jgiebas_HW2

February 1, 2018

1 46-932, Simulation Methods for Option Pricing: Homework 2

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1.1 Question 1

Find the Cholesky decomposition of the following matrix, i.e. a matrix that satisfies

$$AA^T = \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix} = \Sigma$$

Assume matrix A has lower triangular form as follows,

$$A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

Therefore,

$$A^T = \begin{bmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{bmatrix}$$

Then, writing explicitly AA^T and setting equal to Σ

$$AA^T = \begin{bmatrix} a^2 & ab & ad \\ ab & b^2+c^2 & bd+ce \\ ad & bd+ce & d^2+e^2+f^2 \end{bmatrix} = \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix}$$

Solving this system, we find that $(a, b, c, d, e, f) = (1, \rho, \sqrt{1-\rho^2}, \rho^2, \rho\sqrt{1-\rho^2}, \sqrt{1-\rho^2})$ (This can be confirmed by plugging in and performing the matrix multiplication to retrieve Σ.

In order to generate trivariate normals with a particular correlation, can do the following. Generate three standard normals and put them into a vector: $\vec{Z} = (Z_1, Z_2, Z_3)^T$ where $Z_i \sim N(0,1), \forall i$. Then, multiply the matrix A defined above by the vector $\vec{Z}$. The resulting vector $\vec{X} \in \mathbb{R}^{3x1}$ will be trivariate normal with the specified correlation.

1.2 Question 2

```
In [1]: """
        Some useful functions, and importing
        some modules for this question
        """
```

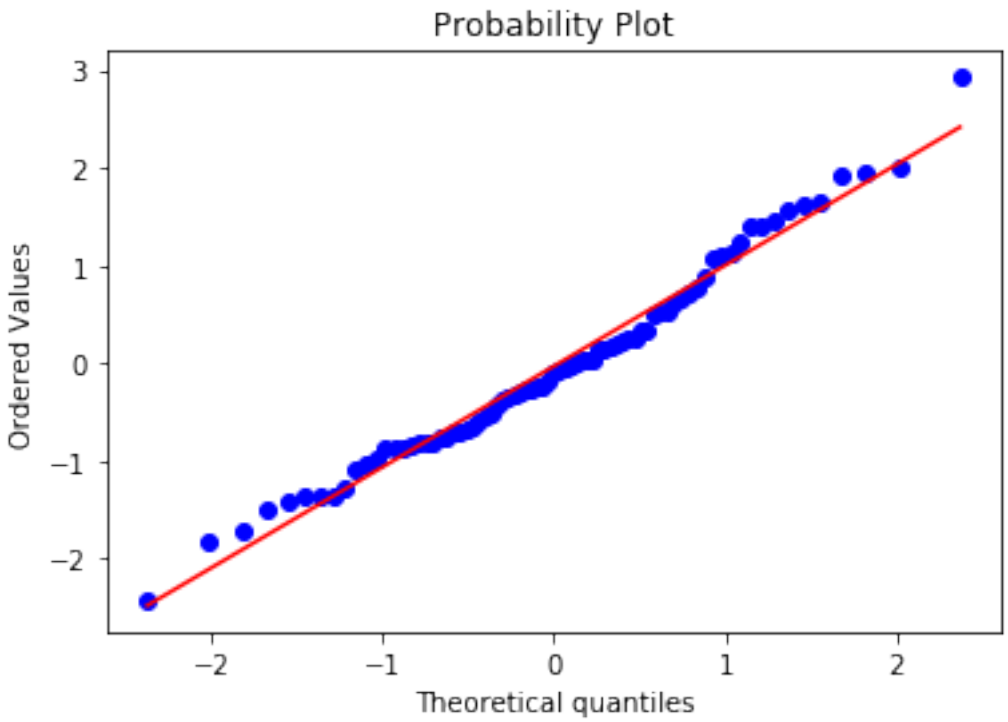
```
# Use numpy's normal RV generator
import numpy as np
import pylab
import scipy.stats as stats
import math
from math import sqrt, exp, pi, fabs, log
import matplotlib

# How to plot normal plots
def get_plt(data):

    stats.probplot(data, dist="norm", plot=pylab)
    pylab
```

1.2.1 Part (a)

Use the rejection method based on the unit exponential distribution (from class)

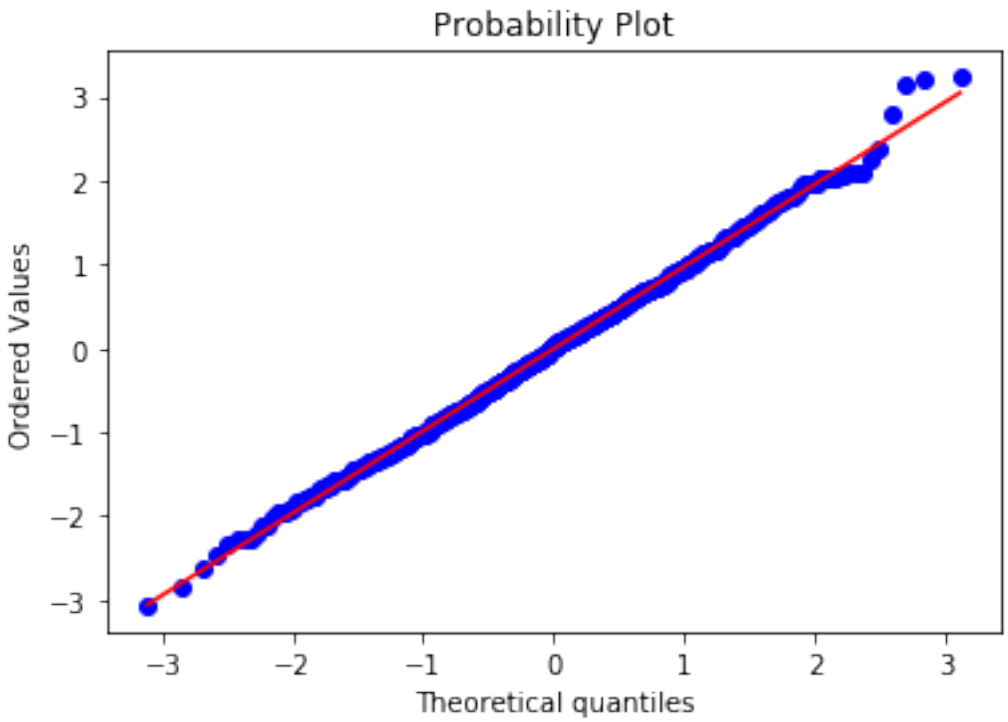
In [2]:
```
# Constant C is a global var
C = sqrt(2*exp(1)/pi)

# Define g(x) function
def g_fn( x ):

    return ( ((1/sqrt(2*pi))*exp(-0.5*(x**2)))/(0.5*C*exp(-1.0*fabs(x))) )


print("This is for n = 100")
x_list = list()
for i in range(100):

    # Generate a uniform
    U = np.random.uniform()

    # Decide which piece of the double-exponential function to use
    if (U < 0.5):
        y = np.log(2*U)
    else:
        y = -1.0*np.log(2-2*U)

    if (np.random.uniform() <= g_fn(y)):
        x_list.append(y)

get_plt(x_list)
```

```
This is for n = 100
```



