CAP 2017, HW 2 due January 31
Solutions & Marking Scheme

Give complete explanations of what you are doing, written in full sentences. Solutions that have all the correct calculations and computations, but lack explanations, will not get full marks!

Each problem is worth 5 marks. Students should write good and complete explanations, written in sentences, so that a fellow student can follow the solution without previously knowing how it goes. For question 2, give 1 point per derivative; this requires both the correct solution and a reasonable attempt at indicating which rules have been used (i.e. no mark just for the solution). For all other questions, use 5/3/1 marking: 5 points for a good solution, maybe a small numerical mistake, 3 points for a solution that has some serious lackings, e.g. a mathematical error, or no text explaining what is happening. 1 point for showing something beyond repeating the problem. You can of course also give 2/4 points if a solution is in between.

1. Calculate the derivative of $f(x) = \sqrt{3x+1}$, using only the definition of the derivative, not any of the rules of differentiation.

   Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} = \lim_{h\to 0} \frac{\left(\sqrt{3(x+h)+1} - \sqrt{3x+1}\right)\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} \\
&= \lim_{h\to 0} \frac{3x+3h+1-(3x+1)}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} = \lim_{h\to 0} \frac{3h}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} \\
&= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)+1} + \sqrt{3x+1}} = \frac{3}{\sqrt{3x+1} + \sqrt{3x+1}} = \frac{3}{2\sqrt{3x+1}}
\end{aligned}$$

2. Calculate the following derivatives, showing your working. Here you can use differentiation rules and known formulas, but must state clearly when you do so - you will obtain no marks if you only give the solution.

   - $\frac{x^2 - 2\sqrt{x}}{x}$

     Solution

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^2-2\sqrt{x}}{x}\right) &= \frac{d}{dx}\left(x - 2x^{-1/2}\right) \\
&= \frac{d}{dx}(x) - \frac{d}{dx}\left(2x^{-1/2}\right) \qquad \text{[difference rule]} \\
&= 1 - 2\left(-\tfrac{1}{2}\right)x^{-3/2} \qquad \text{[power and constant multiple rules]} \\
&= 1 + x^{-\frac{3}{2}}
\end{aligned}$$

     Note: this can also be done using the quotient rule.

   - $x^2 \sin(x)\tan(x)$

     Solution

$$\begin{aligned}
\left(x^2\sin(x)\tan(x)\right)' &= \frac{d}{dx}\left(x^2\right)\sin(x)\tan(x) + x^2\frac{d}{dx}(\sin(x))\tan(x) \\
&\quad + x^2\sin(x)\frac{d}{dx}(\tan(x)) \qquad \text{[product rule]} \\
&= 2x\sin(x)\tan(x) + x^2\cos(x)\tan(x) \\
&\quad + x^2\sin(x)\sec^2(x) \qquad \text{[power rule, trigonometric derivatives]}
\end{aligned}$$

Note: there are many equivalent expressions for the trigonometric parts, e.g. $\left(x^2\sin(x)\tan(x)\right)' = 2x\sin(x)\tan(x) + x^2\sin(x) + x^2\csc(x)\tan^2(x)$.

- $\sqrt[3]{1+\tan(t)}$

  Solution

$$\begin{aligned}\frac{d}{dt}\sqrt[3]{1+\tan(t)} &= \frac{1}{3}\left(1+\tan(t)\right)^{-\frac{2}{3}}\frac{d}{dt}\left(1+\tan(t)\right) && \text{[chain rule]}\\ &= \frac{1}{3}\left(1+\tan(t)\right)^{-\frac{2}{3}}\sec^2(t) && \text{[sum rule, trigonometric derivatives]}\end{aligned}$$

  Note: it is fine to do this in one step, as long as the rules used are clearly stated.

- $x\sin(\frac{1}{x})$

  Solution

$$\begin{aligned}\left(x\sin(\tfrac{1}{x})\right)' &= \frac{d}{dx}(x)\sin(\tfrac{1}{x}) + x\frac{d}{dx}\left(\sin(\tfrac{1}{x})\right) && \text{[product rule]}\\ &= \sin(\tfrac{1}{x}) + x\cos(\tfrac{1}{x})(-x^{-2}) && \text{[chain and power rules]}\\ &= \sin(\tfrac{1}{x}) - \tfrac{1}{x}\cos(\tfrac{1}{x})\end{aligned}$$

- $\left(\frac{v}{v^3+1}\right)^6$

  Solution

$$\begin{aligned}\frac{d}{dv}\left(\frac{v}{v^3+1}\right)^6 &= 6\left(\frac{v}{v^3+1}\right)^5\frac{d}{dv}\left(\frac{v}{v^3+1}\right) && \text{[chain rule]}\\ &= 6\left(\frac{v}{v^3+1}\right)^5\left(\frac{(v^3+1)1 - v(3v^2)}{(v^3+1)^2}\right) && \text{[quotient, sum and power rules]}\\ &= 6\left(\frac{v}{v^3+1}\right)^5\frac{1-2v^3}{(v^3+1)^2} = 6\frac{v^5-2v^8}{(v^3+1)^7}\end{aligned}$$

3. Find the equations of the tangent lines to the curves

   - $y = x + \tan(x)$ at the point (π, π)

     Solution The derivative $\frac{dy}{dx}$ is $1+\sec^2(x)$, hence the slope of the tangent line (m) at $x = x_0 = \pi$ is 2. The full equation for the tangent line is given by $y - y_0 = m(x - x_0)$, i.e. $y - \pi = 2(x - \pi)$ or $y = 2x - \pi$.

   - $y = \frac{\sqrt{x}}{x+1}$ at the point $(4, 0.4)$

     Solution The derivative is given by

$$\frac{dy}{dx} = \frac{1-x}{2\sqrt{x}(x+1)^2},$$

     and hence the slope of the tangent line (m) at $x = x_0 = 4$ is given by $-3/100 = -0.03$. The full equation for the tangent line is given by $y - y_0 = m(x - x_0)$, i.e. $y - 0.4 = -0.03(x - 4)$ or $y = -0.03x + 0.52$.

4. Use linear approximations to estimate the numbers $\sqrt{99.8}$ and $\frac{1}{1002}$.

   Solution We will use the approximation

$$f(x) \approx f(a) + f'(a)(x - a).$$

   For $\sqrt{99.8}$, put $f(x) = \sqrt{x}$, $a = 100$, giving $f(a) = 10$. We have $f'(x) = x^{-1/2}/2$ and so $f'(a) = \frac{1}{20}$. Hence $\sqrt{99.8} \approx 10 + \frac{1}{20}(100 - 99.8) = 9.99$.

   For $\frac{1}{1002}$ put $f(x) = \frac{1}{x}$, and $a = 1000$, giving $f(a) = 0.001$. We have and $f'(x) = -x^{-2}$ and so $f'(a) = -0.000001$. Therefore we obtain $\frac{1}{1002} \approx 0.1 - 0.000001(1002 - 1000) = 0.000998$.