Simulation Methods for Option Pricing
46-932
Homework #3

Due: Thursday, February 8, 2018, 5:30pm

1) Practice on antithetic variables for Black Scholes
Consider a standard Black-Scholes problem, a European call with $S_0 = 100, T = 1, r = .1, \sigma = .1$. The Black-Scholes price for the at-the-money option is \$6.805. Remember to give standard errors for your estimates.

   a) Price this option using standard Monte Carlo methods with a sample size of $n = 1,000$ and compare with the exact price for K = 95, 100, and 105.

   b) Using the same sample size of $n = 1,000$, price the option using the method of antithetic variables for K = 95, 100, 105. Compare you results with those in part a) for all three strikes, i.e. compare the standard errors.

2) Pricing an arithmetic Asian option
Consider an asset price process, $\{S_t,\ t \geq 0\}$ governed by a geometric Brownian motion with drift $\mu = .10$ and volatility $\sigma = .1$ per year. Assume riskless interest rates are $r = .05$ and $S_0 = 100$. Suppose the interval $[0, T]$ is divided into N equal time steps of length $\Delta = \frac{T}{N}$. Consider an European arithmetic Asian option averaged once per time step with strike price $K = 100$. Assume $T = 1$ and $N = 52$. Using the notation that $S_{i\Delta}$ is the asset price at time $i\Delta$, the option payoff is given by $\max(\frac{1}{N}\sum_{i=1}^{N} S_{i\Delta} - K, 0)$.

   (a) Using standard Monte Carlo simulation wtih $n = 1,000$ paths, estimate the price of this option and provide a standard error.

   (b) For each path, construct an antithetic path, average the regular and the antithetic path discounted option payoffs, then provide an estimate of the price and its standard error.

   (c) Estimate the option price using the average final price, $\bar{S}_N$ as a control variable.

   (d) Refer to the material on pages 56-58 of the Course Notes. Estimate the price of this option using the geometric Asian option as a control variable where the price of the geometric Asian option is given by the asymptotic formula ($N \to \infty$), $\sigma^* = \frac{\sigma}{\sqrt{3}}$ and $d^* = \frac{r+d}{2} + \frac{\sigma^2}{12}$ as a control variable.

   (e) Repeat part (d) only using the exact price of the geometric Asian option with $\sigma^* = \sigma\sqrt{\frac{(N+1)(2N+1)}{6N^2}}$ and $d^* = r\frac{N-1}{2N} + d\frac{N+1}{2N} + \sigma^2(\frac{(N+1)(N-1)}{12N^2})$.

   Compare the results in the various parts of this problem, commenting on the effectiveness of the various variance reduction methods.

3) Practice on Control Variables and Importance Sampling.
Consider an asset whose price process is given by a geometric Brownian motion with initial price $S_0 = 100$, drift $\mu = .10$ and volatility $\sigma = 0.2$. Suppose riskless interest rates are $r = .05$. Consider a European call option with expiration $T = 1$ year and various strike prices K. We want to study the benefits of importance sampling when K is large, and the option is out of the money. Note, other variance reduction methods could also be implemented in this problem, but we only focus on importance sampling.

   (a) Using Standard Monte Carlo methods with $n = 10,000$ price this option when $K = 120, 140, 160$.

   (b) Using the put-call parity, reformulate the call pricing problem as a put pricing problem and repeat the previous exercise.

   (c) Repeat part b) using the final price, S_T, as a control variable.

   (d) Repeat this problem, only change the measure so that the option pays off with certainty. That is, rather than using a standard normal distribution to compute the final price, change so that $Z > L$. Compare your results with those in the first two parts.

4) Practice on conditional Monte Carlo and importance sampling: barrier options
Consider Example 4.6.4, p264 of the Glasserman text, pricing a down-and-in barrier option with lower barrier H, strike K, and initial stock price S_0. Assume the evaluation points are equally spaced, $\Delta, 2\Delta, \ldots, m\Delta$ where $\Delta = T/m$. The example prices this option using a double change of measure. The purpose of this exercise is to implement a new variance reduction technique, conditional Monte Carlo, and to compare it with standard Monte Carlo and the method used in the example. The particular focus is on the 4 entries for the first case, $T = 0.25, m = 50$ in Table 4.4 on page 267. Note that the underlying follows a Geometric Brownian Motion process with $r = .05$, $\sigma = 0.15$ and initial value $S_0 = 95$. Define $M = \min_{1 \le k \le m} S(k\Delta)$. Use the same sample size as Glasserman, $n = 100,000$.

   a) Standard Monte Carlo
   Using the standard Monte Carlo method, simulate a path with m time steps and evaluate the payoff $I\{S(T) > K\}I\{M < H\}$. Use the discounted option payoffs to estimate the option price and give its standard error.

   b) Conditional Monte Carlo

      i. Consider a digital option with **no** knock-in barrier, i.e. a payoff function $I\{S(T) > K\}$. Using arbitrage-based-pricing theory, derive a closed-form expression for the price of the digital option. Define the function $P(S_0, K, r, \sigma, T)$ to be this formula, i.e. the price of the digital option.

      ii. Simulate a path as in part (a) sequentially as follows. If $S(k\Delta) \ge H$, $1 \le k \le m$, then assign 0 to the path. Otherwise let τ represent the first time at which $S(\tau\Delta) < H$. At this time point stop generating the path and note that the knock-in condition has been satisfied. Consequently, at that time the option can be explicitly evaluated, i.e. its value is given by $P(S(\tau\Delta), K, r, \sigma, T - \tau\Delta)$. Consequently assign value $e^{-\tau\Delta}P(S(\tau\Delta), K, r, \sigma, T - \tau\Delta)$ to this path. Now estimate the value of the option and its standard error in the usual way.

      iii. Create a table similar to the first 4 lines of Table 4.4 using the ratio of the variances calculated from your results in parts (a) and (c).

5) Discrete versus continuous pricing
Consider a "down and in" call with payoff $\max(0, (S(T) - K)) \cdot I\{M < H\}$. For the continuously monitored barrier case, a closed-form formula exists, see the formula for c_{di} at the bottom of page 533 of Edition 7 of Hull (posted on Blackboard). Suppose $S_0 = K = 100, H = 95, \sigma = .30, r = .10, T = 0.2$, so this corresponds to roughly 50 trading days. Use a sample size of $n = 100,000$. In the problems below compare your results with the continuous case price and compare the variance reductions achieved in b) and c) with a).

   a) Use standard Monte Carlo to estimate the price of the down and in call for $N = 25$ and $N = 50$.

   b) Use conditional Monte Carlo whenever the price path falls below H (if it ever does) to reduce the variance for N = 25 and 50 where N is the number of evaluation points in $[0, T]$. The case of $N \to \infty$ is given by the Hull formula.

   c) The conditional Monte Carlo method done in part (a) is vulnerable to the case in which S_0 and H are far apart, since in this case many paths will not reach the lower barrier and will return a 0 payoff. This problem can be addressed by changing the distribution of the normal random variables from N(0,1) to N(θ, 1) for some $\theta < 0$. This will increase the number of paths that knock-in. This combines the conditional Monte Carlo method with the importance sampling method. Combine the conditional Monte Carlo method with importance sampling by choosing $\theta = -.45$ for $N = 25$ and $\theta = -.30$ for $N = 50$.