CAP 2017, HW 6 due March 7
Solutions & Marking Scheme

Give complete explanations of what you are doing, written in full sentences. Solutions that have all the correct calculations and computations, but lack explanations, will not get full marks!

Each problem is worth 5 marks. Students should write good and complete explanations, written in sentences, so that a fellow student can follow the solution without previously knowing how it goes. For Question 3, please give 1 mark for a correct solution with working. For other questions please use 5/3/1 marking: 5 points for a good solution, maybe a small numerical mistake, 3 points for a solution that has some serious lackings, e.g. a mathematical error, or no text explaining what is happening. 1 point for showing something beyond repeating the problem. You can of course also give 2/4 points if a solution is in between.

1. If f' is continuous on $[a, b]$, show that

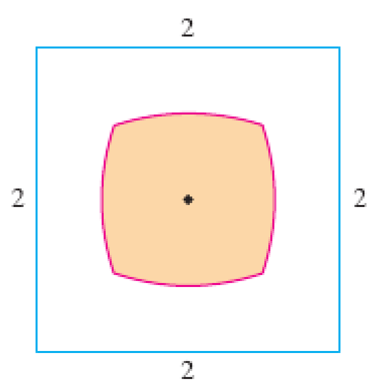
$$2\int_a^b f(x)f'(x)\,dx = [f(b)]^2 - [f(a)]^2.$$

Solution This is an application of the Fundamental Theorem of Calculus.

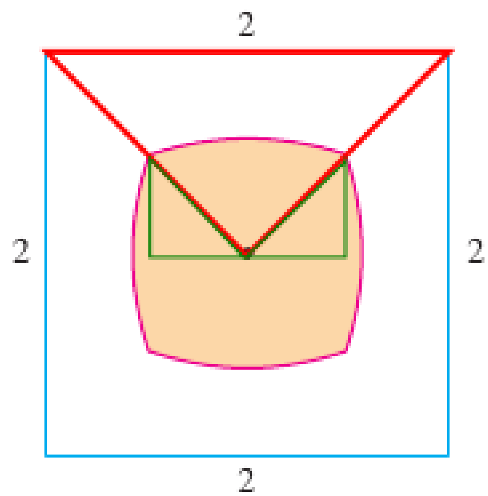
We have
$$\frac{d}{dx}\big(f^2(x)\big) = 2f(x)f'(x)$$
and so by the Fundamental Theorem of Calculus, we obtain
$$\begin{aligned} 2\int_a^b f(x)f'(x) &= \int_a^b \frac{d}{dx}\big(f^2(x)\big)\,dx \\ &= f^2(x)\Big]_a^b \\ &= \big(f(b)\big)^2 - \big(f(a)\big)^2. \end{aligned}$$

2. The figure shows the region inside a square consisting of those points that are closer to the centre than to the sides of the square. The sides of the square have length two. Find the area of the region.

Hint: use the fact that the region is symmetric.



Solution By the symmetry of the problem we need only calculate the area contained in the red triangle in the figure below. The total area of the region will then be four times the area of this sub-region.



Taking the origin to be the centre of the square, the area of interest is bounded by $y = x$ and $y = -x$ and the curve which is an equal distance from the origin to the line $y = 1$. This gives the equation of the curve as

$$y - 1 = \sqrt{x^2 + y^2},$$

or equivalently $y = \frac{1}{2}(1 - x^2)$.

We now need to find the intersection of this curve with the red triangle. The x values are given by the solution of the two equations

$$\begin{aligned} \frac{1}{2}(1 - x^2) &= x \\ \frac{1}{2}(1 - x^2) &= -x. \end{aligned}$$

Solving these two quadratics gives the intersection point coordinates as $(\sqrt{2} - 1, \sqrt{2} - 1)$ and $(1 - \sqrt{2}, \sqrt{2} - 1)$.

It remains to calculate the area of the region. We do this by calculating the area under the curve and subtracting the area of the two green triangles. These each have base of length $\sqrt{2} - 1$ and height also $\sqrt{2} - 1$ and hence the total area of the green triangles is $(\sqrt{2} - 1)^2$.

