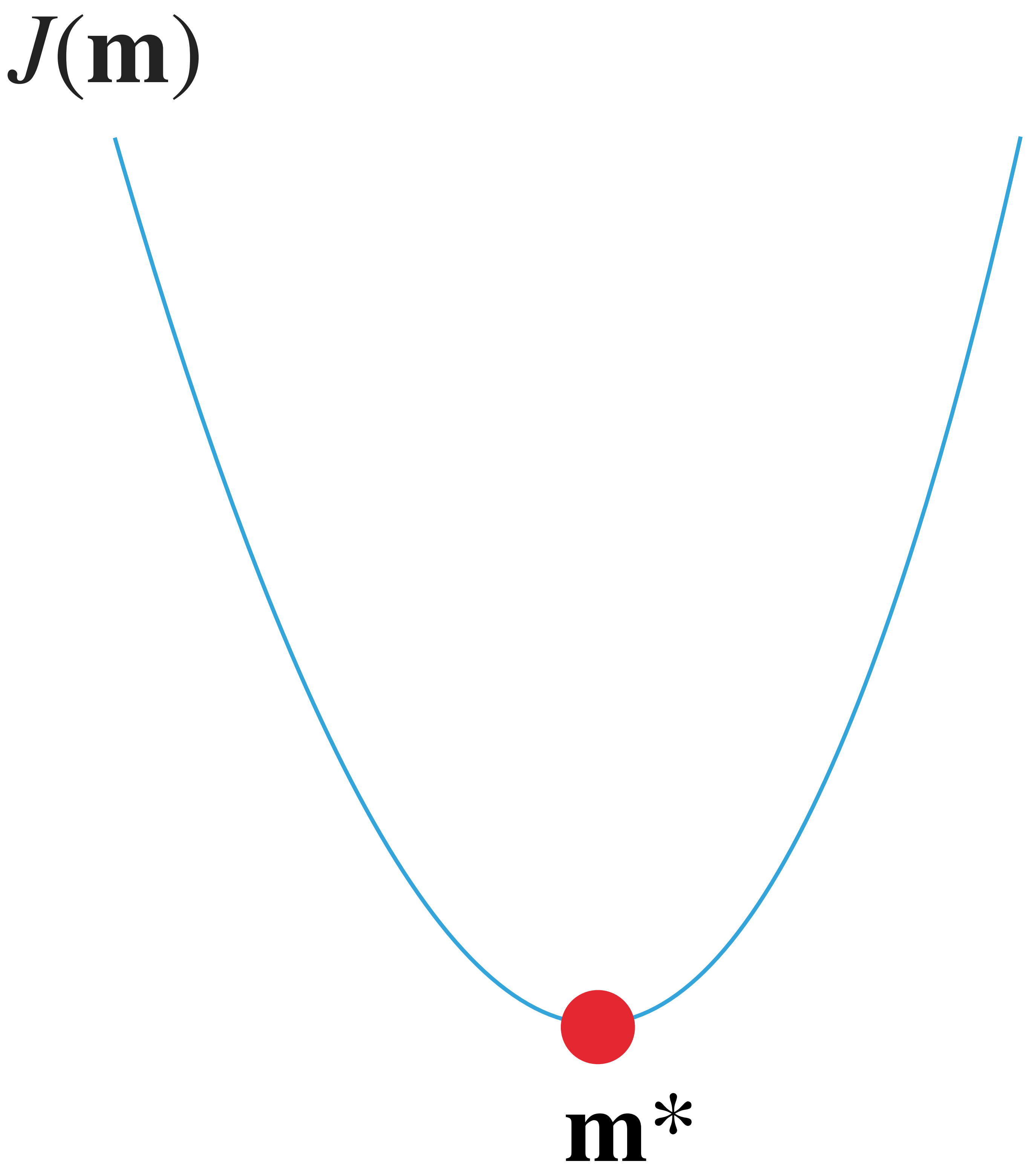
*Assuming convergence, how well constrained are the inverted models for a given dataset?*

*Focus on two interpretations of uncertainty analysis*

- Local resolution analysis
- Linearized Bayesian inversion

# I. Local resolution analysis

Follows the interpretation of Fichtner and Van Leeuwen (2015)



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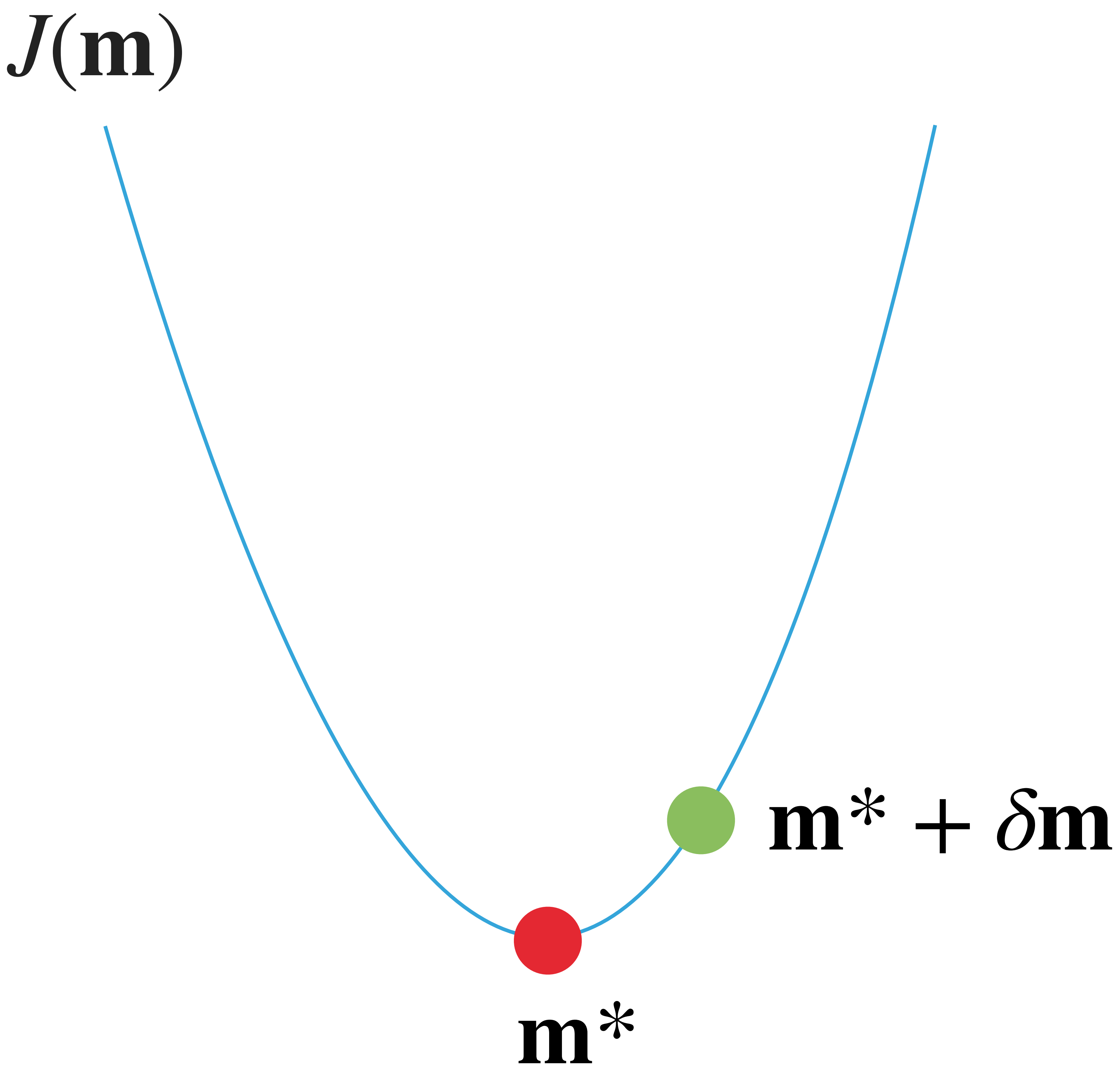

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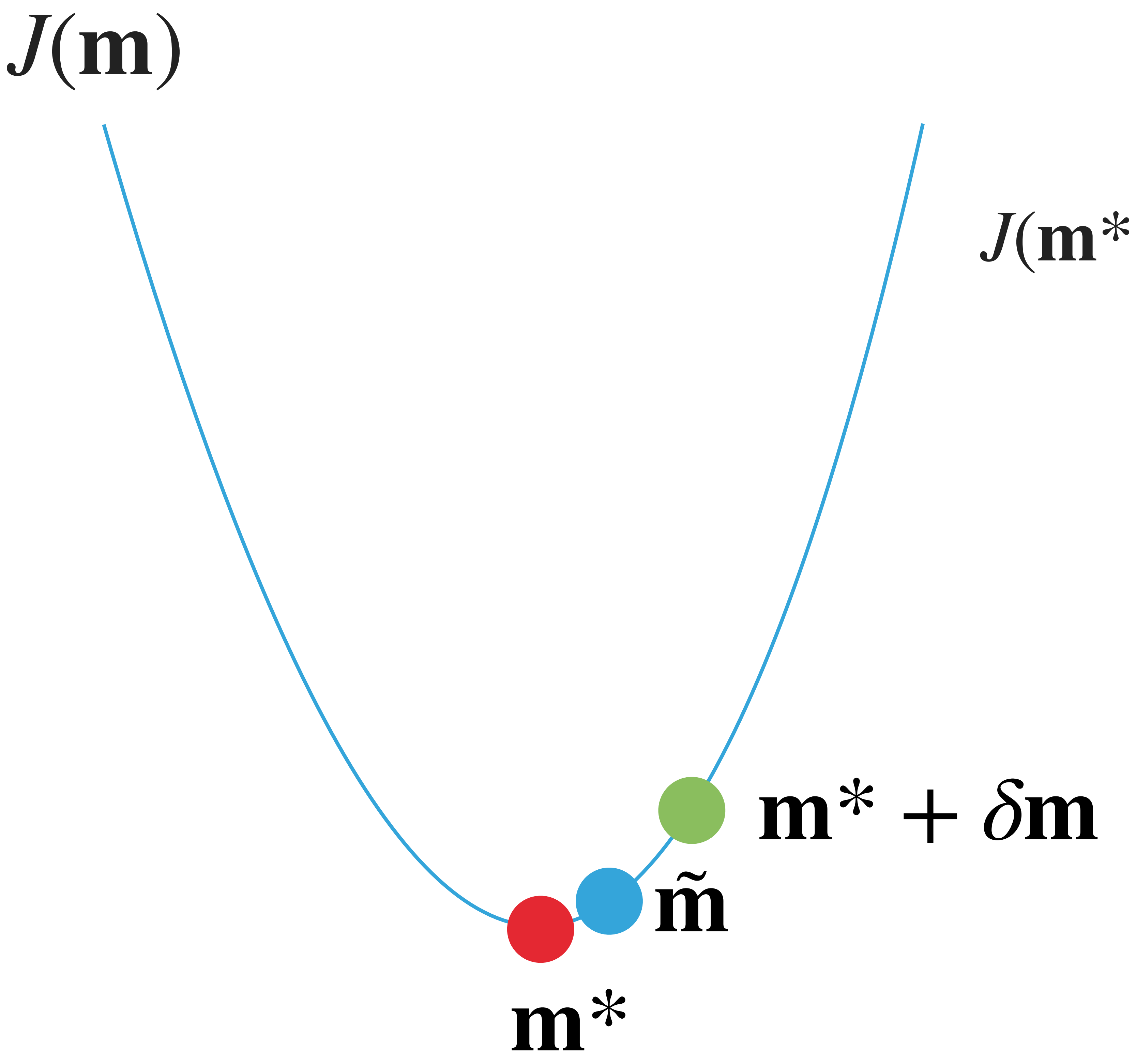


$$J(\mathbf{m}^* + \delta\mathbf{m}) = J(\mathbf{m}^*) + \cancel{\delta\mathbf{m}^T\mathbf{g}(\mathbf{m}^*)} + \frac{1}{2}\delta\mathbf{m}^T\mathbf{H}(\mathbf{m}^*)\delta\mathbf{m}$$

$$\tilde{\mathbf{m}} = (\mathbf{m}^* + \delta\mathbf{m}) - \nu\mathbf{H}\delta\mathbf{m}$$

