Computational Methods
Problem Set 1

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Part 1

A competitive equilibrium for this economy consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for the firm $\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}$ and the household $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ such that

1. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative firm $\{k_t^d, n_t^d, y_t\}_{t=0}^{\infty}$ solves

$$\pi = \max_{\{y_t, k_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t n_t)$$
$$\text{s.t} \quad y_t = F(k_t, n_t) \quad \forall t \geq 0 \quad \text{(F is homogeneous of degree 1)}$$
$$y_t, k_t, n_t \geq 0$$

2. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative household $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t, i_t, x_{t+1}, k_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - n_t)$$
$$\text{s.t} \quad \sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t n_t) + \pi$$
$$x_{t+1} = (1-\delta)x_t + i_t = i_t \quad (\delta = 1) \quad \forall t \geq 0$$
$$0 \leq n_t \leq 1, \quad 0 \leq k_t \leq x_t \quad \forall t \geq 0$$
$$c_t, x_{t+1} \geq 0 \quad \forall t \geq 0$$
$$x_0 = k_0 \quad \text{given}$$

3. Market clearing conditions:

$$y_t = c_t + i_t \quad \text{(Goods market)}$$
$$n_t^d = n_t^s \quad \text{(Labor market)}$$
$$k_t^d = k_t^s \quad \text{(Capital Services Market)}$$

Part 2

The social planner's problem for this economy is

$$w(\overline{k_0}) = \max_{\{c_t, k_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - n_t)$$
$$\text{s.t} \quad F(k_t, n_t) = c_t + k_{t+1}$$
$$c_t \geq 0, \quad k_t \geq 0, \quad 0 \leq n_t \leq 1$$
$$k_0 \leq \overline{k_0} \quad \text{given}$$

Part 3
Since an allocation that maximizes the utility of the representative agent, subject to the technology constraint is a Pareto efficient allocation (by Bolzano - Weierstrass theorem), every Pareto efficient allocation is the solution to the social planner's problem.
Now we need to establish that the competitive equilibrium constitutes a Pareto efficient allocation. By the first welfare theorem, a competitive equilibrium allocation is Pareto efficient. Hence, the equilibrium allocation of consumption, capital, and labor coincides with those of the planner's.
Part 4
We can rewrite the social planner's problem as

$$
\begin{aligned}
w(\overline{k_0}) &= \max_{\{k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(F(k_t, n_t) - k_{t+1}, 1 - n_t) \\
\text{s.t} \quad & 0 \le k_{t+1} \le F(k_t, n_t), \quad 0 \le n_t \le 1 \\
& k_0 = \overline{k_0} \quad \text{given}
\end{aligned}
$$

Bellman equation:

$$
v(k) = \max_{\{0 \le k' \le F(k,n), n\}} U(F(k, n) - k', 1 - n) + \beta v(k')
$$

in which the state variable is k and the control variables are k' and n.
Part 5
Since the consumer does not derive utility from leisure, the optimal labor supply is $n = 1$. Hence, we have $F(k, n) = F(k, 1) = zk^\alpha$. Hence, the social planner's problem is:

$$
\begin{aligned}
w(\overline{k_0}) &= \max_{\{c_t, k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t) \\
\text{s.t} \quad F(k_t) &= c_t + k_{t+1} \\
c_t &\ge 0, \quad k_t \ge 0 \\
k_0 &\le \overline{k_0} \quad \text{given}
\end{aligned}
$$

We have the corresponding Bellman equation:

$$
v(k) = \max_{0 \le k' \le f(k)} \log(zk^\alpha - k') + \beta v(k') \tag{1}
$$

Guess $v(k) = A + B\log(k)$, the FOC yields

$$
\begin{aligned}
\frac{-1}{zk^\alpha - k'} + \frac{\beta B}{k'} &= 0 \\
\Rightarrow k' &= \frac{\beta B z k^\alpha}{1 + \beta B} \qquad (2)
\end{aligned}
$$

