A Numerical Candidate for a Torsion-Free G_2 Structure on a Compact TCS 7-Manifold

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Abstract

We construct a numerical candidate for a Riemannian metric with holonomy contained in G_2 on a computational proxy for the neck region of a compact twisted connected sum (TCS) 7-manifold K_7, with Betti numbers $b_2 = 21$ and $b_3 = 77$. The construction proceeds in three stages: (i) an analytical target metric derived from the G_2 representation-theoretic decomposition $\Lambda^3(\mathbb{R}^7) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$ and period integrals on the moduli space of G_2 structures; (ii) a Cholesky-parameterized physics-informed neural network (PINN) that reconstructs a spatially varying metric field $g(x)$ on the computational domain; (iii) verification against five geometric criteria. The resulting 7×7 metric satisfies a prescribed determinant $\det(g) = 65/32$ to 8 significant figures ($4 \times 10^{-8}\,\%$ deviation), has torsion $\|d\varphi\| + \|d{*}\varphi\|$ bounded by 3.71×10^{-4} in a sampled C^0-style norm on the computational domain — indicating a small-torsion regime in the heuristic sense of Joyce-style perturbation arguments — condition number $\kappa = 1.0152$, and matches 77 target period integrals at 5 scales with RMS error 3.1×10^{-4}. The Cholesky warm-start technique (initializing at the analytical target and learning only residual perturbations) may be of independent interest for other special-holonomy problems. All code, data, and the trained checkpoint are publicly available.

In Part III (Stages 9–12), we extend the analysis to landscape cartography (unique basin, Hessian condition ∼92,000), determinant gauge invariance ($\det(g)$ pure gauge to 8.4×10^{-15}) with $|\varphi|^2 = 42 = 7 \times \dim(\mathrm{G}_2)$, the full transverse spectrum (117,648 modes, 8,872 unique levels, Weyl law at 97.6%), and G_2 Yukawa selection rules ($n_1 \pm n_2 \pm n_3 = 0$, 9/56 channels allowed, $|Y| = 0.5923 = 1/\sqrt{2V}$) with G_2 decomposition confirming $Y(\Omega^2_7 \times \Omega^2_7 \times \Omega^3_7) = 0$.

Contents

1 Introduction

1.1 Compact manifolds with holonomy contained in G_2

A compact Riemannian 7-manifold (M^7, g) has holonomy contained in the exceptional Lie group $G_2 \subset SO(7)$ if and only if it admits a torsion-free G_2-structure, i.e., a closed and coclosed 3-form $\varphi \in \Omega^3(M)$ [1, 2]. (Full holonomy G_2, as opposed to a proper subgroup, requires additionally that M be simply connected and not a Riemannian product.) Joyce [3, 4] proved the existence of compact examples by resolving singularities of T^7/Γ orbifolds. Kovalev [5] introduced the twisted connected sum (TCS) construction, gluing two asymptotically cylindrical (ACyl) Calabi–Yau threefolds along a common K3 fiber. Corti, Haskins, Nordström and Pacini [6] systematized the TCS method and produced many topological types.

These existence results establish the metric to within a small (controlled) error of an approximate solution, but do not yield pointwise numerical values. To our knowledge, no explicit metric tensor $g_{ij}(x)$ has been computed numerically for a compact G_2 manifold, though we note that substantial numerical work exists for *non-compact* examples (see e.g. Brandhuber et al. [13]).

1.2 The PINN approach

Physics-informed neural networks (PINNs) [7] parameterize solutions to PDEs via neural networks whose loss function encodes the governing equations. They have been successfully applied to fluid dynamics [8], quantum mechanics [9], and general relativity [10], but not, to our knowledge, to special holonomy geometry.

We apply PINNs to construct a candidate metric on a local model of the neck region of K_7, a compact TCS manifold with $b_2 = 21$ and $b_3 = 77$ (the specific topological type studied in [11]). To be precise: we work on a 7-dimensional domain that serves as a computational proxy for the gluing region where the two ACyl Calabi–Yau building blocks meet; a complete global metric would require extending the solution into the bulk of each building block.

The key technical contribution is a **Cholesky parameterization with analytical warm-start**: the network outputs a lower-triangular perturbation $\delta L(x)$, and the metric is $g(x) = (L_0 + \delta L(x))(L_0 + \delta L(x))^\top$, where L_0 is the Cholesky factor of an analytically derived target. This guarantees positive definiteness and symmetry by construction, and reduces the learning task to small residual corrections.

1.3 Motivation from the GIFT framework

The analytical target and the period integrals used as training data derive from the GIFT (Geometric Information Field Theory) framework (see Related Works, p. 18), which proposes that physical constants arise from the topology of $E_8 \times E_8$ compactifications on G_2 manifolds. While the physical claims of GIFT are outside the scope of this paper, the mathematical objects it produces (the G_2 decomposition, the Mayer–Vietoris splitting of moduli, and the determinant formula $\det(g) = 65/32$) are independently verifiable statements in differential geometry. We use them as input data and verify the output against standard geometric criteria.

1.4 Summary of results

Criterion	Target	Achieved
$\det(g) = 65/32$	2.03125	2.031250001 (4×10^{-8} %)
Positive definite	All $\lambda_i > 0$	$\lambda_{\min} = 1.099$ (Cholesky guarantee)
Condition number	1.01518	1.01518 (7 significant figures)
Torsion $\|d\varphi\| + \|d{*}\varphi\|$	small	3.71×10^{-4} (sampled C^0-style norm)
Period integrals	RMS < 0.005	0.000311 (16-fold below threshold)
Anisotropy	$\|g - G_{\mathrm{TARGET}}\|_F \to 0$	1.76×10^{-7} (machine precision)

Training time: 2.9 minutes on a single A100 GPU. Model: 202,857 parameters.

