STU44005: Decision Analysis (2020)

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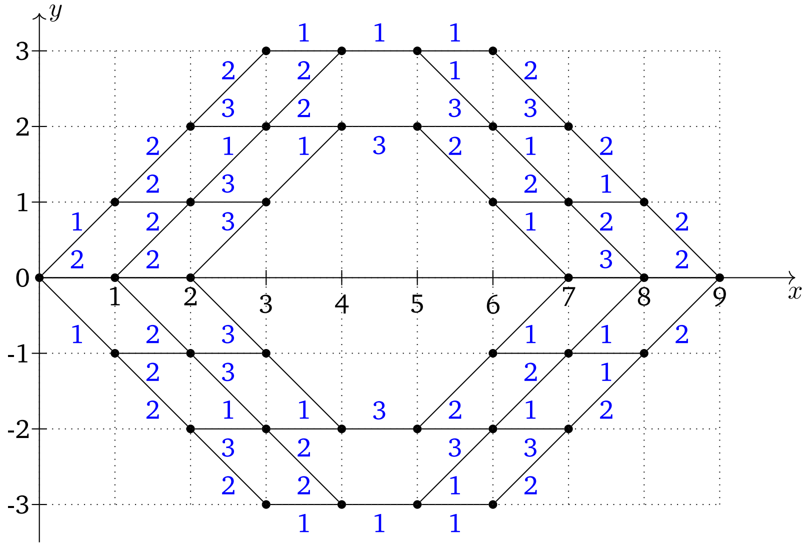
Assessment 1

Problem 1. A driver wants to go from the one side of the city to the other, avoiding the city center. In the figure below you can see all the road-system and the numbers over the arcs represent the time needed to go from a node to the other.



(α) Find the fastest paths starting from the node $(0,0)$, and ending at $(9,0)$.

Solution: We use the backward method of dynamic programming

The costs on the graph symmetric wrt the x-axis, so we need only consider the paths above the x-axis and then *mirror* the solutions.

We define the objective function $f(x,y)$ as the minimal cost from (x,y) to $(9,0)$, i.e.

$$f(x,y) := \min\{u(x,y)+f(x+1,y+1), m(x,y)+f(x+1,y), l(x,y)+f(x+1,y-1)\}$$

Where

- $u(x,y)$ the cost of moving from the node (x,y) to upper-node $(x+1,y+1)$.
- $m(x,y)$ the cost of moving from the node (x,y) to middle-node $(x+1,y)$.
- $l(x,y)$ the cost of moving from the node (x,y) to lower-node $(x+1,y-1)$.

If no such node exists for one of these costs, the value is ∞.

We have the boundary condition $f(9,0)=0$ and we wish to find $f(0,0)$

We will also use decision function $d(x,y)=$ the choice of the next node, to work out the optimal path(s).

$\underline{x=9}$ Points: $(9,0)$

$f(9,0)=0$ (Boundary Condition)

$\underline{x=8}$ Points: $(8,1),\ (8,0)$

$$\begin{aligned} f(8,1) &= \min\{u(8,1)+f(9,2), m(8,1)+f(9,1), l(8,1)+f(9,0)\} \\ &= \min\{\infty,\infty,2+0\} = 2 && d(8,1)=(9,0) \\ f(8,0) &= \min\{u(8,0)+f(9,1), m(8,0)+f(9,0), l(8,0)+f(9,-1)\} \\ &= \min\{\infty,2+0,\infty\} = 2 && d(8,0)=(9,0) \end{aligned}$$

$\underline{x=7}$ Points: $(7,2),\ (7,1),\ (7,0)$

$$\begin{aligned} f(7,2) &= \min\{u(7,2)+f(8,3), m(7,2)+f(8,2), l(7,2)+f(8,1)\} \\ &= \min\{\infty,\infty,2+2\} = 4 && d(7,2)=(8,1) \\ f(7,1) &= \min\{u(7,1)+f(8,2), m(7,1)+f(8,1), l(7,1)+f(8,0)\} \\ &= \min\{\infty,1+2,1+2\} = 3 && d(7,1)=(8,1) \text{ or } (8,0) \\ f(7,0) &= \min\{u(7,0)+f(8,1), m(7,0)+f(8,0), l(7,0)+f(8,-1)\} \\ &= \min\{\infty,3+2,\infty\} = 5 && d(7,0)=(8,0) \end{aligned}$$

$\underline{x=6}$ Points: $(6,3),\ (6,2),\ (6,1)$

$$\begin{aligned} f(6,3) &= \min\{u(6,3)+f(7,4), m(6,3)+f(7,3), l(6,3)+f(7,2)\} \\ &= \min\{\infty,\infty,2+4\} = 6 && d(6,3)=(7,2) \\ f(6,2) &= \min\{u(6,2)+f(7,3), m(6,2)+f(6,2), l(6,2)+f(7,1)\} \\ &= \min\{\infty,3+4,2+3\} = 5 && d(6,2)=(7,1) \\ f(6,1) &= \min\{u(6,1)+f(7,2), m(6,1)+f(7,1), l(6,1)+f(7,0)\} \\ &= \min\{\infty,2+3,2+5\} = 5 && d(6,1)=(7,1) \end{aligned}$$

$\underline{x=5}$ Points: $(5,3),\ (5,2)$

$$\begin{aligned} f(5,3) &= \min\{u(5,3)+f(6,4), m(5,3)+f(6,3), l(5,3)+f(6,2)\} \\ &= \min\{\infty,1+6,1+5\} = 6 && d(5,3)=(6,2) \\ f(5,2) &= \min\{u(5,2)+f(6,3), m(5,2)+f(6,2), l(5,2)+f(6,1)\} \\ &= \min\{\infty,3+5,2+5\} = 6 && d(5,2)=(6,1) \end{aligned}$$

$\underline{x=4}$ Points: $(4,3),\ (4,2)$

$$\begin{aligned} f(4,3) &= \min\{u(4,3)+f(5,4), m(4,3)+f(5,3), l(4,3)+f(5,2)\} \\ &= \min\{\infty,1+6,\infty\} = 7 && d(4,3)=(5,3) \\ f(4,2) &= \min\{u(4,2)+f(5,3), m(4,2)+f(5,2), l(4,2)+f(5,1)\} \\ &= \min\{\infty,3+6,\infty\} = 9 && d(4,2)=(5,2) \end{aligned}$$

$\underline{x=3}$ Points: $(3,3),\ (3,2),\ (3,1)$

$$\begin{aligned} f(3,3) &= \min\{u(3,3)+f(4,4), m(3,3)+f(4,3), l(3,3)+f(4,2)\} \\ &= \min\{\infty,1+7,\infty\} = 8 && d(3,3)=(4,3) \\ f(3,2) &= \min\{u(3,2)+f(4,3), m(3,2)+f(4,2), l(3,2)+f(4,1)\} \\ &= \min\{2+7,2+9,\infty\} = 9 && d(3,2)=(4,3) \\ f(3,1) &= \min\{u(3,1)+f(4,2), m(3,1)+f(4,1), l(3,1)+f(4,0)\} \\ &= \min\{1+9,\infty,\infty\} = 10 && d(3,1)=(4,2) \end{aligned}$$

$\underline{x=2}$ Points: $(2,2),\ (2,1),\ (2,0)$

$$\begin{aligned} f(2,2) &= \min\{u(2,2)+f(3,3), m(2,2)+f(3,2), l(2,2)+f(3,1)\} \\ &= \min\{2+8,3+9,\infty\} = 10 && d(2,2)=(3,3) \\ f(2,1) &= \min\{u(2,1)+f(3,2), m(2,1)+f(3,1), l(2,1)+f(3,0)\} \\ &= \min\{1+9,3+10,\infty\} = 10 && d(2,1)=(3,2) \\ f(2,0) &= \min\{u(2,0)+f(3,1), m(2,0)+f(3,0), l(2,0)+f(3,-1)\} \\ &= \min\{3+10,\infty,\infty\} = 13 && d(2,0)=(3,1) \end{aligned}$$

$\underline{x=1}$ Points: $(1,1),\ (1,0)$

$$
\begin{aligned}
f(1,1) &= \min\{u(1,1)+f(2,2), m(1,1)+f(2,1), l(1,1)+f(2,0)\} \\
&= \min\{2+10, 2+10, \infty\} = 12 \qquad d(1,1) = (2,2) \text{ or } (2,1) \\
f(1,0) &= \min\{u(1,0)+f(2,1), m(1,0)+f(2,0), l(1,0)+f(2,-1)\} \\
&= \min\{2+10, 2+13, \infty\} = 12 \qquad d(1,0) = (2,1)
\end{aligned}
$$

