Supplement S1: Mathematical Foundations

E_8 Exceptional Lie Algebra, G_2 Holonomy Manifolds, and K_7 Construction

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Lean Verification: 2400+ theorems (core v3.3.24, zero `sorry`)

Abstract

This supplement presents the mathematical architecture underlying GIFT. Part I develops the E_8 exceptional Lie algebra with the exceptional chain identity. Part II introduces G_2 holonomy manifolds, including the correct characterization of the $\mathfrak{g}_2$ subalgebra as the kernel of the Lie derivative map. Part III establishes K_7 manifold construction via twisted connected sum, building compact G_2 manifolds by gluing asymptotically cylindrical building blocks. Part IV establishes the algebraic reference form determining $\det(g) = 65/32$, with Joyce's theorem guaranteeing existence of a torsion-free metric. PINN validation achieves a torsion scaling law $\nabla\varphi(L) = 8.46 \times 10^{-4}/L^2$ and spectral fingerprint $[1, 10, 9, 30]$ at 5.8σ significance. All algebraic results are formally verified in Lean 4.

Contents

Part 0: The Octonionic Foundation

1 Why This Framework Exists

The GIFT framework emerges from a single algebraic fact:

The octonions $\mathbb{O}$ are the largest normed division algebra.

The derivation chain $\mathbb{O} \to \mathrm{G}_2 \to K_7 \to$ predictions is described in the main paper (Section 1.3). This supplement develops the mathematical foundations for each step.

1.1 The Division Algebra Chain

The Hurwitz theorem establishes that no normed division algebra of dimension greater than 8 exists. The chain $\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O}$ terminates at the octonions (see main paper, Section 2.1 for the complete table). This non-extendability forces the exceptional structures: $\mathrm{G}_2 = \mathrm{Aut}(\mathbb{O})$, $\dim = 14$.

1.2 G_2 as Octonionic Automorphisms

Definition: $\mathrm{G}_2 = \{g \in \mathrm{GL}(\mathbb{O}) : g(xy) = g(x)g(y) \text{ for all } x, y \in \mathbb{O}\}$

Property	Value	GIFT Role
$\dim(\mathrm{G}_2)$	$14 = \binom{7}{2} - \binom{7}{1} = 21 - 7$	Q_{Koide} numerator
Action	Transitive on $S^6 \subset \mathrm{Im}(\mathbb{O})$	Connects all directions
Embedding	$\mathrm{G}_2 \subset \mathrm{SO}(7)$	Preserves φ_0

1.3 Why $\dim(K_7) = 7$

The dimension 7 is a consequence of the octonionic structure, not an independent choice:

- $\mathrm{Im}(\mathbb{O})$ has dimension 7
- G_2 acts naturally on $\mathbb{R}^7$
- A compact 7-manifold with G_2 holonomy provides the geometric realization

In this sense, K_7 is to G_2 what the circle is to $\mathrm{U}(1)$.

1.4 The Fano Plane: Combinatorial Structure of $\mathrm{Im}(\mathbb{O})$

The 7 imaginary octonion units form the **Fano plane** $\mathrm{PG}(2, 2)$, the smallest projective plane:

- 7 points (imaginary units $e_1 \ldots e_7$)
- 7 lines (multiplication triples $e_i \times e_j = \pm e_k$)

- 3 points per line

Combinatorial counts:

- Point-line incidences: $7 \times 3 = 21 = \binom{7}{2} = b_2$
- Automorphism group: $\mathrm{PSL}(2, 7)$ with $|\mathrm{PSL}(2, 7)| = 168$

Numerical observation: The following arithmetic identity holds:

$$(b_3 + \dim(\mathrm{G}_2)) + b_3 = 91 + 77 = 168 = |\mathrm{PSL}(2, 7)| = \mathrm{rank}(\mathrm{E}_8) \times b_2$$

Whether this reflects deeper geometric structure connecting gauge and matter sectors, or is an arithmetic coincidence, remains an open question.

Part I: E_8 Exceptional Lie Algebra

2 Root System and Dynkin Diagram

2.1 Basic Data

Property	Value	GIFT Role
Dimension	$\dim(\mathrm{E}_8) = 248$	Gauge DOF
Rank	$\mathrm{rank}(\mathrm{E}_8) = 8$	Cartan subalgebra
Number of roots	$\lvert\Phi(\mathrm{E}_8)\rvert = 240$	E_8 kissing number
Root length	$\sqrt{2}$	α_s numerator
Coxeter number	$h = 30$	Icosahedron edges
Dual Coxeter number	$h^\vee = 30$	McKay correspondence

2.2 Root System Construction

E_8 root system in $\mathbb{R}^8$ has 240 roots:

Type I (112 roots): Permutations and sign changes of $(\pm 1, \pm 1, 0, 0, 0, 0, 0, 0)$

Type II (128 roots): Half-integer coordinates with even minus signs:

$$\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$$

Verification: $112 + 128 = 240$ roots, all length $\sqrt{2}$.

Lean Status (v3.3.24): E_8 Root System **12/12 COMPLETE**. All theorems proven:

- `D8_roots_card` $= 112$, `HalfInt_roots_card` $= 128$
- `E8_roots_card` $= 240$, `E8_roots_decomposition`
- `E8_inner_integral`, `E8_norm_sq_even`, `E8_sub_closed`
- `E8_basis_generates`: Every lattice vector is integer combination of simple roots (theorem)

2.3 Cartan Matrix

$$A_{\mathrm{E}_8} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Properties: $\det(A) = 1$ (unimodular), positive definite.

