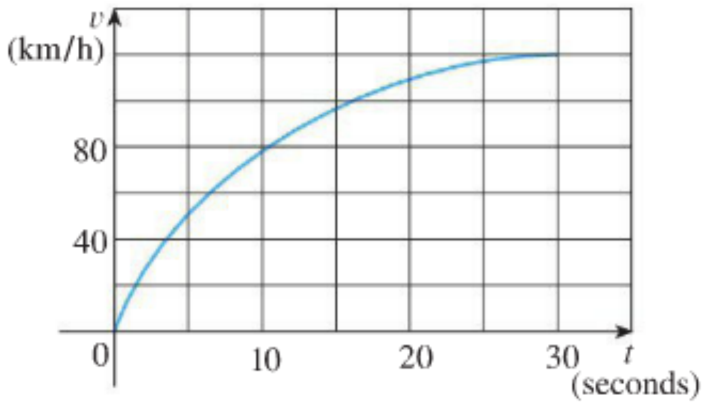
CAP 2017, HW 5 due February 28

Solutions: Use the 5/3/1 marking scheme for all questions.

Give complete explanations of what you are doing, written in full sentences. Solutions that have all the correct calculations and computations, but lack explanations, will not get full marks!

1. (5 marks.) The velocity graph of a car accelerating from rest to a speed of 120 km/h over a period of 30 seconds is shown. Estimate the distance travelled during this period.



Solution We are given the graph of the velocity function of a car accelerating from rest to a speed of 120 km/h over a period of 30 seconds. We would like to estimate the distance traveled by the car in these 30 seconds.

First note that the distance travelled in the first 30 seconds is given by

$$\int_0^{30} v(t)\,dt,$$

by the net change theorem, where $v(t)$ is velocity, i.e. the rate of change of distance over time. So we need to approximate the definite integral

$$\int_0^{30} v(t)\,dt.$$

Note that $v(t)$ is increasing, so the approximation using left endpoints is an under-estimate, and the approximation using right endpoints an over-estimate. So in this situation we would get a better approximation using midpoints. However, since we are only presented the graph of the function we cannot know the exact function value at these midpoints. We will therefore use x_i^* that are near the midpoints, whose function value we can read off (and which should be as close as possible to the velocity at the midpoints). We will use 6 rectangles for our approximation.

- Step 1. Cut up $[0, 30]$ into 6 intervals of equal length $\Delta x = \frac{30-0}{6}s = 5s$. This gives

$$[0, 5], [5, 10], [10, 15], [15, 20], [20, 25] \text{ and } [25, 30].$$

- Step 2. Find all the midpoints $\overline{x}_i$ of these intervals: 2.5, 7.5, 12.5, 17.5, 22.5, 27.5.
- Step 3. Find $f(x_i^*)$, where x_i^* is near $\overline{x}_i$. In this case this would give

$$f(x_1^*) = 20,\ f(x_2^*) = 60,\ f(x_3^*) = 90,\ f(x_4^*) = 100,\ f(x_5^*) = 115,\ f(x_6^*) = 120,$$

all in units of km/h.

- Step 4. To approximate the distance travelled, note that $\Delta x = 5s = \frac{5}{3600}h = \frac{1}{720}h$. We get

$$\int_0^{30} v(t)\,dt \approx \sum_{i=1}^{6} f(x_i^*)\Delta x = \frac{1}{720}(20+60+90+100+115+120) \approx 0.701 \text{ km}.$$

Tutors: there is room for interpretation of suitable points x_i^* and the corresponding values $f(x_i^*)$, e.g. $f(x_1^*)$ could reasonably be taken as 30. Do give marks for estimates which differ from that given here, if arrived at by sensible numbers.

2. (5 marks.) Suppose f has absolute minimum value m and absolute maximum value M. Between what two values must $\int_0^2 f(x)\,dx$ lie? Which property of integrals allows you to make your conclusion?

Solution We claim that

$$2m \leq \int_0^2 f(x)\,dx \leq 2M.$$

To prove this, note that by assumption we have

$$m \leq f(x) \leq M.$$

So by the comparison property for integrals (7, page 278), we have

$$\int_0^2 m\,dx \leq \int_0^2 f(x)\,dx \leq \int_0^2 M\,dx,$$

i.e.

$$2m \leq \int_0^2 f(x)\,dx \leq 2M.$$

One could also use the comparison property (8) directly.

3. (5 marks.) Suppose that a volcano is erupting and readings of the rate $r(t)$ at which solid materials are spewed into the atmosphere are given in the table. The time t is measured in seconds and the units for $r(t)$ are tonnes (metric tons) per second.

t	0	1	2	3	4	5	6
$r(t)$	2	10	24	36	46	54	60

(a) Give upper and lower estimates for the total quantity $Q(6)$ of erupted materials after 6 seconds.

(b) Use the Midpoint Rule to estimate $Q(6)$.

Solution

(a) By the net change theorem, $Q(6) = \int_0^6 r(t)\,dt$. Given that the values of $r(t)$ are increasing, we can make the assumption that $r(t)$ is an increasing function. Thus an upper estimate is R_6, the Riemann sum using right endpoints, and a lower estimate is L_6, the Riemann sum using left endpoints. So we have

$$R_6 = (10+24+36+46+54+60) \text{ tonnes} = 230 \text{ tonnes}$$

as an upper estimate, and

$$L_6 = (2+10+24+36+46+54) \text{ tonnes} = 172 \text{ tonnes}$$

as a lower estimate.

(b) We only have 7 measurements, so to use the midpoint rule, we can cut up the interval $[0, 6]$ into at most 3 pieces. Then $\Delta t = \frac{6-0}{3} = 2$, and the midpoints of the intervals are $\bar{t}_1 = 1$, $\bar{t}_2 = 3$, $\bar{t}_3 = 5$. So

$$M_3 = r(\bar{t}_1)\Delta t + r(\bar{t}_2)\Delta t + r(\bar{t}_3)\Delta t = (10 + 36 + 54)\, 2 \text{ tonnes} = 200 \text{ tonnes}.$$

4. (5 marks.) Find a function f and a number a such that, for all $x > 0$

$$6 + \int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x}.$$

Solution We differentiate this equation on both sides with respect to x, use the FToC part 1) to get an equation

$$\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}}.$$

So

$$f(x) = \frac{x^2}{\sqrt{x}} = x^{\frac{3}{2}}.$$

To find a, we substitute $x = a$ into the original equation to see

$$6 + \int_a^a \frac{f(t)}{t^2} dt = 2\sqrt{a},$$

so $\sqrt{a} = 3$, i.e. $a = 9$.