MA1126: Set Theory Selected Problems

Anthony Gibbons, Daniel Devine

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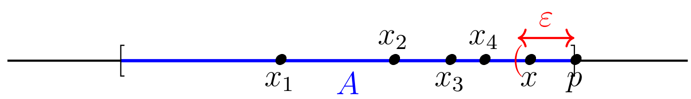
Problem 1: 2016 *Q3*

(a.) *Define an open set in $\mathbb{R}$. Define closed set.*
Solution: *Open set:*
A set $A \subset \mathbb{R}$ is called open if $\forall x \in A$, $\exists \varepsilon > 0$ s.t. $N(x, \varepsilon) \subset A$
Where $N(x, \varepsilon)$ is a neighbourhood $\{y \in \mathbb{R} \mid |x - y| < \varepsilon\}$
Closed set:
A set $A \subset \mathbb{R}$ is closed if $\mathbb{R} \backslash A$ is open, where $\mathbb{R} \backslash A$ is the complement of A in $\mathbb{R}$ $\square$
Prove that A is closed $\iff$ it contains all of its accumulation points.
Solution:
$\Rightarrow$
We need to show that no accumulation points of A are contained in $\mathbb{R} \backslash A$
Accumulation Point: p is an accumulation point of A if
$\forall \varepsilon > 0 \ \exists \ x \in A \backslash \{p\}$ *s.t.* $|x - p| < \varepsilon$



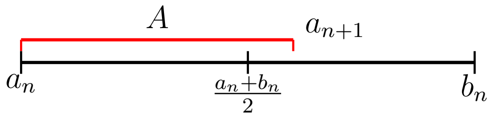
i.e. $\exists$ an infinite sequence $x_1, x_2, x_3, \ldots$ of points of A, distinct from p, that converge to p, with $\lim_{j \to \infty} x_j = p$ but all $x_j \in A$, hence $p \in A$ also
Hence A contains all of its accumulation points
A is closed $\implies$ it contains all of its accumulation points.
$\Leftarrow$ *Now assume A contains all of its accumulation points, then choose $p \in \mathbb{R} \backslash A$, so p is not an accumulation point of A.*
So by definition of accumulation points, $\exists \delta > 0$ s.t. $\{x \in A \mid 0 < |x - p| < \delta\} = \varnothing$
Hence there exists an open ball about p in $\mathbb{R} \backslash A$, so $\mathbb{R} \backslash A$ is open, $\implies$ A is closed
So A is closed $\iff$ it contains all of its accumulation points, as required. $\square$

(b.) *Define Cauchy sequence.*
Solution: *A Cauchy sequence $a_1, a_2, a_3, \ldots$ (infinite sequence) such that $\forall \varepsilon > 0 \exists N \in \mathbb{N} s.t. |a_n - a_m| < \varepsilon \forall n, m \geq N$*

$\square$

(c.) *Prove that every Cauchy Sequence converges in $\mathbb{R}$ is equivalent to the least upper bound axiom*
Solution: *Want C.S. $\implies$ L.U.B and L.U.B $\implies$ C.S. Assume Cauchy convergence.*
Let A be a nonempty subset of $\mathbb{R}$ and suppose A has an upper bound b_1

Since A *is nonempty,* $\exists a_1 \in A$ *and not an upper bound.*
Define sequences $a_1, a_2, a_3, \dots \qquad b_1, b_2, b_3, \dots$ *as follows:*
Check if $\frac{a_n+b_n}{2}$ *is an upper bound of* A*.*
If so, $a_{n+1} = a_n, \quad b_{n+1} = \frac{a_n+b_n}{2}$
Otherwise, $\exists s \in A s.t.\ s > \frac{a_n+b_n}{2}$
Let $a_{n+1} = s, b_{n+1} = b_n$



Then $a_1 \leq a_2 \leq \dots \leq b_2 \leq b_1$ *and* $|b_n - a_n| \to 0$ *as* $n \to \infty$
$\implies$ *Both sequences are Cauchy* $\therefore$ *they converge towards some limit L*
$\implies$ *L is the least upper bound for s* ✓
Note: Every Cauchy sequence is bounded.
Proof:
Take $\varepsilon = 1$*, then*
$\exists\ N \in \mathbb{N} s.t.\ m, n \geq N \implies |x_m - x_n| < 1 \implies |x_m| - |x_n| < 1 \implies |x_m| < 1 + |x_n|$
In general, $|x_n| \leq max\{|x_1|, \dots, |x_n|, 1 + |x_n|\}$
Note also that every sequence has a monotone (strictly increasing) subsequence.

