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THE EFFECT OF WIND ON FLAT-FIRE TRAJECTORIES

Robert L. McCoy

August 1976

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I. INTRODUCTION

In the past, the BRL has frequently been asked to comment on the validity of the classical crosswind deflection formula

$$Z = W_z \left(t - R/V_{x_o}\right) \tag{1}$$

where Z is the wind deflection, W_z is the speed of a constant crosswind acting over the range R, t is the time of flight, and V_{x_o} is the downrange component of muzzle velocity. This formula has been used extensively by riflemen for many years, and questions as to its accuracy have been received from the military commands, captains of the service rifle teams, the National Rifle Association, and most recently by the Army Materiel Systems Analysis Activity.

The answer usually given is that the formula is more accurate than the rifleman's ability to estimate the wind speed. This answer is undoubtedly technically correct in most cases, but the persistence of the question suggests that a more satisfying quantitative answer should be obtained.

In this report the generalized effects of both crosswinds and range winds (head or tail winds) are derived from the basic equation of motion for a projectile acted on by aerodynamic drag and gravity. The effects of space-varying winds along the trajectory are examined, and simple approximate wind corrections for field use are derived and compared with exact and nearly-exact solutions.

II. BASIC EQUATIONS OF MOTION

The projectile is assumed to be a particle, acted on only by the forces of aerodynamic drag and gravity. We choose an earth-fixed XYZ coordinate system, with X pointing downrange from gun to target, Y pointing vertically upward, and Z pointing to the right, when looking downrange. The X-Z plane is chosen tangent to the earth's surface at the launch point, so that the acceleration of gravity will be parallel to the Y axis. (For the relatively short trajectories considered in this report, no corrections for a curved earth will be required.) Figure 1 illustrates the coordinate system and notation used.

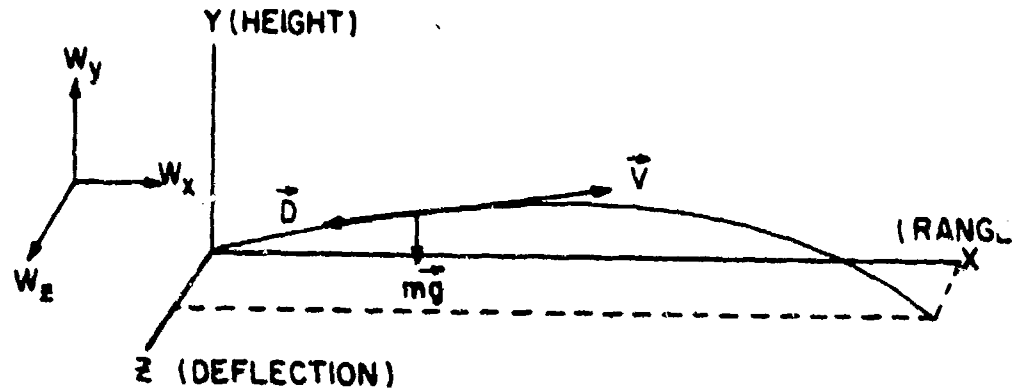

FIGURE 1

The equation of motion for the projectile follows from Newton's second law, and in vector form is

$$m \frac{d\vec{V}}{dt} = \vec{D} + m\vec{g} \tag{2}$$

where $\vec{V}$ is the velocity of the projectile with respect to the earth-fixed axes, $\vec{D}$ is the aerodynamic drag force (parallel to $\vec{V}$ but oppositely directed), and $\vec{g}$ is the acceleration due to gravity.

The drag force is usually written as $\vec{D} = -\frac{1}{2}\rho S C_D V\vec{V}$, where $V = |\vec{V}|$, the scalar magnitude of $\vec{V}$, ρ is the air density, S is a reference area, and C_D is the drag coefficient. However, in the expression for drag, the velocity used must be with respect to air, and if a vector wind $\vec{W}$ is present, the velocity with respect to air is $\vec{V}-\vec{W}$. We now rewrite Equation (2) as

$$m \frac{d\vec{V}}{dt} = -\frac{\rho S C_D}{2} |\vec{V}-\vec{W}| (\vec{V}-\vec{W}) + m\vec{g} \tag{3}$$

Divide both sides of Equation (3) by m, and define $C_D{}^* = \frac{\rho S C_D}{2m}$

$$\frac{d\vec{V}}{dt} = -C_D{}^* |\vec{V}-\vec{W}| (\vec{V}-\vec{W}) + \vec{g} \tag{4}$$

In the XYZ coordinate system, $\vec{V} = (V_x, V_y, V_z)$, $\vec{W} = (W_x, W_y, W_z)$, $\vec{g} = (0, -g, 0)$, and the scalar $|\vec{V}-\vec{W}|$ we will define as

$$\tilde{V} = [(V_x - W_x)^2 + (V_y - W_y)^2 + (V_z - W_z)^2]^{\frac{1}{2}} \quad . \qquad (5)$$

Equation (4) can now be written in component form:

$$\begin{aligned} \dot{V}_x &= - C_D{}^* \tilde{V} (V_x - W_x) \\ \dot{V}_y &= - C_D{}^* \tilde{V} (V_y - W_y) - g \\ \dot{V}_z &= - C_D{}^* \tilde{V} (V_z - W_z) \end{aligned} \qquad (6)$$

where a superscript dot indicates differentiation with respect to time.