$\underline{x=0}$ Final point: $(0,0)$

$$
\begin{aligned}
f(0,0) &= \min\{u(0,0)+f(1,1), m(0,0)+f(1,0)\} \quad \text{ignoring } l(0,0) \\
&= \min\{1+12, 2+12, \infty\} = \underbrace{13}_{\text{Total cost}} \qquad d(0,0) = (1,1)
\end{aligned}
$$

Decision for the routes:

$(0,0) \to (1,1) \to (2,1) \to (3,2) \to (4,3) \to (5,3) \to (6,2) \to (7,1) \to (8,1) \to (9,0)$
$(0,0) \to (1,1) \to (2,2) \to (3,3) \to (4,3) \to (5,3) \to (6,2) \to (7,1) \to (8,1) \to (9,0)$
$(0,0) \to (1,1) \to (2,1) \to (3,2) \to (4,3) \to (5,3) \to (6,2) \to (7,1) \to (8,0) \to (9,0)$
$(0,0) \to (1,1) \to (2,2) \to (3,3) \to (4,3) \to (5,3) \to (6,2) \to (7,1) \to (8,0) \to (9,0)$

We also can reflect along the x-axis to get

$(0,0) \to (1,\text{-}1) \to (2,\text{-}1) \to (3,\text{-}2) \to (4,\text{-}3) \to (5,\text{-}3) \to (6,\text{-}2) \to (7,\text{-}1) \to (8,\text{-}1) \to (9,0)$
$(0,0) \to (1,\text{-}1) \to (2,\text{-}2) \to (3,\text{-}3) \to (4,\text{-}3) \to (5,\text{-}3) \to (6,\text{-}2) \to (7,\text{-}1) \to (8,\text{-}1) \to (9,0)$
$(0,0) \to (1,\text{-}1) \to (2,\text{-}1) \to (3,\text{-}2) \to (4,\text{-}3) \to (5,\text{-}3) \to (6,\text{-}2) \to (7,\text{-}1) \to (8,0) \to (9,0)$
$(0,0) \to (1,\text{-}1) \to (2,\text{-}2) \to (3,\text{-}3) \to (4,\text{-}3) \to (5,\text{-}3) \to (6,\text{-}2) \to (7,\text{-}1) \to (8,0) \to (9,0)$

To get 8 paths, each with total cost 13.

□

(β) A new path from the node $(4,2)$ to $(5,3)$ will open and the time needed to pass from this is $t > 0$. Find the set of the values of t such that: the new optimal paths to include the motion from $(4,2)$ to $(5,3)$ and be faster than what you found in (α).

Solution: We want the new path to be the choice which changes the decision at $f(4,2)$, i.e.

$f(4,2) = \min\{u(4,2)+f(5,3), m(4,2)+f(5,2), l(4,2)+f(5,1)\}$

$= \min\{t+6, 3+6, \infty\}$

This implies that $t < 3$ and the new $f(4,2) = 6+t$,hence $d(4,2) = (5,3)$

We can also look at

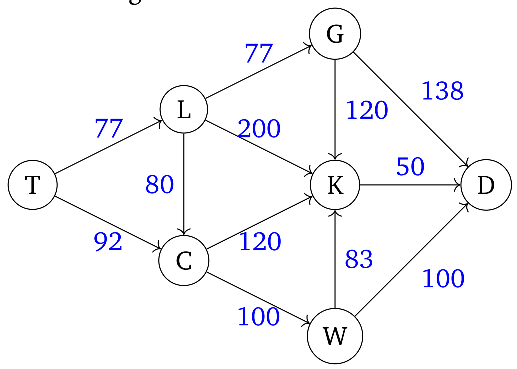
$f(3,2) = \min\{u(3,2)+f(4,3), m(3,2)+f(4,2), l(3,2)+f(4,1)\}$

$= \min\{\infty, 8+t, 2+7\}$ which implies that $t < 1$

We already require $t > 0$ and so $0 < t < 1$

Our total cost would then be strictly between 12 and 13, which is better than the routes from (α).
□

Problem 2. A mathematician starts from Tralee at 6:30am and he drives to Dublin, where he is supposed to teach at 11am. Given that in several highways there are taking place constructions, the necessary time in minutes to go from the several cities to the others are given below:



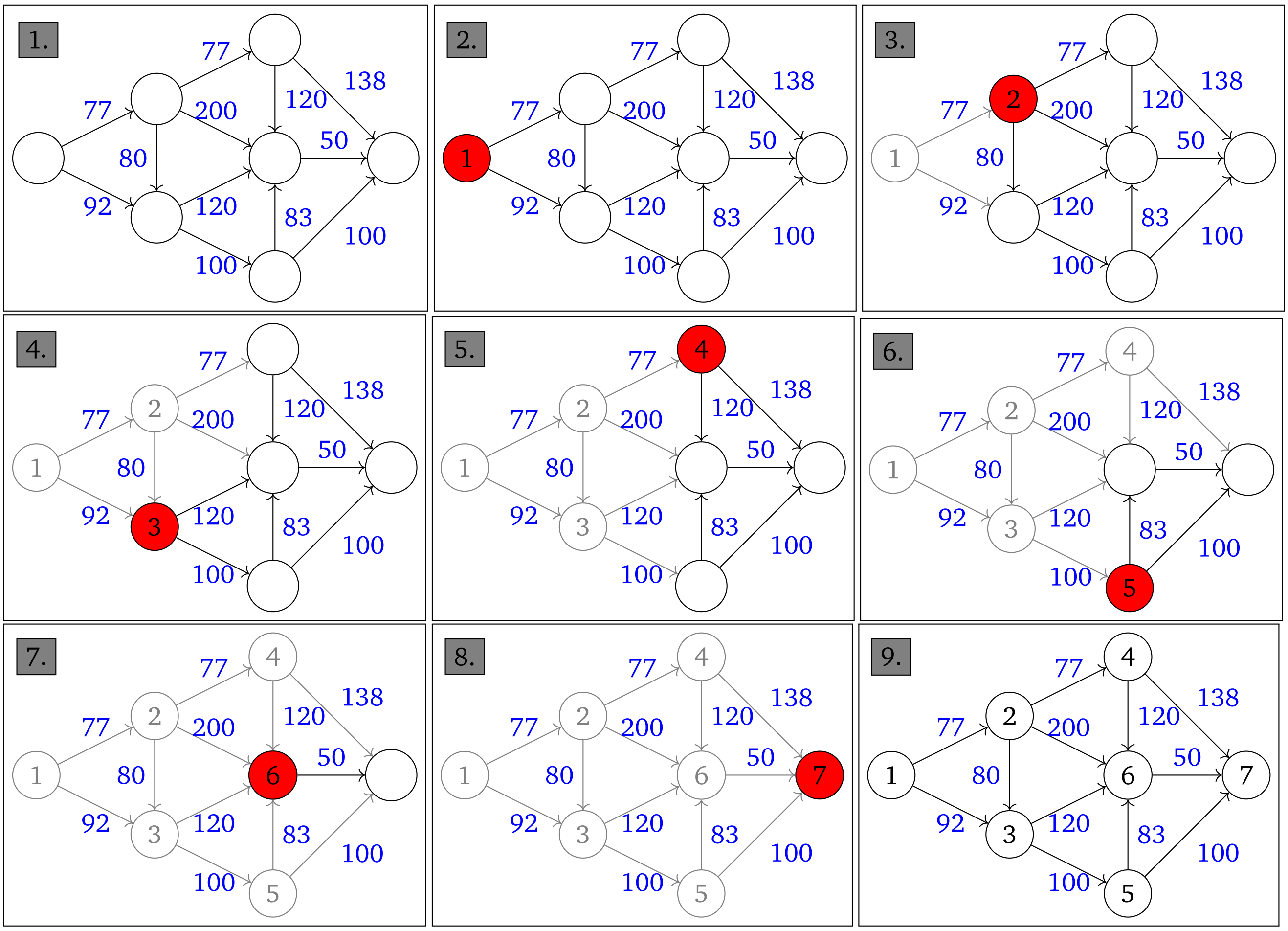
(α) Help him catch his class by finding the fastest path starting from Tralee to Dublin.

Solution:

We have cities $T(0,0)$, $L(1,1)$, $C(1,-1)$, $G(2,2)$, $K(2,0)$, $W(2,-2)$ and $D(3,0)$.

We use the forward method of dynamic programming.

This is a *general network* without circles, and so we can derive a numbering $\{1,\ldots,7\}$ such that the costs α_{ji} implies $j < i$.



Therefore we define the objective function

$f_i :=$ the length of the shortest path from (1) to (i).

