Simulation Methods for Option Pricing
46-932
Homework #1

Due: Thursday, January 25, 2018 5:30pm.

1. Practice on the Probability Integral Transform.

   a) Suppose X is a random variable with a Weibull distribution with shape parameter k and scale parameter λ having a p.d.f. given by

   $$f_X(x) = \begin{cases} k\lambda(\lambda x)^{k-1}e^{-(\lambda x)^k} & \text{if } 0 \leq \text{x}, \\ 0 & \text{if x} < 0. \end{cases}$$

   Give the probability integral transform method for generating this distribution.

   b) (Note, the distribution will arise in the paper by Beaglehole, Dybvig and Zhou). Suppose X is a random variable having the distribution of the maximum of a Brownian Bridge having a c.d.f. given by

   $$F_X(x) = \begin{cases} 1 - e^{-2x(x-b)/h} & \text{if } \max(0, b) \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

   where h and b are given constants. Give the probability integral transform method to generate this distribution. Be careful to ensure that the random variable X you generate satisfies $X \geq \max(0, b)$.

   c) Let X be a random variable having a standard Cauchy distribution, that is X has p.d.f. $f_X(x)$ given by

   $$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \text{ for } -\infty < x < \infty.$$

   Give the probability integral transform method to generate this random variable.

   d) Consider the standard Gumbel distribution having c.d.f. given by

   $$F(x) = \exp(-\exp(-x)), \ -\infty < x < \infty.$$

   This distribution arises in risk management and is known as an *extreme value distribution* (see part ii below).

   i) Give the probability integral transform method to generate one observation from the standard Gumbel distribution.

   ii) Suppose $X_1, \ldots, X_n$ are i.i.d. random variables, each having a standard Gumbel distribution. Let $M_n = \max(X_1, \ldots, X_n)$. Recalling that $P(M_n \leq x) = P((X_1 \leq x) \cap \ldots \cap (X_n \leq x))$ and the X_i, $1 \leq i \leq n$ are independent, find a sequence of constants, $\{a_n\}$ such that $M_n - a_n$ has a standard Gumbel distribution. What does this tell you about the growth of M_n as a function of n?

2. Practice on the Aliasing Algorithm and Introduction to Credit Risk
   This problem is a *highly* simplified version of a problem you will encounter in the MSCF course on credit risk involving rating-based bond valuation. The numerical values for this problem are taken from the paper "A Markov model for the term structure of credit risk spreads," by Jarrow, Lando, and Turnbull (JLT). The paper is posted on Blackboard. The idea is that we want to model the changes in bond ratings over time. JLT model this using a

standard continuous time Markov chain, i.e. a particular bond has one of 8 ratings (AAA = 1, AA = 2, A = 3, BBB = 4, BB = 5, B = 6, CCC = 7, D = 8) where we consider the rating D (8) to refer to "default." Under this model, a bond has one of 8 ratings and holds that rating for an exponential period of time, the parameter of which depends upon the rating. At the end of that holding time, its rating transitions to a new rating according to a transition probability matrix. Table 4, p507 of JLT gives the generator matrix for this Markov chain, Q, given by:

$$Q = \begin{bmatrix} -.1154 & .1019 & .0083 & .0020 & .0031 & 0 & 0 & 0 \\ .0091 & -.1043 & .0787 & .0105 & .0030 & .0030 & 0 & 0 \\ .0010 & .0309 & -.1172 & .0688 & .0107 & .0048 & 0 & .0010 \\ .0007 & .0047 & .0713 & -.1711 & .0701 & .0174 & .0020 & .0049 \\ .0005 & .0025 & .0089 & .0813 & -.2530 & .1181 & .0144 & .0273 \\ 0 & .0021 & .0034 & .0073 & .0568 & -.1929 & .0479 & .0753 \\ 0 & 0 & .0142 & .0142 & .0250 & .0928 & -.4318 & .2856 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

The holding times in each state are exponential with parameters (.1154, .1043, .1172, .1711, .2530, .1929, .4318, 0). The embedded transition probability matrix is

$$P = \begin{bmatrix} 0 & .8838 & .0720 & .0173 & .0269 & 0 & 0 & 0 \\ .0872 & 0 & .7545 & .1007 & .0288 & .0288 & 0 & 0 \\ .0085 & .2637 & 0 & .5870 & .0913 & .0410 & 0 & .0085 \\ .0041 & .0275 & .4167 & 0 & .4097 & .1017 & .0117 & .0286 \\ .0020 & .0099 & .0352 & .3213 & 0 & .4668 & .0569 & .1079 \\ 0 & .0109 & .0176 & .0379 & .2946 & 0 & .2484 & .3906 \\ 0 & 0 & .0329 & .0329 & .0579 & .2149 & 0 & .6614 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

Now it is easy to solve for the transition probabilities for this Markov chain, $p_{ij}(t)$, as they are given as the entries in the matrix $\exp(Qt)$. One can compute them using MATLAB using the matrix exponential function: expm(Q*t). If, however, the holding times in the various states is non-exponential, then a simulation approach is needed. Note, in the credit risk course, you will need to simulate in Markov case because if default occurs before the bond matures, then the exact time of default and the random amount recovered need to be considered.

The alias algorithm can be conveniently used to simulate the transitions, and when each transition occurs, a random variable representing the length of time the rating will stay in that state (with any probability distribution) can be drawn. The homework problem is to simlate this system in the Markov and non-Markov case to work out the probability distribution of the bond rating at time T given the bond has one of the 8 ratings at time 0. This can be conveniently done by keeping track of 2 state variables, the clock and the current rating state. Start the bond out in a particular rating state and set the clock to 0. When the holding time elapses in a rating state, assuming the clock is still less than the target time T, a new rating state is chosen (using, for example, the aliasing algorithm) along with a new holding time for the next rating state. That new holding time can have any distribution, presumably one that best fits the data.

Note that the transition probabilities are different for the different states. It is possible to combine the 8 different alias tables associated with the 8 different rating states into a single table. Since there are 8 states, the transition probabilities distribution will have 8 possible values (at least one of which will have 0 proabability). Thus an alias table would have 7 rows.

By stacking the 8 tables on top of each other, one can create a single table with 56 rows. If the current state is i, $1 \leq i \leq 8$, and row j $(1 \leq j \leq 7)$ is selected, one should use row (currentstate$-1) * 7 + j$ in the 56×3 aliasing matrix.

a) Write a program to generate the transition probabilites for the bond ratings using the Markov model and JLT generator above for T = 5 years. Use a sample size of n = 1,000,000. Compare your simulation results with the exact formula exp(Q*T). Recall that the standard error for a average of counts, $\hat{\pi}$ is $\sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$.

b) Find the bond rating transition probability matrix assuming that the holding times are: $\frac{1}{2}\text{Gamma}(2, \lambda_i)$, where the $\{\lambda_i\}$ are the same as in part (a). Compare with the results of part (a).

Note, there is a document in the Assignments Area of Blackboard in which the matrices Q and P are given along with the holding time vector. You might want to cut-and-paste from this document to avoid having to type in their contents.

3. Studying the Quality of Various Normal Generation Methods.
Using the package or language that you intend to use throughout this course, generate three different sets of observations (n = 100, n = 1,000 and n = 10,000) designed to have a standard normal distribution using the methods listed below, and present a Q-Q (normal) plot for each. Comment on the quality of the methods based on your normal plots.

(a) The normal generator available within the package or language you are using.

(b) The "poor man's" normal generator:
Generate 12 independent standard uniforms, $\{U_i\}_{i=1}^{12}$ and compute $\sum_{i=1}^{12} U_i - 6$.