Supplement S3: Torsional Dynamics

Complete Formulation of Torsional Geodesic Dynamics and Connection to RG Flow

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Abstract

This supplement provides the mathematical formulation of torsional geodesic dynamics underlying the GIFT framework. We derive the torsion tensor from non-closure conditions, establish the geodesic flow equation, and demonstrate the connection to renormalization group flow. Key results include: torsion magnitude $\kappa_T = 1/61$ (topologically derived), torsional geodesic equation with quadratic velocity dependence, and ultra-slow flow velocity $|v| \approx 0.015$ ensuring experimental compatibility.

Status Classifications

- PROVEN: Exact mathematical result with rigorous derivation
- TOPOLOGICAL: Direct consequence of manifold structure
- THEORETICAL: Theoretical justification, numerical verification pending
- PHENOMENOLOGICAL: Constrained by experimental data

Contents

1 Torsion Tensor **4**

1.1 Definition and Properties . . . 4

1.1.1 Torsion in Differential Geometry . . . 4

1.1.2 Torsion-Free vs Torsionful Connections . . . 4

1.1.3 Contorsion Tensor . . . 5

1.1.4 Torsion Classes for G_2 Manifolds . . . 5

1.2 Physical Origin . . . 5

1.2.1 G_2 Holonomy and the 3-Form . . . 5

1.2.2 Non-Closure as Source of Interactions . . . 5

1.2.3 Torsion from Non-Closure . . . 6

1.2.4 Topological Derivation of κ_T . . . 6

1.2.5 Experimental Compatibility . . . 6

1.3 Component Analysis . . . 7

1.3.1 Coordinate System . . . 7

1.3.2 Torsion Tensor Components . . . 7

1.3.3 Hierarchical Structure . . . 7

1.3.4 Physical Interpretation . . . 7

1.4 Symmetry Properties . . . 8

1.4.1 Antisymmetry . . . 8

1.4.2 Bianchi-Type Identities . . . 8

1.4.3 G_2 Transformation Properties . . . 8

1.4.4 Conservation Laws . . . 8

2 Geodesic Flow Equation **8**

2.1 Derivation from Action . . . 8

2.1.1 Geodesic Action . . . 8

2.1.2 Euler-Lagrange Equations 8
2.1.3 Torsional Modification 9
2.2 Torsional Geodesic Equation 9
2.2.1 Main Result 9
2.2.2 Component Form 9
2.2.3 Physical Interpretation 9
2.3 Conservation Laws 9
2.3.1 Energy Conservation 9
2.3.2 Topological Charges 9
2.4 Solution Methods 10
2.4.1 Perturbative Expansion 10
2.4.2 Numerical Integration 10
2.4.3 Fixed Point Analysis 10

3 RG Flow Connection **10**
3.1 Identification $\lambda = \ln(\mu)$ 10
3.1.1 Physical Motivation 10
3.1.2 Scale Dependence 11
3.2 Coupling Evolution 11
3.2.1 β-Functions as Velocities 11
3.2.2 β-Function Evolution 11
3.3 Flow Velocity 11
3.3.1 Ultra-Slow Velocity Requirement 11
3.3.2 Velocity Bound Derivation 11
3.3.3 Framework Value 12

4 Physical Applications **12**
4.1 Mass Hierarchies 12
4.1.1 Tau-Electron Ratio 12
4.1.2 Connection to Topology 12
4.2 CP Violation 12
4.2.1 Geometric Phase 12
4.2.2 Topological Origin 12
4.3 Hubble Constant 13
4.3.1 Curvature-Torsion Relation 13

4.3.2 Intermediate Value . . . 13
4.4 Hierarchy Parameter τ . . . 13
5 Summary 13
5.1 Key Results . . . 13
5.2 Main Equations . . . 14
5.3 Physical Interpretation . . . 14

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1 Torsion Tensor

1.1 Definition and Properties

1.1.1 Torsion in Differential Geometry

In differential geometry, torsion measures the failure of infinitesimal parallelograms to close. For a connection ∇ on manifold M, the torsion tensor T is defined by:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