Our goal is to find f_7, and we also have boundary condition $f_1 = 0$
Our recursive formula for f_i, by the principle of optimization (PO) is

$$f_i := \min_{j<i} \{f_j + \alpha_{ji}\} \qquad \text{for all } i = 1, \ldots, 7$$

Also $\alpha_{ji} := \infty$ whenever there is no path (j) $\to$ (i)

$$\begin{aligned}
f_2 &= f_1 + \alpha_{12} = 77 && d(2) = 1 \\
f_3 &= \min\{f_1 + \alpha_{13}, f_2 + \alpha_{23}\} \\
&= \min\{0 + 92, 77 + 80\} = 92 && d(3) = 1 \\
f_4 &= f_2 + \alpha_{24} \\
&= 77 + 77 = 154 && d(4) = 5 \\
f_5 &= f_3 + \alpha_{35} \\
&= 92 + 100 = 192 && d(5) = 3 \\
f_6 &= \min\{f_2 + \alpha_{26}, f_3 + \alpha_{36}, f_4 + \alpha_{46}, f_5 + \alpha_{56}\} \\
&= \min\{77 + 200, 92 + 120, 154 + 120, 192 + 83\} = 112 && d(6) = 3 \\
f_7 &= \min\{f_4 + \alpha_{47}, f_5 + \alpha_{57}, f_6 + \alpha_{67}\} \\
&= \min\{154 + 138, 192 + 100, 212 + 50\} = \underbrace{262}_{\text{Total Cost}} && d(7) = 6
\end{aligned}$$

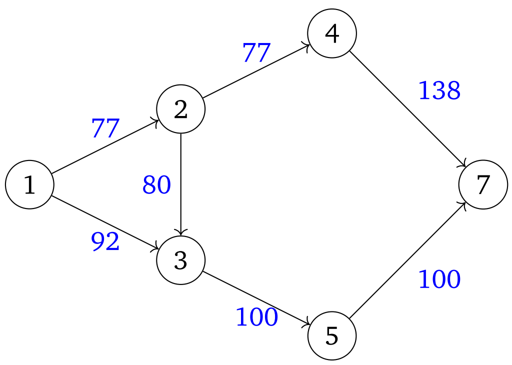
Therefore the fastest path is
(1) $\to$ (3) $\to$ (6) $\to$ (7)
or, for our cities,

Tralee $\to$ Cork $\to$ Kildare $\to$ Dublin

And if he leaves at 6:30am, he will arrive at 6:30am + 262 mins = 10:52am, and so he will be on time. ☐

(β) Is it possible to catch his class, if the specific day, there is a quarantine in Kildare and therefore he cannot pass from this city?

Solution: We cannot include (K) in our route, so our new (numbered) diagram is

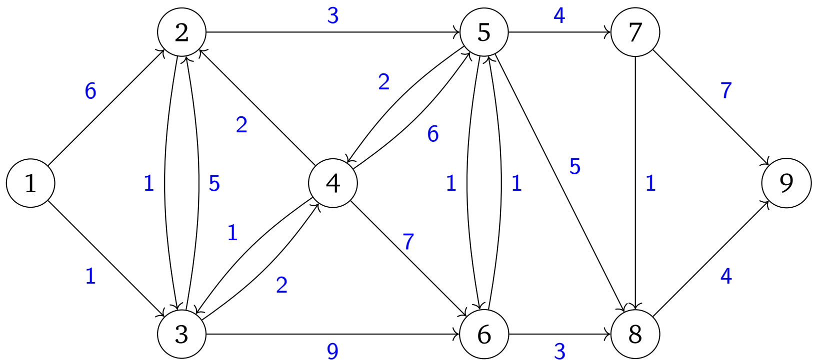


This will only affect f_7 (since we no longer have an f_6.
And so

$$\begin{aligned}
f_7 &= \min\{f_4 + \alpha_{47}, f_5 + \alpha_{57}\} \\
&= \min\{154 + 138, 192 + 100\} = \underbrace{292}_{\text{Total Cost}} \qquad d(7) = 4 \text{ or } 5
\end{aligned}$$

Therefore both fastest paths
(1) $\to$ (3) $\to$ (6) $\to$ (7) and (1) $\to$ (3) $\to$ (6) $\to$ (7)
would take 292 minutes, but 6:30am + 292 mins = 11:22am, so he will be **late**. ☐

Problem 3. Using Dijkstra's method, find the optimal route from the node (1) to the node (9). The values over the arcs represent cost.



Solution: We define $N_i(1) :=$ the set of i nodes that are closest to (1). Our goal is to find the minimum i such that (9) $\in N_i(1)$.

We also define $f_i(j) :=$ length of the shortest path from (1) to (j) over all nodes prior to (j) that belong to $N_i(1)$

And k_i is the i^{th} closest node to (1), so $k_i \notin N_{i-1}(1)$.

$f_{i-1}(k_i) = \min\limits_{j \notin N_{i-1}(1)} (f_{i-1}(j))$

Our recursive formula is then

$$f_i(j) = \begin{cases} f_{i-1}(j), & j \in N_i(1) \\ \min\left\{f_{i-1}(j), f_{i-1}(k_i) + \alpha_{k_i j}\right\}, & j \notin N_i(1) \end{cases}$$

where $\alpha_{k_i j}$ is the cost of going from node (k_i) to node (j) (it is ∞ if there is no such path).

$\underline{i = 1:}$ $N_1(1) = \{1\},\ k_1 = 1$ (always)

$f_1(1) = 0,\ f_1(2) = 6,\ f_1(3) = 1,$
$f_1(4) = \cdots = f_1(9) = \infty$

$\underline{i = 2:}$

Step 1 Finding k_i

$k_2 =$ second closest to 1
$= 3$

In general, $k_2 \notin N_1(1) = \{1\}$

$f_1(k_2) = \min\limits_{k \notin \{1\}} \{f_1(j)\} = f_1(3) = 1$

So $k_2 = 3$ and $N_2(1) = \{1, 3\}$

Step 2 Finding $f_i(j)$, $j \in \{1, 2, \ldots, 9\}$

For $j \in N_2(1) = \{1, 3\}$

$$f_2(j) = f_1(j) \implies \begin{cases} f_2(1) = 0 \\ f_2(3) = 1 \end{cases}$$

For $j \notin N_2(1)$, then $f_2(j) = \min\{f_1(j), f_1(3) + \alpha_{3j}\}$

$f_2(2) = \min\{f_1(2), 1 + \alpha_{32}\}$
$\quad = \min\{6, 1 + 5\} = 6$

$f_2(4) = \min\{f_1(4), f_1(3) + \alpha_{34}\} = 3$

$f_2(6) = \min\{f_1(6), f_1(3) + \alpha_{36}\} = 10$

For the rest, $f_2(4) = f_2(5) = \cdots = f_2(9) = \infty$ since $f_1(j) = \infty,\ j = 5, 7, 8, 9$ and $\alpha_{3j} = \infty$

$\underline{i = 3:}$

$\mathbf{S_1}$ $k_3 \notin \{1, 3\}$

$$f_2(k_3) = \min_{j \notin \{1,3\}} \{f_2(j)\}$$
$$= \min\{f_2(2), f_2(4), \ldots, f_2(9)\} = 3$$
$$\implies k_3 = 4 \text{ and } N_3(1) = \{1, 3, 4\}$$

$\mathbf{S_2}$ $\forall\, j \in N_3(1) \quad f_3(j) = f_2(j) \implies \begin{cases} f_3(1) = 0 \\ f_3(3) = 1 \\ f_3(4) = 3 \end{cases}$

Let $j \notin N_3(1)$, then $f_3(j) = \min\{f_2(j), f_2(4) + \alpha_{4j}\}$

$f_3(2) = \min\{f_2(2), f_2(4) + \alpha_{42}\} = 5$

$f_3(5) = \infty$

$f_3(6) = \min\{f_2(6), f_2(4) + \alpha_{46}\} = 10$

$f_3(7) = \infty$

$f_3(8) = \infty$

$f_3(9) = \infty$

$\underline{i = 4:}$

$\mathbf{S_1}$ $k_4 \notin \{1, 3, 4\}$

$$f_3(k_4) = \min_{j \notin \{1,3,4\}} \{f_3(j)\}$$
$$= \min\{f_2(2), f_2(5), \ldots, f_2(9)\} = 5$$
$$\implies k_4 = 2 \text{ and } N_4(1) = \{1, 2, 3, 4\}$$

$\mathbf{S_2}$ $\forall\, j \in N_4(1) \quad f_4(j) = f_3(j) \implies \begin{cases} f_4(1) = 0 \\ f_4(2) = 5 \\ f_4(3) = 1 \\ f_4(4) = 3 \end{cases}$

