

RECURRENCE FORMULAE FOR SPECTRAL DETERMINANTS

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ABSTRACT. We develop a unified method to study spectral determinants for several different manifolds, including spheres and hemispheres, and projective spaces. This is a direct consequence of an approach based on deriving recursion relations for the corresponding zeta functions, which we are then able to solve explicitly. Apart from new applications such as hemispheres, we also believe that the resulting formulae in the cases for which expressions for the determinant were already known are simpler and easier to compute in general, when compared to those resulting from other approaches.

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1. Introduction

In the last few decades the problem of evaluating the determinant of the Laplace operator on Riemannian manifolds has received considerable attention in the literature. These calculations may be traced back to the work of Minakshisundaram and Pleijel [23], Seeley [30], and Ray and Singer [28], and they are based on a regularization procedure via an associated zeta function. Techniques related to zeta function regularization gained some momentum within the mathematical physics community after Dowker and Critchley's [12], and Hawking's [20] papers from 1976 and 1977, respectively, described possible applications to physics – for a more complete historical account, see [13], for instance; see also [15].

Briefly, given an elliptic differential operator \mathfrak{T} defined on a compact manifold (with or without boundary) with discrete spectrum $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$, where in the case without boundary we leave out the zero eigenvalue, we define the spectral zeta function associated with \mathfrak{T} by

$$\zeta_{\mathfrak{T},M}(s) = \sum_{k=1}^{\infty} (\lambda_k)^{-s},$$

which converges on some half-plane $\Re(s) > \mu$. Under certain conditions, which will be satisfied by the Laplace–Beltrami operator on a compact manifold M (with or without boundary), for instance, the function $\zeta_{\mathfrak{T},M}$ may be continued analytically to a meromorphic function on the whole of the complex plane, while being analytic at zero. We now see that, if it were possible to differentiate the original series with respect to s at zero we would obtain, formally,

$$\zeta'_{\mathfrak{T},M}(0) = - \sum_{k=1}^{\infty} \log(\lambda_k),$$

and thus

$$e^{-\zeta'_{\mathfrak{T},M}(0)} = \prod_{k=1}^{\infty} \lambda_k. \quad (1.1)$$

Clearly this procedure is not justified as such, but it suggests that we may use the analytic continuation of $\zeta_{\mathfrak{T},M}$ to the whole complex plane as a meromorphic function to define the determinant of the operator \mathfrak{T} as

$$\det(\mathfrak{T}, M) := e^{-\zeta'_{\mathfrak{T},M}(0)}, \quad (1.2)$$

where we now use the expression $\zeta_{\mathfrak{T},M}$ to denote this meromorphic function. Also, whenever there is no ambiguity, we will omit the operator both in the index of the zeta function and in the determinant as in (1.2).

Formally, the above definition of $\det(\mathfrak{T}, M)$ is the product of the nonzero eigenvalues of the operator \mathfrak{T} acting on M , and is, in fact, a natural extension to the infinite dimensional setting of a formula that is valid in finite dimensions. While this allows us to make sense of the infinite product appearing in (1.1), the expression on the right-hand side of (1.2) will not, in general, admit a straightforward evaluation. In spite of this, there are several manifolds where this has been done, and for which the determinant has been calculated explicitly. One such example which will be relevant for us are the n -spheres \mathbb{S}^n with the standard metric, which have, in fact, deserved the attention of many authors across a span of more than thirty years, beginning with the work of Vardi in 1988 [33], and followed by several others such as Voros [34], Quine and Choi [27], Kumagai [21], Quine, Heydari and Song [26], Choi and Srivastava [9, 10], Awonusika [1] and Halji [17], with the last two papers dating from 2020.

As a first simple example, consider the case of the circle \mathbb{S}^1 , for which, leaving out the zero eigenvalue, the spectrum is given by $\lambda_{1,k} = k^2$ for $k = 1, 2, \dots$, with multiplicities $m_k^1 = 2$. This yields an associated zeta function given by

$$\zeta_{\mathbb{S}^1}(s) = \sum_{k=1}^{\infty} \frac{2}{k^{2s}} = 2\zeta(2s),$$

where $\zeta(s)$ denotes the Riemann zeta function. Proceeding as described above and taking into consideration that $\zeta'(0) = -\frac{1}{2} \log(2\pi)$, we obtain $\det(\Delta, \mathbb{S}^1) = 4\pi^2$. It is possible to continue in a similar way for higher-dimensional spheres, but not only do the resulting expressions become more involved as n increases, it also becomes clear that in order to find a general formula it will be necessary to make the dependence on the dimension more explicit – for illustration, see the expressions for the determinant for dimensions up to 9, given in Corollary 2.9. This has led the authors of the papers quoted above to try out different procedures to obtain closed-form formulae valid for all n , with the first explicit results having been obtained by Quine and Choi [27], and Kumagai [21]. Finally, we also note that spheres with the standard metric play an important role as critical and extremal metrics for the determinant of the Laplacian among certain classes of metrics [24, 25].

The purpose of the present paper is to provide a unified approach allowing us not only to compute the determinant of \mathbb{S}^n but also of other manifolds such as hemispheres and projective spaces. This is based on shifting the original eigenvalue sequence in order to obtain a *suitable* eigenvalue sequence leading to an appropriate zeta function to which we can then apply the method developed by the first author for the case of the quantum harmonic oscillator [14], namely, the derivation of a recursion formula for the new zeta function. This is then related to the zeta function of the original problem using the techniques devised by Voros [34] – this part of the process has some similarities with Choi's paper [8], where a general formula for the determinant in the odd-dimensional sphere is given. An important step in our method is an explicit formula for the spectral zeta function associated with the shifted eigenvalue sequence (see Lemma 2.3), obtained as a result of being able to determine the solutions of the recursion equations satisfied by the zeta-function. To the best of our knowledge, the derivation of recursion formulae and the solution of the corresponding equations in the context of spectral determinants had not been used previously, although we could trace some examples of the derivation of recursion formulae for partition functions to the work of Camporesi [7]. The recursions obtained also emphasize the dependence of determinants on the dimension. This is a well-known feature, which has been observed on several occasions in the literature – see, for instance [3, 14] for the determinant of the Dirac operator on the n -dimensional sphere and the quantum harmonic oscillator in n dimension, and [24] for extremal problems on spheres.

As a result of this process, we obtain explicit expressions for the determinant of the Laplacian on even- and odd-dimensional spheres, which may be found in Theorems 2.7 and 2.8, respectively. For further reference and comparison purposes, and apart from providing explicit expressions for some low-dimensional spheres explicitly, we also compute the numerical values up to dimension 10 000, showing the first 100 in Table 2.

Due to the flexibility of our method, we are able to apply it to other eigenvalue sequences such as those corresponding to hemispheres (with Dirichlet boundary conditions), and real projective spaces, for instance. The latter case was studied recently in [19], but we believe the results for hemispheres to be new – see [18] for a study of the analytic torsion in that case. The graphs with the values of the determinants as a function of the dimension in all these cases may be seen

in Figures 1, 2, and 3, displaying in a very clear way the dependence of these values on whether the dimension is even or odd. The behaviour observed in these graphs raises several questions related to monotonicity, the precise asymptotic behaviour of the determinants in each case and, on a more speculative side, whether the different limits observed in the only case considered where the manifold has a boundary are a consequence of that fact.