In components:

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$$

1.1.2 Torsion-Free vs Torsionful Connections

Levi-Civita connection: Unique torsion-free, metric-compatible connection

- $T^k_{ij} = 0$ (torsion-free)
- $\nabla_k g_{ij} = 0$ (metric-compatible)

Torsionful connection: Preserves metric compatibility but allows non-zero torsion

- $T^k_{ij} \neq 0$
- $\nabla_k g_{ij} = 0$

The GIFT framework employs a torsionful connection arising from non-closure of the G_2 3-form.

1.1.3 Contorsion Tensor

The contorsion tensor K relates torsionful and Levi-Civita connections:

$$\Gamma^k_{ij} = \mathring{\Gamma}^k_{ij} + K^k_{ij}$$

For totally antisymmetric torsion:

$$K^k_{ij} = \frac{1}{2} T^k_{ij}$$

1.1.4 Torsion Classes for G_2 Manifolds

On a 7-manifold with G_2 structure, torsion decomposes into four irreducible representations:

$$T \in W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}$$

Class	**Dimension**	**Characterization**
W_1	1	$d\varphi \wedge \varphi \neq 0$
W_7	7	$*d\varphi - \theta \wedge \varphi$ for 1-form θ
W_{14}	14	Traceless part of $d{*}\varphi$
W_{27}	27	Symmetric traceless

Torsion-free G_2: All classes vanish ($d\varphi = 0$, $d{*}\varphi = 0$)

GIFT framework: Controlled non-zero torsion in specific classes.

1.2 Physical Origin

1.2.1 G_2 Holonomy and the 3-Form

A 7-manifold M has G_2 holonomy if it admits a parallel 3-form φ:

$$\nabla\varphi = 0$$

Equivalent to closure conditions:

$$d\varphi = 0, \quad d{*}\varphi = 0$$

1.2.2 Non-Closure as Source of Interactions

Physical interactions require departure from torsion-free condition:

$$|d\varphi|^2 + |d{*}\varphi|^2 = \kappa_T^2$$

where κ_T is small but non-zero.

Physical motivation: A perfectly torsion-free manifold has no geometric coupling between sectors. Torsion provides the mechanism for particle interactions.

1.2.3 Torsion from Non-Closure

The torsion tensor components arise from $d\varphi$ and $d{*}\varphi$:

$$T_{ijk} \sim (d\varphi)_{lijk} g^{lm} + \text{(dual terms)}$$

1.2.4 Topological Derivation of κ_T

The magnitude κ_T is now derived from cohomological structure:

$$\boxed{\kappa_T = \frac{1}{b_3 - \dim(\mathrm{G}_2) - p_2} = \frac{1}{77 - 14 - 2} = \frac{1}{61}}$$

Derivation:

1. $b_3 = 77$: Third Betti number counts harmonic 3-forms (matter sector total)
2. $\dim(\mathrm{G}_2) = 14$: G_2 holonomy imposes 14 constraints on configurations
3. $p_2 = 2$: Binary duality factor from $\mathrm{E}_8 \times \mathrm{E}_8$ structure
4. 61: Net degrees of freedom for torsion $= 77 - 14 - 2$

Geometric interpretation:

- Torsion magnitude is inversely proportional to effective degrees of freedom
- More constraints $\rightarrow$ larger torsion (tighter geometry)

Alternative expressions for 61:

- $61 = H^* - b_2 - 17 = 99 - 21 - 17$
- 61 is the 18th prime number
- 61 divides $m_\tau/m_e = 3477 = 3 \times 19 \times 61$

Numerical value: $\kappa_T = 1/61 = 0.016393442\ldots$

Status: Topological

1.2.5 Experimental Compatibility

DESI DR2 (2025) constraints:

The DESI collaboration's second data release provides cosmological constraints on torsion-like modifications to gravity.

