MA1126: Set Theory Selected Problems 2

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Problem 1: *2018 Assignment 5*

1. *Use Zorn's Lemma to prove that for any sets A and B,*

$$A < B, A \sim B \text{ or } B < A$$

Solution: *We want a function* $f : A \to B$ *that is 1-to-1.*
Suppose we have $f : C \to B$ *where* $c \subset A$*, f 1-to-1.*
Let X be the collection of these pairs (C,f)
i.e. $X = \{(C, f) \mid C \subset A, f : C \to B \ 1 - to - 1\}$
Given (C_1, f_1) *and* (C_2, f_2)*, define*
$(C_1, f_1) \leq (C_2, f_2)$ *means*
$C_1 \subset C_2$ *and* $f_2 = f_1$ *on* C_1
so f_2 *is an* <u>*extension*</u> *of* f_1
Check that this is a partial order:
Suppose $\{C_\alpha, f_\alpha\}$ *is a partially ordered subset. We want an upper bound in X*
Let $C = \bigcup C_\alpha$ *and* $f(x) = f_{\alpha_0}(x)$ *if* $x \in C_{\alpha_0}$
This is well defined and 1-to-1 (check)
So by Zorn's Lemma (Every nonempty partially ordered set - in which each totally ordered subset has an upper bound- contains at least one max element) $\exists$ *a maximal element (D,g)*

(1) If $D = A$ *and* $g(D) = B,$ *then* $A \sim B$

(2) If $D = A$ *and* $g(D) \neq B,$ *then* $A < B$

(3) If $D \neq A$ *and* $g(D) = B,$ *then* $B < A$

$g : D \to B$ *is 1-to-1 and onto so* $g^{-1} : B \to D$ *exists and is unique.*
If $D \neq A$ *and* $g(D) \neq B$
Let $a \in A \backslash D,\ b \in B \backslash f(D)$
Let $h : D \cup \{a\} \to f(D) \cup \{b\}$
$\forall d \in D, d \mapsto g(d), a \mapsto b$
then $(D \cup \{a\}, h) > (D, g)$ *but (D,g) was maximal - contradiction*
So any one of the top three statements is true. $\square$