Let $j \notin N_4(1)$, then $f_4(j) = \min\{f_3(j), f_3(2) + \alpha_{2j}\}$

$f_4(5) = \min\{f_3(5), f_3(2) + \alpha_{25}\} = 8$

$f_4(6) = \min\{f_3(6), f_3(2) + \alpha_{26}\} = 10$

$f_4(7) = \infty$

$f_4(8) = \infty$

$f_4(9) = \infty$

$\underline{i = 5:}$

$\mathbf{S_1}$ $k_5 \notin \{1, 2, 3, 4\}$

$$f_4(k_5) = \min_{j \notin \{1,2,3,4\}} \{f_4(j)\}$$
$$= \min\{f_2(5), \ldots, f_2(9)\} = 8$$
$$\implies k_5 = 5 \text{ and } N_5(1) = \{1, 2, 3, 4, 5\}$$

$\mathbf{S_2}$ $\forall\, j \in N_5(1) \quad f_5(j) = f_4(j) \implies \begin{cases} f_5(1) = 0 \\ f_5(2) = 5 \\ f_5(3) = 1 \\ f_5(4) = 3 \\ f_5(5) = 8 \end{cases}$

Let $j \notin N_5(1)$, then $f_5(j) = \min\{f_4(j), f_4(5) + \alpha_{5j}\}$
$f_5(6) = \min\{f_4(6), f_4(5) + \alpha_{56}\} = 9$
$f_5(7) = \min\{f_4(7), f_4(5) + \alpha_{57}\} = 12$
$f_5(8) = \min\{f_4(8), f_4(5) + \alpha_{58}\} = 13$
$f_5(9) = \infty$

$\underline{i = 6:}$

S₁ $k_6 \notin \{1,2,3,4,5\}$

$$f_5(k_6) = \min_{j \notin \{1,\dots,5\}} \{f_5(j)\}$$
$$= \min\{f_2(6), \dots, f_2(9)\} = 9$$

$\implies k_6 = 6$ and $N_6(1) = \{1,2,3,4,5,6\}$

S₂ $\forall\, j \in N_6(1) \quad f_6(j) = f_5(j) \implies \begin{cases} f_6(1) = 0 \\ f_6(2) = 5 \\ f_6(3) = 1 \\ f_6(4) = 3 \\ f_6(5) = 8 \\ f_6(6) = 9 \end{cases}$

Let $j \notin N_6(1)$, then $f_6(j) = \min\{f_5(j), f_5(6) + \alpha_{6j}\}$
$f_6(7) = \min\{f_5(7), f_5(6) + \alpha_{67}\} = 12$
$f_6(8) = \min\{f_5(8), f_5(6) + \alpha_{68}\} = 12$
$f_6(9) = \infty$

$\underline{i = 7:}$

S₁ $k_7 \notin \{1,\dots,6\}$

$$f_6(k_7) = \min_{j \notin \{1,\dots,6\}} \{f_6(j)\}$$
$$= \min\{f_2(7), f_2(8), f_2(9)\} = 12$$

$\implies k_7 = 7$ or $k_7 = 8$

S₂ Therefore

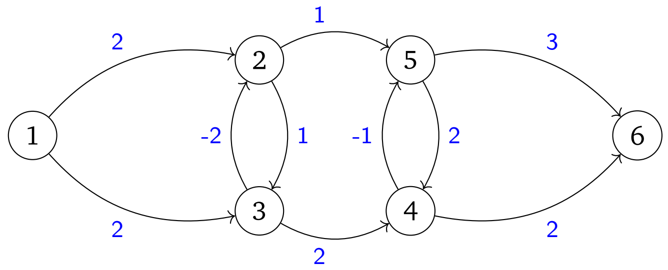
$$f_7(9) = \min \begin{cases} \min\{f_6(9), f_6(7) + \alpha_{79}\} = 19, & k_7 = 7 \\ \min\{f_6(9), f_6(8) + \alpha_{89}\} = 16, & k_7 = 8 \end{cases}$$

So our minimum cost is 16
and our optimal route is

$$1 \to 3 \to 4 \to 2 \to 5 \to 6 \to 8 \to 9$$

□

Problem 4. In the graph below the numbers over the arrows represent the cost from moving from a node to another. The negative values represent profit.



Find the minimum cost route in (at most)

(α) $L = 4$ steps

Solution:

We define our objective function

$f_i(k) :=$ the length of the shortest path from (1) to (k) with at most i arcs (steps)

$1 \leq k \leq 6, \quad 0 \leq i \leq 4$

Target: $f_4(6)$

Boundary: $f_0(k) = \begin{cases} 0, & k = 1 \\ \infty, & k \neq 1 \end{cases}$

in fact, $f_i(1) = 0$ for all $i \geq 0$

Our recursive formula for $2 \leq k \leq 6, \quad 1 \leq i \leq 4$ is

$$f_i(k) = \min_{j \neq k} \{f_{i-1}(j) + \alpha_{jk}\}$$

$\underline{i = 1}$

$$\begin{aligned} f_1(2) &= 2 \qquad d(2) = 1 \\ f_1(3) &= 2 \qquad d(3) = 1 \\ f_1(k) &= \infty \quad k = 4, 5, 6 \end{aligned}$$

$\underline{i = 2}$

$$\begin{aligned} f_2(2) &= \min\{f_1(1) + \alpha_{12}, f_1(3) + \alpha_{32}\} \\ &= \min\{0 + 2, 2 + (-2)\} = 0 && d(2) = 3 \\ f_2(3) &= \min\{f_1(1) + \alpha_{13}, f_1(2) + \alpha_{23}\} \\ &= \min\{0 + 2, 2 + 1\} = 2 && d(3) = 1 \\ f_2(4) &= f_1(3) + \alpha_{34} \\ &= 2 + 2 = 4 && d(4) = 3 \\ f_2(5) &= f_1(2) + \alpha_{25} \\ &= 2 + 1 = 3 && d(5) = 2 \\ f_2(6) &= \infty \end{aligned}$$

$\underline{i = 3}$

$$
\begin{aligned}
f_3(2) &= \min\{f_2(1)+\alpha_{12}, f_2(3)+\alpha_{32}\} \\
&= \min\{0+2, 2+(-2)\} = 0 \qquad && d(2)=3 \\
f_3(3) &= \min\{f_2(1)+\alpha_{13}, f_2(2)+\alpha_{23}\} \\
&= \min\{0+2, 0+1\} = 1 && d(3)=1 \\
f_3(4) &= \min\{f_2(3)+\alpha_{34}, f_2(5)+\alpha_{54}\} \\
&= \min\{2+2, 3+2\} = 4 && d(4)=3 \\
f_3(5) &= \min\{f_2(2)+\alpha_{25}, f_2(4)+\alpha_{45}\} \\
&= \min\{0+1, 4+2\} = 1 && d(5)=2 \\
f_3(6) &= \min\{f_2(4)+\alpha_{46}, f_2(5)+\alpha_{56}\} \\
&= \min\{4+2, 3+3\} = 6 && d(6)=4 \text{ or } 5
\end{aligned}
$$

$\underline{i=4}$

Our goal is $f_4(6)$ so we only need to evaluate that at this step:

$$
\begin{aligned}
f_4(6) &= \min\{f_3(4)+\alpha_{46}, f_3(5)+\alpha_{56}\} \\
&= \min\{4+2, 1+3\} = 4 \qquad && d(6)=5
\end{aligned}
$$

So our optimal route is

$1 \to 3 \to 2 \to 5 \to 6$

with minimal cost 4.

□

(β**)** $L=3$ steps

Solution:

We already found $f_3(6)$ in the previous part of the question, i.e. $f_3(6)=6$ with decision(s) $d(6)=4$ or 5

And so our total cost is 6, and our minimal routes are

$1 \to 2 \to 5 \to 6$ or $1 \to 3 \to 4 \to 6$. □

Problem 5. We need a specific equipment for $T = 4$ years. Each year we may take one of the following decisions:

(i) to keep our equipment

(ii) to replace our equipment with a new one

(iii) to buy new equipment and sell our old one

The following costs may apply:

$k(t) = 1 + 2t =$ the cost of using for one year period a machine of t years old.

$r(t,\tau) = 3 + \tau + t =$ the price for changing our t years old equipment by a new one at the time τ.

$B(\tau) = 5 + \tau =$ the price of buying a new equipment at the time τ.

$s(t,\tau) = 1 - t + \tau =$ the profit of selling our t years old equipment at the time τ.

(α) Find the optimal policy (keep-replace-buy every year), provided that we start having an equipment of $t = 2$ years old.