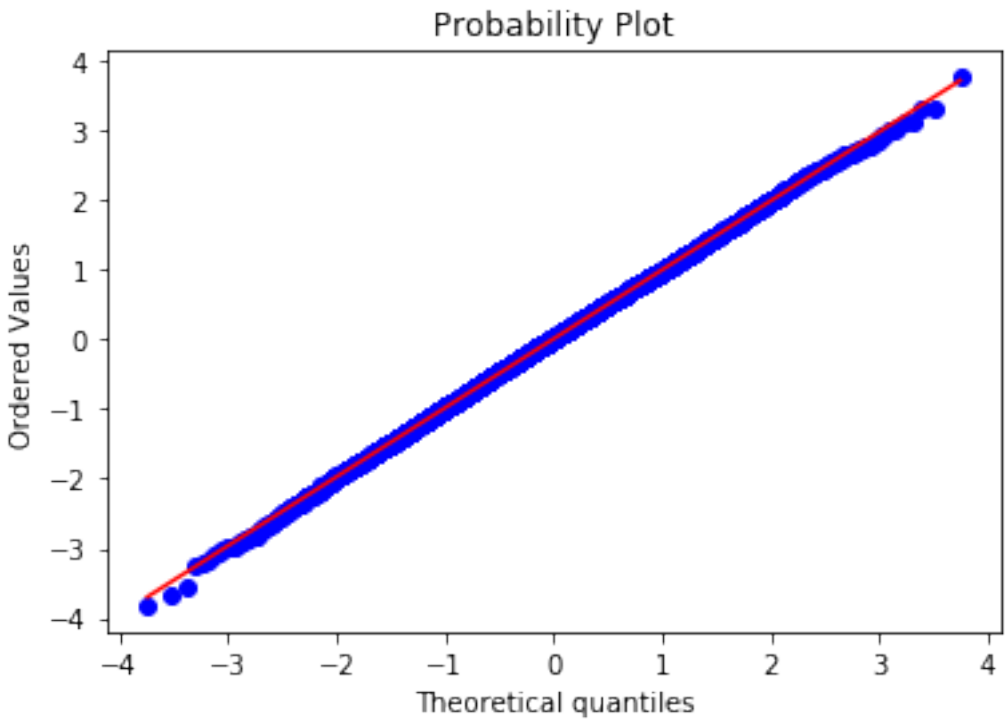

```
In [3]: print("This is for n = 1000")
        x_list = list()

        for i in range(1000):

            # Generate a uniform
            U = np.random.uniform()

            # Decide which piece of the double-exponential function to use
            if (U < 0.5):
                y = np.log(2*U)
            else:
                y = -1.0*np.log(2-2*U)

            if (np.random.uniform() <= g_fn(y)):
                x_list.append(y)

        get_plt(x_list)
```

```
This is for n = 1000
```




```
In [4]: print("This is for n = 10000")
        x_list = list()

        for i in range(10000):

            # Generate a uniform
            U = np.random.uniform()

            # Decide which piece of the double-exponential function to use
            if (U < 0.5):
                y = np.log(2*U)
            else:
                y = -1.0*np.log(2-2*U)

            if (np.random.uniform() <= g_fn(y)):
                x_list.append(y)

        get_plt(x_list)
```

```
This is for n = 10000
```



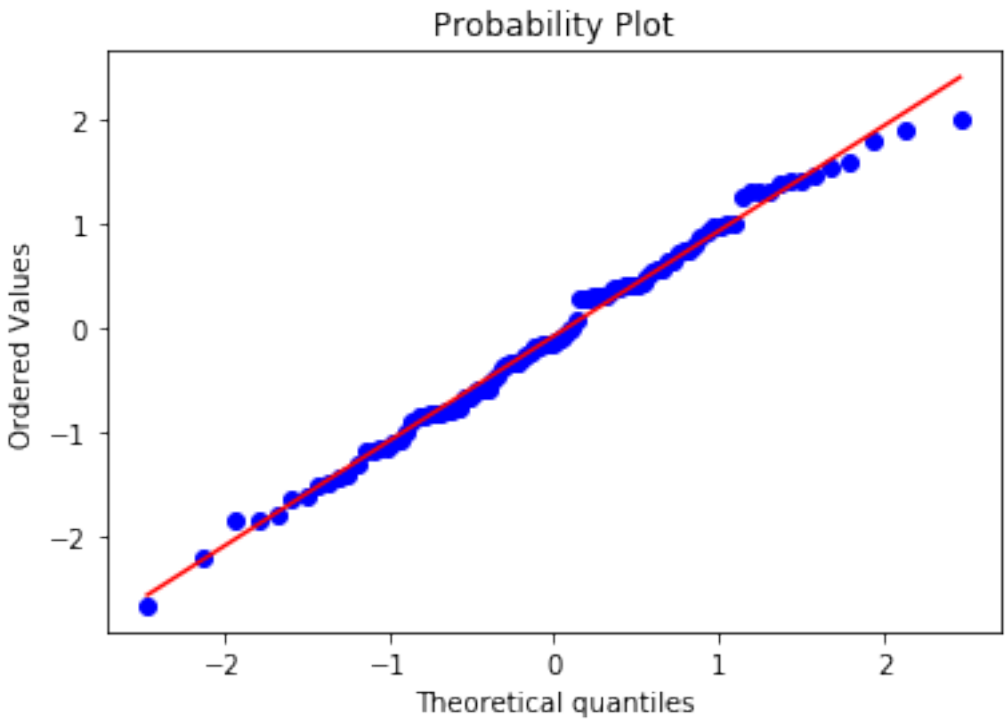
Using the rejection algorithm seems to work quite well. In the $n = 100$ case, there are some deviations from the normal distribution at the extrema; however, as n approaches $10,000$, we see that the deviation of these points from the normal distribution at the extrema is quite small. Hence, it follwows that this is a good algorithm for generating normals.

1.2.2 Part (b)

Use the generalized lambda distribution with $\lambda_1 = 0, \lambda_2 = 0.1975, \lambda_3 = \lambda_4 = 0.1349$

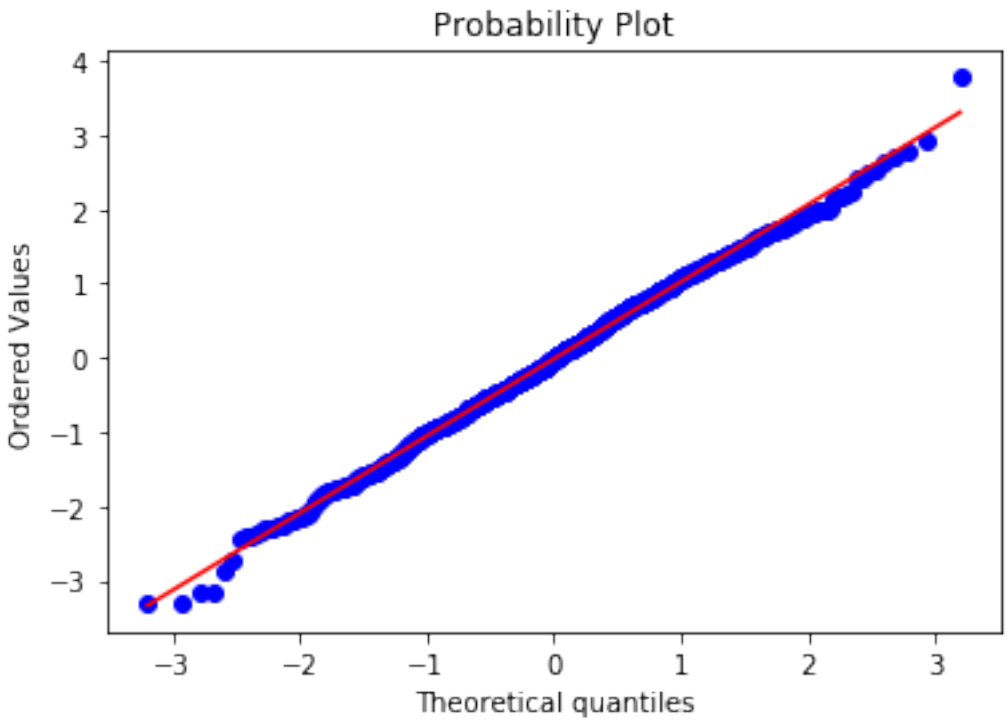
In [5]:
```
# Not even close to sure if this is correct

def f_inv( var ):

    u = np.random.uniform()
    return ( var[0] + ((1/var[1])*(u**var[2] - (1-u)**var[3])) )


print("This is for n = 100")
# Define parameters
vec = [0,0.1975,0.1349,0.1349]

x_list_2 = [f_inv(vec) for i in range(100)]
get_plt(x_list_2)
```

```
This is for n = 100
```