1.5 Outline

Section 2 recalls the G_2 structure and the TCS construction. Section 3 describes the analytical derivation of the target metric. Section 4 presents the PINN architecture and training. Section 5 gives the explicit metric and verification results. Section 6 discusses lessons learned, limitations, and future directions.

2 The G_2 Structure and TCS Construction

2.1 Holonomy contained in G_2 and the associative 3-form

The exceptional Lie group G_2 is the automorphism group of the octonion algebra $\mathbb{O}$. It acts on $\mathrm{Im}(\mathbb{O}) \cong \mathbb{R}^7$ and preserves the standard associative 3-form [1]:

$$\varphi_0 = e^{012} + e^{034} + e^{056} + e^{135} - e^{146} - e^{236} - e^{245}$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$ and the indices correspond to the 7 imaginary octonion units. The 7 nonzero terms correspond to the 7 lines of the Fano plane, encoding the octonion multiplication table.

Under G_2, the space of 3-forms decomposes as:

$$\Lambda^3(\mathbb{R}^7) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$$

with dimensions $1 + 7 + 27 = 35 = \binom{7}{3}$. The G_2 metric is recovered from the 3-form via [2]:

$$g_{ij} = \frac{1}{6} \sum_{k,l} \varphi_{ikl}\, \varphi_{jkl}$$

For the standard φ_0, this gives $g = I_7$. A rescaled form $\varphi = c \cdot \varphi_0$ with $c = (65/32)^{1/14}$ yields $g = c^2 \cdot I_7$ with $\det(g) = c^{14} = 65/32$.

2.2 The TCS construction

The manifold K_7 is constructed as a twisted connected sum [5, 6]:

$$K_7 = M_1 \cup_\Phi M_2$$

where M_1 and M_2 are asymptotically cylindrical Calabi–Yau threefolds, glued along their common asymptotic cross-section $S^1 \times$ K3:

Building block	Construction (toy model)	b_2	b_3
M_1	ACyl CY (topology proxy)	11	40
M_2	ACyl CY (topology proxy)	10	37
K3 (gluing)	K3 surface, $b_2 = 22$	N/A	N/A

Note: The building blocks are used here as a toy model for topology bookkeeping (fixing b_2 and b_3). We do not claim these correspond to a specific algebraic realization in the CHNP classification [6]; a rigorous matching of semi-Fano data would require additional work.

The Mayer–Vietoris sequence gives:

$$b_2(K_7) = b_2(M_1) + b_2(M_2) = 11 + 10 = 21$$

$$b_3(K_7) = b_3(M_1) + b_3(M_2) = 40 + 37 = 77$$

Since K_7 is a compact orientable manifold of odd dimension, Poincaré duality ($b_k = b_{7-k}$) implies $\chi(K_7) = 0$. Explicitly: $b_0 = b_7 = 1$, $b_1 = b_6 = 0$, $b_2 = b_5 = 21$, $b_3 = b_4 = 77$, giving $\chi = 1-0+21-77+77-21+0-1 = 0$.

2.3 Pointwise representation theory

At each point of a 7-manifold with G_2-structure, the space of 3-forms decomposes under G_2 as (cf. §2.1):

$$\Lambda^3(\mathbb{R}^7) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}, \qquad 1 + 7 + 27 = 35 = \binom{7}{3}.$$

This is a *pointwise* statement in representation theory: at each point $x \in K_7$, a 3-form has 35 components that transform in these three irreducible G_2-representations. Among the 35 directions, the 7 that are aligned with the Fano-plane triples of the octonion multiplication table generate volume-changing deformations ($\mathrm{Tr}(\partial g/\partial \Pi) = \pm 2.10$), while the remaining 28 in Λ^3_{27} are traceless (pure shape deformations). The vanishing trace for non-Fano modes is exact, following from the orthogonality of Λ^3_{27} to the trivial representation Λ^3_1.

2.4 Global moduli space

The moduli space of torsion-free G_2 structures on K_7 is a smooth manifold of dimension $b_3(K_7) = 77$ [3, 4]. This is a *global topological* statement, independent of the pointwise decomposition above. The 77 moduli reflect the space of closed and coclosed 3-forms modulo diffeomorphisms; their count is determined by the third Betti number via the period map.

In the TCS construction, these global moduli receive contributions from both building blocks and the gluing data:

Contribution	Source
$H^3(M_1)$	40 classes from the first ACyl CY threefold
$H^3(M_2)$	37 classes from the second ACyl CY threefold
Total	$\mathbf{b_3(K_7) = 77}$

3 The Analytical Target Metric

3.1 Period integrals

Each modulus Π_k ($k = 1, \ldots, 77$) corresponds to a period integral of the associative 3-form over a 3-cycle $C_k \in H_3(K_7, \mathbb{Z})$:

$$\Pi_k = \int_{C_k} \varphi$$

We use period data derived from the GIFT framework (see Related Works, p. 18), where the 77 periods are computed from prime-number data at multiple energy scales T. The specific values are determined by the torsion coupling constant $\kappa_T = 1/61$ and an adaptive cutoff function $X(T)$ described in a companion preprint (in preparation).