Hence the total area A of the shaded region is given by

$$\begin{aligned} A &= 4\Big(\int_{1-\sqrt{2}}^{\sqrt{2}-1} \frac{1}{2}(1 - x^2)\,dx - (\sqrt{2} - 1)^2\Big) \\ &= 4\Big(\frac{1}{2}x - \frac{1}{2}\frac{1}{3}x^3\Big]_{1-\sqrt{2}}^{\sqrt{2}-1}\Big) - 4(\sqrt{2} - 1)^2 \\ &= 4\big((\sqrt{2} - 1) - \frac{1}{3}(\sqrt{2} - 1)^3\big) - 4(\sqrt{2} - 1)^2 \\ &= 4(\sqrt{2} - 1)\big(1 - \tfrac{1}{3}(\sqrt{2} - 1)^2 - \sqrt{2} + 1\big) \\ &= 4(\sqrt{2} - 1)(1 - \sqrt{2}/3) \\ &= \frac{4}{3}(4\sqrt{2} - 5) \approx 0.876. \end{aligned}$$

3. Evaluate the integrals

- $$\int_1^4 \sqrt{t}\ln(t)\,dt$$

  Solution We make the substitution $t = u^2$ and hence $dt = 2u\,du$. So when $t = 4$, $u = 2$ and when $t = 1$, $u = 1$. This gives

  $$\begin{aligned}\int_1^4 \sqrt{t}\ln(t)\,dt &= \int_1^2 2u^2\ln(u^2)\,du\\ &= \int_1^2 4u^2\ln(u)\,du.\end{aligned}$$

  We now integrate by parts (with $f(u) = \ln(u)$ and $g'(u) = u^2$, so $f'(u) = 1/u$ and $g(u) = u^3/3$), giving

  $$\begin{aligned}\int_1^2 4u^2\ln(u)\,du &= \frac{4}{3}u^3\ln(u)\Big]_1^2 - \int_1^2 \frac{4}{3}u^3\frac{1}{u}\,du\\ &= \frac{4}{3}u^3\ln(u)\Big]_1^2 - \int_1^2 \frac{4}{3}u^2\,du\\ &= \frac{32}{3}\ln(2) - \frac{4}{3}\frac{u^3}{3}\Big]_1^2\\ &= \frac{32}{3}\ln(2) - \frac{28}{9} \approx 4.28246.\end{aligned}$$

- $$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr$$

  Solution We substitute $u = 4 + r^2$, which gives $du = 2r\,dr$. So when $r = 0$, $u = 4$ and when $r = 1$, $u = 5$. This gives (noting that $r^2 = u - 4$)

  $$\begin{aligned}\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr &= \frac{1}{2}\int_4^5 \frac{u-4}{\sqrt{u}}\,du\\ &= \frac{1}{2}\int_4^5 u^{1/2}\,du - 2\int_4^5 u^{-1/2}\,du\\ &= \frac{1}{2}\frac{2}{3}u^{3/2}\Big]_4^5 - (2)(2)u^{1/2}\Big]_4^5\\ &= \frac{1}{3}(\sqrt{5}^3 - 8) - 4(\sqrt{5} - 2)\\ &= \frac{16}{3} - \frac{7}{3}\sqrt{5}.\end{aligned}$$

- $$\int_0^1 e^{\sqrt{x}}\,dx$$

  Solution We make the substitution $u = \sqrt{x}$ and hence $dx = 2u\,du$. Note that when $x = 1$, $u = 1$ and similarly when $x = 0$, $u = 0$. This gives

  $$\int_0^1 e^{\sqrt{x}}\,dx = \int_0^1 2ue^u\,du.$$

We now use integration by parts with $f(u) = u$ and $g'(u) = e^u$, giving $f'(u) = 1$ and $g(u) = e^u$:

$$\begin{aligned}\int_0^1 2ue^u\,du &= 2ue^u\Big]_0^1 - 2\int_0^1 e^u\,du \\ &= 2e - 2(e-1) = 2.\end{aligned}$$

