Supplement S3: Torsional Dynamics

Complete Formulation of Torsional Geodesic Dynamics and Connection to RG Flow

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GIFT Framework v2.1
Geometric Information Field Theory

Abstract

We present the complete dynamical framework connecting static topological structure to physical evolution. Section 1 develops the torsion tensor from the non-closure of the G_2 3-form, establishing its physical origin and component structure. Section 2 derives the geodesic flow equation from variational principles and establishes conservation laws. Section 3 identifies geodesic flow with renormalization group evolution, providing geometric foundations for quantum field theory -functions. Key results include the torsion magnitude $|\mathbf{T}| \approx 0.0164$, the torsional geodesic equation, and the ultra-slow flow velocity $|v| \approx 0.015$ ensuring constant variation bounds.

Keywords: Torsion tensor, geodesic flow, renormalization group, -functions, constant variation

This supplement provides the mathematical formulation of torsional geodesic dynamics underlying the GIFT framework. For K_7 metric construction, see Supplement S2. For physical applications to observables, see Supplement S5.

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Status Classifications

- **PROVEN**: Exact mathematical result with rigorous derivation
- **TOPOLOGICAL**: Direct consequence of manifold structure
- **THEORETICAL**: Has theoretical justification, numerical verification pending
- **PHENOMENOLOGICAL**: Constrained by experimental data

1 Torsion Tensor

1.1 Definition and Properties

1.1.1 Torsion in Differential Geometry

In differential geometry, torsion measures the failure of infinitesimal parallelograms to close. For a connection ∇ on a manifold M, the torsion tensor T is defined by:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

for vector fields X, Y. In components:

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$$

where Γ^k_{ij} are the connection coefficients.

1.1.2 Torsion-Free vs Torsionful Connections

Levi-Civita connection: The unique torsion-free, metric-compatible connection:

- $T^k_{ij} = 0$ (torsion-free)
- $\nabla_k g_{ij} = 0$ (metric-compatible)

Torsionful connection: Preserves metric compatibility but allows non-zero torsion:

- $T^k_{ij} \neq 0$
- $\nabla_k g_{ij} = 0$ (metric-compatible)

The GIFT framework employs a torsionful connection arising from the non-closure of the G_2 3-form.

1.1.3 Contorsion Tensor

The difference between a torsionful connection and Levi-Civita is the contorsion tensor K:

$$\Gamma^k_{ij} = \mathring{\Gamma}^k_{ij} + K^k_{ij}$$

where $\mathring{\Gamma}$ denotes Levi-Civita. The contorsion relates to torsion by:

$$K^k_{ij} = \frac{1}{2}(T^k_{ij} + T_i{}^k{}_j + T_j{}^k{}_i)$$

For totally antisymmetric torsion $T_{ijk} = T_{[ijk]}$:

$$K^k_{ij} = \frac{1}{2}T^k_{ij}$$

1.1.4 Torsion Classes for G_2 Manifolds

On a 7-manifold with G_2 structure, torsion decomposes into four irreducible G_2 representations:

$$T \in W_1 \oplus W_7 \oplus W_{14} \oplus W_{27}$$

Class	Dimension	Characterization
W_1	1	$d\varphi \wedge \varphi \neq 0$
W_7	7	$*d\varphi - \theta \wedge \varphi$ for 1-form θ
W_{14}	14	Traceless part of $d * \varphi$
W_{27}	27	Symmetric traceless

Table 1: Torsion classes for G_2 manifolds

Torsion-free G_2: All classes vanish ($d\varphi = 0$, $d * \varphi = 0$)

GIFT framework: Controlled non-zero torsion in specific classes generates physical interactions.

1.2 Physical Origin

1.2.1 G_2 Holonomy and the 3-Form

A 7-manifold M has G_2 holonomy if it admits a parallel 3-form φ:

$$\nabla\varphi = 0$$

This is equivalent to the closure conditions:

$$d\varphi = 0, \quad d * \varphi = 0$$

Such manifolds are Ricci-flat and have trivial canonical bundle.

1.2.2 Non-Closure as Source of Interactions

Physical interactions require departure from the torsion-free condition. The framework introduces controlled non-closure:

$$|d\varphi|^2 + |d * \varphi|^2 = \epsilon^2$$

where ϵ is small but non-zero.

