1 Game Theory Overview

Wikipedia describes game theory as **the study of mathematical models of strategic interaction between rational decision-makers**. It has many applications in economics, logic, computer science, sociology, and more. In this helper hours, we will revisit some of the core tools used in game theory to better understand how rational actors may behave.

1.1 Matrix Notation

The first thing worth visiting is the notation. The games discussed in this class will generally be presented in **matrix form**. In matrix form there are two agents, a row player and a column player. Along each row is an action that the row player may take, and along each column is an action that the column player may take.

At the intersection of these row and column actions are payoffs for both the row and column player if they were to take these actions. In zero sum games, the payoff for the column player = -1 * payoff for the row player, though in the general case, this condition is not met. Note the row/column actions must clearly define how each player will act. In games involving sequential actions, responses to differing observations must be baked into these actions. This will become more clear in the example section.

	C	D
A	1, -1	4, -3
B	0, -2	1, -2

Table 1: Example Payoff Matrix

Consider the matrix from above. The row player has choices of playing action A or B, and the column player has choices of actions C or D. If the row player chose A, and the column player chose D, then the row player would get a payoff 4, and the column player would get a payoff -3.

1.2 Dominance

1.2.1 Strict Dominance

A row α strictly dominates another row β if, no matter what action the column player takes, all row player rewards in row α are strictly greater than the corresponding payoff (in the same column) in row β. In the example matrix from above, Row A strictly dominates Row B. A column strictly dominates another column if, no matter what action the row player uses, all column player reward values in the dominating column are better for the column player than the corresponding payoff (in the same row) in the column being dominated.

1.2.2 Weak Dominance

Weak dominance is similar to strict dominance, but rather than all values needing to be greater than, only one needs to be greater than, and the rest must be greater than or equal to. In other words, if one value is greater, and the rest are the same, it still weakly dominates. If all values are the same, it does not weakly dominate. For example, Col C weakly dominates Col D.

1.2.3 Relevance

What does dominance tell us? Well if a row strictly dominates another row, we are always better off playing the dominating row than the dominated row, so we might as well disregard the dominated row when considering strategies. A similar argument can be applied for strictly dominating columns.

In terms of weak dominance, we know we are at least as well off playing a weakly dominating row or column over whatever it dominates. While we may not be better off, we will never be worse off.

1.3 Equilibrium

An equilibrium occurs whenever both players are content with their strategies against one another. Written more formally, if neither player can improve their reward by changing their strategy, then they are in an equilibrium. All games have an equilibrium. Note, an equilibrium does not need to be a pure strategy, it may be a mixed strategy. A mixed strategy means that rather than playing one action with probability 1.0, play different actions with a distribution of probabilities. For example, in rock paper scissors, a player should play R, P, and S all with equal probability ($\frac{1}{3}$).

2 Example

2.1 Problem Statement

This game involves two players, P1 and P2. The game is played as follows. P1 flips a coin and peeks at it (but does not show anyone else). P1 then makes a claim about what they tossed (H or T). P2 now has the choice to accept or dispute the claim. If the claim is disputed, everyone looks at the coin; if P1 lied, P2 wins, otherwise P1 wins. If the claim is accepted, then it stands and nobody looks at it. P2 then has to toss the coin publicly. If he tosses something better than P1 claimed, P2 wins; if worse, P1 wins; if the same, it's a tie.

2.2 Game Tree

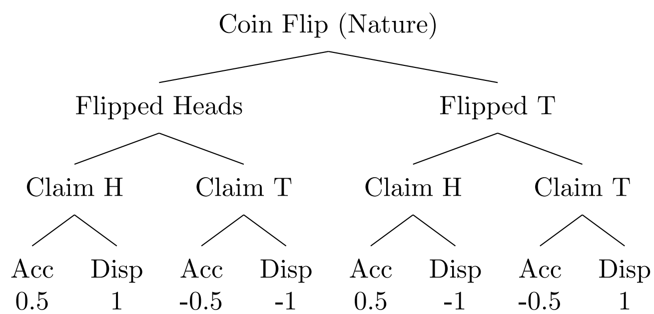

Figure 1: Game Tree

2.3 Matrix Derivation

We will represent the game matrix using similar notation as the Jack-King game from the slide deck. The row player receives a binary observation (from looking at the coin) and has two moves, they can claim either heads or tails: {H, T}. The column player also receives a binary observation (claimed H/T), and has two moves accept or dispute, which we will call {A, D} respectively.

