Deformed logarithm

The mechanism of affinespaces can be generalised to <u>deformed logs</u> (Formalism by J Naudts 2011)

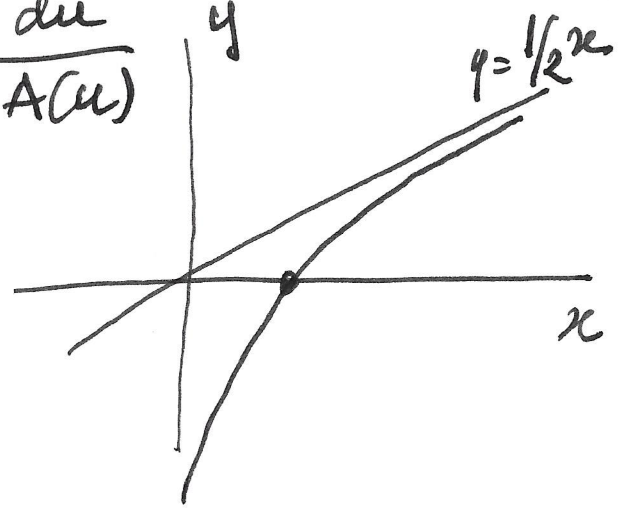
- Given $A : \mathbb{R}_+ \to \mathbb{R}_+$ non-decreasing
- define $\log_A x = \int_1^x \frac{du}{A(u)}$ (is concave)

Example 1 $A(u) = u$ $\log x = \int_1^x \frac{du}{u}$ ordinary log

Example 2 $A(u) = 1$ $\log_A x = \int_1^x \frac{du}{1} = x - 1$

Example 3 (Kaniadakis log 2002; Aitchison (?); Pawlowsky-Glahn (4.20))

$$\frac{d}{dx}\frac{1}{2}\left(x - \frac{1}{x}\right) = \frac{1}{2}\left(1 + \frac{1}{x^2}\right) \quad A(u) = \frac{2u^2}{1+u^2} \quad \log_A x = \int_1^x \frac{du}{A(u)}$$



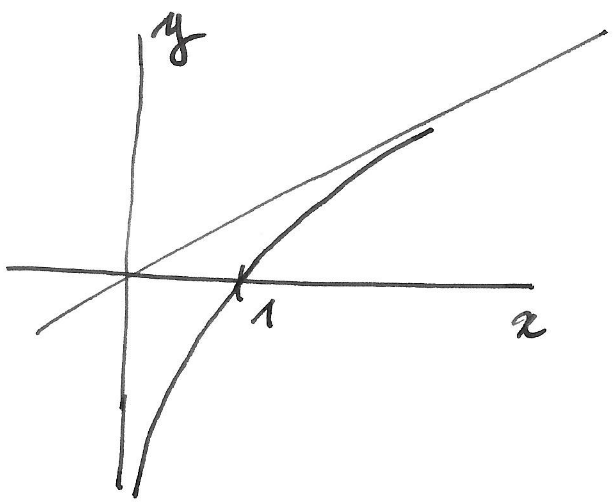
Example 4 (Tsallis log (1988))

$$\log_A x = \frac{x^{1-\alpha} - 1}{1-\alpha} \quad \alpha \neq 1, \text{ for ex. } \alpha = \frac{1}{2}$$

$$\log_A x = 2(x^{1/2} - 1) = \int_1^x \frac{du}{\sqrt{u}}$$

Example 5 (Nigel Newton 2016)

$$\log_A x = \frac{1}{2}(x+\log x-1) = \int_1^x \frac{du}{\frac{2u}{1+u}}$$



<u>deformed exponential</u>

$\exp_A = \log_A^{-1}$ (is convex)

Ex. 1 $\log x = y$ $\quad y = \exp x$

Ex. 2 $\log_A x = x-1$ $\quad \exp_A y = 1+y$

Ex. 3 $\frac{1}{2}\left(x-\frac{1}{x}\right) = y$, $x>0$ $\iff$ $x^2-2yx-1=0$, $x>0$ $\Rightarrow \exp_A y = y+\sqrt{1+y^2}$