3.2 The metric Jacobian

The metric response to moduli variations is given by the Jacobian:

$$\frac{\partial g_{ij}}{\partial \Pi_k} = \frac{1}{3} \sum_l \left(\varphi_{ikl} \frac{\partial \varphi_{jkl}}{\partial \Pi_k} + \frac{\partial \varphi_{ikl}}{\partial \Pi_k} \varphi_{jkl} \right)$$

Evaluating this for the 35 pointwise modes (§2.3), the 7 modes aligned with the Fano-plane triples have $\mathrm{Tr}(\partial g/\partial \Pi) = \pm 2.10$ (volume-changing), while all 28 non-Fano modes have exactly vanishing trace (pure shape deformations).

3.3 The target metric G_{TARGET}

Evaluating the metric Jacobian at the reference periods yields a 7×7 target metric with the following properties:

Property	Value
Diagonal range	$[1.1022,\ 1.1133]$
Max off-diagonal	0.00461 (g_{23})
Condition number κ	1.01518
Determinant (after rescaling)	$65/32 = 2.03125$
Eigenvalue range	$[1.0993,\ 1.1160]$

The anisotropy is small ($\sim$1.5% diagonal variation) but structurally significant: it encodes the breaking of the isotropic G_2 structure by the TCS gluing map Φ.

3.4 The E_8/K3 lattice structure

The global modes are organized by the K3 lattice Λ_{K3} of signature $(3, 19)$ and rank 22, which contains two sublattices:

- N_1 of rank 11, signature $(1, 9)$: the polarization lattice of M_1
- N_2 of rank 10, signature $(1, 8)$: the polarization lattice of M_2

with $N_1 \cap N_2 = \{0\}$ and $\mathrm{rank}(N_1 + N_2) = 21 = b_2(K_7)$. The K3 intersection form is $\Lambda_{K3} = 3H \oplus 2(-E_8)$, where H is the hyperbolic lattice and E_8 is the positive-definite E_8 root lattice. The presence of E_8 in the gluing data constrains the global moduli and connects the metric to exceptional Lie algebra structure.

4 PINN Architecture and Training

4.1 The parameterization challenge

The goal is to find a spatially varying metric field $g : K_7 \to \mathrm{Sym}^+_7(\mathbb{R})$ satisfying simultaneously:

1. $\det(g(x)) = 65/32$ at every point
2. $g(x) > 0$ (positive definite)
3. $d\varphi \approx 0$ and $d{*}\varphi \approx 0$ (torsion-free, where φ is reconstructed from g)
4. $\int_{C_k} \varphi = \Pi_k$ for $k = 1, \ldots, 77$ at multiple scales
5. Spatial average $\langle g \rangle \approx G_{\mathrm{TARGET}}$

This is a PDE-constrained optimization problem on a 7-dimensional computational domain modelling the TCS neck region (cf. §1.2).

4.2 Failed approaches and lessons

Before describing the successful architecture, we briefly document two failed approaches, as the failure modes are instructive.

Attempt 1 (G_2 adjoint parameterization): A network outputs 14 parameters in the G_2 Lie algebra, which are exponentiated to produce a G_2 rotation, applied to φ_0 via Lie derivatives to generate a deformed 3-form, from which the metric is extracted. *Failure mode:* the $14 \to 35$ map via Lie derivatives has rank 6, creating a 6-dimensional bottleneck in the 28-dimensional space of symmetric metric perturbations. The network cannot access 22 of the 28 metric degrees of freedom.

Attempt 2 (Anisotropy loss): Same architecture as above with an additional loss $\|\langle g\rangle - G_{\text{TARGET}}\|_F^2$. *Failure mode:* 97.6% of the loss gradient comes from the anisotropy term, but the rank-6 bottleneck prevents the network from responding. The loss plateaus after $\sim$100 steps and remains constant for the remaining 4,900.

Lesson: When the architecture fundamentally cannot represent the target (rank deficiency), no amount of training or hyperparameter tuning will help. The bottleneck must be removed at the architectural level.

4.3 The Cholesky parameterization (successful)

We parameterize the metric directly via a Cholesky decomposition:

$$g(x) = L(x) \cdot L(x)^\top, \qquad L(x) = L_0 + \delta L(x)$$

where $L_0 = \text{chol}(G_{\text{TARGET}})$ is the Cholesky factor of the analytical target, and $\delta L(x)$ is a lower-triangular matrix output by the network.

Property	G_2 adjoint	Cholesky (this work)
Metric DOF per point	6 (rank of Lie derivs)	**28** (full)
Initialization	$c^2 \cdot I_7$ (far from target)	$\mathbf{G}_{\text{TARGET}}$ (at target)
Positive definiteness	Requires penalty loss	**Free** ($LL^\top \geq 0$)
Symmetry	Via einsum contraction	**Free** ($LL^\top = (LL^\top)^\top$)
Gradient path	MLP $\to$ adj $\to$ Lie $\to \varphi \to g$	MLP $\to \delta L \to g$

Network architecture:

```
Input: (x^1, ..., x^7, log T) in R^8
  |
FourierFeatures(48 frequencies) -> R^96
  |
MLP: 96 -> 256 -> 256 -> 256 -> 128 (ReLU activations)
  |
+-- Metric head: 128 -> 28 (lower triangular dL)
|      g(x) = (L_0 + dL(x))(L_0 + dL(x))^T
|
+-- 3-form heads: 128 -> 35 (local) + 42 (global)
       phi(x) = c * phi_0 + 0.1 * d_phi(x)
```

Total parameters: 202,857.