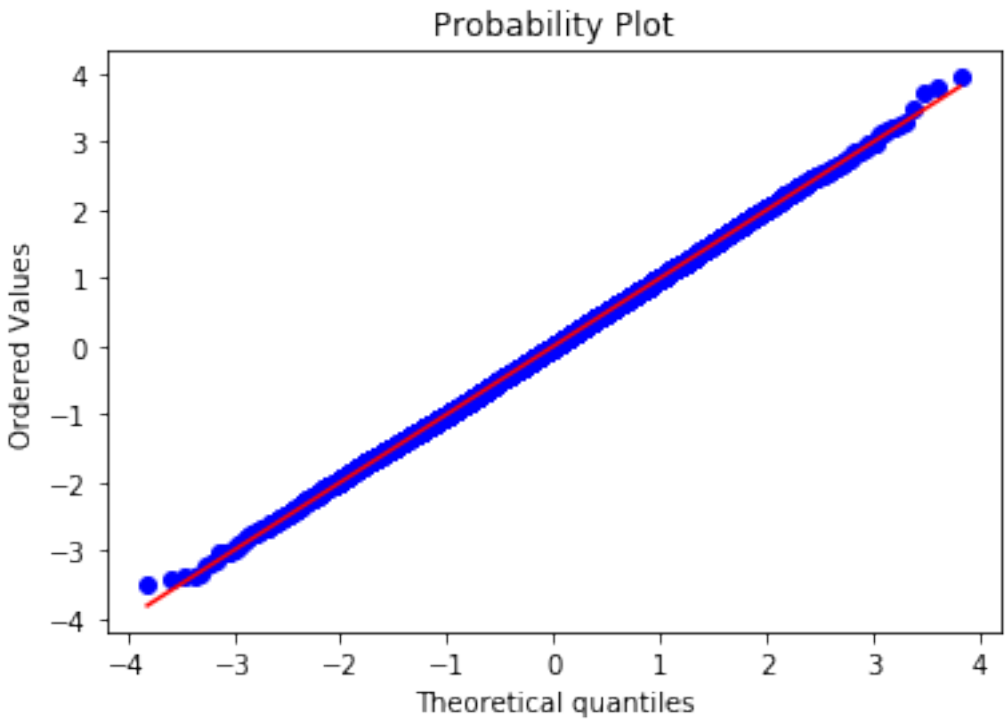

```
In [6]: print("This is for n = 1000")
        # Define parameters
        vec = [0,0.1975,0.1349,0.1349]

        x_list_2 = [f_inv(vec) for i in range(1000)]
        get_plt(x_list_2)
```

```
This is for n = 1000
```




```
In [7]: print("This is for n = 10000")
        # Define parameters
        vec = [0,0.1975,0.1349,0.1349]

        x_list_2 = [f_inv(vec) for i in range(10000)]
        get_plt(x_list_2)
```

```
This is for n = 10000
```



A similar analysis as in part (a) yields that as n approaches $10,000$ the deviations from the normal distribution are minimal and the generalized lambda distribution works very well to generate normals.

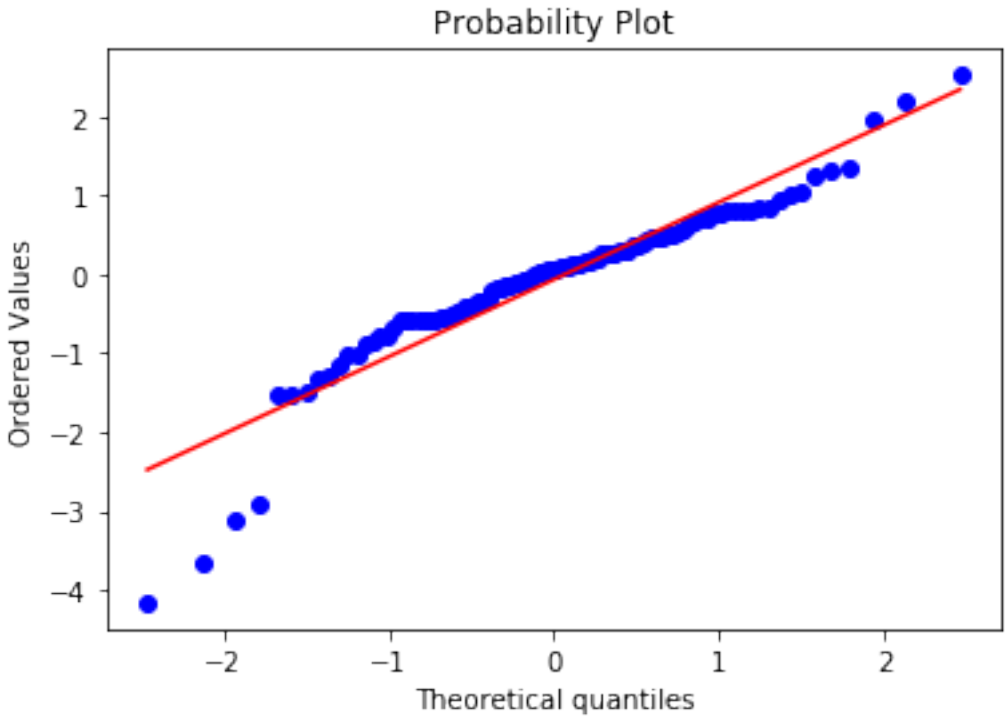
1.2.3 Part (c)

Use the "GS weighted normal" distribution

```
In [8]: # Define parameters to Litterman, Winkelmann (GS) normal mixture
        p   = 0.82
        w_1 = 0.60
        w_2 = 1.98

        # Define the mixture
        def gs():

            u = np.random.uniform()
            if u < p:
                return w_1*np.random.normal()
            else:
                return w_2*np.random.normal()


        # Plot the results
        print("This is for n = 100")
```

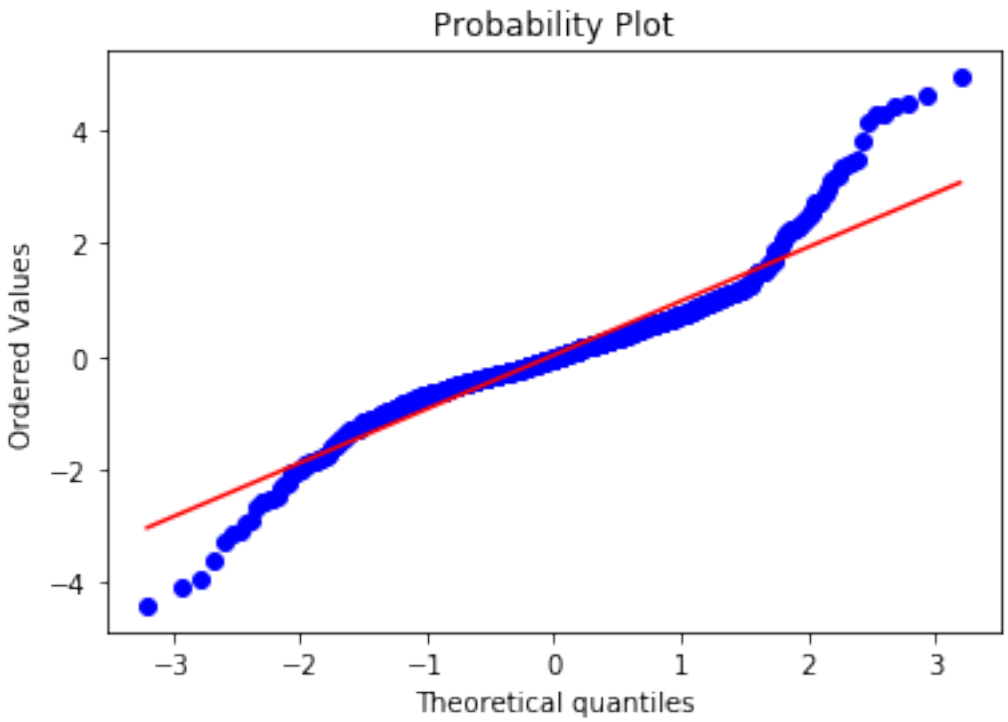
```
        x_list_3 = [gs() for i in range(100)]
        get_plt(x_list_3)
```

```
This is for n = 100
```



