SIAM J. CONTROL OPTIM.
Vol. 56, No. 4, pp. 2463–2484

CONVERGENCE PROPERTIES OF ADAPTIVE SYSTEMS AND THE DEFINITION OF EXPONENTIAL STABILITY*

BENJAMIN M. JENKINS†, ANURADHA M. ANNASWAMY†, EUGENE LAVRETSKY‡, AND TRAVIS E. GIBSON§

Abstract. The convergence properties of adaptive systems in terms of excitation conditions on the regressor vector are well known. With persistent excitation of the regressor vector in model reference adaptive control the state error and the adaptation error are globally exponentially stable or, equivalently, exponentially stable in the large. When the excitation condition, however, is imposed on the reference input or the reference model state, it is often incorrectly concluded that the persistent excitation in those signals also implies exponential stability in the large. The definition of persistent excitation is revisited so as to address some possible confusion in the adaptive control literature. It is then shown that persistent excitation of the reference model only implies local persistent excitation (weak persistent excitation). Weak persistent excitation of the regressor is still sufficient for uniform asymptotic stability in the large, but not exponential stability in the large. We show that there exists an infinite region in the state-space of adaptive systems where the state rate is bounded. This infinite region with finite rate of convergence is shown to exist not only in classic open-loop reference model adaptive systems but also in a new class of closed-loop reference model adaptive systems.

Key words. adaptive control, asymptotic stability, exponential stability, persistence of excitation, weak persistence of excitation

AMS subject classifications. 93C40, 93D20, 37C75, 34D23

DOI. 10.1137/15M1047805

1. Introduction.

It is well known that stability of the origin and asymptotic convergence of the tracking error to zero can be guaranteed in adaptive systems with no restrictions on the external reference input. Asymptotic stability, i.e., convergence of both the tracking error *and* parameter error to zero, can occur only when further conditions of persistent excitation are satisfied. The first known work on asymptotic stability of adaptive systems can be found in [27]. In that work asymptotic stability of adaptive schemes was proven for a class of periodic inputs using results from [26]. The results hinged on a sufficient condition related to the richness of frequency content in the regressor vector of the adaptive system. In the late 70s and early 80s several attempts were made to extend the results of [27] to uniform asymptotic stability. Morgan and Narendra proved necessary and sufficient conditions for uniform asymptotic stability for classes of linear time-varying systems in [31, 32] that are consistent with the structure of adaptive systems. Anderson leveraged techniques developed in [3] to prove the exponential stability of adaptive systems in [1] with Kreisselmeier using similar techniques in [24]. Following these results the persistent excitation conditions for asymptotic stability were moved from the regressor vector to richness conditions on the actual reference model input in references [40, 2, 5, 6, 34]. This was a key step

*Received by the editors November 12, 2015; accepted for publication (in revised form) August 8, 2017; published electronically July 5, 2018. A preliminary version of this paper appears as *Adaptive Control and the Definition of Exponential Stability* in Proceedings of the American Control Conference, 2015.
http://www.siam.org/journals/sicon/56-4/M104780.html

†Department of Mechanical Engineering, MIT, Cambridge, MA 02139 (bjenkins@mit.edu, aanna@mit.edu).
‡Boeing Research and Technology, Huntington Beach, CA 92648 (eugene.lavretsky@boeing.com).
§Brigham and Women's Hospital, Harvard Medical School, Boston, MA 02115 (tgibson@mit.edu, tegibson@bwh.harvard.edu).

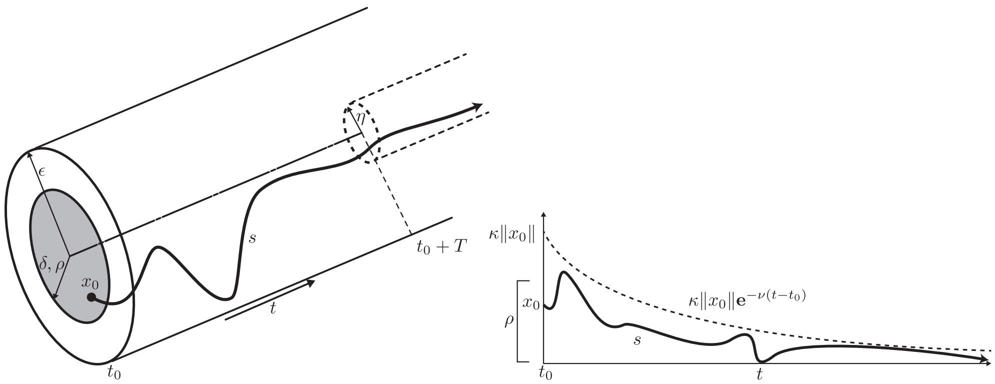
for practical reasons as the control engineer has direct control over the reference input rather than the regressor.

A key distinction exists, however, between the stability properties of the linear time-varying systems studied in [1] and those of the adaptive systems in [32], and forms the starting point for the discussions in this paper. The linear time-varying system in [1] can be shown to be *exponentially stable* under persistent excitation conditions on the underlying regressor. However, once the excitation condition is moved to the reference input, the adaptive systems in [32] can only be shown to be *uniformly asymptotically stable* (UAS). This distinction arises from the endogenous nature of the underlying regressor and is explicitly pointed out in this paper. The degree of persistent excitation of the regressor is dependent on the adaptive system initial conditions. This dependency prevents a uniform correlation between degree of persistent excitation (i.e., rate of exponential convergence) of the adaptive system's internal regressor and the richness of the reference input. The practical implication of this is that the adaptive systems of [32] are not exponentially stable in the large. This distinction between local and global exponential stability is an essential detail when the exponential stability of a system is used to claim robustness properties. Moreover, an infinite region will be shown to exist in the velocity field where the norm of the error velocity is finite. As a result, a subset of the state-space will be shown to exist where the error signals move arbitrarily slowly. Unlike exponentially stable systems, the system's convergence speed decreases as the distance from the equilibrium increases.

The implications of the above property on adaptive systems lie in their robustness. If an unperturbed system is UAS in the large, one cannot guarantee global boundedness of the perturbed system [7, 33]. In contrast, if an unperturbed system is exponentially stable, it is easy to show that the perturbed system will exponentially converge to a compact set whose size is proportional to the size of the perturbation. That is, even with an external input that is sufficiently rich, as the adaptive system only exhibits uniform asymptotic stability in the large, its boundedness in the presence of external disturbances cannot be guaranteed [33], i.e., its robustness is not easy to obtain.

Recently, a new class of adaptive systems has been under discussion (see [14, 13, 11, 12, 8, 9]) which employ a closed-loop in the underlying reference model. These adaptive systems have desirable transient response characteristics such as an improved tracking error whose L-infinty and L-2 norms are small compared to their open-loop counterparts. In addition, the rates of closed-loop signals such as the control input and control parameter have small magnitudes when compared to open-loop reference model (ORM) adaptive systems. In [19, 36], it was shown that the region of slow convergence that is present in the standard adaptive system with ORM [20] is present in this new class of closed-loop reference model (CRM) based adaptive systems as well.

This article is intended to be a cautionary piece and complements the works of [37] and [28] in carefully defining persistent excitation and a weaker condition that is not uniform in initial conditions. Whereas [37] and [28] focus on the various stability results when two different kinds of persistent excitation are studied, we illustrate why, in general, adaptive systems cannot satisfy the original definition of persistent excitation. We pick up where [34] left off and in so doing hope to clarify the true stability properties of adaptive systems. We connect the stability results of general adaptive systems to the region of slow convergence in low dimensional adaptive systems that occur with ORM and CRM. The paper is organized as follows: section 2 reviews the



(a) Stability and asymptotic stability visualized.

(b) Exponential stability visualized.

FIG. 1. *Visual aids for stability discussion.*

definitions for various kinds of stability, section 3 discusses the relationship between persistent excitation and asymptotic stability of adaptive systems, section 4 constructs examples and proves the lack of exponential stability in the large even for low order adaptive systems, section 5 contains simulation results which depict the nature of this slow convergence, and section 6 summarizes our findings.

2. Stability definitions.

Consider a dynamical system defined by the following relations:

$$\begin{aligned} x(t_0) &= x_0, \\ \dot{x}(t) &= f(x(t), t), \end{aligned}$$

where $t \in [t_0, \infty)$ is time and $x \in \mathbb{R}^n$ denotes the state vector. We are interested in systems with equilibrium at $x = 0$, so that $f(0, t) = 0$ for all t. The solution to the differential equation above for $t \geq t_0$ is a transition function $s(t; x_0, t_0)$ such that $\dot{s}(t; x_0, t_0) = f(s(t; x_0, t_0), t)$ and $s(t_0; x_0, t_0) = x_0$. Various definitions of stability now follow [30, 22, 17]. Figure 1 can be used as an aid.

DEFINITION 1 (stability and asymptotic stability). *Letting $t_0 \geq 0$, the equilibrium is*

(i) stable *if for all $\epsilon > 0$ there exists a $\delta(\epsilon, t_0) > 0$ such that $\|x_0\| \leq \delta$ implies $\|s(t; x_0, t_0)\| \leq \epsilon$ for all $t \geq t_0$;*

(ii) attracting *if there exists a $\rho(t_0) > 0$ such that for all $\eta > 0$ there exists an attraction time $T(\eta, x_0, t_0)$ such that $\|x_0\| \leq \rho$ implies $\|s(t; x_0, t_0)\| \leq \eta$ for all $t \geq t_0 + T$;*

(iii) asymptotically stable *if it is stable and attracting;*

(iv) uniformly stable *if the δ in* (i) *is uniform in t_0 and x_0, thus taking the form $\delta(\epsilon)$;*

(v) uniformly attracting *if it is attracting where the ρ and T do not depend on t_0 or x_0 and thus the attracting time takes the form $T(\eta, \rho)$;*

(vi) uniformly asymptotically stable UAS *if it is uniformly stable and uniformly attracting;*

(vii) uniformly bounded *if for all $r > 0$ there exists a $B(r)$ such that $\|x_0\| \leq r$ implies that $\|s(t; x_0, t_0)\| \leq B$ for all $t \geq t_0$;*

(viii) uniformly attracting in the large *if for all* $\rho > 0$ *and* $\eta > 0$ *there exists a* $T(\eta, \rho)$ *such that* $\|x_0\| \le \rho$ *implies* $\|s(t; x_0, t_0)\| \le \eta$ *for all* $t \ge t_0 + T$*;*

(ix) uniformly asymptotically stable in the large (UASL) *if it is uniformly stable, uniformly bounded, and uniformly attracting in the large.*

The definitions of exponential stability are not as prevalent as those above and are assembled below [30, 29].