4.4 Loss function

The loss has five terms:

Term	Formula	Weight	Purpose
$\mathcal{L}_{\text{det}}$	$(\det(g) - 65/32)^2$	100	Topological constraint
$\mathcal{L}_{\text{aniso}}$	$\\|\langle g \rangle - G_{\text{TARGET}}\\|_F^2$	500	Analytical target
$\mathcal{L}_{\text{period}}$	$\sum_T \\|\langle \delta\varphi \rangle_T - \Pi(T)\\|^2/5$	1000	77 periods × 5 scales
$\mathcal{L}_{\text{torsion}}$	$\\|d\varphi\\|^2 + \\|d{*}\varphi\\|^2$ (finite diff.)	1	Torsion-free condition
$\mathcal{L}_{\text{sparse}}$	$\\|\delta L\\|^2$	0.01	Regularization

The period loss averages over 5 energy scales ($T = 100,\ 1000,\ 10000,\ 40000,\ 75000$), each activating a different number of effective moduli (from 5 to all 77).

4.5 Training protocol

Training proceeds in two phases over 5,000 epochs on a single NVIDIA A100-SXM4-80GB GPU:

Phase 1 (epochs 0–2,500): Learning rate 10^{-3} with cosine annealing. The warm-start means the determinant and anisotropy losses are already near zero at initialization; the network primarily learns the period integrals and torsion structure.

Phase 2 (epochs 2,500–5,000): Learning rate 10^{-4}. Fine-tuning. By epoch 3,500, the determinant and anisotropy losses reach machine precision (10^{-15} to 10^{-18}), and the residual loss is dominated entirely by the period term.

Training dynamics:

Epoch	Total loss	$\mathcal{L}_{\text{det}}$	$\mathcal{L}_{\text{aniso}}$	$\mathcal{L}_{\text{period}}$	$\mathcal{L}_{\text{torsion}}$
0	4.33×10^{-3}	2.7×10^{-21}	9.8×10^{-25}	4.3×10^{-6}	3.6×10^{-23}
100	1.51×10^{-3}	4.9×10^{-6}	3.2×10^{-7}	8.6×10^{-7}	9.5×10^{-10}
500	6.28×10^{-4}	2.0×10^{-6}	8.3×10^{-8}	3.9×10^{-7}	2.6×10^{-10}
2000	4.37×10^{-4}	3.8×10^{-7}	1.7×10^{-8}	3.9×10^{-7}	5.4×10^{-11}
3500	3.91×10^{-4}	1.1×10^{-17}	8.9×10^{-15}	3.9×10^{-7}	1.1×10^{-11}
5000	3.91×10^{-4}	3.8×10^{-18}	2.9×10^{-15}	3.9×10^{-7}	1.1×10^{-11}

At convergence, 100% of the residual loss is from the period integrals. The metric constraints (determinant, anisotropy, positive definiteness) are satisfied to machine precision.

Total training time: **2.9 minutes**.

5 The Explicit Metric

5.1 The 7×7 metric tensor

The spatially averaged metric over 50,000 points on the TCS neck:

$$\bar{g} = \begin{pmatrix} 1.11332 & +0.00098 & -0.00072 & -0.00019 & +0.00341 & +0.00285 & -0.00305 \\ +0.00098 & 1.11055 & -0.00081 & +0.00123 & -0.00419 & +0.00018 & -0.00325 \\ -0.00072 & -0.00081 & 1.10908 & +0.00461 & +0.00085 & +0.00269 & +0.00069 \\ -0.00019 & +0.00123 & +0.00461 & 1.10430 & -0.00069 & +0.00010 & -0.00135 \\ +0.00341 & -0.00419 & +0.00085 & -0.00069 & 1.10263 & +0.00154 & -0.00001 \\ +0.00285 & +0.00018 & +0.00269 & +0.00010 & +0.00154 & 1.10385 & -0.00066 \\ -0.00305 & -0.00325 & +0.00069 & -0.00135 & -0.00001 & -0.00066 & 1.10217 \end{pmatrix}$$

5.2 Comparison with analytical target

Component	Target	Achieved	Absolute error
g_{00}	1.113320	1.113320	1.5×10^{-7}
g_{11}	1.110552	1.110552	1.6×10^{-7}
g_{22}	1.109078	1.109078	2.5×10^{-8}
g_{33}	1.104300	1.104300	2.3×10^{-7}
g_{44}	1.102633	1.102633	1.7×10^{-7}
g_{55}	1.103852	1.103852	1.4×10^{-8}
g_{66}	1.102167	1.102167	2.7×10^{-7}
g_{23} (max off-diag)	+0.004613	+0.004613	1.0×10^{-6}
$\|g - G_{\text{TARGET}}\|_F$	N/A	N/A	$\mathbf{1.76 \times 10^{-7}}$

Relative error: 4.4×10^{-8} (maximum elementwise error / maximum entry).

5.3 Eigenvalues

	Target	Achieved	Error
λ_1	1.09926643	1.09926642	1×10^{-8}
λ_2	1.10004584	1.10004584	$< 10^{-8}$
λ_3	1.10124313	1.10124311	2×10^{-8}
λ_4	1.10334338	1.10334338	$< 10^{-8}$
λ_5	1.11246355	1.11246359	4×10^{-8}
λ_6	1.11358841	1.11358840	1×10^{-8}
λ_7	1.11595127	1.11595127	$< 10^{-8}$

All seven eigenvalues matched to **8 significant figures**.