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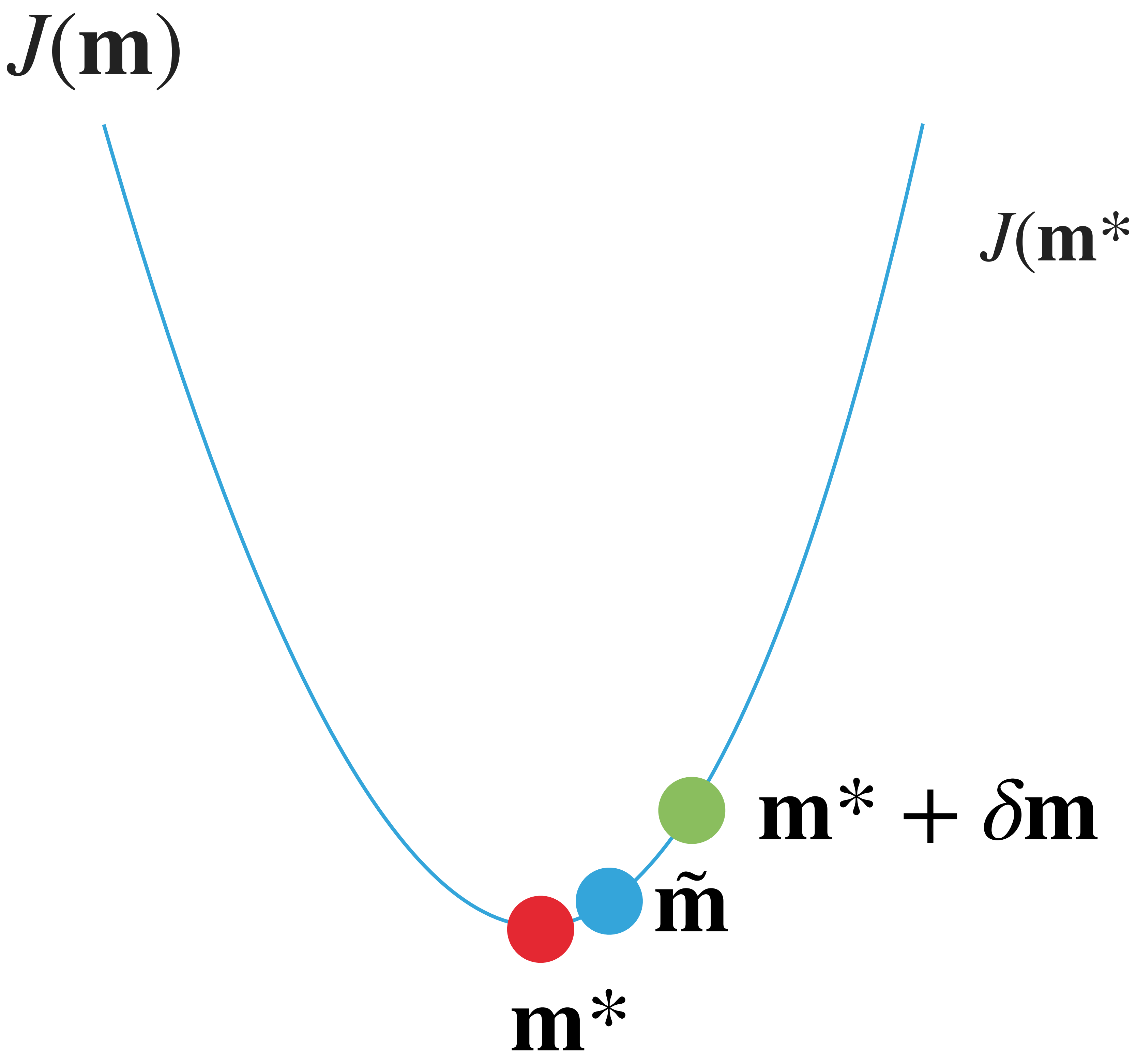


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$$\Downarrow$$

True perturbation smeared by Hessian

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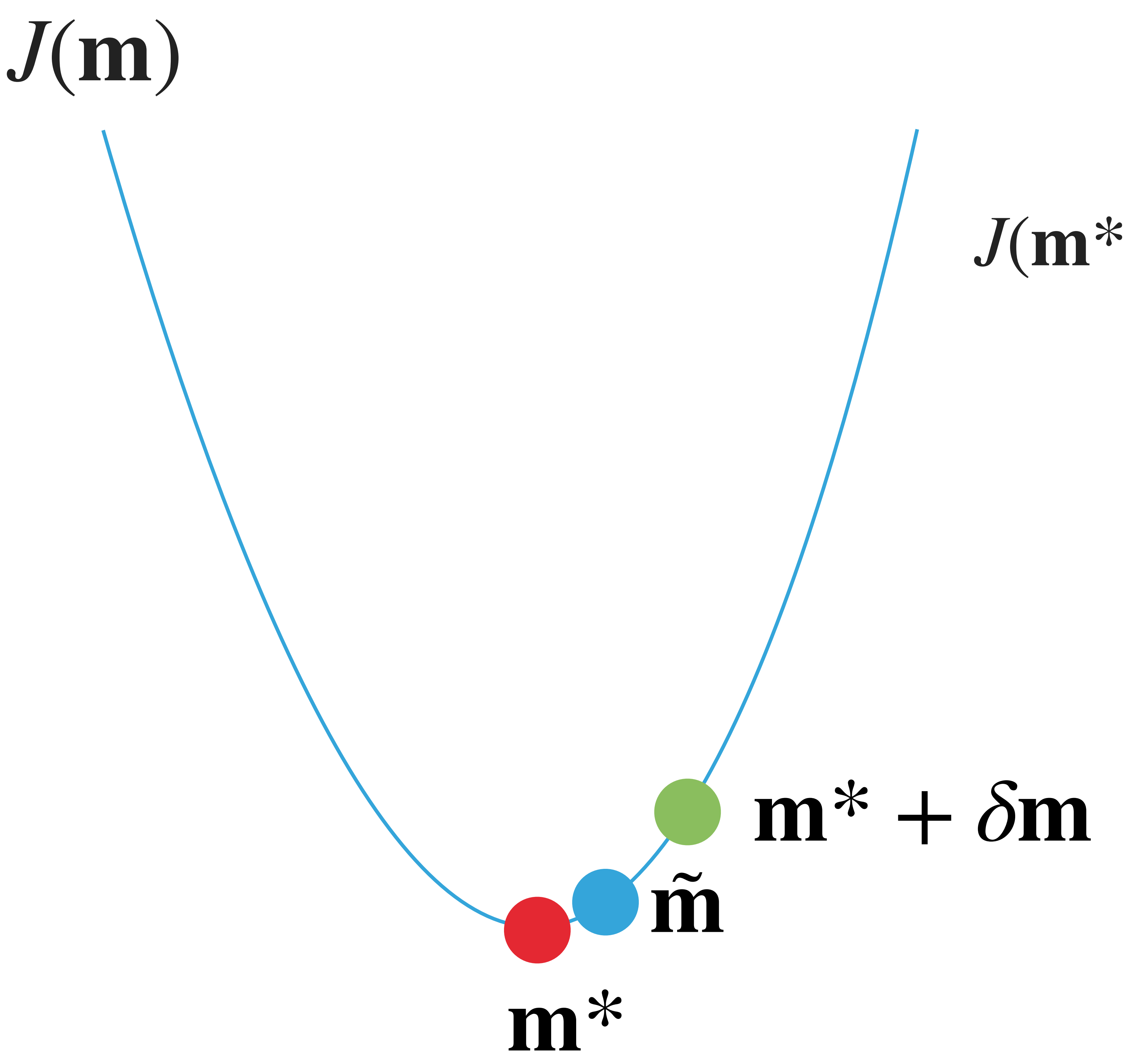


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True perturbation smeared by Hessian

Relies on second-order adjoint method

# II. Bayesian formulation

The Bayesian framework formulates the inverse problem as an inference problem (Tarantola, 2005)

$$\rho(\mathbf{m} \,|\, \mathbf{d}) \propto \rho(\mathbf{d} \,|\, \mathbf{m})\rho(\mathbf{m})$$

Posterior Likelihood Model prior

# II. Bayesian formulation

Assuming Gaussian priors,

$$\rho(\mathbf{m}\,|\,\mathbf{d}) \propto \exp[-\frac{1}{2}(\mathbf{u}(\mathbf{m}) - \mathbf{d})^T C_{\mathbf{D}}^{-1}(\mathbf{u}(\mathbf{m}) - \mathbf{d}) - \frac{1}{2}(\mathbf{m} - \mathbf{m}_0)^T C_{\mathbf{m}}^{-1}(\mathbf{m} - \mathbf{m}_0)]$$

Optimization seeks to maximize log-likelihood

$$\arg\max_{\mathbf{m}} \ \log \rho(\mathbf{m}\,|\,\mathbf{d})$$

# II. Bayesian formulation

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Equivalent to regularized least-squares

$$\arg\min_{\mathbf{m}} \ \frac{1}{2}\|(\mathbf{u}(\mathbf{m}) - \mathbf{d})\|^2_{C_{\mathbf{D}}^{-1}} + \frac{1}{2}\|(\mathbf{m} - \mathbf{m}_0)\|^2_{C_{\mathbf{m}}^{-1}}$$

# Posterior distribution

$$\mathbf{d} \approx \mathbf{u}(\mathbf{m}^*) + \mathbf{G}(\mathbf{m} - \mathbf{m}^*) \qquad (\mathbf{G} = \frac{\partial \mathbf{u}}{\partial \mathbf{m}})$$

Linearizing the modelling operator about **m*** results in a Gaussian posterior

$$\rho(\mathbf{m}\,|\,\mathbf{d}) \propto \exp[-\frac{1}{2}(\mathbf{m} - \mathbf{m}^*)^T \tilde{C}_{\mathbf{m}}^{-1}(\mathbf{m} - \mathbf{m}^*)]$$

With a posterior covariance defined as

$$\tilde{C}_{\mathbf{m}} = (\mathbf{H} + C_{\mathbf{m}}^{-1})^{-1}$$

# Sampling the posterior

Random samples from $\mathcal{N}(\mathbf{m}^*, \tilde{C}_{\mathbf{m}})$ can be drawn via

$$\mathbf{m}_s = \mathbf{m}^* + \tilde{C}_{\mathbf{m}}^{\frac{1}{2}}\mathbf{n}, \quad \mathbf{n} \sim \mathcal{N}(0,1)$$

Cannot readily factor previous expression for $\tilde{C}_{\mathbf{m}}$

# Sampling the posterior

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Use the approach of Bui-Thanh et al. (2014)

$$\tilde{C}_{\mathbf{m}} = C_{\mathbf{m}}^{\frac{1}{2}}(C_{\mathbf{m}}^{\frac{1}{2}}\mathbf{H}C_{\mathbf{m}}^{\frac{1}{2}} + I)^{-1}C_{\mathbf{m}}^{\frac{1}{2}}$$