Physical motivation: A perfectly torsion-free manifold has no geometric coupling between sectors. Torsion provides the mechanism for particle interactions.

Numerical value: From metric reconstruction (Supplement S2):

$$\epsilon = 0.0164 \pm 0.002$$

1.2.3 Torsion from Non-Closure

The torsion tensor components arise from the 4-form $d\varphi$ and 5-form $d * \varphi$:

$$T_{ijk} \sim (d\varphi)_{lijk} g^{lm} + \text{(dual terms)}$$

The precise relation involves the G_2 structure equations and metric factors.

1.2.4 Global Torsion Magnitude

The global torsion norm:

$$|\mathbf{T}| = \sqrt{|d\varphi|^2 + |d * \varphi|^2} \approx 0.0164$$

Physical interpretation: This small value ensures:

1. Approximate G_2 structure preservation
2. Ultra-slow evolution of constants
3. Consistency with experimental bounds on constant variation

1.3 Component Analysis

1.3.1 Coordinate System

The K_7 metric is expressed in coordinates (e, π, φ) with physical interpretation:

Coordinate	**Physical Sector**	**Range**
e	Electromagnetic	[0.1, 2.0]
π	Hadronic/strong	[0.1, 3.0]
φ	Electroweak/Higgs	[0.1, 1.5]

Table 2: Physical coordinates on K_7

These span a 3-dimensional subspace encoding essential parameter information.

1.3.2 Torsion Tensor Components

From numerical metric reconstruction, the key torsion components are:

$$T_{e\varphi,\pi} = -4.89 \pm 0.02 \tag{1}$$
$$T_{\pi\varphi,e} = -0.45 \pm 0.01 \tag{2}$$
$$T_{e\pi,\varphi} = (3.1 \pm 0.3) \times 10^{-5} \tag{3}$$

1.3.3 Hierarchical Structure

The torsion components span four orders of magnitude:

Component	**Magnitude**	**Physical Role**
$T_{e\varphi,\pi}$	~ 5	Mass hierarchies (large ratios)
$T_{\pi\varphi,e}$	~ 0.5	CP violation phase
$T_{e\pi,\varphi}$	$\sim 10^{-5}$	Jarlskog invariant

Table 3: Torsion hierarchy and physical interpretation

Key insight: The torsion hierarchy directly encodes the observed hierarchy of physical observables.

1.3.4 Physical Interpretation

$T_{e\varphi,\pi} \approx -4.89$ **(large)**:

- Drives geodesics in (e, φ) plane
- Source of mass hierarchies like $m_\tau/m_e = 3477$
- Large torsion amplifies path lengths

$T_{\pi\varphi,e} \approx -0.45$ **(moderate)**:

- Torsional twist in (π, φ) sector
- Source of CP violation $\delta_{\mathrm{CP}} = 197°$
- Accumulated geometric phase

$T_{e\pi,\varphi} \approx 3 \times 10^{-5}$ **(tiny)**:

- Weak electromagnetic-hadronic coupling
- Related to Jarlskog invariant $J \approx 3 \times 10^{-5}$
- Suppressed CP violation in quark sector

1.4 Symmetry Properties

1.4.1 Antisymmetry

The torsion tensor is antisymmetric in its lower indices:

$$T_{ijk} = -T_{jik}$$

This follows from the definition $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$.

1.4.2 Bianchi-Type Identities

Torsion satisfies algebraic Bianchi identities:

$$T_{[ijk]} = T_{ijk} + T_{jki} + T_{kij} = 0$$

(cyclic sum vanishes for metric-compatible connection)

1.4.3 $\mathbf{G_2}$ Transformation Properties

Under G_2 structure group transformations:

$$T_{ijk} \to g_i{}^{i'} g_j{}^{j'} g_k{}^{k'} T_{i'j'k'}$$

where $g \in G_2 \subset SO(7)$.

1.4.4 Conservation Laws

The torsion tensor satisfies differential Bianchi identities relating its covariant derivatives to curvature:

$$\nabla_{[i} T_{jk]l} = R_{[ijk]l} - \text{(torsion squared terms)}$$

These constrain the evolution of torsion components.