Substitute back into (1), we have:

$$
\begin{aligned}
A + B\log(k) &= \log(zk^\alpha - \frac{\beta B z k^\alpha}{1 + \beta B}) + \beta(A + B\log(\frac{\beta B z k^\alpha}{1 + \beta B})) \\
\Rightarrow A + B\log(k) &= \log(zk^\alpha) - \log(1 + \beta B) + \beta A + \beta B\log(\frac{\beta B z}{1 + \beta B}) + \beta B\alpha\log(k) \\
\Rightarrow A + B\log(k) &= \log(z) - \log(1 + \beta B) + \beta A + \beta B\log(\frac{\beta B z}{1 + \beta B}) + \alpha\log(k) + \beta B\alpha\log(k)
\end{aligned}
$$

$$\Rightarrow B = \alpha + \beta B \alpha \Rightarrow B = \frac{\alpha}{1-\beta\alpha}$$

$$\Rightarrow A = \frac{1}{1-\beta}\left[\log(z(1-\alpha\beta)) + \frac{\beta\alpha}{1-\beta\alpha}\log(\beta\alpha z)\right]$$

Substitute B into (2), we have:

$$k' = \beta\alpha z k^{\alpha}$$

Part 6
Solve for the steady state value of capital:

$$k = \beta\alpha z k^{\alpha}$$
$$\Rightarrow k = (\beta\alpha z)^{\frac{1}{1-\alpha}}$$

The steady state value of y: $y = F(k) = zk^{\alpha} = z(\beta\alpha z)^{\frac{\alpha}{1-\alpha}}$
The steady state value of r:

$$r = F_k(k,n) = \alpha z k^{\alpha-1} = \alpha z(\beta\alpha z)^{\frac{\alpha-1}{1-\alpha}} = \frac{1}{\beta}$$

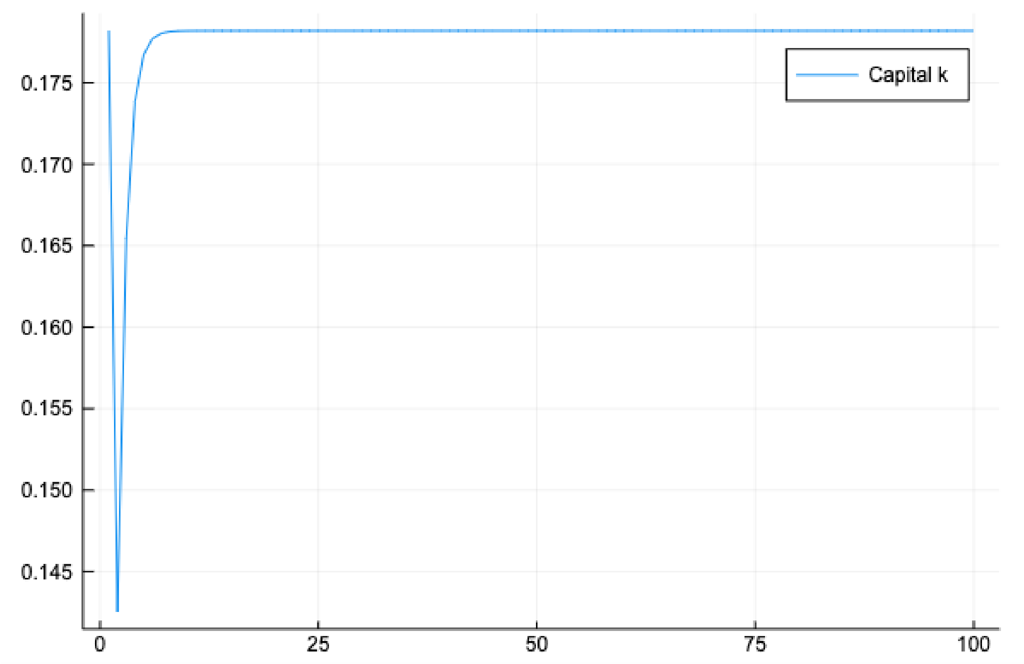
The steady state value of w:

