CAP 2017, HW 4 due February 14
Solutions & Marking Scheme

Give complete explanations of what you are doing, written in full sentences. Solutions that have all the correct calculations and computations, but lack explanations, will not get full marks!

Each problem is worth 5 marks. Students should write good and complete explanations, written in sentences, so that a fellow student can follow the solution without previously knowing how it goes. Please use 5/3/1 marking: 5 points for a good solution, maybe a small numerical mistake, 3 points for a solution that has some serious lackings, e.g. a mathematical error, or no text explaining what is happening. 1 point for showing something beyond repeating the problem. You can of course also give 2/4 points if a solution is in between.

1. Using the guidelines of section 4.4, sketch the graph of the curve $\tan^{-1}\left(\frac{x-1}{x+1}\right)$.

   Solution We proceed as in section 4.4:

   A **Domain**
   Since the domain of $\tan^{-1}(x)$ is all of $\mathbb{R}$ and the domain of $(x-1)/(x+1)$ is $(-\infty,-1)\cup(-1,+\infty)$ (only one zero for the denominator at $x=-1$), the domain of f is

   $$\boxed{(-\inf,-1)\cup(-1,+\inf)=\mathbb{R}\setminus\{-1\}.}$$

   B **Intercepts**
   - If $x=0$ then $f(0)=\tan^{-1}(-1/1)=\tan^{-1}(-1)=-\pi/4$ and so
     $\boxed{y \text{ intercept is } (0,-\pi/4).}$
   - For $f(x)=0$ we need $\frac{x-1}{x+1}=0$ which is if and only if $x=1$,
     $\boxed{x \text{ intercept is } (1,0).}$

   C **Symmetry**
   The function f is neither odd nor even nor periodic.

   D **Asymptotes**
   We have that

   $$\lim_{x\to+\infty} f(x)=\tan^{-1}(1)=\frac{\pi}{4}$$
   $$\lim_{x\to-\infty} f(x)=\tan^{-1}(1)=\frac{\pi}{4}$$

   and hence there is

   $$\boxed{\text{a horizontal asymptote at } y=\tfrac{\pi}{4}.}$$

   [Same horizontal asymptote in both directions.]
   For the vertical asymptote note that f is continuous on its domain and is only undefined at $x=-1$. We have

   $$\lim_{x\to-1^+}\frac{x-1}{x+1}=-\infty$$
   $$\lim_{x\to-1^-}\frac{x-1}{x+1}=+\infty$$

   and so

   $$\lim_{x\to-1^+} f(x)=-\frac{\pi}{2}$$
   $$\lim_{x\to-1^-} f(x)=+\frac{\pi}{2}$$

   so there is

$$\boxed{\text{no vertical asymptote.}}$$

E **Intervals of Increase/Decrease**
We calculate the derivative using the chain rule. Note that

$$\frac{d}{dx}\big(\tan^{-1}(x)\big) = \frac{1}{1+x^2}$$

and by the quotient rule

$$\frac{d}{dx}\Big(\frac{x-1}{x+1}\Big) = \frac{1\cdot(x+1) - 1\cdot(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2}.$$

Hence we find

$$\begin{aligned} f'(x) &= \frac{1}{1+\left(\frac{x-1}{x+1}\right)^2}\frac{2}{(x+1)^2} \\ &= \frac{(x+1)^2}{(x+1)^2+(x-1)^2}\frac{2}{(x+1)^2} \\ &= \frac{2}{2x^2+2} = \frac{1}{x^2+1}. \end{aligned}$$

Hence f is differentiable everywhere on its domain and the derivative is positive everywhere. We therefore see that the graph is

$$\boxed{\text{increasing everywhere.}}$$

F **Local Extrema**
Since $f'(x) > 0$ everywhere on the domain of f, there are no critical points and hence

$$\boxed{\text{no local extrema.}}$$

G **Concavity and points of inflection**
We have, via the chain rule (or quotient rule),

$$f''(x) = \frac{d}{dx}\Big(\frac{1}{x^2+1}\Big) = \frac{-2x}{(x^2+1)^2},$$

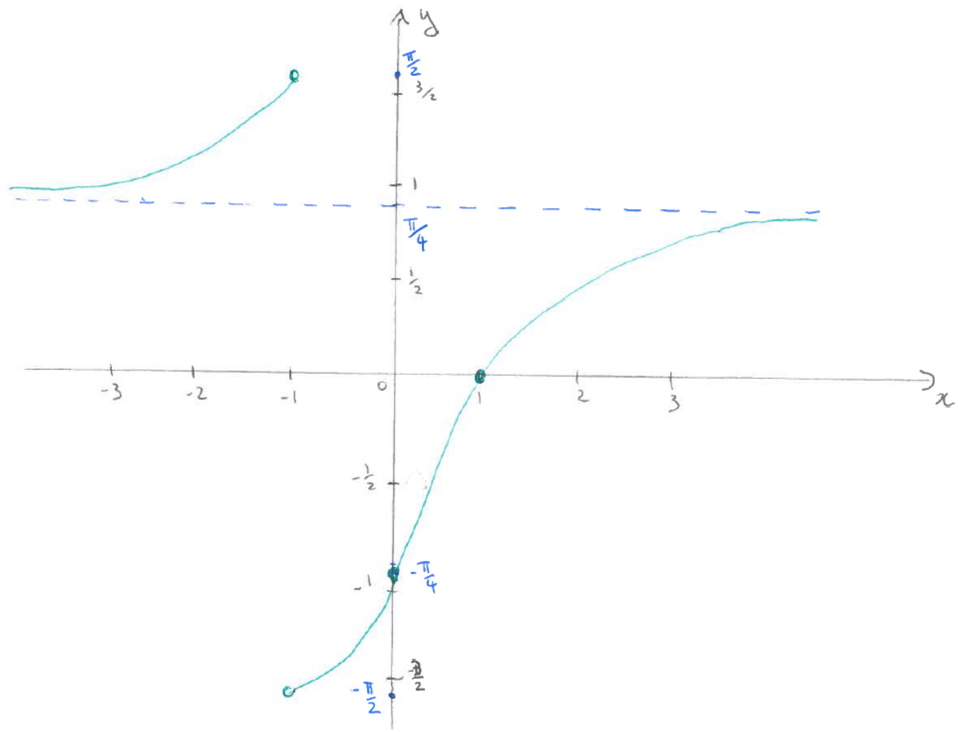
and hence the second derivative exists everywhere on the domain of f.
In particular, we see