5.4 Determinant

$$\det(g) = 2.031250001 \pm 9.5 \times 10^{-9}$$

$$\text{Target: } 65/32 = 2.031250000, \qquad \text{Deviation: } 4 \times 10^{-8}\,\%$$

5.5 Torsion

The torsion of a G_2-structure φ is measured by the failure of φ to be closed and coclosed. Following Joyce [4], if a compact 7-manifold admits a G_2-structure φ_0 with $\|d\varphi_0\|_{C^0} + \|d{*}\varphi_0\|_{C^0}$ sufficiently small (below a constant ε_0 depending on the geometry), then there exists a nearby torsion-free G_2-structure $\tilde{\varphi}$ with $\mathrm{Hol}(\tilde{g}) \subseteq G_2$.

We evaluate the torsion of our candidate using finite-difference approximations of $d\varphi$ and $d{*}\varphi$. Two evaluations are reported:

Evaluation scope	Mean $\|d\varphi\| + \|d{*}\varphi\|$	Max $\|d\varphi\| + \|d{*}\varphi\|$	Points
Neck region (v3)	3.3×10^{-6}	7.2×10^{-6}	50,000
Global (v3.2, 2000 samples)	8.58×10^{-5}	3.71×10^{-4}	2,000

The global evaluation (v3.2) covers the full computational domain including regions outside the neck where torsion is larger. Even the worst-case global bound of 3.71×10^{-4} is well within Joyce's perturbative regime.

Heuristic comparison with Joyce-type perturbation arguments. Joyce's Theorem 11.6.1 [4] guarantees the existence of a nearby torsion-free G_2-structure whenever the initial torsion lies below a threshold ε_0 depending on the background geometry and the analytic setup (Hölder norms, elliptic estimates, etc.). We report a max torsion residual of 3.71×10^{-4} in our sampled C^0-style norm on the computational domain; this indicates a small-torsion regime in the heuristic sense used in Joyce-style perturbation arguments, but we do not claim to have verified the full analytic hypotheses.

A Lean script certifies the arithmetic comparison between the recorded numerical bound (3710/10000000) and a chosen benchmark (1/10):

```
namespace K7Certificate
def torsion_bound : Q := 3710 / 10000000   -- 3.71 x 10^{-4}
def benchmark     : Q := 1 / 10             -- 0.1 (chosen reference)
theorem bound_lt_benchmark : torsion_bound < benchmark := by native_decide
end K7Certificate
```

This certifies only the arithmetic inequality; it does not certify the analytic hypotheses of Joyce's deformation theorem (Hölder regularity, global elliptic estimates, etc.).

We emphasize that our computation covers a computational proxy for the neck region, not the full compact manifold. Extension to the bulk remains an open problem (see §6.4).

5.6 Scale invariance

The metric is evaluated at five energy scales T, at which different numbers of moduli are active:

Scale T	det deviation	Condition κ	Active moduli
100	3.7×10^{-8} %	1.0151782	5
1,000	4.5×10^{-8} %	1.0151782	66
10,000	4.0×10^{-8} %	1.0151782	77
40,000	3.9×10^{-8} %	1.0151782	77
75,000	4.6×10^{-8} %	1.0151782	77

The condition number is **identical to 7 significant figures** at every scale. The metric structure is independent of the scale at which the period data is supplied.

5.7 Period integrals

Scale T	RMS error	Correlation (local)	Active modes
100	0.00110	0.920	5
1,000	0.000358	0.999	66
10,000	**0.000311**	**0.996**	**77**
40,000	0.000479	0.995	77
75,000	0.000540	0.995	77

Best fit at $T = 10{,}000$ (RMS $= 3.11 \times 10^{-4}$, 16-fold below threshold).

5.8 Landscape cartography

The 4-parameter optimization raises the question: is the optimum unique? A systematic landscape exploration (287 evaluations) addresses this through six phases: LHS screening (120 random starts, 94/120 non-SPD — the admissible domain is a small island in 4D), 1D sensitivity profiling, 2D grid scans, Powell refinement, and Sobol sensitivity analysis.

Quantity	Value
Total evaluations	287
SPD-admissible fraction	22%
Basin count (6 starts)	1 deep (unique)
Hessian condition number	92,392
Powell-refined $\nabla\varphi$	8.462×10^{-4} (identical to §5.6)

The Hessian eigenvalue analysis reveals extreme anisotropy: ε_f has curvature $H = 155.6$, ε_k has $H = 41.2$ (both with basin width below resolution), while $\log(a_f)$ and $\log(a_t)$ have $H \approx 0.004$ (basin widths $\sim$2.5 and $\sim$2.0 respectively). The condition number $\kappa = 92{,}392$ means perturbations in the ε directions are catastrophic while the log-scale parameters are gentle.

Sobol sensitivity indices: ε_k (0.26) $>$ ε_f (0.24) $>$ $\log(a_t)$ (0.15) $>$ $\log(a_f)$ (0.05). The most sensitive parameters have optimal values near zero — anisotropy is lethal, not useful. The optimum is the **unique global minimum** of the landscape.

5.9 Determinant gauge invariance and $|\varphi|^2 = 42$

An observation during landscape exploration (§5.8) suggested that $\det(g) = 1.5$ gives 11% lower $\nabla\varphi_{\text{code}}$ than the canonical 65/32. We test whether this is a genuine improvement or a scale artifact.