Definition 2 (exponential stability). *Letting* $t_0 \ge 0$*, the equilibrium is*

(i) exponentially stable *if for every* $\rho > 0$ *there exists* $\nu(\rho) > 0$ *and* $\kappa(\rho) > 0$ *such that* $\|x_0\| \le \rho$ *implies* $\|s(t; x_0, t_0)\| \le \kappa \|x_0\| \mathrm{e}^{-\nu(t-t_0)}$*;*

(ii) exponentially stable in the large (ESL) *if there exists* $\nu > 0$ *and* $\kappa > 0$ *such that* $\|s(t; x_0, t_0)\| \le \kappa \|x_0\| \mathrm{e}^{-\nu(t-t_0)}$ *for all* x_0*.*

Remark 1. ESL implies UASL by choosing $T(\rho, \eta) = \frac{1}{\nu} \log\left(\frac{\kappa\rho}{\eta}\right)$. It is clear that for UASL, T is a function of both ρ and η. But for ESL, T depends only on η/ρ. In other words, if in a system it can be shown that T is a general function of η and ρ and varies even when η/ρ is a constant, then it follows that the associated equilibrium is only UASL and not ESL.

For linear systems, i.e., $\dot{x} = A(t)x$, UAS implies ESL [22, Theorem 3, (C), (D)]. Thus, for linear systems all of the definitions are equivalent. The relationship between these definitions of stability are illustrated in the following implication diagram:

$$\begin{tikzcd}
\text{ESL} \arrow[r, Rightarrow] & \text{ES} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
& \text{UASL} \arrow[r, Rightarrow] & \text{UAS} \arrow[ull, bend left, Rightarrow, "+\ \text{Linear}"]
\end{tikzcd}$$

3. Asymptotic and exponential stability of adaptive systems. We now present two adaptive systems which arise in the context of identification and control. The following definition of persistent excitation is relevant for exponential stability of adaptive systems.

Definition 3 (persistent excitation). *Let* $\omega \in [t_0, \infty) \to \mathbb{R}^p$ *be a time-varying parameter with initial condition defined as* $\omega_0 = \omega(t_0)$*; then the parameterized function of time* $y(t, \omega) : [t_0, \infty) \times \mathbb{R}^p \to \mathbb{R}^m$ *is*

(i) persistent excitation (PE) *if there exists* $T > 0$ *and* $\alpha > 0$ *such that*

$$\int_t^{t+T} y(\tau, \omega) y^{\mathsf{T}}(\tau, \omega) d\tau \succeq \alpha I$$

for all $t \ge t_0$ *and* $\omega_0 \in \mathbb{R}^p$*, and we denote this as* $y(t, \omega) \in \mathrm{PE}$*;*

(ii) weak persistent excitation ($\mathrm{PE}^*(\omega, \Omega)$) *if there exists a compact set* $\Omega \subset \mathbb{R}^p$*,* $T(\Omega) > 0, \alpha(\Omega)$ *such that*

$$\int_t^{t+T} y(\tau, \omega) y^{\mathsf{T}}(\tau, \omega) d\tau \succeq \alpha I$$

for all $\omega_0 \in \Omega$ *and* $t \ge t_0$*, and we denote this as* $y(t, \omega) \in \mathrm{PE}^*(\omega, \Omega)$*.*

The PE definition is well known in the literature [35, 18, 34], while the weak PE, denoted as PE^*, is introduced in this paper and will be used to characterize convergence in adaptive systems.

3.1. Identification in simple algebraic systems [35]. Let $u : [t_0, \infty) \to \mathbb{R}^n$ be the input and $y : [t_0, \infty) \to \mathbb{R}$ be the output of the following algebraic system of equations:

$$y(t) = u^{\mathsf{T}}(t)\theta,$$

where $\theta \in \mathbb{R}^n$ is an unknown parameter. If we assume that u is known and y is measurable, then an estimate of the unknown parameter $\hat{\theta} : [t_0, \infty) \to \mathbb{R}^n$ can be used in constructing an adaptive observer

$$\hat{y}(t) = u^{\mathsf{T}}(t)\hat{\theta}(t),$$

where the update for the estimate of the uncertain parameter is defined as

$$\dot{\hat{\theta}}(t) = -u(t)\left(\hat{y}(t) - y(t)\right).$$

Denoting the parameter error as $\phi(t) = \hat{\theta}(t) - \theta$ the parameter error evolves as

$$\dot{\phi}(t) = -u(t)u^{\mathsf{T}}(t)\phi(t). \tag{1}$$

THEOREM 1. *If $u(t)$ is PE, piecewise continuous, and either* (a) *there exists $\beta > 0$ such that*

$$\int_t^{t+T} u(\tau)u^{\mathsf{T}}(\tau)d\tau \preceq \beta I$$

or (b) *there exists a $u_{\max} > 0$ such that $\|u(t)\| \leq u_{\max}$, then for the dynamics in* (1) *the equilibrium $\phi = 0$ is ESL.*

The proof is given in two flavors: the first follows that of [1] and the second follows that of [35], and then the two methods are compared.

Proof of the theorem following Anderson [1, proof of Theorem 1]. The existence of T, α, and β such that $\alpha I \preceq \int_t^{t+T} u(\tau)u^{\mathsf{T}}(\tau)d\tau \preceq \beta I$ is equivalent to the following system being *uniformly completely observable*: $\Sigma_1 :\ \dot{x}_1 = 0_{n\times n}x_1,\ y_1 = u^{\mathsf{T}}(t)x_1$ [21, Definition (5.23), dual of (5.13)]. This in turn implies that $\Sigma_2 : \dot{x}_2 = -u(t)u^{\mathsf{T}}(t)x_2$, $y_2 = u^{\mathsf{T}}(t)x_2$ is uniformly completely observable as well [3, dual of Theorem 4]. Therefore, there exists α_2 and β_2 such that

$$\alpha_2 I \preceq \int_t^{t+T} \Phi_2^{\mathsf{T}}(\tau, t)u(\tau)u^{\mathsf{T}}(\tau)\Phi_2(\tau, t)d\tau \preceq \beta_2 I, \tag{2}$$

where $\Phi_2(t, t_0)$ is the state transition matrix for Σ_2. Note that the upper bound β is needed to ensure that $\Phi_2(\tau, t)$ is not singular,

$$\det \Phi_2(t, t_0) = \exp\left[-\int_{t_0}^{\mathsf{T}} \operatorname{trace}(u(\tau)u^{\mathsf{T}}(\tau))\, d\tau\right].$$

Let $V(\phi, t) = \frac{1}{2}\phi^{\mathsf{T}}(t)\phi(t)$ and note that Σ_2 and (1) have the same state transition matrix. Thus $\phi(t; t_0) = \Phi_2(t, t_0)\phi(t_0)$. Differentiating V along the system trajectories in (1) we have $\dot{V}(\phi, t; t_0) = -\phi^{\mathsf{T}}(t_0)\Phi_2^{\mathsf{T}}(t, t_0)u(t)u^{\mathsf{T}}(t)\Phi_2(t, t_0)\phi(t_0)$. Using the bound in (2) and integrating as $\int_t^{t+T} \dot{V}(\phi, \tau; t)d\tau$, it follows that $V(t+T) - V(t) \leq -2\alpha_2 V(t)$. Thus $V(t+T) \leq (1 - 2\alpha_2)V(t)$ and therefore the system is UASL and due to linearity it follows that the systems is ESL. ∎

Proof of the theorem following Narendra and Annaswamy [35, proof of Theorem 2.16]. First we note that $u(t)$ being PE is equivalent to

$$\int_t^{t+T} |u^\mathsf{T}(\tau)w|^2 d\tau \geq \alpha$$

holding for any fixed unitary vector w. Let $\tilde{u}(t) \triangleq \frac{u(t)}{u_{\max}}$; then it follows that

$$\begin{aligned}\int_t^{t+T} |u^\mathsf{T}(\tau)w|^2 d\tau &= u_{\max}^2 \int_t^{t+T} |\tilde{u}^\mathsf{T}(\tau)w|^2 d\tau \\ &\leq u_{\max}^2 \int_t^{t+T} |\tilde{u}^\mathsf{T}(\tau)w| d\tau,\end{aligned}$$

where the second line of the above inequality follows due to the fact that $\|\tilde{u}\| \leq 1$ and thus $|\tilde{u}^\mathsf{T}(\tau)w|^2 \leq |\tilde{u}^\mathsf{T}(\tau)w|$. Therefore, u being PE and bounded implies that

$$\frac{\alpha}{u_{\max}} \leq \int_t^{t+T} |u^\mathsf{T}(\tau)w| d\tau. \tag{3}$$

The above bound will be called upon shortly. Moving forward with the proof, consider the Lyapunov candidate $V(\phi, t) = \frac{1}{2}\phi^\mathsf{T}(t)\phi(t)$. Then differentiating along the system directions it follows that $\dot{V}(\phi, t) = -\phi^\mathsf{T}(t)u(t)u^\mathsf{T}(t)\phi(t)$. Integrating $\dot{V}$ and using the Cauchy–Schwarz inequality it follows that

$$\begin{aligned}-\int_t^{t+T} \dot{V}(\phi, \tau) d\tau &= \int_t^{t+T} |u^\mathsf{T}(\tau)\phi(\tau)|^2 d\tau \\ &\geq \frac{1}{T}\left(\int_t^{t+T} |u^\mathsf{T}(\tau)\phi(\tau)| d\tau\right)^2.\end{aligned}$$

The above inequality can equivalently be written as

$$\sqrt{T(V(t) - V(t+T))} \geq \int_t^{t+T} |u^\mathsf{T}(\tau)\phi(\tau)| d\tau. \tag{4}$$

Using the reverse triangle inequality, the right-hand side of the inequality in (4) can be bounded as

$$\int_t^{t+T} |u^\mathsf{T}(\tau)\phi(\tau)| d\tau \geq \int_t^{t+T} |u^\mathsf{T}(\tau)\phi(t)| d\tau - \int_t^{t+T} |u^\mathsf{T}(\tau)[\phi(t) - \phi(\tau)]| d\tau. \tag{5}$$

Using the bound in (3) the first integral on the right-hand side of the above inequality can be bounded as

$$\int_t^{t+T} |u^\mathsf{T}(\tau)\phi(t)| d\tau \geq \|\phi(t)\| \frac{\alpha}{u_{\max}}. \tag{6}$$

