qbs120_ps2_correction_gibran

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Question 1

a. my original solution was correct.
b. my original solution was correct.
c. my original solution was correct.
d. I had a typo on my original solution. Here's the correct one:
   The conditional density of Y|X, defined for $0 \leq y \leq x$, is:

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{\frac{xy}{2}}{\frac{x^3}{4}} \\
&= \frac{2xy}{x^3} \\
&= \frac{2y}{x^2}
\end{aligned}
$$

The conditional density of X|Y, defined for $y \leq x \leq 2$, is:

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{\frac{xy}{2}}{y - \frac{y^3}{4}} \\
&= \frac{2xy}{4y - y^3} \\
&= \frac{2x}{4 - y^2}
\end{aligned}
$$

Question 2

As a standard uniform RV, we know that the density of X_1 is:

$$f_{X_1(x_1)} = 1, 0 \leq x_1 \leq 1$$

We are also told that the conditional density of X_2 is U(X_1, 2):

$$f_{X_1 X_2(x_1,x_2)} = f_{X_1}(x_1 f_{X_2}(x_2|x_1),\ 0 \leq x_1 \leq x_2 \leq 2$$

The joint density can be found directly from these by definition:

$$
\begin{aligned}
f_{X_1X_2(x_1,x_2)} &= f_{X_1}(x_1 f_{X_2}(x_2|x_1) \\
&= 1 * 1/(2 - x1) \\
&= 1/(2 - x_1), with\ 0 \leq x_1 \leq 1 and x_1 \leq x_2 \leq 2
\end{aligned}
$$

Question 3

optional

Question 4

my original solution was correct.

Question 5

We know that the expectation function g(X) for a continuous RV is:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\ dx$$

with $f(x) = \frac{1}{b-a} = \frac{1}{4-1} = \frac{1}{3}$
In this question, $g(X) = 1/X$, with $1 \leq X \leq 4$, so the function becomes:

$$
\begin{aligned}
E[1/X] &= \int_{1}^{4} \frac{1}{X} f(x)\ dx \\
&= \int_{1}^{4} \frac{1}{3X}\ dx \\
&= [ln(x)/3]_1^4 \\
&= ln(4)/3
\end{aligned}
$$

to get $\frac{1}{E[X]}$, we need to compuite the expectation of X:

$$
\begin{aligned}
E[X] &= \int_{1}^{4} X f(x)\ dx \\
&= \int_{1}^{4} \frac{x}{3}\ dx \\
&= [x^2/6]_1^4 \\
&= 16/6 - 1/6 \\
&= 15/6 \\
&= 2.5
\end{aligned}
$$

Thus,

$$1/E[X] = 0.4$$

So, $E[1/X] \neq 1/E[X]$

Question 6

a. Show that $E[Z] = \mu$

We want to find the expectation of a linear function of RVs. With jointly distributed RVs $X_1, X_2, ..., X_n$ and $Y = a\sum_{i=1}^{n} b_i X_i$, E[Y] is a linear function of the $E[X_i]$ (expectation of sum is sum of expectations):

$$E[Y] = a\sum_{i=1}^{n} b_i E[X_i]$$

Plugging in $Z = \alpha X + (1-\alpha)Y$, we get:

$$\begin{aligned} E[Z] &= \alpha E[X] + (1-\alpha)E[Y] \\ &= \alpha\mu + (1-\alpha)\mu \\ &= \mu \end{aligned}$$

b. If X and Y are not independent, what is $E[Z]$?

The result is the same ($E[Z] = \mu$) since Theorem A holds regardless of whether the RVs are independent or not.

c. What is Var(Z)? Does this result hold if X and Y are not independent?

my original solution was correct.

d. Find α in terms of σX and σY to minimize Var(Z).

my original solution was correct.

e. Under what circumstances is it better to use the average (X + Y)/2 than either X or Y alone?

Note that (X+Y)/2 is Z with $\alpha = 1/2$, and the expected value is the same in all cases, per the result from part a). So, the only difference will be the variance and a smaller variance is typically desirable (less uncertainty in the outcome). Using the result for Var(Z) from part b), we know that the variance when $\alpha = 0.5$ is:

$$\begin{aligned} Var(Z) &= \alpha^2\sigma_X^2 + (1 - 2\alpha + \alpha^2)\sigma_Y^2 \\ &= \sigma_X^2/4 + (1 - 1 + 1/4)\sigma_Y^2 \\ &= \sigma_X^2/4 + \sigma_Y^2/4 \end{aligned}$$

When is this less than $Var(X) = \sigma_X^2$? In that case, we would rather use Z than X alone to minimize the variance:

$$\begin{aligned} \sigma_X^2/4 + \sigma_Y^2/4 &< \sigma_X^2 \\ \frac{3\sigma_X^2}{4} &> \sigma_Y^2/4 \\ \frac{\sigma_X^2}{\sigma_Y^2} &> 1/3 \end{aligned}$$

When would we prefer to use Z vs Y alone? That will be when $Var(Z) < Var(Y)$. By symmetry, we know that will be $\frac{\sigma_X^2}{\sigma_Y^2} > 1/3$ or $\frac{\sigma_Y^2}{\sigma_X^2} < 3$.

Combining these inequalities yield us:

$$1/3 < \frac{\sigma_X^2}{\sigma_Y^2} < 3$$

Question 7

get E[XY]

If X and Y are independent, then E[XY] = E[X]E[Y]:

$$E[XY] = E[X]E[Y] = \mu_X \mu_Y$$

get Var(XY)

By definition, here's the formula:

$$Var(XY) = E[(XY)^2] - E[XY]^2$$

We understand that $E[XY] = E[X]E[Y]$. To find $E[(XY)^2] = E[X^2Y^2]$, remember that functions of independent RV are also independent. So, E[X$^{2\text{Y}}$2] = E[Xˆ2]E[Yˆ2]. So,

$$\begin{aligned} Var(XY) &= E[X^2]E[Y^2] - (E[X]E[Y])^2 \\ &= E[X^2]E[Y^2] - E[X]^2E[Y]^2 \end{aligned}$$

$E[X]^2$ and $E[Y]^2$ are just squares of the marginal expectations, so these are okay to include. However, what to do with $E[X^2]E[Y^2]$? The trick here is to recognize that we can re-express $E[X^2]$ as $Var(X) + E[X]^2$, which includes just marginal variance and expectation terms. Using the μ and σ^2 notation, this becomes:

$$\begin{aligned} Var(XY) &= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2 \\ Var(XY) &= \sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \mu_X^2\sigma_Y^2 \end{aligned}$$