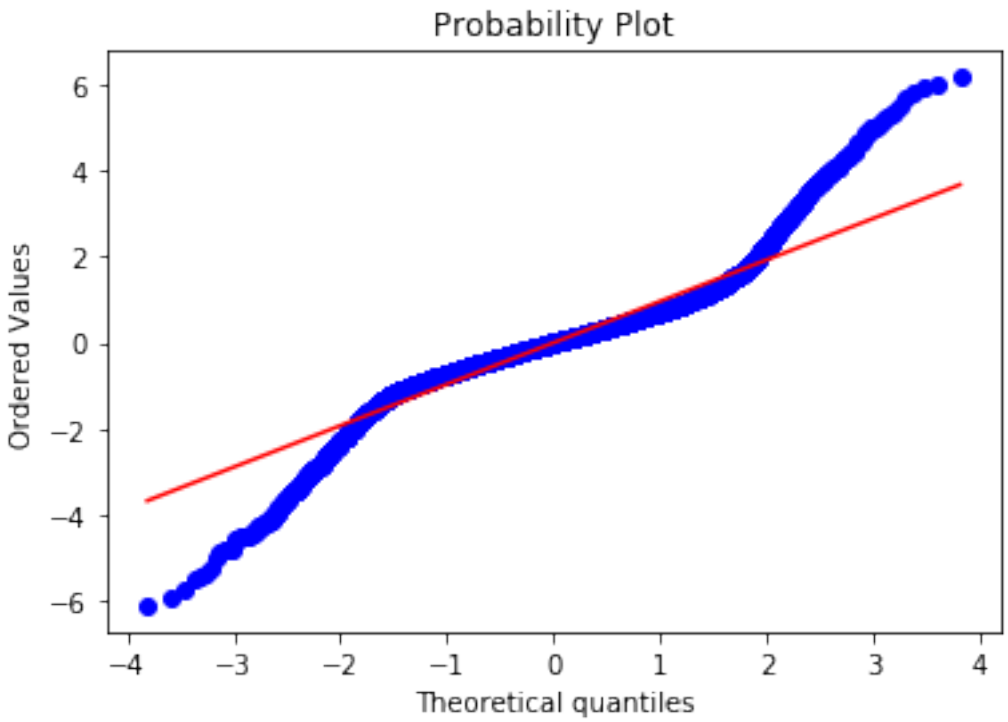
In [9]:
```
        # Plot the results
        print("This is for n = 1000")

        x_list_3 = [gs() for i in range(1000)]
        get_plt(x_list_3)
```

```
This is for n = 1000
```



In [10]:
```
# Plot the results
print("This is for n = 10000")

x_list_3 = [gs() for i in range(10000)]
get_plt(x_list_3)
```

```
This is for n = 10000
```



The "Goldman Sachs" weighted average seems to be the worst of the three methods tested. The deviations from the normal distribution are significant outside of the -2nd and 2nd quantiles. This is clearly not a great algorithm for generating standard normals. The plots somewhat suggest that the $n = 100$ case may be a little better.

1.3 Question 3

1.3.1 Part (a)

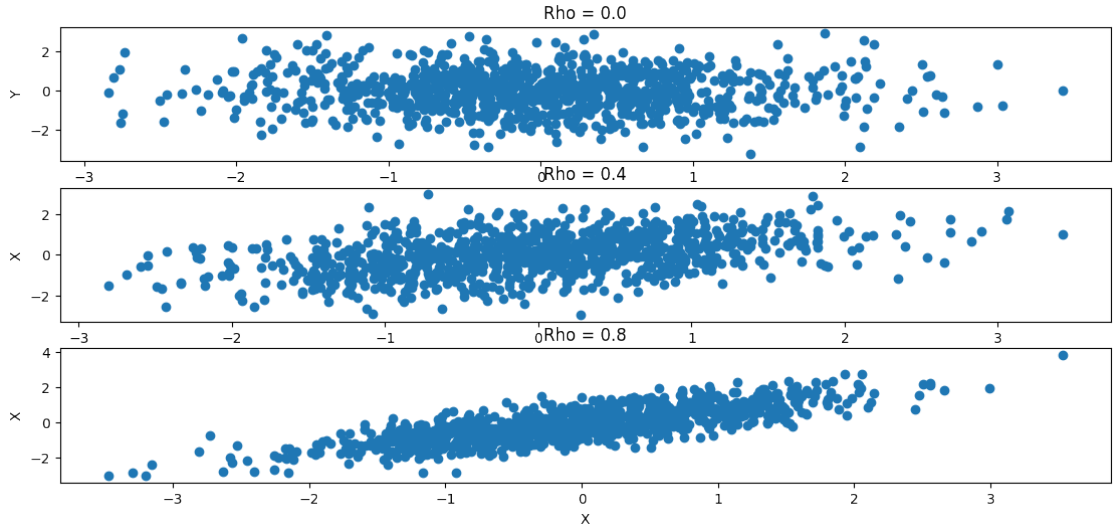
Generate 1,000 bivariate random vars from a standard bivariate normal distribution with correlations = 0, 0.4, and 0.8

```
In [11]: import matplotlib.pyplot as plt
         %matplotlib inline

         fig = plt.figure(figsize=(13.5, 6.0), dpi=100)

         x_list = list()
         y_list = list()

         def bivariate_normal( rho ):

             cov_mat = np.array([[1, rho],[rho, 1]])
             A = np.linalg.cholesky(cov_mat)
```

```
    x_vec = A.dot(np.random.normal(size=2))

    return x_vec[0], x_vec[1]


# Case 1
rho = 0.0
for i in range(1000):
    x, y = bivariate_normal(rho)
    x_list.append(x)
    y_list.append(y)

ax1 = fig.add_subplot(311)
ax1.title.set_text('Rho = 0.0')
plt.xlabel("X")
plt.ylabel("Y")
ax1.scatter(x_list,y_list)

# Case 2
x_list.clear()
y_list.clear()

rho = 0.4
for i in range(1000):
    x, y = bivariate_normal(rho)
    x_list.append(x)
    y_list.append(y)

ax2 = fig.add_subplot(312)
ax2.title.set_text('Rho = 0.4')
plt.xlabel("X")
plt.ylabel("X")
ax2.scatter(x_list,y_list)

# Case 3
x_list.clear()
y_list.clear()

rho = 0.8
for i in range(1000):
    x, y = bivariate_normal(rho)
    x_list.append(x)
    y_list.append(y)

ax3 = fig.add_subplot(313)
ax3.title.set_text('Rho = 0.8')
plt.xlabel("X")
plt.ylabel("X")
```

```
ax3.scatter(x_list,y_list)
```

```
Out[11]: <matplotlib.collections.PathCollection at 0x1112d7278>
```



