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First Order Logic

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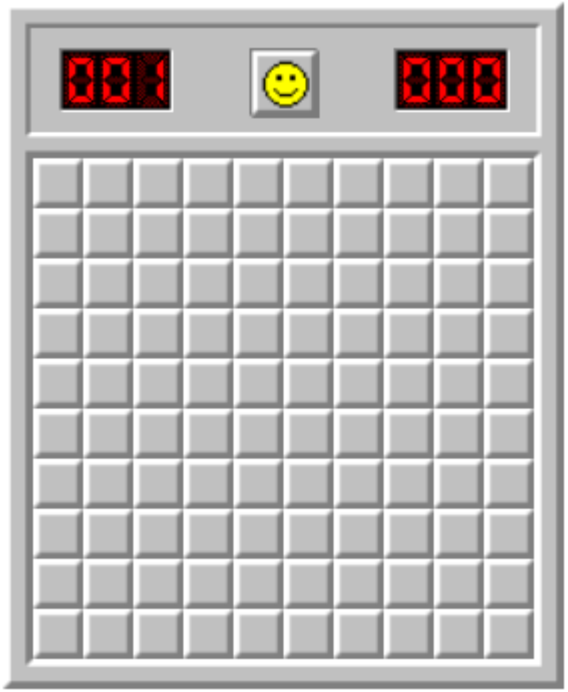
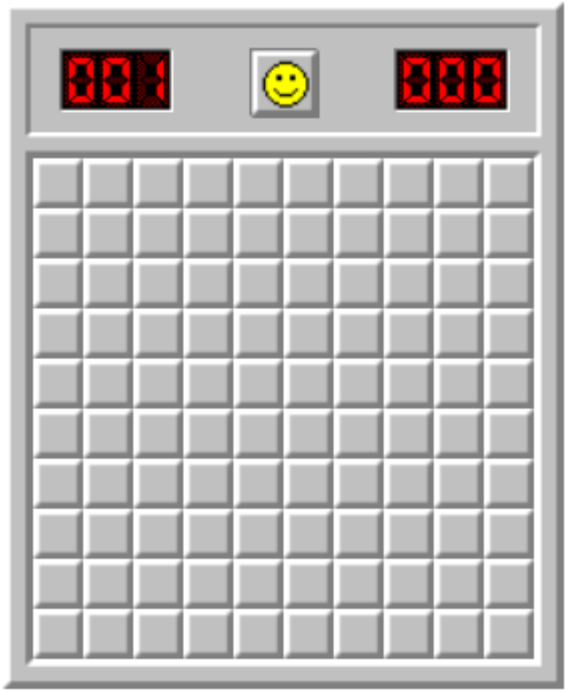
Problems with propositional logic

- For example, for cell (2,3)
  - Landmine_2_3=>number1_1_2
  - Landmine_2_3=>number1_1_3
  - Landmine_2_3=>number1_1_4
  - Landmine_2_3=>number1_2_2
  - Landmine_2_3=>number1_2_4
  - Landmine_2_3=>number1_3_2
  - Landmine_2_3=>number1_3_3
  - Landmine_2_3=>number1_3_4
- Similarly for other cells, resulting in explosion of symbols



First Order Logic

- We will discover the **first order logic** which allows to write
  - landmine(x,y)=>number1(neighbors(x,y))



Syntax of FOL

- Three types of symbols:
  - Constant symbols (capture objects): KingJohn, 2, Dartmouth
  - Predicate symbols (capture relations): Brother, >,...
  - function symbols (capture functions): Sqrt, LeftLegOf
- Connectives: $\wedge | \vee | \Rightarrow | \Leftrightarrow | \neg$ (standard)
- Equality: = Two symbols refer to the same object
- Variables: x, y, z
- Quantifiers: $\forall, \exists$; ways to refer to groups of objects.

