Rotating Consensus and Tracking of Second-order Multi-agent Systems in 3-D under Directed Interaction Topologies

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Abstract—**This paper investigates the rotating motions of second-order multi-agent systems in 3-D under directed interaction topologies. We consider both rotating consensus and consensus tracking algorithms and present convergence conditions. In the case of rotating consensus, sufficient conditions are derived under which all agents rotate around a common point. In the case of rotating consensus tracking, we show that all the agents can rotating around a dynamic virtual leader, when this leader is a neighbor of only a subset of a group of followers and all followers have only local interaction. Tools like matrix theory, linear system theory, and other mathematical skills are used for convergence analysis. Simulation results are provided to illustrate the effectiveness of the theoretical results.**

Keywords—Collective motion, rotating consensus, consensus tracking, cooperative control, multi-agent system.

I. INTRODUCTION

Motivated by current technological advances, a team of autonomous agents that can cooperatively perform complex tasks is rapidly becoming more and more popular. In particular, there has been considerable progress made on robotics and sensor networks, such as surveillance [1], coverage control [2], environmental monitoring [3] and etc.. In many application scenarios, all team members are required to reach an agreement on their states (e.g. positions, phases, velocities and altitudes) by negotiating with their neighbors. Such problems are called consensus, which plays a significant role in order to achieving coordination.

There have been much works on consensus problems of first-order agents, such as consensus under time-varying topology [4], finite-time consensus [5], consensus over random networks [6] and asynchronous consensus [7] have been studied thoroughly. Taking into account the fact that many vehicles such as unmanned aerial vehicles and mobile robots are always controlled directly by their accelerations rather than by their velocities, hence it is also necessary to investigate consensus problems of second-order agents. In [8] and [9], the authors studied conditions on the interaction graph and the control gains for two different consensus algorithms to ensure agreement of both positions and velocities. Moreover, [10] considered the synchronized tracking control for multiple agents with high-order dynamics, whereas the desired trajectory is only available for a portion of the team members. In most of the aforementioned works, they mainly studied the translational properties of the agents. In fact, a class of collective circular motions widely exist in nature including flocks of birds flying along a circular orbit , foraging ants around a piece of rice, a swirling growing epiphyte colony, and panic escaping fish school around a predator. These collective behaviors can be applied to formation flight of satellites, circular mobile sensor networks and so on. However, rare results are derived to generate such motions currently. One of the earliest contributions was given in [11], where circular motions are obtained with a virtual reference beacon. Following this line, more control algorithms were developed to gain collective stable circular motions with allowable equilibrium configurations [12] [13]. In [12], a group of of mobile agents were studied where each agent pursues the leading neighbor along the line of sight rotated by a common offset angle, resulting in a circular motion. In particular, motivated by the applications of autonomous underwater vehicles (AUVs) in oceanographic sampling, a novel rotating formation control problem was solved in [13] to make all agents circle around a common point with some special structures at an unit speed. The aforementioned cyclic pursuit formation controller [12] [13] is based on a fixed network topology, especially, represented by a circulate matrix. The result was extended in [14] by introducing a rotation matrix to an existing second-order consensus protocol and the conditions under which rendezvous, circular patterns, and logarithmic spiral patterns can be achieved were derived, however, the desired collective behavior can be affected by the value of the corresponding rotating Euler angle easily. Along this research line, the latest work is referred to [15] [16] [17] where control protocols were proposed to make all agents surround a common point with a desired formation structure under undirected interaction topologies, in 2D and 3D spaces, respectively.

In this paper, we extend the work of [17] to address the rotating consensus problem for second-order multi-agent systems under directed interaction topologies. Moreover, most of the aforementioned consensus algorithms were often studied either when there does not exist a leader or when the leader is static. Although consensus without a leader is useful in applications such as cooperative rendezvous of a group of vehicles, there are many applications that require a dynamic leader. Examples include formation flying, body guard, and

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coordinated tracking applications. Taking into account the limitations in the aforementioned works, we will address theoretical challenges including a dynamic leader, the objective is that a group of agents rotating around a dynamic leader with local interaction and which can be regarded as the second contribution of this work.

This paper is organized as follows. In Section II, we list some notations and some concepts in graph theory used throughout this note, and the problem formulation as well. Section III states the main results. Simulation results are presented in Section IV to illustrate the theoretical results. Finally, conclusions and future research works are given in Section V.

II. Preliminaries

A. Graph Theory Notions

For a group of n agents, it is a natural way to model the interaction among them by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the agent set and the edge set can be denoted by $\mathcal{V} = \{1, 2, \ldots, n\}$ and $\mathcal{E} \subseteq \mathcal{V}^2$ respectively. An edge denoted as (i, j) means that information can be sent from agent i to agent j, but not necessarily vice verse. That is, agent i is a neighbor of agent j. We use $\mathcal{N}_j$ to denote the neighbor set of agent j, and we also use $\mathcal{A}$ to represent the weighted adjacency matrix, with each entry of $\mathcal{A}$ denoted as a_{ij} defined such that a_{ij} is positive weight if $(j, i) \in \mathcal{E}$, while $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. The Laplacian matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n \times n}$ with $\ell_{ii} = \sum_{j=1, j \neq i}^{n} a_{ij}$ and $\ell_{ij} = -a_{ij}$, $i \neq j$. In particular, we let $a_{ii} = 0$, $i = 1, \ldots, n$, (*i.e.*, agent i is not a neighbor of itself). A directed path is a sequence of edges of the form $(i_1, i_2), (i_2, i_3), \ldots$, where $i_k \in \mathcal{V},\ k = 1, 2, \cdots$. Moreover, a directed graph can have a directed spanning tree when there exists at least one agent that has directed paths to all other agents.