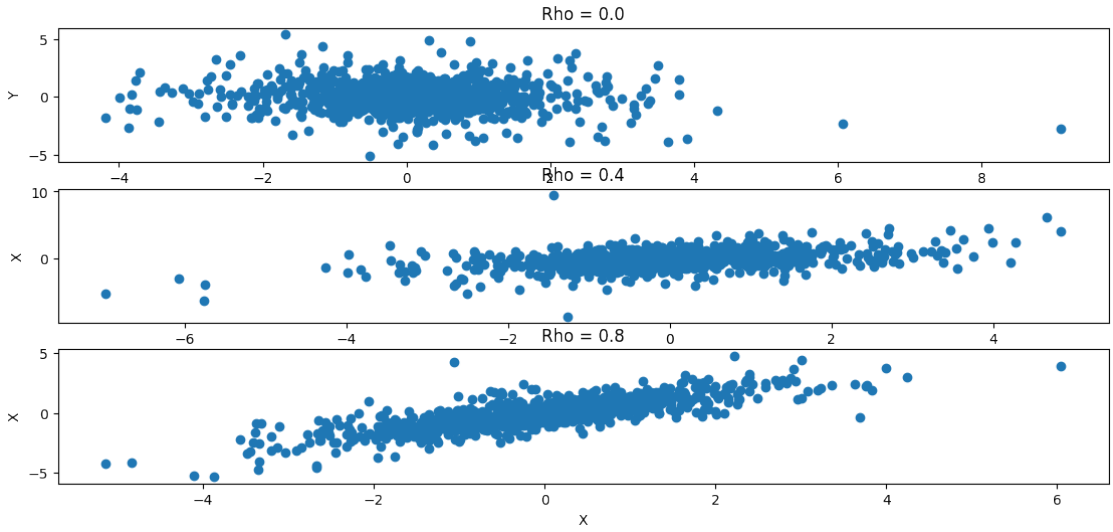
1.3.2 Part (b)

Generate 1,000 bivariate random vars from a standard t(5) distribution with correlations = 0, 0.4, and 0.8

```
In [12]: fig = plt.figure(figsize=(13.5, 6.0), dpi=100)

         # Similar as above, but needed
         def bivariate_normal_2( rho ):

             cov_mat = np.array([[1, rho],[rho, 1]])
             A = np.linalg.cholesky(cov_mat)
             x_vec = A.dot(np.random.normal(size=2))

             return x_vec


         def student_t(rho, df):

             T = np.sqrt(df/np.random.chisquare(df))*bivariate_normal_2(rho)
             return T[0], T[1]


         # Global for all cases
         nu_ = 5
```

```
# Case 1
x_list = list()
y_list = list()

rho = 0.0
for i in range(1000):
    x, y = student_t(rho, nu_)
    x_list.append(x)
    y_list.append(y)

ax1 = fig.add_subplot(311)
ax1.title.set_text('Rho = 0.0')
plt.xlabel("X")
plt.ylabel("Y")
ax1.scatter(x_list,y_list)

# Case 2
x_list = list()
y_list = list()

rho = 0.4
for i in range(1000):
    x, y = student_t(rho, nu_)
    x_list.append(x)
    y_list.append(y)

ax2 = fig.add_subplot(312)
ax2.title.set_text('Rho = 0.4')
plt.xlabel("X")
plt.ylabel("X")
ax2.scatter(x_list,y_list)

# Case 3
x_list = list()
y_list = list()

rho = 0.8
for i in range(1000):
    x, y = student_t(rho, nu_)
    x_list.append(x)
    y_list.append(y)

ax3 = fig.add_subplot(313)
ax3.title.set_text('Rho = 0.8')
plt.xlabel("X")
plt.ylabel("X")
ax3.scatter(x_list,y_list)
```

Out[12]: <matplotlib.collections.PathCollection at 0x1119ff550>



Note: In the t-distribution above, the tails are much heavier as desired. The extremeties reach from approximately -30 to 40 whereas in the normal case the extremes were above a tenth of that (-3,4).

1.3.3 Part (c)

Generate 1,000 bivariate random vars where each marginal follows a exponential(1) distribution, using standard gaussian copula with correlation 0.0, 0.4, 0.8

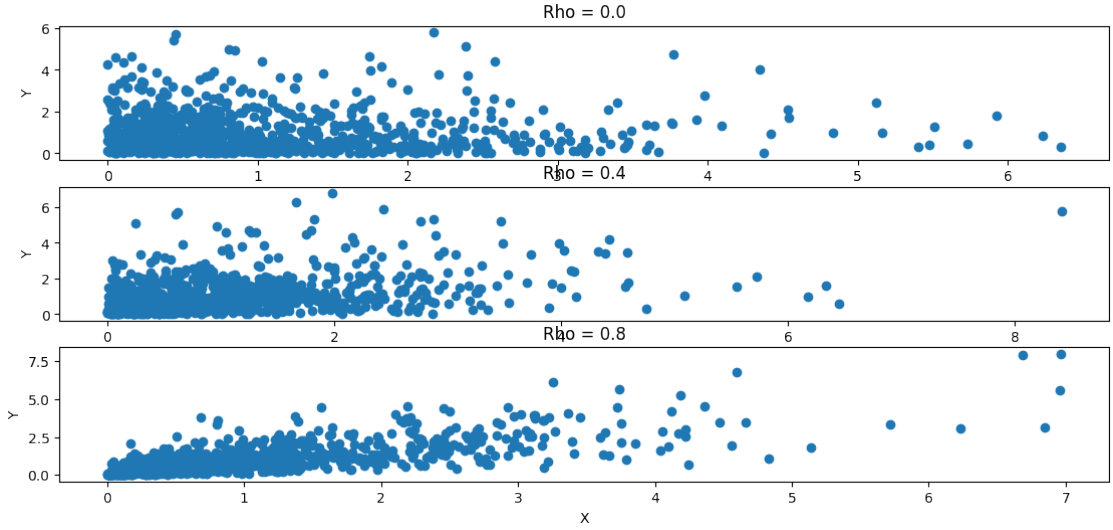
```
In [13]: import scipy
         from scipy.stats import norm

         ## Gaussian Copula
         ## Use the bivariate_normal_2 to generate the Y, in Y = AZ

         fig = plt.figure(figsize=(13.5, 6.0), dpi=100)

         x_list = list()
         y_list = list()

         """
         Case 1
         """
         rho = 0.0
         for i in range(1000):

             x, y = bivariate_normal(rho)
             x_list.append(x)
```

```
    y_list.append(y)

# Define the lists as np.arrays to perform
# element-wise scalar division
x_list = np.array(x_list)
y_list = np.array(y_list)

# Perform the scaling
x_list /= np.std(x_list)
y_list /= np.std(y_list)

# Apply the cdf
x_list = norm.cdf(x_list)
y_list = norm.cdf(y_list)

# Perform PIT for exponential(1) to make all marginal
x_list = -1.0*np.log(x_list)
y_list = -1.0*np.log(y_list)

# Plot
ax1 = fig.add_subplot(311)
ax1.title.set_text('Rho = 0.0')
plt.xlabel("X")
plt.ylabel("Y")
ax1.scatter(x_list,y_list)

"""
Case 2
"""
x_list = list()
y_list = list()

rho = 0.4
for i in range(1000):

    x, y = bivariate_normal(rho)
    x_list.append(x)
    y_list.append(y)

# Define the lists as np.arrays to perform
# element-wise scalar division
x_list = np.array(x_list)
y_list = np.array(y_list)

# Perform the scaling
x_list /= np.std(x_list)
y_list /= np.std(y_list)
```

```
# Apply the cdf
x_list = norm.cdf(x_list)
y_list = norm.cdf(y_list)

# Perform PIT for exponential(1) to make all marginal
x_list = -1.0*np.log(x_list)
y_list = -1.0*np.log(y_list)

# Plot
ax1 = fig.add_subplot(312)
ax1.title.set_text('Rho = 0.4')
plt.xlabel("X")
plt.ylabel("Y")
ax1.scatter(x_list,y_list)

"""
Case 2
"""
x_list = list()
y_list = list()

rho = 0.8
for i in range(1000):

    x, y = bivariate_normal(rho)
    x_list.append(x)
    y_list.append(y)

# Define the lists as np.arrays to perform
# element-wise scalar division
x_list = np.array(x_list)
y_list = np.array(y_list)

# Perform the scaling
x_list /= np.std(x_list)
y_list /= np.std(y_list)

# Apply the cdf
x_list = norm.cdf(x_list)
y_list = norm.cdf(y_list)

# Perform PIT for exponential(1) to make all marginal
x_list = -1.0*np.log(x_list)
y_list = -1.0*np.log(y_list)

# Plot
ax1 = fig.add_subplot(313)
ax1.title.set_text('Rho = 0.8')
```

```
        plt.xlabel("X")
        plt.ylabel("Y")
        ax1.scatter(x_list,y_list)
```

```
Out[13]: <matplotlib.collections.PathCollection at 0x110ee12e8>
```