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Approximate the following with a low-rank approximation

# Sampling the posterior

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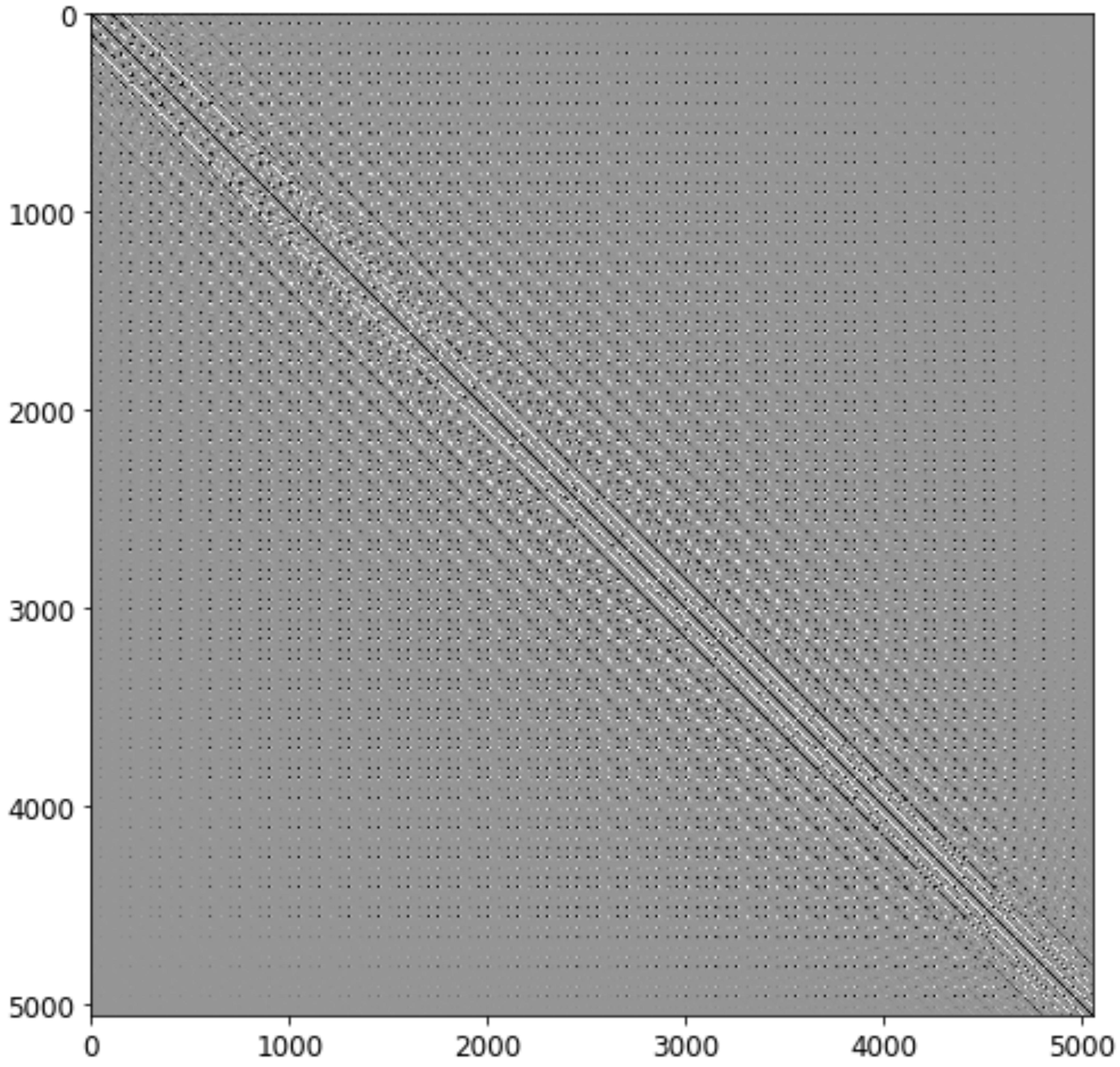
Lanczos iterations require Hessian-vector products

# Fast Hessian-vector products

- Outline two approaches to resolution/uncertainty analysis that feature the Hessian
- Hessian-vector products require costly PDE solves
- Propose a factorization of the Hessian in terms of a superposition of Kronecker products.
- Hessian-vector products involve operations on small matrices

# Kronecker-Factored Hessian

# Structure of the Hessian



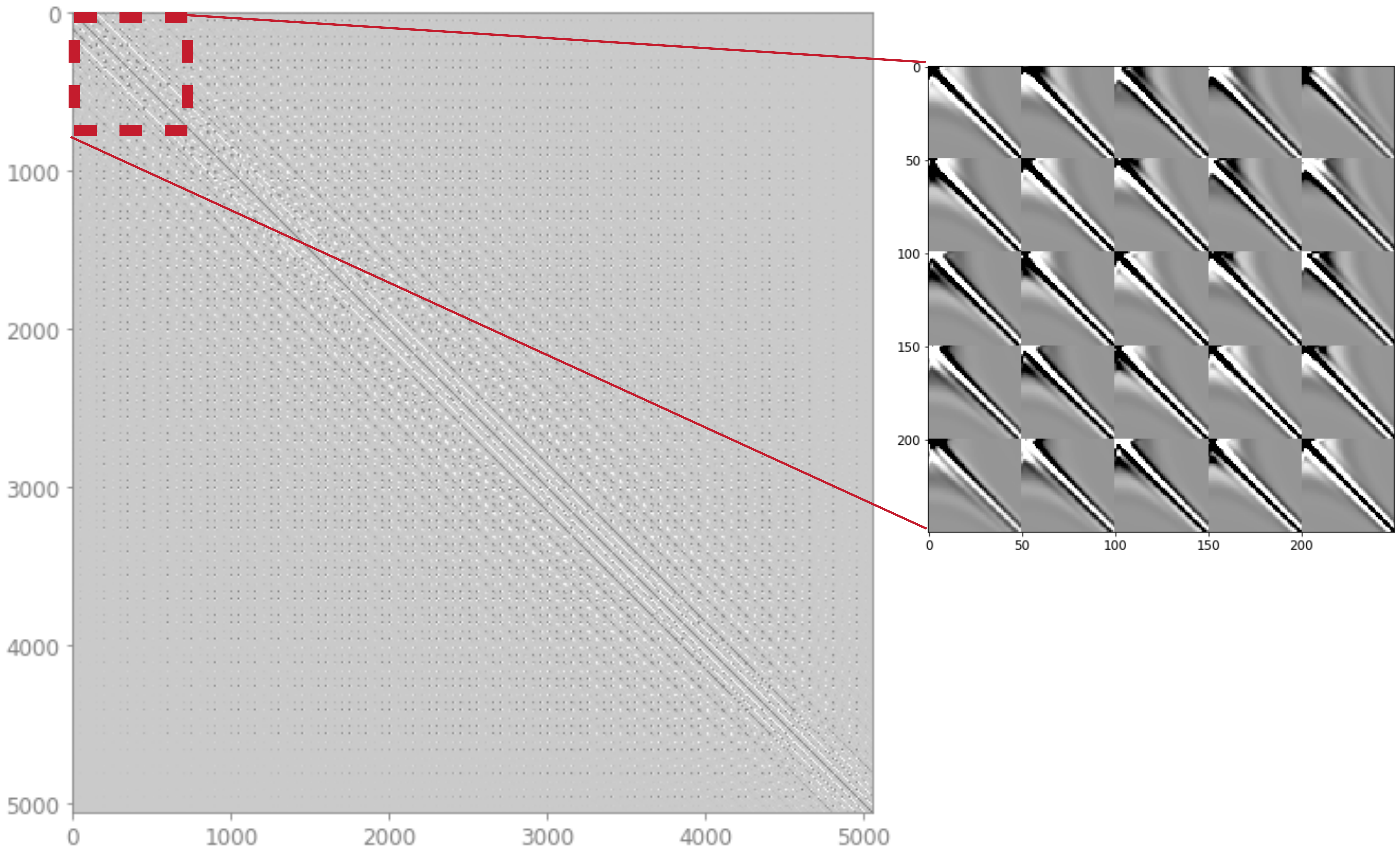

Hessian has a block-banded diagonal structure

Elements typically decay away from the diagonal

**Dimensions:** $n_x n_z$ x $n_x n_z$

(Computed in a homogeneous model of size 101 x 50)