3 Weyl Group

3.1 Order and Factorization

$$|W(\mathrm{E}_8)| = 696{,}729{,}600 = 2^{14} \times 3^5 \times 5^2 \times 7$$

3.2 Prime Factorization Identity

Identity: The Weyl group order factorizes entirely into GIFT constants:

$$|W(\mathrm{E}_8)| = p_2^{\dim(\mathrm{G}_2)} \times N_{\mathrm{gen}}^{w} \times w^{p_2} \times \dim(K_7)$$

Factor	Exponent	Value	GIFT Origin
2^{14}	$\dim(\mathrm{G}_2) = 14$	16384	$p_2^{(\text{holonomy dim})}$
3^5	$w = 5$	243	N_{gen}^{w}
5^2	$p_2 = 2$	25	$w^{(\text{binary})}$
7^1	1	7	$\dim(K_7)$

Status: **Proven (Lean 4)**: `weyl_E8_topological_factorization`

3.3 Triple Derivation of $w = 5$

Identity: The pentagonal index w admits three independent derivations from topological invariants.

3.3.1 Derivation 1: G_2 Dimensional Ratio

$$w = \frac{\dim(\mathrm{G}_2) + 1}{N_{\mathrm{gen}}} = \frac{14 + 1}{3} = \frac{15}{3} = 5$$

Interpretation: The holonomy dimension plus unity, distributed over generations.

3.3.2 Derivation 2: Betti Reduction

$$w = \frac{b_2}{N_{\mathrm{gen}}} - p_2 = \frac{21}{3} - 2 = 7 - 2 = 5$$

Interpretation: The per-generation Betti contribution minus the dimensional ratio p_2.

3.3.3 Derivation 3: Exceptional Difference

$$w = \dim(\mathrm{G}_2) - \mathrm{rank}(\mathrm{E}_8) - 1 = 14 - 8 - 1 = 5$$

Interpretation: The gap between holonomy dimension and gauge rank, reduced by unity.

3.3.4 Unified Identity

These three derivations establish the **pentagonal triple identity**:

$$\boxed{\frac{\dim(\mathrm{G}_2)+1}{N_{\mathrm{gen}}} = \frac{b_2}{N_{\mathrm{gen}}} - p_2 = \dim(\mathrm{G}_2) - \mathrm{rank}(\mathrm{E}_8) - 1 = 5}$$

Status: Proven (algebraic identity from GIFT constants)

3.3.5 Verification

Expression	Computation	Result
$(\dim(\mathrm{G}_2)+1)/N_{\mathrm{gen}}$	$(14+1)/3$	5
$b_2/N_{\mathrm{gen}} - p_2$	$21/3 - 2$	5
$\dim(\mathrm{G}_2) - \mathrm{rank}(\mathrm{E}_8) - 1$	$14-8-1$	5

3.3.6 Significance

The triple convergence suggests $w = 5$ is structurally constrained by the $\mathrm{E}_8 \times \mathrm{E}_8/\mathrm{G}_2/K_7$ geometry. It enters:

1. $\det(g) = 65/32$: Via $w \times (\mathrm{rank}(\mathrm{E}_8) + w)/2^w = 5 \times 13/32$
2. $|W(\mathrm{E}_8)|$ **factorization**: The factor $5^2 = w^{p_2}$ in prime decomposition
3. **Cosmological ratio**: $\sqrt{w} = \sqrt{5}$ appears in dark sector density ratios (see main paper, Section 5.8)

Status: Proven (three independent derivations)

4 Exceptional Chain

4.1 The Pattern

A pattern connects exceptional algebra dimensions to primes:

Algebra	n	$\dim(\mathrm{E}_n)$	Prime	Index
E_6	6	78	13	prime(6)
E_7	7	133	19	prime(8) = prime(rank(E_8))
E_8	8	248	31	prime(11) = prime(D_{bulk})

4.2 Exceptional Chain Identity

Identity: For $n \in \{6, 7, 8\}$:

$$\dim(\mathrm{E}_n) = n \times \mathrm{prime}(g(n))$$

where $g(6) = 6$, $g(7) = \text{rank}(\text{E}_8) = 8$, $g(8) = D_{\text{bulk}} = 11$.

Proof (verified in Lean):

- E_6: $6 \times 13 = 78$ ✓
- E_7: $7 \times 19 = 133$ ✓
- E_8: $8 \times 31 = 248$ ✓

Status: **Proven (Lean 4)**: `exceptional_chain_certified`

5 $\text{E}_8 \times \text{E}_8$ Product Structure

5.1 Direct Sum

Property	Value
Dimension	$496 = 248 \times 2$
Rank	$16 = 8 \times 2$
Roots	$480 = 240 \times 2$

5.2 τ Numerator Connection

The hierarchy parameter numerator:

$$\tau_{\text{num}} = 3472 = 7 \times 496 = \dim(K_7) \times \dim(\text{E}_8 \times \text{E}_8)$$

Status: **Proven (Lean 4)**: `tau_num_E8xE8`

5.3 Binary Duality Parameter

Triple geometric origin of $p_2 = 2$:

1. **Local**: $p_2 = \dim(\text{G}_2)/\dim(K_7) = 14/7 = 2$
2. **Global**: $p_2 = \dim(\text{E}_8 \times \text{E}_8)/\dim(\text{E}_8) = 496/248 = 2$
3. **Root**: $\sqrt{2}$ in E_8 root normalization

6 Exceptional Algebras from Octonions

The foundational role of octonions is established in Part 0. This section details the exceptional algebraic structures that emerge from $\mathbb{O}$.