1.3.4 Part (d)

Generate 1,000 bivariate random vars where each marginal follows a exponential(1) distribution, using standard t-5 copula with correlation 0.0, 0.4, 0.8

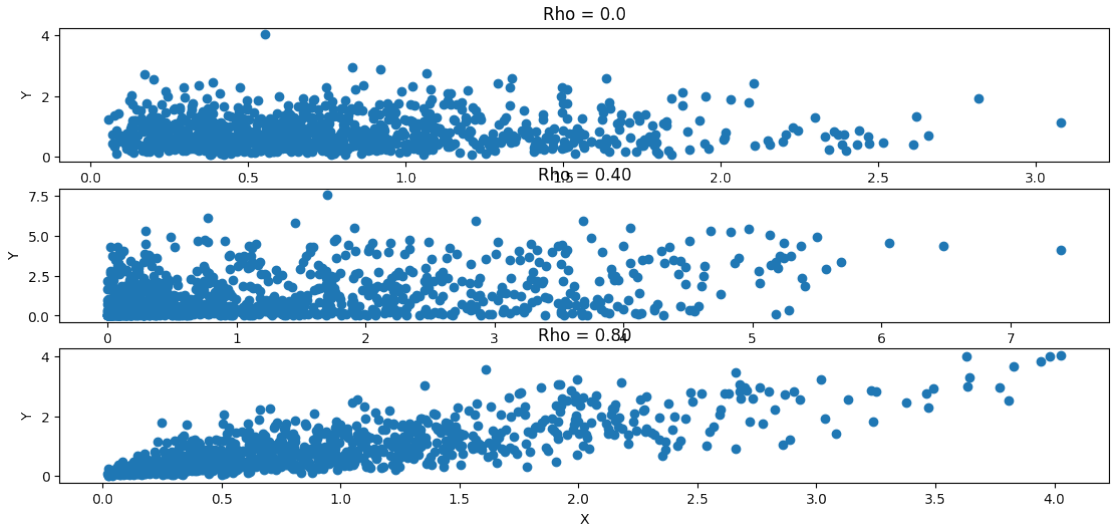
```
In [14]: import scipy
         from scipy.stats import t

         fig = plt.figure(figsize=(13.5, 6.0), dpi=100)

         x_list = list()
         y_list = list()

         """
         Case 1
         """
         rho = 0.0
         for i in range(1000):

             x, y = bivariate_normal(rho)
             x_list.append(x)
             y_list.append(y)

         # Define the lists as np.arrays to perform
```

```
# element-wise scalar division
x_list = np.array(x_list)
y_list = np.array(y_list)

# Get the std of the lists before mutating
x_std = np.std(x_list)
y_std = np.std(y_list)

# Scale them by the chi-squared RV
S = np.random.chisquare(5)
x_list *= np.sqrt(5/S)
y_list *= np.sqrt(5/S)

# Perform the scaling
x_list /= x_std
y_list /= y_std

# Apply the cdf
x_list = t.cdf(x_list, 5)
y_list = t.cdf(y_list, 5)

# Perform PIT for exponential(1) to make all marginal
x_list = -1.0*np.log(x_list)
y_list = -1.0*np.log(y_list)

# Plot
ax1 = fig.add_subplot(311)
ax1.title.set_text('Rho = 0.0')
plt.xlabel("X")
plt.ylabel("Y")
ax1.scatter(x_list,y_list)


"""
Case 2
"""
x_list = list()
y_list = list()

rho = 0.40
for i in range(1000):

    x, y = bivariate_normal(rho)
    x_list.append(x)
    y_list.append(y)

# Define the lists as np.arrays to perform
# element-wise scalar division
```

```
x_list = np.array(x_list)
y_list = np.array(y_list)

# Get the std of the lists before mutating
x_std = np.std(x_list)
y_std = np.std(y_list)

# Scale them by the chi-squared RV
S = np.random.chisquare(5)
x_list *= np.sqrt(5/S)
y_list *= np.sqrt(5/S)

# Perform the scaling
x_list /= x_std
y_list /= y_std

# Apply the cdf
x_list = t.cdf(x_list, 5)
y_list = t.cdf(y_list, 5)

# Perform PIT for exponential(1) to make all marginal
x_list = -1.0*np.log(x_list)
y_list = -1.0*np.log(y_list)

# Plot
ax1 = fig.add_subplot(312)
ax1.title.set_text('Rho = 0.40')
plt.xlabel("X")
plt.ylabel("Y")
ax1.scatter(x_list,y_list)

"""
Case 3
"""
x_list = list()
y_list = list()

rho = 0.80
for i in range(1000):

    x, y = bivariate_normal(rho)
    x_list.append(x)
    y_list.append(y)

# Define the lists as np.arrays to perform
# element-wise scalar division
x_list = np.array(x_list)
y_list = np.array(y_list)
```

```
# Get the std of the lists before mutating
x_std = np.std(x_list)
y_std = np.std(y_list)

# Scale them by the chi-squared RV
S = np.random.chisquare(5)
x_list *= np.sqrt(5/S)
y_list *= np.sqrt(5/S)

# Perform the scaling
x_list /= x_std
y_list /= y_std

# Apply the cdf
x_list = t.cdf(x_list, 5)
y_list = t.cdf(y_list, 5)

# Perform PIT for exponential(1) to make all marginal
x_list = -1.0*np.log(x_list)
y_list = -1.0*np.log(y_list)

# Plot
ax1 = fig.add_subplot(313)
ax1.title.set_text('Rho = 0.80')
plt.xlabel("X")
plt.ylabel("Y")
ax1.scatter(x_list,y_list)
```

Out[14]: <matplotlib.collections.PathCollection at 0x110db8f28>



1.4 Question 4

1.4.1 Part (a)

What is the smallest value of C_n such that $0 < g(x) \leq 1$?

We search for this by identifying the maximum and minimum of $g(x)$.

$$g(x) = \frac{1}{C_n}\frac{f_n(x)}{f_1(x)} = \frac{1}{C_n}\frac{\frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})}\left(1+\frac{x^2}{n}\right)^{-\frac{(n+1)}{2}}}{\frac{\Gamma(1)}{\sqrt{\pi}\Gamma(\frac{1}{2})}\left(1+x^2\right)^{-1}}$$

Using $\Gamma(1) = 1, \Gamma(\frac{1}{2}) = \sqrt{\pi}$, and some basic algebra, we find that we may write,

$$g(x) = \frac{1}{C_n}\left[\sqrt{\frac{\pi}{n}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\right]\left[\left(1+\frac{x^2}{n}\right)^{-\frac{(n+1)}{2}}\left(1+x^2\right)\right]$$

Define $h(x) = \left(1+\frac{x^2}{n}\right)^{-\frac{(n+1)}{2}}\left(1+x^2\right)$. We need to choose C_n such that $g(x)$ is bounded between 0 and 1, therefore we need to know where the extrema of $h(x)$ occur. After taking the derivative of $h(x)$ and setting equal to 0, the extrema are occur at $x = 0, \pm 1$. We recognize that $h(x)$ is even, hence $h(-1) = h(1)$. Also $h(1) > h(0)$, and so it follows that the maximum occurs at $x = \pm 1$. $h(1) = 2(1+\frac{1}{n})^{-1}$. Hence, we choose $C_n = \sqrt{\frac{\pi}{n}}\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}2(1+\frac{1}{n})^{-1}$. This is the smallest value such that $g(x)$ is guarenteed to be bounded by $[0,1]$. **Note:** As mentioned, clearly C_1 should be 1. We see this follows by choosing $n = 1$, and plugging into the above expression for C_n

