Decomposing quiver moduli - a QuiverTools showcase

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[0] Slides and code at https://github.com/giannipetrella/quivertools-showcase

Preamble

Plan

1. What are quiver moduli?
2. What is *QuiverTools*?
3. Teleman inequality and rigidity (in QuiverTools)
4. Hirzebruch–Riemann–Roch and semiorthogonal embeddings (in QuiverTools)

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Aknowledgements
QuiverTools is developed by P. Belmans, H. Franzen and G.P.
This work is supported by the Luxembourg National Research Fund (AFR-17953441)

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Once a *dimension vector* $\mathbf{d}$ is fixed, a representation is identified with a point in the *parameter space*

$$\mathrm{R}(Q, \mathbf{d}) := \bigoplus_{i \to j \in Q_1} \mathrm{Mat}_{d_j \times d_i}(k).$$

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Orbits are precisely isomorphism classes of representations.

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Theorem (King [11])

The (semi)stable locus $\mathrm{R}^{(\theta-\mathrm{sst})\theta-\mathrm{st}}(Q, \mathbf{d})$ *is a* $\mathrm{GL}_{\mathbf{d}}$*-invariant Zariski open which admits a geometric quotient, denoted by* $\mathrm{M}^{(\theta-\mathrm{sst})\theta-\mathrm{st}}(Q, \mathbf{d})$.

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From now on, Q is assumed to be acyclic, for simplicity.

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So we implemented them!

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QuiverTools - quivers

```
julia> Q = mKronecker_quiver(3)
3-Kronecker quiver, with adjacency matrix [0 3; 0 0]

julia>  println("Adjacency matrix: ", Q.adjacency)
        println("arrows: ", arrows(Q))
        println("Number of vertices: ", nvertices(Q))
        println("Number of arrows: ", narrows(Q))
Adjacency matrix: [0 3; 0 0]
arrows: [[1, 2], [1, 2], [1, 2]]
Number of vertices: 2
Number of arrows: 3

julia> is_connected(Q)
true

julia> is_acyclic(Q)
true
```

QuiverTools - quiver moduli

```
julia> M = QuiverModuliSpace(Q, [2, 3])
Moduli space of semistable representations of 3-Kronecker quiver,
with adjacency matrix [0 3; 0 0]
with dimension vector [2, 3] and stability parameter [9, -6]

julia> is_projective(M)
true

julia> is_smooth(M)
true

julia> dimension(M)
6

julia> H = Hodge_polynomial(M)
x^6*y^6 + x^5*y^5 + 3*x^4*y^4 + 3*x^3*y^3 + 3*x^2*y^2 + x*y + 1
```

QuiverTools - quiver moduli

```
julia> Hodge_diamond(M)
7x7 Matrix{Int64}:
 1  0  0  0  0  0  0
 0  1  0  0  0  0  0
 0  0  3  0  0  0  0
 0  0  0  3  0  0  0
 0  0  0  0  3  0  0
 0  0  0  0  0  1  0
 0  0  0  0  0  0  1

julia> P = Poincare_polynomial(M)
L^6 + L^5 + 3*L^4 + 3*L^3 + 3*L^2 + L + 1

julia> Betti_numbers(M) == [1,0,1,0,3,0,3,0,3,0,1,0,1]
true

julia>  println("Picard rank: ", Picard_rank(M))
        println("Index: ", index(M))
Picard rank: 1
Index: 3
```

Harder-Narasimhan stratification

A *slope function* is a function $\mu_\theta : \mathbb{Z}^{Q_0} \to \mathbb{Q}$: $\mu_\theta(\alpha) = \frac{\theta \cdot \alpha}{\sum_i \alpha_i}$.

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A *Harder–Narasimhan type* for Q and $\mathbf{d}$ is a tuple $\mathbf{d}^* = (\mathbf{d}^1, \dots, \mathbf{d}^s)$, such that $\mu(\mathbf{d}^1) > \cdots > \mu(\mathbf{d}^s)$, each $\mathbf{d}^\ell$ admits a semistable representation, and $\sum_{\ell=1}^s \mathbf{d}^\ell = \mathbf{d}$.

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Given a stability parameter θ, every representation V admits a unique *Harder–Narasimhan filtration*,

i.e., $0 = V_0 \subsetneq V_1 \cdots \subsetneq V_s = V$ such that the dimension vectors $\dim(V_\ell / V_{\ell-1})$ form an HN type.