The second integral on the right-hand side of (5) can be bounded as

$$\begin{aligned}\int_t^{t+T} |u^\mathsf{T}(\tau)[\phi(t) - \phi(\tau)]| d\tau &\leq u_{\max} T \sup_{\tau \in [t, t+T]} \|\phi(t) - \phi(\tau)\| \\ &\leq u_{\max} T \int_t^{t+T} \|\dot{\phi}(\tau)\| d\tau \\ &\leq u_{\max}^2 T \int_t^{t+T} \|u^\mathsf{T}(\tau)\phi(\tau)\| d\tau.\end{aligned} \tag{7}$$

The second line in the above inequality follows by the fact that the arc-length between two points in space is always greater than or equal to a straight line between them. The third line in the above inequality follows by substition of the dynamics in (1). Substituting the inequalities in (5)–(7) into (4), it follows that

$$\int_t^{t+T} |u^{\mathsf{T}}(\tau)\phi(\tau)| d\tau \geq \frac{\|\phi(t)\| \frac{\alpha}{u_{\max}}}{1 + u_{\max}^2 T}.$$

Substituting the above bound into (4) and squaring both sides, it follows that

$$V(t+T) \leq \left(1 - \frac{2\alpha^2/u_{\max}^2}{T(1+u_{\max}^2 T)^2}\right) V(t).$$

Therefore the dynamics in (1) are UASL and by linearity this implies ESL as well. $\square$

While the first proof is more generic, the method deployed in the second proof gives direct insight into how the degree of PE, α, and the upper bound, $u_{\max}$, affect the rate of convergence,

$$r_{\text{con}} \triangleq 1 - \frac{2\alpha^2/u_{\max}^2}{T(1+u_{\max}^2 T)^2}. \tag{8}$$

In the method by Anderson the rate of convergence is an existence one given by $(1 - 2\alpha_2)$. No closed form expression is given relating α_2 to the original measures of PE, α and β.[1] It is clear, however, that for fixed T an increase in $u_{\max}$ conservatively implies an increase in β. It is also clear from (8) that an increase in $u_{\max}$ decreases the convergence rate r_{con}. We show below that an increase in β implies a decrease in r_{con}. Recall the Abel–Jacobi–Liouville identity, $\det \Phi_2(t, t_0) = \exp\big[-\int_{t_0}^{\mathsf{T}} \operatorname{trace}(u(\tau)u^{\mathsf{T}}(\tau))\, d\tau\big]$, and thus as β increases, $\det \Phi_2(t, t_0)$ decreases. Now using this fact and the bound in (2) it follows that as β increases α_2 decreases.

Often, adaptive systems generate a dynamic system of the form (1) where $u(\cdot)$ is a function of the parameter estimate itself. For this purpose, a nonlinear system of the form

$$\dot{\phi}(t) = -u(t,\phi)u^{\mathsf{T}}(t,\phi)\phi(t) \tag{9}$$

with $\phi_0 = \phi(t_0)$ needs to be analyzed. This is addressed in the following theorem, where it should be noted that UASL does not imply ESL.

THEOREM 2. *Let $\Omega(r) = \{\phi : \|\phi\| \leq r\}$. If $u(t,\phi) \in \mathrm{PE}^*(\phi, \Omega(r))$ for all r, $u(t)$ is piecewise continuous, and there exists $u_{\max}(r) > 0$ such that $\|u(t,\phi)\| \leq u_{\max}$ for all $\phi_0 \in \Omega(r)$, then ϕ in (9) is UASL.*

Proof. Given that $u(t,\phi) \in \mathrm{PE}^*(\phi, \Omega(r))$, it follows that there exists $T(r)$ and $\alpha(r)$ such that $\int_t^{t+T} |u^{\mathsf{T}}(\tau,\phi)w|^2 d\tau \succeq \alpha(r)$ for all $\phi_0 \in \Omega(r)$. Choosing a Lyapunov candidate as $V(\phi,t) = \frac{1}{2}\phi^{\mathsf{T}}(t)\phi(t)$ and following the same steps as in the proof of Theorem 1 it follows that $V(t + T(r)) \leq r_{\text{con}}(r)V(t)$ for all $\phi_0 \in \Omega(r)$, where

$$r_{\text{con}}(r) = \left(1 - \frac{2\alpha^2(r)/u_{\max}^2(r)}{T(r)(1+u_{\max}^2(r)T(r))^2}\right).$$

[1]If one carefully follows the steps outlined in [1] it may be possible to come up with a closed form relation, but it appears to be nontrivial.

Given that the convergence rate is upper bounded for all $\|\phi_0\| \le r$ and r can be arbitrarily large, the dynamics in (9) are UASL. In order for one to conclude that the dynamics are ESL there would need to exist a constant $0 < \delta < 1$ such that $r_{\text{con}} \le \delta$ for all r. That is, the convergence rate of the Lyapunov function would need to be bounded away from 1 uniformly in initial conditions. This global uniformity is not guaranteed from this analysis and thus it is not possible to conclude ESL. ◻

Remark 2. With Theorem 2, ESL of the general nonlinear system (9) was not explicitly disproven; rather, the functional dependencies between the coefficients imply that one is not able to conclude ESL with the information at hand. The specific dynamics of $u(t,\phi)$ which arise within reference model adaptive control are the subject of the following section and are one such example of $u(t,\phi)$ for which global uniformity of r_{con} is not achieved and therefore ESL is explicitly disproven. In such systems, with $u_{\max}(r)$ and $\alpha(r)$ constant it is shown that $T(r)$ increases in an unbounded fashion as r tends to infinity (or one can fix $T(r)$ but then $\alpha(r)$ decreases as r tends to infinity). The limiting result (in either case) is thus $\lim_{r\to\infty} r_{\text{con}}(r) = 1$.

3.2. Model reference adaptive control. Let $u : [t_0,\infty) \to \mathbb{R}$ be the input and $x : [t_0,\infty) \to \mathbb{R}^n$ the state of a dynamical system

$$\dot{x}(t) = Ax(t) - B\theta^{\mathsf{T}} x(t) + Bu(t), \tag{10}$$

where $A \in \mathbb{R}^{n\times n}$ is known and Hurwtiz and $B \in \mathbb{R}^n$ is known as well, with the parameter $\theta \in \mathbb{R}^n$ unknown. The goal is to design the input so that x follows a reference model state $x_m : [t_0,\infty) \to \mathbb{R}^n$ defined by the linear system of equations

$$\dot{x}_m(t) = Ax_m(t) + Br(t),$$

where $r : [t_0,\infty) \to \mathbb{R}$ is the reference command. Defining the model following error as $e = x - x_m$ the control input $u(t) = \hat{\theta}^{\mathsf{T}}(t)x(t) + r(t)$ achieves this goal when the adaptive parameter $\hat{\theta} : [t_0,\infty) \to \mathbb{R}^n$ is updated as follows:

$$\dot{\hat{\theta}}(t) = -xe^{\mathsf{T}} PB,$$

where $P = P^{\mathsf{T}} \in \mathbb{R}^{n\times n}$ is the positive definite solution to the Lyapunov equation $A^{\mathsf{T}}P + PA = -Q$ for any real $n \times n$ dimensional $Q = Q^{\mathsf{T}} \succ 0$. So as to simplify the notation we let $C \triangleq PB$ and the adaptive system can be compactly represented as

$$\begin{bmatrix} \dot{e}(t) \\ \dot{\phi}(t) \end{bmatrix} = \begin{bmatrix} A & Bx^{\mathsf{T}}(t) \\ -x(t)C^{\mathsf{T}} & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ \phi(t) \end{bmatrix}, \tag{11}$$

where the initial conditions of the model following error and parameter error are denoted as $e_0 = e(t_0)$ and $\phi_0 = \phi(t_0)$. For the dynamics of interest it follows that

$$V(e,\phi) = e^{\mathsf{T}} Pe + \phi^{\mathsf{T}}\phi \tag{12}$$

is a Lyapunov candidate with time derivative along the state trajectories satisfying the inequality, $\dot{V} \le -e^{\mathsf{T}} Qe$. This implies that $e(t)$ and $\phi(t)$ are bounded for all time with

$$\|e\| \le \sqrt{\tfrac{V(e_0,\phi_0)}{P_{\min}}} \text{ and } \|\phi\| \le \sqrt{V(e_0,\phi_0)}, \tag{13}$$

where $P_{\min}$ is the minimum eigenvalue of P. The reference command is bounded by design and thus x_m is bounded and along with the bounds above implies that x is bounded. The boundedness of x and ϕ in turn implies that $\dot{e}$ is bounded for all time. Integration of $\dot{V}$ shows that $e \in \mathcal{L}_2$ with

$$\|e\|_{\mathcal{L}_2} \leq \sqrt{\tfrac{V(e_0,\phi_0)}{Q_{\min}}}, \tag{14}$$

where $Q_{\min}$ is the minimum eigenvalue of Q. From the fact that $e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{e} \in \mathcal{L}_\infty$ it follows that $e \to 0$ as $t \to \infty$ [35, Lemma 2.12]. Before discussing the asymptotic stability of the dynamics in (11) the following lemma is critical in relating PE between the reference model state and the plant state. Let $z = [e^\mathsf{T}, \phi^\mathsf{T}]^\mathsf{T}$; then the dynamics in (11) can be compactly expressed as

$$\dot{z}(t) = \begin{bmatrix} A & Bx^\mathsf{T}(t,z;t_0) \\ -x(t,z;t_0)C^\mathsf{T} & 0 \end{bmatrix} z(t), \tag{15}$$

where we have explicitly denoted x as a function of the state variable z.

