CAP 2017, HW 7 due March 14

Solutions & Marking Scheme

Each question is worth 5 points.

As before, each question is worth 5 points. For question 1, give 1 point per integral, no partial credit. For all other questions please use 5/3/1 marking: 5 points for a good solution, maybe a small numerical mistake, 3 points for a solution that has some serious lackings, e.g. a mathematical error, or no text explaining what is happening. 1 point for showing something beyond repeating the problem. You can of course also give 2/4 points if a solution is in between.

Question 2 is particularly hard, please grade leniently.

The answers to the questions are written down below.

1. (a) Calculate the following integrals

- $$\int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x}\,dx$$

  Solution

  $$\begin{aligned}
  \int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x}\,dx &= \int_{1/3}^{3} \frac{\sqrt{x}}{x(x+1)}\,dx \\
  &= \int_{1/3}^{3} \frac{1}{\sqrt{x}(x+1)}\,dx \\
  &\quad \textit{substitute } u=\sqrt{x} \\
  &= \int_{\sqrt{3}/3}^{\sqrt{3}} \frac{2}{u^2+1}\,du \\
  &= \arctan(u)\Big|_{u=\sqrt{3}/3}^{u=\sqrt{3}} \\
  &= 2\arctan(\sqrt{3}) - 2\arctan(\sqrt{3}/3) = 2\frac{\pi}{3} - 2\frac{\pi}{6} = \frac{\pi}{3}
  \end{aligned}$$

- $$\int \frac{\cos(x)}{\sin^2(x)+\sin(x)}\,dx$$

  Solution

  $$\begin{aligned}
  \int \frac{\cos(x)}{\sin^2(x)+\sin(x)}\,dx &= \int \frac{1}{u^2+u}\,du \quad \textit{by substituting } u=\sin(x) \\
  &= \int \frac{1}{u(u+1)}\,du \\
  &\qquad \text{partial fraction decomposition} \frac{1}{u(u+1)} = \frac{1}{u} + \frac{-1}{u+1} \\
  &= \int \frac{1}{u}\,du + \int \frac{-1}{u+1}\,du \\
  &= \ln|u| - \ln|u+1| + C \\
  &= \ln|\sin(x)| - \ln|\sin(x)+1| + C
  \end{aligned}$$

- $$\int \frac{1}{x^4+1}\,dx$$

Solution

$\int \frac{1}{x^4+1}\,dx$ *use partial fraction decomposition:* $x^4+1 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$

$$\text{and } \frac{1}{x^4+1} = -\frac{1}{2\sqrt{2}}\frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1} + \frac{1}{2\sqrt{2}}\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1}$$

$$= \frac{1}{2\sqrt{2}}\left(-\frac{1}{2}\int \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx + \frac{1}{2}\int \frac{\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx\right.$$
$$\left.+\frac{1}{2}\int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx + \frac{1}{2}\int \frac{\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx\right)$$
$$= \frac{1}{2\sqrt{2}}\left(-\frac{1}{2}\int \frac{d(x^2-\sqrt{2}x+1)}{x^2-\sqrt{2}x+1} + \frac{\sqrt{2}}{2}\int \frac{d(x-\frac{\sqrt{2}}{2})}{(x-\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2}\right.$$
$$\left.+\frac{1}{2}\int \frac{d(x^2+\sqrt{2}x+1)}{x^2+\sqrt{2}x+1} + \frac{\sqrt{2}}{2}\int \frac{d(x+\frac{\sqrt{2}}{2})}{(x+\frac{\sqrt{2}}{2})^2+(\frac{\sqrt{2}}{2})^2}\right)$$
$$= \frac{1}{4\sqrt{2}}\Big(-\ln(x^2-\sqrt{2}x+1) + 2\arctan(\sqrt{2}x-1)$$
$$+\ln(x^2+\sqrt{2}x+1) + 2\arctan(\sqrt{2}x+1)\Big) + C$$

(b) Determine whether the following integrals are convergent or divergent, and evaluate those that are convergent.

- $$\int_{-\infty}^{0} 2^r\,dr$$

Solution

$$\begin{aligned}\int_{-\infty}^{0} 2^r\,dr &= \lim_{t\to-\infty}\int_t^0 2^r\,dr\\ &= \lim_{t\to-\infty}\left(\frac{1}{\ln(2)}2^r\Big|_{r=t}^{r=0}\right)\\ &= \frac{1}{\ln(2)} - \lim_{t\to-\infty}\frac{2^t}{\ln(2)}\\ &= \frac{1}{\ln(2)}\end{aligned}$$

hence the integral converges.