# Structure of the Hessian

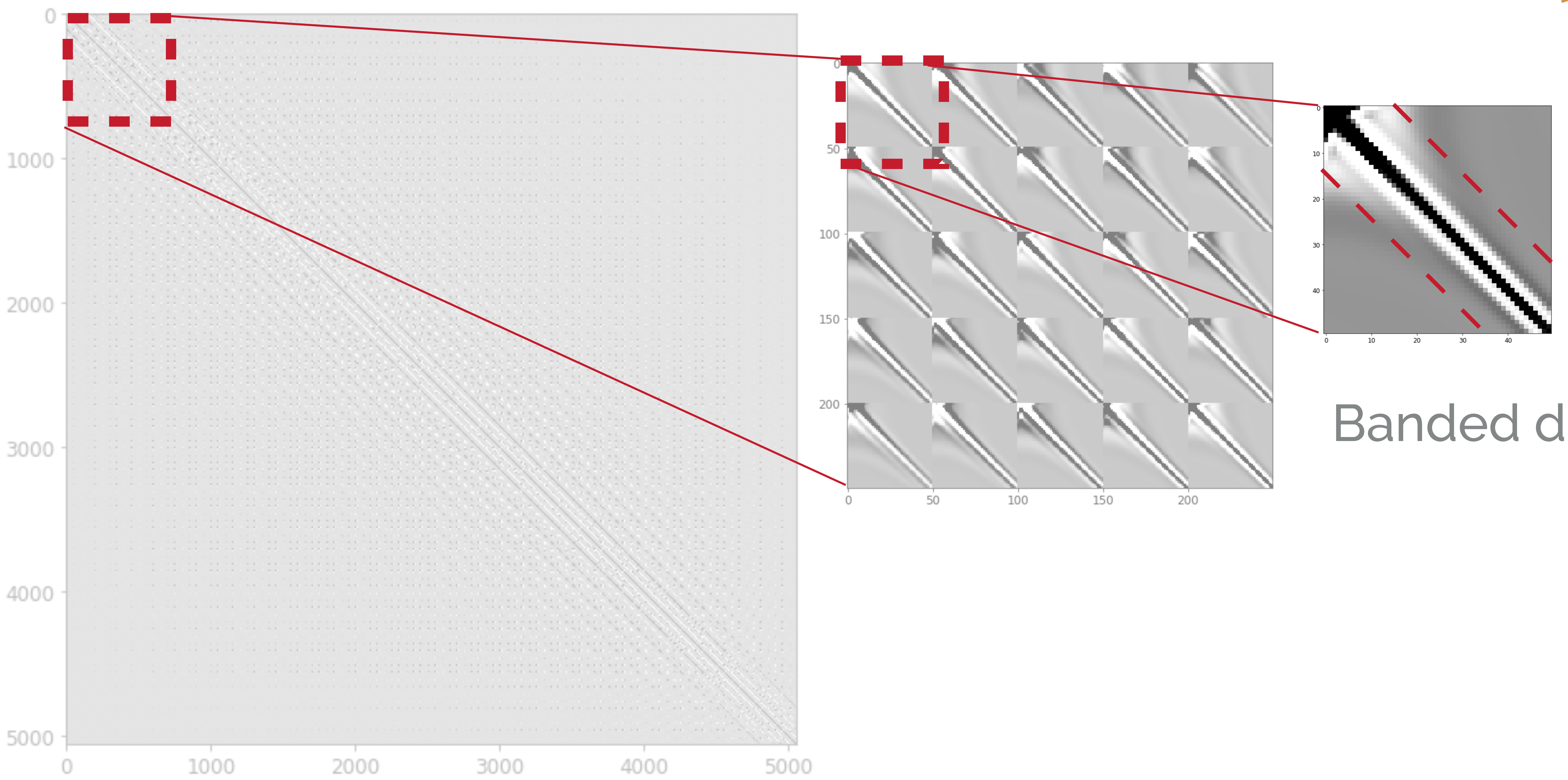



**Block dimensions:**

$n_z \times n_z$

**$n_x$ total blocks**

# Structure of the Hessian



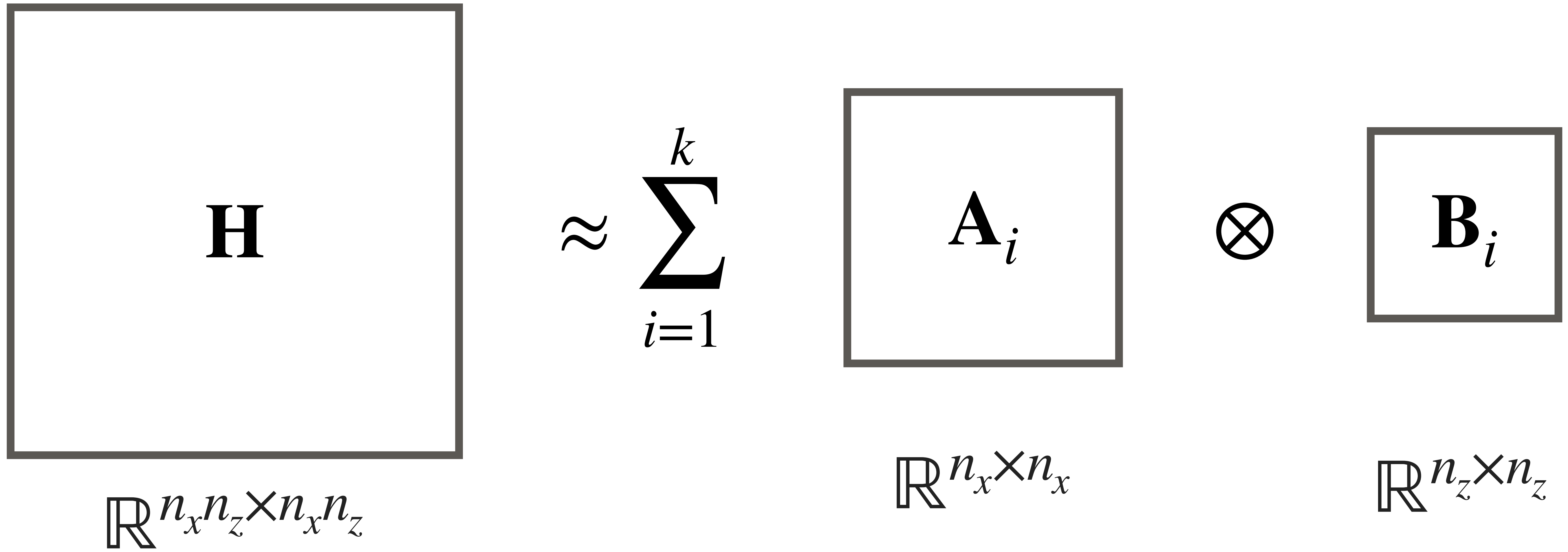

Banded diagonal

# Approximating the Hessian

Propose to approximate the Hessian using a superposition of Kronecker products

$$\mathbf{H} \approx \sum_{i=1}^{k} \mathbf{A}_i \otimes \mathbf{B}_i$$

Factors $\mathbf{A}_i$ and $\mathbf{B}_i$ are small matrices



# The Kronecker product

Suppose **A** and **B** are 2x2 matrices

$$\mathbf{A} = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix}$$

The Kronecker product is defined as

$$\mathbf{H} = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} \otimes \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix} = \begin{bmatrix} a_1 \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} & a_3 \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} \\ a_2 \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} & a_4 \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{bmatrix}$$

# The Kronecker factors

Problem was addressed by Pitsianis (1997)

$$\mathbf{a}\,,\mathbf{b} = \arg\min_{\mathbf{a},\mathbf{b}} \|\tilde{\mathbf{H}} - \mathbf{b}\mathbf{a}^T\|_F^2$$

$$\tilde{\mathbf{H}} = \mathscr{R}(\mathbf{H}) \approx \sum_i^k \mathbf{b}_i \mathbf{a}_i^T$$ where $$\mathbf{a}_i = vec(\mathbf{A}_i) \qquad \mathbf{b}_i = vec(\mathbf{B}_i)$$

Factors are low-rank approximation of the *rearranged Hessian*

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Factors are low-rank approximation of the *rearranged Hessian*

↓

Estimate from SVD

# The rearranged Hessian

Rearrangement operator can be summarized as

$\mathbf{H}$ $\tilde{\mathbf{H}}$

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$$\begin{bmatrix} a_1 \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} & a_3 \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} \\ a_2 \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} & a_4 \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix} \end{bmatrix} \qquad \begin{bmatrix} a_1b_1 & a_2b_1 & a_3b_1 & a_4b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 & a_4b_2 \\ a_1b_3 & a_2b_3 & a_3b_3 & a_4b_3 \\ a_1b_4 & a_2b_4 & a_3b_4 & a_4b_4 \end{bmatrix}.$$

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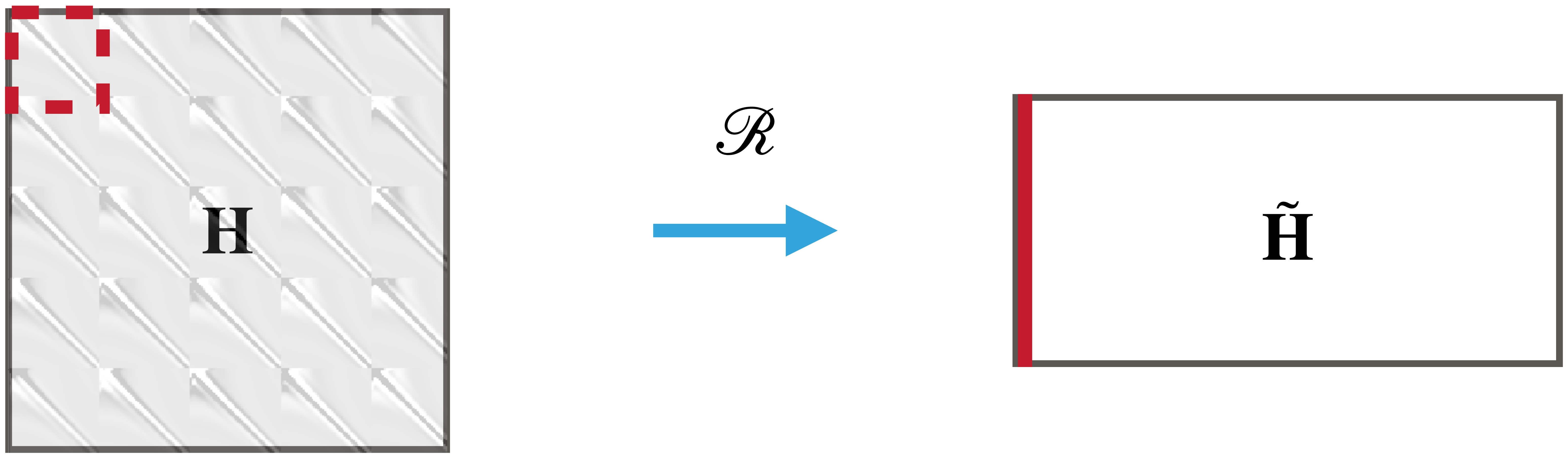


$\mathbb{R}^{n_x n_z \times n_x n_z}$ $\qquad$ $\mathbb{R}^{n_z^2 \times n_x^2}$

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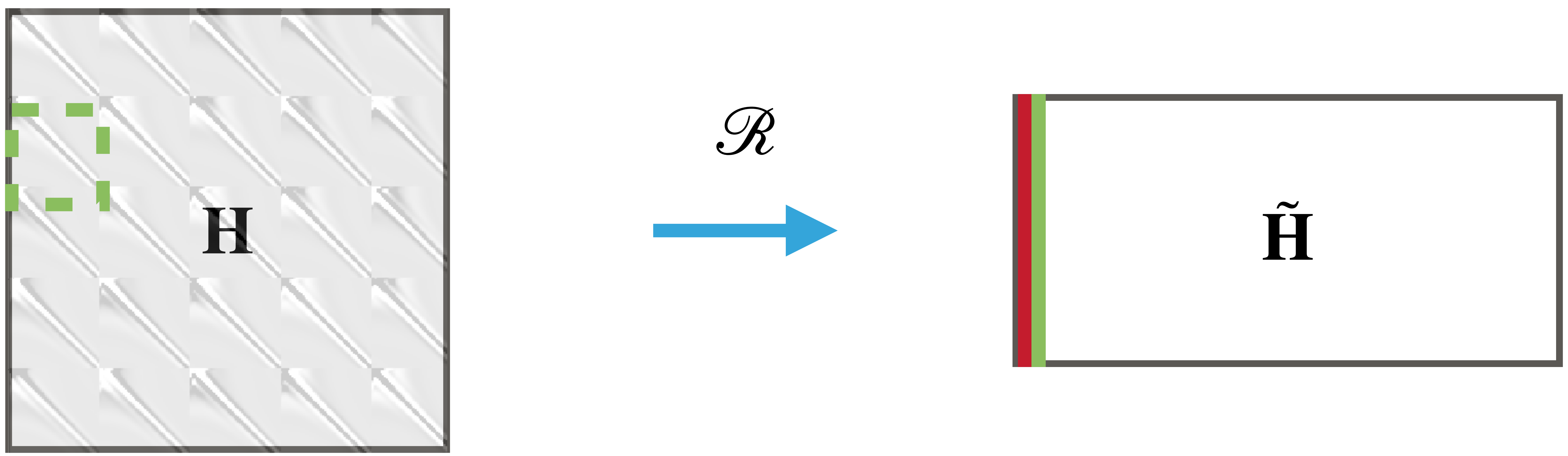