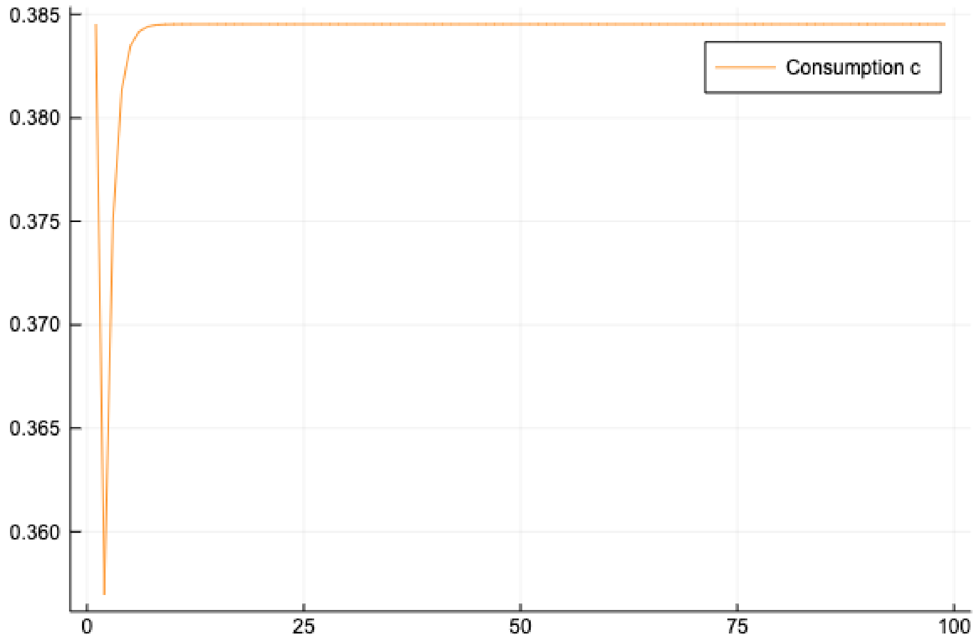
$$w = F_n(k,n) = (1-\alpha)zk^{\alpha} = (1-\alpha)z(\beta\alpha z)^{\frac{\alpha}{1-\alpha}}$$

The steady state value of c:

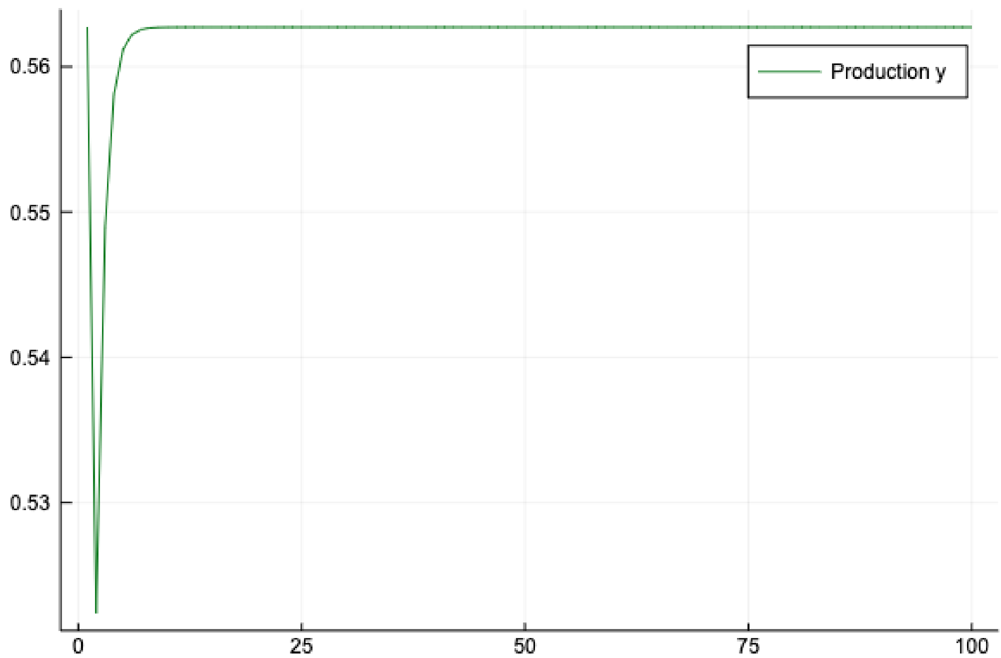
$$c = F(k) - k = F(k) - k = z(\beta\alpha z)^{\frac{\alpha}{1-\alpha}} - (\beta\alpha z)^{\frac{1}{1-\alpha}}$$