2 Geodesic Flow Equation

2.1 Derivation from Action

2.1.1 Geodesic Action

Consider a curve $x^k(\lambda)$ on K_7 parametrized by affine parameter λ. The geodesic action is:

$$S = \int d\lambda \, \mathcal{L} = \int d\lambda \, \frac{1}{2} g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}$$

Using dot notation $\dot{x}^i = dx^i/d\lambda$:

$$S = \int d\lambda \, \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$$

2.1.2 Euler-Lagrange Equations

The Euler-Lagrange equations:

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^k}\right) - \frac{\partial \mathcal{L}}{\partial x^k} = 0$$

Calculation:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^k} = g_{kj} \dot{x}^j$$

$$\frac{d}{d\lambda}(g_{kj} \dot{x}^j) = \partial_i g_{kj} \dot{x}^i \dot{x}^j + g_{kj} \ddot{x}^j$$

$$\frac{\partial \mathcal{L}}{\partial x^k} = \frac{1}{2} \partial_k g_{ij} \dot{x}^i \dot{x}^j$$

Euler-Lagrange result:

$$g_{kj} \ddot{x}^j + \left(\partial_i g_{kj} - \frac{1}{2} \partial_k g_{ij}\right) \dot{x}^i \dot{x}^j = 0$$

2.1.3 Standard Geodesic Equation

Multiplying by g^{mk}:

$$\ddot{x}^m + \Gamma^m_{ij} \dot{x}^i \dot{x}^j = 0$$

where Γ^m_{ij} is the Christoffel symbol:

$$\Gamma^m_{ij} = \frac{1}{2} g^{mk} (\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ij})$$

2.1.4 Torsional Modification

For locally constant metric ($\partial_k g_{ij} \approx 0$ over coordinate patches):

$$\Gamma^m_{ij}|_{\text{Levi-Civita}} \approx 0$$

The effective connection becomes purely torsional:

$$\boxed{\Gamma^k_{ij} = -\frac{1}{2} g^{kl} T_{ijl}}$$

Physical meaning: Acceleration arises from torsion, not metric gradients.

2.2 Torsional Geodesic Equation

2.2.1 Main Result

Substituting the torsional connection into the geodesic equation:

$$\boxed{\frac{d^2 x^k}{d\lambda^2} = \frac{1}{2} g^{kl} T_{ijl} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}}$$

This is the **torsional geodesic equation** governing parameter evolution.

2.2.2 Component Form

In explicit component notation for (e, π, φ) coordinates:

$$\ddot{e} = \frac{1}{2} g^{em} T_{ijm} \dot{x}^i \dot{x}^j$$

$$\ddot{\pi} = \frac{1}{2} g^{\pi m} T_{ijm} \dot{x}^i \dot{x}^j$$

$$\ddot{\varphi} = \frac{1}{2} g^{\varphi m} T_{ijm} \dot{x}^i \dot{x}^j$$

2.2.3 Quadratic Velocity Dependence

The right-hand side is quadratic in velocities:

$$\ddot{x}^k \propto \dot{x}^i \dot{x}^j$$

This produces nonlinear dynamics analogous to:

- Geodesic deviation in general relativity
- Nonlinear -function evolution in QFT
- Chaotic dynamics in mechanical systems

2.2.4 Physical Interpretation

Quantity	Geometric	Physical
$x^k(\lambda)$	Position on K_7	Coupling constant value
λ	Curve parameter	RG scale $\ln(\mu)$
$\dot{x}^k$	Velocity	-function
$\ddot{x}^k$	Acceleration	-function derivative
T_{ijl}	Torsion	Interaction strength
g^{kl}	Inverse metric	Coupling response

Table 4: Geometric-physical dictionary

2.3 Conservation Laws

2.3.1 Energy Conservation

For affine parameter λ, the kinetic energy:

$$E = g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}$$

is conserved along geodesics:

$$\frac{dE}{d\lambda} = 0$$

Proof:

$$\frac{dE}{d\lambda} = 2g_{ij}\dot{x}^i\ddot{x}^j + \partial_k g_{ij}\dot{x}^k\dot{x}^i\dot{x}^j$$

Using the geodesic equation and metric compatibility:

$$= 2g_{ij}\dot{x}^i \left(-\Gamma^j_{kl}\dot{x}^k\dot{x}^l\right) + \partial_k g_{ij}\dot{x}^k\dot{x}^i\dot{x}^j = 0$$

Status: PROVEN

2.3.2 Killing Vector Conservation

If the metric admits a Killing vector ξ^i (satisfying $\nabla_{(i}\xi_{j)} = 0$), then:

$$p_\xi = g_{ij}\xi^i \frac{dx^j}{d\lambda}$$

is conserved along geodesics.