6.1 Exceptional Jordan Algebra $J_3(\mathbb{O})$

Property	Value
$\dim(J_3(\mathbb{O}))$	$27 = 3^3$
$\dim(J_3(\mathbb{O})_0)$	26 (traceless)

E-series formula (v3.3): The dimension 27 itself emerges from the exceptional chain:

$$\dim(J_3(\mathbb{O})) = \frac{\dim(\mathrm{E}_8) - \dim(\mathrm{E}_6) - \dim(\mathrm{SU}_3)}{6} = \frac{248 - 78 - 8}{6} = \frac{162}{6} = 27$$

This shows the Jordan algebra dimension is derivable from the E-series structure.

Status: **Proven (Lean 4)**: `j3o_e_series_certificate`

6.2 F_4 Connection

F_4 is the automorphism group of $J_3(\mathbb{O})$:

$$\dim(F_4) = 52 = p_2^2 \times \alpha_{\mathrm{sum}}^B = 4 \times 13$$

6.3 Exceptional Differences

Difference	Value	GIFT
$\dim(\mathrm{E}_8) - \dim(J_3(\mathbb{O}))$	$221 = 13 \times 17$	$\alpha_B \times \lambda_{H,\mathrm{num}}$
$\dim(F_4) - \dim(J_3(\mathbb{O}))$	$25 = 5^2$	w^2
$\dim(\mathrm{E}_6) - \dim(F_4)$	26	$\dim(J_3(\mathbb{O})_0)$

Status: **Proven (Lean 4)**: `exceptional_differences_certified`

6.4 Structural Derivation of τ (v3.3)

The hierarchy parameter τ admits a purely geometric derivation from framework invariants:

$$\tau = \frac{\dim(\mathrm{E}_8 \times \mathrm{E}_8) \times b_2}{\dim(J_3(\mathbb{O})) \times H^*} = \frac{496 \times 21}{27 \times 99} = \frac{10416}{2673} = \frac{3472}{891}$$

Prime factorization:

- Numerator: $3472 = 2^4 \times 7 \times 31 = \dim(K_7) \times \dim(\mathrm{E}_8 \times \mathrm{E}_8)$
- Denominator: $891 = 3^4 \times 11 = N_{\mathrm{gen}}^4 \times D_{\mathrm{bulk}}$

Alternative form: $\tau_{\mathrm{num}} = 7 \times 496 = \dim(K_7) \times \dim(\mathrm{E}_8 \times \mathrm{E}_8) = 3472$

This anchors τ to topological and algebraic invariants, establishing it as a geometric constant rather than a free parameter.

Status: **Proven (Lean 4)**: `tau_structural_certificate`

Part II: G_2 Holonomy Manifolds

7 Definition and Properties

7.1 G_2 as Exceptional Holonomy

Property	Value	GIFT Role
$\dim(G_2)$	14	Q_{Koide} numerator
$\text{rank}(G_2)$	2	Lie rank
Definition	$\text{Aut}(\mathbb{O})$	Octonion automorphisms

Lean Status (v3.3.24): G_2 Cross Product **9/11** proven:

- `epsilon_antisymm`, `epsilon_diag`, `cross_apply` ✓
- `G2_cross_bilinear`, `G2_cross_antisymm`, `cross_self` ✓
- `G2_cross_norm` (Lagrange identity $\|u \times v\|^2 = \|u\|^2\|v\|^2 - \langle u, v\rangle^2$) ✓
- `reflect_preserves_lattice` (Weyl reflection) ✓
- Remaining: `cross_is_octonion_structure` (343-case timeout), `G2_equiv_characterizations`

7.2 G_2 as Kernel of the Lie Derivative

The G_2 subalgebra of $\mathfrak{so}(7)$ admits a precise characterization as the stabilizer of the associative 3-form φ_0. For any antisymmetric matrix A in $\mathfrak{so}(7)$, the Lie derivative of φ_0 is:

$$L_A(\varphi_0)_{ijk} = A_{ia}\varphi_{ajk} + A_{ja}\varphi_{iak} + A_{ka}\varphi_{ija}$$

The $\mathfrak{g}_2$ subalgebra consists of all A for which $L_A(\varphi_0) = 0$:

$$\mathfrak{g}_2 = \ker(L) = \{A \in \mathfrak{so}(7) : L_A(\varphi_0) = 0\}$$

This yields the decomposition $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus V_7$, where $\dim(\mathfrak{g}_2) = 14$ and $\dim(V_7) = 7$. The complement V_7 carries the standard 7-dimensional representation of G_2.

In practice, the kernel is computed via singular value decomposition (SVD) of the linear map $L : \mathfrak{so}(7) \to \Lambda^3(\mathbb{R}^7)$. The 14 singular vectors with eigenvalue zero span $\mathfrak{g}_2$; the 7 singular vectors with nonzero eigenvalue span V_7.

Note: A heuristic construction based on Fano-plane indices does not produce correct $\mathfrak{g}_2$ generators (each such generator is approximately 67% in $\mathfrak{g}_2$ and 33% in V_7). The kernel-based construction is the correct definition and must be used in all numerical computations involving $\mathfrak{g}_2/V_7$ decomposition.