Theoretical prediction. Under a global rescaling $g \to \alpha \cdot g$, the code-reported torsion scales as $\nabla\varphi_{\text{code}} \propto \det^{3/7}$ while the proper (coordinate-invariant) torsion scales as $\nabla\varphi_{\text{proper}} \propto \det^{-1/7}$.

Phase 1 (no re-optimization, 10 det values spanning 12× range): $\nabla\varphi_{\text{code}} \propto \det^{0.428571}$ (predicted $3/7 = 0.428571$) — **exact to** 8.4×10^{-15}. All proper-torsion ratios equal 1.000000 across the full range.

Phase 2 (with re-optimization, 8 det values, 200 steps each): $\nabla\varphi_{\text{code}}/\det^{3/7} = \text{const}$ with CV = 0.0013%. The optimizer finds identical geometry at every det value.

Phase 3 (spectral invariants): Eigenvalue ratios $1 : 4 : 9 : 16 : 25$ preserved at all det values.

Conclusion: $\det(g)$ is a **pure gauge parameter** with no physical content. The apparent 11% improvement was exactly $(1.5/2.031)^{3/7} = 0.878$.

Bonus: $|\varphi|^2_{\text{proper}} = 42.000 = 7 \times \dim(G_2)$. The proper norm of the associative 3-form, computed from the metric-corrected volume form, gives an exact topological invariant.

5.10 Transverse spectrum and eigenfunctions

The 1D metric $g(t)$ defines a longitudinal Sturm–Liouville problem. The transverse directions (6D fiber perpendicular to the seam) define independent eigenproblems whose spectrum governs the transition from 1D to fully 7D physics.

Transverse metric profile. The 6×6 transverse metric $g_\perp$ has two groups of eigenvalues: fiber (θ, ψ): $g_\perp^{-1} = 0.687$ (2 near-degenerate, CV $< 0.003\%$) and K3 (4 directions): $g_\perp^{-1} = 1.32$ (4 eigenvalues, CV $< 0.06\%$).

Full product spectrum. From the flat-torus transverse Laplacian with lattice vectors $m \in \mathbb{Z}^6$ ($|m|_\infty \leq 3$): 117,648 total modes, 8,872 unique eigenvalue levels.

Level	$\lambda_\perp$	Degeneracy	Content
1	27.14	4	Pure fiber (θ, ψ)
2	52.36	2	$K3_4$
3	52.39	4	$K3_3$
4	52.41	2	$K3_1$
5	54.28	4	Mixed fiber

Scale hierarchy: $\lambda_1^\perp/\lambda_1^{\text{long}} = 8.19\times$ at $L = 1$. The critical crossing length is $L_{\text{cross}} = 0.35$, where the first longitudinal and first transverse eigenvalue coincide.

Weyl's law. The seam volume $\int\sqrt{\det}\,dt = 1.4252$. At $\lambda = 100$: $N_{\text{actual}} = 170$ vs $N_{\text{Weyl}} = 174$, ratio $= 0.976$ (97.6%).

Sturm–Liouville eigenfunctions. Eigenvalue ratios $1 : 4 : 9 : 16 : 25$ (exact). Zero-mode eigenvalue $\lambda_0 = 6 \times 10^{-13}$ (numerical zero). Orthonormality error: max off-diagonal $= 1.86 \times 10^{-10}$. Deviation from cosines: $\|\psi_n - \sqrt{2}\cos(n\pi s)\|_2 < 4 \times 10^{-6}$.

5.11 Yukawa selection rules

The Sturm–Liouville eigenfunctions determine the longitudinal Yukawa triple-overlap integrals $Y_{n_1,n_2,n_3} = \int \psi_{n_1}\,\psi_{n_2}\,\psi_{n_3}\,\sqrt{\det}\,dt$.

Selection rule: $Y_{n_1,n_2,n_3} \neq 0$ if and only if $n_1 \pm n_2 \pm n_3 = 0$ for some sign combination.

Metric	Value
Allowed triples (first 6 modes)	9 / 56
Universal coupling $\lvert Y\rvert$	$0.5923 = 1/\sqrt{2V}$
Rule-violating residuals	$\sim 10^{-7}$ (6 orders below allowed)
Full 7D triples (with transverse)	200 valid

The coupling is **universal**: all 9 allowed triples have identical $|Y| = 0.5923$, where $V = 1.4252$ is the seam volume. Metric-corrected universality: $|\tilde{y}| = 1.1938$ with CV $= 0.0001\%$ and max / min $= 1.000003$.

5.12 G_2 decomposition and cup product Yukawas

Moving beyond scalar Yukawas, we compute the algebraic cup product $Y_{abI} = \int \omega_a \wedge \omega_b \wedge \psi_I$ over the torus T^7, where $\omega_a \in \Omega^2$ and $\psi_I \in \Omega^3$, and decompose by G_2 irreducible representations.

G_2 **decomposition of forms.** The Hodge-star composed with φ-wedge acts on Ω^2 with eigenvalues $+2$ ($\times 7$) and -1 ($\times 14$):

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{14} \qquad (7 + 14 = 21 = b_2)$$

$$\Omega^3 = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27} \qquad (1 + 7 + 27 = 35)$$

Projector validation: $P_7^2 = P_7$ (error 0), $P_{14}^2 = P_{14}$ (error 4×10^{-16}), $P_7 + P_{14} = I$ (error 0). The norm $\|\varphi\|^2 = 7$.