Now Assume Least upper bound axiom is true.
Suppose we have a monotone bounded sequence (increasing)
Since it is bounded, by the L.U.B axiom, then it has a supremum.
$s = \sup_{n \in \mathbb{N}} \{x_n\}$
Claim S is the limit of the sequence.
Let $\varepsilon > 0$*, then* $\exists N \in \mathbb{N}$ *s.t.* $s - \frac{\varepsilon}{2} < x_N \leq x_{N+1} \leq \dots \leq S$
$\implies \forall n \geq N$ *we have* $|x_n - s| < \frac{\varepsilon}{2}$
Now take any x_m *in the original sequence with* $m > N$
$|x_m - x_n| \leq |x_m - s| + |x_n - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
So the sequence is Cauchy ✓ $\square$

Problem 2: 2017 $Q4$

(a.) *Show that* $f : \mathbb{R} \to \mathbb{R}$ *is continuous* $\iff$ *the preimage of any open set* $U, f^{-1}(U)$ *is open*

Solution:

$\Rightarrow$

Assume f *is continuous on* $\mathbb{R}$*, then*

$\forall x_0 \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0 s.t. |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$

In other words, $\forall x_0 \in \mathbb{R}, \forall N(f(x_0), \varepsilon) \exists N(x_0, \delta) s.t. f(N(x_0, \delta)) \subset N(f(x_0), \varepsilon)$

Let U *be open in* $\mathbb{R}$

Let $x_0 \in f^{-1}(U)$

Then $f(x_0) \in U$

$\implies \exists \varepsilon > 0 s.t. N(f(x_0), \varepsilon) \subset U$

$\implies \exists N(x_0, \delta) s.t. f(N(x_0, \delta)) \subset N(f(x_0), \varepsilon) \subset U$

$\implies N(x_0, \delta) \subset f^{-1}(U)$

x_0 *is an interior point of* $f^{-1}(U) \implies f^{-1}(U)$ *is open.*

$\Leftarrow$

$f^{-1}(U)$ *for any* U *open.*

Given $N(f(x_0), \varepsilon)$*, this is open.* $\implies f^{-1}(N(f(x_0), \varepsilon))$ *is open.*

and $x_0 \in f^{-1}(N(f(x_0), \varepsilon))$ *so* x_0 *is an interior point*

$\implies \exists \delta > 0 s.t. N(x_0, \delta) \subset f^{-1}(N(f(x_0), \varepsilon))$

$\implies f(N(x_0, \delta) \subset N(f(x_0), \varepsilon) \implies f$ *is continuous.* $\square$

(b.) *Prove that the continuous image of a compact subset of R is compact.*

Solution: *Suppose* $A \subset \mathbb{R}$ *is compact, and* f *is continuous*

Suppose $f(A) \subset \bigcup_\alpha U_\alpha, U_\alpha open$

Then $A \subset f^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha f^{-1}(U_\alpha)$

By part (a.), each of the $f^{-1}(U_\alpha)$ *is open.*

And A *is compact* $\implies \exists$ *a finite subcover of* A

.e. $A \subset f^{-1}(U_{\alpha_1}) \cup f^{-1}(U_{\alpha_2}) \cup \dots \cup f^{-1}(U_{\alpha_n})$

$\implies f(A) \subset U_{\alpha_1} \cup U_{\alpha_1} \cup \dots U_{\alpha_n} \implies f(A)$ *is compact.* $\square$

(c.) *Use part (b.) to prove the Extreme Value Theorem*

Solution:

