Fisher Information

$\theta \mapsto q(\theta)$ smooth curve $\quad \overset{*}{q}(\theta) = \frac{d}{dt} \log q(\theta) \quad E_{q(t)}(\overset{*}{q}(\theta)) = 0$

Fisher information $\quad I(\theta) = Var(\overset{*}{q}(\theta)) = \mathbb{E}_{q(t)}(\overset{*}{q}(\theta)^2)$

Gradient of $\phi : \mathcal{S}(X) \to \mathbb{R} \quad \frac{d}{d\theta} \phi(q(\theta)) = \langle \operatorname{Grad} \phi(q(\theta)), \overset{*}{q}(\theta) \rangle_{q(\theta)}$

Cramer-Rao inequality

$$\left(\frac{d}{d\theta} \phi(q(\theta)) \right)^2 = \left(\langle \operatorname{Grad} \phi(q(\theta)), \overset{*}{q}(\theta) \rangle_{q(\theta)} \right)^2$$

$$\leq \mathbb{E}_{q(\theta)}(\operatorname{Grad} \phi(q(\theta))^2) \, \mathbb{E}_{q(\theta)}((\overset{*}{q}(\theta))^2)$$

$$\boxed{\mathbb{E}_{q(\theta)}(\operatorname{Grad} \phi(q(\theta))^2) \geq \left(\frac{d}{d\theta} \phi(q(\theta)) \right)^2 I(\theta)^{-1}}$$

Ex. 1 $\phi(q) = E_q(t)$ "t is an unbiased estimator of $\phi(q)$"

$\mathrm{Grad}\,\phi(q) = t - E_q(t) = t - \phi(q)$ "error"

$E_q\left(\left(\mathrm{Grad}\,\phi(q)\right)^2\right) = \mathrm{Var}_q(t)$ "error variance"

C-R: $\mathrm{Var}_{q(\theta)}(t) \geq \left(\frac{d}{d\theta}\phi(q(\theta))\right)^2 I(\theta)^{-1}$

Ex. 2 $\phi(q) = -E_q(\log q)$ 'Entropy'

$\mathrm{Grad}\,\phi(q) = -\log q - (-E_q(\log q))$ $\Rightarrow$ Statistical Physics

C-R.: $\mathrm{Var}_{q(\theta)}(\log q(\theta)) \geq \left(\frac{d}{d\theta}\phi(q(\theta))\right)^2 I(\theta)^{-1}$