Equations (6) are a precise statement of Newton's second law for the particle trajectory, and are in suitable form for accurate solution by numerical integration. The non-linearity introduced through the expression for $\tilde{V}$ leads to difficulty in finding analytical solutions; a more practical approach is to find a sufficiently accurate linear approximation to $\tilde{V}$ for the particular case we wish to solve.

As an example of the problems encountered with exact analytical solutions, we will solve Equations (6) for the case of a constant cross-wind.

Let $s = \int_0^t \tilde{V}\, dt$, $W_x = 0$ and rewrite the first and third of Equations (6) with s as the independent variable.

$$\begin{aligned} V_x{}' &= - C_D{}^* V_x \\ V_z{}' &= - C_D{}^* (V_z - W_z) \end{aligned} \qquad (7)$$

where the superscript prime (') means $\frac{d(\ )}{ds}$.

The general solution is

$$V_x = V_{x_o}\, e^{-\int_o^s C_D^*\, ds_1}$$

$$V_z = e^{-\int_o^s C_D^*\, ds_1} \int_o^s W_z C_D^*\, e^{\int_o^{s_1} C_D^*\, ds_2}\, ds_1 \tag{8}$$

Since W_z is constant, and $e^{-\int_o^s C_D^*\, ds_1} = V_x/V_{x_o}$

$$V_z = W_z \left[1 - \frac{V_x}{V_{x_o}}\right] \tag{9}$$

Hence the deflection Z, due to the constant crosswind, W_z, is given by

$$Z = \int_o^t V_z dt = W_z \left[t - \frac{R}{V_{x_o}}\right] \quad , \tag{10}$$

since $R = \int_o^t V_x dt$.

Equation (10) does confirm the classical formula for crosswind deflection; however, the time of flight, t, in Equation (10), is not time of flight to range R for the no-wind case. The t in Equation (10) is time of flight to arc length s, for the specific crosswind case considered. In general, neither s nor t is known a priori along the actual wind trajectory, and they can only be found by solving Equations (6) accurately by numerical integration.

If the trajectories considered are restricted to those for which range, R, is insignificantly different from arc length, s, and the time of flight to range R is essentially the same for the actual wind and no-wind cases, then Equation (10) becomes a practical gunnery formula. We will now show that the class of trajectories for which Equation (10) is useful consists of all trajectories fulfilling the "flat-fire" condition.

The scalar $\tilde{V}$ can be written

$$\tilde{V} = (V_x - W_x) \left[1 + \left(\frac{V_y - W_y}{V_x - W_x}\right)^2 + \left(\frac{V_z - W_z}{V_x - W_x}\right)^2\right]^{1/2} \tag{11}$$

Let $\varepsilon_y = \dfrac{V_y - W_y}{V_x - W_x}$, $\varepsilon_z = \dfrac{V_z - W_z}{V_x - W_x}$

Substituting these definitions into Equation (11), and expanding in binomial series,

$$\tilde{V} = (V_x - W_x)\ [1 + 1/2(\varepsilon_y{}^2 + \varepsilon_z{}^2) - 1/8(\varepsilon_y{}^2 + \varepsilon_z{}^2)^2 + 1/16(\varepsilon_y{}^2 + \varepsilon_z{}^2)^3 + \ .\ .\ .\ ] \tag{12}$$

If $(V_x - W_x)$ is to be an adequate approximation to $\tilde{V}$, the quantity $(\varepsilon_y{}^2 + \varepsilon_z{}^2)$ must be negligible in comparison with unity; we will assume $(\varepsilon_y{}^2 + \varepsilon_z{}^2) \leq 10^{-2}$. This requirement means that $(V_x - W_x)$ approximates $\tilde{V}$ to less than a half of one percent error.

For high velocity projectiles employed in a direct-fire role, the ratio of any component of wind velocity to V_x will usually not exceed 10^{-2}. Since V_z cannot exceed W_z in magnitude, the term $\varepsilon_z{}^2$ is of order 10^{-4}, and may be neglected. Hence, $(V_x - W_x)$ will be a sufficiently accurate approximation to $\tilde{V}$ if $\varepsilon_y{}^2 \leq 10^{-2}$. Since $\left|\dfrac{W_y}{V_x}\right|$ will not exceed 10^{-2}, this means that $\left|\dfrac{V_y}{V_x}\right| \leq 10^{-1}$, or the initial angle of departure must be restricted to be within about five degrees of the horizontal. (In practice, gun elevation angles as large as ten or even fifteen degrees can be tolerated without incurring serious errors in the classical wind deflection formula.)