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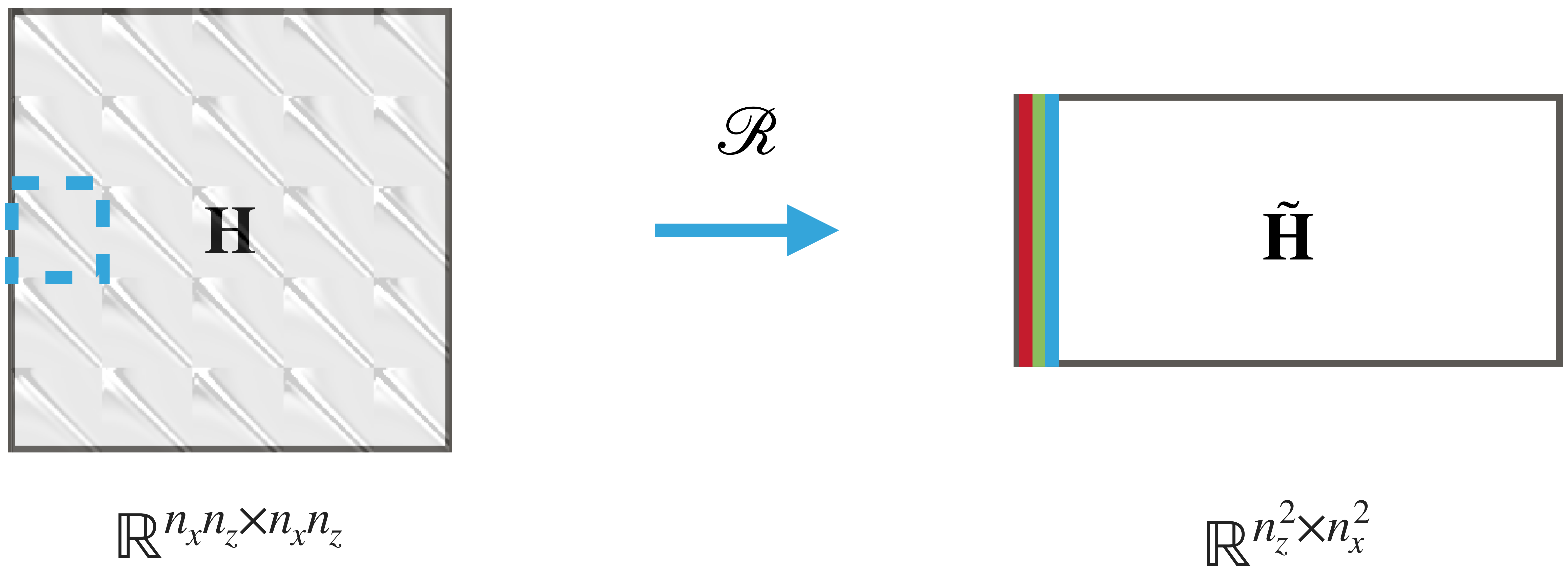


# Hessian-vector products

The Kronecker product possesses a useful property

$$(\mathbf{A} \otimes \mathbf{B})\mathbf{m} = vec(\mathbf{B}\mathbf{M}\mathbf{A}^T), \quad \mathbf{m} = vec(\mathbf{M})$$

# Hessian-vector products

The Kronecker product possesses a useful property

$$\overset{\mathbb{R}^{n_x n_z \times n_x n_z}}{\boxed{\mathbf{A}\otimes\mathbf{B}}}\ \overset{\mathbb{R}^{n_x n_z \times 1}}{\boxed{\mathbf{m}}} = vec(\overset{\mathbb{R}^{n_z \times n_z}}{\boxed{\mathbf{B}}}\ \overset{\mathbb{R}^{n_z \times n_x}}{\boxed{\mathbf{M}}}\ \overset{\mathbb{R}^{n_x \times n_x}}{\boxed{\mathbf{A}^T}})$$

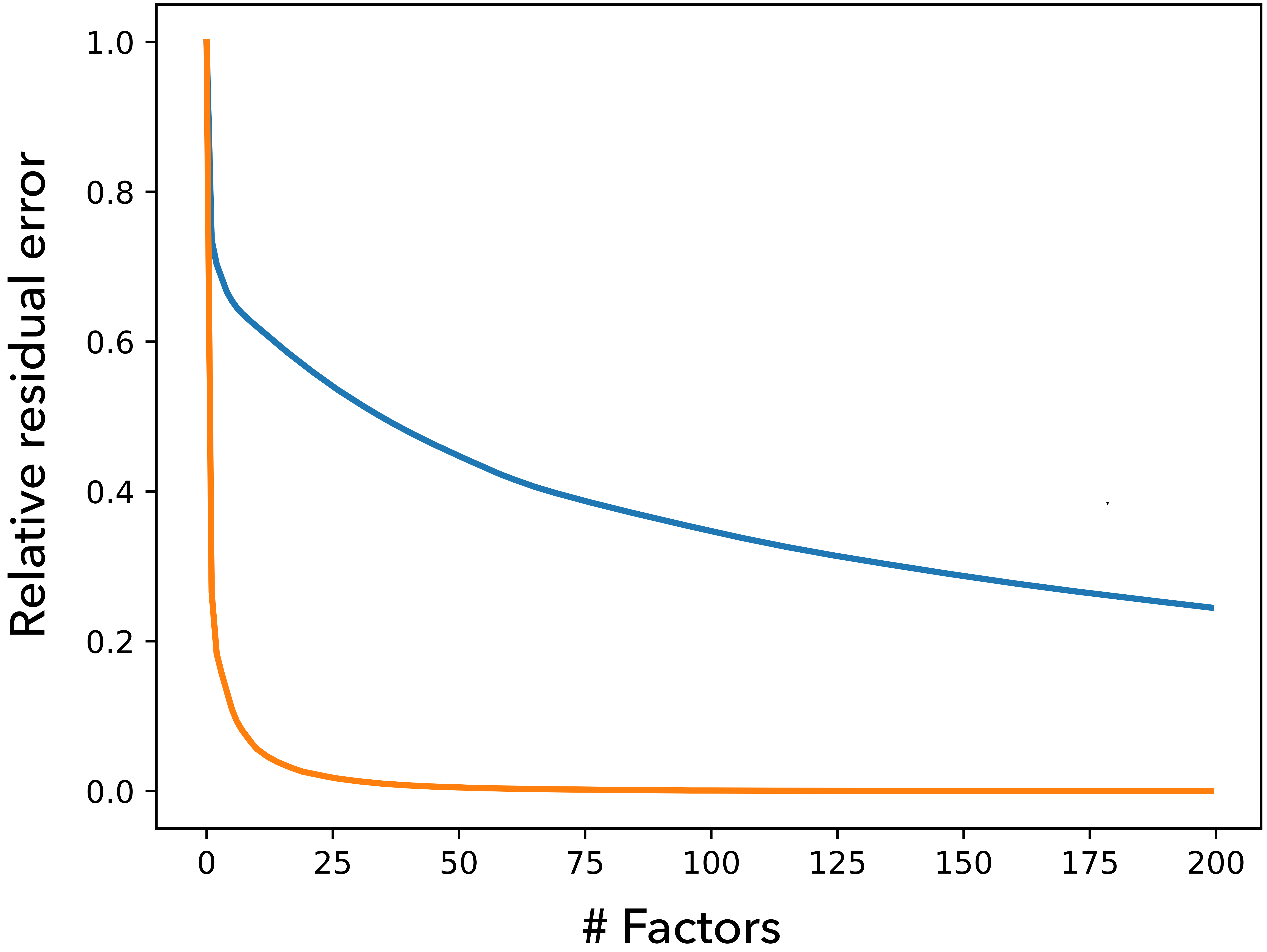
# Hessian-vector products

This allows a Hessian-vector product to be approximated as

$$\mathbf{Hm} \approx \sum_{i=1}^{k} vec(\mathbf{B}_i \mathbf{M} \mathbf{A}_i^T) \quad \longrightarrow \quad \sum_{i=1}^{k} vec\Big(\overset{\mathbb{R}^{n_z \times n_z}}{\boxed{\mathbf{B}_i}}\ \overset{\mathbb{R}^{n_z \times n_x}}{\boxed{\quad \mathbf{M} \quad}}\ \overset{\mathbb{R}^{n_x \times n_x}}{\boxed{\quad \mathbf{A}_i^T \quad}}\Big)$$

**Can approximate Hessian-vector products using operations with small matrices**

# Approximation error



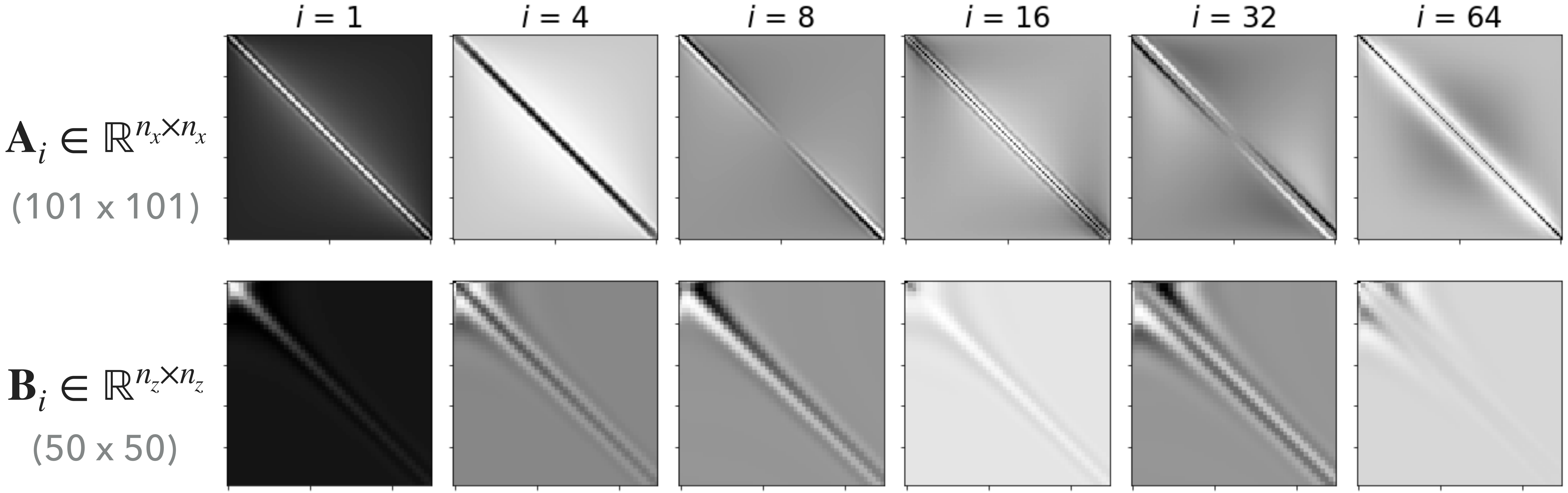

Error in low-rank approximation

$$\frac{\|\mathbf{H} - \sum_i^k \lambda_i \mathbf{u}_i \mathbf{v}_i^T\|_F}{\|\mathbf{H}\|_F}$$

Error in Kronecker approximation

$$\frac{\|\mathbf{H} - \sum_i^k \mathbf{A}_i \otimes \mathbf{B}_i\|_F}{\|\mathbf{H}\|_F}$$

# Kronecker factors



# Estimating the factors

**Small problems:** Direct SVD of rearranged Hessian

**Medium/Large problems:** Estimate factors using low-rank matrix completion of rearranged Hessian

$$\tilde{\mathbf{A}}, \tilde{\mathbf{B}} = \underset{\tilde{\mathbf{A}} \in \mathbb{R}^{n_z{}^2 \times k},\ \tilde{\mathbf{B}} \in \mathbb{R}^{n_x{}^2 \times k}}{\arg\min} \; ||P_\Omega(\tilde{\mathbf{H}} - \tilde{\mathbf{B}}\tilde{\mathbf{A}}^T)||_F^2, \qquad \tilde{\mathbf{A}} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k]$$