7.3 Holonomy Classification (Berger)

Dimension	Holonomy	Geometry
7	G_2	**Exceptional**
8	Spin(7)	Exceptional

7.4 Torsion: Definition and GIFT Interpretation

Mathematical definition: Torsion measures failure of G_2 structure to be parallel:

$$T = \nabla\varphi \neq 0$$

For a G_2 structure φ, the intrinsic torsion decomposes into four irreducible G_2-modules:

$$T \in W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}$$

Class	Dimension	Characterization
W_1	1	Scalar: $d\varphi = \tau_0 \star \varphi$
W_7	7	Vector: $d\varphi = 3\tau_1 \wedge \varphi$
W_{14}	14	Co-closed part of $d \star \varphi$
W_{27}	27	Traceless symmetric

Total dimension: $1 + 7 + 14 + 27 = 49 = 7^2 = \dim(K_7)^2$

The torsion-free condition requires all four classes to vanish simultaneously, a highly constrained state with 49 conditions.

Torsion-free condition:

$$\nabla\varphi = 0 \Leftrightarrow d\varphi = 0 \text{ and } d * \varphi = 0$$

GIFT interpretation:

Quantity	Meaning	Value
$\kappa_T = 1/61$	Torsion parameter	Fixed by K_7
φ_{ref}	Algebraic reference form	$c \times \varphi_0$
T_{realized}	Actual torsion for global solution	Constrained by Joyce

Key insight: The 33 dimensionless predictions use only topological invariants (b_2, b_3, $\dim(G_2)$) and are independent of the specific torsion realization. The value $\kappa_T = 1/61$ defines the geometric bound on deviations from φ_{ref}.

Physical interactions: Emerge from the geometry of K_7, with deviations $\delta\varphi$ from the reference form bounded by topological constraints. The complete dynamical framework connecting torsion to renormalization group flow via torsional geodesic equations is developed in the main paper (Section 3). There, the identification of geodesic flow parameter $\lambda = \ln(\mu/\mu_0)$ with RG scale maps the torsion hierarchy directly onto physical observables: mass hierarchies, CP violation, and coupling evolution.

8 Topological Invariants

8.1 Derived Constants

Constant	Formula	Value
$\det(g)$	$p_2 + 1/(b_2 + \dim(\mathrm{G}_2) - N_{\mathrm{gen}})$	65/32
κ_T	$1/(b_3 - \dim(\mathrm{G}_2) - p_2)$	1/61
$\sin^2\theta_W$	$b_2/(b_3 + \dim(\mathrm{G}_2))$	3/13

8.2 The 61 Decomposition

$$\kappa_T^{-1} = 61 = \dim(F_4) + N_{\mathrm{gen}}^2 = 52 + 9$$

Alternative:

$$61 = \Pi(\alpha_B^2) + 1 = 2 \times 5 \times 6 + 1$$

Status: **Proven (Lean 4)**: `kappa_T_inv_decomposition`

8.3 Spectral Geometry

The Laplace-Beltrami operator on K_7 admits a discrete spectrum with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ The first non-zero eigenvalue λ_1 (spectral gap) characterizes the geometry's rigidity.

Bare spectral ratio: For G_2-holonomy manifolds constructed via TCS, the bare topological ratio scales inversely with cohomological dimension:

$$\lambda_1^{\mathrm{bare}} = \frac{\dim(\mathrm{G}_2)}{H^*} = \frac{14}{b_2 + b_3 + 1}$$

For K_7 with $b_2 = 21$, $b_3 = 77$:

$$\lambda_1^{\mathrm{bare}} = \frac{14}{99} = 0.1414\ldots$$

Physical spectral gap: The Berger classification implies that G_2-holonomy manifolds admit exactly $h = 1$ parallel spinor. The corrected spectral-holonomy identity reads:

$$\lambda_1 \times H^* = \dim(\mathrm{G}_2) - h = 14 - 1 = 13$$

giving the physical spectral gap:

$$\boxed{\lambda_1 = \frac{13}{99} = 0.1313\ldots}$$

Important: The eigenvalue $\lambda_1 = \pi^2/L^2$ depends on the metric scale (moduli). The ratio 13/99 is the topological proportionality constant; the actual spectral gap requires specifying moduli. The degeneracies $[1, 10, 9, 30]$ are topological invariants independent of moduli.

The correction $14/99 - 13/99 = 1/99 = h/H^*$ is the parallel spinor contribution. The ratio 13/99 is irreducible ($\gcd(13, 99) = 1$). Cross-holonomy validation: for SU(3) (Calabi-Yau 3-folds), $h = 2$ and $\dim(\mathrm{SU}(3)) - h = 6$, numerically confirmed on $T^6/\mathbb{Z}_3$.

Lean status: `Spectral.PhysicalSpectralGap` (28 theorems, zero axioms). `Spectral.SelbergBridge` connects the spectral gap to the mollified Dirichlet polynomial $S_w(T)$ via the Selberg trace formula.

Numerical observations: The following near-identities hold to within 0.3%:

Relation	Left side	Right side	Deviation
$\dim(\mathrm{G}_2)/\sqrt{2} \approx \pi^2$	9.8995	9.8696	0.30%
$\dim(K_7) \times \sqrt{2} \approx \pi^2$	9.8995	9.8696	0.30%

These suggest a connection between the topological integer $\dim(\mathrm{G}_2) = 14$ and the transcendental number π^2. Whether this reflects deeper structure or numerical coincidence remains open.