Solution:

$\mathbf{S_1}$ $\tau = 0$ $\tau = 4$

$f(t,\tau) :=$ the minimum cost for using the car of age t from τ to T.

Actions and costs (PO)):

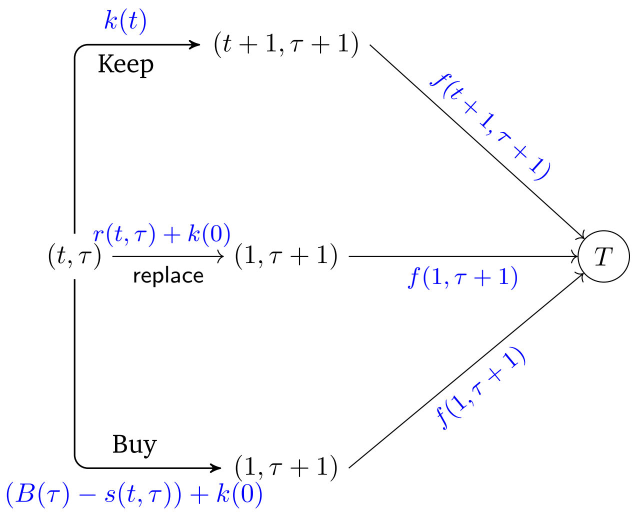
- **Keep**
  $k(t) + f(t+1, \tau+1)$
- **Replace**
  $r(t) + k(0) + f(1, \tau+1)$
- **Buy**
  $B(\tau) + k(0) - s(t,\tau) + f(1,\tau+1)$

Our objective function to optimize is, for $1 \le \tau < 4$ and $t \ge 1$

$$f(t,\tau) = \min\{1 + 2t + f(t+1,\tau+1), 3 + \tau + t + 1 + f(1,\tau+1), 5 + \tau + 1 - (1 - t + \tau) + f(1,\tau+1)\}$$
$$= \min \begin{cases} 1 + 2t + f(t+1,\tau+1), \\ 4 + \tau + t + f(1,\tau+1), \\ 5 + t + f(1,\tau+1) \end{cases}$$

Want to find $f(2,0)$ (we start with 2 year old equipment)

We have boundary condition: $f(t,T) = -s(t,T) = t - T - 1 = t - 5$



$\mathbf{S_2}$ $\underline{\tau = 4}$ $f(1,4) = -4,\ f(2,4) = -3, \ldots, f(6,4) = 1$

S₃ $\underline{\tau = 3}$ Since we start with a 2 y.o. piece of equipment, either we've kept the equipment the whole time, so $t = 2 + 3 = 5$, or we bought/replaced it at some point, so $1 \le t \le 5$

$$\begin{aligned}
f(1,3) &= \min\{k(1) + f(2,4), r(1,3) + k(0) + f(1,4), (B(3) - s(1,3)) + k(0) + f(1,4)\} \\
&= \min\{3 - 3, 7 + 1 - 4, 8 - 3 + 1 - 4\} = 0 && \text{keep} \\
f(2,3) &= \min\{k(2) + f(3,4), r(2,3) + k(0) + f(1,4), (B(3) - s(2,3)) + k(0) + f(1,4)\} \\
&= \min\{5 - 2, 8 + 1 - 4, 8 - 2 + 1 - 4\} = 3 && \text{keep or buy} \\
f(3,3) &= \min\{k(3) + f(4,4), r(3,3) + k(0) + f(1,4), (B(3) - s(3,3)) + k(0) + f(1,4)\} \\
&= \min\{7 - 1, 9 + 1 - 4, 8 - 1 + 1 - 4\} = 4 && \text{buy} \\
f(4,3) &= \min\{k(4) + f(5,4), r(4,3) + k(0) + f(1,4), (B(3) - s(4,3)) + k(0) + f(1,4)\} \\
&= \min\{9 + 0, 10 + 1 - 4, 8 + 0 + 1 - 4\} = 5 && \text{buy} \\
f(5,3) &= \min\{k(5) + f(6,4), r(5,3) + k(0) + f(1,4), (B(3) - s(5,3)) + k(0) + f(1,4)\} \\
&= \min\{11 + 1, 11 + 1 - 4, 8 + 1 + 1 - 4\} = 6 && \text{buy}
\end{aligned}$$

$\underline{\tau = 2}$ by the same argument as above, $1 \le t \le 4$

$$\begin{aligned}
f(1,2) &= \min\{k(1) + f(2,3), r(1,2) + k(0) + f(1,3), (B(2) - s(1,2)) + k(0) + f(1,3)\} \\
&= \min\{3 + 3, 6 + 1 + 0, 7 - 2 + 1 + 0\} = 6 && \text{keep or buy} \\
f(2,2) &= \min\{k(2) + f(3,3), r(2,2) + k(0) + f(1,3), (B(2) - s(2,2)) + k(0) + f(1,3)\} \\
&= \min\{5 + 4, 7 + 1 + 0, 7 - 1 + 1 + 0\} = 7 && \text{buy} \\
f(3,2) &= \min\{k(3) + f(4,3), r(3,2) + k(0) + f(1,3), (B(2) - s(3,2)) + k(0) + f(1,3)\} \\
&= \min\{7 + 5, 8 + 1 + 0, 7 + 0 + 1 + 0\} = 8 && \text{buy} \\
f(4,2) &= \min\{k(4) + f(5,3), r(4,2) + k(0) + f(1,3), (B(2) - s(4,2)) + k(0) + f(1,3)\} \\
&= \min\{9 + 6, 9 + 1 + 0, 7 + 1 + 1 + 0\} = 9 && \text{buy}
\end{aligned}$$

$\underline{\tau = 1}, \quad 1 \le t \le 3$

$$\begin{aligned}
f(1,1) &= \min\{k(1) + f(2,2), r(1,1) + k(0) + f(1,2), (B(1) - s(1,1)) + k(0) + f(1,2)\} \\
&= \min\{3 + 7, 5 + 1 + 6, 6 - 1 + 1 + 6\} = 10 && \text{keep} \\
f(2,1) &= \min\{k(2) + f(3,2), r(2,1) + k(0) + f(1,2), (B(1) - s(2,1)) + k(0) + f(1,2)\} \\
&= \min\{5 + 8, 6 + 1 + 6, 6 + 0 + 1 + 6\} = 13 && \text{any choice} \\
f(3,1) &= \min\{k(3) + f(4,2), r(3,1) + k(0) + f(1,2), (B(1) - s(3,1)) + k(0) + f(1,2)\} \\
&= \min\{7 + 9, 7 + 1 + 6, 6 + 1 + 1 + 6\} = 14 && \text{replace or buy}
\end{aligned}$$

$\underline{\tau = 0}$ goal at $t = 2$ year old equipment we start with

$$\begin{aligned}
f(2,0) &= \min\{k(2) + f(3,1), r(2,0) + k(0) + f(1,1), (B(1) - s(2,0)) + k(0) + f(1,1)\} \\
&= \min\{5 + 14, 5 + 1 + 10, 5 + 1 + 1 + 10\} = 16 && \text{replace}
\end{aligned}$$

S₄ Decision: Policy

$\tau = 0:$ replace $\rightarrow (1,1)$
$\tau = 1:$ keep $\rightarrow (2,2)$
$\tau = 2:$ buy $\rightarrow (1,3)$
$\tau = 3:$ keep sell next year

□

(β) Prove that the strategy above remains the same, given that we start having an equipment of $t \ge 1$ years old.

Solution:

We again have the objective function for $1 \le \tau < 4$

$$f(t,\tau) = \min \begin{cases} 1 + 2t + f(t+1, \tau+1), \\ 4 + \tau + t + f(1, \tau+1), \\ 5 + t + f(1, \tau+1) \end{cases}$$

With the same boundary condition $f(t,4) = t - 5$
We wish to find $f(t,0)$ for arbitrary $t \geq 1$.

