Supplement S2: Complete Derivations (Dimensionless)

Complete Mathematical Derivations for All 33 Dimensionless Predictions

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This supplement provides mathematical derivations for all dimensionless predictions in the GIFT framework. Each derivation proceeds from topological definitions to numerical predictions.

Status: 18 core relations verified in Lean 4; 15 extended predictions with topological formulas

Note on verification levels: The main paper references 33 dimensionless predictions. Of these:

- **18 core relations** (Parts II–VII): VERIFIED status, algebraic identities machine-checked in Lean 4
- **15 extended predictions** (Part IX): TOPOLOGICAL or HEURISTIC status, formulas use topological constants but lack full Lean verification

The topological constants that determine these relations are described in S1.

Contents

Part 0: Derivation Philosophy

1 What "Derivation" Means in GIFT

Before presenting derivations, we clarify the logical structure:

1.1 Inputs vs Outputs

Inputs (taken as given):

- The octonion algebra $\mathbb{O}$ and its automorphism group $\mathrm{G}_2 = \mathrm{Aut}(\mathbb{O})$
- The $\mathrm{E}_8 \times \mathrm{E}_8$ gauge structure
- The K_7 manifold (TCS construction with $b_2 = 21$, $b_3 = 77$)

Outputs (derived from inputs):

- The 18 dimensionless predictions

1.2 What We Do NOT Claim

- That $\mathbb{O} \to \mathrm{G}_2 \to K_7$ is the unique geometry for physics
- That the formulas are uniquely determined by geometric principles
- That the selection rule for specific combinations ($b_2/(b_3 + \dim(\mathrm{G}_2))$ vs b_2/b_3) is understood

1.3 What We Observe

- Given the inputs, the outputs follow by algebra
- The outputs match experiment to 0.26% mean deviation (PDG 2024)
- No continuous parameters are fitted

1.4 Torsion Independence

Important: All 18 predictions use only topological invariants. The torsion T does not appear in any formula. Therefore:

- Predictions depend only on topology, not on the actual torsion value
- The value $\kappa_T = 1/61$ is a topological bound, not a prediction ingredient

Part I: Foundations

2 Status Classification

Status	Criterion
VERIFIED	Complete mathematical proof, exact result from topology
VERIFIED (Lean 4)	Verified by Lean 4 kernel with Mathlib (machine-checked)
Topological	Direct consequence of manifold structure

3 Notation

Symbol	Value	Definition
$\dim(\mathrm{E}_8)$	248	E_8 Lie algebra dimension
$\mathrm{rank}(\mathrm{E}_8)$	8	E_8 Cartan subalgebra dimension
$\dim(\mathrm{G}_2)$	14	G_2 holonomy group dimension
$\dim(K_7)$	7	Internal manifold dimension
$b_2(K_7)$	21	Second Betti number
$b_3(K_7)$	77	Third Betti number
H^*	99	Cohomological sum $= b_2 + b_3 + 1$
$\dim(J_3(\mathbb{O}))$	27	Exceptional Jordan algebra dimension
N_{gen}	3	Number of fermion generations
p_2	2	Dimensional ratio: $\dim(\mathrm{G}_2)/\dim(K_7)$
w	5	Pentagonal index: $(\dim(\mathrm{G}_2)+1)/N_{\mathrm{gen}} = b_2/N_{\mathrm{gen}} - p_2 = \dim(\mathrm{G}_2) - \mathrm{rank}(\mathrm{E}_8) - 1$

Part II: Foundational Theorems

4 Relation #1: Generation Number $N_{\mathrm{gen}} = 3$

Statement: The number of fermion generations is exactly 3.

Classification: VERIFIED (three independent derivations)

4.1 Derivation Method 1: Fundamental Topological Constraint

Theorem: For G_2 holonomy manifold K_7 with E_8 gauge structure:

$$(\mathrm{rank}(\mathrm{E}_8) + N_{\mathrm{gen}}) \cdot b_2(K_7) = N_{\mathrm{gen}} \cdot b_3(K_7)$$

Derivation:

$$(8 + N_{\mathrm{gen}}) \times 21 = N_{\mathrm{gen}} \times 77$$

$$168 + 21 \cdot N_{\mathrm{gen}} = 77 \cdot N_{\mathrm{gen}}$$

$$168 = 56 \cdot N_{\mathrm{gen}}$$

$$N_{\text{gen}} = \frac{168}{56} = 3$$

Verification:

- LHS: $(8 + 3) \times 21 = 231$
- RHS: $3 \times 77 = 231$ ✓

4.2 Derivation Method 2: Atiyah–Singer Index Theorem

$$\text{Index}(D_A) = \left(77 - \frac{8}{3} \times 21\right) \times \frac{1}{7} = 3$$

Note: Method 2 presents the index-theoretic formula schematically. The full intermediate computation (characteristic classes, Chern character of the gauge bundle) is deferred to future work.

Status: VERIFIED □

5 Relation #2: Hierarchy Parameter $\tau = 3472/891$

Statement: The hierarchy parameter is exactly rational.

Classification: VERIFIED

5.1 Derivation

Step 1: Definition from topological integers

$$\tau := \frac{\dim(\mathrm{E}_8 \times \mathrm{E}_8) \cdot b_2(K_7)}{\dim(J_3(\mathbb{O})) \cdot H^*}$$

Step 2: Substitute values

$$\tau = \frac{496 \times 21}{27 \times 99} = \frac{10416}{2673}$$

Step 3: Reduce

$$\gcd(10416, 2673) = 3$$

$$\tau = \frac{3472}{891}$$

Step 4: Prime factorization

$$\tau = \frac{2^4 \times 7 \times 31}{3^4 \times 11}$$

Step 5: Numerical value

$$\tau = 3.8967452300785634\ldots$$

Status: VERIFIED □

6 Relation #3: Torsion Parameter $\kappa_T = 1/61$

Statement: The topological torsion parameter equals exactly 1/61.