Lemma 2.1: [4] Let $\mathcal{L}$ be the nonsymmetric Laplacian matrix associated with weighted directed graph $\mathcal{G}$. Then $\mathcal{L}$ has at least one zero eigenvalue and all other eigenvalues have positive real parts. Furthermore, $\mathcal{L}$ has exactly one zero eigenvalue and all its nonzero eigenvalues has positive real parts if and only if the directed graph $\mathcal{G}$ has a directed spanning tree. In addition, there exists $\mathbf{1}_n$ satisfying $\mathcal{L}\mathbf{1}_n = \mathbf{0}$ and $\mathbf{p} \in \mathbb{R}^n$, satisfying $\mathbf{p} > \mathbf{0}$, $\mathbf{p}^T\mathcal{L} = \mathbf{0}$ and $\mathbf{p}^T\mathbf{1}_n = 1$. That is, $\mathbf{1}_n$ and $\mathbf{p}$ are, respectively, the right and left eigenvectors of $\mathcal{L}$ associated with the zero eigenvalue.

B. Coordinates Transformation

In this subsection, we introduce some information about transformation of coordinates. Consider two Castesian coordinates systems with a common origin denoted as $S_1(x, y, z)$ and $S_2(i, j, k)$. Then given an arbitrary vector p expressed in terms of the two different axes, we can represent this vector by (x_p, y_p, z_p) in S_1 and (i_p, j_p, k_p) in S_2. By simple calculation, there exists a particular linear transformation which can be written in a matrix form as

$$[x_p, y_p, z_p]^T = R[i_p, j_p, k_p]^T$$

where $R \in \mathbb{R}^3$ is formulated as the rotation matrix of S_2 with respect to S_1, and also note that $RR^T = R^T R = I_3$.

C. Problem Formulation

Consider the multi-agent system consisting of n agents with the following dynamics:

$$\dot{r}_i = v_i, \quad \dot{v}_i = u_i, \quad i = 1, \ldots, n, \tag{1}$$

where $r_i \in \mathbb{R}^3$ and $v_i \in \mathbb{R}^3$ are, respectively, the position vector and velocity vector of the ith agent, and $u_i \in \mathbb{R}^3$ is the control input vector. In this paper, it should be noted that all vectors are in an inertial Castesian coordinates system denoted by S_0 throughout this paper unless otherwise stated.

In 3-D, when all agents reach consensus while surrounding a common point on a plane, we can define the normal of this plane as a specified unit vector $i_\rho \in \mathbb{R}^3$. Without loss of generality, we assume that all agents finally move in the counterclockwise direction. Moreover, for convenience of discussions, we introduce a new Castesian coordinate system S_n such that the third coordinate axis is parallel to the unit vector i_ρ and S_n and S_0 share the same origin. The rotation matrix from S_0 to S_n is denoted as $R_{n0} \in \mathbb{R}^3$.

Definition 2.1: [17] The multi-agent system (1) reaches rotating consensus if

$$\begin{aligned}
&\lim_{t \to +\infty} i_\rho^T v_i(t) = 0, \\
&\lim_{t \to +\infty} [\dot{v}_i(t) - \omega R_{n0}^T R^\perp R_{n0} v_i(t)] = 0, \\
&\lim_{t \to +\infty} [r_i(t) - r_k(t)] = 0, \\
&\lim_{t \to +\infty} [v_i(t) - v_k(t)] = 0
\end{aligned} \tag{2}$$

for all i, k, where $\omega \in \mathbb{R}$ is a positive constant which is actually the angular velocity, and $R^\perp = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
The aforementioned definition guarantees that all agents finally reach consensus while surrounding a common point in the counterclockwise direction with an identical angular velocity ω on a plane perpendicular to the vector i_ρ. For the simplicity of following discussions, we assume that $\omega = 1$.

III. Main Results

A. Leaderless Rotating Consensus of Second-order Networks under Directed Interaction

In this subsection, we will firstly show the results proposed in [17] can be extended to directed interaction topologies. Considering the multi-agent system (1), a rotating consensus protocol is studied in [17] as

$$u_i = u_{i1} + u_{i2}, \tag{3}$$

where

$$u_{i1} = R_{n0}^T \bar{R}^\perp R_{n0} v_i$$

and

$$\begin{aligned}
u_{i2} = &-\sum_{j=1}^{n} a_{ij} \left[v_i - v_j\right] \\
&- \sum_{j=1}^{n} a_{ij} \left[(r_i + R_{n0}^T R^\perp R_{n0} v_i) - (r_j + R_{n0}^T R^\perp R_{n0} v_j)\right]
\end{aligned}$$

with

$$\bar{R}^{\perp}=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{bmatrix}$$

for all $i=1,\ldots,n$, where a_{ij} is the (i,j)th entry of the weighted adjacency matrix $\mathcal{A}_n$ associated with the directed graph $\mathcal{G}$. Noted from the protocol (3), the first part is a local feedback term that generates desirable circular motion on a plane perpendicular to the vector i_ρ, while the second part is designed for eliminating the disagreement on velocities and centers of circular motion.