Part 7
(a) Capital decreases to 80 % of its steady state value

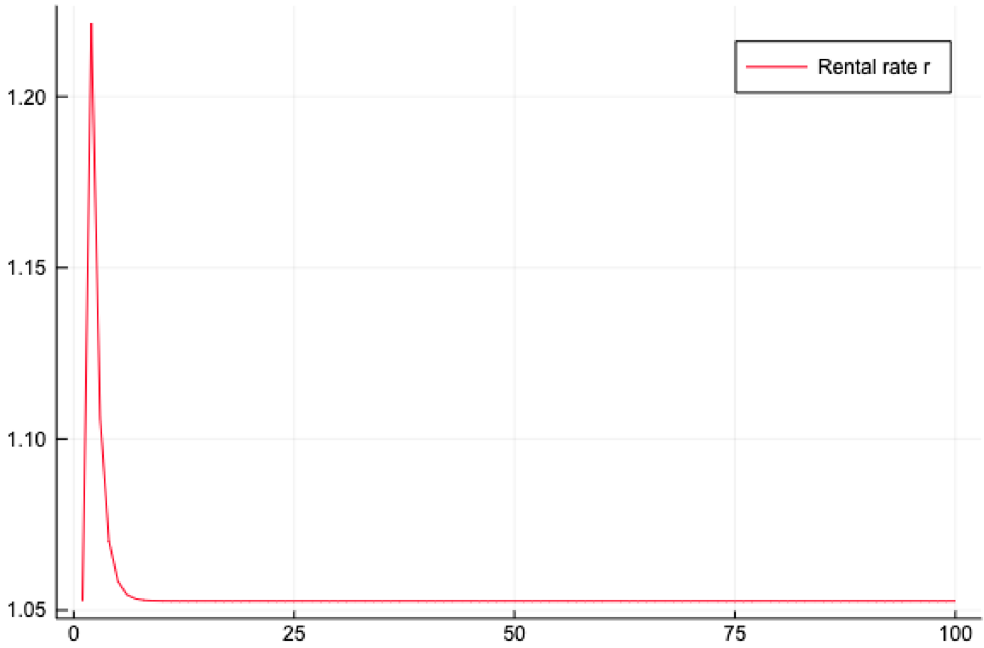


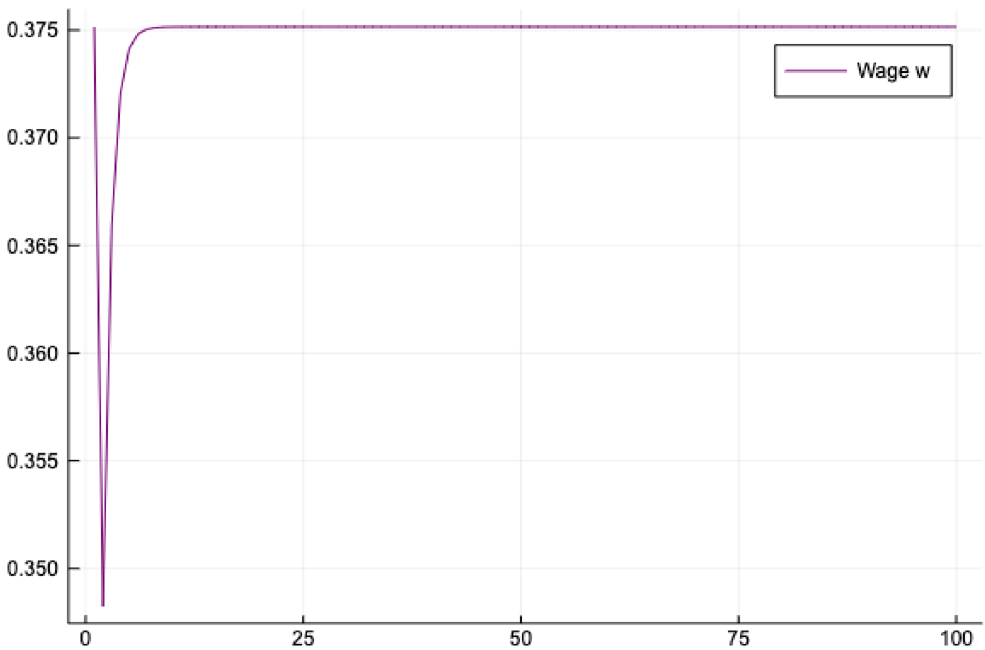


0.385
0.380
0.375
0.370