We will now introduce the flat-fire conditions into Equations (6). The vertical wind component, W_y, can be neglected with no loss in generality since, for flat fire, the effect of a W_y is analogous to that of a W_z, but in the vertical plane. Setting $\tilde{V} = V_x - W_x$, and dropping W_y, Equations (6) become

$$\begin{aligned} \dot{V}_x &= - C_D{}^* \ (V_x - W_x)^2 \\ \dot{V}_y &= - C_D{}^* \ (V_x - W_x)\ V_y - g \\ \dot{V}_z &= - C_D{}^* \ (V_x - W_x)\ (V_z - W_z) \end{aligned} \tag{13}$$

Equations (13) will now be used to derive the effects of crosswinds and range winds on flat-fire trajectories.

III. EFFECT OF CROSSWIND ON THE TRAJECTORY

We assume no range wind and a constant crosswind, W_z, acting over the entire trajectory. The first and third equations of (13) simplify to

$$\dot{V}_x = - C_D^* V_x^2$$
$$\dot{V}_z = - C_D^* V_x (V_z - W_z) \tag{14}$$

We first rewrite Equations (14) with X as the new independent variable.

$$V_x' + C_D^* V_x = 0$$
$$V_z' + C_D^* V_z = C_D^* W_z \tag{15}$$

where for the remainder of this report, the superscript prime (') will indicate differentiation with respect to X. At t = 0, X = 0, $V_x = V_{x_o}$, and with no loss in generality, V_{z_o} will be assumed zero. The general solution of (15) is

$$V_x = V_{x_o} e^{-\int_o^X C_D^* dr_1}$$
$$V_z = e^{-\int_o^X C_D^* dr_1} \int_o^X W_z C_D^* e^{\int_o^{r_1} C_D^* dr_2} dr_1 \tag{16}$$

Since W_z is constant, and $e^{-\int_o^X C_D^* dr_1} = \frac{V_x}{V_{x_o}}$,

$$V_z = (\frac{V_x}{V_{x_o}}) W_z [e^{\int_o^X C_D^* dr_1} - 1] = W_z [1 - \frac{V_x}{V_{x_o}}] \quad (1$$

The deflection Z, due to the constant crosswind is

$$Z = \int_0^t V_z dt_1 = W_z \left[t - \frac{R}{V_{x_0}}\right] \tag{18}$$

Equation (18) appears identical to Equation (10), but in (18) the time, t, is measured to range R rather than arc length, and since $\tilde{V} = V_x$ (remember, $W = 0$), t also represents the time along a no-wind trajectory. The quantity $(t - R/V_{x_0})$ is often called "lag time", since it is physically the time difference, or lag, between actual flight time and the time to the same range in a vacuum.

To demonstrate the application of Equation (18), we will do two cases as example problems. The first case selected is a low velocity 40mm projectile with a maximum range of approximately 500 metres. The second case is the 7.62mm NATO bullet at ranges out to one kilometre. For both cases, a constant crosswind speed of 5 metres/second is assumed to act over the entire range. The following tables compare the wind deflection predicted by Equation (18) with those obtained from numerical integration of Equations (6).

TABLE I

Low Velocity 40mm Projectile, Muzzle Vel. = 76 Metres/Second

Gun Elevation (Degrees)	Range (Metres)	Time of Flight (Sec)	(R/V_{x_0}) (Sec)	[Eq. (18)] Defl., Z (Metres)	[Numerical Solution] Defl., Z (Metres)
5	100	1.3606	1.3173	.217	.217
10	190	2.6935	2.5319	.808	.810
15	260	3.8478	3.5324	1.577	1.581
20	330	5.1448	4.6086	2.681	2.688
30	410	7.1594	6.2128	4.733	4.746

TABLE 2

High Velocity 7.62mm NATO Bullet (M80), Muzzle Vel. = 840 Metres/Second

Range (Metres)	Time of Flight (Sec)	(R/V_{x_o}) (Sec)	[Eq. (18)] Defl., Z (Metres)	[Numerical Solution] Defl., Z (Metres)
200	.2622	.2382	.120	.120
400	.5835	.4764	.536	.535
600	.9887	.7146	1.371	1.370
800	1.5148	.9528	2.810	2.810
1000	2.1588	1.1910	4.839	4.839

Table 1 indicates that rather severe violations of the flat-fire restriction do not invalidate the use of Equation (18), and Table 2 shows that for a very nearly flat-fire projectile the classical formula gives, for practical purposes, an exact result. We will now examine the effects of space-varying crosswinds on flat-fire trajectories.

For a non-constant crosswind, the second of Equations (15) can be integrated either by parts or by a simple numerical quadrature. However, the differential equation is linear, and an easier approach is to superpose a series of solutions for constant crosswinds starting at different downrange distances.

Consider a constant crosswind commencing at $X = X_1$, and continuing downrange. This is shown in Figure 2.