Compute samples preferentially according to the structure

# Estimating the factors

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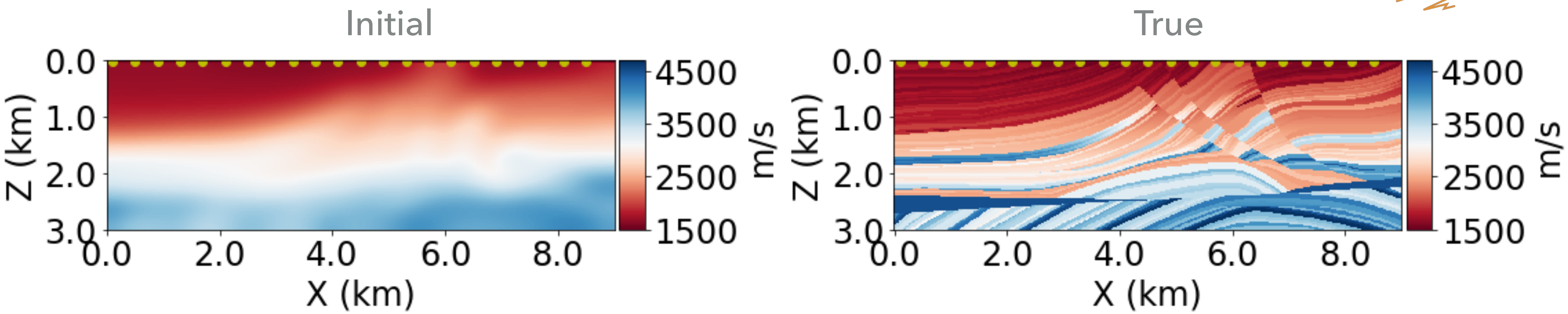
Requires PDE solves - Limit cost by using receiver Green's functions ($N_s$ + $N_r$)

# Numerical Experiments

# Marmousi model

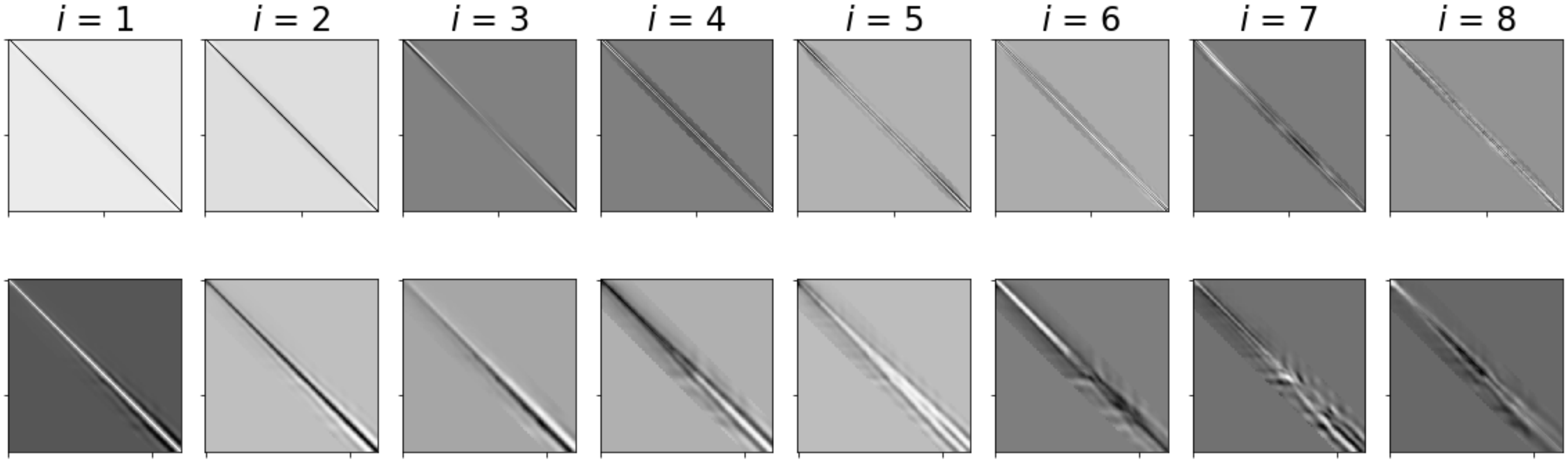



- Time-domain acoustic modelling (900 x 300)
- 22 sources, 225 receivers distributed at surface
- Multi-scale inversion (3-5Hz, -7Hz, -9Hz)
- 30 preconditioned NLCG iterations per frequency band

# Kronecker factors

$\mathbf{A}_i \in \mathbb{R}^{n_x \times n_x}$
(900 x 900)

$\mathbf{B}_i \in \mathbb{R}^{n_z \times n_z}$
(300 x 300)

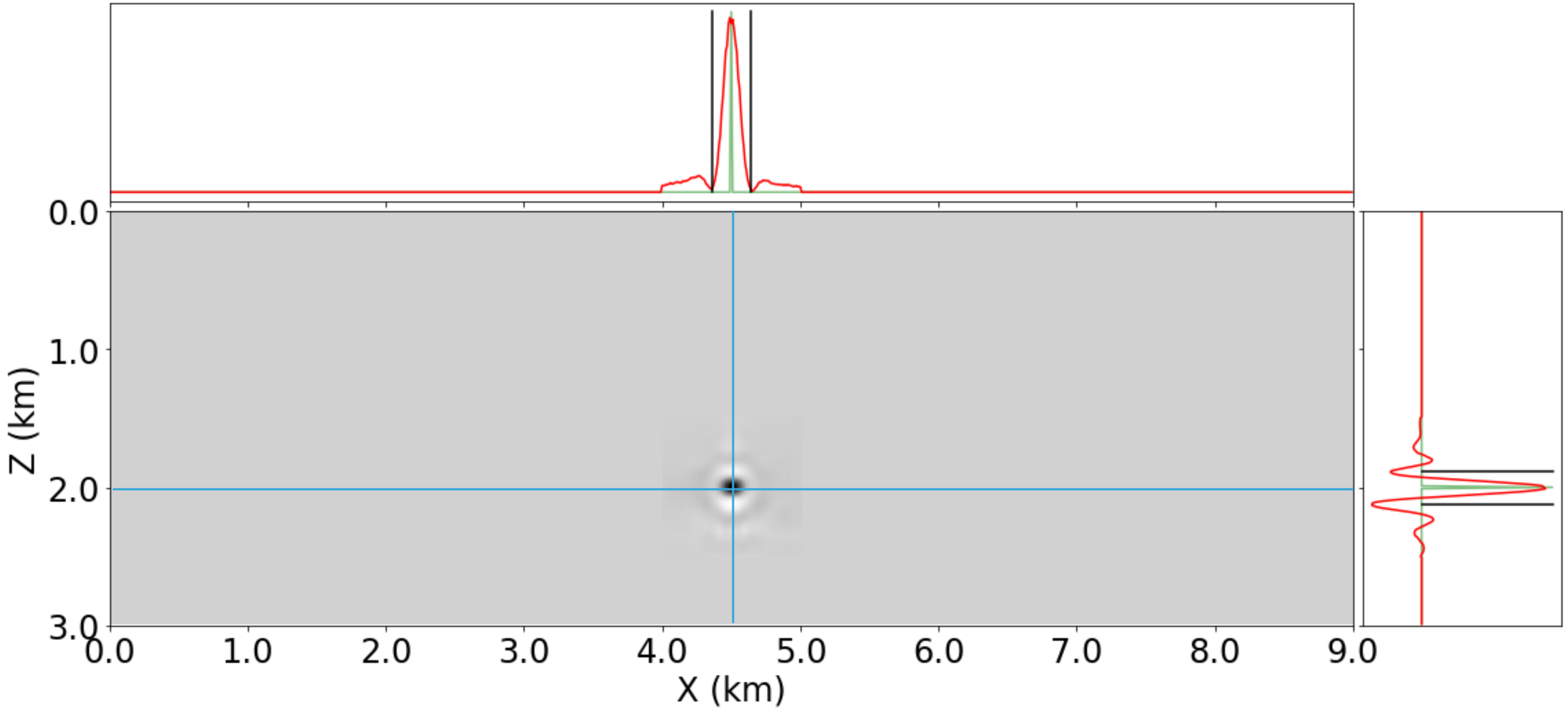



Estimated from 0.2% of the total number of elements in the Hessian ($7.29 \times 10^{10}$)

# Local Resolution Analysis

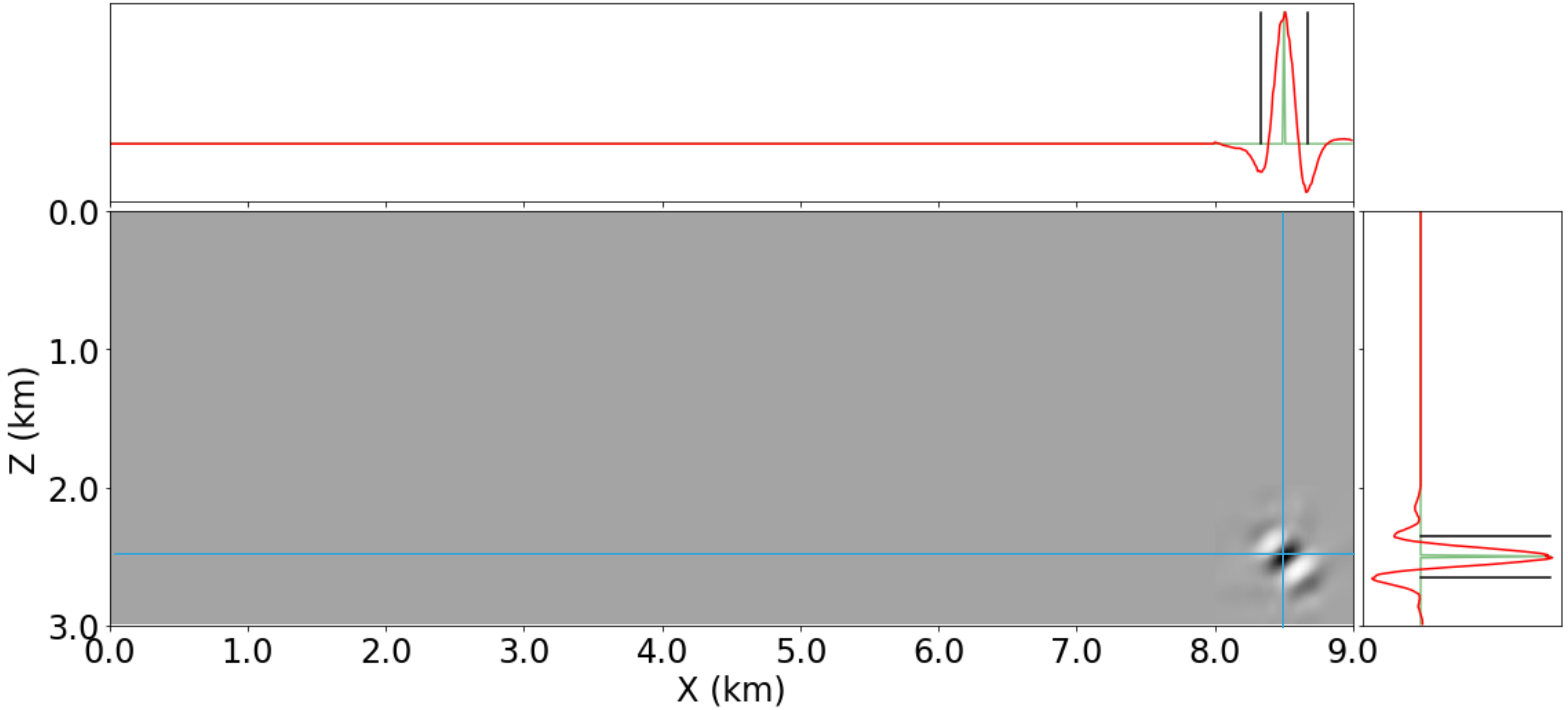
# Probing the subsurface

Estimate local resolution lengths by applying the Hessian to point scatterers at various locations