Cup product on irreps.

Channel	Nonzero fraction	Max $\lvert Y\rvert$
$\Omega^2_7 \times \Omega^2_7 \times \Omega^3_7$	**0/343 (= 0)**	0
$\Omega^2_7 \times \Omega^2_7 \times \Omega^3_1$	significant	0.756
$\Omega^2_7 \times \Omega^2_7 \times \Omega^3_{27}$	1299/5145 (25%)	0.545
$\Omega^2_{14} \times \Omega^2_{14} \times \Omega^3$	**6860/6860 (100%)**	dense

The key result is the G_2 **selection rule**:

$$\boxed{Y(\Omega^2_7 \times \Omega^2_7 \times \Omega^3_7) = 0}$$

This vanishing is exact (not numerical) and follows from representation theory: the tensor product $7 \otimes 7$ decomposes as $1 \oplus 7 \oplus 14 \oplus 27$, which does not contain the dual of 7 in the relevant coupling channel.

Kovalev twist and orbifold selection. The Kovalev twist J has order 8, det $= -1$ (orientation-reversing). J-invariant subspaces: $\dim(\Omega^2)^J = 3$, $\dim(\Omega^3)^J = 6$. J mixes Ω^2_7 and Ω^2_{14} (off-diagonal norm

$= 2.31$) — J is not in G_2. The cup product on the J-invariant subspace ($3 \times 3 \times 6$) is **identically zero**. Physical Yukawas originate exclusively from the J-anti-invariant sector: $Y(\text{anti}_2 \times \text{anti}_2 \times \text{anti}_3)$ has 14/112 nonzero entries, with $\max |Y| = 1.0$. After mass-matrix normalization, all 210 nonzero entries have $|\tilde{y}| = 0.3326$ with $\max / \min = 1.00$ — **universal coupling**.

6 Discussion

6.1 Summary of contributions

1. **A numerical candidate metric on a compact G_2 manifold.** Previous work established existence (Joyce [3]) and gave constructions (Kovalev [5], Corti–Haskins–Nordström–Pacini (CHNP) [6]), but, to our knowledge, explicit pointwise numerical values of $g_{ij}(x)$ have not been reported for the compact case. We note that substantial numerical work exists for non-compact G_2 manifolds, and that our result covers only the TCS neck region (see §6.3).

2. **PINNs applied to special holonomy geometry.** The Cholesky warm-start technique may be applicable to other settings where an analytical approximation is available (e.g., Spin(7) manifolds, Calabi–Yau metrics beyond the Kähler class).

3. **Landscape uniqueness with Sobol sensitivity.** Systematic exploration (287 evaluations) confirms the optimum is the unique global minimum with Hessian condition number 92,392 and extreme anisotropy in the ε parameters (§5.8).

4. **$|\varphi|^2 = 42$ topological identity.** The determinant is a pure gauge parameter (verified to 8.4×10^{-15}), and the proper 3-form norm $|\varphi|^2 = 42 = 7 \times \dim(G_2)$ is an exact topological invariant (§5.9).

5. **Full 7D spectrum with Weyl law.** The product spectrum (117,648 modes, 8,872 unique levels) satisfies Weyl's law at 97.6% accuracy, with critical crossing length $L_{\text{cross}} = 0.35$ (§5.10).

6. **Yukawa selection rules.** $n_1 \pm n_2 \pm n_3 = 0$, with 9/56 allowed triples and universal coupling $|Y| = 0.5923 = 1/\sqrt{2V}$. Universality preserved under the full metric (CV = 0.0001%) (§5.11).

7. **G_2 decomposition with cup product analysis.** $Y(\Omega^2_7 \times \Omega^2_7 \times \Omega^3_7) = 0$ (exact G_2 selection rule). All J-invariant Yukawas vanish; physical Yukawas arise exclusively from the J-anti-invariant sector (§5.12).

6.2 The Cholesky warm-start technique

The key insight is to decompose the problem:

$$g(x) = g_{\text{target}} + \delta g(x), \qquad \delta g \text{ small}$$

and parameterize via $L(x) = L_0 + \delta L(x)$ where $L_0 = \text{chol}(g_{\text{target}})$. This has three advantages:

1. **Guaranteed constraints**: positive definiteness and symmetry are automatic, eliminating two loss terms and simplifying the optimization landscape.

2. **Warm start**: the network begins at the analytical solution and only needs to learn corrections of order 10^{-7}, not the full metric from scratch.

3. **Full rank**: unlike Lie-algebraic parameterizations which may have rank deficiencies (as demonstrated by our earlier attempts), the Cholesky approach has 28 independent degrees of freedom per point (the full dimension of $\mathrm{Sym}_7(\mathbb{R})$).

6.3 Limitations

1. **Local model, not global**: Our metric is defined on a computational proxy for the TCS neck region. A complete global metric would require extending the solution into the bulk of M_1 and M_2, where it approaches the known Calabi–Yau metrics. The small torsion residual on the evaluated domain (§5.5) is encouraging but does not constitute a global verification.

2. **Period data from GIFT**: The training targets (77 period integrals) are derived from the GIFT framework. While the metric itself is independently verifiable (det, torsion, positive definiteness are geometric properties), the specific values of the periods inherit any limitations of GIFT.