0.365
0.360
0
25
50
75
100
Consumption c

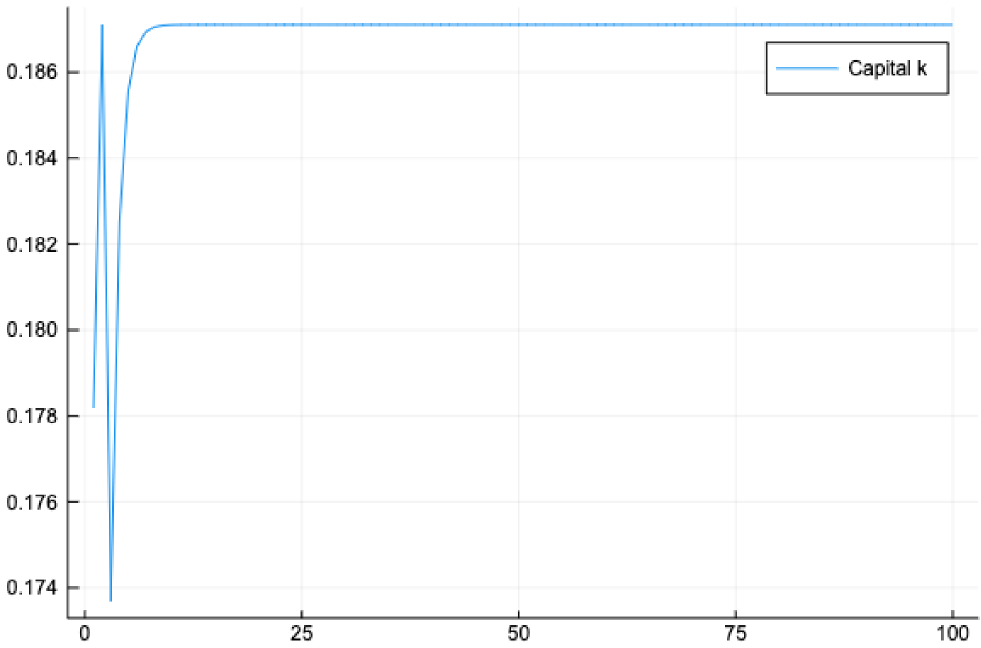


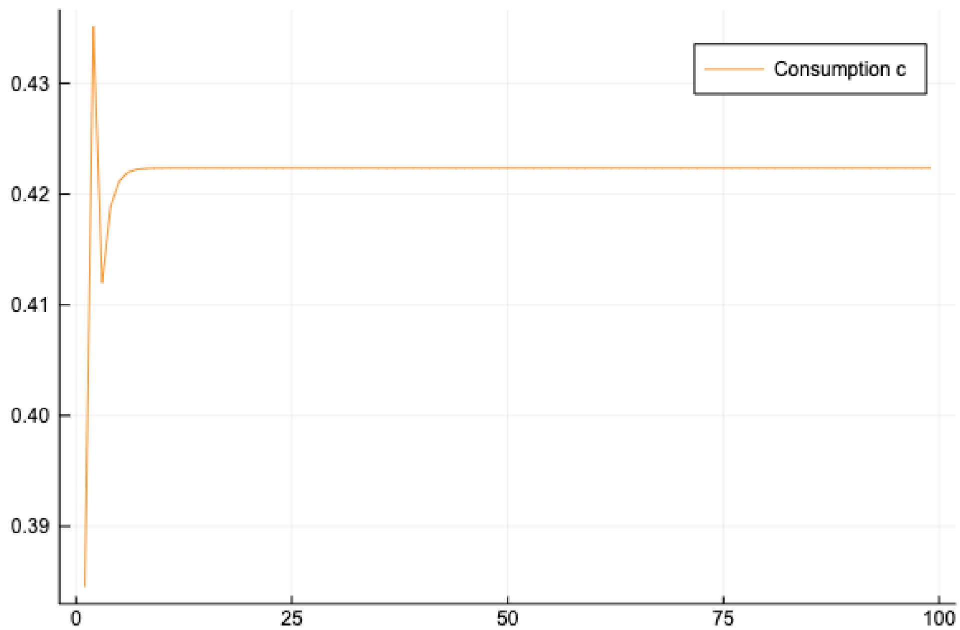
0.56
0.55
0.54
0.53
0
25
50
75
100
Production y

