Financial Data Science Homework Set2

Jordan Giebas

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Question 2

We estimate the parameters α and β of the $\Gamma(\alpha, \beta)$ distribution using the method of moments below. For the following, let $X \sim \Gamma(\alpha, \beta)$, and equate the theoretical moments with the sample mean and variance.

$$\mathbb{E}\left[X\right] = \frac{\alpha}{\beta} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$$

$$\mathbb{E}\left[X^2\right] - (\mathbb{E}\left[X\right])^2 = \frac{\alpha(\alpha+1)}{\beta^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = S^2$$

Manipulating the first and plugging into the second we find that,

$$\tfrac{\alpha}{\beta} = \bar{X} \implies \tfrac{\alpha^2}{\bar{X}^2} = \beta^2 \implies \tfrac{\alpha(\alpha+1)}{\frac{\alpha^2}{\bar{X}^2}} = S^2$$

Solving for α, and then using this result to solve for β, we find that

$$\alpha = \frac{\bar{X}}{S^2}$$

$$\beta = \frac{\bar{X}^2}{S^2}$$

Question 3

Note that: $bias = \mathbb{E}\left[\widehat{\theta}\right] - \theta$. Substituting the definition of bias into the hint the rest follows,

$$\begin{aligned} MSE(\widehat{\theta}) &= \mathbb{E}\left[(\widehat{\theta} - \theta)^2\right] \\ &= \mathbb{E}\left[((\widehat{\theta} - \mathbb{E}(\widehat{\theta})) + bias(\widehat{\theta}))^2\right] \\ &= \mathbb{E}\left[(\widehat{\theta} - \mathbb{E}(\widehat{\theta}))^2\right] + 2\,\mathbb{E}\left[(\widehat{\theta} - \mathbb{E}(\widehat{\theta}))bias(\widehat{\theta})\right] + \mathbb{E}\left[bias^2(\widehat{\theta})\right] \\ &= \mathbb{E}\left[(\widehat{\theta} - \mathbb{E}(\widehat{\theta}))^2\right] + 0 + bias^2(\widehat{\theta}) \\ &= Var(\widehat{\theta}) + bias^2(\widehat{\theta}) \end{aligned}$$

Question 4

Part(a)

To show that $\widehat{\mu}$ is an unbiased estimator of μ, we must show that $\mathbb{E}\left[\widehat{\mu}\right] = \mu$.

$$\begin{aligned} \mathbb{E}\left[\widehat{\mu}\right] &= \mathbb{E}\left[\sum_{i=1}^{n} \omega_i X_i\right] \\ &= \sum_{i=1}^{n} \omega_i\,\mathbb{E}\left[X_i\right] \\ &= \sum_{i=1}^{n} \omega_i \mu \\ &= \mu \sum_{i=1}^{n} \omega_i \end{aligned}$$

Therefore note that if $\sum_{i=1}^{n} \omega_i = 1$, then it is clear that $\mathbb{E}\left[\widehat{\mu}\right] = \mu$ and $\widehat{\mu}$ is an unbiased estimator.

Part(b)

To minimize the variance while enforcing the unbiased constraint that $\sum_{i=1}^{n} \omega_i = 1$, we follow the hint.

$$
\begin{aligned}
V(\widehat{\mu}) &= \sum_{i=1}^{n} \omega_i^2 \sigma_i^2 \\
&= \sum_{i=1}^{n-1} \omega_i^2 \sigma_i^2 + \omega_n^2 \sigma_n^2 \\
&= \sum_{i=1}^{n-1} \omega_i^2 \sigma_i^2 + (1 - \sum_{i=1}^{n-1} \omega_i)^2 \sigma_n^2 \\
&= \sum_{i=1}^{n-1} \omega_i^2 \sigma_i^2 + \sigma_n^2 (1 - 2\sum_{i=1}^{n-1} \omega_i + (\sum_{i=1}^{n-1} \omega_i)^2)
\end{aligned}
$$

To minimize the expression, we set the partial derivative w.r.t. w_i equal to zero and solve.

$$
\begin{aligned}
V(\widehat{\mu}) &= \sum_{i=1}^{n-1} \omega_i^2 \sigma_i^2 + \sigma_n^2 (1 - 2\sum_{i=1}^{n-1} \omega_i + (\sum_{i=1}^{n-1} \omega_i)^2) \\
\frac{\partial V(\widehat{\mu})}{\partial \omega_i} &= 2\omega_i \sigma_i^2 - 2\sigma_n^2 + 2\sigma_n^2 \sum_{i=1}^{n-1} \omega_i \\
0 &= \omega_i \sigma_i^2 - \sigma_n^2 + \sigma_n^2 (1 - \omega_n) \\
\omega_i &= \frac{\sigma_n^2}{\sigma_i^2} \omega_n
\end{aligned}
$$

Summing each side $\forall i \in \{1, 2, ..., n\}$ and solving for ω_n, we find that

$$
\omega_n = \frac{1}{\sigma_n^2 \sum_{i=1}^{n} \frac{1}{\sigma_i^2}}
$$

Plugging this back into our previous equation, we finally find the the ω_i that will minimize the variance is given as follows

$$
\omega_i = \frac{\sigma_n^2}{\sigma_i^2} \frac{1}{\sigma_n^2 \sum_{i=1}^{n} \frac{1}{\sigma_i^2}} = \frac{\frac{1}{\sigma_i^2}}{\sum_{j=1}^{n} \frac{1}{\sigma_j^2}}
$$