Denote $\xi=[r_1^T,v_1^T,\ldots,r_n^T,v_n^T]$, substituting (3) to (1) yields

$$\dot{\xi}=\left(I_n\otimes A-\mathcal{L}\otimes B\right)\xi, \tag{4}$$

where $\mathcal{L}$ is the Laplacian associated with the directed graph $\mathcal{G}$, $A=\begin{bmatrix}0 & I_3\\ 0 & R_{n0}^T\bar{R}^{\perp}R_{n0}\end{bmatrix}$ and $B=\begin{bmatrix}0 & 0\\ I_3 & R_{n0}^T R^{\perp}R_{n0}\end{bmatrix}$. By applying a variable substitution $\bar{\xi}=(I_{2n}\otimes R_{n0})\xi=[\bar{r}_1^T,\bar{v}_1^T,\ldots,\bar{r}_n^T,\bar{v}_n^T]$, the closed loop system (4) turns into

$$\dot{\bar{\xi}}=\left(I_n\otimes\bar{A}-\mathcal{L}\otimes\bar{B}\right)\bar{\xi}, \tag{5}$$

where $\bar{A}=\begin{bmatrix}0 & I_3\\ 0 & \bar{R}^{\perp}\end{bmatrix}$ and $\bar{B}=\begin{bmatrix}0 & 0\\ I_3 & R^{\perp}\end{bmatrix}$. Furthermore, noting that $\bar{r}=[\bar{r}_x,\bar{r}_y,\bar{r}_z]^T$ and $\bar{v}=[\bar{v}_x,\bar{v}_y,\bar{v}_z]^T$, let $\varphi_1=[\bar{r}_{1x},\bar{r}_{1y},\bar{v}_{1x},\bar{v}_{1y},\ldots,\bar{r}_{nx},\bar{r}_{ny},\bar{v}_{nx},\bar{v}_{ny}]^T$ and $\varphi_2=[\bar{r}_{1z},\bar{v}_{1z},\ldots,\bar{r}_{nz},\bar{v}_{nz}]^T$, then system (5) can be decomposed into two separated subsystems given as

$$\dot{\varphi}_1=\left(I_n\otimes E-\mathcal{L}\otimes F\right)\varphi_1 \tag{6}$$

and

$$\dot{\varphi}_2=\left(I_n\otimes\bar{E}-\mathcal{L}\otimes\bar{F}\right)\varphi_2, \tag{7}$$

where $E=\begin{bmatrix}0 & I_2\\ 0 & J\end{bmatrix}$, $F=\begin{bmatrix}0 & 0\\ I_2 & I_2+J\end{bmatrix}$, $J=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$, $\bar{E}=\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}$ and $\bar{F}=\begin{bmatrix}0 & 0\\ 1 & 2\end{bmatrix}$.

Remark 3.1: Noting that the original multi-agent system (4) has been decoupled into two separated subsystems (6) and (7) by means of linear transformation, then the analysis of properties of (6) and (7) can be equivalent to those of (4). So it is obvious to see that the system (4) satisfies with definition (2) if and only if (6) and (7) satisfy with definiton (2). Since the third coordinate axis is parallel to the vector i_ρ, then for all $\bar{v}_{iz}, i=1,\ldots,n$ tend to zero as $t\to+\infty$. Furthermore, (6) and (7) hold that $\lim_{t\to+\infty}([\bar{r}_{ix},\bar{r}_{iy},\bar{r}_{iz}]^T-[\bar{r}_{jx},\bar{r}_{jy},\bar{r}_{jz}]^T)=0$, $\lim_{t\to+\infty}([\bar{v}_{ix},\bar{v}_{iy}]^T-[\bar{v}_{jx},\bar{v}_{jy}]^T)=0$, $\lim_{t\to+\infty}\bar{v}_{iz}=0$, and $\lim_{t\to+\infty}\left([\dot{\bar{v}}_{ix},\dot{\bar{v}}_{iy}]^T-J[\bar{v}_{ix},\bar{v}_{iy}]^T\right)=0$ for any $i=1,\ldots,n$ and $j=1,\ldots,n$.

Suppose that the directed graph $\mathcal{G}$ has a spanning tree. Note from Lemma 2.1 that eigenvalues of $\mathcal{L}$ can be label as $0<\lambda_2<\lambda_3,\ldots,\lambda_n$ and there exists a singular linear transformation W such that $W^{-1}LW=\mathrm{diag}\{0,\lambda_2,\lambda_3,\ldots,\lambda_n\}$.

Lemma 3.1: If the directed graph $\mathcal{G}$ has a spanning tree, then $\lim_{t\to+\infty}(\bar{r}_{iz}-\bar{r}_{jz})=0$ and $\lim_{t\to+\infty}\bar{v}_{iz}=0$ for any i,j. That is, all agents finally converge to a plane perpendicular to i_ρ. Furthermore,

$$\lim_{t\to+\infty}\bar{r}_{iz}=\mathbf{p}^T\bar{\mathbf{r}}_{z0}+\mathbf{p}^T\bar{\mathbf{v}}_{z0}$$

for each $i=1,\ldots,n$, where $\mathbf{p}$ is defined in Lemma 2.1, $\bar{\mathbf{r}}_z=[\bar{r}_{1z},\bar{r}_{2z},\ldots,\bar{r}_{nz}]^T$, $\bar{\mathbf{v}}_z=[\bar{v}_{1z},\bar{v}_{2z},\ldots,\bar{v}_{nz}]^T$, and $\bar{\mathbf{r}}_{z0}$ and $\bar{\mathbf{v}}_{z0}$ are the initial states of $\bar{\mathbf{r}}_z$ and $\bar{\mathbf{v}}_z$, respectively.