Syntax of FOL: Connectives & Complex Sentences

- **Complex Sentences** are formed in the same way, using the same logical connectives, as in propositional logic

- The **Logical Connectives**:
  - $\Leftrightarrow$ biconditional
  - $\Rightarrow$ implication
  - $\wedge$ and
  - $\vee$ or
  - $\neg$ negation

- **Semantics** for these logical connectives are the same as we already know from propositional logic.

Examples

- Brother(Richard, John) $\wedge$ Brother(John, Richard)

- King(Richard) $\vee$ King(John)

- King(John) => $\neg$ King(Richard)

(Semantics of complex sentences are the same as in propositional logic)

Syntax of FOL

- Three types of **symbols**:
  - Constant symbols (capture objects): KingJohn, 2, Dartmouth
  - Predicate symbols (capture relations): Brother, >,...
  - function symbols (capture functions): Sqrt, LeftLegOf
- **Connectives**: $\wedge \mid \vee \mid \Rightarrow \mid \Leftrightarrow \mid \neg$ (standard)
- Equality: = Two symbols refer to the same object
- Variables: x, y, z
- Quantifiers: $\forall, \exists$; ways to refer to groups of objects.

Syntax of FOL: Variables

- **Variables** range over objects in the world.

- (A **variable not bound by a quantifier** is called **free**.)
  - All variables we will use are bound by a quantifier.

Universal quantification

- Universal (∀)
  - Sentence is true for all values of x in the domain of variable x.
  - Conjunction of all sentences obtained by substitution of an object for the quantified variable

- $\forall x\ human(x) \Rightarrow mammal(x)$
  - What it really means (universal instantiation):

    human(John)⇒mammal(John)

    (∧) human(Alice)⇒mammal(Alice)

    (∧) human(laptop)⇒mammal(laptop)

    …

Is this a correct sentence?

- $\forall x\ human(x) \land mammal(x)$

Discussion

Common mistake for universal quantification

- Common mistake is to use AND as main connective
  $\forall$ x human(x) $\wedge$ mammal(x)
  - This means everything is human and a mammal!
  - (human(Jerry) $\wedge$ mammal(Jerry $\wedge$ (human(laptop) $\wedge$ mammal(laptop)) $\wedge$ …
- **Note that => is the natural connective to use with $\forall$.**

Existential quantifiers

- Existential ($\exists$)
  - Sentence is true for some value of x in the domain of variable x
  - Is equivalent to disjunction of all sentences obtained by substitution of an object for the quantified variable.

- “some humans are male”
  - $\exists x\ human(x) \land male(x)$
  - Means there is an x who is a human and is a male
  - What it really means (existential instantiation):

    (human(Jerry) ∧ male(Jerry)) ∨

    (human(laptop) ∧ male(laptop)) ∨ …

- “Some pig can fly” $\exists$ x pig(x) => fly(x) (correct?)

Common mistake for existential quantifiers

- Common mistake is to use => as main connective
- "Some pig can fly" $\exists$ x pig(x) => fly(x) (wrong)
  - This is true if there is something not a pig!
    (pig(Jerry) => fly(Jerry)) $\vee$
    (pig(laptop) => fly(laptop)) $\vee$ …
- **Note that $\wedge$ is the natural connective to use with $\exists$.**

Combining Quantifiers – Order (Scope)

The order of “like” quantifiers does not matter.

$$\forall x\ \forall y\ P(x, y) \equiv \forall y\ \forall x\ P(x, y)$$

$$\exists x\ \exists y\ P(x, y) \equiv \exists y\ \exists x\ P(x, y)$$

Like nested ANDs and ANDs in a logical sentence

Combining Quantifiers – Order (Scope)

The order of "unlike" quantifiers is important.

Like nested ANDs and ORs in a logical sentence.

$\forall$ x $\exists$ y Loves(x,y)

- For everyone ("all x") there is someone ("exists y") whom they love.
- There might be a different y for each x (y is inside the scope of x)

$\exists$ y $\forall$ x Loves(x,y)

- There is someone ("exists y") whom everyone loves ("all x").
- Every x loves the same y (x is inside the scope of y)

Parentheses can clarify: $\exists$ y ($\forall$ x Loves(x,y))

Properties of quantifiers

- $\forall x\ P(x)$ when negated becomes ?
- $\exists\ x\ P(x)$ when negated becomes ?