Suppose f *is continuous on* $[a, b]$,

$[a, b]$ *is compact by Heine-Borel.*

Then f([a,b]) is compact by part (b.) $\implies$ *it is closed and bounded.*

bounded $\implies$ *LUB and GLB exist*

closed $\implies$ *LUB and GLB belong to* $f([a, b])$

So $\exists x_0, x \in [a, b]$ *s.t.* $f(x_0) = \max_{x\in[a,b]} f(x) \qquad f(x_1) = \min_{x\in[a,b]} f(x)$

This is the Extreme Value Theorem. $\square$

Problem 3: 2017 $Q3$

(a.) Prove or disprove

1. $U_n open \ \forall n \geq 1 \implies \bigcap_{n\geq1} U_n$ *open.*

   Solution: $U_n open \not\Longrightarrow \bigcap_{n\geq1} U_n$ *open.*

   e.g. let $U_n = (-\frac{1}{n}, \frac{1}{n})$
   $\bigcap_{n\geq1} U_n = \{0\}$ *which is not open.* $\square$

2. U_n *open* $\forall n \geq 1 \implies \bigcup_{n\geq1} U_n$ *open.*

   Solution: *Let* $x \in \bigcup_{n\geq1} U_n \implies x \in U_{n_i}$ *for some* n_i
   $\implies \exists\varepsilon > 0$ *s.t.* $N(x, \varepsilon) \subset U_{n_i}$
   $\implies N(x, \varepsilon) \subset \bigcup_{n\geq1} U_n$
   $\implies \bigcup_{n\geq1} U_n$ *is open.*

   $\square$

3. $U_n closed \ \forall n \geq 1 \implies \bigcap_{n\geq1} U_n$ *closed.*

   Solution: U_n *closed* $\implies (U_n)^C open$
   $\implies \bigcup_{n\geq1} (U_n)^C$ *is open, from above.*
   $\implies (\bigcup_{n\geq1} (U_n)^C)^C$
   $\implies \bigcap_{n\geq1} (U_n^C)^C = \bigcap_{n\geq1} (U_n)$ *closed.* $\square$

4. $U_n closed \ \forall n \geq 1 \implies \bigcup_{n\geq1} U_n$ *closed.*

   Solution: $U_n closed \not\Longrightarrow \bigcup_{n\geq1} U_n$ *closed.*

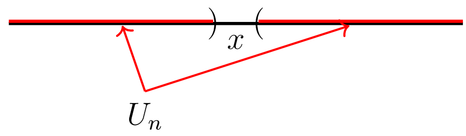
   e.g. $O_n = \{\frac{1}{n}\}$ *0 is an accumulation point not included in* $\bigcup_{n\geq1} U_n$
   $\implies$ *not closed (since a set is closed* $\iff$ *it contains all its accumulation points.*

   $\square$

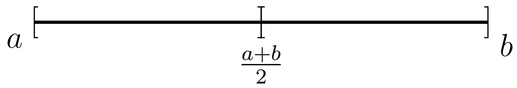
(b.) Prove a subset of R is compact $\iff$ *it is closed and bounded*
Solution:
$\Rightarrow$
Suppose $A \subset \mathbb{R}$ *Suppose it is not bounded*
Let $U_n = (-n, n)$ *Then* $A \subset \bigcup U_n = \mathbb{R}$
But a finite subcover cannot be found $\implies$ *A is not compact*
So not bounded $\implies$ *not compact*

Taking the contrapositive compact $\implies$ *bounded.*
Suppose $A \subset \mathbb{R}$ *and not closed* $\exists x$ *an accumulation point of A not in A*

Let $U_n = (-\infty, x - \frac{1}{n}) \cup (x + \frac{1}{n}, +\infty)$ *be a cover of A*



But a finite subcover cannot be found $\implies$ *A not compact*
So not closed $\implies$ *not compact*
Taking the contrapositive compact $\implies$ *closed.*
$\Leftarrow$
Now suppose A is closed and bounded.
Case 1: $A = [a, b]$
Suppose $A \subset \bigcup_\alpha U_\alpha$ *and a finite number of* U_α *don't cover* $A = I_1$