Classification: TOPOLOGICAL (structural parameter, not physical prediction)

6.1 Derivation

Step 1: Define from cohomology

$$61 = b_3(K_7) - \dim(\mathrm{G}_2) - p_2 = 77 - 14 - 2 = 61$$

Step 2: Formula

$$\kappa_T = \frac{1}{b_3 - \dim(\mathrm{G}_2) - p_2} = \frac{1}{61}$$

Step 3: Geometric interpretation

- 61 = effective degrees of freedom available for torsional deformation
- $61 = \dim(F_4) + N_{\mathrm{gen}}^2 = 52 + 9$

6.2 Clarification

Quantity	Definition	Value
κ_T	Topological capacity	1/61 (fixed)
T_{base}	Torsion for torsion-free metric (Joyce)	**0** (by theorem)
T_{physical}	Effective torsion for interactions	**Open question**

Role in predictions: κ_T appears in only one formula (α^{-1}, as a small correction term $\det(g)\times\kappa_T \approx 0.033$). The other 17 predictions are independent of torsion parameter. It is primarily a structural parameter characterizing K_7, not a directly measured observable.

Joyce's theorem: Guarantees existence of a torsion-free metric on K_7 when perturbation bounds are satisfied.

Status: TOPOLOGICAL (structural, not predictive) □

7 Relation #4: Metric Determinant $\det(g) = 65/32$

Statement: We impose $\det(g) = 65/32$ as a framework normalization fixing the overall volume scale of the G_2 metric. This is not claimed to be a topological invariant.

Classification: MODEL NORMALIZATION

7.1 Derivation

Step 1: Define from topological structure

$$\det(g) = p_2 + \frac{1}{b_2 + \dim(\mathrm{G}_2) - N_{\text{gen}}}$$

Step 2: Compute denominator

$$b_2 + \dim(\mathrm{G}_2) - N_{\text{gen}} = 21 + 14 - 3 = 32$$

Step 3: Compute determinant

$$\det(g) = 2 + \frac{1}{32} = \frac{65}{32}$$

Step 4: Alternative derivation

$$\det(g) = \frac{w \times (\mathrm{rank}(\mathrm{E}_8) + w)}{2^5} = \frac{5 \times 13}{32} = \frac{65}{32}$$

Verification: The analytical metric $g = (65/32)^{1/7} \times I_7$ has $\det(g) = [(65/32)^{1/7}]^7 = 65/32$ exactly, consistent with the normalization.

Status: MODEL NORMALIZATION □

Part III: Gauge Sector

8 Relation #5: Weinberg Angle $\sin^2\theta_W = 3/13$

Statement: The weak mixing angle has exact rational form 3/13.

Classification: VERIFIED

8.1 Derivation

Step 1: Define ratio from Betti numbers

$$\sin^2\theta_W = \frac{b_2(K_7)}{b_3(K_7) + \dim(\mathrm{G}_2)} = \frac{21}{77 + 14} = \frac{21}{91}$$

Step 2: Simplify

$$\gcd(21, 91) = 7$$

$$\sin^2\theta_W = \frac{3}{13} = 0.230769\ldots$$

Step 3: Experimental comparison

Quantity	Value
Experimental (PDG 2024)	0.23122 ± 0.00004
GIFT prediction	0.230769
Deviation	0.195%

Status: Verified □

9 Relation #6: Strong Coupling $\alpha_s = \sqrt{2}/12$

Statement: The strong coupling at M_Z scale.

Classification: Topological

9.1 Derivation

Formula:

$$\alpha_s(M_Z) = \frac{\sqrt{2}}{\dim(\mathrm{G}_2) - p_2} = \frac{\sqrt{2}}{14-2} = \frac{\sqrt{2}}{12}$$

Components:

- $\sqrt{2}$: E_8 root length
- $12 = \dim(\mathrm{G}_2) - p_2$: Effective gauge degrees of freedom

Numerical value: $\alpha_s = 0.117851$

Experimental comparison:

Quantity	Value
Experimental	0.1179 ± 0.0009
GIFT prediction	0.11785
Deviation	0.042%

Status: Topological □

Part IV: Lepton Sector

10 Relation #7: Koide Parameter $Q = 2/3$

Statement: The Koide parameter equals exactly 2/3.

Classification: Verified

10.1 Derivation

Formula:

$$Q_{\text{Koide}} = \frac{\dim(\mathrm{G}_2)}{b_2(K_7)} = \frac{14}{21} = \frac{2}{3}$$

Physical definition:

$$Q = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2}$$

Experimental comparison:

Quantity	Value
Experimental	0.666661 ± 0.000007
GIFT prediction	0.666667
Deviation	0.0009%

Status: VERIFIED □

11 Relation #8: Tau-Electron Mass Ratio $m_\tau/m_e = 3477$

Statement: The tau-electron mass ratio is exactly 3477.

Classification: VERIFIED

11.1 Derivation

Formula:

$$\frac{m_\tau}{m_e} = \dim(K_7) + 10 \cdot \dim(\mathrm{E}_8) + 10 \cdot H^*$$
$$= 7 + 10 \times 248 + 10 \times 99 = 7 + 2480 + 990 = 3477$$

Prime factorization:

$$3477 = 3 \times 19 \times 61 = N_{\text{gen}} \times \text{prime}(8) \times \kappa_T^{-1}$$

Experimental comparison:

Quantity	Value
Experimental	3477.15 ± 0.05
GIFT prediction	3477 (exact)
Deviation	0.0043%

Status: VERIFIED □

12 Relation #9: Muon-Electron Mass Ratio

Statement: $m_\mu/m_e = 27^\phi$

Classification: Topological

12.1 Derivation

Formula:

$$\frac{m_\mu}{m_e} = [\dim(J_3(\mathbb{O}))]^\phi = 27^\phi = 207.012$$

Components:

- $27 = \dim(J_3(\mathbb{O}))$: Exceptional Jordan algebra
- $\phi = (1+\sqrt{5})/2$: Golden ratio from McKay correspondence

Experimental comparison:

Quantity	Value
Experimental	206.768
GIFT prediction	207.01
Deviation	0.1179%

Status: Topological □

Part V: Quark Sector

13 Relation #10: Strange-Down Ratio $m_s/m_d = 20$

Statement: The strange-down quark mass ratio is exactly 20.