Ex. 4 $2(x^{1/2}-1)=y$ $\quad \exp_A y = \left(1+\frac{1}{2}y\right)^2$

Ex. 5 $\frac{1}{2}(x+\log x-1)=y$ $\iff$ $1+2y = \log(xe^x)$

From $\exp_A \log_A x = x$ $\Rightarrow$ $\boxed{\exp_A'(y) = A(\exp_A y)}$

Exponential family (non-parametric)

$\mathcal{X}$ finite state space, p probability function, u random variable

$$q = \exp_A(u - K_p(u) + \log_A p)$$

is a probability function if $K_p(u)$ is the solution of

$$\sum_x \exp_A\left(u(x) - \underset{K}{\cancel{K_p}}(u) + \log_A p(x)\right) = 1$$

Compute the derivative $dK_p(u)[h] = \frac{d}{dt} K_p(u+th)\Big|_{t=0}$

$$0 = \sum_x A\left(\exp_A(u(x) - K_p(u) + \log_A p(x))\right)\left(h(x) - dK_p(u)[h]\right)$$

$$= \sum_x q(x)\left(h(x) - dK_p(u)[h]\right) \quad \text{that is}$$

$$\boxed{\mathbb{E}_{\tilde q}(h) = dK_p(u)[h]} \qquad \tilde q = \frac{A(q)}{\sum_x A(q(x))}$$

Choose $\mathbb{E}_{\tilde p}(u) = 0$

Affine space

Define the Bundle of the escort probability function

$$\left\{ (q,v) \mid q \in \mathcal{P}(\mathcal{X}),\ \mathbb{E}_{\tilde{q}}(v) = 0 \right\}$$

and the charts

$$s_p(q) = \log_{\{A\}} q - \log_{\{A\}} p - \mathbb{E}_{\tilde{p}}\left(\log_{\{A\}} q - \log_{\{A\}} p\right)$$

Example: (Kaniadakis)

$$s_p(q) = \frac{1}{2}\left(q - \frac{1}{q}\right) - \frac{1}{2}\left(p - \frac{1}{p}\right) -$$

$$\sum_x \frac{2p(x)^2}{1+p(x)^2}\left(\frac{1}{2}\left(q(x) - \frac{1}{q(x)}\right) - \frac{1}{2}\left(p(x) - \frac{1}{2}p(x)\right)\right)$$

$$\times\ 1 \Big/ \sum_x \frac{2p(x)^2}{1+p(x)^2}$$

Divergence

$$s_p^{-1}(u) = \exp_A(u - K_p(u) + \log_A p) \qquad u \in B_p$$

The deformed KL divergence is

$$D(p \| q) = \mathbb{E}_{\tilde{p}}(\log_A p - \log_A q)$$

Example: (N. Newton)

$$\sum_x \frac{2p(x)}{1+p(x)}\left((p(x) - q(x)) + \log \frac{p(x)}{q(x)}\right)$$

Score

The ~~[illegible]~~ velocity (affine) is

$$\frac{d}{dt} S_p(q(t)) = \frac{d}{dt}\left(\log_A q(t) - \log_A p - \mathbb{E}_{\tilde{p}}\left(\log_A q(t) - \log_A p\right)\right)$$

$$= \frac{\dot{q}(t)}{A(q(t))} - \mathbb{E}_{\tilde{p}}\left(\frac{\dot{q}(t)}{A(q(t))}\right)$$

the score is the velocity in the moving frame, $p = q(t)$

$$\frac{D}{dt}\left(\frac{\dot{q}(t)}{A(q(t))} - \mathbb{E}_{\tilde{q}(t)}\left(\frac{\dot{q}(t)}{A(q(t))}\right)\right) = \frac{\dot{q}(t)}{A(q(t))}$$

$$\mathbb{E}_{\tilde{q}(t)}\left(\frac{\dot{q}(t)}{A(q(t))}\right) = \sum \frac{\dot{q}(x;t)}{A(q(x;t))} \frac{A(q(x;t))}{\sum_y A(q(y;t))} = 0$$