Due to its nature, we believe this method to be applicable to other situations such as complex and quaternionic projective spaces, and to hemispheres with Neumann boundary conditions, among others. These also include the quantum harmonic oscillator, closely related to the Dirac operator on spheres, which was studied in [14], and to which we return now to determine a closed-form solution of the recursion formulae derived there for the corresponding zeta function. The fact that in this case the asymptotic behaviour of the determinant in the dimension is exponentially decreasing, while we expect the remaining cases analysed here to have an algebraic behaviour, would indicate that our approach is not restricted to determinants with a specific behaviour.

The structure of the paper is as follows. In the next section we collect the necessary results and describe the procedure using the case of the n -sphere as an example. In Section 3 we apply our method to the other examples already mentioned above. Appendix A contains some useful facts about central factorial numbers which are used throughout the paper, while in Appendix B we provide some tables with numerical values for reference.

2. Method description: the case of \mathbb{S}^n

2.1. Eigenvalues and a tale of two zeta functions. Let $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ be the n -dimensional unit sphere with the standard metric induced by the \mathbb{R}^{n+1} Euclidean norm. The spectrum of the Laplace operator Δ on \mathbb{S}^n is well known and given by [2]

$$\omega_{n,k} = k(k + n - 1)$$

for $k = 0, 1, 2, \dots$, with multiplicities

$$m_k^n := \text{mult}(\omega_{n,k}) = \binom{n+k}{k} - \binom{n+k-2}{k-2} = (2k+n-1) \frac{(k+n-2)!}{k!(n-1)!} \quad (2.1)$$

Consider now the associated zeta function

$$\zeta_{\mathbb{S}^n}(s) = \sum_{k=1}^{\infty} (\omega_{n,k})^{-s} = \sum_{k=1}^{\infty} \frac{m_k^n}{(k(k+n-1))^s}$$

In order to compute $\zeta'_{\mathbb{S}^n}(0)$, we will consider a shift of the eigenvalues by a constant yielding a perfect square and a more manageable zeta function. To this end, define

$$\lambda_{n,k} := \omega_{n,k} + \lambda_n = \left(k + \frac{n-1}{2}\right)^2,$$

the eigenvalue sequence obtained after shifting $\omega_{n,k}$ by $\lambda_n = \left(\frac{n-1}{2}\right)^2$, and consider its associated zeta function

$$\zeta_{\mathbb{S}^n}(s, \lambda_n) = \sum_{k=1}^{\infty} (\omega_{n,k} + \lambda_n)^{-s} = \sum_{k=1}^{\infty} (\lambda_{n,k})^{-s} \quad (2.2)$$

It is clear that $\zeta_{\mathbb{S}^n}(s, 0) = \zeta_{\mathbb{S}^n}(s)$ and, for simplicity, in what follows we use a prime to denote the derivative with respect to the variable s , namely, $\zeta'(s, a) = \frac{\partial \zeta(s, a)}{\partial s}$. More important, this

new zeta function yields an associated determinant which is now possible to calculate in any dimension, and from which the original determinant may then be retrieved.

2.2. Relating the determinants of $\zeta_{\mathbb{S}^n}(s)$ and $\zeta_{\mathbb{S}^n}(s, \lambda_n)$: the method of Voros.

In 1987 Voros studied several functions associated with infinite increasing sequences of real numbers [34]. This was later extended to the more general case of complex number sequences by Quine, Heydari and Song [26]. Of particular relevance here are sequences formed by the eigenvalues of elliptic operators, including the Laplacian, and one important issue is to know when a certain sequence may be what is referred to in [26] as *zeta regularizable*, that is, when the corresponding zeta function has a meromorphic continuation with at most simple poles, to a right-half-plane containing the origin, and is analytic at the origin.

One possible way to address these issues, which was used in [34], uses techniques from analytic number theory, including the analytical continuation of Mellin transforms and the Weierstrass canonical product $E(\lambda)$ associated with the sequence $\{\lambda_k\}$, defined by

$$E(\lambda) := \prod_{k=1}^{\lfloor \mu \rfloor} \left\{ \left(1 - \frac{\lambda}{\lambda_k} \right) \exp \left(\frac{\lambda}{\lambda_k} + \frac{\lambda^2}{2\lambda_k^2} + \cdots + \frac{\lambda^{\lfloor \mu \rfloor}}{\lfloor \mu \rfloor \lambda_k^{\lfloor \mu \rfloor}} \right) \right\}, \quad (2.3)$$

where μ denotes the abscissa of (absolute) convergence of $\zeta_{\mathfrak{T},M}$, and the sum in the exponent is considered to vanish when $\mu < 1$.

In the following theorem we collect a result from [34, pp. 447] in an appropriate form to be used by us in the sequel.

Theorem 2.1 (Voros [34]). *Suppose that the sequence of eigenvalues $\{\lambda_k\}$ associated with the operator \mathfrak{T} acting on M is a monotonically increasing sequence of real numbers. Given a constant λ , the relationship between the determinants associated with the sequences $\{\lambda_k - \lambda\}$ and $\{\lambda_k\}$ is given by*

$$\zeta'_{\mathfrak{T},M}(0) = \zeta'_{\mathfrak{T},M}(0, -\lambda) + \sum_{m=1}^{\lfloor \mu \rfloor} FP[\zeta_{\mathfrak{T},M}(m, -\lambda)] \frac{\lambda^m}{m} + \sum_{m=2}^{\lfloor \mu \rfloor} c_{-m} H_{m-1} \frac{\lambda^m}{m!} - \log(E(\lambda))$$

where an empty sum (i.e. the case where $\mu < 1$) is to be considered zero, the Harmonic numbers H_n are given by

$$H_n = \sum_{k=1}^n \frac{1}{k},$$

the finite part (FP) is defined by

$$FP[f(s)] = \begin{cases} f(s), & \text{if } s \text{ is not a pole of } f \\ \lim_{\epsilon \rightarrow 0} \left(f(s + \epsilon) - \frac{\text{Res}(f, s)}{\epsilon} \right), & \text{if } s \text{ is a pole of } f \end{cases} \quad (2.4)$$

and

$$c_{-m} = \text{Res}(\zeta_{\mathfrak{T},M}, m) \Gamma(m) \quad (2.5)$$

where $\Gamma(z)$ denotes the Gamma function.

Remark 2.1. For a different formula for the coefficients c_{-m} see Section 2.6.

For our purposes, and for ease of reference, consider the following corollary of Theorem 2.1.