Properties of quantifiers

- $\forall x\ P(x)$ when negated becomes $\exists x\ \neg P(x)$
- $\exists\ x\ P(x)$ when negated becomes $\forall x\ \neg P(x)$
- Example
  - $\forall x$ sleep(x)
    - It means everybody sleeps
  - If negated, it becomes $\exists x\ \neg$sleep(x)
    - There is somebody who doesn’t sleep

Properties of quantifiers

- $\forall x\ P(x)$ is logically equivalent to $\equiv \neg\exists x\ \neg P(x)$
- $\exists\ x\ P(x)$ is logically equivalent to $\equiv \neg\forall x\ \neg P(x)$
- Example
  - $\forall x$ sleep(x)
    - It means everybody sleeps
  - $\neg\exists x\ \neg$sleep(x)
    - There is nobody who doesn’t sleep

Connections between Quantifiers

In effect:

- $\forall$ is a conjunction over the universe of objects
- $\exists$ is a disjunction over the universe of objects

Thus, DeMorgan’s rules can be applied

De Morgan's Law for Quantifiers

De Morgan's Rule

$P \wedge Q \equiv \neg (\neg P \vee \neg Q)$
$P \vee Q \equiv \neg (\neg P \wedge \neg Q)$

$\neg (P \wedge Q) \equiv (\neg P \vee \neg Q)$
$\neg (P \vee Q) \equiv (\neg P \wedge \neg Q)$

Generalized De Morgan's Rule

$\forall x\, P(x) \equiv \neg \exists x \neg P(x)$
$\exists x\, P(x) \equiv \neg \forall x \neg P(x)$

$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$
$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$

AND/OR Rule is simple: if you bring a negation inside a disjunction or a conjunction, always switch between them ($\neg$ OR → AND $\neg$; $\neg$ AND → OR $\neg$).

QUANTIFIER Rule is similar: if you bring a negation inside a universal or existential, always switch between them ($\neg \exists \rightarrow \forall \neg$; $\neg \forall \rightarrow \exists \neg$).

Example of sentences with quantifiers

- **"All persons are mortal."**

  [Use: Person(x), Mortal (x)]

  $\forall x \; Person(x) \Rightarrow Mortal(x)$

- **Equivalent Forms:**

  $\forall x \; \neg Person(x) \vee Mortal(x)$

- **Common Mistakes:**

  $\forall x \; Person(x) \wedge Mortal(x)$

Example

Example of sentences with quantifiers

- **“Sissy has a sister who is a cat.”**

  [Use: Sister(Sissy, x), Cat(x)]

  $\exists x \; \text{Sister}(\text{Sissy}, x) \wedge \text{Cat}(x)$

- **Common Mistakes:**

  $\exists x \; \text{Sister}(\text{Sissy}, x) \Rightarrow \text{Cat}(x)$

Example

Example of sentences with quantifiers

- **"For every food, there is a person who eats that food."**

  [Use: Food(x), Person(y), Eats(y, x)]

  $\forall x\ \exists y\ Food(x) \Rightarrow [\ Person(y) \wedge Eats(y, x)\ ]$

- **Equivalent Forms:**

  $\forall x\ Food(x) \Rightarrow \exists y\ [\ Person(y) \wedge Eats(y, x)\ ]$

  $\forall x\ \exists y\ \neg Food(x) \vee [\ Person(y) \wedge Eats(y, x)\ ]$

  $\forall x\ \exists y\ [\ \neg Food(x) \vee Person(y)\ ] \wedge [\neg Food(x) \vee Eats(y, x)\ ]$

  $\forall x\ \exists y\ [\ Food(x) \Rightarrow Person(y)\ ] \wedge [\ Food(x) \Rightarrow Eats(y, x)\ ]$