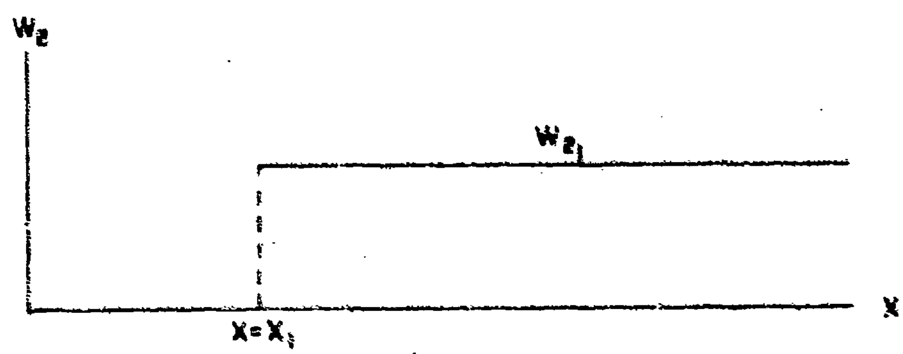

FIGURE 2

The solution of Equations (15) with the crosswind of Figure 2 is

$$V_x = V_{x_o}\, e^{-\int_o^X C_D^*\, dr_1}$$

$$V_z = e^{-\int_o^X C_D^*\, dr_1}\, W_{z_1} \int_{X_1}^{X} C_D^*\, e^{\int_o^{r_1} C_D^*\, dr_2}\, dr_1 \tag{19}$$

Hence $V_z = W_{z_1}\left[1 - \frac{V_x}{V_{x_1}}\right]$, where $V_{x_1} = V_x$ at $X = X_1$.

For $R > X_1$, the deflection at the target due to the wind of Figure 2 is

$$Z(R) = W_{Z_1}\left[t(R) - t(X_1) - \frac{R-X_1}{V_{X_1}}\right] \tag{20}$$

If a constant crosswind $[-\overline{W}_{z_{(1,2)}}]$ commencing at $X = X_2$ is added to a constant crosswind $\overline{W}_{z_{(1,2)}}$ commencing at $X = X_1$, the net wind acting on the trajectory is shown in Figure 3.

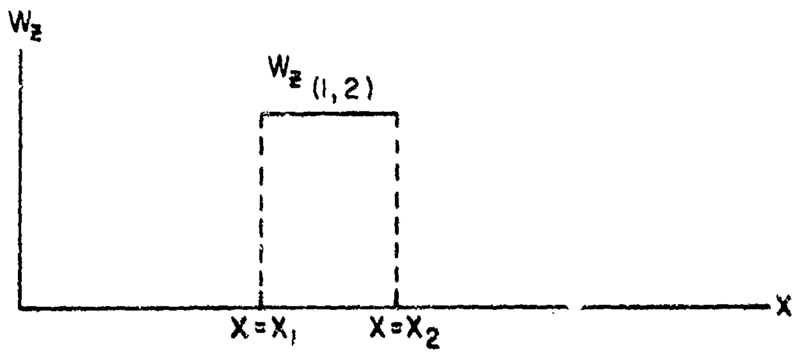

FIGURE 3

The deflection at the target due to the crosswind of Figure 3 is

$$Z(R) = \overline{W}_{z_{(1,2)}} \left\{ \left[t(R) - t(X_1) - \frac{R - X_1}{V_{x_1}} \right] - \left[t(R) - t(X_2) - \frac{R-X_2}{V_{x_2}} \right] \right\} \qquad (21)$$

for $R \geqslant X_2$.

If wind $\overline{W}_{z_{(1,2)}}$ acts over the entire range, $X_1 = 0$, $X_2 = R$, and Equation (21) reduces to Equation (18). Since any downrange variation of W_z can be approximated by a series of constant winds acting over short intervals, Equation (21) provides a general method of determining the effect of variable crosswinds on flat-fire trajectories.

Equation (21) states that the effect of crosswind over any interval of downrange distance is proportional to the amount of change in lag time over that interval, and suggests an answer to the question: "At what point along the trajectory is the projectile most sensitive to crosswind?". The answer is: the point at which the rate of change of lag time with respect to distance is a maximum.

Let $\mu_{(1,2)} = \left[t(R) - t(X_1) - \frac{R-X_1}{V_{x_1}} \right] - \left[t(R) - t(X_2) - \frac{R-X_2}{V_{x_2}} \right]$.

Let $X_2 = X_1 + \Delta X$, where ΔX is a very small distance, so that $\mu(X_1)$ is the crosswind sensitivity to a unit crosswind, acting over distance ΔX at distance X_1 from the gun.

From Equation (15), $\frac{\Delta V_x}{\Delta X_1} \approx - C_D{}^*(X_1) V_{x_1}$, and $V_{x_2} \approx V_{x_1} [1 - u]$, where $u = C_D{}^*(X_1)\Delta X$. Thus:

$$t(X_1 + \Delta X) - t(X_1) \approx \frac{\Delta X}{\frac{1}{2}[V_{x_1} + V_{x_2}]} \approx \frac{\Delta X}{V_{x_1}} \left[\frac{1}{1 - u/2} \right] \text{ , and hence:}$$

$$\mu(X_1) \approx \frac{1}{V_{x_1}} \left[\frac{\Delta X}{1 - u/2} + \frac{R - X_1 - \Delta X}{1 - u} - (R - X_1) \right] \ .$$

or

$$\mu(X_1) \approx \frac{C_D^*(X_1)\Delta X}{V_{x_1}} \left\{\frac{(R-X_1) - \frac{1}{2}\Delta X[1 + C_D^*(X_1)(R-X_1)]}{1 - 3u/2 + u^2/2}\right\}$$