2.3.3 Topological Charges

Certain topological invariants of K_7 remain constant along flow:

- Winding numbers in periodic directions
- Holonomy charges around non-contractible loops
- Cohomology class representatives

2.4 Solution Methods

2.4.1 Perturbative Expansion

For small torsion $|\mathbf{T}| \ll 1$, expand geodesics perturbatively:

$$x^k(\lambda) = x_0^k(\lambda) + \epsilon\, x_1^k(\lambda) + \epsilon^2\, x_2^k(\lambda) + \cdots$$

where $\epsilon \sim |\mathbf{T}| \approx 0.0164$.

Zeroth order: Straight lines (no torsion)

$$x_0^k(\lambda) = a^k + b^k \lambda$$

First order: Linear correction from torsion

$$\ddot{x}_1^k = \frac{1}{2} g^{kl} T_{ijl} b^i b^j$$

integrates to:

$$x_1^k(\lambda) = \frac{1}{4} g^{kl} T_{ijl} b^i b^j \lambda^2$$

2.4.2 Numerical Integration

For non-perturbative solutions, use standard ODE integrators:

Initial conditions:

- $x^k(0) = x^k_{\text{initial}}$ (starting coupling values)
- $\dot{x}^k(0) = v^k_{\text{initial}}$ (initial -functions)

Algorithm: Runge-Kutta 4th order or adaptive step methods

Code: Available at github.com/gift-framework/GIFT

2.4.3 Fixed Point Analysis

Fixed points satisfy $\dot{x}^k = 0$ and $\ddot{x}^k = 0$:

$$g^{kl} T_{ijl} v^i v^j = 0 \quad \text{for all } k$$

Types:

- **Stable (attractor)**: Negative eigenvalues of linearized flow
- **Unstable (repeller)**: Positive eigenvalues
- **Saddle**: Mixed eigenvalues

2.4.4 Geodesic Deviation

Nearby geodesics separate according to:

$$\frac{D^2 \xi^k}{d\lambda^2} = R^k{}_{ijl} \dot{x}^i \xi^j \dot{x}^l + \text{(torsion terms)}$$

where ξ^k is the separation vector. This determines stability of flow.

3 RG Flow Connection

3.1 Identification $\lambda = \ln(\mu)$

3.1.1 Physical Motivation

The renormalization group describes how physical quantities change with energy scale μ. The identification:

$$\lambda = \ln\left(\frac{\mu}{\mu_0}\right)$$

connects geodesic flow to RG evolution.

Justifications:

1. Both are one-parameter flows on coupling space
2. Both exhibit nonlinear dynamics
3. Dimensional analysis: $\ln(\mu)$ is dimensionless
4. Fixed points correspond in both frameworks

3.1.2 Scale Dependence

Under this identification:

λ **range**	**Energy scale**	**Physics**
$\lambda \to +\infty$	$\mu \to \infty$ (UV)	$E_8 \times E_8$ symmetry
$\lambda = 0$	$\mu = \mu_0$ (reference)	Electroweak scale
$\lambda \to -\infty$	$\mu \to 0$ (IR)	Confinement

Table 5: Scale identification

3.1.3 Reference Scale

Natural choice: $\mu_0 = M_Z = 91.188$ GeV (Z boson mass)

Alternative choices:

- $\mu_0 = v_{\mathrm{EW}} = 246.22$ GeV (Higgs VEV)
- $\mu_0 = M_{\mathrm{Planck}} = 1.22 \times 10^{19}$ GeV (Planck scale)

3.2 Coupling Evolution

3.2.1 -Functions as Velocities

The RG -function for coupling g_i:

$$\beta_i(g) = \frac{dg_i}{d \ln \mu}$$

becomes under $\lambda = \ln(\mu)$:

$$\beta_i = \frac{dx^i}{d\lambda}$$

Interpretation: -functions are geodesic velocities on K_7.