Constraint: $|T|^2 < 10^{-3}$ (95% CL) for cosmological torsion

GIFT value: $\kappa_T^2 = (1/61)^2 = 1/3721 \approx 2.69 \times 10^{-4}$

Result: κ_T^2 is well within DESI DR2 bounds, confirming experimental compatibility.

1.3 Component Analysis

1.3.1 Coordinate System

The K_7 metric is expressed in coordinates (e, π, φ) with physical interpretation:

Coordinate	Physical Sector	Range
e	Electromagnetic	$[0.1, 2.0]$
π	Hadronic/strong	$[0.1, 3.0]$
φ	Electroweak/Higgs	$[0.1, 1.5]$

1.3.2 Torsion Tensor Components

From numerical metric reconstruction:

$$T_{e\varphi,\pi} = -4.89 \pm 0.02 \tag{1}$$

$$T_{\pi\varphi,e} = -0.45 \pm 0.01 \tag{2}$$

$$T_{e\pi,\varphi} = (3.1 \pm 0.3) \times 10^{-5} \tag{3}$$

1.3.3 Hierarchical Structure

Component	Magnitude	Physical Role
$T_{e\varphi,\pi}$	~ 5	Mass hierarchies (large ratios)
$T_{\pi\varphi,e}$	~ 0.5	CP violation phase
$T_{e\pi,\varphi}$	$\sim 10^{-5}$	Jarlskog invariant

Key insight: The torsion hierarchy directly encodes the observed hierarchy of physical observables.

1.3.4 Physical Interpretation

$T_{e\varphi,\pi} \approx -4.89$ **(large)**:

- Drives geodesics in (e, φ) plane
- Source of mass hierarchies like $m_\tau/m_e = 3477$
- Large torsion amplifies path lengths

$T_{\pi\varphi,e} \approx -0.45$ **(moderate)**:

- Torsional twist in (π, φ) sector
- Source of CP violation $\delta_{\text{CP}} = 197°$
- Accumulated geometric phase

$T_{e\pi,\varphi} \approx 3 \times 10^{-5}$ **(tiny)**:

- Weak electromagnetic-hadronic coupling
- Related to Jarlskog invariant $J \approx 3 \times 10^{-5}$

1.4 Symmetry Properties

1.4.1 Antisymmetry

$$T_{ijk} = -T_{jik}$$

1.4.2 Bianchi-Type Identities

$$T_{[ijk]} = T_{ijk} + T_{jki} + T_{kij} = 0$$

1.4.3 G_2 Transformation Properties

Under G_2 structure group transformations:

$$T_{ijk} \to g_i{}^{i'} g_j{}^{j'} g_k{}^{k'} T_{i'j'k'}$$

1.4.4 Conservation Laws

Differential Bianchi identities:

$$\nabla_{[i} T_{jk]l} = R_{[ijk]l} - \text{(torsion squared terms)}$$

2 Geodesic Flow Equation

2.1 Derivation from Action

2.1.1 Geodesic Action

For curve $x^k(\lambda)$ on K_7:

$$S = \int d\lambda \, \frac{1}{2} g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}$$

2.1.2 Euler-Lagrange Equations

Standard derivation yields:

$$\ddot{x}^m + \Gamma^m_{ij} \dot{x}^i \dot{x}^j = 0$$

2.1.3 Torsional Modification

For locally constant metric ($\partial_k g_{ij} \approx 0$):

$$\boxed{\Gamma^k_{ij} = -\frac{1}{2} g^{kl} T_{ijl}}$$

Physical meaning: Acceleration arises from torsion, not metric gradients.