For most projectiles, C_D^* is of order 10^{-3}, hence $C_D^*(X_1)R$ and $C_D^*(X_1)X_1$ can be neglected in comparison with R or X_1. The sensitivity of the trajectory to a unit crosswind at $X = X_1$, acting over the small distance ΔX is well approximated by

$$\mu(X_1) = \frac{u}{V_{x_1}} (R-X_1) \tag{22}$$

At the target, $X_1 = R$, and Equation (22) tells us that crosswind at the target has no effect at all on the trajectory. [Equation (21) gives the same result, if X_1 is chosen slightly less than R, and $X_2 = R$.]

Equation (22) still does not tell us directly where the point of maximum sensitivity is, although the factor $(R - X_1)$ suggests that it is closer to the gun than to the target. If we assume C_D^* is constant, i.e., independent of X, $V_{x_1} = V_{x_o} e^{-C_D^* X_1}$, and we find

$$\mu(X_1) = \frac{u}{V_{x_o}} e^{C_D^* X_1} (R - X_1) \tag{23}$$

Although C_D^* does vary slowly with distance, Equation (23) is useful in establishing the general shape of $\mu(X_1)$ versus X_1. If $|C_D^* X_1| \ll 1$, the term $e^{C_D^* X_1} \approx 1$; hence for very low drag projectiles the maximum sensitivity occurs at or near the gun. For high drag projectiles, the point of maximum sensitivity moves out from the gun, but the curve eventually falls to zero at the target for any level of drag. The general shapes of $\mu(X_1)$ for high and low drag projectiles are shown in Figure 4.

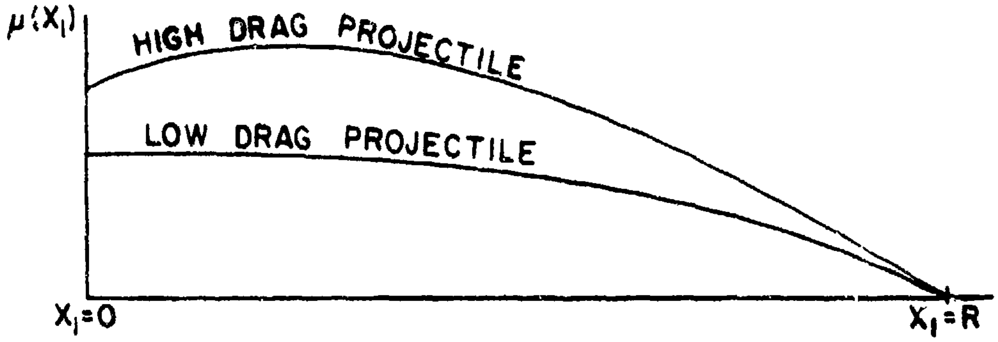

FIGURE 4

To further illustrate the method of application of Equation (21), the effect of two hypothetical variable crosswinds on the trajectory of the 7.62mm M80 bullet will be calculated. The crosswinds are shown in Figure 5.

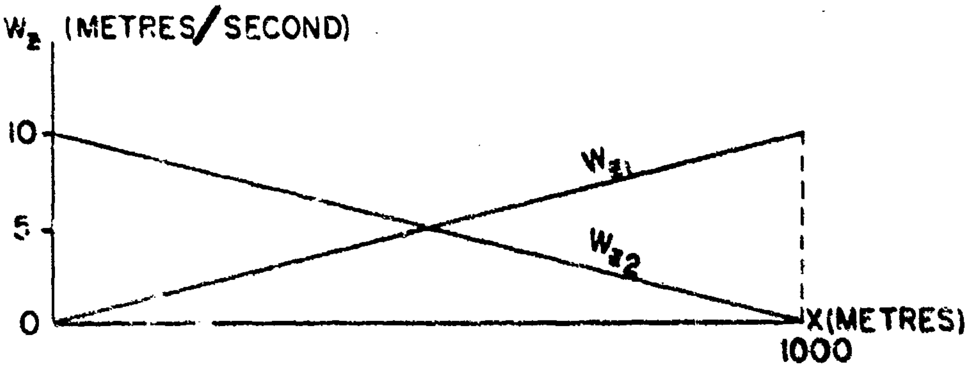

FIGURE 5

The first crosswind, W_{z_1}, is zero at the gun, but has grown to 10 metres/second at the target; the second, W_{z_2}, is 10 metres/second at the gun, and decreases to no wind at target. Both winds average 5 metres/second over the whole range, but the second crosswind, W_{z_2}, will have a larger effect on the trajectory.