- $$\int \tan^2(x)\sec(x)\,dx$$

Solution We first use the trigonometric identity $\tan^2(x) = \sec^2(x) - 1$ to give

$$\int \tan^2(x)\sec(x)\,dx = \int \sec^3(x)\,dx - \int \sec(x)\,dx.$$

The first integral is done in Example 8 of Section 6.2 (integrate by parts, use the above identity and rearrange once you recover the original integral on the right hand side). The second integral is standard (see page 320). This gives

$$\begin{aligned}\int \tan^2(x)\sec(x)\,dx &= \frac{1}{2}\big(\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|\big) - \ln|\sec(x)+\tan(x)| + C \\ &= \frac{1}{2}\big(\sec(x)\tan(x) - \ln|\sec(x)+\tan(x)|\big) + C.\end{aligned}$$

- $$\int \frac{1}{\sqrt{t^2-6t+13}}\,dt$$

Solution We first complete the square in the denominator, giving

$$\int \frac{1}{\sqrt{t^2-6t+13}}\,dt = \int \frac{1}{\sqrt{(t-3)^2+4}}\,dt.$$

We then use the substitution $u = t-3$, which gives $du = dt$, and so

$$\int \frac{1}{\sqrt{t^2-6t+13}}\,dt = \int \frac{1}{\sqrt{u^2+4}}\,du.$$

Now make the substitution $u = 2\tan(t)$ and $du = 2\sec^2(t)\,dt$. Note that we may take $t \in (-\pi/2, \pi/2)$ and so, in particular, $\sec(t) > 0$. This gives

$$\begin{aligned}\int \frac{1}{\sqrt{t^2-6t+13}}\,dt &= \int \frac{2\sec^2(t)}{2\sec t}\,dt. \\ &= \int \sec(t)\,dt \\ &= \ln|\sec(t)+\tan(t)| + C,\end{aligned}$$

where in the last step we used a standard integral (page 320).
We finally undo the substitutions, first by using a trigonometric identity, to give

$$\begin{aligned}\int \frac{1}{\sqrt{t^2-6t+13}}\,dt &= \ln|\sqrt{\tan^2(t)+1} + \tan(t)| + C \\ &= \ln|\sqrt{\tfrac{u^2}{4}+1} + \tfrac{u}{2}| + C \\ &= \ln\left(\frac{\sqrt{u^2+4}+u}{2}\right) + C \\ &= \ln(\sqrt{u^2+4}+u) - \ln(2) + C \\ &= \ln(\sqrt{t^2-6t+13}+t-3) + \tilde{C}.\end{aligned}$$

Note that this is also equal to $\sinh^{-1}\left(\frac{t-3}{2}\right) + C$ and full marks should be given for any solution (with working) that gets this solution.

4. A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is m, the fuel is consumed at rate r, and the exhaust gases are ejected with constant velocity v_e (relative to the rocket). A model for the velocity of the rocket at time t is given by the equation

$$v(t) = -gt - v_e \ln\left(\frac{m - rt}{m}\right),$$

where g is the acceleration due to gravity and t is not too large. If $g = 9.8$ m/s^2, $m = 30,000$ kg, $r = 160$ kg/s, and $v_e = 3000$ m/s, find the height above ground of the rocket one minute after liftoff.

Solution We have that the distance, or height, is the integral of the velocity. Hence, since the height above ground at time zero is zero, we have

$$\begin{aligned}
\text{height after 60 s} &= \int_0^{60} \Big[-9.8t - 3000 \ln\Big(1 - \frac{160}{30,000}t\Big)\Big]\, dt \\
&= \frac{-9.8}{2}t^2\Big]_0^{60} - 3000 \int_0^{60} \ln\Big(1 - \frac{160}{30,000}t\Big)\, dt
\end{aligned}$$

We make the substitution $u = 1 - 160/30,000t$ and so $dt = -3000/16\, du$ and when $t = 0$, $u = 1$ and when $t = 60$, $u = 17/25$:

$$\begin{aligned}
\text{height after 60 s} &= \frac{-9.8}{2}t^2\Big]_0^{60} + \frac{(3000)^2}{16} \int_1^{17/25} \ln(u)\, du \\
&= -9.8(1800) + \frac{(3000)^2}{16}\big(u\ln(u) - u\big)\Big]_1^{17/25} \\
&= -9.8(1800) + \frac{(3000)^2}{16}\Big(\frac{17}{25}\ln\left(\tfrac{17}{25}\right) - \frac{17}{25} + 1)\Big) \\
&\approx 14,844.1 \text{ m.}
\end{aligned}$$

Note that the integral of $\ln(u)$ is given in Example 2 on page 313.