- **Common Mistakes:**

  $\forall x\ \exists y\ [\ Food(x) \wedge Person(y)\ ] \Rightarrow Eats(y, x)$

  $\forall x\ \exists y\ Food(x) \wedge Person(y) \wedge Eats(y, x)$

Example

Example of sentences with quantifiers

- **"Every person eats some food."**

  [Use: Person (x), Food (y), Eats(x, y)]

  $\forall x\ \exists y\ Person(x) \Rightarrow [\ Food(y) \wedge Eats(x, y)\ ]$

- **Equivalent Forms:**

  $\forall x\ Person(x) \Rightarrow \exists y\ [\ Food(y) \wedge Eats(x, y)\ ]$

  $\forall x\ \exists y\ \neg Person(x) \vee [\ Food(y) \wedge Eats(x, y)\ ]$

  $\forall x\ \exists y\ [\ \neg Person(x) \vee Food(y)\ ] \wedge [\ \neg Person(x) \vee Eats(x, y)\ ]$

- **Common Mistakes:**

  $\forall x\ \exists y\ [\ Person(x) \wedge Food(y)\ ] \Rightarrow Eats(x, y)$

  $\forall x\ \exists y\ Person(x) \wedge Food(y) \wedge Eats(x, y)$

Example

Example of sentences with quantifiers

- **"Some person eats some food."**

  [Use: Person (x), Food (y), Eats(x, y)]

  $\exists x\ \exists y\ \text{Person}(x) \wedge \text{Food}(y) \wedge \text{Eats}(x, y)$

- **Common Mistakes:**

  $\exists x\ \exists y\ [\ \text{Person}(x) \wedge \text{Food}(y)\ ] \Rightarrow \text{Eats}(x, y)$

Example

Example of sentences with quantifiers

- **“Everyone has a favorite food.”**

  [Use: Person(x), Food(y), Favorite(y, x)]

- **Equivalent Forms:**
- $\forall x\ \exists y\ \text{Person}(x) \Rightarrow [\ \text{Food}(y) \wedge \text{Favorite}(y, x)\ ]$
- $\forall x\ \text{Person}(x) \Rightarrow \exists y\ [\ \text{Food}(y) \wedge \text{Favorite}(y, x)\ ]$
- $\forall x\ \exists y\ \neg\text{Person}(x) \vee [\ \text{Food}(y) \wedge \text{Favorite}(y, x)\ ]$
- $\forall x\ \exists y\ [\ \neg\text{Person}(x) \vee \text{Food}(y)\ ] \wedge [\ \neg\text{Person}(x) \vee \text{Favorite}(y, x)\ ]$
- $\forall x\ \exists y\ [\text{Person}(x) \Rightarrow \text{Food}(y)\ ] \wedge [\ \text{Person}(x) \Rightarrow \text{Favorite}(y, x)\ ]$

- **Common Mistakes:**
- $\forall x\ \exists y\ [\ \text{Person}(x) \wedge \text{Food}(y)\ ] \Rightarrow \text{Favorite}(y, x)$
- $\forall x\ \exists y\ \text{Person}(x) \wedge \text{Food}(y) \wedge \text{Favorite}(y, x)$

Example

Equality

- $term_1 = term_2$ is true

  if and only if $term_1$ and $term_2$ refer to the same object

- E.g., definition of *Sibling* in terms of *Parent*, using = is:

$$\forall x,y\ Sibling(x,y) \Leftrightarrow$$
$$[\neg(x = y) \wedge$$
$$\exists m,f\ \ \neg(m = f) \wedge Parent(m,x) \wedge Parent(f,x)$$
$$\wedge\ Parent(m,y) \wedge\ Parent(f,y)]$$

Semantics

- sentences + (model, interpretation) ↦ true/false
- interpretation specifies exactly which objects, relations, and functions are referred to by the constant, predicate, and function symbols.

  '=' sign is used

Models

- A set of true/false values for every relation among objects. (Think of a set of directed edges, with different colors for each relation, of graph.)

How many models?