1.4.2 Part (b)

Compute C_2, C_3, C_5, C_∞

Below, I write a function to compute C_n for $n \in \mathbb{N}$.

```
In [15]: import numpy as np
         import scipy.special as ss


         def c_n( n ):

             return ( 2*(np.sqrt(np.pi/float(n))*(ss.gamma((n+1)/2.0)/ss.gamma(n/2.0)))/(1+(1/


         for n_ in [2,3,5]:

             outstr = "C_" + str(n_) + ": "
             print(outstr, c_n(n_))
```

```
C_2:  1.48096097939
C_3:  1.73205080757
C_5:  1.98761598
```

For the C_∞ case, we use the fact that the t-distribution approaches the normal distribution as the degrees of freedom tends toward infinity. We solve in a similar fashion,

$$g(x) = \frac{1}{C_\infty}\frac{\phi(x)}{f_1(x)} = \frac{1}{C_\infty}\frac{\frac{1}{\sqrt{2\pi}}exp\big(-0.5*x^2\big)}{\frac{\Gamma(1)\left(1+x^2\right)^{-1}}{\sqrt{\pi}\Gamma(\frac{1}{2})}} = \frac{1}{C_\infty}\sqrt{\frac{\pi}{2}}e^{-\frac{1}{2}x^2}\left(1+x^2\right)$$

By a similar process as above, we find that the maximum value of $e^{\frac{-1}{2}x^2}\left(1+x^2\right)$ is at 1. Hence, we choose $C_\infty = \sqrt{\frac{2\pi}{e}}$

```
In [16]: print("C_\infty: ", np.sqrt((2*np.pi)/np.exp(1)))
```

```
C_\infty:  1.52034690107
```

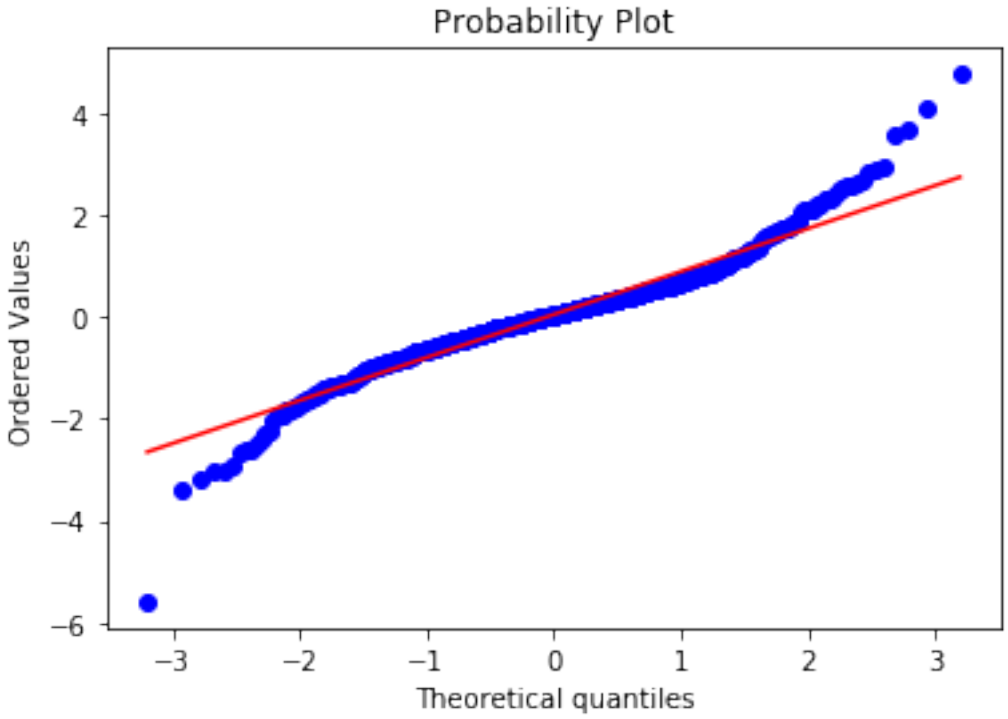
1.4.3 Part (c)

Use the rejection algorithm to generate m observations that follow the t-distribution with n degree of freedom. Report $m = 1000$ observations for each $n = 1, 3, 5, 10, 30$.

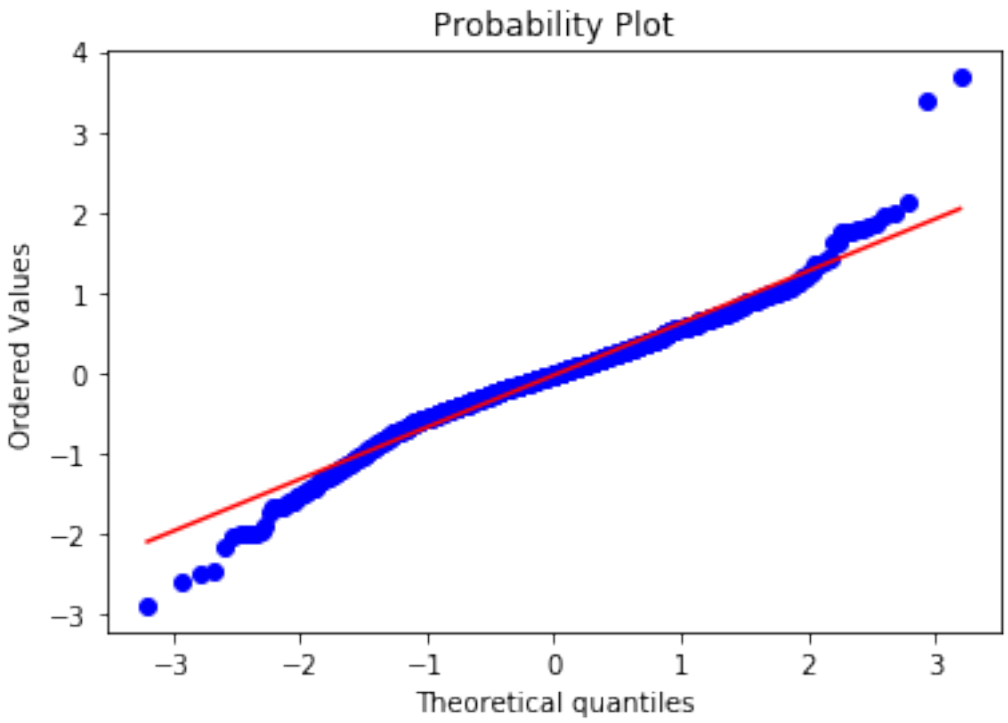
```
In [17]: """
         f_n(x) = C_n * f_1(x) * g(x)

         Note: f_1(x) is the Cauchy distribution. We
         wrote how to construct the probability integral
         transform of such a rv on last homework

         Rejection Algorithm:
             1. Generate a random variable, Y, using
                the probability integral transform
                where the density is given by f_1(x).
             2. Generate a uniform random variable
             3. Test if U <= g(Y)
                 a. True : Return Y
                 b. False: Repeat

         Do this until you have m observations
         """

         # Definition of g(x) given above
         def g(x, n):

             return ( ((1+(x**2/float(n)))**(-1.0*((n-1)/2.0)))/(1+x**2) )


         # Probability integral transform for Cauchy
         def gen_Y():
```

```
    u = np.random.uniform()
    return ( np.tan(np.pi*(u - 0.5)) )

def rej_algo(m, n):

    rv_list = []
    while (len(rv_list) != m):

        # Generate uniform, and Y
        U = np.random.uniform()
        Y = gen_Y()

        # condition
        if (U <= g(Y, n)):
            rv_list.append(Y)

    return rv_list
```