Consider $[a, \frac{a+b}{2}]$ *and* $[\frac{a+b}{2}, b]$
One of these subintervals which we will call I_2*, also does not have a finite subcover.*
Divide I_2 *in half, one of these will not have a finite subcover* $\ldots$ *etc.*
We have $I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_n \supset \ldots$ *infinite series of nested subintervals.*
with each I_n *closed and bounded interval that does not have a finite subcover of* U_α
By the Nested Intervals Theorem, $\bigcap I_n \neq \varnothing$
Let $x \in \bigcap I_n$*, so* $x \in U_{\alpha_i}$ *for some* α_i
U_{α_i} *open, then* $\exists \varepsilon > 0$ *s.t.* $N(x, \varepsilon) \subset U_{\alpha_i}$
But$|I_n| \to 0$ *and if* $|I_n| < \varepsilon$*, then* $x \in I_n \implies I_n \subset N(x, \varepsilon)$
$\implies I_n$ *has a finite subcover* $\Rightarrow\Leftarrow$ *Contradiction*
Then A is closed
Case 2:
A is closed and bounded but not necessarily an interval.
If $A \subset \bigcup U_\alpha \qquad \exists [a, b]$ *s.t.* $A \subset [a, b]$
and if we consider all U_α *and* A^C*, we have an open cover of* $[a, b]$*.*
$\implies \exists$ *a finite subcover* $U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ *(*$\cup A^C$ *possibly)*
But $A \cup A^C = \varnothing \therefore U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \supset A$ *covers A* $\square$

Problem 4: *Page* 16 *Q*48
we define $\oplus$ *to denote exclusive or, so* $p \oplus q \equiv (p \vee Q) \wedge \neg(p \wedge q)$
Hence the truth table for exclusive or is as follows:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

a. *find simpler statement forms that are logically equivalent to* $p \oplus p$ *and* $(p \oplus p) \oplus p$

Solution:

p	$(p \oplus p)$	$(p \oplus p) \oplus p$
T	F	T
F	F	F

So $(p \oplus p) \equiv$ *contradiction,* $\quad (p \oplus p) \oplus p \equiv p$ $\square$

b. *Is* $(p \oplus q) \oplus r \equiv (p \oplus q) \oplus r$*? Justify your answer.*

Solution:

p	q	r	$(p \oplus q)$	$(p \oplus q) \oplus r$	$(q \oplus r)$	$p \oplus (q \oplus r)$
T	T	T	F	T	F	T
T	T	F	F	F	T	F
T	F	T	T	F	T	F
T	F	F	T	T	F	T
F	T	T	T	F	F	F
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

So $(p \oplus q) \oplus r \equiv (p \oplus q) \oplus r$ $\square$

c. *Is* $(p \oplus q) \wedge r \equiv (p \wedge q) \oplus (q \wedge r)$*? Justify your answer.*

Solution:

p	q	r	$(p \oplus q)$	$(p \oplus q) \wedge r$	$(p \wedge r)$	$(q \wedge r)$	$(p \wedge r) \oplus (q \wedge r)$
T	T	T	F	F	T	T	F
T	T	F	F	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	F	F	F	F
F	T	T	T	T	F	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

So $(p \oplus q) \wedge r \equiv (p \wedge q) \oplus (q \wedge r)$ $\square$

Problem 5: 2018 *Q2*

(a.) *Define what it means for 2 sets to have the same cardinal number.*

Solution: *2 sets X and Y have the same cardinality if $\exists$ a bijective map $f : X \to Y$ i.e. a 1-1 correspondence.* $\square$

Define Cardinal Number.

Solution: *The cardinal number of a set X, $\#X$ is the number of elements in the set X*

e.g. $\#\{1, 2, 4, 6, 9\} = 5$ $\square$

Prove that $\mathbb{Q}$ is countable but $\mathbb{R}$ is not.