- For each binary relation, there are n^2 possible object pairs (2-tuples), n^3 possible ternary relations, n^k possible k-ary relations.
- That's just the number of tuples. Each can be true or false. So for each relation, we get a factor of 2^(n^k) models.
- n might be infinite. (Maybe the objects are natural numbers, which can be described in FOL.)

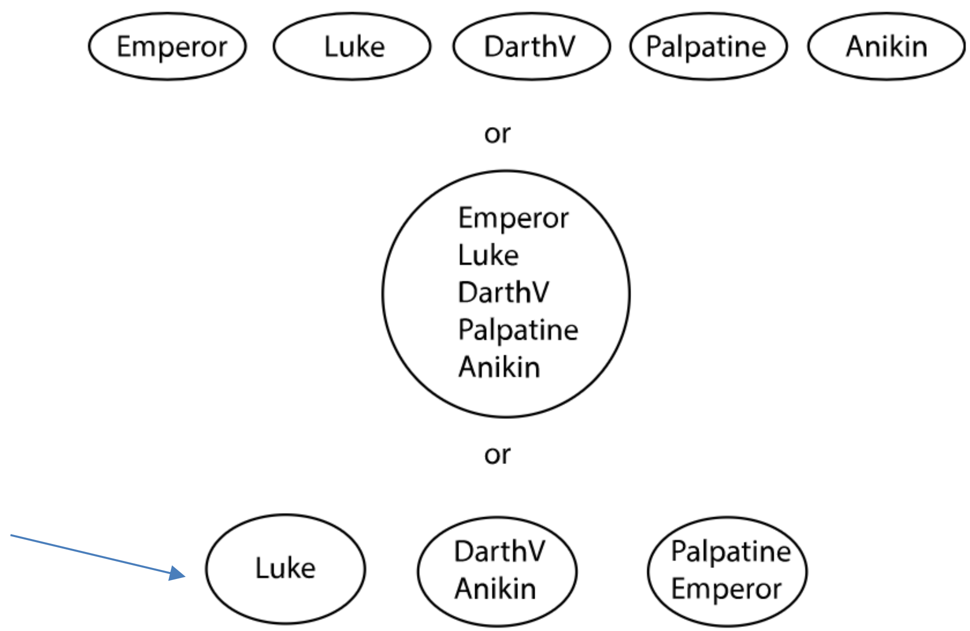
Discussion

Interpretation

- Computational complexity gets even worse

Example

- Symbols: Luke, DarthVader, Emperor, Palpatine, Anikin. Five symbols, but how many objects?



Intended interpretation

Syntactic Ambiguity

- FOL provides many ways to represent the same thing.
- E.g., "Ball-5 is red."
  - HasColor(Ball-5, Red)
    - Ball-5 and Red are objects related by HasColor.
  - Red(Ball-5)
    - Red is a unary predicate applied to the Ball-5 object.
  - HasProperty(Ball-5, Color, Red)
    - Ball-5, Color, and Red are objects related by HasProperty.
  - ColorOf(Ball-5) = Red
    - Ball-5 and Red are objects, and ColorOf() is a function.
  - HasColor(Ball-5(), Red())
    - Ball-5() and Red() are functions of zero arguments that both return an object, which objects are related by HasColor.
  - …
- This can GREATLY confuse a pattern-matching reasoner.
  - Especially if multiple people collaborate to build the KB, and they all have different representational conventions.

Syntactic Ambiguity – Partial solution

- FOL can be TOO expressive, can offer TOO MANY choices

- Likely confusion, especially for **teams** of Knowledge Engineers

- Different team members can make different representation choices
  - E.g., represent "Ball43 is Red." as:
    - a property (= adjective)? E.g., "Red(Ball43)" ?
    - an object (= noun)? E.g., "Red = Color(Ball43))" ?
    - a predicate (= verb)? E.g., "HasProperty(Ball43, Red)" ?

