Simulation Methods for Option Pricing
46-932
Homework #4

Due: Thursday, February 15, 2018

Problem #1 Practice on Conditional Monte Carlo
Assume the price process for an underlying is given by the Hull-White stochastic volatility model, i.e.

$$dS = rSdt + vSdW_1, \quad dv^2 = \alpha v^2 dt + \psi v^2 dW_2.$$

Use an Euler (first order) discretization scheme for the paths of the underlying and construct $n = 10,000$ paths in both parts below.

Use the paths as the basis for a standard Monte Carlo simulation to estimate the price of a standard European call with strike K. You should do this using standard Monte Carlo and conditional Monte Carlo. Take as parameter values: $S_0 = K = 100, r = .05, T = 1, v^2(0) = .04$ and try $N = 50$. Consider 2 cases: $(\alpha, \psi) = (.10, .10), (.10, 1.0)$.

a) Use standard Monte Carlo techniques to estimate the call price and provide standard errors.

b) Recall Exercise 5.4 page 253 in *Stochastic Calculus for Finance II: Continuous Time Models* by Steve Shreve. This exercise shows that one can price a European call on an underlying governed by Brownian motion with time varying but deterministic volatility, $\sigma(t),\ 0 \le t \le T$ using the Black Scholes formula where the constant volatility σ is replace by $\sqrt{\frac{1}{T}\int_0^T \sigma^2(t)dt}$.

   Suppose one simulates a volatility path $\{\sigma(i\Delta),\ 0 \le i \le N\}$ where $\Delta = \frac{T}{N}$ and approximates $\sqrt{\frac{1}{T}\int_0^T \sigma^2(t)dt}$ by $\sqrt{\frac{1}{N}\sum_{i=1}^N \sigma^2(i\Delta)}$. Now, for a given volatility path, the volatility function becomes time varying but known. Consequently, a call option can be priced by the Black-Scholes formula with the volatility replaced by $\sqrt{\frac{1}{N}\sum_{i=1}^N \sigma^2(i\Delta)}$ without ever simulating the underlying price path.

   Simulate n volatility paths and price the option as described above on each volatility path. Average and compute the standard error. This is the conditional Monte Carlo method for computing the option price under a stochastic volatility model. Compare with the results for standard Monte Carlo in part (a).

Problem #2 Practice on interest rate derivatives and CIR
Consider a Cox-Ingersoll-Ross (CIR) square root diffusion spot-rate model (refer to Glasserman pp120-125)

$$dr(t) = \alpha(b - r(t))dt + \sigma\sqrt{r(t)}dW(t),$$

on $[0, T]$ where $\alpha = .2, \sigma = .1, b = .05$, and $r(0) = .04$. For $n = 1,000$ paths and $N = 50$ time steps:

a) Find the price of a zero coupon bond paying \$1 at time $T = 1$.

b) A **caplet** on a principal of L over the time interval $[t, t + \delta]$ with rate R provides a payoff of $L\delta \max(0, r(t) - R)$ at time t. Assuming $t = 1$, $\delta = 1/12$, $L = 1$, $R = .05$, and the same CIR model given above applies, find the value of the caplet assuming the current time is 0. Note, the definition of the caplet in this problem where the payoff is at time t rather than time $t+\delta$

is not typical but is used to avoid the nuisance of having the time-step increment, $\frac{t}{N}$, not be an even multiple of the interval length δ.

Note: A **cap** is just a collection (portfolio) of caplets, so pricing caps as well as floors, swaps, swaptions, etc. is just a simple extension of the program in part b). While the transition density is known for the CIR model, one can use a discretization scheme to evaluate interest rate derivatives for other spot rate models.

Problem #3 Replicating Broadie and Glasserman "Greeks" methodology.
Refer to Table 1 on page 276 of "Estimating security price derivatives using simulation," by Broadie and Glasserman, which can be downloaded in the Course Documents area of Blackboard). The relevant results are given in the table below. Consider the first column ($S_0 = 90$) only. Replicate the 6 simulation results for Delta and Vega (mean and standard error) using the resimulation, pathwise and likelihood ratio methods. Use the same parameter values as those used by Broadie and Glasserman given at the bottom of the table. In particular $r = .1, K = 100, \delta = .03, \sigma = 0.25, T = 0.2, n = 10,000, h = .0001$.

You may use the formulas given in equations (32) for delta and (7) for vega in the paper or the formulas from Example 7.2.1 pp388-389 in Glasserman's text (with $r+\frac{1}{2}\sigma^2$ replaced by $r-\delta+\frac{1}{2}\sigma^2$).

In all cases use the final price, S_T as the control variable.

The desired results from Table 1 for delta are:

	Delta Est.	Delta Std Err	Vega Est.	Vega Std Err
Resimulation estimate	0.217	0.005	11.640	0.268
Resimulation with control	0.221	0.003	11.887	0.175
Pathwise estimate	0.217	0.005	11.640	0.268
Pathwise with control	0.221	0.003	11.887	0.175
Likelihood estimate	0.215	0.008	11.490	0.672
Likelihood with control	0.220	0.006	11.857	0.600

Problem #4 Applying Broadie and Glasserman to Digital Options.
Consider a simple European digital option with payoff at T of $+1$ if and only if $S_T \geq K$. Assume the asset price process is a geometric Brownian motion under the risk-neutral measure:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

In the simulation, assume $S_0 = 95, K = 100, r = .05, \sigma = .20, T = 1, n = 10,000$ and in implementing the resimulation method use $h = .0001$.

a) Using standard risk-neutral pricing theory calculate for general parameter values the price of the digital option and the delta of the option. Recall that you found a closed form expression for the function P of the price of a digital option in Problem #3 in Assignment #3. The Greeks (like delta) can be obtained by simply differentiating this formula on the respective parameters.

b) Use the resimulation method to estimate the delta.

c) Use the likelihood method to estimate the delta.