CAP 2016, HW 3 due February 2

Solutions & Marking Scheme

Give complete explanations of what you are doing, written in full sentences. Solutions that have all the correct calculations and computations, but lack explanations, will not get full marks!

1. (5 points) Under certain circumstances a rumour spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}},$$

where $p(t)$ is the proportion of the population that knows the rumor at time t (in days) and a and k are positive constants.

(a) Find $\lim_{t\to\infty} p(t)$. What does this mean for the rumor?

(b) Find the rate of spread of the rumor.

(c) Find the inverse function of $p(t)$ and give an interpretation of the meaning.

(d) Graph p for the case $a = 10, k = 0.5$ and use your graph to estimate how long it will take for 80% of the population to hear the rumor. Can you also calculate this time?

Solution

(a) (1pt) First note that

$$\begin{aligned} \lim_{t\to\infty} (1 + a\,e^{-kt}) &= 1 + a \lim_{t\to\infty} e^{-kt} \\ &= 1 + a\ 0 \\ &= 1. \end{aligned}$$

From this we get, by the quotient rule, that

$$\lim_{t\to\infty} \frac{1}{1 + a\,e^{-kt}} = \frac{1}{\lim_{t\to\infty} 1 + a\,e^{-kt}} = \frac{1}{1} = 1.$$

This means that the rumour will gradually spread through the whole population. To be precise, this means that for every proportion of the population < 1, there will be a finite time when the rumour has spread through a bigger part of the population (though at no finite time has it spread completely).

(b) (2pts) The rate of spread is the rate of change of the proportion of the population that knows the rumour. This is given by the derivative:

$$\begin{aligned} p'(t) &= \left(\frac{1}{1 + a\,e^{-kt}}\right)' \\ &= \frac{-(1 + a\,e^{-kt})'}{(1 + a\,e^{-kt})^2} \\ &= \frac{ake^{-kt}}{(1 + a\,e^{-kt})^2} \end{aligned}$$

(c) (1pt) In order to find the inverse function we need to solve the equation

$$s = \frac{1}{1 + ae^{-kt}}$$

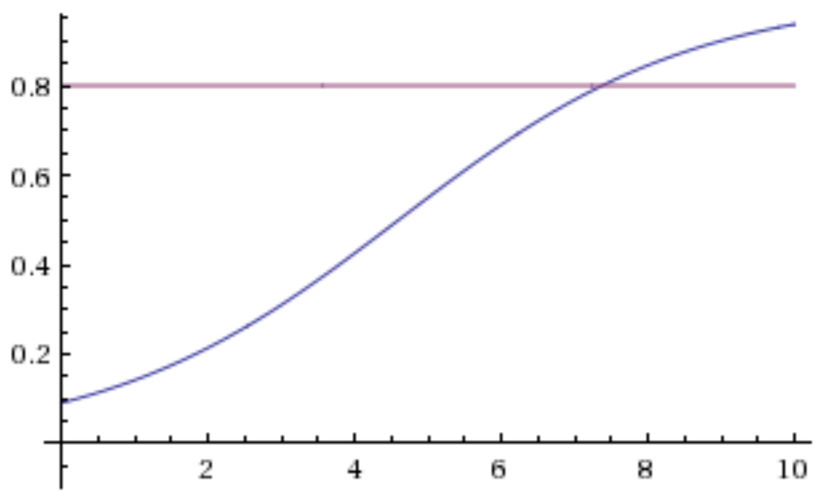
for t. This gives

$$(1 + a\,e^{-kt})\,s = 1$$
$$\Updownarrow$$
$$s + s\,a\,e^{-kt} = 1$$
$$\Updownarrow$$
$$s\,a\,e^{-kt} = 1 - s$$
$$\Updownarrow$$
$$e^{-kt} = \frac{1-s}{s\,a}$$
$$\Updownarrow$$
$$-kt = \ln\left(\frac{1-s}{s\,a}\right)$$
$$\Updownarrow$$
$$t = \frac{-1}{k}\ln\left(\frac{1-s}{s\,a}\right).$$

So the inverse function is

$$p^{-1}(s) = \frac{-1}{k}\ln\left(\frac{1-s}{s\,a}\right).$$

(d) (1pt)



From the graph one can estimate that the rumour will reach 80% of the population between 7 and 8 days. When we calculate it directly we get

$$p^{-1}(0.8) = -\frac{1}{0.5}\ln\left(\frac{1}{0.8*10} - \frac{1}{10}\right) \approx 7.37.$$

2. (5 points) Bismuth-210 has a half-life of 5.0 days.

   (a) A sample originally has a mass of 800 mg. Find a formula for the mass remaining after t days.

   (b) Find the mass remaining after 30 days.

   (c) When is the mass reduced to 1 mg?

   (d) Sketch a graph of the mass function.