(a) $\underline{\tau = 3}$

$$\begin{aligned} f(t,3) &= \min\left\{k(t) + f(t+1,4), r(t,3) + k(0) + f(1,4), (B(3) - s(t,3)) + k(0) + f(1,4)\right\} \\ &= \min\left\{1 + 2t + f(t+1,4), 7 + t + f(1,4), 5 + t + f(1,4)\right\} \\ &= \min\left\{1 + 2t + t + 1 - 5, 7 + t + 1 - 5, 5 + t + 1 - 5\right\} \\ &= \min\left\{3t - 3, t + 3, t + 1\right\} \\ &= \begin{cases} 0, & t = 1 \quad \text{keep} \\ 3. & t = 2 \quad \text{keep or buy} \\ t + 1 & t \geq 3 \quad \text{buy} \end{cases} \end{aligned}$$

(b) $\underline{\tau = 2}$

$$\begin{aligned} f(t,2) &= \min\left\{k(t) + f(t+1,3), r(t,2) + k(0) + f(1,3), (B(2) - s(t,2)) + k(0) + f(1,3)\right\} \\ &= \min\left\{1 + 2t + f(t+1,3), 6 + t + 0, 5 + t + 0\right\} \end{aligned}$$

But we know

$$f(t+1,3) = \begin{cases} 3 & t = 1 \\ t + 2 & t \geq 2 \end{cases}$$

And so

$$\begin{aligned} f(t,2) &= \min\left\{1 + 2t + f(t+1,3), 6 + t, 5 + t\right\} \\ &= \begin{cases} \min\left\{1 + 2 + 3, 6 + 1, 5 + 1\right\}, & t = 1 \\ \min\left\{1 + 2t + t + 2, 6 + t, 5 + t\right\}, & t \geq 2 \end{cases} \\ &= \begin{cases} \min\left\{6, 7, 6\right\}, & t = 1 \\ \min\left\{3 + 3t, 6 + t, 5 + t\right\}, & t \geq 2 \end{cases} \\ &= \begin{cases} 6 & t = 1 \quad \text{keep or buy} \\ t + 5 & t \geq 2 \quad \text{buy} \end{cases} \end{aligned}$$

(c) $\underline{\tau = 1}$

$$\begin{aligned} f(t,1) &= \min\left\{k(t) + f(t+1,2), r(t,1) + k(0) + f(1,2), (B(1) - s(t,1)) + k(0) + f(1,2)\right\} \\ &= \min\left\{1 + 2t + f(t+1,2), 5 + t + 6, 5 + t + 6\right\} \end{aligned}$$

But we know

$$f(t+1,2) = t + 6 \qquad t \geq 1$$

And so

$$\begin{aligned} f(t,1) &= \min\left\{1 + 2t + t + 6, 11 + t, 11 + t\right\} \\ &= \min\left\{7 + 3t, 11 + t, 11 + t\right\} \\ &= \begin{cases} 10 & t = 1 \quad \text{keep} \\ 13 & t = 2 \quad \text{any choice} \\ 11 + t & t \geq 3 \quad \text{replace or buy} \end{cases} \end{aligned}$$

$\underline{\tau = 0}$ goal is $f(t,0)$

$$\begin{aligned} f(t,0) &= \min\left\{k(t) + f(t+1,1), r(t,0) + k(0) + f(1,1), (B(0) - s(t,0)) + k(0) + f(1,1)\right\} \\ &= \min\left\{1 + 2t + f(t+1,1), 4 + t + 10, 5 + t + 10\right\} \\ &= \min\left\{1 + 2t + f(t+1,1), 14 + t, 15 + t\right\} \end{aligned}$$

But we know

$$f(t+1,1) = \begin{cases} 13 & t = 1 \\ 12 + t & t \geq 2 \end{cases}$$

And so

$$f(t,0) = \begin{cases} \min\{1+2t+13, 14+t, 15+t\} & t=1 \\ \min\{1+2t+12+t, 14+t, 15+t\} & t \geq 2 \end{cases}$$
$$= \begin{cases} \min\{14+2t, 14+t, 15+t\} & t=1 \\ \min\{13+3t, 14+t, 15+t\} & t \geq 2 \end{cases}$$
$$= 14+t \qquad t \geq 1 \qquad \text{replace}$$

(d) Decision: Policy

$\tau = 0:$ replace $\quad \rightarrow (1,1)$
$\tau = 1:$ keep $\quad \rightarrow (2,2)$
$\tau = 2:$ buy $\quad \rightarrow (1,3)$
$\tau = 3:$ keep $\quad$ sell next year

Therefore we replace the car at year 1 regardless of its age $t \geq 1$
And so the policy is the same for all $t \geq 1$. Part (α) is just a particular case we validated above.

□

Assessment 2

Problem 1. A Student works on his final year project and he is planing his study for the next 6 weeks. He will spend each week entirely in any of the following processes:

1) writing his thesis.
2) studying an extra result to add in his thesis.
3) trying to develop a new original result.

He estimated together with his supervisor the benefits (in some hypothetical units) of each week that can be spent in each of the above processes.

The table below summarizes the benefits $\Phi_i(y)$ of spending $y \in \{0,\dots,6\}$ weeks in the i^{th} process, $i = 1,2,3$.

$\Phi_i(y)\backslash y =$	0	1	2	3	4	5	6
$\Phi_1(y)$	0	1	2	3	4	5	6
$\Phi_2(y)$	0	0	2	2	3	4	5
$\Phi_3(y)$	0	0	0	5	5	6	7

(α) Decide how the Student has to allocate the 6 study weeks in the three processes in order to maximize his total benefit.

Solution:

$\mathbf{S_1}$ We have the stages $v = 1,2,3$

Let $f_v(y) :=$ the maximum contribution in the processes $v,\dots,3$ when y weeks are available.
Goal is $f_1(6)$

$\mathbf{S_2}$ $f_v(y) = \max\limits_{0\le y_v\le y}\left\{\Phi_v(y_v) + f_{v+1}(y-y_v)\right\}, \quad 0\le y\le 6,\ 1\le v<3$

$f_3(y) = \Phi_3(y),\ 0\le y\le 6$

$\mathbf{S_3}$ $\underline{v=2}$

$$\begin{aligned}
f_2(y) &= \max_{0\le y_2\le y}\left\{\Phi_2(y_2) + f_3(y-y_2)\right\}\\
&= \max_{0\le y_2\le y}\left\{\Phi_2(y_2) + \Phi_3(y-y_2)\right\}, \quad 0\le y\le 6\\
f_2(0) &= \max\{\Phi_2(0)+\Phi_3(0)\} = 0\\
f_2(1) &= \max\{\Phi_2(0)+\Phi_3(1), \Phi_2(1)+\Phi_3(0)\}\\
&= \max\{0+0, 0+0\} = 0\\
f_2(2) &= \max\{\Phi_2(0)+\Phi_3(2), \Phi_2(1)+\Phi_3(1), \Phi_2(2)+\Phi_3(0)\}\\
&= \max\{0+0, 0+0, 2+0\} = 2\\
f_2(3) &= \max\{\Phi_2(0)+\Phi_3(3), \Phi_2(1)+\Phi_3(2), \Phi_2(2)+\Phi_3(1), \Phi_2(3)+\Phi_3(0)\}\\
&= \max\{0+5, 0+0, 2+0, 2+0\} = 5\\
f_2(4) &= \max\{\Phi_2(0)+\Phi_3(4), \Phi_2(1)+\Phi_3(3), \Phi_2(2)+\Phi_3(2), \Phi_2(3)+\Phi_3(1), \Phi_2(4)+\Phi_3(0)\}\\
&= \max\{0+5, 0+5, 2+0, 2+0, 3+0\} = 5\\
f_2(5) &= \max\{\Phi_2(0)+\Phi_3(5), \Phi_2(1)+\Phi_3(4), \Phi_2(2)+\Phi_3(3), \Phi_2(3)+\Phi_3(2), \Phi_2(4)+\Phi_3(1), \Phi_2(5)+\Phi_3(0)\}\\
&= \max\{0+6, 0+5, 2+5, 2+0, 3+0, 4+0\} = 7\\
f_2(6) &= \max\{\Phi_2(0)+\Phi_3(6), \Phi_2(1)+\Phi_3(5), \dots, \Phi_2(5)+\Phi_3(1), \Phi_2(6)+\Phi_3(0)\}\\
&= \max\{0+7, 0+6, 2+5, 2+5, 3+0, 4+0, 5+0\} = 7
\end{aligned}$$

$\mathbf{S_4}$ $\underline{v=1}$

$$\begin{aligned}
f_1(6) &= \max_{0\le y_1\le 6}\left\{\Phi_1(y_1) + f_2(6-y_1)\right\}\\
&= \max\{\Phi_1(0)+f_2(6), \Phi_1(1)+f_2(5), \dots, \Phi_1(5)+f_2(1), \Phi_1(6)+f_2(0)\}\\
&= \max\{0+7, 1+7, 2+5, 3+5, 4+2, 5+0, 6+0\} = 8
\end{aligned}$$

$\mathbf{S_5}$ Policy

We have two maximal policies:

i. Process 1: $y_1 = 1$ weeks ($f_2(5)$)
   Process 2: $y_2 = 2$ weeks ($f_1(3)$)
   Process 3: $y_3 = 3$

ii. Process 1: $y_1 = 3$ weeks ($f_2(3)$)
   Process 2: $y_2 = 0$ weeks ($f_1(3)$)
   Process 3: $y_3 = 3$

Both have maximized benefit of 8 units.