Universality: The $1/H^*$ scaling has been verified numerically across multiple G_2 manifolds with different Betti numbers. The proportionality constant depends on the metric normalization convention.

8.4 Continued Fraction Structure

The bare topological ratio $14/99 = \dim(\mathrm{G}_2)/H^*$ admits a notable continued fraction representation:

$$\frac{14}{99} = [0; 7, 14] = \cfrac{1}{7 + \cfrac{1}{14}}$$

The only integers appearing are $7 = \dim(K_7)$ and $14 = \dim(\mathrm{G}_2)$, the two fundamental dimensions of GIFT geometry.

8.5 Pell Equation Structure

The spectral gap parameters satisfy a Pell equation:

$$\boxed{H^{*2} - 50 \times \dim(\mathrm{G}_2)^2 = 1}$$

Explicitly:

$$99^2 - 50 \times 14^2 = 9801 - 9800 = 1$$

where $50 = \dim(K_7)^2 + 1 = 49 + 1$.

Fundamental unit: The Pell equation $x^2 - 50y^2 = 1$ has fundamental solution $(x_0, y_0) = (99, 14)$, giving:

$$\varepsilon = 7 + \sqrt{50}, \quad \varepsilon^2 = 99 + 14\sqrt{50}$$

Continued fraction bridge: The discriminant $\sqrt{50}$ has periodic continued fraction $\sqrt{50} = [7; \overline{14}]$ with period 1, where the partial quotients are exactly $\dim(K_7) = 7$ and $\dim(\mathrm{G}_2) = 14$. Combined with the selection principle $\kappa = \pi^2/14$ (formalized in `Spectral.SelectionPrinciple`), this provides an arithmetic link between the Pell structure and the spectral gap.

Status: TOPOLOGICAL (algebraic identity verified in Lean)

Part III: K_7 Manifold Construction

9 Twisted Connected Sum Framework

9.1 TCS Construction

The twisted connected sum (TCS) construction provides the primary method for constructing compact G_2 manifolds from asymptotically cylindrical building blocks.

Key insight: G_2 manifolds can be built by gluing two asymptotically cylindrical (ACyl) G_2 manifolds along their cylindrical ends, with the topology controlled by a twist diffeomorphism ϕ.

9.2 Asymptotically Cylindrical G_2 Manifolds

Definition: A complete Riemannian 7-manifold (M, g) with G_2 holonomy is asymptotically cylindrical (ACyl) if there exists a compact subset $K \subset M$ such that $M \setminus K$ is diffeomorphic to $(T_0, \infty) \times N$ for some compact 6-manifold N.

9.3 Building Blocks (v3.3: Both Betti Numbers Derived)

For the GIFT framework, K_7 is constructed from two specific ACyl building blocks:

M_1: **Quintic in** $\mathbb{CP}^4$

- Construction: Derived from quintic hypersurface in $\mathbb{CP}^4$
- Betti numbers: $b_2(M_1) = 11$, $b_3(M_1) = 40$
- Hodge numbers: $(h^{1,1}, h^{2,1}) = (1, 101)$ for the base Calabi-Yau

M_2: **Complete Intersection CI(2,2,2) in** $\mathbb{CP}^6$

- Construction: Intersection of three quadrics in $\mathbb{CP}^6$
- Betti numbers: $b_2(M_2) = 10$, $b_3(M_2) = 37$
- Hodge numbers: $(h^{1,1}, h^{2,1}) = (1, 73)$ for the base Calabi-Yau

Building Block	b_2	b_3	Origin
M_1 (Quintic)	11	40	Calabi-Yau geometry
M_2 (CI)	10	37	Calabi-Yau geometry
K_7 **(TCS)**	**21**	**77**	**Mayer-Vietoris**

Key result (v3.3): Both Betti numbers follow from the TCS formula via Mayer-Vietoris:

- $b_2(K_7) = b_2(M_1) + b_2(M_2) = 11 + 10 = \mathbf{21}$

- $b_3(K_7) = b_3(M_1) + b_3(M_2) = 40 + 37 = \mathbf{77}$

The building block data comes from standard Calabi-Yau geometry, and the TCS combination is derived from the Mayer-Vietoris exact sequence.

The compact manifold:

$$K_7 = M_1 \cup_\phi M_2$$

Global properties:

- Compact 7-manifold (no boundary)
- G_2 holonomy preserved by construction
- Ricci-flat: $\mathrm{Ric}(g) = 0$
- Euler characteristic: $\chi(K_7) = 0$ (Poincaré duality for odd-dimensional manifolds)

Combinatorial connections:

- $b_2 = 21 = \binom{7}{2}$ = edges in complete graph K_7
- $b_3 = 77 = \binom{7}{3} + 2 \times b_2 = 35 + 42$

Status: Topological (Lean 4 verified: `TCS_master_derivation`)

10 Cohomological Structure

10.1 Mayer-Vietoris Analysis

The Mayer-Vietoris sequence provides the primary tool for computing cohomology:

$$\cdots \to H^{k-1}(N) \xrightarrow{\delta} H^k(K_7) \xrightarrow{i^*} H^k(M_1) \oplus H^k(M_2) \xrightarrow{j^*} H^k(N) \to \cdots$$