Lemma 3. *For the dynamics in* (15) *if $x_m(t)$ is PE with an α and T such that $\int_t^{t+T} x_m(\tau)x_m^\mathsf{T}(\tau)d\tau \succeq \alpha I$, and there exists a β such that $\|x_m(t)\| \leq \beta$, then $x(t,z)$ is* $\mathrm{PE}^*(z, Z(\zeta))$ *with $Z(\zeta) = \{z : V(z) \leq \zeta\}$ for all $\zeta > 0$ with the following bounds holding:*

$$\int_t^{t+pT} x(\tau)x^\mathsf{T}(\tau)d\tau \succeq \alpha' I \tag{16}$$

with $p > p_{\min}$ where

$$\sqrt{p_{\min}} \triangleq \frac{\left(\sqrt{\frac{\zeta}{P_{\min}}} + 2\beta\right)\sqrt{T\frac{\zeta}{Q_{\min}}}}{\alpha} \tag{17}$$

and

$$\alpha' \triangleq p\alpha - \left(\sqrt{\tfrac{\zeta}{P_{\min}}} + 2\beta\right)\sqrt{pT\tfrac{\zeta}{Q_{\min}}}. \tag{18}$$

Before going to the proof of this lemma a few comments are in order. First, note that the state variable z contains both the model following error e and the parameter error ϕ. Therefore, what is being said is that there is PE* of x for all initial conditions e_0 and ϕ_0 in the compact regions defined by the level sets of the Lyapunov function $V(z) = e^\mathsf{T}Pe + \phi^\mathsf{T}\phi$. Furthermore, because these conditions hold for arbitrarily large level sets, i.e., ζ can be arbitrarily large, PE* of x is achieved for any initial condition $z_0 \in \mathbb{R}^{2n}$. However, because the parameters in the PE bound in (16), namely, p, are not uniform in z_0 it cannot be concluded that x is PE.

Proof. This proof follows closely that of [5, Theorem 3.1]. For any fixed unitary vector w, consider the following equality: $(x_m^\mathsf{T}w)^2 - (x^\mathsf{T}w)^2 = -(x^\mathsf{T}w - x_m^\mathsf{T}w)(x^\mathsf{T}w + x_m^\mathsf{T}w)$. Using the definition of e, the bound in (13) for e, and the bound β in the statement of the lemma, it follows that

$$(x_m^\mathsf{T}w)^2 - (x^\mathsf{T}w)^2 \leq \|e\|\left(\sqrt{\tfrac{V(z_0)}{P_{\min}}} + 2\beta\right).$$

Moving $(x_m^\mathsf{T}w)^2$ to the right-hand side, multiplying by -1, and integrating from t to $t + pT$ where p is defined just above (17)

$$\int_t^{t+pT} (x^\mathsf{T}(\tau)w)^2 d\tau \geq \int_t^{t+pT} (x_m^\mathsf{T}(\tau)w)^2 d\tau - \left(\sqrt{\tfrac{V(z_0)}{P_{\min}}} + 2\beta\right) \int_t^{t+pT} \|e(\tau)\| d\tau.$$

Applying Cauchy–Schwarz to the integral on the right-hand side and using the fact that $\int_t^{t+T} (x_m^\mathsf{T}(\tau)w)^2 d\tau \geq \alpha$ we have that

$$\int_t^{t+pT} (x^\mathsf{T}(\tau)w)^2 d\tau \geq p\alpha - \left(\sqrt{\tfrac{V(z_0)}{P_{\min}}} + 2\beta\right) \sqrt{pT \int_t^{t+pT} \|e(\tau)\|^2 d\tau}.$$

Applying the bound in (14) for the $\mathcal{L}_2$ norm of e, it follows that

$$\int_t^{t+pT} (x^\mathsf{T}(\tau)w)^2 d\tau \geq p\alpha - \left(\sqrt{\tfrac{V(z_0)}{P_{\min}}} + 2\beta\right) \sqrt{pT \tfrac{V(z_0)}{Q_{\min}}}.$$

For all $z_0 \in Z(\zeta)$ it follows that $V(z_0) \leq \zeta$ and therefore

$$p\alpha - \left(\sqrt{\tfrac{V(z_0)}{P_{\min}}} + 2\beta\right) \sqrt{pT \tfrac{V(z_0)}{Q_{\min}}} \geq \alpha'.$$

It follows directly that $\int_t^{t+pT} (x^\mathsf{T}(\tau)w)^2 d\tau \geq \alpha'$ for all $t \geq t_0$ and $z_0 \in Z(\zeta)$. ∎

Remark 3. The main takeaway from this lemma is that for a given α and T such that $\int_t^{t+T} x_m(\tau)x_m^\mathsf{T}(\tau)d\tau \succeq \alpha I$ and for a fixed α' such that $\int_t^{t+pT} x(\tau)x^\mathsf{T}(\tau)d\tau \succeq \alpha' I$, as the size of the level set $V(z) = \zeta$ is increased, p must also increase. This can be seen directly through (17) where $p_{\min}$ increases with increasing ζ. Thus, as p increases, the time window pT over which the excitation is measured increases as well.

Remark 4. We note that there is nothing in the above lemma that requires time to be continuous and thus the aforementioned relationship between PE and PE* via an $\mathcal{L}_2$ condition also holds for discrete time systems via an equivalent ℓ_2 relationship.

THEOREM 4. *If $r(t)$ is piecewise continuous and bounded, and $x_m(t)$ is PE and uniformly bounded, then the equilibrium of the dynamics in* (15) *is UASL.*

Proof. Given that $x_m \in$ PE it follows from Lemma 3 that $x(t, z) \in \mathrm{PE}^*(z, Z(\zeta))$ for any ζ where $Z(\zeta) = \{z : V(z) \leq \zeta\}$ and the Lyapunov function V is defined in (12). From (13) it follows that all signals are bounded. Furthermore given that r is piecewise continuous and bounded it follows from (10) that $\dot{x}$ is piecewise continuous. Therefore $x \in \mathcal{P}_{[t_0,\infty)}$; see the definition of piecewise smooth in Definition 4 the appendix. With $x(t, z) \in \mathrm{PE}^*(z, Z(\zeta)) \cap \mathcal{P}_{[t_0,\infty)}$ for any fixed ζ applying [31, Theorem 5] it follows that the dynamics of interest are UAS. Given that the above results hold for any $\zeta > 0$, the dynamics of interest are therefore UASL. Due to the fact that PE bounds for x do not hold globally uniformly in the initial condition z_0 one is not able to conclude ESL from this analysis. ∎

We can in fact state something even stronger and will give a proof by example in the following section (following Theorem 7).

THEOREM 5. *The reference command $r(t)$ being piecewise continuous and bounded, and the reference model state $x_m(t)$ being uniformly bounded and PE, are not sufficient for the equilibrium of the dynamics in* (15) *to be ESL.*

Remark 5. Stated more informally, if the input to the reference model is sufficiently rich, then the output of the reference model will be persistently exciting, but then the plant state will only be weakly persistently exciting. Note then that the main Theorem of [6] should be slightly weakened in its claim.[2]

4. Lack of exponential stability in the large for adaptive systems. In this section two examples are presented to illustrate rigorously by example the implication made in Theorem 4, i.e., PE of the reference model does not imply exponential stability in the large of the adaptive system and thus proves Theorem 5. This is performed by constructing an invariant unbounded region in the state space of the direct adaptive system where the rate of change per unit time of the system state is finite. It is this feature which implies a lack of exponential stability in the large.

The first example is identical to the dynamics in (3.2), but with a learning gain added to the update law. The second example is a modified version of classic direct adaptive control with an error feedback term in the reference model [14, 9]. To distinguish the two systems we characterize them by their reference models and refer to the first system as ORM adaptive control and the second as CRM adaptive control. The CRM adaptive system has been added due to recent interest in transient properties of adaptive systems, with the class of CRM systems portraying smoother trajectories as compared to their ORM counterpart [14, 9].

4.1. Scalar ORM adaptive control with PE reference state. The following scalar dynamics are nearly identical to those in section 3.2; however, we repeat them herein with $A = a < 0$ and $B = b > 0$ to emphasize that they are scalars. Let $u : [t_0, \infty) \to \mathbb{R}$ be the input, $x : [t_0, \infty) \to \mathbb{R}$ the plant state, $x_m : [t_0, \infty) \to \mathbb{R}$ the reference state, and $r : [t_0, \infty) \to \mathbb{R}$ the reference input to the following set of differential equations:

$$\dot{x}(t) = ax(t) - b\theta x(t) + bu(t), \tag{19}$$

$$\dot{x}_m(t) = ax_m(t) + br(t), \tag{20}$$

with the parameter $\theta \in \mathbb{R}$ unknown. For ease of exposition, in both this section and the next, we will assume that $r(t)$ is a nonzero constant, i.e., $r(t) \equiv \bar{r}$, $\bar{r} \neq 0$. The control input is defined as $u(t) = \hat{\theta}(t)x(t) + r(t)$ with $\hat{\theta} : [t_0, \infty) \to \mathbb{R}$ updated as follows:

$$\dot{\hat{\theta}}(t) = -\gamma x e, \tag{21}$$

where $e = x - x_m$ and $\gamma > 0$ is a tuning gain.

As before, the error dynamics can be compactly expressed in vector form as $z(t) = [e(t), \phi(t)]^{\mathsf{T}}$. A sufficient condition for the uniform asymptotic stability of the above system is that the reference input remain a nonzero constant for all time. This can be proved using Theorem 4. Given that for a constant reference command the above dynamics are also autonomous we give the same result using the well-known invariance principle from Lasalle and Krasovskii [23, 25, 4].

Theorem 6. *For the system defined in* (19)–(21) *and* $r(t) \equiv \bar{r}$, $\bar{r} \neq 0$, $z = 0$ *is UASL.*

[2]Sufficient richness of the reference input does not imply persistence of excitation of the regressor vector. The authors of [5, 6] are careful in proving that richness of the reference input only implies exponential convergence. The careful wording of *convergence*, however, was changed to exponential *stability* countless times elsewhere in the literature. It was then inappropriately concluded that uniform asymptotic stability in the large is equivalent to exponential stability in the large for adaptive systems.

Proof. Define the Lyapunov function

$$V(e,\phi) = e^2 + \frac{1}{\gamma}\phi^2, \tag{22}$$

then $\dot{V}(e,\phi) = 2ae^2$. Since $V > 0$ for all $z \neq 0$, $\dot{V} \leq 0$ for all $z \in \mathbb{R}^2$, and $V \to \infty$ as $z \to \infty$ the equilibrium at the origin is uniformly stable and uniformly bounded. Given that the system is autonomous, it follows from the invariance principle that the origin is UASL. ∎

We are now going to construct an unbounded invariant region as discussed at the beginning of this section. The reference model state initial condition is chosen as $x_m(t_0) = \bar{x}$, where

$$\bar{x} \triangleq \frac{-b\bar{r}}{a} > 0 \tag{23}$$

so that $x_m(t) = \bar{x}$ for all time. Then the error dynamics are completely described by the second order dynamics

$$\dot{z}(t; x_0, a, b, \gamma, \bar{x}) = \begin{bmatrix} \dot{e}(t) \\ \dot{\phi}(t) \end{bmatrix} = \begin{bmatrix} ae(t) + b\phi(t)(e(t)+\bar{x}) \\ -\gamma e(t)(e(t)+\bar{x}) \end{bmatrix} \tag{24}$$

with $s(t; t_0, z(t_0))$ the transition function for the dynamics above.