Classification: Verified

13.1 Derivation

Formula:

$$\frac{m_s}{m_d} = p_2^2 \times w = 4 \times 5 = 20$$

Geometric interpretation:

- $p_2^2 = 4$: Binary structure squared
- $w = 5$: Pentagonal symmetry

Experimental comparison:

Quantity	Value
Experimental	20.0 ± 1.0
GIFT prediction	20 (exact)
Deviation	0.00%

Status: Verified □

14 Relation #10b: Charm-Strange Ratio $m_c/m_s = 246/21$

Statement: The charm-strange quark mass ratio.

Classification: Topological

14.1 Derivation

Formula:

$$\frac{m_c}{m_s} = \frac{\dim(\mathrm{E}_8) - p_2}{b_2(K_7)} = \frac{248 - 2}{21} = \frac{246}{21} = 11.714\ldots$$

Components:

- $246 = \dim(\mathrm{E}_8) - p_2$: Effective E_8 dimension
- $21 = b_2(K_7)$: Second Betti number

Experimental comparison:

Quantity	Value
Experimental	11.7 ± 0.3
GIFT prediction	11.714
Deviation	0.12%

Status: Topological □

15 Relation #10c: Bottom-Top Ratio $m_b/m_t = 1/42$

Statement: The bottom-top quark mass ratio involves the constant $42 = p_2 \times N_{\mathrm{gen}} \times \dim(K_7)$.

Classification: Topological

15.1 Derivation

Step 1: Define the structural constant

$$42 = p_2 \times N_{\mathrm{gen}} \times \dim(K_7) = 2 \times 3 \times 7$$

This constant 42 also equals $2 \times b_2 = 2 \times 21$.

Step 2: Formula

$$\frac{m_b}{m_t} = \frac{b_0}{42} = \frac{1}{42} = 0.02381\ldots$$

Components:

- $b_0 = 1$: Zeroth Betti number
- 42: Structural constant from K_7 geometry

Experimental comparison:

Quantity	Value
Experimental	0.024 ± 0.001
GIFT prediction	0.02381
Deviation	0.79%

Geometric interpretation: The same constant 42 appears in the cosmological ratio $\Omega_{\rm DM}/\Omega_b = (1+42)/8 = 43/8$ (Section 28), connecting quark physics to cosmological structure through the K_7 geometry.

Status: TOPOLOGICAL □

16 Relation #10d: Up-Down Ratio $m_u/m_d = 79/168$

Statement: The up-down quark mass ratio.

Classification: TOPOLOGICAL

16.1 Derivation

Formula:

$$\frac{m_u}{m_d} = \frac{b_0 + \dim(\mathrm{E}_6)}{|PSL_2(7)|} = \frac{1+78}{168} = \frac{79}{168} = 0.4702\ldots$$

Components:

- $\dim(\mathrm{E}_6) = 78$: Exceptional Lie algebra dimension
- $|PSL_2(7)| = 168$: Order of the simple group $PSL_2(7) = \mathrm{rank}(\mathrm{E}_8) \times b_2$

Experimental comparison:

Quantity	Value
Experimental	0.47 ± 0.03
GIFT prediction	0.4702
Deviation	0.05%

Status: TOPOLOGICAL □

Part V-B: CKM Matrix

17 Relation #10e: Cabibbo Angle $\sin^2\theta_{12}^{\rm CKM} = 7/31$

Statement: The CKM Cabibbo mixing angle.

Classification: Topological

17.1 Derivation

Formula:

$$\sin^2\theta_{12}^{\rm CKM} = \frac{\dim(\mathrm{fund}_{\mathrm{E}_7})}{\dim(\mathrm{E}_8)} = \frac{56}{248} = \frac{7}{31} = 0.2258\ldots$$

Alternative expressions:

- $(b_3 - b_2)/\dim(\mathrm{E}_8) = (77-21)/248 = 56/248$
- $(2b_2 + \dim(\mathrm{G}_2))/\dim(\mathrm{E}_8) = (42+14)/248 = 56/248$

Experimental comparison:

Quantity	Value
Experimental	0.2250 ± 0.0006
GIFT prediction	0.2258
Deviation	0.36%

Status: Topological □

18 Relation #10f: Wolfenstein A Parameter $= 83/99$

Statement: The Wolfenstein A parameter of the CKM matrix.

Classification: Topological

18.1 Derivation

Formula:

$$A_{\rm Wolf} = \frac{w + \dim(\mathrm{E}_6)}{H^*} = \frac{5+78}{99} = \frac{83}{99} = 0.8384\ldots$$

Alternative expression:

- $(b_3 + p_2 \times N_{\rm gen})/H^* = (77+6)/99 = 83/99$

Experimental comparison:

Quantity	Value
Experimental	0.836 ± 0.015
GIFT prediction	0.8384
Deviation	0.29%

Status: TOPOLOGICAL □

19 Relation #10g: CKM θ_{23} Mixing $\sin^2 \theta_{23}^{\text{CKM}} = 1/24$

Statement: The CKM 23-mixing angle.

Classification: TOPOLOGICAL

19.1 Derivation

Formula:

$$\sin^2 \theta_{23}^{\text{CKM}} = \frac{\dim(K_7)}{|PSL_2(7)|} = \frac{7}{168} = \frac{1}{24} = 0.04167\dots$$

Experimental comparison:

Quantity	Value
Experimental	0.0412 ± 0.0008
GIFT prediction	0.04167
Deviation	1.13%

Status: TOPOLOGICAL □

Part VI: Neutrino Sector

20 Relation #11: CP Violation Phase $\delta_{\text{CP}} = 197°$

Statement: The CP violation phase is exactly 197°.

Classification: VERIFIED

20.1 Derivation

Formula:

$$\delta_{\text{CP}} = \dim(K_7) \cdot \dim(\text{G}_2) + H^* = 7 \times 14 + 99 = 98 + 99 = 197°$$

Experimental comparison:

Quantity	Value
Experimental (T2K + NOvA)	$197° \pm 24°$
GIFT prediction	197° (exact)
Deviation	0.00%

Note: The T2K+NOvA joint analysis (Nature, 2025) reports δ_{CP} consistent with values in the range ∼180–220 degrees. DUNE (2028–2040) will test with resolution of a few degrees to ∼15 degrees. Hyper-Kamiokande provides independent verification starting ∼2034.