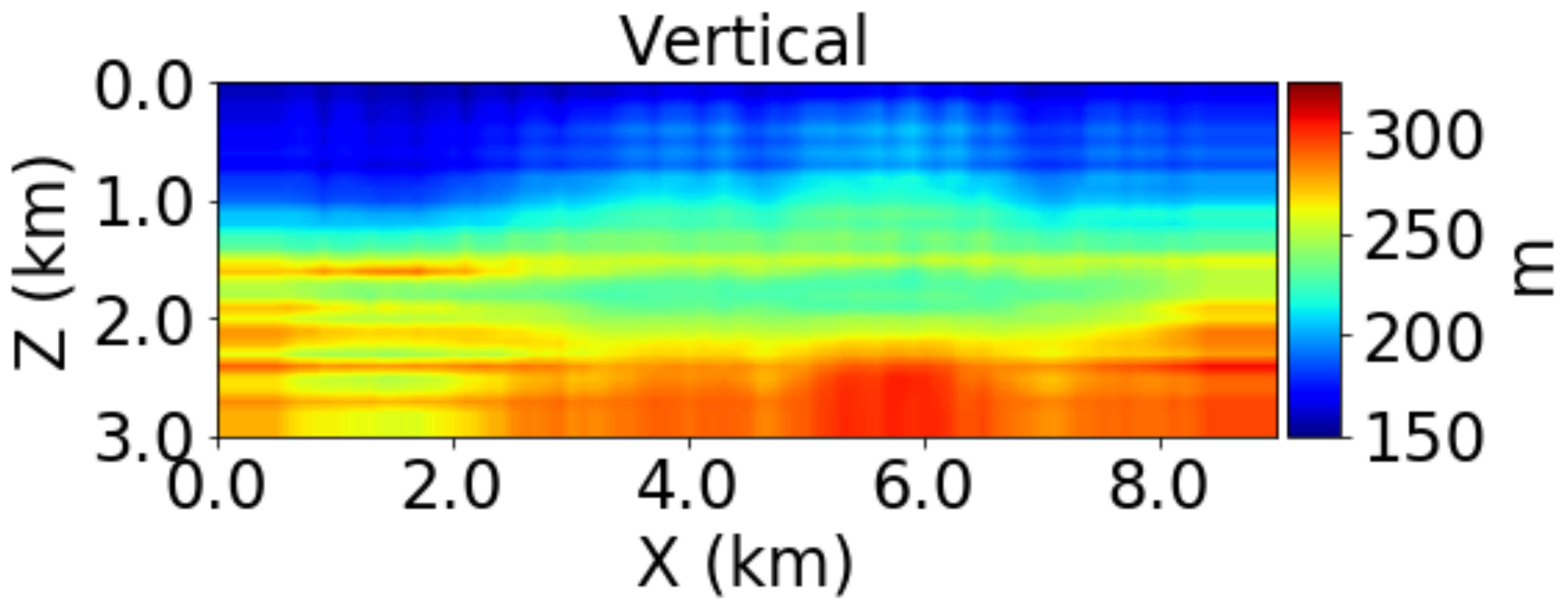
# Probing the subsurface

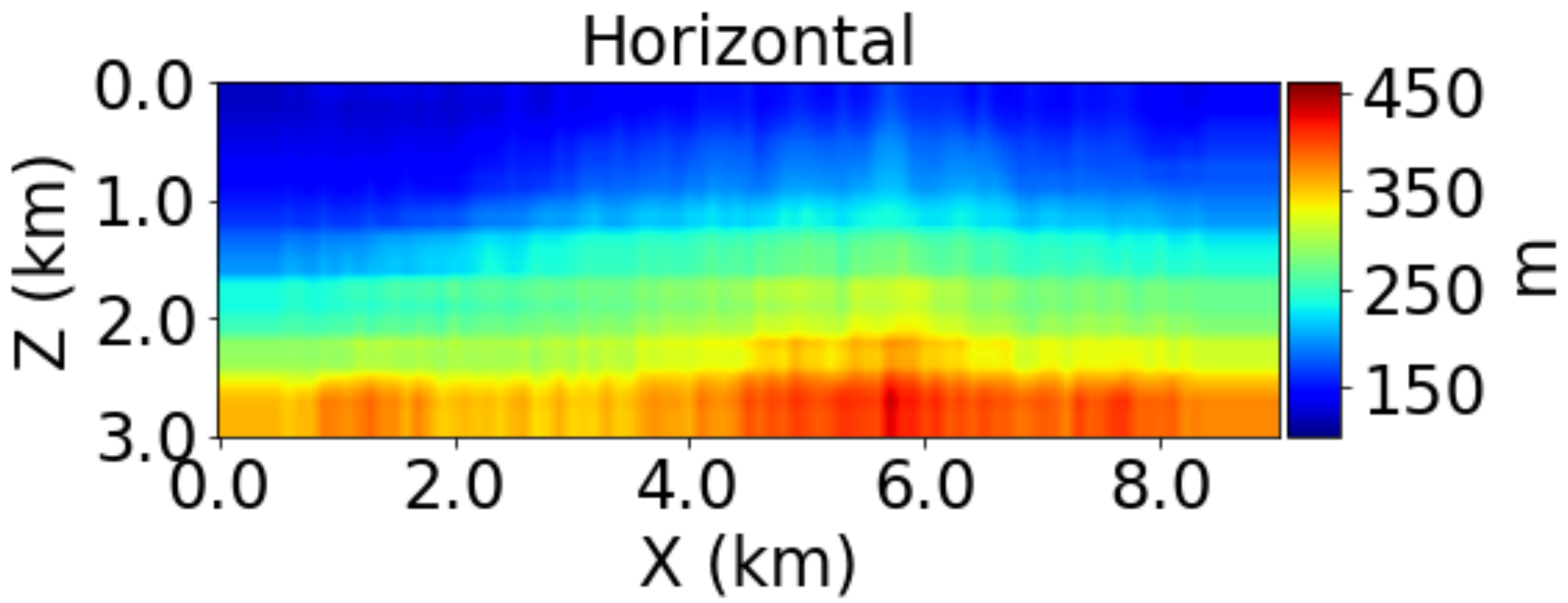
Vertical and horizontal resolution lengths become less meaningful when scatterers are not isotropically smeared



# Resolution lengths

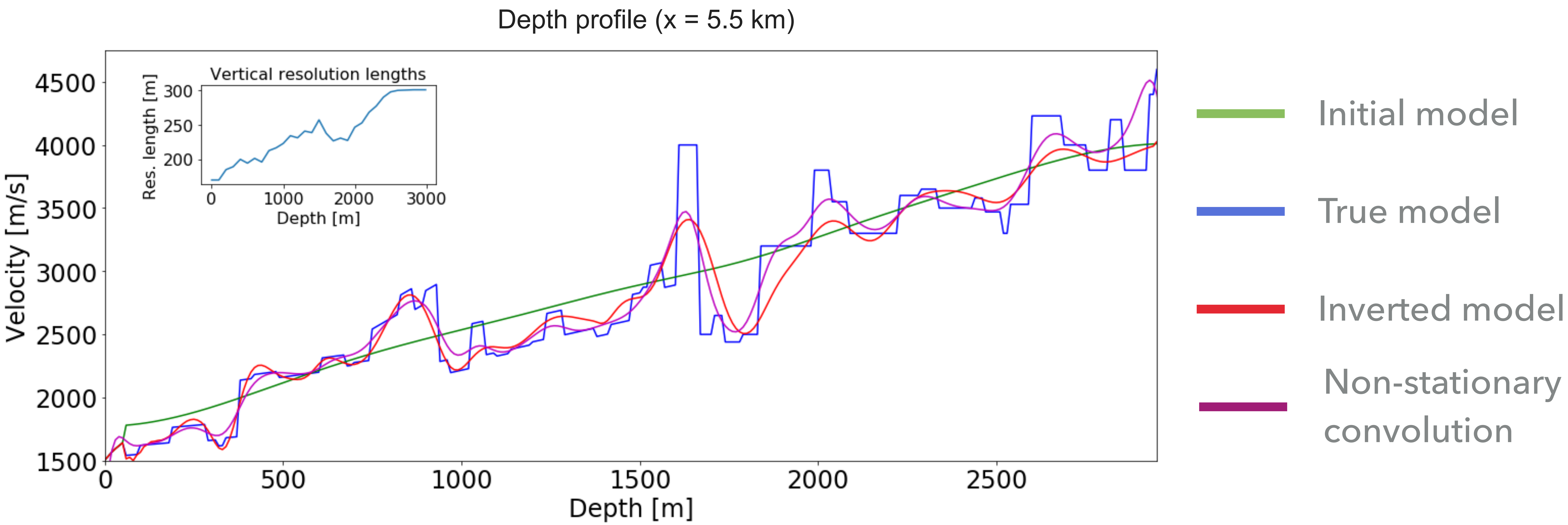
Probe the subsurface to generate maps of horizontal and vertical resolution lengths (interpolated, queried every 10 m).



# Non-stationary convolution

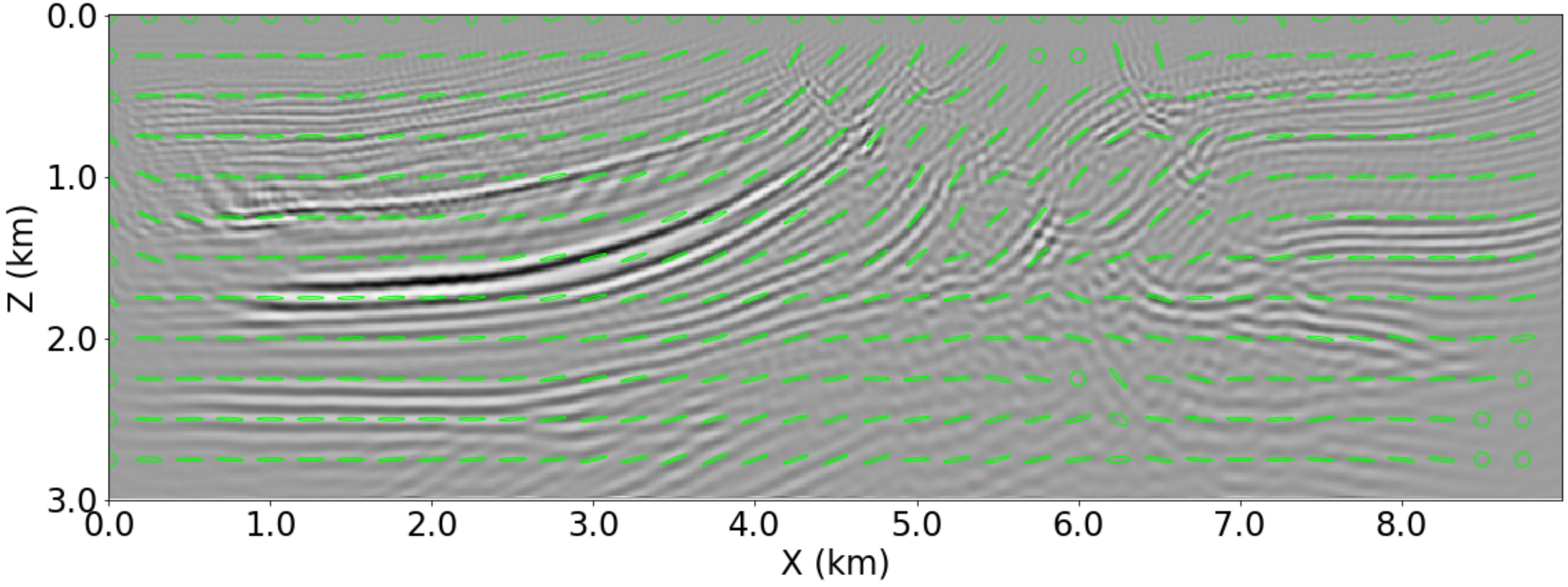
Perform 1D non-stationary convolution on the true model with a series of Gaussian filters parametrized by the variable resolution lengths