Standard (no-wind) and the variable crosswind cases were solved by numerical integration of Equations (6), and the numerical solutions are taken to be correct. For application of Equation (21), the distance to target (1000 metres) was divided into 100 metre intervals; the down-range distance, velocity and time for the no-wind case are listed in Table 3. The lag time for each distance and the sensitivity, μ, and average crosswind speed for each interval were then computed, and the total deflection at the target determined as the sum of deflections produced over each interval.

TABLE 3

7.62mm NATO Bullet (M80), with Variable Crosswinds

X_1 (M)	V_x (M/Sec)	t (Sec)	$\left[t(R)-t(X_1) - \frac{(R-X_1)}{V_{x_1}}\right]$	$\bar{\mu}$	$\bar{W}_{z_1}$	$\bar{\mu}\bar{W}_{z_1}$	$\bar{W}_{z_2}$	$\bar{\mu}\bar{W}_{z_2}$
0	839.60	0	.9678					
				.1109	0.5	.0555	9.5	1.0536
100	764.62	.1248	.8569					
				.1147	1.5	.1721	8.5	.9750
200	693.03	.2622	.7422					
				.1187	2.5	.2968	7.5	.8903
300	624.37	.4142	.6235					
				.1227	3.5	.4295	6.5	.7976
400	558.40	.5835	.5008					
				.1247	4.5	.5612	5.5	.6859
500	495.51	.7736	.3761					
				.1229	5.5	.6760	4.5	.5531
600	436.24	.9887	.2542					
				.1155	6.5	.7508	3.5	.4043
700	381.12	1.2339	.1577					
				.0908	7.5	.6810	2.5	.2270
800	334.97	1.5148	.0469					
				.0381	8.5	.3239	1.5	.0572
900	309.27	1.8267	.0088					
				.0088	9.5	.0836	0.5	.0044
1000	293.62	2.1588	0					

$\Sigma\bar{\mu}\bar{W}_{z_1} = 4.030M$ $\Sigma\bar{\mu}\bar{W}_{z_2} = 5.648M$

The numerical solution gives $Z(1000M) = 4.022M$ for W_{z_1} and $Z(1000M) = 5.657M$ for W_{z_2}. (A constant W_z of 5 metres/second would have produced 4.839 metres deflection, from Table 1.) The small

discrepancies in the results of Equation (21) are due to the rather coarse computing interval of 100 metres; however, for a practical calculation, the result shows that an even larger interval could have been used with satisfactory accuracy for field purposes.

IV. EFFECT OF RANGE WIND ON THE TRAJECTORY

We assume no crosswind, and a constant range wind, W_x, acting over the entire trajectory. The first and second equations of (13) are

$$\dot{V}_x = - C_D^* (V_x - W_x)^2$$

$$\dot{V}_y = - C_D^* (V_x - W_x) V_y - g$$

If the independent variable is changed to $X = \int_o^t V_x dt$ in the first equation of (13),

$$V_x' + C_D^* V_x = C_D^* W_x \left(2 - \frac{W_x}{V_x}\right)$$

We now assume that V_x is at least two orders of magnitude larger than W_x, and our differential equation for downrange velocity simplifies to

$$V_x' + C_D^* V_x = 2C_D^* W_x \tag{24}$$

At $t = 0$, $X = 0$, $V_x = V_{x_o}$, and the general solution is

$$V_x = V_{x_o} e^{-\int_o^X C_D^* dr_1} + 2W_x \left[1 - e^{-\int_o^X C_D^* dr_1}\right] \tag{25}$$

The first term on the right-hand side of (25) is the drag-induced velocity decay experienced by the projectile in the absence of range wind. The second term shows that a tail wind ($+W_x$) adds an increment of downrange velocity whose magnitude is zero at the gun, and increases to <u>twice</u> the wind speed at very long ranges! [A head wind ($-W_x$) would subtract an equivalent downrange velocity increment.]

For a given range, R, we now denote values of velocity, time and height for a trajectory with range wind as $[V_x]$, $[V_y]$, $[t]$, $[Y]$; the

same symbol without the bracket will mean the equivalent value along a no-wind trajectory.

If W_x is identically zero, $V_x = V_{x_o} e^{-\int_o^X C_D^* \, dr_1}$, and we immediately have an expression for $[V_x]$.

$$[V_x] = V_x + 2W_x \left[1 - \frac{V_x}{V_{x_o}}\right] \qquad (26)$$

To find the effect of a range wind on time of flight to range X, we need the average downrange velocity for the wind case.

$$[V_x]_{(AV)} = \frac{1}{X} \int_o^X [V_x] dr_1 = \frac{1}{X} \left[\int_o^X V_x dr_1 + 2W_x X - \frac{2W_x}{V_{x_o}} \int_o^X V_x dr_1 \right]$$

But $\frac{1}{X} \int_o^X V_x dr_1 = V_{x_{(AV)}}$, and we find

$$[V_x]_{(AV)} = V_{x_{(AV)}} + 2W_x \left[1 - \frac{V_{x_{(AV)}}}{V_{x_o}}\right] \qquad (27)$$

Since $t = \frac{R}{V_{x_{(AV)}}}$, and $[t] = \frac{R}{[V_x]_{(AV)}}$,

$$\frac{1}{[t]} = \frac{1}{t} \left[1 - \frac{2W_x}{V_{x_o}}\right] + \frac{2W_x}{R}, \text{ or}$$

$$[t] = t / \left[1 + 2W_x \left(\frac{t}{R} - \frac{1}{V_{x_o}}\right)\right] \qquad (28)$$

Thus a tail wind results in a lower time of flight to a fixed range, as we would expect.