Corollary 2.2. Assume that $\left|\frac{\lambda}{\lambda_k}\right| < 1$ for all $k \in \mathbb{N}$. Then

$$\begin{aligned} \zeta'_{\mathfrak{T},M}(0) &= \zeta'_{\mathfrak{T},M}(0, -\lambda) + \sum_{m=1}^{\lfloor \mu \rfloor} FP[\zeta_{\mathfrak{T},M}(m, -\lambda)] \frac{\lambda^m}{m} + \sum_{m=2}^{\lfloor \mu \rfloor} c_{-m} H_{m-1} \frac{\lambda^m}{m!} \\ &\quad + \sum_{m=\lfloor \mu \rfloor+1}^{\infty} \zeta_{\mathfrak{T},M}(m, -\lambda) \frac{\lambda^m}{m} \end{aligned}$$

Proof. Applying logarithms to both sides of (2.3) yields

$$\begin{aligned} -\log(E(\lambda)) &= \sum_{k=1}^{\infty} \left[-\log\left(1 - \frac{\lambda}{\lambda_k}\right) - \sum_{m=1}^{\lfloor \mu \rfloor} \frac{\lambda^m}{m \lambda_k^m} \right] \\ &= \sum_{k=1}^{\infty} \left[\sum_{m=1}^{\infty} \frac{\lambda^m}{m \lambda_k^m} - \sum_{m=1}^{\lfloor \mu \rfloor} \frac{\lambda^m}{m \lambda_k^m} \right] \\ &= \sum_{m=\lfloor \mu \rfloor+1}^{\infty} \zeta_{\mathfrak{T},M}(m, -\lambda) \frac{\lambda^m}{m} \end{aligned}$$

where, in the first step, we can exchange the order of summation of the infinite series since both series are absolutely convergent due to the assumption $\left|\frac{\lambda}{\lambda_k}\right| < 1$ for all k ; this also allows us to use the logarithm series expansion in the second line. \square

2.3. Recursions. We shall now introduce the crucial step underlying our approach. Following Theorem A in [14], in which a two-term recursion formula in the dimension was derived for the spectral determinant of the quantum harmonic oscillator, we show that a similar recursion may also be obtained in our first example, i.e the Laplace operator on \mathbb{S}^n . For this case, the zeta function $\zeta_{\mathbb{S}^n}(s, \lambda_n)$ defined in (2.2) takes the form

$$\zeta_{\mathbb{S}^n}(s, \lambda_n) = \sum_{k=1}^{\infty} \frac{m_k^n}{\left(k + \frac{n-1}{2}\right)^{2s}}. \quad (2.6)$$

Lemma 2.3. The zeta function $\zeta_{\mathbb{S}^n}(s, \lambda_n)$ in (2.6) satisfies the following two-term recursion

$$\zeta_{\mathbb{S}^{n+2}}(s, \lambda_{n+2}) = \frac{\zeta_{\mathbb{S}^n}(s-1, \lambda_n) - \left(\frac{n-1}{2}\right)^2 \zeta_{\mathbb{S}^n}(s, \lambda_n)}{n(n+1)} - \left(\frac{2}{n+1}\right)^{2s}$$

with initial conditions

$$\begin{aligned} \zeta_{\mathbb{S}^1}(s, \lambda_1) &= 2\zeta(2s) \\ \zeta_{\mathbb{S}^2}(s, \lambda_2) &= (4^s - 2)\zeta(2s-1) - 4^s \end{aligned}$$

Proof. Consider the multiplicities m_k^n given in (2.1) and note that

$$\begin{aligned} m_k^{n+2} &= (2k+n+1) \frac{(k+n)!}{k!(n+1)!} \\ &= m_{k+1}^n \frac{(k+n)(k+1)}{n(n+1)} \end{aligned}$$

$$\begin{aligned}
&= m_{k+1}^n \frac{k^2 + k(n+1) + n}{n(n+1)} \\
&= m_{k+1}^n \frac{(2k+n+1)^2 - (n-1)^2}{4n(n+1)}.
\end{aligned}$$

Substituting this in the expression for $\zeta_{\mathbb{S}^n}(s, \lambda_n)$ given by (2.6) yields

$$\begin{aligned}
\zeta_{\mathbb{S}^{n+2}}(s, \lambda_{n+2}) &= \sum_{k=1}^{\infty} \frac{m_k^{n+2}}{\left(k + \frac{n+1}{2}\right)^{2s}} \\
&= \sum_{k=1}^{\infty} \frac{m_{k+1}^n}{\left(k + \frac{n+1}{2}\right)^{2s}} \frac{(2k+n+1)^2 - (n-1)^2}{4n(n+1)} \\
&= \sum_{k=1}^{\infty} \frac{m_{k+1}^n}{\left(k + \frac{n+1}{2}\right)^{2s}} \frac{(k + \frac{n+1}{2})^2 - (\frac{n-1}{2})^2}{n(n+1)} \\
&= \left[\sum_{k=1}^{\infty} \frac{m_{k+1}^n}{\left(k + \frac{n+1}{2}\right)^{2(s-1)}} - \left(\frac{n-1}{2}\right)^2 \sum_{k=1}^{\infty} \frac{m_{k+1}^n}{\left(k + \frac{n+1}{2}\right)^{2s}} \right] \frac{1}{n(n+1)} \\
&= \left[\sum_{k=2}^{\infty} \frac{m_k^n}{\left(k + \frac{n-1}{2}\right)^{2(s-1)}} - \left(\frac{n-1}{2}\right)^2 \sum_{k=2}^{\infty} \frac{m_k^n}{\left(k + \frac{n-1}{2}\right)^{2s}} \right] \frac{1}{n(n+1)} \\
&= \frac{\zeta_{\mathbb{S}^n}(s-1, \lambda_n) - (\frac{n-1}{2})^2 \zeta_{\mathbb{S}^n}(s, \lambda_n)}{n(n+1)} - \left(\frac{2}{n+1}\right)^{2s}.
\end{aligned}$$

□

Even though $\{\lambda_{n,k}\}$ is not the original eigenvalue sequence associated with the Laplacian on \mathbb{S}^n , its relevance should be clear by now. In the preceding section we presented a method due to Voros by which we are able to relate the functional determinant of zeta functions associated with different sequences, as long as these are only shifted by a constant. In fact we have that

$$\begin{aligned}
-\log [\det(\mathbb{S}^n)] = \zeta'_{\mathbb{S}^n}(0) &= \zeta'_{\mathbb{S}^n}(0, \lambda_n) + \sum_{m=1}^{|\mu|} FP[\zeta_{\mathbb{S}^n}(m, \lambda_n)] \frac{(\lambda_n)^m}{m} + \sum_{m=2}^{|\mu|} c_{-m} H_{m-1} \frac{(\lambda_n)^m}{m!} \\
&\quad + \sum_{m=|\mu|+1}^{\infty} \zeta_{\mathbb{S}^n}(m, \lambda_n) \frac{(\lambda_n)^m}{m}
\end{aligned}$$

The recursion in the preceding lemma will enable us to derive an explicit formula for $\zeta_{\mathbb{S}^n}(s, \lambda_n)$, specifically, a finite weighted sum of Riemann zeta functions $\zeta(s)$. The properties of such weights are induced by the recursion and shall be dealt with in Appendix A. As mentioned in the Introduction, this recursion clearly shows a dual behavior on the dimension.

2.4. Solution to the recursion: explicit expressions for $\zeta_{\mathbb{S}^n}(s, \lambda_n)$. We are now able to solve the recursion in Lemma 2.3 to obtain more manageable expressions for $\zeta_{\mathbb{S}^n}(s, \lambda_n)$ in even and odd dimensions separately.

Theorem 2.4. *Let $n \in \mathbb{N}$. The zeta function $\zeta_{\mathbb{S}^n}(s, \lambda_n)$ defined in (2.6) satisfies the following identities*

$$\begin{aligned}\zeta_{\mathbb{S}^{2n+1}}(s, \lambda_{2n+1}) &= \sum_{i=1}^n \bar{u}(n, i) \zeta(2(s-i)) - n^{-2s} \\ \zeta_{\mathbb{S}^{2n}}(s, \lambda_{2n}) &= \sum_{i=1}^n \bar{v}(n, i) (4^s - 2^{2i-1}) \zeta(2s-2i+1) - \left(\frac{2n-1}{2}\right)^{-2s}\end{aligned}$$

where $\bar{u}(n, i)$ and $\bar{v}(n, i)$ are defined in (A.5) and (A.6) respectively.