Proof: For system (7), pre-multiplying and post-multiplying $W^{-1}\otimes I_2$ and $W\otimes I_2$ yields

$$\begin{aligned}&(W^{-1}\otimes I_2)\left(I_n\otimes\bar{E}-\mathcal{L}\otimes\bar{F}\right)(W\otimes I_2)\\ =&I_n\otimes\bar{E}-\mathrm{diag}\{0,\lambda_2,\lambda_3,\ldots,\lambda_n\}\otimes\bar{F}\\ =&\mathrm{diag}\{\bar{E},\bar{E}-\lambda_2\bar{F},\bar{E}-\lambda_3\bar{F},\ldots,\bar{E}-\lambda_n\bar{F}\}.\end{aligned}$$

It is easy to be calculated that $\bar{E}$ has two simple eigenvalues 0 and -1 and all eigenvalues of $\bar{E}-\lambda_i\bar{F}, i=2,\ldots,n$, have negative real parts. Which turns out that $I_n\otimes\bar{E}-\mathcal{L}\otimes\bar{F}$ has exactly one simple zero eigenvalue and all other $2n-1$ eigenvalues are on the open Left Half Plane, then we will have $\bar{r}_{iz}\to\bar{r}_{jz}$ and $\bar{v}_{iz}\to 0$ for all i,j as $t\to+\infty$. Nextly, by pre-multiplying $I_n\otimes\bar{E}-\mathcal{L}\otimes\bar{F}$ with $\mathbf{p}^T\otimes[0\quad 1]$, it can be found that $\mathbf{p}^T\dot{\bar{\mathbf{v}}}_z=-\mathbf{p}^T\bar{\mathbf{v}}_z$ which implies $\mathbf{p}^T\bar{\mathbf{v}}_z=e^{-t}\mathbf{p}^T\bar{\mathbf{v}}_{z0}$, where $\mathbf{p}$ is defined in Lemma 2.1, $\bar{\mathbf{v}}_z=[\bar{v}_{1z},\bar{v}_{2z},\ldots,\bar{v}_{nz}]^T$, and $\bar{\mathbf{v}}_{z0}$ is the initial state of $\bar{\mathbf{v}}_z$. Hence, $\mathbf{p}^T\bar{\mathbf{r}}_z=\mathbf{p}^T\bar{\mathbf{r}}_{z0}+\int_0^t\mathbf{p}^T\bar{\mathbf{v}}_z(\tau)d\tau$, and which indicates $\mathbf{p}^T\bar{\mathbf{r}}_z=\mathbf{p}^T\bar{\mathbf{r}}_{z0}+\mathbf{p}^T\bar{\mathbf{v}}_{z0}$. ■

Lemma 3.2: [17] All roots of the equation

$$s^4+2\lambda s^3+[\lambda^2+2\lambda+(1-\lambda)^2]s^2+2\lambda^2 s+\lambda^2=0 \tag{8}$$

have negative real parts, where $s,\lambda\in\mathbb{R}$ and $\lambda>0$.

Lemma 3.3: If the directed graph $\mathcal{G}$ has a spanning tree, then

$$\lim_{t\to+\infty}\left\|[\bar{v}_{ix},\bar{v}_{iy}]^T\right\|=\left\|[\mathbf{p}^T\bar{\mathbf{v}}_{x0},\mathbf{p}^T\bar{\mathbf{v}}_{y0}]^T\right\|$$

and

$$\begin{aligned}\lim_{t\to+\infty}\left([\bar{r}_{ix},\bar{r}_{iy}]^T+J[\bar{v}_{ix},\bar{v}_{iy}]^T\right)=&[\mathbf{p}^T\bar{\mathbf{r}}_{x0},\mathbf{p}^T\bar{\mathbf{r}}_{y0}]^T\\ &+J[\mathbf{p}^T\bar{\mathbf{v}}_{x0},\mathbf{p}^T\bar{\mathbf{v}}_{y0}]^T\end{aligned}$$

for any $i=1,\ldots,n$, where $\bar{\mathbf{r}}_x=[\bar{r}_{1x},\bar{r}_{2x},\ldots,\bar{r}_{nx}]^T$, $\bar{\mathbf{v}}_x=[\bar{v}_{1x},\bar{v}_{2x},\ldots,\bar{v}_{nx}]^T$, $\bar{\mathbf{r}}_y=[\bar{r}_{1y},\bar{r}_{2y},\ldots,\bar{r}_{ny}]^T$, $\bar{\mathbf{v}}_y=[\bar{v}_{1y},\bar{v}_{2y},\ldots,\bar{v}_{ny}]^T$, and $\bar{\mathbf{r}}_{x0}$, $\bar{\mathbf{v}}_{x0}$, $\bar{\mathbf{r}}_{y0}$ and $\bar{\mathbf{v}}_{y0}$ are the initial states of $\bar{\mathbf{r}}_x$, $\bar{\mathbf{v}}_x$, $\bar{\mathbf{r}}_y$ and $\bar{\mathbf{v}}_y$, respectively.