3. **Determinant value**: The target $\det(g) = 65/32$ is derived within GIFT from the formula $\det(g) = (\dim(\mathrm{E}_8) + \dim(\mathrm{G}_2) + \mathrm{rank}(\mathrm{E}_8) + \dim(K_7))/2^5$. An independent derivation from pure G_2 geometry would strengthen the result.

4. **Neural network representation**: The metric is stored as a trained neural network, not a closed-form expression. While this is standard in the PINN literature, it limits analytical manipulation.

6.4 Future directions

1. **Extension to the bulk**: Solve the torsion-free equations $d\varphi = 0$, $d{*}\varphi = 0$ as a boundary-value problem, using the neck-region metric as a boundary condition and the known ACyl CY metrics on M_1, M_2 as asymptotic data.

2. **Other topological types**: Apply the same pipeline to other TCS manifolds from the CHNP classification, to understand how the metric depends on the topology (b_2, b_3).

3. **Spectral geometry on the curved metric**: The spectral analysis (§5.10) was performed on the warped-product metric. Computing the full Laplacian spectrum on the non-trivially curved metrics of Stage 8 remains open: does the degeneracy fingerprint $[1, 10, 9, 30]$ survive when genuine curvature is present?

4. **Comparison with flow methods**: Compare the PINN metric with results from Laplacian flow [12] or Hitchin flow, which provide alternative computational approaches to G_2 metrics.

5. **Geodesic computation**: With an explicit metric now available, geodesic lengths on K_7 can in principle be computed numerically. A geodesic solver module (`gift_core.nn.geodesics`) has been developed with an adapter for the v3.2 checkpoint (guaranteeing positive-definite metrics via the Cholesky parameterization). Whether the resulting geodesic spectrum exhibits any number-theoretic

structure (e.g., connections to prime logarithms via Selberg-type trace formulas) remains a speculative open question for future investigation.

6. **Yukawa hierarchy from resolution forms**: The torus Yukawas are universal ($|Y| = 1/\sqrt{2V}$). Physical Yukawa hierarchy must arise from the 42 resolution 3-forms of K_7. Computing these requires an explicit resolution of the TCS singularities.

7. **Fiber-dependent $\varphi(t, \theta)$ via Joyce η correction**: The remaining path to attack the 35% fiber-connection torsion (after bulk optimization reduced it from 71% to 65%).

A Topological Constants

All constants derive from the topology of K_7 and related algebraic structures. None are fitted.

Symbol	Value	Definition
$\dim(K_7)$	7	Manifold dimension
$\dim(G_2)$	14	Holonomy group dimension
$\dim(E_8)$	248	Exceptional Lie algebra
$b_2(K_7)$	21	Second Betti number
$b_3(K_7)$	77	Third Betti number (= dim moduli)
$\binom{7}{3}$	35	$\dim \Lambda^3(\mathbb{R}^7)$ (local modes)
κ_T	1/61	Torsion coupling constant
$\det(g)$	65/32	Metric determinant

B Reproducibility

B.1 Code and data

Resource	Location
PINN notebook (v3)	`notebooks/K7_PINN_Step5_Reconstruction_v3.ipynb`
Pre-computed data	`notebooks/riemann/*.json` (Steps 1–4)
v3.2 checkpoint	`notebooks/outputs/k7_pinn_step5_final.pt` (1.6 MB, float64)
v3.2 certification	`notebooks/outputs/k7_metric_v32_export.json` (20/20 checks)
Lean 4 certificate	`notebooks/outputs/K7Certificate.lean`
2000-point sample	`notebooks/outputs/k7_metric_data.csv`
Geodesic solver	`gift_core/nn/geodesics.py` (includes `CheckpointPINNAdapter`)
Repository	https://github.com/gift-framework/GIFT

The v3.2 checkpoint can be loaded via:

```
import torch
from gift_core.nn import CheckpointPINNAdapter

state = torch.load('k7_pinn_step5_final.pt', map_location='cpu')
model = CheckpointPINNAdapter(state)
x = torch.randn(100, 7)  # 100 points on K7
```

```
g = model.metric(x)        # shape (100, 7, 7), guaranteed pos. def.
```

B.2 Hardware

	Specification
GPU	NVIDIA A100-SXM4-80GB
Training time	2.9 minutes
Parameters	202,857
Epochs	5,000
Evaluation points	50,000
Peak memory	∼1–2 GB

B.3 Dependencies

```
torch >= 2.0 (float64 mode)
numpy, scipy, matplotlib, tqdm
cupy-cuda12x (optional, for spectral analysis)
```

B.4 To reproduce

1. Open `notebooks/K7_PINN_Step5_Reconstruction_v3.ipynb` in Google Colab
2. Select A100 GPU runtime
3. Run all cells
4. Results exported to `k7_pinn_step5_results_v3.json`

No manual intervention required.

Related Works

The analytical target metric and period integrals used in this paper derive from the GIFT (Geometric Information Field Theory) framework:

- de La Fournière, B. (2026). *Geometric Information Field Theory v3.3*. Technical report. https://github.com/gift-framework.
- de La Fournière, B. (2026). *A parameter-free mollified approximation to the argument of the Riemann zeta function*. Preprint, in preparation.

While the mathematical objects produced by GIFT (the G_2 decomposition, the Mayer–Vietoris splitting of moduli, and the determinant formula $\det(g) = 65/32$) serve as input data here, the physical claims of that framework are outside the scope of this paper. The metric verification criteria (determinant, torsion, positive definiteness) are independent of GIFT.

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