The invariant set is constructed by first defining three one-dimensional manifolds $\mathsf{S}_1, \mathsf{S}_2, \mathsf{S}_3$ and three preliminary subsets of $\mathbb{R}^2$ which we will denote $\mathsf{P}_1, \mathsf{P}_2, \mathsf{P}_3$, and finally three regions $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ are defined whose union is our invariant set of interest. Use Figure 2 to help visualize these regions. We begin by defining the surface

$$\mathsf{S}_1 \triangleq \left\{ [e,\phi]^\mathsf{T} \mid e = -\bar{x} \right\}. \tag{25}$$

The region $\mathsf{P}_1 \subset \mathbb{R}^2$ and the second surface S_2 are defined as

$$\begin{aligned} &\mathsf{P}_1 \triangleq \left\{ [e,\phi]^\mathsf{T} \;\middle|\; \phi < \frac{a}{b} \right\}, \\ &\mathsf{S}_2 \triangleq \left\{ [e,\phi]^\mathsf{T} \;\middle|\; e = \frac{(a-b\phi)\bar{x}}{a+b\phi}, [e,\phi]^\mathsf{T} \in \mathsf{P}_1 \right\}. \end{aligned} \tag{26}$$

Similarly a second subset of the error-space $\mathsf{P}_2 \subset \mathbb{R}^2$ and a third surface S_3 are defined as

$$\begin{aligned} &\mathsf{P}_2 \triangleq \left\{ [e,\phi]^\mathsf{T} \;\middle|\; \frac{a}{b} \leq \phi < 0 \right\}, \\ &\mathsf{S}_3 \triangleq \left\{ [e,\phi]^\mathsf{T} \mid e = 0, [e,\phi]^\mathsf{T} \in \mathsf{P}_2 \right\}. \end{aligned}$$

We now define regions M_1 and M_2 as

$$\mathsf{M}_1 \triangleq \left\{ [e,\phi]^\mathsf{T} \;\middle|\; -\bar{x} < e < \frac{(a-b\phi)\bar{x}}{a+b\phi}, [e,\phi]^\mathsf{T} \in \mathsf{P}_1 \right\}, \tag{27}$$

$$\mathsf{M}_2 \triangleq \left\{ [e,\phi]^\mathsf{T} \mid -\bar{x} < e < 0, [e,\phi]^\mathsf{T} \in \mathsf{P}_2 \right\}. \tag{28}$$

From these definitions, we note that the surfaces S_1 and S_2 form the two sides of the region M_1. Similarly, S_1 and S_3 form the sides of the region M_2. In order to complete

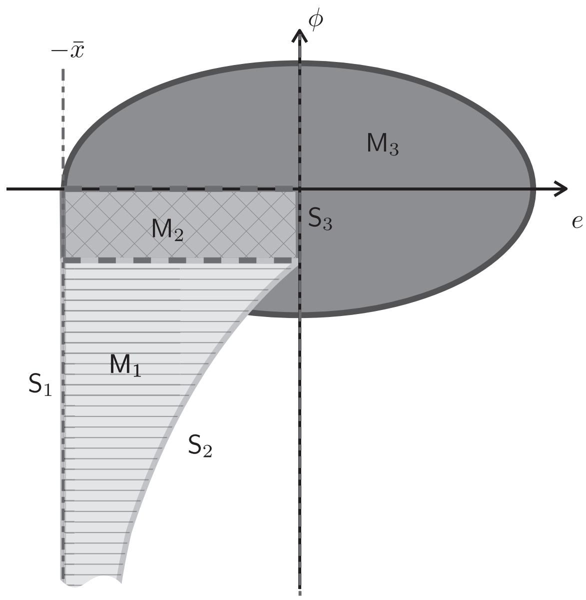


FIG. 2. *The three regions* M_1 *(horizontal lines),* M_2 *(crosshatch), and* M_3 *(solid) whose union results in the invariant set* M_0*.*

the invariant set a third region is defined using the Lyapunov function in (22) which gives us the convex bounded region

$$\mathsf{M}_3 \triangleq \left\{ [e,\phi]^\mathsf{T} \,\middle|\, e^2 + \frac{1}{\gamma}\phi^2 < \bar{x}^2 \right\}. \tag{29}$$

The union of the three regions is defined as

$$\mathsf{M}_0 \triangleq \mathsf{M}_1 \cup \mathsf{M}_2 \cup \mathsf{M}_3. \tag{30}$$

The following theorem will show three facts. First, the error velocities within M_0 are finite and bounded even though M_0 is unbounded. Second, M_0 is an invariant set. Last, a lower limit on the time of convergence is given as a function of the initial condition $z(t_0)$ and the ratio $\frac{\|s(t_1;t_0,z(t_0))\|}{\|s(t_0;t_0,z(t_0))\|}$ for some $t_1 \geq t_0$. The conclusion to be arrived at is that the system is UASL and not ESL.

THEOREM 7. *For the error dynamics* $z(t)$ *with* $r(t) = \bar{r}$ *and* $x_m(t_0) = \bar{x}$ *with* M_0 *defined in* (30) *and* $s(t;t_0,z(t_0))$ *the transition function of the differential equation* (24)*, the following hold:*

(i) $\|\dot{z}\| \leq d_z$ *for all* $z \in \mathsf{M}_0$*, where*

$$d_z \triangleq \sqrt{(|a\bar{x}| + 2|b\sqrt{\gamma}\bar{x}^2|)^2 + (2\gamma\bar{x}^2)^2}.$$

(ii) M_0 *is an invariant set.*

(iii) *A trajectory beginning at* $z(t_0) \in \mathsf{M}_0$ *will converge to a fraction of its original magnitude at time* t_1*, with*

$$T \geq \frac{\|z(t_0)\|(1-c)}{d_z}, \tag{31}$$

where $c = \frac{\|s(t_1;t_0,z(t_0))\|}{\|s(t_0;t_0,z(t_0))\|}$ *and* $T = t_1 - t_0$.

Proof of (i). From the definition of M_1 in (27) and M_2 in (28), and the definition of $\dot{\phi}$ in (24), it follows that $|\dot{\phi}(z)| \leq \gamma\frac{\bar{x}^2}{4}$ for all $z \in \mathsf{M}_1 \cup \mathsf{M}_2$. Similarly, from the definition of M_3 in (29) it follows that $|\dot{\phi}(z)| \leq 2\gamma\bar{x}^2$ for all $z \in \mathsf{M}_3$. Therefore

$$|\dot{\phi}(z)| \leq 2\gamma\bar{x}^2 \tag{32}$$

for all $z \in \mathsf{M}_0$, where M_0 is defined in (30).

From the definition of $\dot{e}$ in (24) and the definitions of M_1, M_2, and M_3 it follows that $|\dot{\phi}(z)| \leq |a\bar{x}|$ for all $z \in \mathsf{M}_1 \cup \mathsf{M}_2$ and $|\dot{\phi}(z)| \leq |a\bar{x}| + 2b\sqrt{\gamma}\bar{x}^2$ for all $z \in \mathsf{M}_3$. Therefore

$$|\dot{e}(z)| \leq |a\bar{x}| + 2b\sqrt{\gamma}\bar{x}^2 \tag{33}$$

for all $z \in \mathsf{M}_0$. From the bounds in (32) and (33) for $\dot{\phi}$ and $\dot{e}$, respectively, Theorem 7(i) follows. ∎

Proof of (ii). In order to evaluate the behavior of the trajectories on the surfaces S_1, S_2, and S_3, normal vectors are defined along the surfaces that point toward M_0. The normal vectors are

$$\hat{n}_1 = [1, 0]^\mathsf{T}, \quad \hat{n}_2(z) = \left[\frac{-\partial e}{\partial \phi}, 1\right]^\mathsf{T}_{z \in \mathsf{S}_2}, \quad \text{and} \quad \hat{n}_3 = [-1, 0]^\mathsf{T},$$

where $\frac{\partial e}{\partial \phi} = \frac{-2b\bar{x}a}{(a+b\phi)^2}$. We then find that $\hat{n}_i^\mathsf{T}(z)\dot{z}(z) \geq 0$ for $z \in \mathsf{S}_i$ and $i = 1, 2, 3$. From the general stability proof of the adaptive system with Lyapunov function $V = e^2 + \frac{1}{\gamma}\phi^2$ once within M_3 a trajectory cannot leave it. ∎

Proof of (iii). For a trajectory to traverse from $z(t_0)$ to a magnitude less than $c\|z(t_0)\|$ (such that $\|s(t_1)\| \leq c\|s(t_0)\|$) it must travel at least a distance $\|z(t_0)\|(1-c)$ over which it has a maximum rate of d_z; therefore

$$T \geq \frac{\|z(t_0)\|(1-c)}{d_z}.$$ ∎

Proof of Theorem 5. The results from Theorem 7 illustrate that for an input which provides PE of the reference model, there exists an unbounded region where the adaptive system is UASL and not ESL. For the system to possess ESL, the lower bound in (31) needs to be dependent only on c and independent of $z(t_0)$; see Remark 1 with c analogous to η/ρ. The lower bound on T is therefore sufficient to prove that ESL is not possible. ∎

It can also be shown that the learning rate, $\dot{\phi}$, of the adaptive parameter tends to zero as the initial adaptive parameter error $\phi(t_0)$ tends to negative infinity inside M_1. In the previous theorem we only showed that $\dot{\phi}$ is uniformly bounded for all initial conditions inside the larger set M_0. Thus, not only is ESL impossible, there is an unbounded region in the base of M_1 where adaptation occurs at a slower and slower rate the deeper the initial condition starts in the trough of M_1. This effect is visualized through simulation examples in a later section.

COROLLARY 8. *For the error dynamics* $z(t)$ *defined by the differential equation in* (24) *with* $r(t) = \bar{r}$ *and* $x_m(t_0) = \bar{x}$ *it follows that* $\dot{\phi}(e, \phi) \to 0$ *as* $\phi \to -\infty$ *with* $[e, \phi]^\mathsf{T} \in \mathsf{M}_1$.