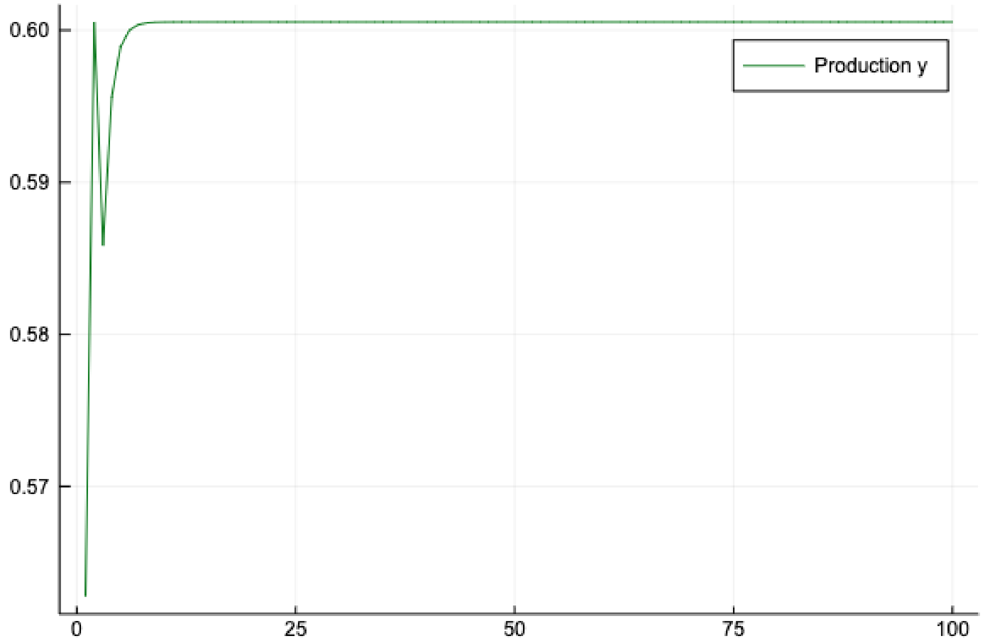

Rental rate r
1.20
1.15
1.10
1.05
0
25
50
75
100

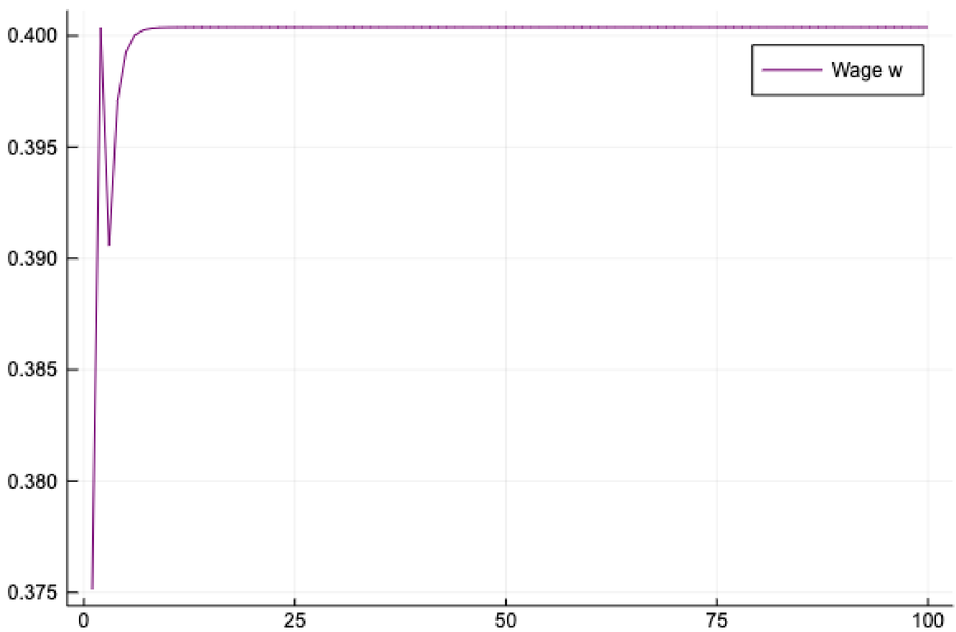

Wage w
0.375
0.370
0.365
0.360
0.355
0.350
0
25
50
75
100

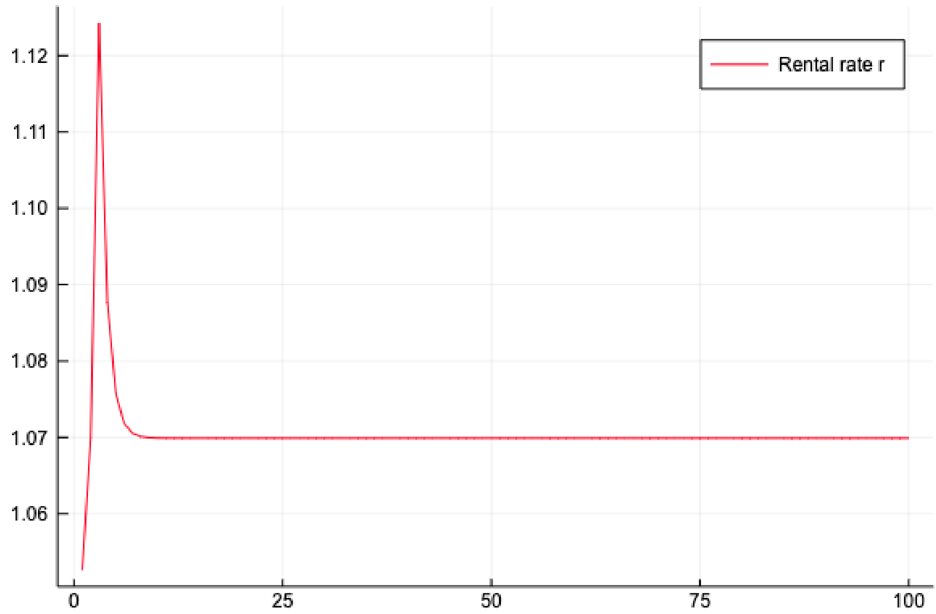
(b) Productivity increases permanently by 5%






Production y
0.60
0.59
0.58
0.57
0
25
50
75
100


Wage w
0.400
0.395
0.390
0.385
0.380
0.375
0
25
50
75
100


1.12
1.11
1.10
1.09
1.08
1.07
1.06
0
25
50
75
100
Rental rate r