Proof: For system (6), noting that the graph $\mathcal{G}$ has a directed spanning tree, pre-multiplying and post-multiplying $W^{-1}\otimes I_4$ and $W\otimes I_4$ yields

$$\begin{aligned}&(W^{-1}\otimes I_4)\left(I_n\otimes E-\mathcal{L}\otimes F\right)(W\otimes I_4)\\ =&I_n\otimes E-\mathrm{diag}\{0,\lambda_2,\lambda_3,\ldots,\lambda_n\}\otimes F\\ =&\mathrm{diag}\{E,E-\lambda_2 F,E-\lambda_3 F,\ldots,E-\lambda_n F\}.\end{aligned}$$

Then by calculating the characteristic polynomial of $E-\lambda_i F$, it follows that $\det(sI-E+\lambda_i F)=s^4+2\lambda_i s^3+[\lambda_i^2+2\lambda_i+(1-\lambda_i)^2]s^2+2\lambda_i^2 s+\lambda_i^2=0, i=2,\ldots,n$, which implies that all eigenvalues of $E-\lambda_i F$ lies on the open Left Half Plane by Lemma 3.2. Moreover, by pre-multiplying $(I_n\otimes E-\mathcal{L}\otimes F)$ with $\mathbf{p}^T\otimes[0\quad J]$, it yields $[\mathbf{p}^T\dot{\bar{\mathbf{v}}}_x,\mathbf{p}^T\dot{\bar{\mathbf{v}}}_y]^T=J[\mathbf{p}^T\bar{\mathbf{v}}_x,\mathbf{p}^T\bar{\mathbf{v}}_y]^T$, which implies $[\mathbf{p}^T\bar{\mathbf{v}}_x, \mathbf{p}^T\bar{\mathbf{v}}_y]^T = e^{Jt}[\mathbf{p}^T\bar{\mathbf{v}}_{x0}, \mathbf{p}^T\bar{\mathbf{v}}_{y0}]^T$. Thus, it yields $\lim_{t\to+\infty}\left\|[\mathbf{p}^T\bar{\mathbf{v}}_x, \mathbf{p}^T\bar{\mathbf{v}}_y]^T\right\| = \left\|[\mathbf{p}^T\bar{\mathbf{v}}_{x0}, \mathbf{p}^T\bar{\mathbf{v}}_{y0}]^T\right\|$. That is, $[\mathbf{p}^T\bar{\mathbf{r}}_x, \mathbf{p}^T\bar{\mathbf{r}}_y]^T = [\mathbf{p}^T\bar{\mathbf{r}}_{x0}, \mathbf{p}^T\bar{\mathbf{r}}_{y0}]^T + \int_0^t e^{J\tau}[\mathbf{p}^T\bar{\mathbf{v}}_{x0}, \mathbf{p}^T\bar{\mathbf{v}}_{y0}]^T d\tau$. Also note that $\left(I_n \otimes \bar{E} - \mathcal{L} \otimes \bar{F}\right)\left(\mathbf{1}^T \otimes [I_2, \mathbf{0}]\right)^T = \mathbf{0}$ which indicates $\lim_{t\to+\infty}([\bar{r}_{ix}, \bar{r}_{iy}]^T - [\bar{r}_{jx}, \bar{r}_{jy}]^T) = 0$ and $\lim_{t\to+\infty}([\bar{v}_{ix}, \bar{v}_{iy}]^T - [\bar{v}_{jx}, \bar{v}_{jy}]^T) = 0$ for each $i = 1, \dots, n$. By some calculation, it yields that $\lim_{t\to+\infty}\left\|[\bar{v}_{ix}, \bar{v}_{iy}]^T\right\| = \left\|[\mathbf{p}^T\bar{\mathbf{v}}_{x0}, \mathbf{p}^T\bar{\mathbf{v}}_{y0}]^T\right\|$ and $\lim_{t\to+\infty}\left([\bar{r}_{ix}, \bar{r}_{iy}]^T + J[\bar{v}_{ix}, \bar{v}_{iy}]^T\right) = [\mathbf{p}^T\bar{\mathbf{r}}_{x0}, \mathbf{p}^T\bar{\mathbf{r}}_{y0}]^T + J[\mathbf{p}^T\bar{\mathbf{v}}_{x0}, \mathbf{p}^T\bar{\mathbf{v}}_{y0}]^T$. ■

Then the main result of this subsection can be summarized by the following theorem.

Theorem 3.2: Consider a multi-agent system consisting of n second-order agents under fixed directed graph $\mathcal{G}$. If the directed graph has a spanning tree, then the multi-agent system (1) with distributed algorithm (3) reaches rotating consensus. Moreover,

$$\begin{aligned}&\lim_{t\to+\infty}[r_i(t) + R_{n0}^T R^{\perp} R_{n0} v_i(t)]\\ =&R_{n0}^T\begin{bmatrix}[\mathbf{p}^T\bar{\mathbf{r}}_{x0}, \mathbf{p}^T\bar{\mathbf{r}}_{y0}]^T + J[\mathbf{p}^T\bar{\mathbf{v}}_{x0}, \mathbf{p}^T\bar{\mathbf{v}}_{y0}]^T\\ \mathbf{p}^T\bar{\mathbf{r}}_{z0} + \mathbf{p}^T\bar{\mathbf{v}}_{z0}\end{bmatrix},\end{aligned}$$

and

$$\lim_{t\to+\infty}\|v_i\| = \left\|[\mathbf{p}^T\bar{\mathbf{v}}_{x0}, \mathbf{p}^T\bar{\mathbf{v}}_{y0}]^T\right\|$$

for all $i = 1, \dots, n$.