Solution:

1. *Define* $\mathbb{Q}_+ =$

$$\begin{tikzcd}
s_{1,1} \arrow[r] & s_{1,2} \arrow[dl] & s_{1,3} \arrow[r] & s_{1,4} \arrow[dl] & \cdots \\
s_{2,1} \arrow[d] & s_{2,2} \arrow[ur] & s_{2,3} \arrow[dl] & s_{2,4} & \cdots \\
s_{3,1} \arrow[ur] & s_{3,2} \arrow[dl] & s_{3,3} & s_{3,4} & \cdots \\
s_{4,1} \arrow[d] & s_{4,2} & s_{4,3} & s_{4,4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}$$

Where $s_{ij} = \frac{i}{j}, j \neq 0$, *i.e.*

$$\begin{tikzcd}
\frac{1}{1} \arrow[r] & \frac{1}{2} \arrow[dl] & \frac{1}{3} \arrow[r] & \frac{1}{4} \arrow[dl] & \cdots \\
\frac{2}{1} \arrow[d] & \frac{2}{2} \arrow[ur] & \frac{2}{3} \arrow[dl] & \frac{2}{4} & \cdots \\
\frac{3}{1} \arrow[ur] & \frac{3}{2} \arrow[dl] & \frac{3}{3} & \frac{3}{4} & \cdots \\
\frac{4}{1} \arrow[d] & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}$$

We define the list in the diagonalising order as defined in the diagram, where we skip the rational numbers already in the list, e.g. $\frac{2}{2} = \frac{1}{1}$ $\quad \frac{8}{4} = \frac{2}{1}$

Hence $\mathbb{Q}_+$ is countable. Similarly, $\mathbb{Q}_-$ is countable. Hence $\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_-$ is countable.

2. *We show below in part (c.) that $\prod_{i=1}^{\infty}\{0,1\} \sim \mathbb{P}(\mathbb{N})$ and that $\#\mathbb{P}(\mathbb{N}) > \#\mathbb{N}$*
   Claim the interval $(0,1)$ is not countable. If not, then $\mathbb{R}$ is not countable.
   Assume $\mathbb{R}$ is countable. We consider the countable list of decimal expansions

$$\begin{array}{lcccccc} a^1 = & \cdot & a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a^2 = & \cdot & a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ a^3 = & \cdot & a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\ a^4 = & \cdot & a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\ \vdots & . & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Where $0 \le a_{ij} \le 9 \ \forall\ i,j \ge 1$ Define a decimal expansion

$$b = \cdot(a_{11}+1)\ (a_{22}+1)\ (a_{33}+1)\ (a_{44}+1)\ \cdots\ (a_{nn}+1)\ \cdots$$

But $9 \to 1$ as we don't want a string of zeroes
This is a real number by completeness of the real line, This is not equal to any of the decimal expansions above since it is different to each of the a^i in the i^{th} place.
Hence $(0,1)$ is not countable $\therefore \mathbb{R}$ is not countable.

□

(b.) *State the Schroeder-Bernstein Theorem and Cantor's Theorem.*
Solution: *<u>Schroeder-Bernstein Theorem</u>*
If $\exists f : X \to Y$ one to one
If $\exists g : Y \to X$ one to one
Then $\exists h : X \to Y$ one to one and onto.
i.e. $\#X \le \#Y$ and $\#Y \le \#X \implies \#X = \#Y$

<u>Cantor's Theorem</u>
For any set A, $\#A < \#\mathbb{P}(A)$ i.e. $n < 2^n$
i.e. $\nexists f : A \to \mathbb{P}(A)$ one to one and onto. □
Prove Cantor's Theorem.
Solution:

Suppose There exists $f : A \to \mathbb{P}(A)$ one to one and onto.
Let $B = \{x \in A \mid x \notin f(x)\} \therefore B \subset A$
$B \subset A \implies B \in \mathbb{P}(A)$
f surjective $\implies \exists\ y$ s.t. $f(y) = B$
Suppose $y \in B, y \in f(y) \implies y \notin B \Rightarrow\Leftarrow$ contradiction.
Suppose $y \notin B, y \notin f(y) \implies y \in B \Rightarrow\Leftarrow$ contradiction.
$\therefore f$ does not exist. □