Proof. For fixed ϕ and an e such that $[e, \phi]^{\mathsf{T}} \in \mathsf{M}_1$, which we will assume from this point forward in the proof, it follows that $-\bar{x} \leq e \leq \frac{a-b\phi}{a+b\phi}\bar{x}$ per the definition of M_1 in (27). Written another way,

$$e = -\bar{x} + \Delta, \tag{34}$$

where $\Delta \in [0, \frac{2a}{a+b\phi}\bar{x}]$. Substitution of (34) into the definition of $\dot{\phi}$ from (24) it follows that

$$\dot{\phi} = -\gamma(\bar{x} + \Delta)^2 + \gamma\bar{x}(\bar{x} - \Delta).$$

After expanding and canceling terms the above equation reduces to

$$\dot{\phi} = -\gamma\big(3\bar{x}\Delta + \Delta^2\big). \tag{35}$$

From the fact that $\Delta \leq \frac{2a}{a+b\phi}\bar{x}$ it follows that $\lim_{\phi\to-\infty} \Delta = 0$ (recall that $a < 0$). Using this limiting value of Δ and (35) it follows that $\lim_{\phi\to-\infty} \dot{\phi} = 0$ when $[e, \phi]^{\mathsf{T}} \in \mathsf{M}_1$. ∎

This corollary helps connect the results from this section back to our definitions of PE and PE* and to Remark 3. While it is possible for $x_m \in$ PE our analysis technique only allowed us to conclude that $x \in$ PE*. This was characterized by the fact that in order for x to maintain the same level of excitation, which we referred to as α' in (16), the time window over which the excitation was measured, pT in (16), would have to increase as the norm of the initial conditions of the system increased. This is precisely what is occurring in the bottom of M_1. In the bottom of this region it follows by definition that $|x| \leq \Delta$, which tends to zero as $\phi(t_0)$ decreases to negative infinity, and all the while the speed at which the state can leave this region is decreasing as well.

4.2. Scalar CRM adaptive system. We now consider a modified adaptive system in which the reference model contains a feedback loop with the state error. The plant is the same as that in (19) with an identical control law and the same update law as that in (21). The reference model, however, is now defined as

$$\dot{x}_m(t) = ax_m(t) + br(t) - \ell e(t), \tag{36}$$

where $\ell < 0$. In the CRM setting the reference model is now able to *meet the plant halfway*. The burden of error minimization is not entirely put on the adaptive controller and the reference model trajectory is softened while still asymptotically converging to the ORM reference model. This results in smoother transients [14, 9].

Throughout this section it is assumed that $\bar{r} > 0$ is a constant; however, no longer does $x_m(t) = \bar{x}$ for all time. Unlike in the ORM cases, the reference model dynamics cannot be ignored. The resulting system can be represented as

$$\dot{z}(t; x_0, a, b, \gamma, \bar{r}, \ell) = \begin{bmatrix} \dot{x}_m(t) \\ \dot{e}(t) \\ \dot{\phi}(t) \end{bmatrix} = \begin{bmatrix} ax_m(t) + b\bar{r} - \ell e(t) \\ (a+\ell)e(t) + b\phi(t)x(t) \\ -\gamma e(t)x(t) \end{bmatrix}. \tag{37}$$

We will show that this modified adaptive system cannot be ESL and for the specific $r(t)$ chosen is UASL.

THEOREM 9. *For the system defined in* (37) *with* $r(t) \equiv \bar{r}$, $\bar{r} \neq 0$ *the equilibrium of* z *is UASL.*

Proof. Consider the Lyapunov candidate in (22) and differentiating along the dynamics in (37) it follows that $\dot{V}(e,\phi) = 2(a+\ell)e^2$. Since $V > 0$ for all $z \neq 0$, $\dot{V} \leq 0$ for all $[e\ \phi]^{\mathsf{T}} \in \mathbb{R}^2$, and $V \to \infty$ as $z \to \infty$, it follows that $z = [\bar{x}, 0, 0]^{\mathsf{T}}$ is uniformly stable in the large. Since the system is autonomous it follows from the invariance principle that $z = [\bar{x}, 0, 0]^{\mathsf{T}}$ is UASL as well. ∎

Now a number of regions in the state-space ($\mathbb{R}^3$) are defined which allow the construction and proof of this subsection's main result which mirrors the results of Theorem 7. In particular, three regions will be defined. It will then be shown that a specific region M_0, the union of these three regions, will remain invariant. As this region M_0 is infinite and the vector field defined by (37) has a finite maximum velocity, we can conclude that CRM adaptive systems do not posses exponential stability in the large but are at best UASL.

Define a subset of the state-space, $\mathsf{P}_1 \subset \mathbb{R}^3$,

$$\mathsf{P}_1 \triangleq \left\{ [x_m, e, \phi]^{\mathsf{T}} \left| \phi < \frac{a+\ell}{b}, \frac{b\bar{r}}{a+\ell} \leq x_m \leq \bar{x}, [x_m, e, \phi]^{\mathsf{T}} \in \mathbb{R}^3 \right. \right\},$$

and within the subset P_1 a region

$$\mathsf{M}_1 \triangleq \left\{ [x_m, e, \phi]^{\mathsf{T}} \left| -x_m \leq e \leq \frac{x_m(a+\ell+b\phi)}{a+\ell-b\phi}, [x_m, e, \phi]^{\mathsf{T}} \in \mathsf{P}_1 \right. \right\}.$$

Define a second subset of the state-space, $\mathsf{P}_2 \subset \mathbb{R}^3$,

$$\mathsf{P}_2 \triangleq \left\{ [x_m, e, \phi]^{\mathsf{T}} \left| \frac{a+\ell}{b} \leq \phi < 0, \frac{b\bar{r}}{a+\ell} \leq x_m \leq \bar{x}, [x_m, e, \phi]^{\mathsf{T}} \in \mathbb{R}^3 \right. \right\},$$

and within this subset a region

$$\mathsf{M}_2 \triangleq \left\{ [x_m, e, \phi]^{\mathsf{T}} \left| -x_m \leq e \leq 0, [x_m, e, \phi]^{\mathsf{T}} \in \mathsf{P}_2 \right. \right\}.$$

A third region is defined as

$$\mathsf{M}_3 \triangleq \left\{ [x_m, e, \phi]^{\mathsf{T}} \left| e^2 + \frac{1}{\gamma}\phi^2 \leq \bar{x}^2, 0 \leq x_m \leq 2\bar{x}, [x_m, e, \phi]^{\mathsf{T}} \in \mathbb{R}_3 \right. \right\}.$$

The union of these three M regions is then the invariant set M_0, defined as

$$\mathsf{M}_0 \triangleq \mathsf{M}_1 \cup \mathsf{M}_2 \cup \mathsf{M}_3. \tag{38}$$

The three regions are shown in Figure 3. Four surfaces of this region will be used in the proof of the following theorem:

$$\mathsf{S}_1 \triangleq \left\{ [x_m, e, \phi]^{\mathsf{T}} \left| e = \frac{x_m(a+\ell+b\phi)}{a+\ell-b\phi}, [x_m, e, \phi]^{\mathsf{T}} \in \mathsf{P}_1 \right. \right\}, \tag{39}$$

$$\begin{aligned} \mathsf{S}_2 &\triangleq \left\{ [x_m, e, \phi]^{\mathsf{T}} \left| e = -x_m, [x_m, e, \phi]^{\mathsf{T}} \in \mathsf{P}_1 \cup \mathsf{P}_2 \right. \right\}, \\ \mathsf{S}_3 &\triangleq \left\{ [x_m, e, \phi]^{\mathsf{T}} \left| e = 0, [x_m, e, \phi]^{\mathsf{T}} \in \mathsf{P}_2 \right. \right\}, \\ \mathsf{S}_4 &\triangleq \left\{ [x_m, e, \phi]^{\mathsf{T}} \left| e^2 + \frac{1}{\gamma}\phi^2 = \bar{x}^2, [x_m, e, \phi]^{\mathsf{T}} \in \mathsf{M}_0 \right. \right\}. \end{aligned} \tag{40}$$

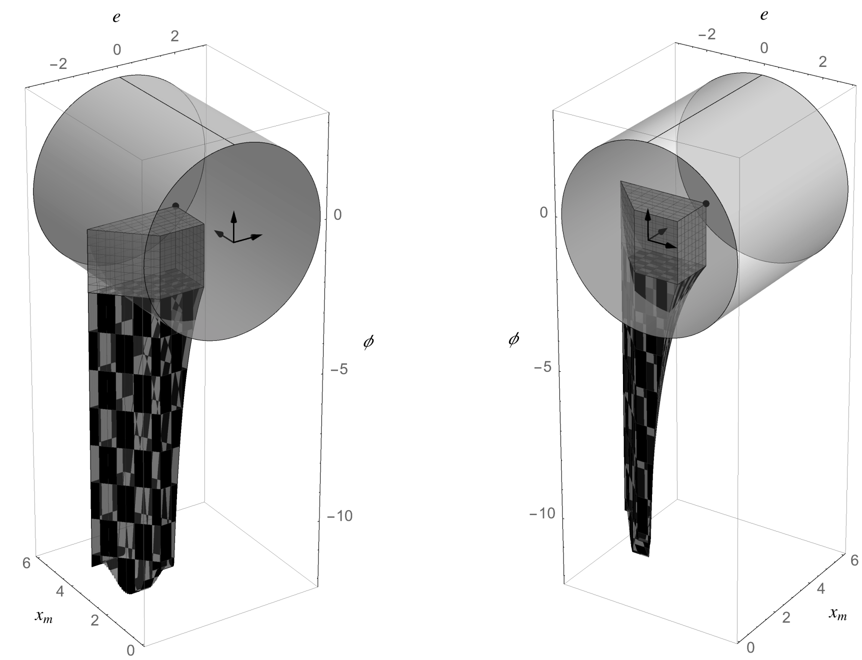


FIG. 3. *The three regions* M_1 *(checker),* M_2 *(crosshatch), and* M_3 *(solid) whose union results in the invariant set* M_0*.*

THEOREM 10. *For the error dynamics* $z(t)$ *with* $r(t) = \bar{r}$, M_0 *as defined in* (38), *and* $s(t; t_0, z(t_0))$ *the transition function of the differential equation* (37), *the following hold:*

(i) $\|\dot{z}\| \leq d_z$ *for all* $z \in \mathsf{M}_0$, *where*

$$d_z \triangleq \sqrt{(|(a+\ell)\bar{x}| + 2b\sqrt{\gamma}\bar{x}^2)^2 + (2\gamma\bar{x}^2)^2 + (|(a+\ell)\bar{x}| + \bar{r})^2}.$$

(ii) M_0 *is an invariant set.*

(iii) *A trajectory beginning at* $z(t_0) \in \mathsf{M}_0$ *will converge to a fraction of its original magnitude at time* t_1, *with*

$$T \geq \frac{\|z(t_0)\|(1-c)}{d_z}, \tag{41}$$

where $c = \frac{\|s(t_1;t_0,z(t_0))\|}{\|s(t_0;t_0,z(t_0))\|}$ *and* $T = t_1 - t_0$.