□

(β) If the Student would like to have some progress in all the above three processes, what is the optimal strategy that he has to follow when he is allocating his study-weeks?

Solution: Only the first option has some amount of weeks (and some progress) allocated to each process, so the policy $y_1 = 1$ week for process 1 (writing thesis), $y_2 = 2$ weeks for process 2 (study) and $y_3 = 3$ weeks for process 3 (develop original work). □

Problem 2. The Head of a Department is building a new program and he will use Teaching fellows, Assistant Professors and Associate Professors to build the several necessary modules.

The salary (in some money units) of a scientist from any of the three categories $(s_i,\ i = 1, 2, 3)$ and the amount of their contributions $(c_i,\ i = 1, 2, 3)$ (in some units of contribution) are summarized in the next table:

Rank	i	s_i	c_i
TF	1	3	5
Assis.	2	6	12
Assoc.	3	8	16

(α) Given that the available budget equals with 17 money units and that the Head of the Department of course tries to maximize the total contribution, express the problem in terms of a typical problem of Dynamic Programming.

Solution: We will tackle this as an Optimal Load problem, as in the course notes (in particular, this is similar to Exercise 12 from the notes).

- Total budget is $W = 17$
- We have $n = 3$ categories of scientists, $i = 1$: Teaching Fellow (TF), $i = 2$: Assistant Professor (Assis.), $i = 3$: Associate Professor (Assoc.)
- Let s_i be their salaries
- Let c_i be their level of contribution
- Let x_i be the number of employees of the i^{th} type that the HoD will hire
  Clearly, x_i will be an integer.

We then have further notation

- $p_i = x_i c_i$ is the contribution from the i^{th} type of scientists.
- $x_i s_i$ is the cost of hiring the i^{th} type of scientists.

We have the restriction $x_i s_i \leq 17$ for all $i = 1, 2, 3$.

$\square$

(β) Find the optimal hiring policy that the HoD should follow, using a method of Dynamic Programming.

Solution:

$\mathbf{S_1}$ We have $1 \leq i \leq 3$ and at any point, we have remaining weight w, where $0 \leq w \leq 17 = W$.

We define the objective function

$f_i(w) :=$ maximum profit from scientist types $i, \ldots, 3$ when the remaining budget is w units.

Our goal is $f_1(17)$.

$\mathbf{S_2}$ $\underline{i = 3}$ $\quad 0 \leq w \leq 17$

$$x_n = x_3 = \left[\frac{w}{s_3}\right] = \left[\frac{w}{8}\right] = \begin{cases} 0 & 0 \leq w < 8 \\ 1 & 8 \leq w < 16 \\ 2 & 16 \leq w \leq 17 \end{cases}$$

$$f_3(w) = \left[\frac{w}{8}\right] \cdot 16 = \begin{cases} 0 & 0 \leq w < 8 \\ 16 & 8 \leq w < 16 \\ 32 & 16 \leq w \leq 17 \end{cases}$$

S₃ $i = 1, 2 \quad 0 \leq w \leq 17$

We recursively define the objective function

$$f_i(w) = \max_{x_i} \{x_i c_i + f_{i+1}(w - x_i s_i)\} \qquad x_i \in \left\{0, \ldots, \left[\frac{w}{s_i}\right]\right\}$$

S₄ $\underline{i = 2} \implies s_2 = 6 \implies \left[\frac{w}{6}\right] = \begin{cases} 0 & 0 \leq w < 6 \\ 1 & 6 \leq w < 12 \\ 2 & 12 \leq w \leq 17 \end{cases}$

$0 \leq w < 6 \quad x_2 = 0 \quad f_2(w) = 0 + f_3(w) = 0$
$6 \leq w < 12 \quad x_2 \in \{0, 1\}$
$f_2(w) = \max\{0 + f_3(w), 12 + f_3(w - 6)\}$
$6 \leq w < 8 \quad f_2(w) = \max\{0, 12\} = 12$
$8 \leq w < 12 \quad f_2(w) = \max\{16, 12\} = 16$
$12 \leq w < 17 \quad x_2 \in \{0, 1, 2\}$
$f_2(w) = \max\{0 + f_3(w), 12 + f_3(w - 6), 24 + f_3(w - 12)\}$
$12 \leq w < 14 \quad f_2(w) = \max\{16, 12, 24\} = 24$
$14 \leq w < 16 \quad f_2(w) = \max\{16, 28, 24\} = 28$
$16 \leq w \leq 17 \quad f_2(w) = \max\{32, 28, 24\} = 32$

S₅ $\underline{i = 1} \implies s_1 = 3 \implies \left[\frac{w}{3}\right] = \begin{cases} 0 & 0 \leq w < 3 \\ 1 & 3 \leq w < 6 \\ 2 & 6 \leq w \leq 9 \\ 3 & 9 \leq w < 12 \\ 4 & 12 \leq w \leq 15 \\ 5 & 15 \leq w \leq 17 \end{cases}$

We only need to evaluate our goal, $f_1(17)$

$$\begin{aligned} f_1(17) &= \max\{0 + f_2(w), 5 + f_2(w - 3), 10 + f_2(w - 6), 15 + f_2(w - 9), 20 + f_2(w - 12), 25 + f_2(w - 15)\} \\ &= \max\{32, 5 + 28, 10 + 16, 15 + 16, 20 + 0, 25 + 0\} = 33 \end{aligned}$$

S₆ **Decision: Hiring policy**

Maximal/optimal contribution $= 33$

$$\begin{array}{lll} \text{Hire} & x_1 = 1 & \text{Teaching Fellow} \\ & \downarrow & \\ & f_2(14) & \\ & \downarrow & \\ & x_2 = 1 & \text{Assistant Professor} \\ & \downarrow & \\ & f_3(8) & \\ & \downarrow & \\ & x_3 = 1 & \text{Associate Professor} \end{array}$$

□

(γ) What is the maximum number of Teaching Fellows that makes sense to be hired (no Dynamic Programming is needed here)?

Solution: Since Assistant Professors cost exactly twice as much per unit as Teaching Fellows, and the average contribution per unit cost is $\frac{12}{6} = 2$ for Assis., but only $\frac{5}{3}$ for TF, we should hire at most one TF (as advised by the optimal hiring policy).

Two TF would contribute 10, while one Assis. would contribute 12, therefore it is only reasonable **one** Teaching fellow at most. □

(δ) Can you propose a different hiring strategy to the HoD, if you had a budget availability of 18 money units (again no DP is necessary here)?

Solution: We could hire three Assistant professors, costing $3 \times 6 = 18$ money units, and contribute $3 \times 12 = 36$ units to the department. $\square$

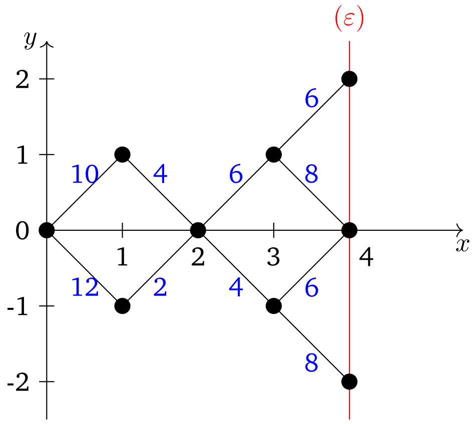
(ε) Ask the financial services of the University for a budget extension of just one unit, justifying your request in the way you believe.

Solution: If we had an increased budget of $17 + 1 = 18$ monetary units, we could **hire three Assistant Professors** and get a contribution of 36 units (as in the above part). This is an **increase of three units** from the optimal hiring policy we established for the current budget in part (β), which had total contribution of 33.

In terms of contribution per unit cost of the hiring budget, the optimal policy in part (β) is $33/17 = 1.94$, while the contribution per unit cost of our proposed budget is $36/18 = 2$.

In conclusion, an increased budget to 18 monetary units would both increase the total contribution of the department, and its contribution per unit cost. $\square$

Problem 3. The values on the arcs of the following graph, represent the cost of the motion of a mobile from a node to the other. From a command center we give orders to the mobile's driver, but she executes our orders with probability $p = 0.6$ (otherwise, she follows the opposite direction, from what we have commanded).



Find the optimal command policy for minimizing the expected cost with the following methods

(α) optimal control with feedback.

