Problem set 2

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Part 1
A competitive equilibrium for this economy consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for the firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ and the household $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ such that

1. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative firm $\{k_t^d, l_t^d, y_t\}_{t=0}^{\infty}$ solves

$$\pi = \max_{\{y_t,k_t,l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t l_t)$$
$$\text{s.t} \quad y_t = F(k_t, l_t) \quad \forall t \geq 0 \quad \text{(F is homogeneous of degree 1)}$$
$$y_t, k_t, l_t \geq 0$$

2. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative household $\{c_t, i_t, x_{t+1}, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t,i_t,x_{t+1},k_t,l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \chi\frac{l_t^{1+\eta}}{1+\eta}\right]$$
$$\text{s.t} \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t l_t) + \pi$$
$$x_{t+1} = (1-\delta)x_t + i_t \quad \forall t \geq 0$$
$$0 \leq l_t \leq 1, \quad 0 \leq k_t \leq x_t \quad \forall t \geq 0$$
$$c_t, x_{t+1} \geq 0 \quad \forall t \geq 0$$
$$x_0 = k_0 \quad \text{given}$$

3. Market clearing conditions:

$$y_t = c_t + i_t \quad \text{(Goods market)}$$
$$l_t^d = l_t^s \quad \text{(Labor market)}$$
$$k_t^d = k_t^s \quad \text{(Capital Services Market)}$$

Part 2 (Assume Cobb-Douglas production function: $y_t = z k_t^{\alpha} l_t^{1-\alpha}$)
By solving firm's problem, we have:

$$r_t = F'_{k_t}(k_t, n_t) = z\alpha\left(\frac{k_t}{l_t}\right)^{\alpha-1}, \quad w_t = F'_{l_t}(k_t, l_t) = z(1-\alpha)\left(\frac{k_t}{l_t}\right)^{\alpha}$$

Solve the household's problem: (recall $\pi = 0$ as a result of the firm's problem)

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \chi\frac{l_t^{1+\eta}}{1+\eta}\right] + \lambda\left[\sum_{t=0}^{\infty} p_t\left(r_t k_t + w_t l_t - c_t - k_{t+1} + (1-\delta)k_t\right)\right]$$

Taking FOCs:

$$\begin{aligned} c_t: &\quad \beta^t c_t^{-\sigma} = \lambda p_t && (1) \\ l_t: &\quad \beta^t \chi l_t^{\eta} = \lambda p_t w_t && (2) \\ k_{t+1}: &\quad p_t = p_{t+1}(1-\delta + r_{t+1}) && (3) \end{aligned}$$

From (1) and (2):

$$\frac{c_t^{-\sigma}}{\chi l_t^{\eta}} = \frac{1}{w_t} = \frac{1}{z(1-\alpha)k_t^{\alpha} l_t^{-\alpha}} \tag{4}$$

From (1) and (3):

$$\frac{1}{\beta}\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma} = 1-\delta + r_{t+1} = 1-\delta + z\alpha k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} \tag{5}$$

In steady state, we have:

$$\begin{aligned} &c_t = c_{t+1} = c \\ \Rightarrow &\frac{1}{\beta} = 1-\delta + z\alpha k^{\alpha-1} l^{1-\alpha} \\ \Rightarrow &\frac{k}{l} = \left[\frac{1-\beta+\beta\delta}{\beta z\alpha}\right]^{\dfrac{1}{\alpha-1}} && (6) \end{aligned}$$

From the resource constraint, we get:

$$c = Ak^{\alpha} l^{1-\alpha} - k + (1-\delta)k \Rightarrow \frac{c}{l} = z\left(\frac{k}{l}\right)^{\alpha} - \delta\left(\frac{k}{l}\right) \tag{7}$$

From (4), we have:

$$\begin{aligned} &\chi l^{\eta} = c^{-\sigma} z(1-\alpha)\left(\frac{k}{l}\right)^{\alpha} \\ \Rightarrow &\chi l^{\eta+\sigma} = \left(\frac{c}{l}\right)^{-\sigma} z(1-\alpha)\left(\frac{k}{l}\right)^{\alpha} \\ \Rightarrow &\chi l^{\eta+\sigma} = \left[z\left(\frac{k}{l}\right)^{\alpha} - \delta\left(\frac{k}{l}\right)\right]^{-\sigma} z(1-\alpha)\left(\frac{k}{l}\right)^{\alpha} \\ \Rightarrow &l = \left[\frac{1}{\chi}\left[z\left(\frac{k}{l}\right)^{\alpha} - \delta\left(\frac{k}{l}\right)\right]^{-\sigma} z(1-\alpha)\left(\frac{k}{l}\right)^{\alpha}\right]^{\dfrac{1}{\eta+\sigma}} \end{aligned}$$

From the firm's problem:

$$r_t = z\alpha\left(\frac{k}{l}\right)^{\alpha-1}; \quad w_t = z(1-\alpha)\left(\frac{k}{l}\right)^{\alpha}$$

Part 3
The social planner's problem for this economy is

$$\begin{aligned} w(\overline{k_0}) &= \max_{\{c_t,k_t,l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t, l_t) \\ \text{s.t} \quad F(k_t, l_t) &= c_t + k_{t+1} - (1-\delta)k_t \\ delc_t \geq 0, \quad k_t &\geq 0, \quad 0 \leq n_t \leq 1 \\ k_0 &\leq \overline{k_0} \quad \text{given} \end{aligned}$$

We can rewrite the social planner's problem as

$$\begin{aligned} & w(\overline{k_0}) = \max_{\{k_{t+1},l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(F(k_t, l_t) + (1-\delta)k_t - k_{t+1}, l_t) \\ & \text{s.t} \quad 0 \leq k_{t+1} \leq F(k_t, l_t) + (1-\delta)k_t, \quad 0 \leq l_t \leq 1 \\ & \qquad k_0 = \overline{k_0} \quad \text{given} \end{aligned}$$

Bellman equation:

$$v(k) = \max_{\{0 \leq k' \leq F(k,l)+(1-\delta)k, l\}} U(F(k,l) + (1-\delta)k - k', l) + \beta v(k')$$

Part 4 (See the codes)
Assume $\beta = 0.95$ and $\delta = 1$. From the result in part 2, we have $\chi = 39.6426$.
Part 5