Proof.

- Odd dimensional case:

♦ Induction on n .

◊ *Base case:* $n = 1$

In view of Lemma 2.3, $\zeta_{\mathbb{S}^3}(s, \lambda_3) = \zeta(2(s-1))-1$ which agrees with Theorem 2.4 since $\bar{u}(1, 1) = 1$.

◊ *Induction step:*

Following the recursion in Lemma 2.3 for $\zeta_{\mathbb{S}^{2n+1}}(s, \lambda_{2n+1})$ we obtain

$$\begin{aligned}\zeta_{\mathbb{S}^{2n+1}}(s, \lambda_{2n+1}) &= \frac{\zeta_{\mathbb{S}^{2n-1}}(s-1, \lambda_{2n-1}) - (n-1)^2 \zeta_{\mathbb{S}^{2n-1}}(s, \lambda_{2n-1})}{2n(2n-1)} - n^{-2s} \\ &= \frac{\sum_{i=1}^{n-1} \bar{u}(n-1, i) \zeta(2(s-i-1)) - (n-1)^2 \sum_{i=1}^{n-1} \bar{u}(n-1, i) \zeta(2(s-i))}{2n(2n-1)} - n^{-2s} \quad \text{Induction Hypothesis} \\ &= \frac{\sum_{i=1}^n \bar{u}(n-1, i-1) \zeta(2(s-i)) - (n-1)^2 \sum_{i=1}^n \bar{u}(n-1, i) \zeta(2(s-i))}{2n(2n-1)} - n^{-2s} \quad \text{Proposition A.1 (iv) and (v)} \\ &= \left[\sum_{i=1}^n \frac{\bar{u}(n-1, i-1) - (n-1)^2 \bar{u}(n-1, i)}{2n(2n-1)} \zeta(2(s-i)) \right] - n^{-2s} \\ &= \sum_{i=1}^n \bar{u}(n, i) \zeta(2(s-i)) - n^{-2s} \quad \text{Proposition A.3 (ii)}\end{aligned}$$

- Even dimensional case: Analogous to the odd dimensional case by means of Proposition A.5. \square

From Theorem 2.4 we conclude that $\zeta_{\mathbb{S}^{2n+1}}(s, \lambda_{2n+1})$ is defined for $\Re(s) > n + \frac{1}{2}$ and it can be meromorphically continued to a function of the whole complex plane, since it depends uniquely on the zeta function $\zeta(s)$, with exactly n poles at $\{\frac{3}{2}, \frac{5}{2}, \dots, \frac{2n+1}{2}\}$. Analogously, we conclude that $\zeta_{\mathbb{S}^{2n}}(s, \lambda_{2n})$ is defined for $\Re(s) > n$ and can be analytically continued to a meromorphic function of the whole complex plane with exactly n poles at $\{1, 2, \dots, n\}$.

2.5. Recovering $\zeta_{\mathbb{S}^n}(s)$ and the determinant. We are now able to combine the expressions obtained in Theorem 2.4 and Voros method, namely, Corollary 2.2, to obtain simple and efficiently computable expressions for the determinant of the Laplacian Δ on the odd (\mathbb{S}^{2n+1}) and even (\mathbb{S}^{2n}) dimensional spheres separately.

In the following proposition we recall some properties of the Riemann and Hurwitz zeta functions, $\zeta(s)$ and $\zeta(s, a)$, respectively, and of the Bernoulli numbers B_n , all of which are well-known and may be found in [9, Chapter 2].

Proposition 2.5. *Let $p, n \in \mathbb{N}$ and consider the Bernoulli numbers, B_n . The following identities hold true.*

- (i) $\zeta(s) = \zeta(s, 1) = \frac{1}{(2^s - 1)} \zeta(s, \frac{1}{2})$
- (ii) $\zeta(s, a) = \zeta(s, a + n) + \sum_{j=0}^{n-1} (j+a)^{-s}$
- (iii) $\zeta'(s, n) = \zeta'(s) + \sum_{j=1}^{n-1} j^{-s} \log(j)$
- (iv) $\zeta'(s, n + \frac{1}{2}) = \log(2) 2^s \zeta(s) + (2^s - 1) \zeta'(s) + \sum_{j=1}^n \left(\frac{2j-1}{2}\right)^{-s} \log\left(\frac{2j-1}{2}\right)$
- (v) $\zeta'(0) = -\frac{1}{2} \log(2\pi)$
- (vi) $\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$
- (vii) $\sum_{k=1}^n k^p = \frac{1}{p+1} \sum_{j=0}^p \binom{p+1}{j} B_j n^{p+1-j}$ Faulhaber's formula

These will be used multiple times throughout. It will also prove useful to consider the following identities which can be found in [9, pp. 258, identities (64) and (67)].

Proposition 2.6. *Let $i \in \mathbb{N}_0$ and $|t| < |a|$. The following identities hold true.*

$$\begin{aligned} (i) \quad \sum_{m=1-i}^{\infty} \frac{\zeta(2m, a)}{m+i} t^{2m+2i} &= \sum_{k=0}^{2i} \binom{2i}{k} \left[\zeta'(-k, a-t) + (-1)^k \zeta'(-k, a+t) \right] t^{2i-k} - 2\zeta'(-2i, a) \\ (ii) \quad \sum_{\substack{m=-i \\ i \neq 0}}^{\infty} \frac{\zeta(2m+1, a)}{m+i+1} t^{2m+2i+2} &= \sum_{k=0}^{2i+1} \binom{2i+1}{k} \left[\zeta'(-k, a-t) - (-1)^k \zeta'(-k, a+t) \right] t^{2i-k+1} \\ &\quad - \frac{t^{2i+2}}{i+1} [\psi(2i+2) - \psi(a) + \gamma] - 2\zeta'(-2i-1, a) \end{aligned}$$

where $\psi(z)$ denotes the digamma function defined as the logarithmic derivative of Γ . For $n \in \mathbb{N}$ we have

$$\psi(n) = H_{n-1} - \gamma \tag{2.7}$$

where γ is the Euler-Mascheroni constant.

In particular, these will allow us to compute the Weiestrass canonical product $E(\lambda)$ defined by (2.3) and reduce it to a finite sum of more manageable functions.