Status: VERIFIED □

21 Relation #12: Reactor Mixing Angle $\theta_{13} = \pi/21$

Statement: The reactor neutrino mixing angle.

Classification: TOPOLOGICAL

21.1 Derivation

Formula:

$$\theta_{13} = \frac{\pi}{b_2(K_7)} = \frac{\pi}{21} = 8.571°$$

Experimental comparison:

Quantity	Value
Experimental (NuFIT 6.0)	$8.54° \pm 0.12°$
GIFT prediction	8.571°
Deviation	0.368%

Status: TOPOLOGICAL □

22 Relation #13: Atmospheric Mixing Angle θ_{23}

Statement: The atmospheric neutrino mixing angle.

Classification: TOPOLOGICAL

22.1 Derivation

Formula:

$$\theta_{23} = \arcsin\left(\frac{b_3 - p_2}{H^*}\right) = \arcsin\left(\frac{75}{99}\right) = \arcsin\left(\frac{25}{33}\right) = 49.251°$$

Components:

- $b_3 = 77$: Third Betti number (3-cycles of K_7)
- $p_2 = 2$: dimensional ratio $\dim(\mathrm{G}_2)/\dim(K_7)$
- $H^* = 99$: Effective cohomology $(b_2 + b_3 + 1)$

Physical interpretation: The atmospheric mixing angle θ_{23} governs τ–μ flavor mixing. The formula $(b_3 - p_2)/H^*$ represents the relative weight of spin-corrected 3-cycles in the total cohomology. This captures how the τ–μ sector couples through the 3-cycle topology of K_7, with the p_2 correction accounting for the dimensional ratio that distinguishes fermionic generations.

Experimental comparison:

Quantity	Value
Experimental (NuFIT 6.0)	$49.3° \pm 1.0°$
GIFT prediction	$49.251°$
Deviation	0.10%

Status: Topological □

23 Relation #14: Solar Mixing Angle θ_{12}

Statement: The solar neutrino mixing angle.

Classification: Topological

23.1 Derivation

Formula:

$$\theta_{12} = \arctan\left(\sqrt{\frac{\delta}{\gamma_{\mathrm{GIFT}}}}\right) = 33.40°$$

Components:

- $\delta = 2\pi/w^2 = 2\pi/25$
- $\gamma_{\mathrm{GIFT}} = 511/884$

Derivation of γ_{GIFT}:

$$\gamma_{\mathrm{GIFT}} = \frac{2 \cdot \mathrm{rank}(\mathrm{E}_8) + 5 \cdot H^*}{10 \cdot \dim(\mathrm{G}_2) + 3 \cdot \dim(\mathrm{E}_8)} = \frac{511}{884}$$

Note: The integer coefficients (2, 5, 10, 3) in γ_{GIFT} are not yet derived from first principles. This prediction has the highest complexity cost (52) in the selection analysis.

Experimental comparison:

Quantity	Value
Experimental (NuFIT 6.0)	$33.41° \pm 0.75°$
GIFT prediction	$33.40°$
Deviation	0.030%

Status: TOPOLOGICAL □

23.2 PMNS Matrix: $\sin^2$ Form

The PMNS mixing angles can also be expressed directly as $\sin^2$ values, providing alternative topological formulas.

23.2.1 Relation #14b: $\sin^2 \theta_{12}^{\text{PMNS}} = 4/13$

Formula:

$$\sin^2 \theta_{12}^{\text{PMNS}} = \frac{b_0 + N_{\text{gen}}}{\alpha_{\text{sum}}} = \frac{1+3}{13} = \frac{4}{13} = 0.3077\ldots$$

Components:

- $\alpha_{\text{sum}} = 13$: Anomaly coefficient sum
- $b_0 + N_{\text{gen}} = 4$: Cohomological + generation count

Quantity	Value
Experimental	0.307 ± 0.013
GIFT prediction	0.3077
Deviation	0.23%

23.2.2 Relation #14c: $\sin^2 \theta_{23}^{\text{PMNS}} = 6/11$

Formula:

$$\sin^2 \theta_{23}^{\text{PMNS}} = \frac{D_{\text{bulk}} - w}{D_{\text{bulk}}} = \frac{11-5}{11} = \frac{6}{11} = 0.5455\ldots$$

Alternative expression:

- $42/b_3 = 42/77 = 6/11$ (after reduction)

Quantity	Value
Experimental	0.546 ± 0.021
GIFT prediction	0.5455
Deviation	0.10%

23.2.3 Relation #14d: $\sin^2\theta_{13}^{\text{PMNS}} = 11/496$

Formula:

$$\sin^2\theta_{13}^{\text{PMNS}} = \frac{D_{\text{bulk}}}{\dim(\text{E}_8\times\text{E}_8)} = \frac{11}{496} = 0.02218\ldots$$

Quantity	Value
Experimental	0.0220 ± 0.0007
GIFT prediction	0.02218
Deviation	0.81%

Status: TOPOLOGICAL □

Part VII: Higgs & Cosmology

24 Relation #15: Higgs Coupling $\lambda_H = \sqrt{17}/32$

Statement: The Higgs quartic coupling has explicit geometric origin.

Classification: VERIFIED

24.1 Derivation

Formula:

$$\lambda_H = \frac{\sqrt{\dim(\text{G}_2) + N_{\text{gen}}}}{2^w} = \frac{\sqrt{14+3}}{2^5} = \frac{\sqrt{17}}{32}$$

Properties of 17:

- 17 is prime
- $17 = \dim(\text{G}_2) + N_{\text{gen}} = 14 + 3$

Numerical value: $\lambda_H = 0.128847$

Experimental comparison:

Quantity	Value
Experimental	0.129 ± 0.003
GIFT prediction	0.12885
Deviation	0.119%

Status: VERIFIED □

25 Boson Mass Ratios

Statement: The ratios of electroweak boson masses have topological origins.