Proof of (i). Each component of the vector field is bounded:

$$\begin{aligned} |\dot{\phi}(z)| &\leq 2\gamma x_0^2, \\ |\dot{e}(z)| &\leq |(a+\ell)\bar{x}| + 2b\sqrt{\gamma}\bar{x}^2, \\ |\dot{x}_m(z)| &\leq |(a+\ell)\bar{x}| + \bar{r} \end{aligned}$$

when $z \in \mathsf{M}_0$, and thus $\|\dot{z}\| \leq d_z$. ∎

Proof of (ii). In order to evaluate the behavior of the trajectories on the surfaces of M_1 and M_2, normal vectors are defined along the surfaces. The normal vectors $\hat{n}_2$ and $\hat{n}_3$ have trivial definitions easily determined by inspection. The normal vector $\hat{n}_1$ is constructed using the cross product of two tangential vectors $\hat{n}_1 = \hat{t}_1 \otimes \hat{t}_2$, where

$$\hat{t}_1 = \left[1 \;\; 0 \;\; \frac{\partial e}{\partial x_m}\right]^{\mathsf{T}}_{z\in\mathsf{S}_1} \quad \text{and} \quad \hat{t}_2 = \left[1 \;\; \frac{\partial e}{\partial x_m} \;\; 0\right]^{\mathsf{T}}_{z\in\mathsf{S}_1}.$$

It follows directly that $\hat{n}_i^{\mathsf{T}}(z)\dot{z}(z) \geq 0$ for $z \in \mathsf{S}_i$ and $i = 1, 2, 3$. From the stability analysis in the proof of Theorem 9 we know that S_4 is simply a level set of the Lyapunov function and thus M_4 is invariant. Therefore no trajectory can exit M_0, making it an invariant set. ◻

Proof of (iii). This proof is identical to the proof of item (iii) in Theorem 7. ◻

Just as with an ORM, with a CRM the dynamics are at best UASL. The region of slow convergence is present in CRM adaptive control as well and a similar corollary holds.

COROLLARY 11. *For the error dynamics $z(t)$ defined by the differential equation in* (37) *with $r(t) = \bar{r}$ it follows that $\dot{\phi}(x_m, e, \phi) \to 0$ as $\phi \to -\infty$ with $[x_m, e, \phi]^{\mathsf{T}} \in \mathsf{M}_1$.*

5. Simulation examples. Simulations are now presented for the ORM adaptive system and the CRM adaptive system. The main purpose of these simulations is to illustrate the invariance of their respective M_0, and the slow convergence, especially the sluggish phenomenon that is treated in Corollaries 8 and 11. Before continuing to the results we need to distinguish between the surfaces in the ORM and CRM cases and define two new surfaces. First, let the following two surfaces in the ORM case be redefined as $\mathsf{S}_{O1} = \mathsf{S}_1$ and $\mathsf{S}_{O2} = \mathsf{S}_2$, where S_1 and S_2 are defined in (25) and (26). Similarly for the CRM, $\mathsf{S}_{C1} = \mathsf{S}_1$ and $\mathsf{S}_{C2} = \mathsf{S}_2$, where S_1 and S_2 are defined in (39) and (40). The two new surfaces to be defined pertain to the condition $\dot{e} = 0$. For ORMs this surface is defined as

$$\mathsf{S}_{O5} \triangleq \left\{ [e, \phi]^{\mathsf{T}} \;\middle|\; e = \frac{-\bar{x}b\phi}{a + b\phi} \right\}$$

and for the CRMs a similar curve is defined as

$$\mathsf{S}_{C5} \triangleq \left\{ [e, \phi]^{\mathsf{T}} \;\middle|\; e = \frac{-x_m b\phi}{a + b\phi}, x_m = \frac{\ell e - b\bar{r}}{a} \right\},$$

where the second equation in the definition of S_{C5} is derived from (36) by setting $\dot{x}_m = 0$. Nine initial states are chosen specifically for each system, defined in Tables 1 and 2. Rather than defining numerical values for each initial condition, we choose them as points of intersection between two unique surfaces. The values of the parameters for the simulations are as follows:

$$a = -1, \qquad \ell = -1, \qquad \gamma = 1, \qquad b = 1, \qquad r = 3. \tag{42}$$

Figure 4(a) contains the two-dimensional phase portrait for trajectories of the ORM adaptive system resulting from each of the initial conditions of Table 1. Figure 4(b) contains the two-dimensional projection of the three-dimensional phase space trajectories of the CRM adaptive system resulting from each of the initial conditions of Table 2. Before we proceed, we observe that in both Figures 4(a) and 4(b), there is

TABLE 1

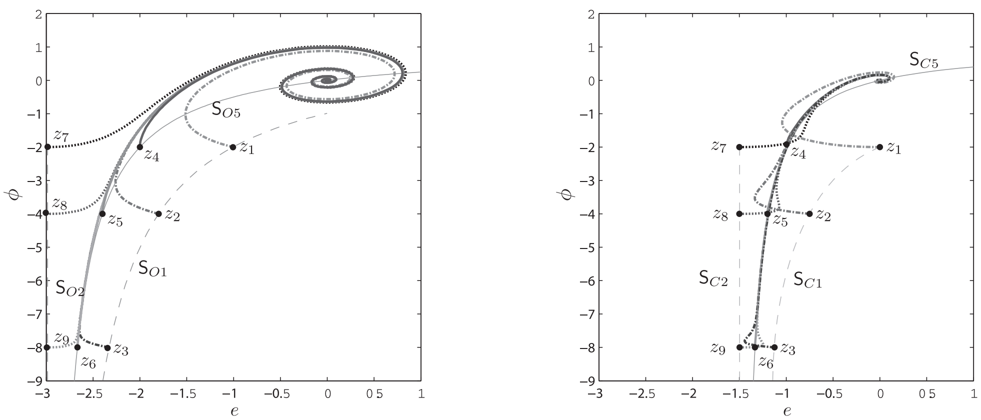
Initial conditions $z_i, i = 1, 2, \ldots, 9$, for the ORM example system. Each initial condition, z_i, is the point of intersection of the two indicated surfaces in the corresponding row and column.

	S_{O1}	S_{O5}	S_{O2}
$\phi = -2$	z_1	z_4	z_7
$\phi = -4$	z_2	z_5	z_8
$\phi = -8$	z_3	z_6	z_9

TABLE 2

Initial conditions $z_i, i = 1, 2, \ldots, 9$, for the CRM example system. Each initial condition, z_i, is the point of intersection of the two indicated surfaces in the corresponding row and column.

	S_{C1}	S_{C5}	S_{C2}
$\phi = -2$	z_1	z_4	z_7
$\phi = -4$	z_2	z_5	z_8
$\phi = -8$	z_3	z_6	z_9



(a) 2D phase portrait for the ORM adaptive system with initial conditions defined in Table 1.

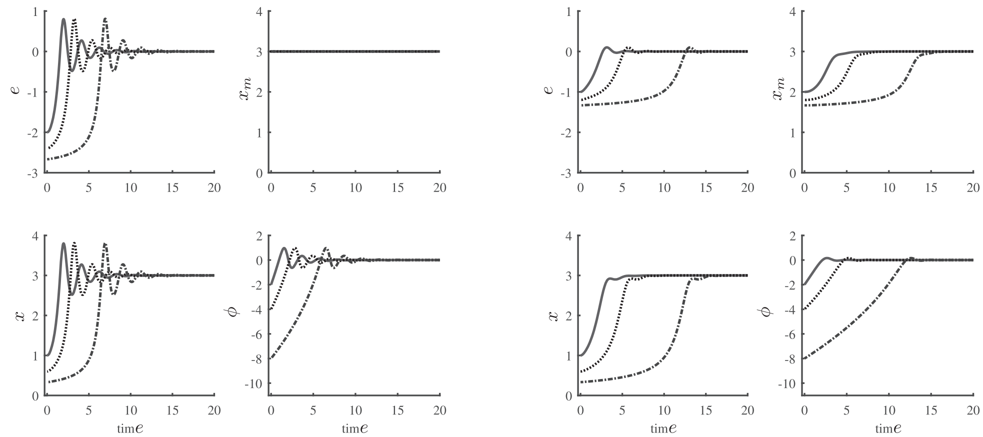
(b) 2D projection of the 3D phase portrait for the CRM adaptive system with initial conditions defined in Table 2.

FIG. 4. *Phase portraits of the ORM and CRM adaptive systems.*

an attractor that all initial conditions converge to. This attractor partially coincides with S_{O5} and S_{C5}. We focus on those initial conditions that are closest to these attractors that are common to both ORM and CRM adaptive systems, which are given by initial conditions z_4, z_5, and z_6. With these initial conditions we next discuss the region of slow convergence in both adaptive systems.

We present time responses of e, ϕ, x, and x_m for the ORM system in Figure 5(a) and the CRM adaptive system in Figure 5(b), respectively, for the initial conditions z_4, z_5, and z_6. Defining T_s as the settling time beyond which $\|z(t_0) - z(\infty)\|$ reduces to 5% of its initial value, we have that $T_s \in \{5.37,\ 5.62,\ 8.19\}$ for these three initial conditions for the ORM system and $T_s \in \{3.69,\ 5.85,\ 12.74\}$ for the CRM system. Notice that although $\frac{z(t_0)}{z(t_0+T_s)}$ is identical for all three trajectories, T_s increases as $\|z(t_0)\|$ increases, implying that the system is not exponentially stable in the large.

Trajectories initialized at both z_5 and z_6 demonstrate the slow convergence described in this paper, which is characterized by the nearly flat portion of the response



(a) Simulation of the the ORM adaptive system.

(b) Simulation of the the CRM adaptive system.

FIG. 5. *Time series trajectories of the ORM and CRM adaptive systems for intial conditions* z_4 *(solid),* z_5 *(dot), and* z_6 *(dash-dot) as defined in Tables* 1 *and* 2.

of e and x prior to convergence. From the third initial condition, z_6, the exacerbated sluggish effect in the CRM adaptive system can clearly be seen. The error convergence for large initial conditions is even slower compared to that of the ORM system. It was observed that this convergence became slower as $|\ell|$ was increased further. It should be noted that these convergence properties coexist with the absence of the oscillatory behavior in the CRM in comparison to the ORM. That is, the introduction of the feedback gain ℓ helps in producing a *smooth* adaptation but not a *fast* adaptation. Increasing γ along with ℓ can keep convergence times similar to those of the ORM while maintaining reduced oscillations.