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Theorem (Reineke [12])

The paremeter space $\mathrm{R}(Q, \mathbf{d})$ *admits a stratification into smooth, disjoint, locally closed subsets* $S_{\mathbf{d}^*}$*, each corresponding to a HN type. The trivial type* $(\mathbf{d})$ *corresponds to the semistable locus* $\mathrm{R}^{\theta-\mathrm{sst}}(Q, \mathbf{d})$.

HN types in QuiverTools

```
julia> Q = mKronecker_quiver(3);

julia> M = QuiverModuliSpace(Q, [2, 3]);

julia> HN = all_HN_types(M, unstable=true)
7-element Vector{Vector{AbstractVector{Int64}}}:
 [[1, 1], [1, 2]]
 [[2, 2], [0, 1]]
 [[2, 1], [0, 2]]
 [[1, 0], [1, 3]]
 [[1, 0], [1, 2], [0, 1]]
 [[1, 0], [1, 1], [0, 2]]
 [[2, 0], [0, 3]]
```

Teleman quantization

Fact:
To each HN type $\mathbf{d}^*$ is associated a 1-PS of $\mathrm{GL}_{\mathbf{d}}$, denoted by $\lambda_{\mathbf{d}^*}$. On each stratum $S_{\mathbf{d}^*}$, this 1-PS acts on $\det(\mathcal{N}^{\vee}_{S_{\mathbf{d}^*}\,/\,\mathrm{R}})|_{S^{\lambda_{\mathbf{d}^*}}_{\mathbf{d}^*}}$, and the *weight* of this action is denoted by $\eta_{\mathbf{d}^*}$.

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Let F be a coherent sheaf on $\mathrm{R}(Q, \mathbf{d})$, with an action of $\mathrm{GL}_{\mathbf{d}}$ that descends it to the quotient. Denote the descent of F to $\mathrm{M}^{\theta-\mathrm{st}}(Q, \mathbf{d})$ by $\mathcal{F}$.

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Theorem (Teleman quantization [3, 7])

If, for all the HN strata in $\mathrm{R}(Q, \mathbf{d})$, *the strict inequality*

$$\max W(F, \mathbf{d}^*) < \eta_{\mathbf{d}^*}$$

holds, then for all $k > 0$, $\mathrm{H}^k(\mathrm{M}^{\theta-\mathrm{st}}(Q, \mathbf{d}), \mathcal{F}) = 0$.

Teleman quantization with QuiverTools

```
julia> eta = all_Teleman_bounds(M)
Dict{Vector{AbstractVector{Int64}}, Int64}:
 [[2, 2], [0, 1]]         => 60
 [[2, 1], [0, 2]]         => 100
 [[1, 0], [1, 2], [0, 1]] => 100
 [[1, 0], [1, 3]]         => 360
 [[1, 0], [1, 1], [0, 2]] => 270
 [[1, 1], [1, 2]]         => 15
 [[2, 0], [0, 3]]         => 270
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Rigidity for quiver moduli

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Using the *universal families* for M, i.e., some special vector bundles $\{\mathcal{U}_i\}_{i\in Q_0}$, one can construct the *standard exact sequence* [4]:

$$0 \to \mathcal{O}_{\mathrm{M}} \to \bigoplus_{i\in Q_0} \mathcal{U}_i^\vee \otimes \mathcal{U}_i \to \bigoplus_{i\to j\in Q_1} \mathcal{U}_i^\vee \otimes \mathcal{U}_j \to \mathcal{T}_{\mathrm{M}} \to 0. \qquad (1)$$

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With some homological algebra as in [3], one sees that, if for all $k > 0$ and all $i, j \in Q_0$ we can show that $\mathrm{H}^k(\mathrm{M}, \mathcal{U}_i^\vee \otimes \mathcal{U}_j) = 0$, then $\mathrm{H}^k(\mathrm{M}, \mathcal{T}_{\mathrm{M}}) = 0$.

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In other words, the moduli space $\mathrm{M}^{\theta-\mathrm{sst}}(Q, \mathbf{d}^*)$ is *rigid*.

Teleman inequality with QuiverTools