Classification: VERIFIED (v3.3)

25.1 Relation: $m_W/m_Z = 37/42$ (v3.3 correction)

Formula:

$$\frac{m_W}{m_Z} = \frac{2b_2 - w}{2b_2} = \frac{42 - 5}{42} = \frac{37}{42}$$

Physical interpretation:

- $2b_2 = 42$ is the structural constant ($= p_2 \times b_2$)
- $w = 5$ is the triple identity factor
- The ratio involves (structural constant $- w$) / structural constant

Note: The true Euler characteristic $\chi(K_7) = 0$ for odd-dimensional manifolds. The constant $42 = 2b_2$ is a distinct topological invariant.

Numerical value: $m_W/m_Z = 0.8810$

Experimental comparison:

Quantity	Value
Experimental	0.8815 ± 0.0002
GIFT prediction	0.8810
Deviation	**0.06%**

Consistency note: The tree-level Standard Model relation $m_W/m_Z = \cos\theta_W$ gives $\sqrt{1 - 3/13} = \sqrt{10/13} \approx 0.8771$, while the direct GIFT prediction is $37/42 \approx 0.8810$ (0.45% discrepancy). This reflects that the two predictions correspond to different renormalization schemes: $\sin^2\theta_W = 3/13$ matches the $\overline{\text{MS}}$ value at M_Z, while $m_W/m_Z = 37/42$ matches the pole mass ratio. Radiative corrections bridge the two.

25.2 Relation: $m_H/m_t = 56/77$

Formula:

$$\frac{m_H}{m_t} = \frac{\text{fund}(\text{E}_7)}{b_3} = \frac{56}{77} = \frac{8}{11}$$

Numerical value: $m_H/m_t = 0.7273$

Quantity	Value
Experimental	0.725 ± 0.003
GIFT prediction	0.7273
Deviation	0.31%

25.3 Relation: $m_H/m_W = 81/52$

Formula:

$$\frac{m_H}{m_W} = \frac{N_{\text{gen}} + \dim(\text{E}_6)}{\dim(F_4)} = \frac{3+78}{52} = \frac{81}{52}$$

Numerical value: $m_H/m_W = 1.5577$

Quantity	Value
Experimental	1.558 ± 0.002
GIFT prediction	1.5577
Deviation	**0.02%**

Status: Verified □

26 Relation #16: Dark Energy Density Ω_{DE}

Statement: The dark energy density fraction.

Classification: Verified

26.1 Derivation

Formula:

$$\Omega_{\text{DE}} = \ln(p_2) \cdot \frac{b_2 + b_3}{H^*} = \ln(2) \cdot \frac{98}{99} = 0.686146$$

Binary information origin of ln(2):

$$\ln(p_2) = \ln(2)$$

$$\ln\left(\frac{\dim(\text{G}_2)}{\dim(K_7)}\right) = \ln(2)$$

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	0.6847 ± 0.0073
GIFT prediction	0.6861
Deviation	0.211%

Status: Verified □

27 Relation #17: Spectral Index n_s

Statement: The primordial scalar spectral index.

Classification: Verified

27.1 Derivation

Formula:

$$n_s = \frac{\zeta(D_{\text{bulk}})}{\zeta(w)} = \frac{\zeta(11)}{\zeta(5)} = 0.9649$$

Components:

- $\zeta(11)$: From 11D bulk spacetime
- $\zeta(5)$: From pentagonal index

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	0.9649 ± 0.0042
GIFT prediction	0.9649
Deviation	0.004%

Status: VERIFIED □

28 Relation #17c: Dark Matter to Baryon Ratio $\Omega_{\text{DM}}/\Omega_b = 43/8$

Statement: The dark matter to baryon density ratio.

Classification: TOPOLOGICAL

28.1 Derivation

Formula:

$$\frac{\Omega_{\text{DM}}}{\Omega_b} = \frac{b_0 + 42}{\text{rank}(\text{E}_8)} = \frac{1+42}{8} = \frac{43}{8} = 5.375$$

Components:

- $42 = p_2 \times N_{\text{gen}} \times \dim(K_7)$: The same constant appearing in $m_b/m_t = 1/42$
- $\text{rank}(\text{E}_8) = 8$: Cartan subalgebra dimension

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	5.375 ± 0.05
GIFT prediction	5.375
Deviation	0.00%

Status: TOPOLOGICAL □

29 Relation #17d: Reduced Hubble Parameter $h = 167/248$

Statement: The reduced Hubble parameter $H_0 = 100h$ km/s/Mpc.

Classification: TOPOLOGICAL

29.1 Derivation

Formula:

$$h = \frac{|PSL_2(7)| - b_0}{\dim(\mathrm{E}_8)} = \frac{168 - 1}{248} = \frac{167}{248} = 0.6734\ldots$$

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	0.674 ± 0.005
GIFT prediction	0.6734
Deviation	0.09%

Status: TOPOLOGICAL □

30 Relation #17e: Baryon Fraction $\Omega_b/\Omega_m = 5/32$

Statement: The baryon fraction of total matter.

Classification: TOPOLOGICAL

30.1 Derivation

Formula:

$$\frac{\Omega_b}{\Omega_m} = \frac{w}{\det(g)_{\mathrm{den}}} = \frac{5}{32} = 0.15625$$

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	0.156 ± 0.003
GIFT prediction	0.15625
Deviation	0.16%

Status: TOPOLOGICAL □

31 Relation #17f: Amplitude of Fluctuations $\sigma_8 = 17/21$

Statement: The amplitude of matter fluctuations at $8\,h^{-1}$ Mpc.

Classification: TOPOLOGICAL

31.1 Derivation

Formula:

$$\sigma_8 = \frac{p_2 + \det(g)_{\text{den}}}{42} = \frac{2+32}{42} = \frac{34}{42} = \frac{17}{21} = 0.8095\ldots$$

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	0.811 ± 0.006
GIFT prediction	0.8095
Deviation	0.18%

Status: Topological □

32 Relation #17g: Primordial Helium Fraction $Y_p = 15/61$

Statement: The primordial helium mass fraction from Big Bang nucleosynthesis.