(c.) *Prove $\prod_{i=1}^{\infty}\{0,1\} \sim \mathbb{P}(\mathbb{N})$*
Solution: *This means there exists a bijective map $f : \prod_{i=1}^{\infty}\{0,1\} \to \mathbb{P}(\mathbb{N})$*
$\prod_{i=1}^{\infty}\{0,1\}$ is a sequence of 0's and 1's. e.g.$\{1,0,1,0,0,\dots\}$
$A \subset \mathbb{N} \to f(A) \in \prod_{i=1}^{\infty}\{0,1\}$

$$e.g.\ \{1,3,7\} = \begin{array}{cccccccccccc} \{ & 1, & 0, & 1, & 0, & 0, & 0, & 1, & 0, & 0, & \dots & \} \\ & 1 & \cancel{2} & 3 & \cancel{4} & \cancel{5} & \cancel{6} & 7 & \cancel{8} & \cancel{9}, & \dots & \end{array}$$

So the sequence $\{f(A)\}_i = \begin{cases} 1, & if\ i \in A \\ 0, & if\ i \notin A \end{cases}$

Is f bijective?

Injective: Does $f(A_1) = f(A_2) \implies A_1 = A_2$?

Let $j \in A_1 \implies f(A_1)_j = 1 = f(A_2)_j \implies j \in A_2$

So $A_1 \subset A_2$

By symmetry, $A_2 \subset A_1$ also

$\implies A_1 = A_2 \therefore$ *injective.*

Surjective: given $b = \{b_1, b_2, b_2, \dots\} \in \prod_{i=1}^{\infty}\{0,1\}$

Does there exist $A \subset \mathbb{N}$ s.t. $f(A) = b$?

We know $j \in A \iff b_j = 1$

$f(A)_j - 1 \iff j \in A \iff b_j = 1 \forall j$

$\therefore f(A)_j = b_j \forall j$

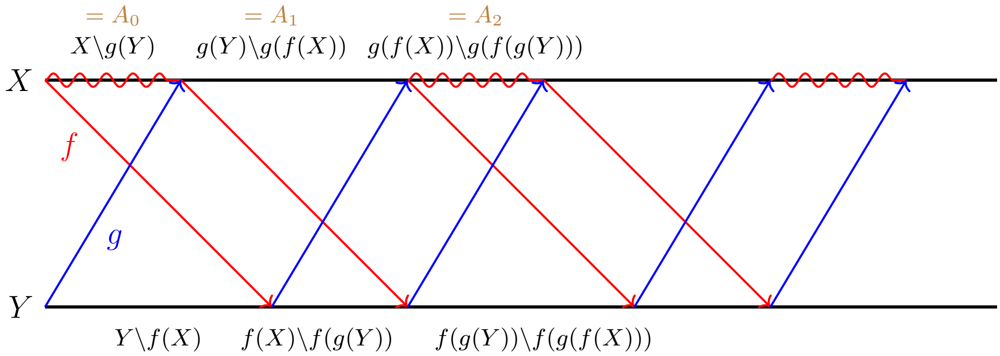
$\implies f(A) = b \therefore$ *surjective*

Hence f is bijective, and $\prod_{i=1}^{\infty}\{0,1\} \sim \mathbb{P}(\mathbb{N})$ $\square$

Extra: *Prove the Schroeder-Bernstein Theorem*

Solution:

In the below diagram we use the fact that $f(X \backslash Y) = f(X) \backslash f(Y)$



$X = X\backslash g(Y) \ \cup \ g(Y)\backslash g(f(X)) \ \cup \ g(f(X))\backslash g(f(g(Y))) \ \cup \ \dots$

$Y = Y\backslash f(X) \ \cup \ f(X)\backslash f(g(Y)) \ \cup \ f(g(Y))\backslash f(g(f(X))) \ \cup \ \dots$

$A_{n+1} = g(f(A_n))$ *and* $A_\infty = \bigcup_{n=0}^{\infty} A_n$

Define

$$h(x) = \begin{cases} f(x), & if\ x \in A_\infty \\ g^{-1}(x), & otherwise \end{cases}$$

We can easily show h is onto since all of Y is used

But is h injective?

We are also using the result that if h is injective on A_1, A_2, then h is injective on $A_1 \cup A_2$ if $f(A_1) \cap f(A_2) = \varnothing$.

Hence A_∞ is the union of disjoint A_n, and h is injective on all the A_n since f is. so h is injective on the A_∞.

Similarly h is injective $\forall x \notin A_\infty$ by injectivity of g.

So h is injective $\forall x \in X$

$\implies$ *h is bijective, as required* $\square$