Proof: Combining Lemma 3.1 and Lemma 3.3 yields

$$\begin{aligned}&\lim_{t\to+\infty}[r_i(t) + R_{n0}^T R^{\perp} R_{n0} v_i(t)]\\ =&\lim_{t\to+\infty} R_{n0}^T\left[\bar{r}_i + R^{\perp}\bar{v}_i\right]\\ =&\lim_{t\to+\infty} R_{n0}^T\begin{bmatrix}[\bar{r}_{ix}, \bar{r}_{iy}]^T + J[\bar{v}_{ix}, \bar{v}_{iy}]^T\\ \bar{r}_{iz} + \bar{v}_{iz}\end{bmatrix}\\ =&R_{n0}^T\begin{bmatrix}[\mathbf{p}^T\bar{\mathbf{r}}_{x0}, \mathbf{p}^T\bar{\mathbf{r}}_{y0}]^T + J[\mathbf{p}^T\bar{\mathbf{v}}_{x0}, \mathbf{p}^T\bar{\mathbf{v}}_{y0}]^T\\ \mathbf{p}^T\bar{\mathbf{r}}_{z0} + \mathbf{p}^T\bar{\mathbf{v}}_{z0}\end{bmatrix},\end{aligned}$$

and

$$\begin{aligned}&\lim_{t\to+\infty}\|v_i\|\\ =&\lim_{t\to+\infty}\left\|[\bar{v}_{ix}, \bar{v}_{iy}, \bar{v}_{iz}]^T\right\|\\ =&\left\|[\mathbf{p}^T\bar{\mathbf{v}}_{x0}, \mathbf{p}^T\bar{\mathbf{v}}_{y0}]^T\right\|.\end{aligned}$$

■

Remark 3.3: Compared with [17], which requires that the associated interaction graph to be undirected and connected, we show this condition can be extended to directed graph with a spanning tree by the theoretical proofs. Furthermore, when the directed communication topology contains a directed spanning tree, the final consensus equilibrium is equal to the weighted average of the initial conditions of those agents rather than arithmetic average derived in [17].

B. Rotating Consensus Tracking of Second-order Networks with Partial Access to the Virtual Leader

In this subsection, we will propose a rotating consensus tracking protocol to make all agents finally rotating around a dynamic target on a plane perpendicular to the vector i_ρ. Here, we assume that the dynamic reference states r_d v_d, and $\dot{v}_d$, are time-varying and available to only a portion of all agents, and for simplicity, we denote $\zeta_i = r_i + R_{n0}^T R^{\perp} R_{n0} v_i$. The consensus tracking protocol is proposed as:

$$u_i = u_{i1} + u_{i2}, \tag{9}$$

where

$$u_{i1} = R_{n0}^T \bar{R}^{\perp} R_{n0} v_i$$

and

$$u_{i2} = \frac{1}{\sum_{j=1}^{n+1} a_{ij}} \sum_{j=1}^{n+1} a_{ij}\left\{R_{n0}^T R^{\top} R_{n0}\dot{\zeta}_j - (v_i - v_j) - (\zeta_i - \zeta_j)\right\}$$

with

$$R^{\top} = \begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

where $a_{ij}, i = 1, \dots, n, j = 1, \dots, n+1$, is the (i, j)th entry of the adjacency matrix $\mathcal{A}_{n+1}$, $a_{i(n+1)}$ is 0 if agent i has no access to the virtual leader and a positive scalar otherwise, and $v_{n+1} = v_d$, $\dot{v}_{n+1} = \dot{v}_d$, $\zeta_{n+1} = \zeta_d$, $\dot{\zeta}_{n+1} = \dot{\zeta}_d$. Apparently, the $(n+1)$th row of $\mathcal{A}_{n+1}$ are all zeros.

We have the following result for consensus tracking with a dynamic virtual leader.

Theorem 3.4: Let $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{(n+1)\times(n+1)}$, where a_{ij} and $a_{i(n+1)}$, $i = 1, \dots, n, j = 1, \dots, n$, are defined in (9) and $a_{(n+1)i} = 0$ for $i = 1, \dots, n+1$. With the control protocol (9), $r_i \to r_d$ and $v_i \to v_d$ asymptotically if and only if the graph $\mathcal{G}$ has a directed spanning tree.

Proof: First, it is easy to see $\dot{\zeta}_i = R_{n0}^T R^{\perp} R_{n0} u_{i2}$ such that (9) can be written as

$$\dot{\zeta}_i = \frac{1}{\sum_{j=1}^{n+1} a_{ij}} \sum_{j=1}^{n+1}\left\{a_{ij}\dot{\zeta}_j - a_{ij} R_{n0}^T R^{\perp} R_{n0}\left[(v_i - v_j) + (\zeta_i - \zeta_j)\right]\right\},$$

which implies that

$$(\mathcal{L} \otimes I_3)\dot{\zeta} = -(\mathcal{L} \otimes R_{n0}^T R^{\perp} R_{n0})v - (\mathcal{L} \otimes R_{n0}^T R^{\perp} R_{n0})\zeta,$$

where $\zeta = [\zeta_1^T, \dots, \zeta_{n+1}^T]^T$ and $v = [v_1^T, \dots, v_{n+1}^T]$. By applying variable substitutions $\bar{\zeta} = (I_{n+1} \otimes R_{n0})\zeta$ and $\bar{v} = (I_{n+1} \otimes R_{n0})v$, then it yields

$$(\mathcal{L} \otimes I_3)\dot{\bar{\zeta}} = -(\mathcal{L} \otimes R^{\perp})\bar{v} - (\mathcal{L} \otimes R^{\perp})\bar{\zeta}.$$