Solution:

$\mathbf{S_1}$ The Optimal Expectation Function (OEF)

$f(x, y) :=$ minimum expected cost from (x, y) to (ε)

$\mathbf{S_2}$ Boundary Condition (BC) at stage $x = 3$

$f(3, -3) = f(3, -1) = f(3, 1) = f(3, 3) = 0$

$\mathbf{S_3}$ Auxiliary functions

- $u(x, y) =$ the cost of going from node (x, y) to the upper node $(x + 1, y + 1)$
- $l(x, y) =$ the cost of going from node (x, y) to the lower node $(x + 1, y - 1)$
- $U(x, y) =$ the minimum expected cost from (x, y) to (ε) when the command is $u(x, y)$ or use upper path.
- $L(x, y) =$ the minimum expected cost from (x, y) to (ε) when the command is $l(x, y)$ or use lower path.

Focus on $U(x, y)$

Command: up $\begin{cases} \text{executed:} & p\,(u(x, y) + f(x + 1, y + 1)) \\ \text{not executed:} & (1 - p)\,(l(x, y) + f(x + 1, y - 1)) \end{cases}$

$U(x, y) = p(u + f(x + 1, y + 1)) + (1 - p)(l + f(x + 1, y - 1))$

Focus on $L(x, y)$

$L(x, y) = p(l + f(x + 1, y - 1)) + (1 - p)(u + f(x + 1, y + 1))$

$\mathbf{S_4}$ Recursive formula for OEF, using optimization principle.

$$f(x, y) = \min \{U(x, y), L(x, y)\}$$

$\mathbf{S_5}$ Stage $x = 3$

Nodes $(3, 1)$, $(3, -1)$

- Node $(3,1)$
  $$\begin{aligned}U(3,1) &= p(u(3,1)+f(4,2))+(1-p)(l(3,1)+f(4,0))\\ &= 0.6\cdot 6+0.4\cdot 8\\ &= 6.8\\ L(3,1) &= p(l(3,1)+f(4,0))+(1-p)(u(3,1)+f(4,2))\\ &= 0.6\cdot 8+0.4\cdot 6\\ &= 7.2\end{aligned}$$
  $\implies f(3,1)=\min\{6.8,7.2\}=6.8$
  Command: *up*
- Node $(3,-1)$
  Since $u(3,-1)=u(3,1)$ and $l(3,-1)=l(3,1)$ we get the same $U(x,y)$ and $L(x,y)$, hence $f(3,-1)=\min\{6.8,7.2\}=6.8$
  Command: *up*

Stage $x=2$

- Node $(2,0)$
  $$\begin{aligned}U(2,0) &= p(u(2,0)+f(3,1))+(1-p)(l(2,0)+f(3,-1))\\ &= 0.6\,(6+6.8)+0.4\,(4+6.8)\\ &= 0.6\cdot 12.8+0.4\cdot 10.8\\ &= 12\\ L(2,0) &= p(l(2,0)+f(3,-1))+(1-p)(u(2,0)+f(3,1))\\ &= 0.6\,(4+6.8)+0.4\,(6+6.8)\\ &= 11.6\end{aligned}$$
  $\implies f(2,0)=\min\{12,11.6\}=11.6$
  Command: *down*

Stage $x=1$ Nodes $(1,1)$, $(1,-1)$

Since at each of these nodes, we have only one choice of path, we assume we cannot go wrong with our command (cannot go *off-path*).

- Node $(1,1)$ $f(1,1)=4+f(2,0)=4+11.6=15.6$
  Command: *down*
- Node $(1,-1)$ $f(1,-1)=2+f(2,0)=2+11.6=13.6$
  Command: *up*

Stage $x=0$

- Node $O(0,0)$
  $$\begin{aligned}U(0,0) &= p(u(0,0)+f(1,1))+(1-p)(l(0,0)+f(1,-1))\\ &= 0.6\,(10+15.6)+0.4\,(12+13.6)\\ &= 25.6\\ U(0,0) &= p(l(0,0)+f(1,-1))+(1-p)(u(0,0)+f(1,1))\\ &= 0.6\,(12+13.6)+0.4\,(10+15.6)\\ &= 25.6\end{aligned}$$
  $\implies f(0,0)=\min\{25.6,25.6\}=25.6$
  Command: *any*

S$_6$ Optimal Command Policy

$(0,0)$	any
$(1,1)$	down
$(1,-1)$	up
$(2,0)$	down
$(3,1)$	up
$(3,-1)$	up

□

(β) open cycle, where paths containing the same direction three times in row (UUU or LLL) are not possible and therefore not commanded.

Solution:

S₁ Possible command sequences:

ULUU	ULUL	ULLU	ULLL
LUUU	LUUL	LULU	LULL

But we cannot give commands ULLL or LUUU since we cannot give the same command 3 times in a row.

So we have commands

ULUU	ULUL	ULLU	
	LUUL	LULU	LULL

We study each seperately.

We assume that if a command is taken, which was not executed (opposite direction taken) and hence the same direction is taken 3 times in a row (e.g. command given is ULLU but the final direction is not executed, so the actual path taken is ULLL) then this is possible.

We also assume the first two commands UL or LU are taken together, since there is only one choice of path at $x = 1$ (as discussed in the previous part).

S₂ Focus on ULUU (commands: up, down, up, up)

Path	Cost	Probability
ULUU	$10 + 4 + 6 + 6 = 26$	p^3
ULUL	$10 + 4 + 6 + 8 = 28$	$p^2(1-p)$
ULLU	$10 + 4 + 4 + 6 = 24$	$p^2(1-p)$
ULLL	$10 + 4 + 4 + 8 = 26$	$p(1-p)^2$
LUUU	$12 + 2 + 6 + 6 = 26$	$p^2(1-p)$
LUUL	$12 + 2 + 6 + 8 = 28$	$p(1-p)^2$
LULU	$12 + 2 + 4 + 6 = 24$	$p(1-p)^2$
LULL	$12 + 2 + 4 + 8 = 26$	$(1-p)^3$

Then

$$\begin{aligned}\mathbb{E}(\text{ULUU}) &= \sum(\text{Cost})(\text{probability})\\ &= 26 \times 0.6^3 + 28 \times 0.6^2 \times 0.4 + 24 \times 0.6^2 \times 0.4 + 26 \times 0.6 \times 0.4^2\\ &+ 26 \times 0.6^2 \times 0.4 + 28 \times 0.6 \times 0.4^2 + 24 \times 0.6 \times 0.4^2 + 26 \times 0.4^3\\ &= 26\end{aligned}$$

Focus on ULUL (commands: up, down, up, down)

Similarly we can work out the cost of taking each path and their corresponding probabilities

$$\begin{aligned}\mathbb{E}(\text{ULUL}) &= 28 \times 0.6^3 + 26 \times 0.6^2 \times 0.4 + 26 \times 0.6^2 \times 0.4 + 24 \times 0.6 \times 0.4^2\\ &+ 28 \times 0.6^2 \times 0.4 + 26 \times 0.6 \times 0.4^2 + 26 \times 0.6 \times 0.4^2 + 24 \times 0.4^3\\ &= 26.4\end{aligned}$$

Focus on ULLU (commands: up, down, down, up)

$$\begin{aligned}\mathbb{E}(\text{ULLU}) &= 24 \times 0.6^3 + 26 \times 0.6^2 \times 0.4 + 26 \times 0.6^2 \times 0.4 + 24 \times 0.6 \times 0.4^2\\ &+ 24 \times 0.6^2 \times 0.4 + 26 \times 0.6 \times 0.4^2 + 26 \times 0.6 \times 0.4^2 + 28 \times 0.4^3\\ &= 25.6\end{aligned}$$

Focus on LUUL (commands: down, up, up, down)

Notice that this will have the same expected cost as ULUL, since the first two steps UL and LU both have cost 14.

Therefore $\mathbb{E}(\text{LUUL}) = 26.4$

Focus on LULU (commands: down, up, down, up)

Similarly, this will have the same expected cost as ULLU, since the first two steps UL and LU both have cost 14.

Therefore $\mathbb{E}(\text{LULU}) = 25.6$

Focus on LULL (commands: down, up, down, down)

$$\begin{aligned}\mathbb{E}(\text{LULL}) &= 26 \times 0.6^3 + 24 \times 0.6^2 \times 0.4 + 28 \times 0.6^2 \times 0.4 + 26 \times 0.6 \times 0.4^2 \\ &\quad + 26 \times 0.6^2 \times 0.4 + 24 \times 0.6 \times 0.4^2 + 28 \times 0.6 \times 0.4^2 + 24 \times 0.4^3 \\ &= 26\end{aligned}$$

S_3 Minimum expected cost

$\min\{\mathbb{E}(\text{ULUU}), \mathbb{E}(\text{ULUL}), \mathbb{E}(\text{ULLU}), \mathbb{E}(\text{LUUL}), \mathbb{E}(\text{LULU}), \mathbb{E}(\text{LULL})\} = 25.6$

Optimal command sequence: ULLU or LULU

□