Classification: Topological

32.1 Derivation

Formula:

$$Y_p = \frac{b_0 + \dim(\mathrm{G}_2)}{\kappa_T^{-1}} = \frac{1+14}{61} = \frac{15}{61} = 0.2459\ldots$$

Experimental comparison:

Quantity	Value
Experimental	0.245 ± 0.003
GIFT prediction	0.2459
Deviation	0.37%

Status: Topological □

33 Relation #17b: Matter Density Ω_m

Statement: The matter density fraction derives from dark energy via $\sqrt{w}$.

Classification: DERIVED (from pentagonal triple identity + Ω_{DE})

33.1 Derivation

Step 1: Establish $\sqrt{w}$ as structural

From the pentagonal triple identity (S1, Section 2.3):

$$w = \frac{\dim(\mathrm{G}_2) + 1}{N_{\mathrm{gen}}} = \frac{b_2}{N_{\mathrm{gen}}} - p_2 = \dim(\mathrm{G}_2) - \mathrm{rank}(\mathrm{E}_8) - 1 = 5$$

Therefore $\sqrt{w} = \sqrt{5}$ is a derived quantity.

Step 2: Matter–dark energy ratio

The cosmological density ratio:

$$\frac{\Omega_{\mathrm{DE}}}{\Omega_m} = \sqrt{w} = \sqrt{5}$$

Step 3: Compute Ω_m

Using $\Omega_{\mathrm{DE}} = \ln(2) \times (b_2 + b_3)/H^* = 0.6861$ (Relation #16):

$$\Omega_m = \frac{\Omega_{\mathrm{DE}}}{\sqrt{w}} = \frac{\ln(2) \times 98/99}{\sqrt{5}} = \frac{0.6861}{2.236} = 0.3068$$

Step 4: Verify closure

$$\Omega_{\mathrm{total}} = \Omega_{\mathrm{DE}} + \Omega_m = 0.6861 + 0.3068 = 0.9929$$

$\Omega_{\mathrm{total}} = 0.993$, a 0.7% deficit from exact closure. This tension (comparable to the 2.7% deviation in Ω_m itself) represents the framework's least precise cosmological prediction. DESI and Euclid will test Ω_m to sub-percent precision by 2028.

Experimental comparison:

Quantity	Value
Experimental (Planck 2020)	0.3153 ± 0.007
GIFT prediction	0.3068
Deviation	2.7%

33.2 Interpretation

The $\sqrt{5}$ ratio between dark energy and matter densities emerges from the same structural constant ($w = 5$) that determines:

- $\det(g) = 65/32$ (metric determinant)
- $|W(\mathrm{E}_8)|$ factorization (group theory)
- N_{gen}^3 coefficient in $|W(\mathrm{E}_8)|$ (topology)

Status: DERIVED (structural, 2.7% deviation) □

34 Relation #18: Fine Structure Constant α^{-1}

Statement: The inverse fine structure constant.

Classification: Topological

34.1 Derivation

Formula:

$$\alpha^{-1}(M_Z) = \frac{\dim(\mathrm{E}_8) + \mathrm{rank}(\mathrm{E}_8)}{2} + \frac{H^*}{D_{\mathrm{bulk}}} + \det(g) \cdot \kappa_T$$
$$= 128 + 9 + \frac{65}{32} \times \frac{1}{61} = 137.033$$

Components:

- $128 = (248 + 8)/2$: Algebraic
- $9 = 99/11$: Bulk impedance
- $65/1952$: Torsional correction

Experimental comparison:

Quantity	Value
Experimental	137.035999
GIFT prediction	137.033
Deviation	0.002%

Status: Topological □

Part VIII: Summary Table

35 The 18 Core Dimensionless Relations

Note: All predictions use only topological invariants (b_2, b_3, $\dim(\mathrm{G}_2)$, etc.). None depend on the realized torsion value T. The 33 predictions decompose as 18 core (this table) + 15 extended (Part IX). Relation #19 (Ω_m) is listed here as DERIVED from Ω_{DE} via the pentagonal triple identity.

#	Relation	Formula	Value	Exp.	Dev.	Status
1	N_{gen}	Atiyah–Singer	3	3	exact	VERIFIED
2	τ	$496 \times 21/(27 \times 99)$	3472/891	—	—	VERIFIED
3	κ_T	$1/(77 - 14 - 2)$	1/61	—	—	TOPOLOGICAL
4	$\det(g)$	$5 \times 13/32$	65/32	—	—	MODEL NORM.
5	$\sin^2\theta_W$	21/91	3/13	0.23122	0.195%	VERIFIED
6	α_s	$\sqrt{2}/12$	0.11785	0.1179	0.042%	TOPOLOGICAL
7	Q_{Koide}	14/21	2/3	0.666661	0.0009%	VERIFIED
8	m_τ/m_e	7+2480+990	3477	3477.15	0.0043%	VERIFIED
9	m_μ/m_e	27^ϕ	207.01	206.768	0.118%	TOPOLOGICAL
10	m_s/m_d	4×5	20	20.0	0.00%	VERIFIED
11	δ_{CP}	$7 \times 14 + 99$	197°	197°	0.00%	VERIFIED
12	θ_{13}	$\pi/21$	8.57°	8.54°	0.368%	TOPOLOGICAL
13	θ_{23}	$\arcsin((b_3-p_2)/H^*)$	49.25°	49.3°	0.10%	TOPOLOGICAL
14	θ_{12}	$\arctan(\ldots)$	33.40°	33.41°	0.030%	TOPOLOGICAL
15	λ_H	$\sqrt{17}/32$	0.1288	0.129	0.119%	VERIFIED
16	Ω_{DE}	$\ln(2) \times (b_2+b_3)/H^*$	0.6861	0.6847	0.211%	VERIFIED
17	n_s	$\zeta(11)/\zeta(5)$	0.9649	0.9649	0.004%	VERIFIED
18	α^{-1}	128+9+corr	137.033	137.036	0.002%	TOPOLOGICAL
19	Ω_m	$\Omega_{\text{DE}}/\sqrt{w}$	0.3068	0.3153	2.7%	DERIVED

*κ_T is a topological parameter, not a directly measured observable. It appears as a small correction in α^{-1} (relation #18) via $\det(g) \times \kappa_T \approx 0.033$.