Note that $\bar{\zeta}_i = \bar{r}_i + R^{\perp}\bar{v}_i$, then it follows that

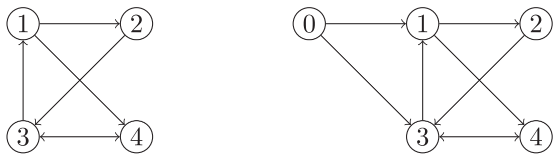
$$(\mathcal{L} \otimes R^{\perp})\dot{\bar{v}} = -(\mathcal{L} \otimes \begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 3\end{bmatrix})\bar{v} - (\mathcal{L} \otimes R^{\perp})\bar{r},$$

which indicates that

$$(\mathcal{L} \otimes J)\dot{\bar{v}}_{xy} = -(\mathcal{L} \otimes J)\bar{v}_{xy} - (\mathcal{L} \otimes J)\bar{r}_{xy}$$

and

$$\mathcal{L}\dot{\bar{v}}_z = -3\mathcal{L}\bar{v}_z - \mathcal{L}\bar{r}_z,$$



(a) (b)

Fig. 1. Network topology for four agents: (a) Interaction between four agents without a dynamic leader; (b) Interaction between four agents with a dynamic leader.

where $\bar{r}_{xy} = [\bar{r}_{1x}, \bar{r}_{1y}, \bar{r}_{2x}, \bar{r}_{2y}, \ldots, \bar{r}_{(n+1)x}, \bar{r}_{(n+1)y}]^T$, $\bar{v}_{xy} = [\bar{v}_{1x}, \bar{v}_{1y}, \bar{v}_{2x}, \bar{v}_{2y}, \ldots, \bar{v}_{(n+1)x}, \bar{v}_{(n+1)y}]^T$, $\bar{r}_z = [\bar{r}_{1z}, \bar{r}_{2z}, \ldots, \bar{r}_{(n+1)z}]^T$ and $\bar{v}_z = [\bar{v}_{1z}, \bar{v}_{2z}, \ldots, \bar{v}_{(n+1)z}]^T$. It is obvious to see $(\mathcal{L} \otimes J)\bar{r}_{xy} \to 0$, $(\mathcal{L} \otimes J)\bar{v}_{xy} \to 0$, $\mathcal{L}\bar{r}_z \to 0$ and $\mathcal{L}\bar{v}_z \to 0$ which implies that $[\bar{r}_{ix}, \bar{r}_{ix}]^T \to [\bar{r}_{jx}, \bar{r}_{jx}]^T$, $[\bar{v}_{ix}, \bar{v}_{ix}]^T \to [\bar{v}_{jx}, \bar{v}_{jx}]^T$, $\bar{r}_{iz} \to \bar{r}_{jz}$ and $\bar{v}_{iz} \to \bar{v}_{jz}$, $\forall i, j \in 1, \ldots, n+1$, if and only if the graph of $\mathcal{G}$ has a directed spanning tree, which in turns implies that $r_i \to r_d$ and $v_i \to v_d$ since the node r_d is the root. ■

Remark 3.5: In real applications, the derivatives of the neighbors information states $\dot{\zeta}_j$ can be calculated by using numerical differentiation. For example, $\dot{\zeta}_j$ can be approximated by $(\zeta_j(kT) - \zeta_j(kT - T))/T$, where T is the sampling period, and k is the discrete-time index.

IV. Simulations

Numerical simulations will be presented to illustrate the theoretical results derived in this paper. We consider a group of four vehicles with directed interaction graph $\mathcal{G}$ shown in Fig. 1(a). Note that $\mathcal{G}$ has a spanning tree. For the leaderless consensus problem, the adjacency matrix associated with $\mathcal{G}$ is defined as follows

$$\mathcal{A}_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1.5 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 1 \\ 0.5 & 0 & 1 & 0 \end{bmatrix},$$

It can be calculated that $\mathbf{p} = [0.2750, 0.1000, 0.3750, 0.2500]^T$. For the leader-following case, the adjacency matrix is given as

$$\mathcal{A}_5 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0.5 \\ 1.5 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 1 & 0.6 \\ 0.5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Firstly, we present simulation results to illustrate Theorem 3.2. The initial states of the multi-agent system is taken as $[r_1^T, r_2^T, r_3^T, r_4^T, v_1^T, v_2^T, v_3^T, v_4^T]^T = [-1, 1, -4, -2, -1, -4, -4, -1, -2, -4, -2, -2, -3, -2, 0, 0, 2, 1, 0, -3, 1, -3, 0, -2]^T$ and i_ρ is taken as $[\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}]^T$. Fig. 2 shows that the trajectories of all agents converge to a common circular orbit, which indicates that all the agents eventually move around a common point on a plane perpendicular to the vector i_ρ as time evolving. According to Theorem 3.2, it can also be calculated that the common circular point and the rotating radius are, respectively,

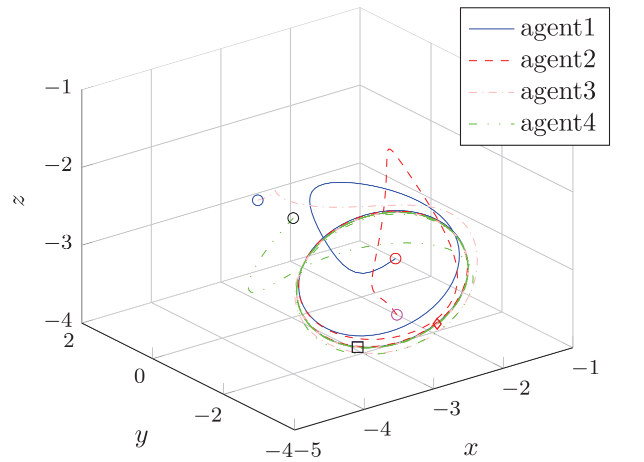


Fig. 2. Trajectories of the four agents using (3). Circles denote the starting positions of agents, diamonds denote the snapshots of the agents at $t = 30$, while squares denote the ending positions of agents at $t = 100$.