2.5.1. The case of \mathbb{S}^{2n+1} .

It is now clear that $FP[\zeta_{\mathbb{S}^{2n+1}}(m, \lambda_{2n+1})] = \zeta_{\mathbb{S}^{2n+1}}(m, \lambda_{2n+1})$ from (2.4) and that $c_{-m}^{2n+1} = 0$ from (2.5) for $m \in \mathbb{N}$. From Corollary 2.2 we obtain

$$\zeta'_{\mathbb{S}^{2n+1}}(0) = \zeta'_{\mathbb{S}^{2n+1}}(0, \lambda_{2n+1}) + \sum_{m=1}^{\infty} \zeta_{\mathbb{S}^{2n+1}}(m, \lambda_{2n+1}) \frac{(\lambda_{2n+1})^m}{m} \tag{2.8}$$

where $\lambda_{2n+1} = n^2$. Differentiating $\zeta_{\mathbb{S}^{2n+1}}(s, \lambda_{2n+1})$ in Theorem 2.4 at $s = 0$ yields

$$\zeta'_{\mathbb{S}^{2n+1}}(0, \lambda_{2n+1}) = \sum_{i=1}^n 2\bar{u}(n, i)\zeta'(-2i) + \log(n^2) \quad (2.9)$$

In view of (2.8), and again using the expression for $\zeta_{\mathbb{S}^{2n+1}}(s, \lambda_{2n+1})$ in Theorem 2.4 consider the following tedious yet simple computations

$$\begin{aligned} & \sum_{m=1}^{\infty} \zeta_{\mathbb{S}^{2n+1}}(m, \lambda_{2n+1}) \frac{(\lambda_{2n+1})^m}{m} = \sum_{m=1}^{\infty} \frac{n^{2m}}{m} \left[\sum_{i=1}^n \bar{u}(n, i)\zeta(2(m-i)) - n^{-2m} \right] \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \left[\sum_{i=1}^n \bar{u}(n, i)n^{2m} \left(\zeta(2(m-i), n+1) + \sum_{k=1}^n k^{-2m+2i} \right) - 1 \right] \quad \text{Proposition 2.5 (ii)} \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \left[\sum_{i=1}^n \bar{u}(n, i)n^{2m}\zeta(2(m-i), n+1) + \sum_{i=1}^n \bar{u}(n, i)n^{2i} \right. \\ &\quad \left. + n^{2m} \sum_{k=1}^{n-1} k^{-2m} \sum_{i=1}^n \bar{u}(n, i)k^{2i} - 1 \right] \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{i=1}^n \bar{u}(n, i)n^{2m}\zeta(2(m-i), n+1) \quad \text{Proposition A.3 (iii) and (iv)} \\ &= \sum_{i=1}^n \bar{u}(n, i) \sum_{m=1}^{\infty} \frac{\zeta(2(m-i), n+1)n^{2m}}{m} \\ &= \sum_{i=1}^n \bar{u}(n, i) \sum_{m=1-i}^{\infty} \frac{\zeta(2m, n+1)n^{2m+2i}}{m+i} \\ &= \sum_{i=1}^n \bar{u}(n, i) \left(\sum_{k=0}^{2i} \binom{2i}{k} \left[\zeta'(-k) + (-1)^k \zeta'(-k, 2n+1) \right] n^{2i-k} \right. \\ &\quad \left. - 2\zeta'(-2i, n+1) \right) \quad \text{Proposition 2.6 (i)}_{a=n+1, t=n} \\ &= \sum_{i=1}^n \bar{u}(n, i) \left(\sum_{k=0}^{2i} \binom{2i}{k} \zeta'(-k)n^{2i-k} \left[(1 + (-1)^k) + (-1)^k \sum_{j=1}^{2n} j^k \log(j) \right] \right. \\ &\quad \left. - 2\zeta'(-2i) - 2 \sum_{j=1}^{2n} j^{2i} \log(j) \right) \quad \text{Proposition 2.5 (iii)} \\ &= \sum_{i=1}^n \bar{u}(n, i) \left(\sum_{k=0}^i 2 \binom{2i}{2k} \zeta'(-2k)n^{2i-2k} + \sum_{j=1}^{2n} \log(j)n^{2i} \sum_{k=0}^{2i} \binom{2i}{k} \left(\frac{-j}{n} \right)^k \right. \\ &\quad \left. - 2\zeta'(-2i) + \sum_{j=1}^{2n} j^{2i} \log(j^2) \right) \\ &= \sum_{i=1}^n \bar{u}(n, i) \left(\sum_{k=0}^i 2 \binom{2i}{2k} \zeta'(-2k)n^{2i-2k} - 2\zeta'(-2i) + \sum_{j=1}^{2n} \log(j)(n-j)^{2i} \right) \end{aligned}$$

$$\begin{aligned}
& - \sum_{j=1}^n j^{2i} \log(j^2) \Big) \\
& = \sum_{i=1}^n \bar{u}(n, i) \left(\sum_{k=0}^i 2 \binom{2i}{2k} \zeta'(-2k) n^{2i-2k} - 2\zeta'(-2i) + \sum_{j=1}^{n-1} \log \left(1 - \frac{j^2}{n^2} \right) j^{2i} \right. \\
& \quad \left. - \log \left(\frac{n}{2} \right) n^{2i} \right) \\
& = \sum_{i=1}^n \bar{u}(n, i) \sum_{k=0}^i 2 \binom{2i}{2k} \zeta'(-2k) n^{2i-2k} - \sum_{i=1}^n 2\bar{u}(n, i) \zeta'(-2i) - \log \left(\frac{n}{2} \right) \tag{Proposition A.3 (iii) and (iv)}
\end{aligned}$$

Theorem 2.7. Let $n \in \mathbb{N}$. The determinant of the Laplacian Δ on odd-dimensional spheres \mathbb{S}^{2n+1} satisfies the following identity

$$-\log [\det(\mathbb{S}^{2n+1})] = \zeta'_{\mathbb{S}^{2n+1}}(0) = \frac{2}{(2n)!} \sum_{k=1}^n \zeta'(-2k) [s(2n, 2k) + s(2n+1, 2k+1)] + \log \left(\frac{n}{\pi} \right)$$

where $s(n, k)$ are the Stirling numbers of the first kind as defined in Proposition A.1.

Proof. Using (2.8), (2.9) and the previous result we obtain

$$\begin{aligned}
\zeta'_{\mathbb{S}^{2n+1}}(0) & = \sum_{i=1}^n 2\bar{u}(n, i) \sum_{k=0}^i 2 \binom{2i}{2k} \zeta'(-2k) n^{2i-2k} + \log(2n) \\
& = \sum_{i=1}^n 2\bar{u}(n, i) \sum_{k=1}^i 2 \binom{2i}{2k} \zeta'(-2k) n^{2i-2k} + 2\zeta'(0) \sum_{i=1}^n \bar{u}(n, i) n^{2i} + \log(2n) \\
& = \sum_{i=1}^n 2\bar{u}(n, i) \sum_{k=1}^i 2 \binom{2i}{2k} \zeta'(-2k) n^{2i-2k} + \log \left(\frac{n}{\pi} \right) \\
& = \sum_{k=1}^n 2\zeta'(-2k) \sum_{i=k}^n 2 \binom{2i}{2k} \bar{u}(n, i) n^{2i-2k} + \log \left(\frac{n}{\pi} \right) \\
& = \frac{2}{(2n)!} \sum_{k=1}^n \zeta'(-2k) [s(2n, 2k) + s(2n+1, 2k+1)] + \log \left(\frac{n}{\pi} \right)
\end{aligned}$$

where the second step follows by Proposition 2.5 (v) and Proposition A.3 (iii) and the last step follows from Proposition A.3 (v). \square