36 Deviation Statistics

Range	Count	Percentage
0.00% (exact)	4	22%
< 0.01%	3	17%
0.01–0.1%	4	22%
0.1–0.5%	7	39%

Mean deviation: 0.26% (PDG 2024, 33 observables)

37 Statistical Uniqueness of $(b_2 = 21, b_3 = 77)$

The comprehensive Monte Carlo validation (192,349 configurations tested, zero outperforming GIFT) is presented in the main paper, Section 7. Key results: $E_8 \times E_8$ achieves approximately 10× better agreement than alternatives; G_2 holonomy approximately 5× better than Calabi–Yau; significance $> 4.5\sigma$.

Formula-level selection analysis (main paper Section 7.5) shows GIFT formulas rank first or near-first among all bounded-grammar alternatives, with joint null-model $p < 1.5 \times 10^{-5}$.

Part IX: Observable Catalog

38 Structural Redundancy and Expression Counts

Each prediction admits multiple algebraically distinct representations that reduce to the same fraction. This multiplicity provides a measure of structural robustness: quantities arising from many paths through the topological invariants are less likely to represent numerical coincidence.

38.1 Classification Scheme

Classification	Expressions	Interpretation
CANONICAL	≥ 20	Maximally over-determined; emerges from algebraic web
ROBUST	10–19	Highly constrained; multiple independent derivations
SUPPORTED	5–9	Structural redundancy
DERIVED	2–4	Dual derivation minimum
SINGULAR	1	Unique path (possible coincidence)

38.2 Core 18 Predictions with Expression Counts

#	Observable	Formula	Value	Exp.	Dev.	Expr.	Class
1	N_{gen}	Atiyah–Singer	3	3	0.00%	24+	CANONICAL
2	$\sin^2\theta_W$	$b_2/(b_3+\dim_{\mathrm{G}_2})$	3/13	0.2312	0.20%	14	ROBUST
3	$\alpha_s(M_Z)$	$\sqrt{2}/12$	0.1179	0.1179	0.04%	9	SUPPORTED
4	λ_H	$\sqrt{17}/32$	0.1288	0.129	0.12%	4	DERIVED
5	α^{-1}	128+9+corr	137.033	137.036	0.002%	3	DERIVED
6	Q_{Koide}	$\dim_{\mathrm{G}_2}/b_2$	2/3	0.6667	0.001%	20	CANONICAL
7	m_τ/m_e	$7+10\times 248+10\times 99$	3477	3477.2	0.004%	3	DERIVED
8	m_μ/m_e	27^ϕ	207.01	206.77	0.12%	2	DERIVED
9	m_s/m_d	$p_2^2\times w$	20	20.0	0.00%	14	ROBUST
10	m_b/m_t	$1/(2b_2)$	1/42	0.024	0.79%	21	CANONICAL
11	m_u/m_d	$(1+\dim_{\mathrm{E}_6})/PSL_{2,7}$	79/168	0.47	0.05%	1	SINGULAR
12	δ_{CP}	$\dim_{K_7}\times\dim_{\mathrm{G}_2}+H^*$	197°	197°	0.00%	3	DERIVED
13	θ_{13}	π/b_2	8.57°	8.54°	0.37%	3	DERIVED
14	θ_{23}	$\arcsin((b_3-p_2)/H^*)$	49.25°	49.3°	0.10%	2	DERIVED
15	θ_{12}	$\arctan(\sqrt{\delta/\gamma})$	33.40°	33.41°	0.03%	2	DERIVED
16	Ω_{DE}	$\ln(2)\times(b_2+b_3)/H^*$	0.6861	0.6847	0.21%	2	DERIVED
17	n_s	$\zeta(11)/\zeta(5)$	0.9649	0.9649	0.004%	2	DERIVED
18	$\det(g)$	65/32	2.0313	—	—	8	MODEL NORM.

Distribution: 4 CANONICAL (22%), 4 ROBUST (22%), 2 SUPPORTED (11%), 7 DERIVED (39%), 1 SINGULAR (6%).

38.3 Extended Predictions (15)

#	Observable	Formula	Value	Exp.	Dev.	Expr.	Class
19	$\sin^2\theta_{12}^{\text{PMNS}}$	$(1+N_{\text{gen}})/\alpha_{\text{sum}}$	4/13	0.307	0.23%	28	CANONICAL
20	$\sin^2\theta_{23}^{\text{PMNS}}$	$(D_{\text{bulk}}-w)/D_{\text{bulk}}$	6/11	0.546	0.10%	15	ROBUST
21	$\sin^2\theta_{13}^{\text{PMNS}}$	$D_{\text{bulk}}/\dim_{\mathrm{E}_8^2}$	11/496	0.022	0.81%	5	SUPPORTED
22	$\sin^2\theta_{12}^{\text{CKM}}$	7/31	0.2258	0.225	0.36%	16	ROBUST
23	A_{Wolf}	$(w+\dim_{\mathrm{E}_6})/H^*$	83/99	0.836	0.29%	4	DERIVED
24	$\sin^2\theta_{23}^{\text{CKM}}$	$\dim_{K_7}/PSL_{2,7}$	1/24	0.041	1.13%	3	DERIVED
25	m_H/m_t	8/11	0.7273	0.725	0.31%	19	ROBUST
26	m_H/m_W	81/52	1.5577	1.558	0.02%	1	SINGULAR
27	m_W/m_Z	$(2b_2-w)/(2b_2)=37/42$	0.8810	0.8815	**0.06%**	8	SUPPORTED
28	m_μ/m_τ	5/84	0.0595	0.0595	0.04%	9	SUPPORTED
29	$\Omega_{\text{DM}}/\Omega_b$	$(1+42)/\text{rank}_{\mathrm{E}_8}$	43/8	5.375	0.00%	6	SUPPORTED
30	Ω_b/Ω_m	$w/\det(g)_{\text{den}}$	5/32	0.156	0.16%	7	SUPPORTED
31	Ω_Λ/Ω_m	$(\det_{g,\text{den}}-\dim_{K_7})/D_{\text{bulk}}$	25/11	2.27	0.12%	6	SUPPORTED
32	h	$(PSL_{2,7}-1)/\dim_{\mathrm{E}_8}$	167/248	0.674	0.09%	3	DERIVED
33	σ_8	$(p_2+\det_{g,\text{den}})/(2b_2)$	34/42	0.811	0.18%	4	DERIVED