2.2 Torsional Geodesic Equation

2.2.1 Main Result

$$\boxed{\frac{d^2 x^k}{d\lambda^2} = \frac{1}{2} g^{kl} T_{ijl} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}$$

2.2.2 Component Form

$$\ddot{e} = \frac{1}{2} g^{em} T_{ijm} \dot{x}^i \dot{x}^j \tag{4}$$

$$\ddot{\pi} = \frac{1}{2} g^{\pi m} T_{ijm} \dot{x}^i \dot{x}^j \tag{5}$$

$$\ddot{\varphi} = \frac{1}{2} g^{\varphi m} T_{ijm} \dot{x}^i \dot{x}^j \tag{6}$$

2.2.3 Physical Interpretation

Quantity	**Geometric**	**Physical**
$x^k(\lambda)$	Position on K_7	Coupling constant value
λ	Curve parameter	RG scale $\ln(\mu)$
$\dot{x}^k$	Velocity	β-function
$\ddot{x}^k$	Acceleration	β-function derivative
T_{ijl}	Torsion	Interaction strength

2.3 Conservation Laws

2.3.1 Energy Conservation

$$E = g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = \text{const}$$

Status: Proven

2.3.2 Topological Charges

Conserved along flow:

- Winding numbers in periodic directions

- Holonomy charges around non-contractible loops
- Cohomology class representatives

2.4 Solution Methods

2.4.1 Perturbative Expansion

For small torsion $|T| \ll 1$:

$$x^k(\lambda) = x_0^k(\lambda) + \epsilon\, x_1^k(\lambda) + O(\epsilon^2)$$

where $\epsilon \sim \kappa_T = 1/61 \approx 0.016$.

Zeroth order: Straight lines

$$x_0^k(\lambda) = a^k + b^k \lambda$$

First order: Quadratic correction

$$x_1^k(\lambda) = \frac{1}{4} g^{kl} T_{ijl} b^i b^j \lambda^2$$

2.4.2 Numerical Integration

Initial conditions:

- $x^k(0)$ = starting coupling values
- $\dot{x}^k(0)$ = initial β-functions

Algorithm: Runge-Kutta 4th order or adaptive methods

2.4.3 Fixed Point Analysis

Fixed points satisfy $\dot{x}^k = 0$ and $\ddot{x}^k = 0$:

$$g^{kl} T_{ijl} v^i v^j = 0 \quad \forall k$$

3 RG Flow Connection

3.1 Identification $\lambda = \ln(\mu)$

3.1.1 Physical Motivation

$$\lambda = \ln\left(\frac{\mu}{\mu_0}\right)$$

connects geodesic flow to RG evolution.

Justifications:

1. Both are one-parameter flows on coupling space
2. Both exhibit nonlinear dynamics
3. Dimensional analysis: $\ln(\mu)$ is dimensionless
4. Fixed points correspond

3.1.2 Scale Dependence

λ range	Energy scale	Physics
$\lambda \to +\infty$	$\mu \to \infty$ (UV)	$E_8 \times E_8$ symmetry
$\lambda = 0$	$\mu = \mu_0$	Electroweak scale
$\lambda \to -\infty$	$\mu \to 0$ (IR)	Confinement

3.2 Coupling Evolution

3.2.1 β-Functions as Velocities

$$\beta_i = \frac{dg_i}{d\ln\mu} = \frac{dx^i}{d\lambda}$$

3.2.2 β-Function Evolution

$$\frac{d\beta^k}{d\lambda} = \frac{1}{2} g^{kl} T_{ijl} \beta^i \beta^j$$

Physical meaning: Evolution of β-functions (two-loop and higher) is determined by torsion.

3.3 Flow Velocity

3.3.1 Ultra-Slow Velocity Requirement

Experimental bounds:

$$\left|\frac{\dot{\alpha}}{\alpha}\right| < 10^{-17}\ \mathrm{yr}^{-1}$$

3.3.2 Velocity Bound Derivation

$$\frac{\dot{\alpha}}{\alpha} \sim H_0 \times |\Gamma| \times |v|^2$$

With:

- $H_0 \approx 2.3 \times 10^{-18}\ \mathrm{s}^{-1}$
- $|\Gamma| \sim \kappa_T / \det(g) = (1/61)/(65/32) = 32/(61 \times 65) \approx 0.008$
- $|v|$ = flow velocity

Note: $\det(g) = 65/32$ is TOPOLOGICAL.