2.5.2. The case of \mathbb{S}^{2n} .

From Corollary 2.2 together with the previous insights we obtain

$$\begin{aligned}
\zeta'_{\mathbb{S}^{2n}}(0) & = \zeta'_{\mathbb{S}^{2n}}(0, \lambda_{2n}) + \sum_{m=1}^n FP[\zeta'_{\mathbb{S}^{2n}}(0, \lambda_{2n})] \frac{(\lambda_{2n})^m}{m} + \sum_{m=2}^n c_{-m}^{2n} H_{m-1} \frac{(\lambda_{2n})^m}{m!} \\
& \quad + \sum_{m=n+1}^{\infty} \zeta'_{\mathbb{S}^{2n}}(m, \lambda_{2n}) \frac{(\lambda_{2n})^m}{m} \tag{2.10}
\end{aligned}$$

where $\lambda_{2n} = \left(\frac{2n-1}{2}\right)^2$. Unlike the odd-dimensional case, the components of (2.10) need further care. Differentiating $\zeta'_{\mathbb{S}^{2n}}(s, \lambda_{2n})$ in Theorem 2.4 at $s = 0$ yields

$$\zeta'_{\mathbb{S}^{2n}}(0, \lambda_{2n}) = \sum_{i=1}^n \bar{v}(n, i) (\log(4)\zeta(-2i+1) + (2 - 2^{2i})\zeta'(-2i+1)) - 2 \log\left(\frac{2}{2n-1}\right) \quad (2.11)$$

Consider the coefficients c_{-m}^{2n} defined by (2.5) and note that, using Theorem 2.4, we obtain

$$\begin{aligned} c_{-m}^{2n} &= \text{Res}(\zeta_{\mathbb{S}^{2n}}(s, \lambda_{2n}), m) \Gamma(m) \\ &= (4^m - 2^{2m-1}) \bar{v}(n, m) \text{Res}(\zeta(2s-2m+1), m) (m-1)! \\ &= 2^{2m-1} \bar{v}(n, m) \text{Res}(\zeta(2s-1), 1) (m-1)! \\ &= 2^{2m-2} \bar{v}(n, m) (m-1)! \end{aligned}$$

which leads to, in view of (2.10),

$$\sum_{m=2}^n c_{-m}^{2n} H_{m-1} \frac{(\lambda_{2n})^m}{m!} = \sum_{m=2}^n \bar{v}(n, m) \frac{(2n-1)^{2m}}{4m} H_{m-1} \quad (2.12)$$

Furthermore, consider the definition of the finite part as given by (2.4) and note that, using Theorem 2.4, we obtain

$$\begin{aligned} FP[\zeta_{\mathbb{S}^{2n}}(m, \lambda_{2n})] &= \sum_{i=1}^n \bar{v}(n, i) (4^m - 2^{2i-1}) FP[\zeta(2m-2i+1)] - \left(\frac{2}{2n-1}\right)^{2m} \\ &= \sum_{\substack{i=1 \\ i \neq m}}^n \underbrace{\bar{v}(n, i) (4^m - 2^{2i-1}) \zeta(2m-2i+1)}_{f_i(m)} - \left(\frac{2}{2n-1}\right)^{2m} \\ &\quad + \lim_{\epsilon \rightarrow 0} \left\{ f_m(m+\epsilon) + \frac{\text{Res}(f_m, m)}{\epsilon} \right\} \end{aligned}$$

Using the fact that $\zeta(s)$ satisfies (see [32, pp. 219])

$$\lim_{\epsilon \rightarrow 0} \left\{ \zeta(1+\epsilon) - \frac{1}{\epsilon} \right\} = \gamma$$

we see that the last term in the previous equation simplifies as follows

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \left\{ f_m(m+\epsilon) - \frac{\text{Res}(f_m, m)}{\epsilon} \right\} &= \lim_{\epsilon \rightarrow 0} \left\{ \bar{v}(n, m) (2^{2m+2\epsilon} - 2^{2m-1}) \zeta(1+2\epsilon) - \frac{\bar{v}(n, m) 2^{2m-1}}{2\epsilon} \right\} \\ &= \bar{v}(n, m) 2^{2m-1} \lim_{\epsilon \rightarrow 0} \left\{ (2^{2\epsilon+1} - 1) \left(\zeta(1+2\epsilon) - \frac{1}{2\epsilon} \right) + \frac{2^{2\epsilon}-1}{\epsilon} \right\} \\ &= \bar{v}(n, m) 2^{2m-1} (\gamma + \log(4)) \end{aligned}$$

Consider now, in virtue of (2.10), the following

$$\sum_{m=n+1}^{\infty} \zeta_{\mathbb{S}^{2n}}(m, \lambda_{2n}) \frac{(\lambda_{2n})^m}{m} + \sum_{m=1}^n FP[\zeta_{\mathbb{S}^{2n}}(m, \lambda_{2n})] \frac{(\lambda_{2n})^m}{m} =$$

$$\begin{aligned}
&= \sum_{m=1}^{\infty} \frac{1}{m} \left[\sum_{\substack{i=1 \\ i \neq m}}^n \bar{v}(n, i) (4^m - 2^{2i-1}) \zeta(2m - 2i + 1) \left(\frac{2n-1}{2} \right)^{2m} - 1 \right] \\
&\quad + \sum_{i=1}^n \bar{v}(n, i) \frac{(2n-1)^{2i}}{2i} (\gamma + \log(4))
\end{aligned}$$

We will now handle the infinite sum in the last equation by means of a procedure similar to that used in the odd-dimensional case.

$$\begin{aligned}
&\sum_{m=1}^{\infty} \frac{1}{m} \left[\sum_{\substack{i=1 \\ i \neq m}}^n \bar{v}(n, i) (4^m - 2^{2i-1}) \zeta(2m - 2i + 1) \left(\frac{2n-1}{2} \right)^{2m} - 1 \right] \\
&= \sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{\substack{i=1 \\ i \neq m}}^n \bar{v}(n, i) \zeta(2m - 2i + 1) (2n-1)^{2m} \right. \\
&\quad \left. - 2^{2i-1} \sum_{\substack{i=1 \\ i \neq m}}^n \bar{v}(n, i) \zeta(2m - 2i + 1) \left(\frac{2n-1}{2} \right)^{2m} - 1 \right) \\
&= \sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{\substack{i=1 \\ i \neq m}}^n \bar{v}(n, i) \zeta(2m - 2i + 1, 2n) (2n-1)^{2m} \right. \\
&\quad \left. - 2^{2i-1} \sum_{\substack{i=1 \\ i \neq m}}^n \bar{v}(n, i) \zeta(2m - 2i + 1, n) \left(\frac{2n-1}{2} \right)^{2m} \right) \\
&\quad + \sum_{m=1}^{\infty} \frac{1}{m} \left[\sum_{\substack{i=1 \\ i \neq m}}^n \bar{v}(n, i) (2n-1)^{2m} \left(\sum_{k=1}^{n-1} (2k)^{-2m+2i-1} - \sum_{k=1}^{2n-1} k^{-2m+2i-1} \right) - 1 \right] \\
&= \sum_{i=0}^{n-1} \bar{v}(n, i+1) \left(\sum_{\substack{m=-i \\ m \neq 0}}^{\infty} \frac{\zeta(2m+1, 2n)}{m+i+1} (2n-1)^{2m+2i+2} \right. \\
&\quad \left. - 2^{2i+1} \sum_{\substack{m=-i \\ m \neq 0}}^{\infty} \frac{\zeta(2m+1, n)}{m+i+1} \left(\frac{2n-1}{2} \right)^{2m+2i+2} \right) \\
&\quad + \sum_{m=1}^{\infty} \frac{1}{m} \left[(2n-1)^{2m} \sum_{k=1}^{n-1} (2k-1)^{-2m} \sum_{i=1}^n \bar{v}(n, i) (2k-1)^{2i-1} \right. \\
&\quad \left. + \sum_{i=1}^n \bar{v}(n, i) (2n-1)^{2i-1} - 1 \right] - \left(\frac{H_{n-1}}{2} - H_{2n-1} \right) \sum_{i=1}^n \bar{v}(n, i) \frac{(2n-1)^{2i}}{i} \\
&= \sum_{i=0}^{n-1} \bar{v}(n, i+1) \left(\sum_{\substack{m=-i \\ m \neq 0}}^{\infty} \frac{\zeta(2m+1, 2n)}{m+i+1} (2n-1)^{2m+2i+2} \right)
\end{aligned}$$