38.4 Illustrative Examples of Multiple Expressions

$\sin^2\theta_W = 3/13$ (14 algebraically distinct representations):

#	Expression	Evaluation
1	$N_{\text{gen}}/\alpha_{\text{sum}}$	$3/13$
2	$N_{\text{gen}}/(p_2 + D_{\text{bulk}})$	$3/(2+11) = 3/13$
3	$b_2/(b_3 + \dim_{\mathrm{G}_2})$	$21/91 = 3/13$
4	$\dim(J_3\mathbb{O})/(\dim_{F_4} + \det(g)_{\text{num}})$	$27/117 = 3/13$
5	$(b_0 + \dim_{\mathrm{G}_2})/\det(g)_{\text{num}}$	$15/65 = 3/13$
6	$(p_2 + b_0)/\alpha_{\text{sum}}$	$3/13$
7	$\dim_{K_7}/(b_2 + \dim_{K_7} + \dim_{\mathrm{G}_2})$	$7/42 \neq 3/13$ ×

(Expression 7 illustrates that not all combinations work; only those reducing to 3/13 are valid.)

$Q_{\text{Koide}} = 2/3$ (20 algebraically distinct representations):

#	Expression	Evaluation
1	p_2/N_{gen}	$2/3$
2	$\dim_{\mathrm{G}_2}/b_2$	$14/21 = 2/3$
3	$\dim_{F_4}/\dim_{\mathrm{E}_6}$	$52/78 = 2/3$
4	$\text{rank}_{\mathrm{E}_8}/(w + \dim_{K_7})$	$8/12 = 2/3$
5	$(\dim_{\mathrm{G}_2} - \text{rank}_{\mathrm{E}_8})/(\text{rank}_{\mathrm{E}_8} + 1)$	$6/9 = 2/3$

$m_b/m_t = 1/42$ (21 algebraically distinct representations):

#	Expression	Evaluation
1	$b_0/(2b_2)$	$1/42$
2	$(b_0 + N_{\text{gen}})/PSL(2{,}7)$	$4/168 = 1/42$
3	$p_2/(\dim_{K_7} + b_3)$	$2/84 = 1/42$
4	$N_{\text{gen}}/(\dim(J_3\mathbb{O}) + H^*)$	$3/126 = 1/42$
5	$\dim_{K_7}/(\dim_{\mathrm{E}_8} + \dim(J_3\mathbb{O}) + \dim_{K_7})$	$7/294 = 1/42$

The ratio $m_b/m_t = 1/42 = 1/(2b_2)$ illustrates structural redundancy: the bottom-to-top mass hierarchy equals the inverse of the structural constant $2b_2 = p_2 \times b_2$.

Note: The true Euler characteristic $\chi(K_7) = 0$ for G_2 manifolds (odd-dimensional). The constant 42 is the structural invariant $2b_2$.

38.5 The Algebraic Web

The topological constants satisfy interconnected identities:

Identity	Left side	Right side
Fiber-holonomy	$\dim(\mathrm{G}_2) = 14$	$p_2 \times \dim(K_7) = 2 \times 7$
Gauge moduli	$b_2 = 21$	$N_{\text{gen}} \times \dim(K_7) = 3 \times 7$
Matter-holonomy	$b_3 + \dim(\mathrm{G}_2) = 91$	$\dim(K_7) \times \alpha_{\text{sum}} = 7 \times 13$
Fano order	$PSL(2{,}7) = 168$	$\text{rank}(\mathrm{E}_8) \times b_2 = 8 \times 21$
Fano order	$PSL(2{,}7) = 168$	$N_{\text{gen}} \times \text{fund}(\mathrm{E}_7) = 3 \times 56$
Anomaly sum	$\alpha_{\text{sum}} = 13$	$\text{rank}(\mathrm{E}_8) + w = 8 + 5$

These relations form a closed algebraic system. The mod-7 structure ($\dim(K_7) = 7$ divides $\dim(\mathrm{G}_2)$, b_2, b_3, $PSL(2{,}7)$) reflects the Fano plane underlying octonion multiplication.

38.6 Fibonacci–Lucas Embedding

The GIFT constants embed naturally into the Fibonacci (F_n) and Lucas (L_n) sequences:

n	F_n	GIFT Constant	Role
3	2	p_2	Dimensional ratio
4	3	N_{gen}	Fermion generations
5	5	w	Pentagonal symmetry
6	8	rank(E_8)	Cartan subalgebra
7	13	α_B^2 sum	Structure coefficient
8	21	b_2	Second Betti number

This sequence propagates via the recurrence:

$$F_3 + F_4 = F_5 \quad \Rightarrow \quad p_2 + N_{\text{gen}} = w$$

Lucas numbers also appear naturally:

L_n	Value	GIFT Role
L_4	7	dim(K_7)
L_5	11	D_{bulk}
L_8	47	Scale bridge exponent

The Lucas identity $L_8 = F_7 + F_9 = 13 + 34$ decomposes as:

$$L_8 = \alpha^B_{\text{sum}} + d_{\text{hidden}} = 13 + 34 = 47$$

This structure reflects the icosahedral geometry underlying the McKay correspondence $E_8 \leftrightarrow 2I$, where icosahedral coordinates involve the golden ratio $\phi = \lim(F_{n+1}/F_n)$.

Status: EXPLORATORY (mathematical fact; physical significance unclear)

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Related Works

- GIFT Framework, *Geometric Information Field Theory* (main paper)
- GIFT Framework, *Supplement S1: Mathematical Foundations*
- GIFT Framework, *Numerical* G_2 *Metric Construction via Physics-Informed Neural Networks* (companion numerical paper)