6. Conclusions. In this paper, precise definitions of asymptotic and exponential stability are reviewed and a definition of weak persistent excitation is introduced, which is initial condition dependent. With these definitions it has been shown that when PE conditions are imposed on the reference model, it results in weak persistent excitation of the adaptive system. The implication of this weak PE is that the speed of convergence is initial condition dependent, resulting in UASL of the origin in the underlying error system. Exponential stability *in the large* cannot be proven and claims of robustness should be based on the UASL property.

While some of the preliminary results outlined in this work apply to both continuous and discrete time dynamics, the question of whether weak persistence of excitation in the underlying plant state excludes ESL in discrete time adaptive control is still an open one. The methods proposed in this work are not directly applicable to the analysis of discrete time adaptive systems due to the fact that the space of stabilizing gains in discrete time is compact as compared to the unbounded region, which exists in continuous time systems.

This work does, however, directly relate to many active areas of research in adaptive control and the results of this work should be heeded as we move forward. Transient performance is one such area. In order for adaptive control to be deployed in a real system, its transient behavior needs to be fully characterized. Through this and other studies on CRM adaptive control, it is clear that there is a fundamental trade-off in adaptive control when it comes to smooth transients and adaptive parameter convergence. One cannot simply have both smooth and fast adaptation in these control systems.

These results also have implications in the machine learning community, especially when large datasets are analyzed and the learning is performed with sequential examples (because the dataset is too large to analyze in a single shot) [15]. In those paradigms the learning is inherently dynamical. Dynamics also appear explicitly in other areas of machine learning such as adversarial training [16], bayesian optimization [38], and reinforcement learning [39]. In all three of the above examples the way in which the "state space" is explored is initial condition dependent, and that is what connects those learning paradigms to ours.

Appendix A. Definitions.

DEFINITION 4 (piecewise smooth function [40]). *Let $\mathcal{C}_\delta$ be a set of points in $[t_0, \infty)$ for which there exists a $\delta > 0$ such that for all $t_1, t_2 \in \mathcal{C}_\delta$, $t_1 \neq t_2$ implies $|t_1 - t_2| \geq \delta$. Then $\mathcal{P}_{[t_0,\infty)}$ is defined as the class of real valued functions on $[t_0, \infty)$ such that for every $u \in \mathcal{P}_{[t_0,\infty)}$, there corresponds some δ and $\mathcal{C}_\delta$ such that*

(i) *$u(t)$ and $\dot{u}(t)$ are continuous and bounded on $[t_0, \infty) \setminus \mathcal{C}_\delta$ and*

(ii) *for all $t_1 \in \mathcal{C}_\delta$, $u(t)$ and $\dot{u}(t)$ have finite limits as $t \to t_1^+$ and $t \to t_1^-$*

REFERENCES

[1] B. D. O. Anderson, *Exponential stability of linear equations arising in adaptive identification*, IEEE Trans. Automat. Control, 22 (1977), pp. 83–88.

[2] B. D. O. Anderson and C. R. Johnson, Jr., *Exponential convergence of adaptive identification and control algorithms*, Automatica, 18 (1982), pp. 1–13.

[3] B. D. O. Anderson and J. B. Moore, *New results in linear system stability*, SIAM J. Control, 7 (1969), pp. 398–414.

[4] E. A. Barbashin and N. N. Krasovskii, *On the stability of motion as a whole*, Dokl. Akad. Nauk SSSR, 86 (1952), pp. 453–456.

[5] S. Boyd and S. Sastry, *On parameter convergence in adaptive control*, Systems Control Lett., 3 (1983), pp. 311–319.

[6] S. Boyd and S. S. Sastry, *Necessary and sufficient conditions for parameter convergence in adaptive control*, Automatica, 22 (1986), pp. 629–639.

[7] C. Desoer, R. Liu, and L. Auth, *Linearity vs. nonlinearity and asymptotic stability in the large*, IEEE Trans. Circuit Theory, 12 (1965), pp. 117–118, https://doi.org/10.1109/TCT.1965.1082383.

[8] T. Gibson, Z. Qu, A. Annaswamy, and E. Lavretsky, *Adaptive output feedback based on closed-loop reference models*, in IEEE Trans. Automat. Control, 60 (2015), pp. 2728–2733, https://doi.org/10.1109/TAC.2015.2405295.

[9] T. E. Gibson, *Closed-Loop Reference Model Adaptive Control: With Application to Very Flexible Aircraft*, Ph.D. thesis, Massachusetts Institute of Technology, 2014.

[10] T. E. Gibson and A. M. Annaswamy, *Adaptive control and the definition of exponential stability*, in Proceedings of the 2015 American Control Conference, 2015, pp. 1549–1554, https://doi.org/10.1109/ACC.2015.7170953.

[11] T. E. Gibson, A. M. Annaswamy, and E. Lavretsky, *Improved transient response in adaptive control using projection algorithms and closed loop reference models*, in Proceedings of the AIAA Guidance Navigation and Control Conference, 2012.

[12] T. E. Gibson, A. M. Annaswamy, and E. Lavretsky, *Closed-loop reference model adaptive control, Part* I: *Transient performance*, in Proceedings of the American Control Conference, 2013.

[13] T. E. Gibson, A. M. Annaswamy, and E. Lavretsky, *Closed-loop reference models for output–feedback adaptive systems*, in Proceedings of the European Control Conference, 2013.

[14] T. E. Gibson, A. M. Annaswamy, and E. Lavretsky, *On adaptive control with closed-loop reference models: Transients, oscillations, and peaking*, IEEE Access, 1 (2013), pp. 703–717.

[15] I. Goodfellow, Y. Bengio, and A. Courville, *Deep Learning*, MIT Press, Cambridge, MA, 2016, http://www.deeplearningbook.org.
[16] I. Goodfellow, J. Pouget-Abadie, M. Mirza, B. Xu, D. Warde-Farley, S. Ozair, A. Courville, and Y. Bengio, *Generative adversarial nets*, in Advances in Neural Information Processing Systems, 2014, pp. 2672–2680.
[17] W. Hahn, *Stability of Motion*, Springer-Verlag, New York, 1967.
[18] P. Ioannou and J. Sun, *Robust Adaptive Control*, Dover, New York, 2013.
[19] B. Jenkins, T. Gibson, A. Annaswamy, and E. Lavretsky, *Convergence properties of adaptive systems with open-and closed-loop reference models*, in Proceedings of the AIAA Guidance Navigation and Control Conference, 2013.
[20] B. Jenkins, T. Gibson, A. Annaswamy, and E. Lavretsky, *Uniform asymptotic stability and slow convergence in adaptive systems*, in Proceedings of the IFAC International Workshop on Adaptation and Learning in Control and Signal Processing, Vol. 11, 2013, pp. 446–451.
[21] R. E. Kalman, *Contributions to the theory of optimal control*, Bol. Soc. Math. Mex., 5 (1960), pp. 102–119.
[22] R. E. Kalman and J. E. Bertram, *Control systems analysis and design via the 'second method' of Liapunov,* I. *Continuous-time systems*, J. Basic Engineering, 82 (1960), pp. 371–393.
[23] N. Krasovskii, *Stability of Motion*, Stanford University Press, Stanford, CA, 1963.
[24] G. Kreisselmeier, *Adaptive observers with exponential rate of convergence*, IEEE Trans. Automat. Control, 22 (1977), pp. 2–8.
[25] J. LaSalle, *Some extensions of Liapunov's second method*, IRE Trans. Circuit Theory, 7 (1960), pp. 520–527, https://doi.org/10.1109/TCT.1960.1086720.
[26] J. P. Lasalle, *Asymptotic stability criteria*, in Hydrodynamic Instability, Proc. Sympos. Appl. Math. 13, AMS, Providence, RI, 1962, pp. 299–307.
[27] P. M. Lion, *Rapid identification of linear and nonlinear systems*, AIAA J., 5 (1967), pp. 1835–1842.
[28] A. Loría and E. Panteley, *Uniform exponential stability of linear time-varying systems: Revisited*, Systems Control Lett., 47 (2002), pp. 13–24.
[29] I. Malkin, *On stability in the first approximation*, Sb. Nauch. Trudov Kazanskogo Aviac. Inst., 3 (1935).
[30] J. S. Massera, *Contributions to stability theory*, Ann. Math., 64 (1956), pp. 182–206.
[31] A. Morgan and K. Narendra, *On the stability of nonautonomous differential equations* $\dot{x} = [A + B(t)]x$*, with skew symmetric matrix* $B(t)$, SIAM J. Control Optim., 15 (1977), pp. 163–176.
[32] A. Morgan and K. Narendra, *On the uniform asymptotic stability of certain linear nonautonomous differential equations*, SIAM J. Control Optim., 15 (1977), pp. 5–24.
[33] K. S. Narendra and A. M. Annaswamy, *Robust adaptive control in the presence of bounded disturbances*, IEEE Trans. Automat. Control, 31 (1986), pp. 306–315.
[34] K. S. Narendra and A. M. Annaswamy, *Persistent excitation in adaptive systems*, Internat. J. Control, 45 (1987), pp. 127–160.
[35] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*, Dover, New York, 2005.
[36] O. Nouwens, A. M. Annaswamy, and E. Lavretsky, *Analysis of slow convergence regions in adaptive systems*, in 2016 American Control Conference (ACC), Boston, MA, 2016, pp. 6995–7000.
[37] E. Panteley, A. Loría, and A. Teel, *Relaxed persistency of excitation for uniform asymptotic stability*, IEEE Trans. Automat. Control, 46 (2001), pp. 1874–1886.
[38] B. Shahriari, K. Swersky, Z. Wang, R. P. Adams, and N. de Freitas, *Taking the human out of the loop: A review of bayesian optimization*, Proc. IEEE, 104 (2016), pp. 148–175.
[39] R. S. Sutton, A. G. Barto, and R. J. Williams, *Reinforcement learning is direct adaptive optimal control*, IEEE Control Systems, 12 (1992), pp. 19–22.
[40] J. S.-C. Yuan and W. M. Wonham, *Probing signals for model reference identification*, IEEE Trans. Automat. Control, 22 (1977), pp. 530–538.