# Linearized Bayesian Inversion

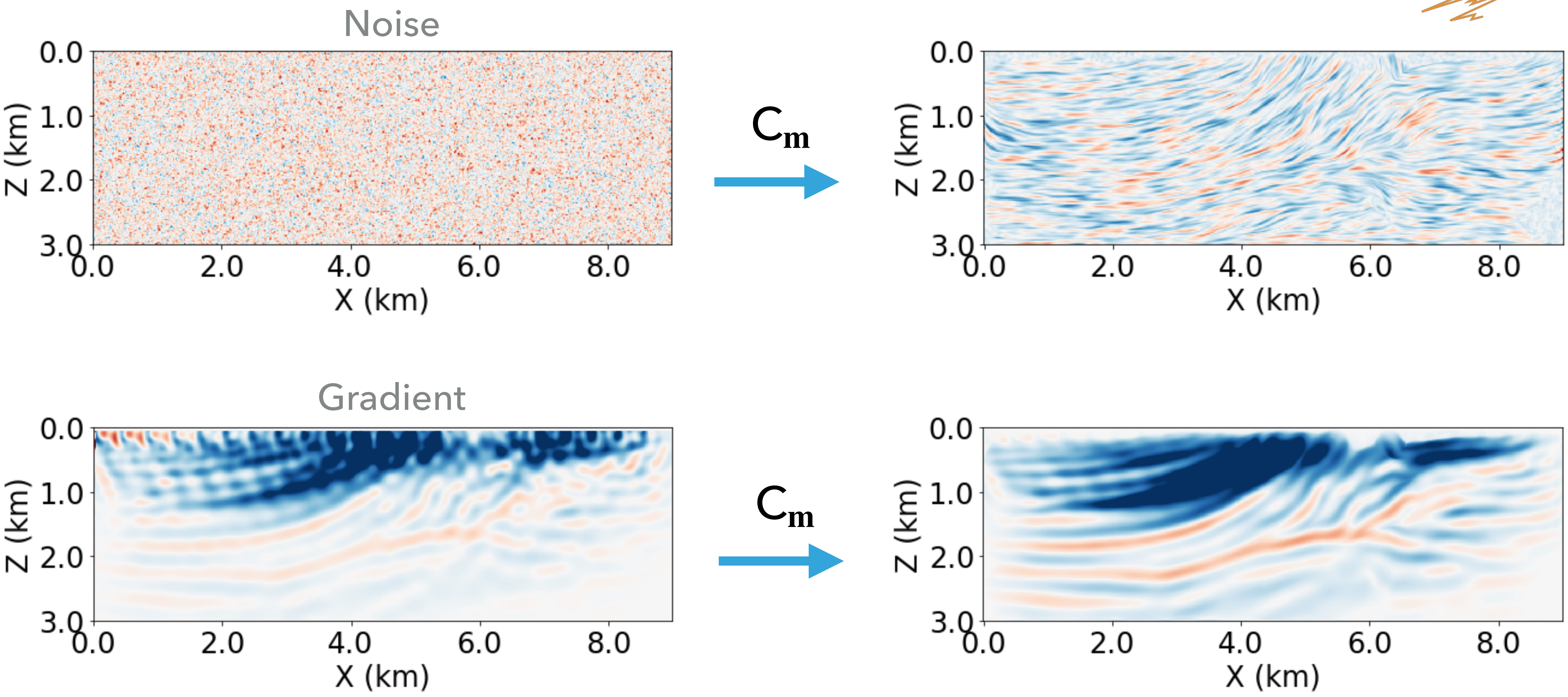
# Prior covariance - $C_m$

Use an anisotropic Matern covariance (Hale, 2014)



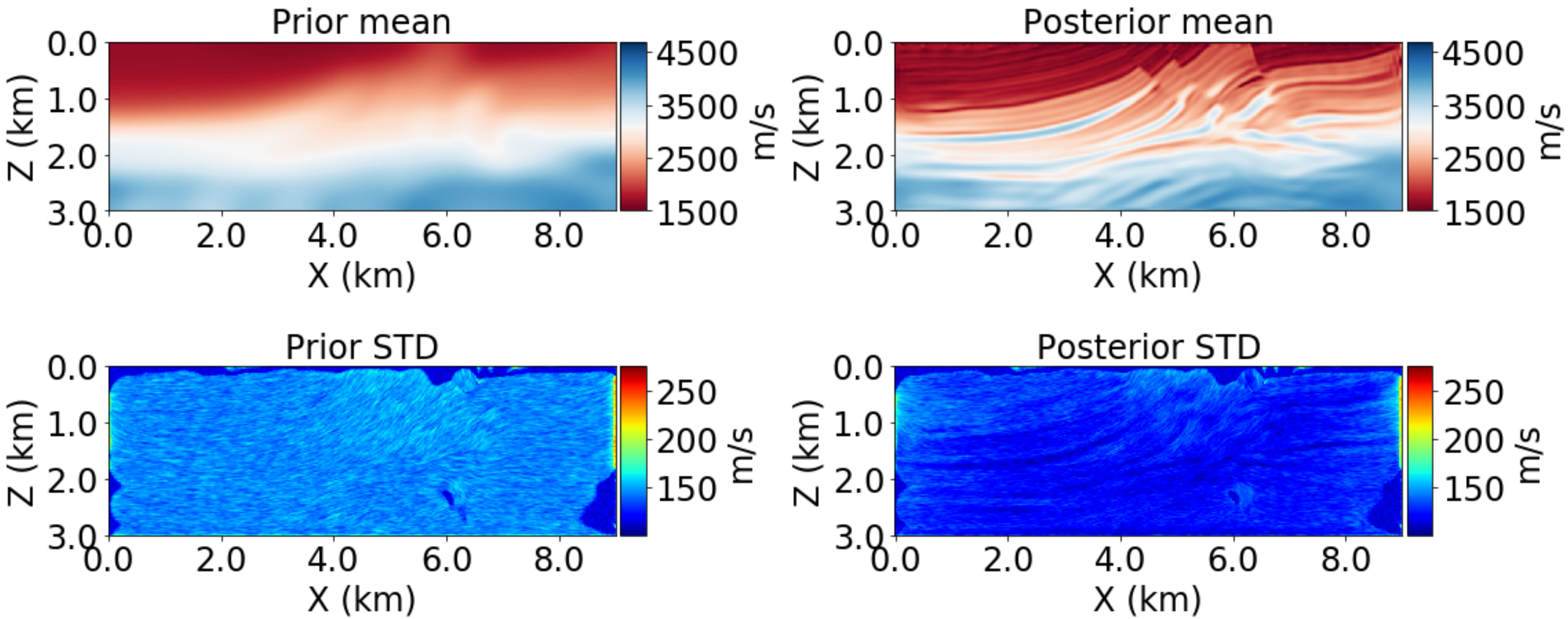

Allows for structure-oriented smoothing informed by a seismic image

# Shaping covariance (Hale, 2014)

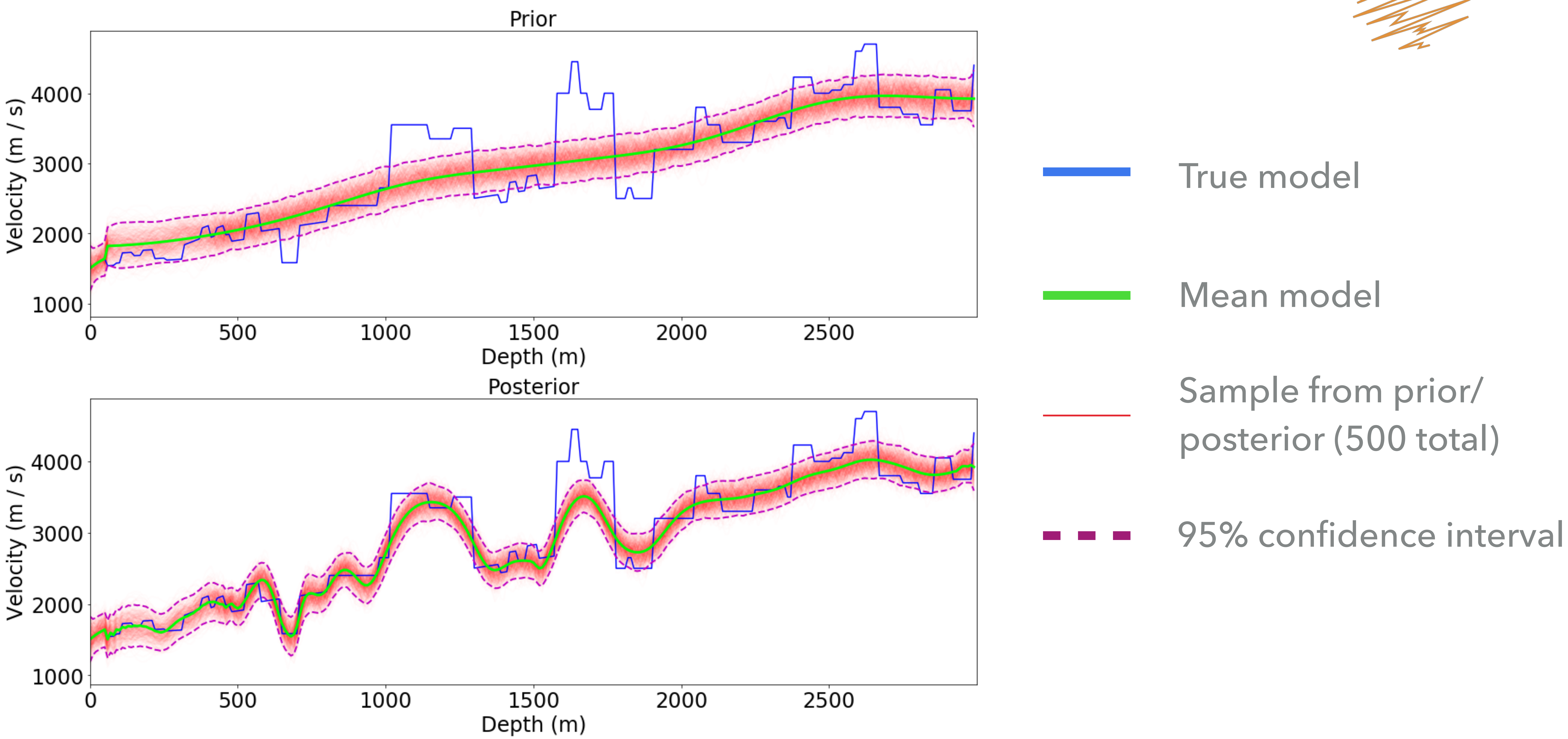


# Comparing distributions

Sample means and standard deviations computed over 500 random samples



# Depth profile comparison (x = 6.4 km)



# Conclusions

- Propose a novel **factorization of the Hessian** as a **superposition of Kronecker products**
- Use Kronekcer factorization for **efficient Hessian-vector products**
- Presented two forms of resolution/uncertainty analysis using fast Hessian-vector products
- Extensions to 3D are challenging as they involve Tensor-completion (via Canonical/Parafac decomposition)

# Acknowledgements

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