```
julia> W = all_weights_endomorphisms_universal_bundle(M)
Dict{Vector{AbstractVector{Int64}}, Vector{Int64}}:
  [[2, 2], [0, 1]]          => [0, 15, -15, 0]
  [[2, 1], [0, 2]]          => [0, 10, -10, 0]
  [[1, 0], [1, 2], [0, 1]]  => [0, 10, 15, -10, 0, 5, -15, -5, 0]
  [[1, 0], [1, 3]]          => [0, 45, -45, 0]
  [[1, 0], [1, 1], [0, 2]]  => [0, 15, 30, -15, 0, 15, -30, -15, 0]
  [[1, 1], [1, 2]]          => [0, 5, -5, 0]
  [[2, 0], [0, 3]]          => [0, 15, -15, 0]

julia> all(maximum(W[hntype]) < eta[hntype] for hntype in HN)
true
```

What is a derived category?

Idea: we want to study the category of quasicoherent $\mathcal{O}_{\mathrm{M}}$-modules, $QCoh(M)$.
More specifically, we want to understand their cohomology - so we "only want objects in $QCoh(\mathrm{M})$ up to quasi-isomorphism".

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It's actually the category of *complexes of objects* in $QCoh(\mathrm{M})$ with formal inverses to quasi-isomorphisms...

Semiorthogonal embeddings 1/2

To understand $D^b(\mathrm{M})$, we want a *semiorthogonal decomposition*.

Definition

A *semiorthogonal decomposition* (SOD) is a sequence of (full) subcategories where "there are no morphisms or extensions from right to left":

$$\mathcal{C} = \langle A_1, A_2, \ldots, A_n \rangle,$$

where $\mathrm{Hom}(V, W) = \mathrm{Ext}(V, W) = 0$ for all $V \in A_i$, $W \in A_{<i}$.

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For moduli spaces of vector bundles on a curve C, $\mathcal{M}_C(r, \mathcal{L})$, this is done in various degrees of generality by [5, 9, 10], [2] and [8]. The "recipe" is to define a functor $D^b(C) \to D^b(\mathcal{M}_C(r, \mathcal{L}))$, show that it is fully faithful, twist it to embed several copies of $D^b(C)$ into $D^b(\mathcal{M}_C(r, \mathcal{L}))$ in a particular order and lastly show that these copies are semiorthogonal.

Semiorthogonal embeddings 2/2

For quiver moduli, Belmans–Franzen [4] defined a pseudo-Fourier–Mukai transform

$$\Phi_{\mathcal{U}} : D^b(Q) \to D^b(M) : \ V \mapsto \mathcal{U} \otimes^L_{kQ} V,$$

and show that under reasonable assumptions it is fully faithful. In particular, a copy of $D^b(Q)$ is embedded in $D^b(M)$.

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Let r be the index of M, i.e., the largest divisor of K_{M} in $Pic(\mathrm{M})$, and define $H := -\frac{1}{r}K_{\mathrm{M}}$.
Under what conditions, if any, is the following partial decomposition semiorthogonal?

$$\langle \Phi_{\mathcal{U}}, \mathcal{O}_{\mathrm{M}}, \Phi_{\mathcal{U}(H)}, \mathcal{O}_{\mathrm{M}}(H), \ldots \Phi_{\mathcal{U}((r-1)H)}, \mathcal{O}_{\mathrm{M}}((r-1)H), \ldots \rangle$$

Semiorthogonal embeddings 2/2

After *some* homological algebra, this amounts to checking that for all $k \geq 0$, for all $0 \leq n_1 < n_2 \leq r-1$ and for all $i, j \in Q_0$,

$$\mathrm{H}^k(\mathrm{M}, \mathcal{U}_i^\vee \otimes \mathcal{U}_j \otimes \mathcal{O}((n_1 - n_2)H)) = 0, \tag{2}$$

$$\mathrm{H}^k(\mathrm{M}, \mathcal{U}_i^\vee \otimes \mathcal{O}(n_1 - n_2)H) = 0, \ \textit{and} \tag{3}$$

$$\mathrm{H}^k(\mathrm{M}, \mathcal{O}(n_1 - n_2)H) = 0, \tag{4}$$

and that for all $0 \leq n_1 \leq n_2 \leq r-1$,

$$\mathrm{H}^k(\mathrm{M}, \mathcal{U}_i \otimes \mathcal{O}(n_1 - n_2)H) = 0. \tag{5}$$

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To check these vanishings we combine Teleman quantization and Hirzebruch–Riemann–Roch computations, enabled by the results on Chow rings of quiver moduli in [1, 6].

SOD in QuiverTools

Example

$$Q = \begin{tikzcd} \overset{1}{\circ} \arrow[r, bend left] \arrow[r] \arrow[r, bend right] & \overset{2}{\circ} \end{tikzcd} \quad \mathbf{d} = (2,3),\ \theta = (9,-6).$$

SOD in QuiverTools

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```
julia> r, n = index(M), dimension(M)
(3, 6)

julia> H = all_weights_irreducible_component_canonical(M);

julia> for hntype in keys(H)
           H[hntype] *= -1
       end

julia> H
Dict{Vector{AbstractVector{Int64}}, Int64}:
 [[2, 2], [0, 1]]          => -30
 [[2, 1], [0, 2]]          => -40
 [[1, 0], [1, 2], [0, 1]]  => -40
 [[1, 0], [1, 3]]          => -135
 [[1, 0], [1, 1], [0, 2]]  => -105
 [[1, 1], [1, 2]]          => -5
 [[2, 0], [0, 3]]          => -90
```