- PARTIAL SOLUTION:
  - An upon-agreed **ontology** that settles these questions
  - Ontology = what exists in the world & how it is represented
  - The Knowledge Engineering teams agrees upon an ontology BEFORE they begin encoding knowledge

Summary

- First order logic to represent also objects and relations
  - Syntax includes sentences, predicate symbols, function symbols, constant symbols, variables, quantifiers
- Nested quantifiers
  - Order of unlike quantifiers matters (the outer scopes the inner)
    - Like nested ANDs and ORs
  - Order of like quantifiers does not matter
    - like nested ANDS and ANDs
- Semantics needs also interpretation

Next

- How do we make inference with FOL?

Logical agent with FOL

- Sentences are added to a knowledge base using TELL
  - TELL(KB, King(John)) .
  - TELL(KB, Person(Richard))
  - TELL(KB, $\forall$ x King(x) $\Rightarrow$ Person(x)) .
- We can ask questions of the knowledge base using ASK. E.g.,
  - ASK(KB, King(John)) returns true.
- Query that is logically entailed by the knowledge base should be answered affirmatively.
  - E.g., given the two preceding assertions, the query ASK(KB, Person(John)) should also return true.
- We can ask quantified queries, such as
  - ASK(KB, $\exists$ x Person(x)) .
  - True answer, but not very helpful. It is like answering “Can you tell me the time?” with “Yes.”
- ASKVARS returns what value of x makes the sentence true
  - ASKVARS(KB, Person(x))
  - E.g., there will be two answers: {x/John} and {x/Richard} -- answer called a **substitution** or **binding list**.

FOL Version of Wumpus World

- Suppose a wumpus-world agent is using an FOL KB and perceives a smell and a breeze (but no glitter) at t = 5:

- Typical percept sentence:
  TELL(KB, Percept([Stench,Breeze,not Glitter, None, None],t=5))

- Actions:
  Turn(Right), Turn(Left), Forward, Shoot, Grab, Release, Climb

- To determine best action, construct query:

  ASK(∃ a BestAction(a,5))

  ASKVARS(BestAction(a,5))

- Inference to return {a/Grab}

Knowledge Base for Wumpus World

- Perception
  - $\forall$s,g,x,y,t Percept([s,Breeze,g,x,y],t) $\Rightarrow$ Breeze(t)
  - $\forall$s,b,x,y,t Percept([s,b,Glitter,x,y],t) $\Rightarrow$ Glitter(t)
- Reflex action
  - $\forall$t Glitter(t) $\Rightarrow$ BestAction(Grab,t)
- Reflex action with internal state
  - $\forall$t Glitter(t) $\wedge\neg$Holding(Gold,t) $\Rightarrow$ BestAction(Grab,t)

Holding(Gold,t) can not be observed: keep track of change.

Deducing hidden properties

Environment definition:

$\forall$x,y,a,b *Adjacent*([x,y],[a,b]) $\Leftrightarrow$
[a,b] $\in$ {[x+1,y], [x-1,y],[x,y+1],[x,y-1]}

Properties of locations:

$\forall$s,t *At*(Agent,s,t) $\wedge$ Breeze(t) $\Rightarrow$ Breezy(s)

Squares are breezy near a pit:

- Diagnostic rule---infer cause from effect
  $\forall$s Breezy(s) $\Leftrightarrow$ $\exists$ r Adjacent(r,s) $\wedge$ Pit(r)
- Causal rule---infer effect from cause (model based reasoning)
  $\forall$r Pit(r) $\Rightarrow$ [$\forall$s Adjacent(r,s) $\Rightarrow$ Breezy(s)]

Knowledge engineering in FOL

1. Identify the task
2. Assemble the relevant knowledge
3. Decide on a vocabulary of predicates, functions, and constants
4. Encode general knowledge about the domain
5. Encode a description of the specific problem instance
6. Pose queries to the inference procedure and get answers
7. Debug the knowledge base

An interpretation maps all symbols in KB onto matching symbols in a possible world. All possible interpretations gives a combinatorial explosion of mappings. Your job, as a Knowledge Engineer, is to write the axioms in KB so **they are satisfied only under the intended interpretation in your own real world.**