- $$\int_{-\infty}^{\infty} \cos(\pi t)\,dt$$

Solution

$$\begin{aligned}\int_{-\infty}^{\infty}\cos(\pi t)\,dt &= \int_{-\infty}^{0}\cos(\pi t)\,dt + \int_0^{\infty}\cos(\pi t)\,dt\\ &= \lim_{s\to-\infty}\int_s^0\cos(\pi t)\,dt + \int_0^{\infty}\cos(\pi t)\,dt\end{aligned}$$

Now for the first term we have

$$\begin{aligned}\lim_{s\to-\infty}\int_s^0 \cos(\pi t)\,dt &= \lim_{s\to-\infty}\left(\frac{1}{\pi}\sin(\pi t)\Big|_{t=s}^{t=0}\right)\\ &= \lim_{s\to-\infty}\frac{-1}{\pi}\sin(\pi t)\end{aligned}$$

and since this does not converge, the whole original integral $\int_{-\infty}^{\infty}\cos(\pi t)\,dt$ does not converge.

2. Astronomers use a technique called *stellar stereography* to determine the density of stars in a star cluster from the observed (two-dimensional) density that can be analysed from a photograph. Suppose that in a spherical cluster of radius R the density of stars depends only on the distance r from the centre of the cluster. If the perceived star density is given by $y(s)$, where s is the observed planar distance from the centre of the cluster, and $x(r)$ is the actual density, it can be shown that

$$y(s) = \int_s^R \frac{2r}{\sqrt{r^2-s^2}}x(r)\,dr.$$

If the actual density of stars in a cluster is $x(r) = \frac{1}{2}(R-r)^2$, show that the perceived density is

$$y(s) = s^2R\left[\ln(s) - \ln(R+\sqrt{R^2-s^2})\right] + \sqrt{R^2-s^2}\left(\frac{2}{3}s^2+\frac{1}{3}R^2\right).$$

Solution Note to markers: this is perhaps the hardest integral students are asked to do. Last time we checked Wolfram Alpha was not able to calculate it. Please grade leniently.

We have to calculate the improper integral

$$y(s) = \int_s^R \frac{\cancel{2}r}{\sqrt{r^2-s^2}} \frac{1}{\cancel{2}}(R-r)^2\,dr = \lim_{\epsilon\to s^+} \int_\epsilon^R \frac{r}{\sqrt{r^2-s^2}}(R-r)^2\,dr$$

Use trig substitution $r = s\,\sec(u)$

$$= \lim_{\epsilon\to s^+} \int_{\arccos(\frac{s}{\epsilon})}^{\arccos(\frac{s}{R})} \frac{s\sec(u)\,(R - s\sec(u))\,s\sec(u)\tan(u)}{\sqrt{s^2(\sec^2(u)-1)}}\,du$$

$$= \lim_{\epsilon\to s^+} \int_{\arccos(\frac{s}{\epsilon})}^{\arccos(\frac{s}{R})} \frac{s^2\sec^2(u)\,(R - s\sec(u))^2\tan(u)}{s\tan(u)}\,du$$

$$= \lim_{\epsilon\to s^+} \int_{\arccos(\frac{s}{\epsilon})}^{\arccos(\frac{s}{R})} s\sec^2(u)(R - s\sec(u))^2\,du$$

$$= \lim_{\epsilon\to s^+} \int_{\arccos(\frac{s}{\epsilon})}^{\arccos(\frac{s}{R})} s\sec^2(u)(R^2 - 2Rs\sec(u) + s^2\sec^2(u))\,du$$

$$= \lim_{\epsilon\to s^+} \left(sR^2 \int_{\arccos(\frac{s}{\epsilon})}^{\arccos(\frac{s}{R})} \sec^2(u)\,du - 2s^2R \int_{\arccos(\frac{s}{\epsilon})}^{\arccos(\frac{s}{R})} \sec^3(u)\,du + s^3 \int_{\arccos(\frac{s}{\epsilon})}^{\arccos(\frac{s}{R})} \sec^4(u)\,du \right)$$