SOD in QuiverTools - Teleman inequality

```
julia> chi = [-1, 1]; Ui = all_weights_universal_bundle(M, chi)
Dict{Vector{AbstractVector{Int64}}, Vector{Int64}}:
 [[2, 2], [0, 1]]           => [15, 0]
 [[2, 1], [0, 2]]           => [20, 10]
 [[1, 0], [1, 2], [0, 1]]   => [25, 15, 10]
 [[1, 0], [1, 3]]           => [90, 45]
 [[1, 0], [1, 1], [0, 2]]   => [60, 45, 30]
 [[1, 1], [1, 2]]           => [5, 0]
 [[2, 0], [0, 3]]           => [45, 30]

julia> UidualUj = all_weights_endomorphisms_universal_bundle(M);
```

SOD in QuiverTools - Teleman inequality

```
julia> all(maximum(UidualUj[hn]) - H[hn] < eta[hn] for hn in HN)
true

julia> all(maximum(-Ui[hn]) - H[hn] < eta[hn] for hn in HN)
true

julia> all(-H[hn] < eta[hn] for hn in HN)
true

julia> b = true

julia> for t in 0:r-2
        b=b && all(maximum(Ui[hn])-t*H[hn] < eta[hn] for hn in HN)
       end

julia> b
true
```

SOD in QuiverTools - Hirzebruch–Riemann–Roch

```
julia> CH, CHvars = Chow_ring(M);

julia> x11, x12, x21, x22, x23 = CHvars;

julia> w=(Euler_form(Q,d,[1,0])-Euler_form(Q,[1,0],d))*x11+
          (Euler_form(Q,d,[0,1])-Euler_form(Q,[0,1],d))*x21;

julia> truncated_exp(x, n) = sum(x^i/factorial(i) for i in 0:n)

julia> Hbundle = truncated_exp(-1//r * w, n)

julia> Hbundle_dual = truncated_exp(1//r * w, n)
```

SOD in QuiverTools - Hirzebruch–Riemann–Roch

```
julia> u1 = 2 + x21 + 1//2 *x11^2 − x12 + 1//6 *x11*x12 −
       1//2 *x23 − 1//8 *x23*x11 + 1//12 *x12^2 −
       1//80 * x23*x12 − 1//720 * x23^2;

julia> u1star = 2 − x21 + 1//2*x11^2 − x12 − 1//6*x11*x12 +
       1//2*x23 − 1//8*x23*x11 + 1//12*x12^2 +
       1//80 * x23*x12 − 1//720 * x23^2;

julia> u2 = 3 + x11 + 1//2*x11^2 − x22 − 1//2*x22*x11+
       2//3*x11*x12 + 1//8*x23*x11 + 1//12*x22*x12−
       1//4*x12^2 + 1//120*x23*x12;

julia> u2star = 3 − x11 + 1//2*x11^2 − x22 + 1//2*x22*x11−
       2//3*x11*x12 + 1//8*x23*x11 + 1//12*x22*x12−
       1//4*x12^2 − 1//120*x23*x12;

julia> bundles, dual_bundles = [u1, u2], [u1star, u2star];
```

SOD in QuiverTools - Hirzebruch–Riemann–Roch

```
julia> m = 3;

julia> for s in 1:m-1
    println([integral(M, u*v*Hbundle_dual^s)
            for u in dual_bundles for v in bundles])
    println([integral(M, u*Hbundle_dual^s)
            for u in dual_bundles])
    println([integral(M, Hbundle_dual^s)])
    end
[0, 0, 0, 0]
[0, 0]
[0]
[0, 0, 0, 0]
[0, 0]
[0]

julia> println([integral(M, u) for u in bundles])
       println([integral(M, u*Hbundle_dual) for u in bundles])
[0, 0]
[0, 0]
```

Thank you for your attention!

Slides and code at
https://github.com/giannipetrella/quivertools-showcase

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