$$\begin{aligned}
& - 2^{2i+1} \sum_{\substack{m=-i \\ m \neq 0}}^{\infty} \frac{\zeta(2m+1, n)}{m+i+1} \left(\frac{2n-1}{2} \right)^{2m+2i+2} \\
& - \left(\frac{H_{n-1}}{2} - H_{2n-1} \right) \sum_{i=1}^n \bar{v}(n, i) \frac{(2n-1)^{2i}}{i}
\end{aligned}$$

where the second step follows from Proposition 2.5 (ii) and, in the last step, we used Proposition A.5 (iii) and (iv) to conclude the terms inside the square brackets vanish. In order to simplify the two infinite sums inside the parenthesis, consider the second equation in Proposition 2.6 (ii) applied twice ($a = 2n$, $t = 2n-1$ and $a = n$, $t = \frac{2n-1}{2}$ resp.) to obtain

$$\begin{aligned}
& \sum_{\substack{m=-i \\ m \neq 0}}^{\infty} \frac{\zeta(2m+1, 2n)}{m+i+1} (2n-1)^{2m+2i+2} - 2^{2i-1} \sum_{\substack{m=-i \\ m \neq 0}}^{\infty} \frac{\zeta(2m+1, n)}{m+i+1} \left(\frac{2n-1}{2} \right)^{2m+2i+2} \\
& = \sum_{k=0}^{2i+1} \binom{2i+1}{k} \left[\zeta'(-k) + (-1)^{k+1} \zeta'(-k, 4n-1) \right] (2n-1)^{2i+1-k} \\
& \quad - \frac{(2n-1)^{2i+2}}{i+1} (\psi(2i+2) - \psi(2n) + \gamma) - 2\zeta'(-2i-1, 2n) \\
& \quad - 2^{2i+1} \left(\sum_{k=0}^{2i+1} \binom{2i+1}{k} \left[\zeta' \left(-k, \frac{1}{2} \right) + (-1)^{k+1} \zeta' \left(-k, 2n - \frac{1}{2} \right) \right] \right. \\
& \quad \times \left. \left(\frac{2n-1}{2} \right)^{2i+1-k} - \frac{(2n-1)^{2i+2}}{i+1} (\psi(2i+2) - \psi(n) + \gamma) - 2\zeta'(-2i-1, n) \right) \\
& = \sum_{k=0}^i 2 \binom{2i+1}{2k+1} (2n-1)^{2i-2k} (2^{2k+1} \zeta'(-2k-1) - \log(2) \zeta(-2k-1)) \\
& \quad + \sum_{j=1}^{2n-1} \log \left(\frac{2j-1}{2} \right) (2n-2j)^{2i+1} - \sum_{j=1}^{4n-2} \log(j) (2n-j-1)^{2i+1} \\
& \quad + \frac{(2n-1)^{2i+2}}{i+1} \left(\frac{\psi(2i+2)}{2} - \psi(2n) + \frac{\psi(n)}{2} + \frac{\gamma}{2} \right) \\
& \quad + (2^{2i+2} - 2) \zeta'(-2i-1) + 2 \left(\sum_{j=1}^{n-1} \log(j) (2j)^{2i+1} - \sum_{j=1}^{2n-1} \log(j) j^{2i+1} \right)
\end{aligned}$$

where Proposition 2.5 (iii) was used in the last step.

Theorem 2.8. *Let $n \in \mathbb{N}$. The determinant of the Laplacian on even-dimensional spheres satisfies the following identity*

$$\begin{aligned}
-\log \left[\det \left(\mathbb{S}^{2n} \right) \right] = \zeta'_{\mathbb{S}^{2n}}(0) & = \frac{2}{(2n-1)!} \sum_{k=1}^n \zeta'(-2k+1) (s(2n-1, 2k-1) + s(2n, 2k)) \\
& + \sum_{i=1}^n \bar{v}(n, i) \frac{(2n-1)^{2i}}{2i} \left(\frac{H_{i-1}}{2} - H_{2i-1} \right) + \log(2n-1)
\end{aligned}$$

where $s(n, k)$ are the Stirling numbers of the first kind as defined by equation (A.1), and $\bar{v}(n, k)$ are as defined by formula (A.6).

Proof. Using (2.11), (2.12) and the previous result together with (2.10) we obtain

$$\begin{aligned}
\zeta'_{S^{2n}}(0) &= \sum_{i=1}^n \bar{v}(n, i) \left[\frac{(2n-1)^{2i}}{4i} H_{i-1} + \frac{(2n-1)^{2i}}{2i} (\gamma + \log(4)) \right. \\
&\quad + \sum_{k=1}^i 2 \binom{2i-1}{2k-1} (2n-1)^{2i-2k} (2^{2k-1} \zeta'(-2k+1) - \log(2) \zeta(-2k+1)) \\
&\quad + \frac{(2n-1)^{2i+2}}{i+1} \left(\frac{\psi(2i+2)}{2} - \psi(2n) + \frac{\psi(n)}{2} + \frac{\gamma}{2} \right) \\
&\quad - \left(\frac{H_{n-1}}{2} - H_{2n-1} \right) \frac{(2n-1)^{2i}}{i} \\
&\quad + \log(4) \zeta(-2i+1) + (2-2^{2i}) \zeta'(-2i+1) \\
&\quad + \sum_{j=1}^{2n-1} \log \left(\frac{2j-1}{2} \right) (2n-2j)^{2i+1} - \sum_{j=1}^{4n-2} \log(j) (2n-j-1)^{2i+1} \\
&\quad \left. + 2 \sum_{j=1}^{n-1} \log(j) (2j)^{2i+1} - 2 \sum_{j=1}^{2n-1} \log(j) j^{2i+1} \right] - 2 \log \left(\frac{2}{2n-1} \right) \\
&= \sum_{i=1}^n \bar{v}(n, i) \left[\log(4) \zeta(-2i+1) + \frac{(2n-1)^{2i}}{2i} \left(\log(4) + H_{2i-1} - \frac{H_{i-1}}{2} \right) \right. \\
&\quad + \sum_{k=1}^i 2 \binom{2i-1}{2k-1} (2n-1)^{2i-2k} (2^{2k-1} \zeta'(-2k+1) - \log(2) \zeta(-2k+1)) \\
&\quad \left. - \log(4) \sum_{j=1}^{n-1} (2j)^{2i-1} \right] - \log \left(\frac{2n-1}{2} \right) \\
&= \frac{2}{(2n-1)!} \sum_{k=1}^n \zeta'(-2k+1) (s(2n-1, 2k-1) + s(2n, 2k)) \\
&\quad + \sum_{i=1}^n \bar{v}(n, i) \left[-\log(4) \sum_{k=1}^i \binom{2i-1}{2k-1} (2n-1)^{2i-2k} \zeta(-2k+1) + \log(4) \zeta(-2i+1) \right. \\
&\quad \left. + \frac{(2n-1)^{2i}}{2i} \left(\log(4) + H_{2i-1} - \frac{H_{i-1}}{2} \right) - \log(4) \sum_{j=1}^{n-1} (2j)^{2i-1} \right] + \log \left(\frac{2n-1}{2} \right) \\
&= \frac{2}{(2n-1)!} \sum_{k=1}^n \zeta'(-2k+1) (s(2n-1, 2k-1) + s(2n, 2k)) \\
&\quad + \sum_{i=1}^n \bar{v}(n, i) \frac{(2n-1)^{2i}}{2i} \left(\frac{H_{i-1}}{2} - H_{2i-1} \right) + \log(2n-1)
\end{aligned}$$