If the independent variable is changed to X in the second equation of (13),

$$V_y' + C_D^* \left(1 - \frac{W_x}{V_x}\right) V_y = - \frac{g}{V_x} \qquad (29)$$

Since V_x is assumed to be much larger than W_x, this equation reduces to

$$V_y' + C_D^* V_y = -\frac{g}{V_x} \qquad (30)$$

The general solution for the range wind case is

$$[V_y] = V_{y_o} e^{-\int_o^X C_D^* dr_1} - g e^{-\int_o^X C_D^* dr_1} \left[\int_o^X \frac{1}{[V_x]} e^{\int_o^{r_1} C_D^* dr_2} dr_1\right] \qquad (31)$$

But $V_x = V_{x_o} e^{-\int_o^X C_D^* dr_1}$, and we get the result

$$[V_y] = \left(\frac{V_{y_o}}{V_{x_o}}\right) V_x - g V_x \left[\int_o^X \frac{dr_1}{V_x [V_x]}\right] \qquad (32)$$

Now $V_x [V_x] = V_x^2 \left[1 + \frac{2W_x}{V_x} - \frac{2W_x}{V_{x_o}}\right]$, and since terms of order $\frac{W_x}{V_x}$ can be neglected, $V_x [V_x] \approx V_x^2$. Hence $V_y = \left(\frac{V_{y_o}}{V_{x_o}}\right) V_x - g V_x \int_o^X \frac{dr_1}{V_x^2}$, and we find

$$[V_y] \approx V_y \qquad (33)$$

Equation (33) tells us that for small ratio of range wind speed to downrange projectile speed, a range wind does not affect the vertical component of velocity at a given range. Hence, at a fixed range R to target the impact height on the target for the range wind case is simply the impact height for the no-wind case at time [t].

$$[Y]_{(R)} = Y_{[t]} \qquad (34)$$

For small ([t] - t), Equation (34) gives

$$[Y]_{(R)} = Y_{(R)} + V_{y_{(R)}} ([t] - t) \qquad (35)$$

Substituting for [t] from Equation (28) and simplifying gives the alternate form,

$$[Y]_{(R)} = Y_{(R)} - V_{y_{(R)}} \left[\frac{2W_x t \left(t - \frac{R}{V_{x_o}}\right)}{R + 2W_x \left(t - \frac{R}{V_{x_o}}\right)}\right] \qquad (36)$$

Equation (35) or (36) tells us that a constant tailwind acting over the entire range will decrease the trajectory height for distances shorter than the point of maximum ordinate, and increase the height for ranges beyond the point of maximum ordinate; i.e., a tailwind produces a "flatter" trajectory. A constant headwind would obviously have the reverse effect.

A method for determining the effect of a variable range wind is immediately suggested by noting the similarity of Equation (24) and the second equation of (15). Consider a constant range wind commencing at $X = X_1$, and continuing down range. Integrating Equation (24) with constant range wind W_{x_1}, starting at $X = X_1$, we find

$$[V_x] = V_{x_o} e^{-\int_o^X C_D^* \, dr_1} + 2W_{x_1} e^{-\int_o^X C_D^* \, dr_1} \int_{X_1}^X C_D^* \, e^{\int_o^{r_1} C_D^* \, dr_2} dr_1 \qquad (37)$$

Since $V_x = V_{x_o} e^{-\int_o^X C_D^* \, dr}$,

$$[V_x] = V_x + 2W_{x_1} \left[1 - \frac{V_x}{V_{x_1}}\right] , \; X > X_1 \qquad (38)$$

If a constant range wind $[-\overline{W}_{x_{(1,2)}}]$ commencing at $X = X_2$ is added to a constant range wind $\overline{W}_{x_{(1,2)}}$ commencing at $X = X_1$, the net range wind acting on the trajectory is similar to the crosswind of Figure 5. The effect on $[V_x]$ of $\overline{W}_{x_{(1,2)}}$ acting from $X = X_1$ to $X = X_2$ is

$$[V_x] = V_x + 2\overline{W}_{x_{(1,2)}} \left[\left(1 - \frac{V_x}{V_{x_1}}\right) - \left(1 - \frac{V_x}{V_{x_2}}\right)\right] \quad \text{for } R \geqslant X_2 \qquad (39)$$

To determine the point along the trajectory at which the downrange velocity is most sensitive to a range wind, we will follow an argument similar to that used in deriving Equation (22).