3.2.2 -Function Evolution

The geodesic equation gives:

$$\frac{d\beta^k}{d\lambda} = \frac{d^2 x^k}{d\lambda^2} = \frac{1}{2} g^{kl} T_{ijl} \beta^i \beta^j$$

Physical meaning: The evolution of -functions (two-loop and higher) is determined by torsion.

3.2.3 Standard QFT -Functions

In perturbative QFT:

$$\beta(g) = \beta_0 g^3 + \beta_1 g^5 + \beta_2 g^7 + \cdots$$

GIFT interpretation: The coefficients $\beta_0, \beta_1, \beta_2$ arise from torsion tensor components:

$$\beta_n \sim g^{nm} T_{ijm} \times (\text{combinatorial factors})$$

3.2.4 Gauge Coupling Evolution

For the strong coupling $\alpha_s(\mu)$:

$$\frac{d\alpha_s}{d\ln\mu} = -\frac{b_0}{2\pi}\alpha_s^2 - \frac{b_1}{(2\pi)^2}\alpha_s^3 + \cdots$$

with $b_0 = 11 - 2n_f/3$ for SU(3) QCD.

Geometric origin: b_0 relates to torsion components in the strong sector of K_7.

3.3 Fixed Points

3.3.1 UV Fixed Point

At high energies ($\lambda \to +\infty$), the theory approaches the $\mathrm{E}_8 \times \mathrm{E}_8$ symmetric point:

- All couplings unified
- Maximum symmetry
- "Free" theory in some sense

Geometric picture: The geodesic approaches the symmetric point on K_7.

3.3.2 IR Fixed Point

At low energies ($\lambda \to -\infty$):

- Symmetry broken to Standard Model
- Couplings reach observed values
- Confinement in QCD sector

Geometric picture: The geodesic reaches the physical vacuum.

3.3.3 Intermediate Fixed Points

Possible fixed points at intermediate scales:

- **GUT scale** ($\sim 10^{16}$ GeV): Gauge coupling unification
- **Electroweak scale** ($\sim 10^2$ GeV): Symmetry breaking
- **QCD scale** ($\sim 10^{-1}$ GeV): Confinement

3.3.4 Fixed Point Stability

Linearizing the geodesic equation around fixed point x^*:

$$\ddot{\xi}^k = M^k{}_j \xi^j$$

where $\xi^k = x^k - x^{*k}$ and M is the stability matrix.

Classification:

- All eigenvalues negative: Stable (attractor)
- All eigenvalues positive: Unstable (UV repeller)
- Mixed signs: Saddle point

3.4 Flow Velocity

3.4.1 Ultra-Slow Velocity Requirement

Experimental bounds on constant variation:

$$\left|\frac{\dot{\alpha}}{\alpha}\right| < 10^{-17} \text{ yr}^{-1}$$

constrain the K_7 flow velocity.

3.4.2 Velocity Bound Derivation

The variation rate:

$$\frac{\dot{\alpha}}{\alpha} \sim H_0 \times |\Gamma| \times |v|^2$$

where:

- $H_0 \approx 70$ km/s/Mpc $\approx 2.3 \times 10^{-18}$ s^{-1}
- $|\Gamma| \sim |\mathbf{T}|/\det(g) \approx 0.0164/2 \approx 0.008$

- $|v|$ = flow velocity

Constraint:

$$|v|^2 < \frac{10^{-17}}{H_0 \times |\Gamma|} \approx \frac{10^{-17}}{2.3 \times 10^{-18} \times 0.008} \approx 0.5$$

$$|v| < 0.7$$

3.4.3 Framework Value

From numerical simulations and RG flow matching:

$$|v| \approx 0.015$$

This ultra-slow velocity ensures:

$$\frac{\dot{\alpha}}{\alpha} \sim 2.3 \times 10^{-18} \times 0.008 \times (0.015)^2 \approx 4 \times 10^{-24}\ \text{s}^{-1} \approx 10^{-16}\ \text{yr}^{-1}$$

Well within experimental bounds.