$$\begin{aligned} f''(x) &> 0 \text{ if } x < 0 \\ f''(x) &= 0 \text{ if } x = 0 \\ f''(x) &< 0 \text{ if } x > 0. \end{aligned}$$

Therefore we have

$$\boxed{f(x) \text{ is concave upward on } (-\infty,-1) \text{ and } (-1,0)}$$
$$\boxed{\text{there is an inflection point at } x=0 \text{ (i.e. at } (0,-\pi/4))}$$
$$\boxed{f(x) \text{ is concave downward on } (0,+\infty).}$$

H **Sketch**



2. The manager of a 100-unit apartment complex knows from experience that all units will be occupied if the rent is £800 per month. A market survey suggests that, on average, one additional unit will remain vacant for each £10 increase in rent. What rent should the manager charge to maximise profit?

   Solution If the amount charged per unit is

   $$\text{£}(800 + 10n)$$

   then the number of vacant apartments will be n.

   Therefore the total amount of rent collected will be

   $$r(n) = \text{£}(800 + 10n)(100 - n) = \text{£}(80,000 + 200n - 10n^2).$$

   Treating this as a function of n, we want to find the value of n that maximizes r.

   Differentiating we find that

   $$r'(n) = 200 - 20n$$

   and it is easy to see that there is only one critical point, namely $n = 10$. Differentiating again gives

   $$r''(n) = -20 < 0$$

   and hence this critical point is a maximum.

   Inserting this value gives the rent that maximizes the profit:

   $$\boxed{\text{£}(800 + 10\text{n}) = \text{£}900.}$$

   [Note: since the maximum occurs at an integer value there is no problem with having used a ‘continuous’ approach to what is actually a ‘discrete’ problem.]

3. Two balls are thrown upward from the edge of a cliff 140 meters above the ground. The first is thrown with a speed of 15 m/s, the other is thrown a second later with a speed of 8 m/s. Do the balls ever pass each other?

   [Hint: look at Example 6 in Section 4.7].

   Solution

   Let $h_1(t)$ and $h_2(t)$ be the heights of the first and second balls above the ground (bottom of the cliff), respectively.

   While the balls are in the air, they satisfy the differential equation

$$\frac{d^2h}{dt^2} = \frac{dv}{dt} = -9.8 \text{ m/s}^2$$

   where v is the velocity. [Change in distance over time is velocity, and change in velocity over time is acceleration.]

   They also satisfy the conditions

$$h_1(0) = 140 \text{ m} \tag{1}$$
$$h_2(1) = 140 \text{ m} \tag{2}$$
$$\frac{dh_1}{dt}(0) = 15 \text{ m/s} \tag{3}$$
$$\frac{dh_2}{dt}(1) = 8 \text{ m/s} \tag{4}$$

   We now antidifferentiate the equation for $\frac{d^2h_1}{dt^2}$ to find

$$\frac{dh_1}{dt}(t) = -9.8t \text{ m/s}^2 + C$$

   and using (3) gives $C = 15$ m/s. Inserting this value and antidifferentiating again gives

$$h_1(t) = -9.8\frac{t^2}{2} \text{ m/s}^2 + 15t \text{ m/s} + D.$$

   Using (1) gives $D = 140$ m and so the equation for the height of the first ball is

$$h_1(t) = -9.8\frac{t^2}{2} \text{ m/s}^2 + 15t \text{ m/s} + 140 \text{ m}.$$

   A very similar calculation shows that

$$h_2(t) = -9.8\frac{(t-1)^2}{2} \text{ m/s}^2 + 8(t-1) \text{ m/s} + 140 \text{ m}.$$

   Setting $h_1(t) = h_2(t)$ and rearranging gives

$$-4.9\left(t^2 - (t-1)^2\right) + 15t - 8t + 8 = 0$$

   or, after some simplification

$$-2.8t \text{ m/s} + 12.9 \text{ m} = 0 \Rightarrow t = \frac{-12.9}{-2.8} \text{ s} \approx 4.6s.$$

   At $t^* = \frac{-12.9}{2.8}$ s we find that

$$h_1(t^*) = h_2(t^*) = 105.101 \text{ m}$$

   so the balls are indeed still in the air.

4. At 2:00 PM a car's speedometer reads 50 km/h. At 2:10 PM it reads 65 km/h. Show that at some time between 2:00 and 2:10 the acceleration is exactly 90 km/h^2.

Solution This is an application of the mean value theorem.

We start by computing the average rate of acceleration:

$$\begin{aligned} \text{average acceleration} &= \frac{\text{speed(2:10 PM)} - \text{speed(2:00 PM)}}{\text{2:10 PM} - \text{2:00 PM}} \\ &= \frac{(65-50)\ \text{km/h}}{10\ \text{minutes}} = \frac{15\ \text{km/h}}{1/6\ \text{h}} \\ &= 90\ \text{km/h}^2. \end{aligned}$$

Hence by the mean value theorem there must be a time c between 2:00 PM and 2:10 PM where the acceleration is exactly 90 km/h^2.