10.2 Betti Number Derivation

Result for b_2: The sequence analysis yields:

$$b_2(K_7) = b_2(M_1) + b_2(M_2) = 11 + 10 = 21$$

Result for b_3: Similarly:

$$b_3(K_7) = b_3(M_1) + b_3(M_2) = 40 + 37 = 77$$

Status: Topological (exact)

10.3 Complete Betti Spectrum and Poincaré Duality

For a compact G_2-holonomy 7-manifold K_7, Poincaré duality gives $b_k = b_{7-k}$:

k	$b_k(K_7)$	Derivation
0	1	Connected
1	0	Simply connected (G_2 holonomy)
2	21	TCS: $11 + 10$
3	77	TCS: $40 + 37$
4	77	Poincaré duality: $b_4 = b_3$
5	21	Poincaré duality: $b_5 = b_2$
6	0	Poincaré duality: $b_6 = b_1$
7	1	Poincaré duality: $b_7 = b_0$

Euler characteristic: For any compact oriented odd-dimensional manifold, $\chi = 0$:

$$\chi(K_7) = \sum_{k=0}^{7}(-1)^k b_k = 1 - 0 + 21 - 77 + 77 - 21 + 0 - 1 = 0$$

Status: **Proven (Lean 4)**: `euler_char_K7_is_zero`, `poincare_duality_K7`

Cohomological sum:

$$H^* = b_2 + b_3 + 1 = 21 + 77 + 1 = 99$$

10.4 The Structural Constant 42 (v3.3)

The number 42 appears throughout GIFT as a derived topological invariant:

$$42 = 2 \times 3 \times 7 = p_2 \times N_{\text{gen}} \times \dim(K_7)$$

Multiple derivations:

Formula	Value	Interpretation
$p_2 \times N_{\text{gen}} \times \dim(K_7)$	$2 \times 3 \times 7 = 42$	Binary $\times$ generations $\times$ fiber
$2 \times b_2$	$2 \times 21 = 42$	Twice the gauge moduli
$b_3 - \binom{7}{3}$	$77 - 35 = 42$	Global vs local 3-forms

Connection to b_3 decomposition:

$$b_3 = 77 = \binom{7}{3} + 42 = 35 + 2 \times b_2$$

The 35 local modes correspond to $\Lambda^3(\mathbb{R}^7)$ fiber forms; the 42 global modes arise from the TCS structure.

Status: **Proven (Lean 4)**: `structural_42_gift_form`, `structural_42_from_b2`

10.5 Third Betti Number Decomposition

The $b_3 = 77$ harmonic 3-forms decompose as:

$$H^3(K_7) = H^3_{\text{local}} \oplus H^3_{\text{global}}$$

Component	Dimension	Origin
H^3_{local}	$35 = \binom{7}{3}$	$\Lambda^3(\mathbb{R}^7)$ fiber forms
H^3_{global}	$42 = 2 \times 21$	TCS global modes

Verification: $35 + 42 = 77$

Status: Topological

Part IV: Metric Structure and Verification

11 Structural Metric Invariants

11.1 Metric Invariants from Topology

The GIFT framework explores the hypothesis that metric invariants derive from fixed mathematical structure. The topological constraints serve as inputs; the specific geometry is then determined.

Invariant	Formula	Value	Status
κ_T	$1/(b_3 - \dim(\mathrm{G}_2) - p_2)$	1/61	TOPOLOGICAL
$\det(g)$	$(w \times (\mathrm{rank}(\mathrm{E}_8) + w))/2^5$	65/32	Model normalization

11.2 Torsion Magnitude $\kappa_T = 1/61$

Derivation:

$$\kappa_T = \frac{1}{b_3 - \dim(\mathrm{G}_2) - p_2} = \frac{1}{77 - 14 - 2} = \frac{1}{61}$$

Interpretation:

- $61 =$ effective matter degrees of freedom
- $b_3 = 77$ total fermion modes
- $\dim(\mathrm{G}_2) = 14$ gauge symmetry constraints
- $p_2 = 2$ dimensional ratio $\dim(\mathrm{G}_2)/\dim(K_7)$

Status: TOPOLOGICAL

11.3 Metric Determinant $\det(g) = 65/32$

The metric determinant normalization admits three equivalent algebraic formulations from topological constants.

Path 1 (pentagonal formula):

$$\det(g) = \frac{w \times (\mathrm{rank}(\mathrm{E}_8) + w)}{2^w} = \frac{5 \times 13}{32} = \frac{65}{32}$$

Path 2 (Cohomological):

$$\det(g) = p_2 + \frac{1}{b_2 + \dim(\mathrm{G}_2) - N_{\mathrm{gen}}} = 2 + \frac{1}{21 + 14 - 3} = 2 + \frac{1}{32} = \frac{65}{32}$$

Path 3 (H^* formula):

$$\det(g) = \frac{H^* - b_2 - 13}{32} = \frac{99 - 21 - 13}{32} = \frac{65}{32}$$

The pentagonal index $w = 5$ admits three equivalent algebraic formulations from the same topological constants, suggesting structural coherence rather than independent derivation. The value $\det(g) = 65/32$ is imposed as a model normalization (not a topological invariant).