We now use $\int \sec^2(u)\,du = \tan(u) + C,$

from §*6.2 example 8 we have* $\int \sec^3(u)\,du = \frac{1}{2}(\sec(u)\tan(u) + \ln|\sec(u)+\tan(u)|) + C$

and finally $\int \sec^4(u)\,du = \int (1+\tan^2(u))\,du\tan'(u)\,du = \tan(u) + \frac{\tan^3(u)}{3} + C$

hence we have

$$= \lim_{\epsilon\to s^+} \left\{ sR^2\tan(u)\Big|_{\arccos(\frac{s}{\epsilon})}^{u=\arccos(\frac{s}{R})} - s^2R\,(\sec(u)\tan(u) + \ln|\sec(u)+\tan(u)|)\Big|_{\arccos(\frac{s}{\epsilon})}^{u=\arccos(\frac{s}{R})} \right.$$
$$\left. + s^3\left(\tan(u) + \frac{\tan^3(u)}{3}\right)\Big|_{\arccos(\frac{s}{\epsilon})}^{u=\arccos(\frac{s}{R})} \right\}$$

$$= \lim_{\epsilon\to s^+} \left\{ \frac{sR^2\sqrt{1-\frac{s^2}{R^2}}}{s/R} - \frac{sR^2\sqrt{1-\frac{s^2}{\epsilon^2}}}{s/\epsilon} \right.$$

$$-s^2R\left(\frac{R}{s}\frac{\sqrt{1-\frac{s^2}{R^2}}}{s/R} + \ln\left|\frac{R}{s} + \frac{\sqrt{1-\frac{s^2}{R^2}}}{s/R}\right| - \frac{\epsilon}{s}\frac{\sqrt{1-\frac{s^2}{\epsilon^2}}}{s/\epsilon} - \ln\left|\frac{\epsilon}{s} + \frac{\sqrt{1-\frac{s^2}{\epsilon^2}}}{s/\epsilon}\right| \right)$$

$$\left. +s^3\left(\frac{\sqrt{1-\frac{s^2}{R^2}}}{s/R} + \frac{1}{3}\left(\frac{\sqrt{1-\frac{s^2}{R^2}}}{s/R}\right)^3 - \frac{\sqrt{1-\frac{s^2}{\epsilon^2}}}{s/\epsilon} - \frac{1}{3}\left(\frac{\sqrt{1-\frac{s^2}{\epsilon^2}}}{s/\epsilon}\right)^3 \right) \right\}$$

$$= \frac{sR^2\sqrt{1-\frac{s^2}{R^2}}}{s/R} - s^2R\left(\frac{R}{s}\frac{\sqrt{1-\frac{s^2}{R^2}}}{s/R} + \ln\left|\frac{R}{s} + \frac{\sqrt{1-\frac{s^2}{R^2}}}{s/R}\right| \right) + s^3\left(\frac{\sqrt{1-\frac{s^2}{R^2}}}{s/R} + \frac{1}{3}\left(\frac{\sqrt{1-\frac{s^2}{R^2}}}{s/R}\right)^3 \right)$$

$$= R^2\sqrt{R^2-s^2} - R^2\sqrt{R^2-s^2} - s^2R\ln\left(\frac{1}{s}\left(R+\sqrt{R^2-s^2}\right)\right) + s^2\sqrt{R^2-s^2} + \frac{1}{3}\left((R^2-s^2)\sqrt{R^2-s^2}\right)$$

$$= s^2R\left[\ln(s) - \ln(R+\sqrt{R^2-s^2})\right] + \sqrt{R^2-s^2}\left(\frac{2}{3}s^2 + \frac{1}{3}R^2\right).$$

3. Use the comparison theorem to determine whether the integral

$$\int_0^\infty \frac{\arctan(x)}{2+e^x}\,dx$$

is convergent or divergent.

Solution First remark that

$$\frac{\arctan(x)}{2+e^x} \geq 0, \text{ for all } x \geq 0.$$

Moreover, we have that

$$\frac{\pi/2}{e^x} \geq \frac{\arctan(x)}{2+e^x}.$$

Since

$$\begin{aligned}\int_0^\infty \frac{\pi/2}{e^x}\,dx &= \lim_{t\to\infty}\left(\frac{\pi}{2}\int_0^t e^{-x}\,dx\right)\\ &= \lim_{t\to\infty}\left(-\frac{\pi}{2}e^{-x}\Big|_{x=0}^{x=t}\right)\\ &= \frac{\pi}{2}\end{aligned}$$

(and in particular converges), we can can conclude by the comparison theorem that $\int_0^\infty \frac{\arctan(x)}{2+e^x}\,dx$ converges.