Solution

(a) (2pts)Given an original sample of size of $m_0 = 800$ mg, we want to find a formula for the mass after t days. As discussed in the book, radioactive decay like this is modelled by the formula for exponential decay:

$$m(t) = m_0 e^{k\,t}.$$

Since we are told that Bismuth-210 has a half life of 5.0 days, we have

$$m(5) = \frac{m_0}{2},$$

from which we can deduce

$$m_0 e^{k\,5} = \frac{m_0}{2} \Rightarrow e^{k\,5} = \frac{1}{2} \Rightarrow k\,5 = \ln\left(\frac{1}{2}\right) \Rightarrow k = \frac{\ln(0.5)}{5}.$$

Hence in our case, we have that

$$m(t) = 800 e^{\frac{ln(0.5)}{5}t}$$

gives the radioactive mass (in mg) after t time (in days).

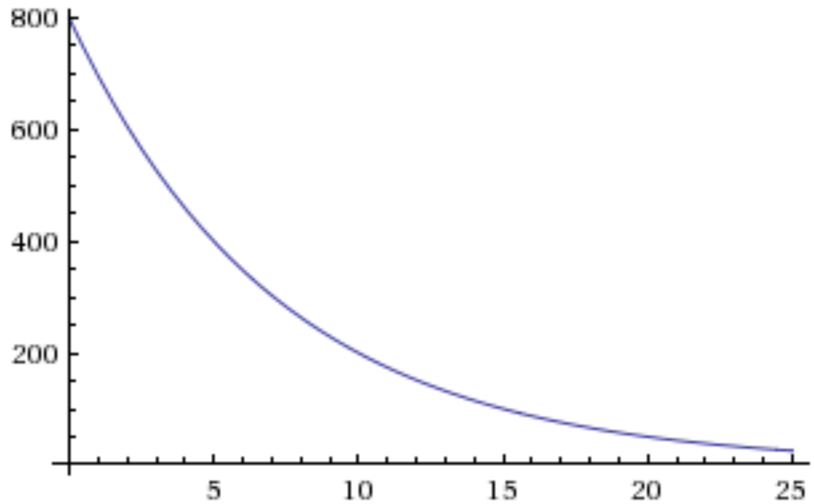
(b) (1pt) To find the mass remaining after 30 days, we can evaluate $m(30)$, or we can simply observe that in 30 days the mass halves 6 times, so we have

$$m(30) = \frac{1}{2^6} 800mg = 12.5mg.$$

(c) (1pt) To find when $m(t) = 1mg$, we have to set $m(t) = 1$, and solve for t. This gives

$$\begin{gathered}
800 e^{\frac{\ln(0.5)}{5}t} = 1 \\
\Updownarrow \\
e^{\frac{\ln(0.5)}{5}t} = \frac{1}{800} \\
\Updownarrow \\
\frac{\ln(0.5)}{5} t = \ln\left(\frac{1}{800}\right) \\
\Updownarrow \\
t = \frac{5\ln(1/800)}{\ln(0.5)} \approx 48.21
\end{gathered}$$

(d) (1pt)



3. (5 points) If f'' is continuous, show that

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$

Solution (5/3/1 marking scheme) Since f is differentiable it is continuous, and hence we have

$$\begin{aligned} \lim_{h \to 0} f(x+h) - 2f(x) + f(x-h) &= \left(\lim_{h \to 0} f(x+h)\right) - 2f(x) + \left(\lim_{h \to 0} f(x-h)\right) \\ &= f(x) - 2f(x) + f(x) = 0. \end{aligned}$$

Since of course also $\lim_{h \to 0} h^2 = 0$, we can apply l'Hospital's rule to get

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h}.$$

Since f'' exists f' is differentiable and hence continuous, and once again we have

$$\lim_{h \to 0} (f'(x+h) - f'(x-h)) = f'(x) - f'(x) = 0.$$

Since also $\lim_{h \to 0} 2h = 0$ we can apply l'Hospital's rule again, and finally get

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} &= \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} \\ &= \lim_{h \to 0} \frac{f''(x+h) + f''(x-h)}{2} \\ &= f''(x). \end{aligned}$$

4. (5 points) Suppose that $3 \leq f'(x) \leq 5$ for all values of x, where f is a function defined on all of the real numbers and differentiable everywhere. Show that

$$18 \leq f(8) - f(2) \leq 30.$$

Solution (5/3/1 marking scheme) This is a classical application of the mean value theorem. Since f is differentiable everywhere, it is continuous on $[2, 8]$ and differentiable on $(2, 8)$. Hence the conditions of the mean value theorem apply, and hence we know that there exists a $c \in (2, 8)$ such that

$$f'(c) = \frac{f(8) - f(2)}{8 - 2}.$$

Since we also know that $3 \leq f'(c) \leq 5$, we have that

$$3 \leq \frac{f(8) - f(2)}{6} \leq 5,$$

which of course gives

$$18 \leq f(8) - f(2) \leq 30.$$