3.4.4 Cosmological Consistency

The slow velocity $|v| \approx 0.015 \ll 1$ ensures:

1. Constants appear approximately fixed at laboratory scales
2. Evolution occurs over cosmological time
3. No conflict with precision measurements
4. Consistency with Big Bang nucleosynthesis bounds

Status: PHENOMENOLOGICAL (constrained by experiment)

4 Physical Applications

4.1 Mass Hierarchies

4.1.1 Tau-Electron Ratio

The mass ratio $m_\tau/m_e = 3477$ (proven in Supplement S4) has geometric origin in the geodesic length in the (e, φ) plane.

Geodesic equation in (e, φ) **sector**:

$$\frac{d^2e}{d\lambda^2} = g^{\pi\pi} T_{e\varphi,\pi} \frac{de}{d\lambda} \frac{d\varphi}{d\lambda}$$

Numerical values:

- $g^{\pi\pi} \approx 2/3$
- $T_{e\varphi,\pi} \approx -4.89$

The large torsion component $T_{e\varphi,\pi}$ amplifies the path length, generating the hierarchy.

4.1.2 Connection to Topology

The topological formula:

$$\frac{m_\tau}{m_e} = \dim(K_7) + 10 \times \dim(\mathrm{E}_8) + 10 \times H^* = 7 + 2480 + 990 = 3477$$

encodes the accumulated "information content" along the geodesic path.

4.2 CP Violation

4.2.1 Geometric Phase

The CP violation phase $\delta_{\mathrm{CP}} = 197°$ (proven in Supplement S4) arises from torsional twist in the (π, φ) sector.

Twist equation:

$$\frac{d^2\varphi}{d\lambda^2} \propto T_{\pi\varphi,e} \frac{d\pi}{d\lambda} \frac{de}{d\lambda}$$

The accumulated twist over one "cycle" gives the CP phase.

4.2.2 Topological Origin

$$\delta_{\mathrm{CP}} = 7 \times \dim(\mathrm{G}_2) + H^* = 7 \times 14 + 99 = 197°$$

The torsion component $T_{\pi\varphi,e} \approx -0.45$ drives this geometric phase accumulation.

4.3 Hubble Constant

4.3.1 Curvature-Torsion Relation

The Hubble constant emerges from:

$$H_0^2 \propto R \cdot |\mathbf{T}|^2$$

where:

- $R \approx 1/54$: Effective scalar curvature
- $|\mathbf{T}| \approx 0.0164$: Torsion magnitude

4.3.2 Intermediate Value

The framework predicts:

$$H_0 \approx 69.8 \text{ km/s/Mpc}$$

This intermediate value between CMB (67.4) and local (73.0) measurements suggests potential geometric resolution of the Hubble tension.

5 Summary

This supplement established the torsional geodesic dynamics of the GIFT framework:

Key Results

Result	**Value**	**Status**
Torsion magnitude	$\lvert\mathbf{T}\rvert \approx 0.0164$	THEORETICAL
$T_{e\varphi,\pi}$	-4.89	THEORETICAL
$T_{\pi\varphi,e}$	-0.45	THEORETICAL
$T_{e\pi,\varphi}$	$\sim 3 \times 10^{-5}$	THEORETICAL
Flow velocity	$\lvert v\rvert \approx 0.015$	PHENOMENOLOGICAL
$\dot{\alpha}/\alpha$ bound	$< 10^{-16}$ yr^{-1}	PHENOMENOLOGICAL

Table 6: Key dynamical results

Main Equations

Torsional connection:

$$\Gamma^k_{ij} = -\frac{1}{2} g^{kl} T_{ijl}$$

Geodesic equation:

$$\frac{d^2 x^k}{d\lambda^2} = \frac{1}{2} g^{kl} T_{ijl} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}$$

RG identification:

$$\lambda = \ln(\mu/\mu_0), \quad \beta^i = \frac{dx^i}{d\lambda}$$

Physical Interpretation

The framework provides geometric foundations for:

- Mass hierarchies from geodesic lengths
- CP violation from torsional twist
- RG flow from geodesic evolution
- Constant stability from ultra-slow velocity

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