Simulation Methods for Option Pricing
46-932
Homework #2

Due: Thursday, February 1, 2018, 5:30pm

1) Practice on the Cholesky decomposition
   Consider the following matrix:

$$\begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix}.$$

   Find the Cholesky decomposition and indicate how one could simulation a sequence of trivariate normal random variables with the above covariance matrix.

2) Studying the Quality of Various Normal Generation Methods (continued).
   Using the package or language that you intend to use throughout this course, generate three different sets of observations (n = 100, n = 1,000 and n = 10,000) designed to have a standard normal distribution using the methods listed below, and present a Q-Q (normal) plot for each. Comment on the quality of the methods based on your normal plots.

   (a) The rejection method based on the unit exponential distribution presented in class.

   (b) The generalized lambda distribution with $\lambda_1 = 0$ $\lambda_2 = 0.1975$, $\lambda_3 = \lambda_4 = 0.1349$.

   (c) The "Goldman Sachs weighted normal" distribution described in Chapter 1.

3) Practice on generating bivariate data and copulas

   (a) Generate 1,000 bivariate random variables from a standard bivariate-normal distribution with correlation 0, 0.4, and 0.8. Show scatterplots of the data for each case and comment on the way in which those plots change as the correlation increases.

   (b) Generate 1,000 bivariate random variables from a standard bivariate-t_5 distribution with correlation 0, 0.4, and 0.8. Show scatterplots of the data for each case and comment on the way in which those plots change as the correlation increases. Compare with the plots from the previous part of this problem.

   (c) Generate 1,000 bivariate random variables each component of which has an exponential(1) marginal distribution using a standard Gaussian copula with correlation 0, 0.4, and 0.8. Show scatterplots of the data for each case and comment on the way in which those plots change as the correlation increases.

   (d) Generate 1,000 bivariate random variables each component of which has an exponential(1) marginal distribution using a standard t_5 copula with correlation 0, 0.4, and 0.8. Show scatterplots of the data for each case and comment on the way in which those plots change as the correlation increases. Compare with the scatterplots in the previous part of this problem.

4) Generating t-variates using rejection
The t-distribution is often used to model data having heavier tails than the normal distribution. The p.d.f. for a t-distribution with n degrees of freedom is given by:

$$f_n(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})}(1+\frac{x^2}{n})^{-(n+1)/2}, \quad \text{for } -\infty < x < \infty,$$

where $\Gamma(\cdot)$ is the gamma function. Note, for positive integers, $\Gamma(n) = (n-1)!$, for $x > 1$, $\Gamma(x) = (x-1)\Gamma(x-1)$, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. A t-distribution with 1 degree of freedom is the Cauchy distribution. Note also that as $n \to \infty$, the t-distribution converges to the standard normal distribution.

The t-distribution for an integer number of degrees of freedom can be generated using a rejection algorithm. Assuming that Cauchy distributed random variables are easy to generate (see homework problem 1.1c), we can factor the t p.d.f. into

$$f_n(x) = C_n \cdot f_1(x)[\frac{f_n(x)}{C_n \cdot f_1(x)}] = C_n \cdot f_1(x) \cdot g(x).$$

Recall that C_n must be chosen so that $0 < g(x) \leq 1$ for $-\infty < x < \infty$.

(a) Give an expression for the smallest value of C_n that can be used in the rejection algorithm for generating t-variates with n degrees of freedom.

(b) Obviously, $C_1 = 1$. Compute C_2, C_3, C_5, C_∞. For C_∞ it is easiest to define $g(x) = \frac{\phi(x)}{C_\infty \cdot f_1(x)}$ where $\phi(x)$ is the standard normal p.d.f.

(c) Write a rejection algorithm to generate m random variables having a t-distribution with n degrees of freedom and use it to generate $m = 1000$ observations and present normal Q-Q plots for the t-distribution with 1, 3, 5, 10, and 30 degrees of freedom.

5) Practice on antithetic variables for Black Scholes
Consider a standard Black-Scholes problem, a European call with $S_0 = 100, T = 1, r = .05, \sigma = .1$. The Black-Scholes price for the at-the-money option is \$6.805. Remember to give standard errors for your estimates.

a) Price this option using standard Monte Carlo methods with a sample size of $n = 1,000$ and compare with the exact price for K = 95, 100, and 105.

b) Using the same sample size of $n = 1,000$, price the option using the method of antithetic variables for K = 95, 100, 105. Compare you results with those in part a) for all three strikes, i.e. compare the standard errors.