CAP 2017 – Solutions to HW 1

Each problem is worth 5 marks. Students should write good and complete explanations, written in sentences, so that a fellow student can follow the solution without previously knowing how it goes. For questions 1 and 4, use 5/3/1 marking: 5 points for a good solution, maybe a small numerical mistake, 3 points for a solution that has some serious lackings, e.g. a mathematical error, or no text explaining what is happening. 1 point for showing something beyond repeating the problem.

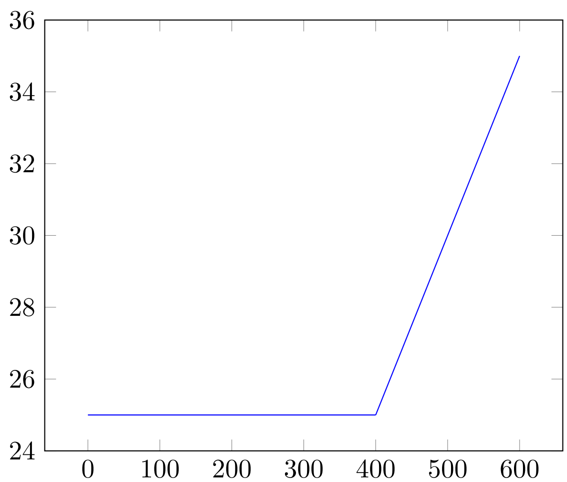
1. A cell phone plan has a basic charge of £25 a month. The plan includes 400 free minutes and charges 5 pence for each additional minute of usage. Write the monthly cost C as a function of the number x of minutes used and graph C as a function of x for $0 \leq x \leq 600$.

   Solution Let x denote the number of minutes used. When $0 \leq x \leq 400$, the cost of the plan is just the basic charge, $£25$. When $x \geq 400$, the cost is

   $$\begin{aligned} C(x) &= \text{basic charge} + (\text{number of additional minutes used}).(\text{charge per additional minutes}) \\ &= £(25 + (x - 400)0.05). \end{aligned}$$

   So $C(x)$ is the piecewise defined function

   $$C(x) = \begin{cases} £25 & \text{if } 0 \leq x \leq 400; \\ £(25 + (x - 400)0.05 & if 400 \leq x \leq 600. \end{cases}$$



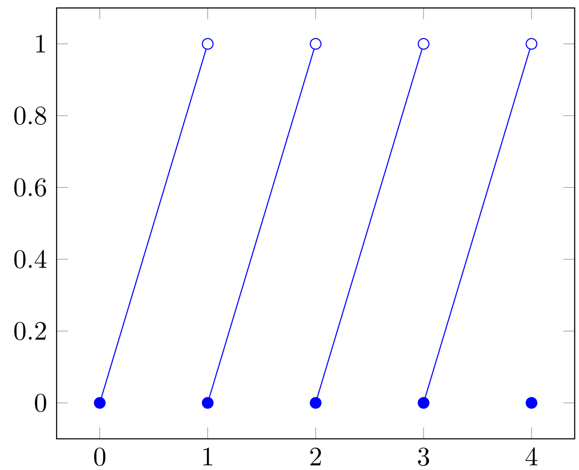
2. (5 Marks) Let $f(x) = x - \lfloor x \rfloor$.

   a) Sketch the graph of f.

   b) If n is an integer, evaluate (i) $\lim_{x \to n^-} f(x)$, (ii) $\lim_{x \to n^+} f(x)$.

   c) For what values of a does $\lim_{x \to a} f(x)$ exist?

   Solution a) (1 mark)



For b) (2 marks), we have $\lim_{x\to n^-} f(x) = 1$ and $\lim_{x\to n^+} f(x) = 0$. When x is close to n but smaller than n, then $\lfloor x \rfloor = n-1$. When x is close to n but larger than n then $\lfloor x \rfloor = n$.

For c) (2 marks) note that it follows from $b)$ that for an integer n, the right-hand limit and left-hand limit do not agree, so the limit does not exist when a is an integer. When a is not an integer, then the righ-hand limit and left-hand limit exist and agree, thus the limit exists. This should be explained. For example, one could say "When a is not an integer, and x is close to a, then $\lfloor x \rfloor = \lfloor a \rfloor$."

3. a) Suppose f and g are even functions. What can you say about $f+g$ and fg?

   b) What if f and g are both odd?

   c) Prove that every function defined on the real line can be written as the sum of an even and an odd function.

Solution For $a)$ (1 mark), we have $(f+g)(-x) = f(-x)+g(-x) = f(x)+g(x) = (f+g)(x)$, where the first and third equality are the definition of $f+g$ and the second equality uses the assumption that f and g are even. Thus $f+g$ is even. Similarly, we have $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$, so fg is even.

For $b)$ (2 marks), we have $(f+g)(-x) = f(-x)+g(-x) = -f(x)-g(x) = -(f(x)+g(x)) = -((f+g)(x))$, so the sum of two odd functions is odd. On the other hand, $(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = (-1)^2 f(x)g(x) = (fg)(x)$, so the product of two odd functions is even.

For $c)$ (2 marks), let us denote our arbitrary function by $f(x)$. If we put

$$F_1(x) = \frac{f(x)+f(-x)}{2} \quad \text{and} \quad F_2(x) = \frac{f(x)-f(-x)}{2},$$

then we have

$$F_1(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(x)+f(-x)}{2} = F_1(x),$$

hence $F_1(x)$ is even. Similarly we have

$$F_2(-x) = \frac{f(-x)-f(-(-x))}{2} = -\frac{f(x)-f(-x)}{2} = -F_2(x),$$

hence $F_2(x)$ is odd. Since it is straightforward that

$$f(x) = F_1(x) + F_2(x),$$

we have written $f(x)$ as the sum of an odd and an even function. Remark that these are the only possibilities for $F_1(x)$ and $F_2(x)$.

4. Use the Intermediate Value Theorem to prove that there is a positive number c such that $c^2 = 2$.

   Solution Let $f(x) = x^2$. Then f is continuous (it is a polynomial). We have that $f(1) = 1$ and $f(2) = 4$. Since 2 is between 1 and 4, we can apply the Intermediate Value theorem to see that there is $c \in (1, 2)$ where $f(c) = 2$. But $f(c) = c^2$, so we have established the existence of c with $c^2 = 2$.