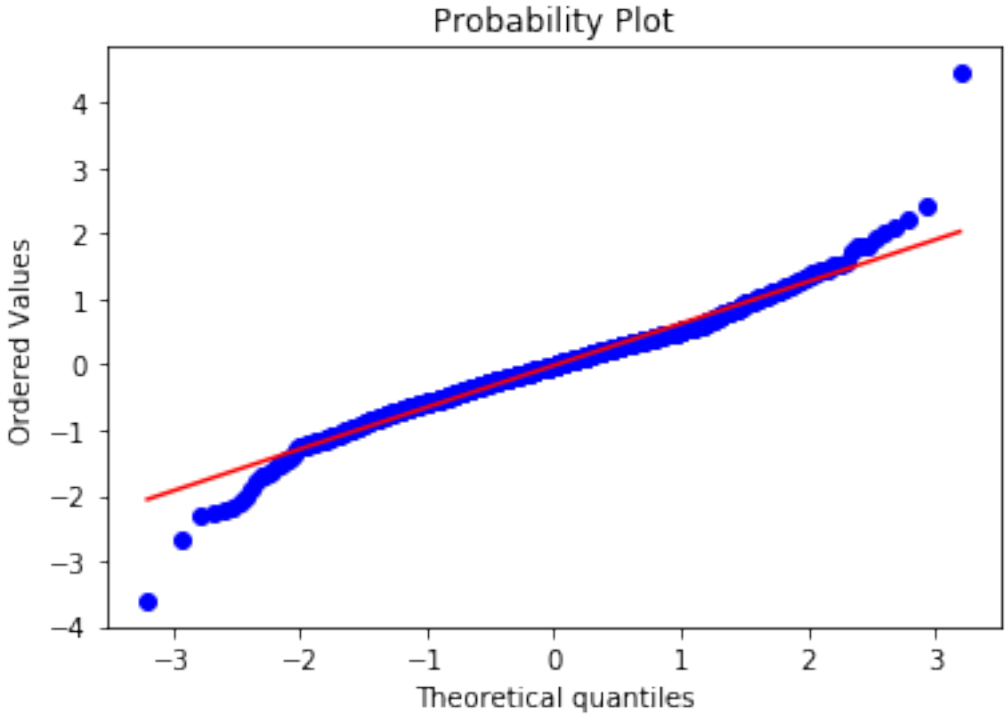
In [18]:
```
# Case 1: m=1000, n=1
get_plt(rej_algo(1000,1))
```



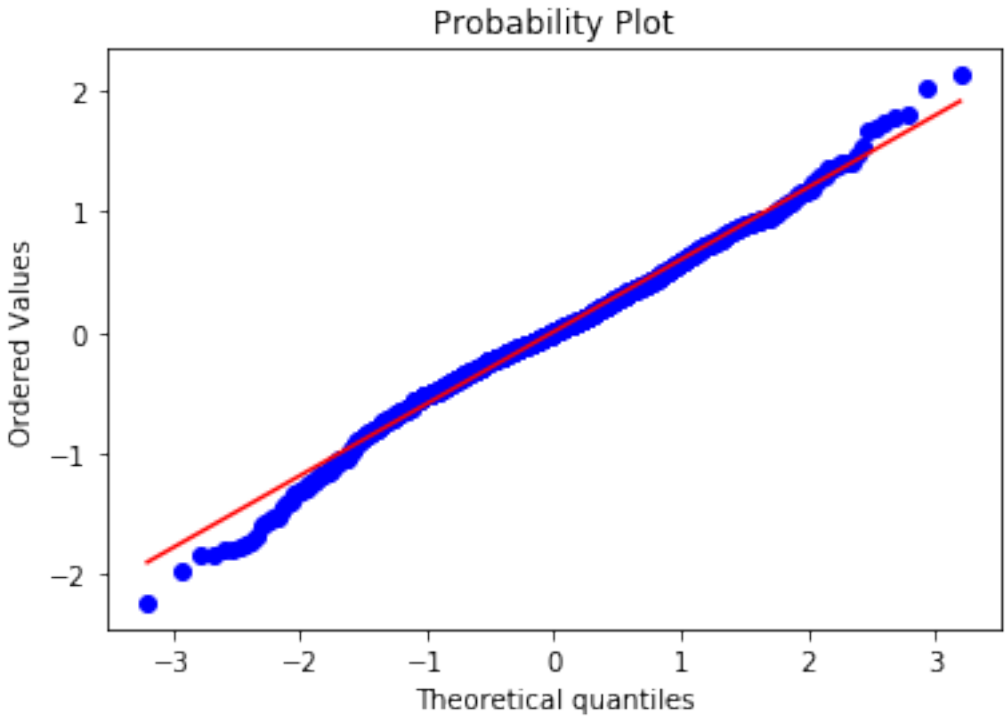
In [19]:
```
# Case 1: m=1000, n=3
get_plt(rej_algo(1000,3))
```



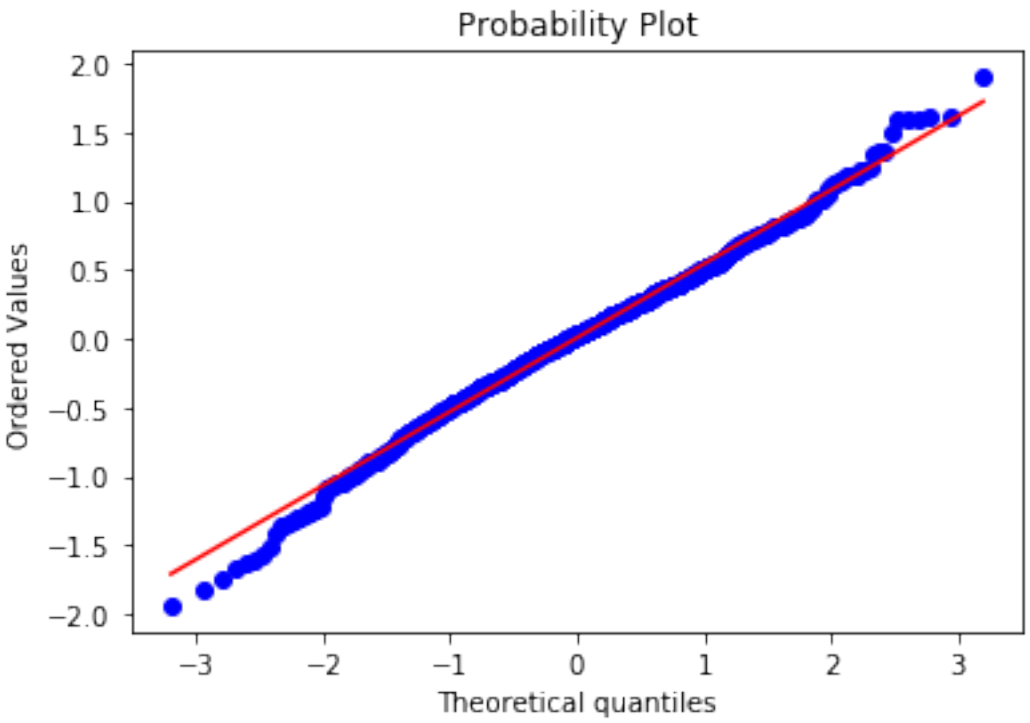
In [20]:
```
# Case 1: m=1000, n=5
get_plt(rej_algo(1000,5))
```



In [21]:
```
# Case 1: m=1000, n=10
get_plt(rej_algo(1000,10))
```



In [22]:
```
# Case 1: m=1000, n=30
get_plt(rej_algo(1000,30))
```



These plots exactly suggest what we'd like. The t-distribution places more weight on the tails when the degrees of freedom is smaller, and as they approach infinity, less weight is placed on the tails and the t-distribution is asymptotically the normal. The above plots confirm this: when the degrees of freedom is smaller, the extremeties (or tails) deviate from the normal distribution significantly, whereas the when the degrees of freedom is large, they tighten around the normal distribution.