4. (a) Find the number a such that the line $x = a$ bisects the area under the curve $y = 1/x^2$, $1 \leq x \leq 4$.

Solution We have that

$$\begin{aligned}\int_1^4 \frac{1}{x^4}\,dx &= \frac{-1}{x}\Big|_{x=1}^{x=4}\\ &= \frac{3}{4}\end{aligned}$$

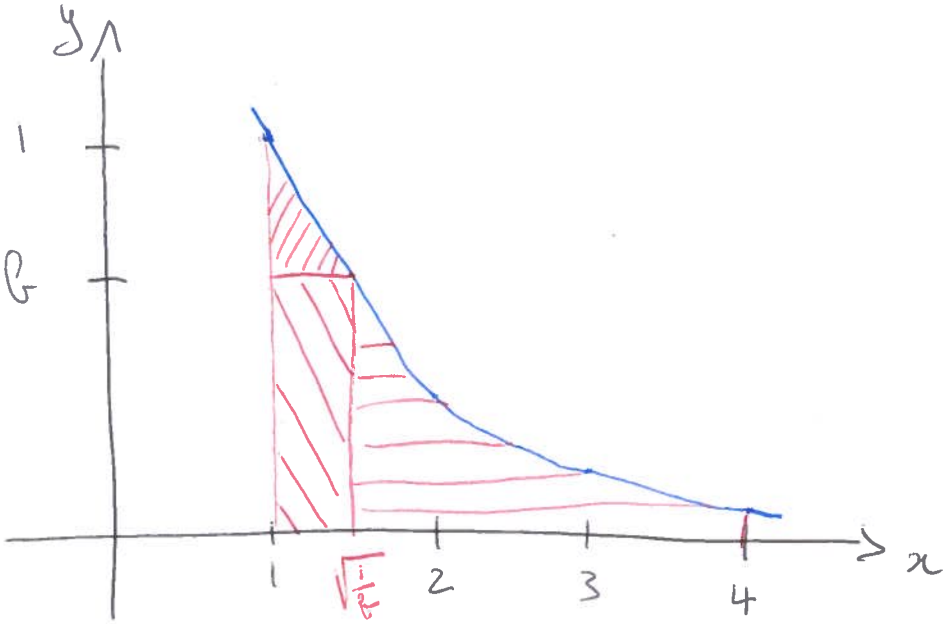
We therefore want to find a such that

$$\frac{3}{8} = \frac{1}{2}\frac{3}{4} = \int_1^a \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_{x=1}^{x=a} = 1 - \frac{1}{a}.$$

Hence $\frac{1}{a} = \frac{5}{8}$, and $a = \frac{8}{5}$.

(b) Find the number b such that the line $y = b$ bisects the area in part (a).

Solution It is easiest to divide the area under the graph of $y = \frac{1}{x^2}$ between $x = 1$ and $x = 4$ in 3 parts, as in the picture.



We will refer to the top left part as area A, the bottom left part as area B, and the right part as area C. We want

$$\begin{aligned} \text{area A} &= \text{area B} + \text{area C} \\ &= \frac{1}{2}\frac{3}{4} = \frac{3}{8} \end{aligned}$$

Now,

$$\begin{aligned} \text{area B} &= \left(\int_1^{\sqrt{\frac{1}{b}}} \frac{1}{x^2}\,dx\right) - b\left(\sqrt{\frac{1}{b}} - 1\right) \\ &= \frac{-1}{x}\bigg|_{x=1}^{x=\sqrt{\frac{1}{b}}} - \sqrt{b} + b \\ &= -\sqrt{b} + 1 - \sqrt{b} + b \\ &= 1 - 2\sqrt{b} + b \\ &= (1-\sqrt{b})^2. \end{aligned}$$

Hence

$$(1-\sqrt{b})^2 = \frac{3}{8}.$$

As we want $0 < b < 1$ we have $0 < \sqrt{b} < 1$, and therefore

$$1 - \sqrt{b} = \sqrt{\frac{3}{8}} = \frac{1}{2}\sqrt{\frac{3}{2}}.$$

Hence

$$\sqrt{b} = 1 - \frac{1}{2}\sqrt{\frac{3}{2}} \text{ and } b = \frac{11}{8} - \sqrt{\frac{3}{2}} \approx 0.15.$$