Using this, every row can be comprised of two variables XY, where X indicates the response when a heads is observed and Y represents the response when a tails is observed. For example the TH row strategy means the column player always claims to have the opposite of what they actually have.

Every column can also be comprised of two variables NM, where N indicates the response when heads was claimed by P1, and M indicates the response when tails was claimed by P1. For example, DD will dispute regardless of if P1 claims heads or tails. Using this, we arrive at the following matrix:

	AA	AD	DA	DD
HH	0.5, -0.5	0.5, -0.5	0, 0	0, 0
HT	0, 0	0.75, -0.75	0.25, -0.25	1, -1
TH	0, 0	-0.25, 0.25	-0.75, 0.75	-1, 1
TT	-0.5, 0.5	0, 0	-0.5, 0.5	0, 0

Table 2: Payoff Matrix

2.4 Equilibrium Determination

Assume we number columns from left to right 1, 2, 3, 4 and rows 1, 2, 3, 4 from top to bottom. Do any rows strictly dominate? Locate and remove them.

Row 1 strictly dominates row 3. Row 2 strictly dominates row 4.

	AA	AD	DA	DD
HH	0.5, -0.5	0.5, -0.5	0, 0	0, 0
HT	0, 0	0.75, -0.75	0.25, -0.25	1, -1
TH	X	X	X	X
TT	X	X	X	X

What about columns? Repeat the same procedure:

Col 3 strongly dominates Col 2

	AA	AD	DA	DD
HH	0.5, -0.5	X	0, 0	0, 0
HT	0, 0	X	0.25, -0.25	1, -1
TH	X	X	X	X
TT	X	X	X	X

Any equilibrium present after removing a weakly dominated row/col will still be present in the matrix that never removed it, so lets try that. Are there any instances of weak dominance? If so, remove them.

Col 3 weakly dominates Col 4.

	AA	AD	DA	DD
HH	0.5, -0.5	X	0, 0	X
HT	0, 0	X	0.25, -0.25	X
TH	X	X	X	X
TT	X	X	X	X

Now, lets solve for a mixed strategy equilibrium.

To make P1 indifferent between HH and HT, we need utility for HH to equal utility for HT:

$$0.5P(AA) + 0 = 0 + 0.25(1 - P(AA))$$

Or P(AA) $= \frac{1}{3}$. To make P2 indifferent between AA and DA, we need for the utility of AA to equal the utility of DA

$$-0.5P(HH) + 0 = 0 + -0.25(1 - P(HH))$$

Or P(HH) $= \frac{1}{3}$.

2.5 Equilibrium

Based on our calculations from above, one equilibrium to the game is:

$$P(HH) = \frac{1}{3}$$

$$P(HT) = \frac{2}{3}$$

$$P(AA) = \frac{1}{3}$$

$$P(DA) = \frac{2}{3}$$

2.6 Intuition

Lets take a look back at the dominated rows and columns, and try to interpret why they make sense.

Say for example, claiming tails when you flipped heads. Both of the strategies that performed this action were dominated. Why might this be the case? Well, you always lose or tie when you claim tails after receiving a heads, where as you always win or tie when you claim heads. Clearly, claiming heads is the better choice in such a case.

Why might accepting when someone claims tails be a good choice for P2? Well, P2 will always tie or win when they accept in this case. But they may be wrong if they dispute.

3 Follow Up

3.1 Weak Dominance

We used the fact that equilibriums present after removing weakly dominated rows and columns are still present in the original matrix, why does this make intuitive sense?

3.2 Pure Strategy

Is there a pure strategy equilibrium for this game? How do you know?

3.3 Simple Game Tree To Matrix Code

For those interested in a programmatic way of going from a game tree to a game theory matrix, see code in github repo for this week which does it for the example problem.