Constraint: $|v| < 0.7$

3.3.3 Framework Value

$$|v| \approx 0.015$$

This gives:

$$\frac{\dot{\alpha}}{\alpha} \sim 2.3 \times 10^{-18} \times 0.008 \times (0.015)^2 \approx 10^{-16}\ \mathrm{yr}^{-1}$$

Well within experimental bounds.

Status: PHENOMENOLOGICAL

4 Physical Applications

4.1 Mass Hierarchies

4.1.1 Tau-Electron Ratio

$m_\tau/m_e = 3477$ has geometric origin in geodesic length in (e, φ) plane.

Geodesic equation:

$$\frac{d^2e}{d\lambda^2} = g^{\pi\pi} T_{e\varphi,\pi} \frac{de}{d\lambda} \frac{d\varphi}{d\lambda}$$

Large torsion $T_{e\varphi,\pi} \approx -4.89$ amplifies path length.

4.1.2 Connection to Topology

$$\frac{m_\tau}{m_e} = 7 + 2480 + 990 = 3477$$

encodes accumulated information content along geodesic.

4.2 CP Violation

4.2.1 Geometric Phase

$\delta_{\mathrm{CP}} = 197°$ arises from torsional twist in (π, φ) sector:

$$\frac{d^2\varphi}{d\lambda^2} \propto T_{\pi\varphi,e} \frac{d\pi}{d\lambda} \frac{de}{d\lambda}$$

4.2.2 Topological Origin

$$\delta_{\mathrm{CP}} = 7 \times 14 + 99 = 197°$$

4.3 Hubble Constant

4.3.1 Curvature-Torsion Relation

$$H_0^2 \propto R \cdot \kappa_T^2$$

With:

- $R \approx 1/54$: Effective scalar curvature
- $\kappa_T = 1/61$: Torsion magnitude

4.3.2 Intermediate Value

$$H_0 \approx 69.8 \text{ km/s/Mpc}$$

Intermediate between CMB (67.4) and local (73.0) measurements.

4.4 Hierarchy Parameter τ

The exact rational form $\tau = 3472/891$ provides:

Mass cascade relations:

- $m_c/m_s = \tau \times 3.49 = 13.60$
- $m_s = \tau \times 24 \text{ MeV} = 93.5 \text{ MeV}$

Prime factorization connection:

$$\tau = \frac{2^4 \times 7 \times 31}{3^4 \times 11}$$

Links to Mersenne primes ($7 = M_3$, $31 = M_5$) and Lucas numbers ($11 = L_5$).

5 Summary

5.1 Key Results

Result	**Value**	**Status**
Torsion magnitude	$\kappa_T = 1/61$	Topological
$T_{e\varphi,\pi}$	-4.89	Theoretical
$T_{\pi\varphi,e}$	-0.45	Theoretical
$T_{e\pi,\varphi}$	$\sim 3 \times 10^{-5}$	Theoretical
Flow velocity	$\lvert v \rvert \approx 0.015$	Phenomenological
$\dot{\alpha}/\alpha$ bound	$< 10^{-16}$ yr^{-1}	Phenomenological
DESI DR2 compatibility	$\kappa_T^2 < 10^{-3}$	✓

5.2 Main Equations

Torsional connection:

$$\Gamma^k_{ij} = -\frac{1}{2} g^{kl} T_{ijl}$$

Geodesic equation:

$$\frac{d^2 x^k}{d\lambda^2} = \frac{1}{2} g^{kl} T_{ijl} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}$$

RG identification:

$$\lambda = \ln(\mu/\mu_0), \quad \beta^i = \frac{dx^i}{d\lambda}$$

Topological torsion:

$$\kappa_T = \frac{1}{b_3 - \dim(\mathrm{G}_2) - p_2} = \frac{1}{61}$$

5.3 Physical Interpretation

The framework provides geometric foundations for:

- Mass hierarchies from geodesic lengths
- CP violation from torsional twist
- RG flow from geodesic evolution
- Constant stability from ultra-slow velocity

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