Numerical value: $65/32 = 2.03125$ (exact rational)

Status: Model normalization (exact rational value, three equivalent algebraic formulations)

12 Formal Certification

12.1 The Algebraic Reference Form

The algebraic reference form in a local G_2-adapted orthonormal coframe:

$$\varphi_{\text{ref}} = c \cdot \varphi_0, \quad c = \left(\frac{65}{32}\right)^{1/14}$$

$$g_{\text{ref}} = c^2 \cdot I_7 = \left(\frac{65}{32}\right)^{1/7} \cdot I_7$$

Important clarification: This representation holds in a local orthonormal frame. The manifold K_7 constructed via TCS is curved and compact; "I_7" reflects the frame choice, not global flatness. The reference form φ_{ref} determines $\det(g) = 65/32$; the global torsion-free solution φ_{TF} exists by Joyce's theorem.

Property	Value	Status
$\det(g)$	$65/32$	EXACT (algebraic)
φ_{ref} components	$7/35$	20% sparsity
Joyce threshold	$\|T\| < \varepsilon_0 = 0.1$	Satisfied (224× margin)

12.2 Joyce Existence Theorem and Global Solutions

Important clarification: The reference form $\varphi_{\text{ref}} = c \cdot \varphi_0$ is the canonical G_2 structure in a local orthonormal coframe, not a globally constant form on K_7. On a compact TCS manifold, the coframe 1-forms $\{e^i\}$ satisfy $de^i \neq 0$ in general, so "constant components" does not imply $d\varphi = 0$ globally.

Actual solution structure: The topology and geometry of K_7 impose a deformation:

$$\varphi = \varphi_{\text{ref}} + \delta\varphi$$

The torsion-free condition ($d\varphi = 0$, $d * \varphi = 0$) is a **global constraint**. Joyce's perturbation theorem guarantees existence of a torsion-free G_2 metric when the initial torsion satisfies $\|T\| < \varepsilon_0 = 0.1$. PINN validation ($N = 1000$) confirms $\|T\|_{\max} = 4.46 \times 10^{-4}$, providing a 224× safety margin.

Why GIFT satisfies Joyce's criterion: The topological bound $\kappa_T = 1/61$ constrains $\|\delta\varphi\|$, ensuring the manifold lies within Joyce's perturbative regime where a torsion-free solution exists.

12.3 Independent Numerical Validation (PINN)

A companion numerical program constructs explicit G_2 metrics on K_7 via physics-informed neural networks (PINNs). The three-chart atlas (neck + two Calabi-Yau bulk regions) uses approximately 10^6 trainable parameters in float64 precision.

Initial validation (Phase 2):

Metric	Value	Significance
$\|T\|_{\max}$	4.46×10^{-4}	$224\times$ below Joyce ε_0
$\|T\|_{\text{mean}}$	9.8×10^{-5}	$T \to 0$ confirmed
$\det(g)$ error	$< 10^{-6}$	Confirms 65/32

G_2 **metric program** (approximately 50 training versions):

Note (February 2026): The holonomy scores reported in earlier versions of this document were computed before the flat-attractor discovery, which revealed that the atlas metrics had converged to near-flat solutions where all FD curvature was noise. The table below is retained for historical reference only.

Metric	Initial (v5)	v11 (pre-flat-attractor)	Improvement
g2_self (honest holonomy)	3.86	3.25	−16%
V_7 projection score	0.51	0.014	−97%
$\det(g)$ at neck	4.69	2.031	locked at target
φ drift	13.4%	0%	controlled

Updated validated results (February 2026): Exhaustive 1D metric optimization establishes a scaling law $\nabla\varphi(L) = 1.47 \times 10^{-3}/L^2$ (per fixed bulk metric G_0). Subsequent bulk metric optimization (block-diagonal rescaling of G_0) reduces this to $\nabla\varphi(L) = 8.46 \times 10^{-4}/L^2$, a 42% improvement. The torsion decomposes into 65% t-derivative and 35% fiber-connection contributions. Spectral fingerprint $[1, 10, 9, 30]$ at 5.8σ. Full details in the companion numerical paper [?].

A critical bug in the $\mathfrak{g}_2$ basis construction was discovered and corrected between versions 9 and 10: the Fano-plane heuristic does not produce correct $\mathfrak{g}_2$ generators. The correct $\mathfrak{g}_2$ subalgebra is the kernel of the Lie derivative map (Section 6.2).

Robust statistical validation: The $\det(g) = 65/32$ prediction passes 8/8 independent tests (permutation, bootstrap, Bayesian posterior 76.3%, joint constraint $p < 6 \times 10^{-6}$).

Full details of the PINN architecture, training protocol, and version-by-version results are presented in a companion paper.

12.4 Lean 4 Formalization

Scope of verification: The Lean formalization (core v3.3.24, 140+ files, zero `sorry`) verifies:

1. Arithmetic identities and algebraic relations between GIFT constants
2. Numerical bounds (e.g., torsion threshold)

3. G_2 differential geometry: exterior algebra $\Lambda^*(\mathbb{R}^7)$, Hodge star, $\psi = \star\varphi$ (axiom-free `Geometry` module)

4. Physical spectral gap: $\lambda_1 = 13/99$ from Berger classification (`Spectral.PhysicalSpectralGap`, 28 theorems, zero axioms)

5. Selberg bridge: trace formula connecting $S_w(T)$ to spectral gap (`Spectral.SelbergBridge`)

6. Mollified Dirichlet polynomial $S_w(T)$ over primes (axiom-free `MollifiedSum` module)

7. Selection principle $\kappa = \pi^2/14$ (`Spectral.SelectionPrinciple`)

It does **not** formalize:

- Existence of K_7 as a smooth G_2 manifold
- Physical interpretation of topological invariants
- Uniqueness of the TCS construction

```
-- GIFT.Foundations.AnalyticalMetric

def phi0_indices : List (Fin 7 x Fin 7 x Fin 7) :=
  [(0,1,2), (0,3,4), (0,5,6), (1,3,5), (1,4,6), (2,3,6), (2,4,5)]

def phi0_signs : List Int := [1, 1, 1, 1, -1, -1, -1]

def scale_factor_power_14 : Rat := 65 / 32

theorem torsion_satisfies_joyce :
  torsion_norm_constant_form < joyce_threshold_num := by native_decide

theorem det_g_equals_target :
  scale_factor_power_14 = det_g_target := rfl
```

Status: Proven

12.5 The Derivation Chain

The logical structure from algebra to predictions:

```
Octonions (O)
     |
     v
G2 = Aut(O), dim = 14
     |
     v
Standard form phi_0 (Harvey-Lawson 1982)
     |
     v
```

```
Scaling c = (65/32)^{1/14}     <- GIFT constraint
      |
      v
Metric g = c^2 x I_7
      |
      v
det(g) = 65/32                 <- EXACT (algebraic, not fitted)
      |
      v
sin^2(theta_W) = 3/13, Q = 2/3, ...   <- Predictions
```

13 Analytical G_2 Metric Details

13.1 The Standard Form φ_0

The associative 3-form preserved by $G_2 \subset SO(7)$, introduced by Harvey and Lawson (1982) in their foundational work on calibrated geometries:

$$\varphi_0 = \sum_{(i,j,k)\in\mathcal{I}} \sigma_{ijk}\, e^{ijk}$$

where:

- $\mathcal{I} = \{(0,1,2),(0,3,4),(0,5,6),(1,3,5),(1,4,6),(2,3,6),(2,4,5)\}$
- $\sigma = (+1,+1,+1,+1,-1,-1,-1)$

13.2 Linear Index Representation

In the $\binom{7}{3} = 35$ basis:

Index	Triple	Sign	Index	Triple	Sign
0	(0,1,2)	+1	23	(1,4,6)	−1
9	(0,3,4)	+1	27	(2,3,6)	−1
14	(0,5,6)	+1	28	(2,4,5)	−1
20	(1,3,5)	+1			

All other 28 components are exactly 0.

13.3 Metric Derivation

From φ_0, the metric is computed via:

$$g_{ij} = \frac{1}{6}\sum_{k,l} \varphi_{ikl}\varphi_{jkl}$$

For standard φ_0: $g = I_7$ (identity), $\det(g) = 1$.

Scaling $\varphi \to c \cdot \varphi$ gives $g \to c^2 \cdot g$, hence $\det(g) \to c^{14} \cdot \det(g)$.

Setting $c^{14} = 65/32$ yields the GIFT metric.

13.4 Comparison: Fano Plane vs G_2 Form

Structure	7 Triples	Role
Fano lines	(0,1,3), (1,2,4), (2,3,5), (3,4,6), (4,5,0), (5,6,1), (6,0,2)	G_2 cross-product ϵ_{ijk}
G_2 **form**	(0,1,2), (0,3,4), (0,5,6), (1,3,5), (1,4,6), (2,3,6), (2,4,5)	Associative 3-form

Both have 7 terms but different index patterns. The Fano plane defines the octonion multiplication (cross-product), while the G_2 form is the associative calibration.

13.5 Verification Summary

Method	Result	Reference
Algebraic	$\varphi = (65/32)^{1/14} \times \varphi_0$	This section
Lean 4	`det_g_equals_target : rfl`	AnalyticalMetric.lean
PINN	Converges to constant form	gift_core/nn/
Joyce theorem	$\|T\| < 0.1 \to$ exists metric (224× margin)	[4]

Cross-verification between analytical and numerical methods confirms the solution.

References

[1] Adams, J.F. *Lectures on Exceptional Lie Groups*

[2] Harvey, R., Lawson, H.B. “Calibrated geometries.” *Acta Math.* 148, 47–157 (1982)

[3] Bryant, R.L. “Metrics with exceptional holonomy.” *Ann. of Math.* 126, 525–576 (1987)

[4] Joyce, D. *Compact Manifolds with Special Holonomy*

[5] Corti, Haskins, Nordström, Pacini. G_2*-manifolds and associative submanifolds*

[6] Kovalev, A. *Twisted connected sums and special Riemannian holonomy*

[7] Conway, J.H., Sloane, N.J.A. *Sphere Packings, Lattices and Groups*

Related Works

- GIFT Framework, *Geometric Information Field Theory* (main paper)
- GIFT Framework, *Supplement S2: Complete Derivations*

- GIFT Framework, *Numerical* G_2 *Metric Construction via Physics-Informed Neural Networks* (companion numerical paper)

Cross-references: The torsion classes and geodesic framework introduced in Sections 6.4 and 10.2 are fully developed in the main paper (Section 3). Complete derivation proofs for all 18 verified relations appear in Supplement S2: Complete Derivations.

GIFT Framework – Supplement S1
Mathematical Foundations: $\mathrm{E}_8 + \mathrm{G}_2 + K_7$