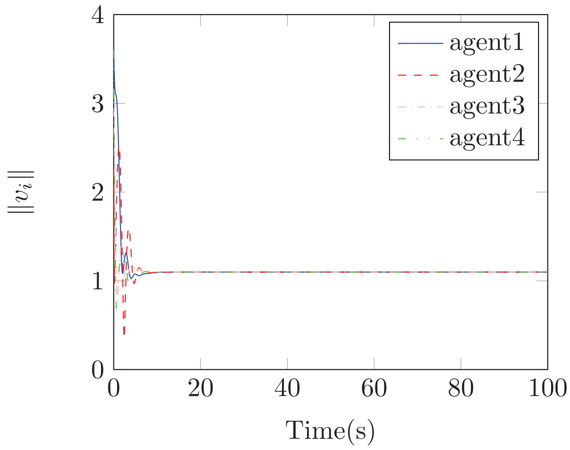


Fig. 3. Radius evolving of the four agents using (3). Note that radius of each agent reach consensus eventually.

$[-3.1516, -2.8495, -2.8245]^T$ and 1.0984. Fig. 3 shows the convergency of radius of four agents, which is consistent with Theorem 3.2.

Nextly, we give the leader following results to illustrate Theorem 3.4. Note from Fig. 1(b) that only agent 1 and agent 3 are access to the virtual leader. We let the reference states be $r_d(t) = R_{n0}^T[\rho\cos t, \rho\sin t, 0]^T + [1, 0, t]^T$ and $v_d(t) = R_{n0}^T[-\rho\sin t, \rho\cos t, 0]^T + [0, 0, 1]^T$ and the initial states and i_ρ be, respectively, $[r_1^T, r_2^T, r_3^T, r_4^T, v_1^T, v_2^T, v_3^T, v_4^T]^T = [-1, 1, -4, -2, -1, -4, -4, -1, -2, -4, -2, -2, -3, -2, 0, 0, 2, 1, 0, -3, 1, -3, 0, -2]^T$ and $[\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}]^T$. Fig. 4 shows that the trajectories of all agents controlled under protocol (9). It can be seen that all the agents will rotating around with the virtual dynamic leader. Fig. 5 shows that the convergence of velocities of all agents with partial access to the virtual reference, which illustrates the effectiveness of Theorem 3.4 proposed.

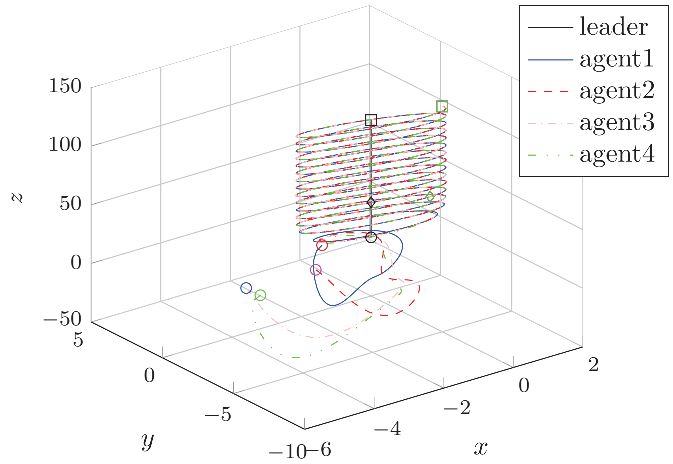


Fig. 4. Trajectories evolving of the four agents by using (9). Circles denote the starting positions of agents, diamonds denote the snapshots of the agents at $t = 30$, while squares denote the ending positions of agents at $t = 100$.

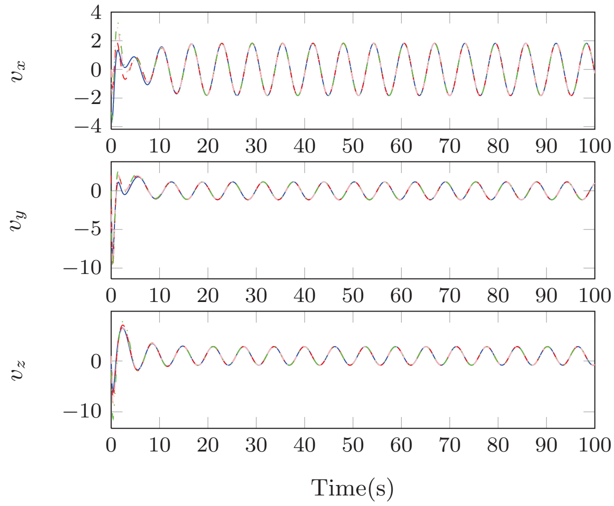


Fig. 5. Velocities evolving of the four agents by using (9). Note that velocities of each agent reach consensus eventually.

V. Conclusion

In this paper, we investigate collective rotating motions of second-order agents under directed interaction. We first consider rotating consensus algorithm and derive sufficient conditions under which a team of agents achieve coordination on their positions and velocities while rotating around a common point. Then we propose a rotating consensus tracking protocol and give sufficient conditions to make all agents rotating around a dynamic virtual leader. Finally, Simulation results were provided to illustrate the theoretical results. It should be noted that though we extend the interaction topology between agents from undirected graph to directed, while the communication graph is fixed, and issues such as dynamic switching topologies and time delays will pose many challenging problems that warrant further research. Moreover, due to the unsymmetrical property of the associated Laplacian matrix, so we can not analyze the rotating formation control problems under directed graph by the same method like [15], and this could also be our future research interest.

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