Let $\nu_{(1,2)} = [1 - \frac{V_x}{V_{x_1}}] - [1 - \frac{V_x}{V_{x_2}}]$, so that $\nu_{(1,2)}$ is the sensitivity of $[V_x]$ to a unit range wind acting over the interval $X = X_1$ to $X = X_2$. Let $X_2 = X_1 + \Delta X$, where ΔX is a very small distance.

$$\nu(X_1) = \frac{V_x}{V_{x_2}} - \frac{V_x}{V_{x_1}}$$

In deriving Equation (22), we showed that $V_{x_2} \approx V_{x_1}$ $[1 - u]$, and so we find

$$\nu(X_1) \approx u\,[\frac{V_x}{V_{x_1}}] \tag{40}$$

Since the maximum value of V_x occurs at the gun, and the minimum value is at the target, the interpretation of Equation (40) is somewhat easier than that of Equation (22). Equation (40) says the maximum sensitivity of downrange velocity to range wind occurs at the target, and the minimum sensitivity occurs at the gun! There is no value of X_1 for which $\nu(X_1)$ is zero; however, Equation (40) suggests that variation in range wind along the trajectory is much less important than an equivalent variation in crosswind.

Finally, we will show that the effect on a flat-fire trajectory of any range wind is at least an order of magnitude less than the effect of an identical crosswind. This fact, plus the conclusions reached from Equation (40), will allow us to dispose of the range wind problem.

From Equation (18) we recall that the effect of a constant crosswind, W, acting over the entire range is to produce a horizontal deflection, $\Delta Z = W(t - \frac{R}{V_{x_o}})$. The effect of an equal range wind on impact height is $\Delta Y = [Y] - Y = -V_{Y(R)}\ [\frac{2Wt\ (t - \frac{R}{V_{x_o}})}{X + 2W\ (t - \frac{R}{V_{x_o}})}]$.

Eliminating $W\ (t - \frac{R}{V_{x_o}})$ between these expressions, and taking the absolute value of ΔY,

$$|\Delta Y| = V_{Y_{(R)}} \left[\frac{2t\Delta Z}{R+2\Delta Z}\right] \tag{41}$$

Now, t is approximately $\left(\frac{R}{V_{X_o}}\right)$, and ΔZ is at least two orders of magnitude less than R. Thus

$$\left|\frac{\Delta Y}{\Delta Z}\right| \sim \frac{2V_{Y_{(R)}}}{V_{X_o}} \tag{42}$$

For a flat-fire trajectory, V_y is restricted to be an order of magnitude less than V_x, hence ΔY cannot exceed 20% of ΔZ. Thus, for practical purposes the rifleman is justified in neglecting the effects of range winds.

As a final check on the validity of Equation (36), an additional trajectory for the 7.62mm M80 bullet was calculated, with a constant 5 metres/second tailwind. For a no-wind trajectory, zeroed at 1000 metres range ($Y_{1000 \text{ metres}} = 0$), the value of V_y at 1000 metres is -10.02 metres/second, and the other required values from the no-wind case are given in Table 3 of Section III.

$$[Y]_{(1000 \text{ metres})} = 0 + 10.02 \left[\frac{2(5)(2.1588)(.9678)}{1000 + 2(5)(.9678)}\right] = 0.207 \text{ metre}$$

The numerical solution gives $[Y]_{(1000 \text{ metres})} = .221$ metre and demonstrates the accuracy of Equation (36). Since a constant 5 metres/second crosswind would have produced a deflection of 4.839 metres at the same range, it is obvious that the rifleman should concentrate his attention on the crosswind and disregard the range wind.

V. CONCLUSIONS

The basic differential equations of motion for a projectile acted on by aerodynamic drag and gravity are derived and linearized by means of the flat-fire assumption. The linearized equations are then used to find the effects of generalized range winds and cross winds on flat-fire trajectories.

The classical crosswind deflection formula is found to be correct. An extension of the classical formula is derived for non-constant

crosswinds, and a method for determining the effect of any given crosswind is demonstrated.

The effect of range wind on a flat-fire trajectory is shown to be insignificant in comparison with the crosswind effect.

LIST OF SYMBOLS

C_D	Drag coefficient
C_D^*	Retardation factor ($C_D^* = \rho S C_D / 2m$)
D	Drag force, opposed to velocity
g	Acceleration due to gravity
m	Mass of projectile
r	Dummy variable of integration
R	Range to target
s	Arc length along trajectory
S	Reference area of projectile
t	Time of flight
u	Dimensionless retardation factor ($u = C_D^* \Delta X$)
V	Velocity of projectile
V_x, V_y, V_z	Components of projectile velocity along the coordinate axes
$\tilde{V}$	Speed of projectile with respect to air
W	Wind velocity
W_x, W_y, W_z	Components of wind velocity along the coordinate axes
X, Y, Z	Orthogonal coordinate system (earth-fixed)
$\Delta(\ )$	An incremental change in the quantity ( )
α	Crosswind sensitivity factor
ω	Range wind sensitivity factor
ρ	Density of air

LIST OF SYMBOLS (continued)

$(\dot{\ })$	Derivative of ( ) with respect to time
$(\ )'$	Derivative of ( ) with respect to distance
$[(\ )]$	Indicates the value of ( ) for a trajectory with a range wind

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