where, in the second step, both (2.7) and Proposition A.6 (*i*) and (*ii*) are used and, in the third step, we employ Proposition A.5 (*v*) and finally, in the last step, we use Proposition 2.5 (*vi*) and (*vii*). \square

Using the expressions for the determinant obtained in Theorem 2.7 and Theorem 2.8 we present the first few values explicitly and numerically in the following corollary.

Corollary 2.9. *The determinant of the Laplacian on \mathbb{S}^n ($n = 2, 3, 4, 5, 6, 7, 8, 9$) are given by*

$$\begin{aligned}\det(\mathbb{S}^2) &= e^{\frac{1}{6}} A^4 \\ &= 3.19531\dots \\ \det(\mathbb{S}^3) &= \pi \exp\left[\frac{\zeta(3)}{2\pi^2}\right] \\ &= 3.33885\dots \\ \det(\mathbb{S}^4) &= \frac{1}{3} \exp\left[\frac{83}{144} - \frac{2\zeta'(-3)}{3}\right] A^{\frac{13}{3}} \\ &= 1.73694\dots \\ \det(\mathbb{S}^5) &= \frac{\pi}{2} \exp\left[\frac{23\zeta(3)}{24\pi^2} - \frac{\zeta(5)}{8\pi^4}\right] \\ &= 1.76292\dots \\ \det(\mathbb{S}^6) &= \frac{1}{5} \exp\left[\frac{1381}{2160} - 2\zeta'(-3) - \frac{\zeta'(-5)}{30}\right] A^{\frac{149}{30}} \\ &= 1.29002\dots \\ \det(\mathbb{S}^7) &= \frac{\pi}{3} \exp\left[\frac{949\zeta(3)}{720\pi^2} - \frac{13\zeta(5)}{24\pi^4} + \frac{\zeta(7)}{32\pi^6}\right] \\ &= 1.22252\dots \\ \det(\mathbb{S}^8) &= \frac{1}{7} \exp\left[\frac{4730849}{7257600} - \frac{1199\zeta'(-3)}{360} - \frac{71\zeta'(-5)}{360} - \frac{\zeta'(-7)}{1260}\right] A^{\frac{383}{70}} \\ &= 1.05041\dots \\ \det(\mathbb{S}^9) &= \frac{\pi}{4} \exp\left[\frac{16399\zeta(3)}{10080\pi^2} - \frac{2087\zeta(5)}{1920\pi^4} + \frac{31\zeta(7)}{128\pi^6} - \frac{\zeta(9)}{128\pi^8}\right] \\ &= 0.94673\dots\end{aligned}$$

where $A := e^{\frac{1}{12} - \zeta'(-1)}$ is the Glaisher-Kinkelin's constant.

In Appendix B we provide the numerical values for determinants up to dimension 100 for reference. However, the above formulae allow us to compute much higher dimensions and in Figure 1 we show a graph with the values up to dimension 10000. The two distinct sets of points correspond to even and odd dimensions.

2.6. Discrepancies in the numerical values in the literature. There are some discrepancies in the literature for the values of the determinant of the sphere, namely, between those obtained by Kumagai [21] and Quine and Choi [27], and those obtained by Choi and Srivastava in [9]. Commenting upon these discrepancies in [9, Section 5.5], Choi and Srivastava trace them back to the values used for the coefficients c_{-m} in these papers, as the differences coincide with

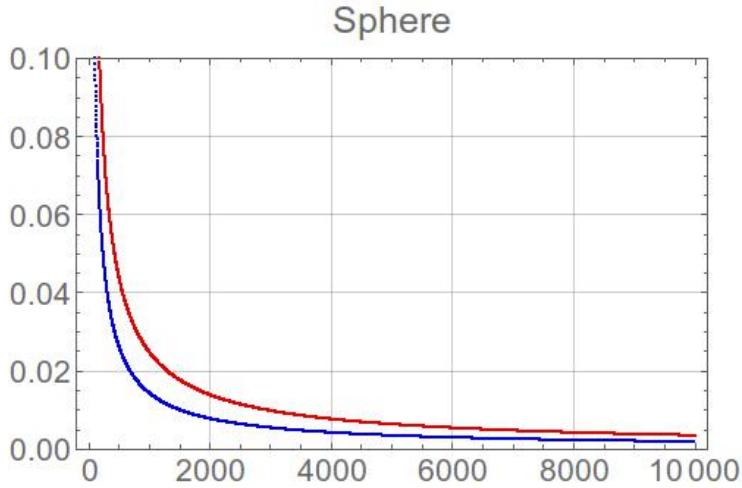


FIGURE 1. The first 10 000 values for $\det(\mathbb{S}^n)$ (odd dimensions: blue; even dimensions: red).

some of the values of c_{-m} used. They then suggest that the way in which these values are calculated should be examined carefully. The formulae for these coefficients are originally given in Voros [34, pp. 444] as in formula (2.5) above, and this is what we have used in this paper. We believe that there is a misprint in the simplified formula given in [34, equation (3.3)], and that the correct formula for positive m should read as

$$c_{-m} = (-1)^m \lim_{s \rightarrow m} \frac{\zeta_{\mathfrak{T},M}(s)}{\Gamma(1-s)}.$$

To see this, note that following the usual procedure using the Mellin transform of the partition function associated with an eigenvalue sequence λ_k , namely,

$$\Theta(t) = \sum_{k=0}^{\infty} e^{-t\lambda_k},$$

we obtain

$$\begin{aligned} \zeta_{\mathfrak{T},M}(s) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \Theta(t) dt \\ &= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \Theta(t) dt + \frac{1}{\Gamma(s)} \int_1^{+\infty} t^{s-1} \Theta(t) dt \end{aligned}$$

where the rightmost term is analytic in s . Following the notation used in [34], the expansion of the partition function at 0^+ for the Laplace-Beltrami operator on a compact manifold (with or without boundary) may be written as

$$\Theta(t) \sim \sum_{k=0}^{+\infty} c_{i_k} t^{i_k},$$

where the indexes i_k form an increasing sequence of real numbers growing to infinity – for more details on the coefficients c_{i_k} and i_k see, for instance, [16] and the references therein. We now

sketch the remaining part of the argument, which still follows a standard approach. Writing the above as

$$\begin{aligned}\zeta_{\mathfrak{T},M}(s) &= \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left[\Theta(t) - \sum_{k=0}^p c_{i_k} t^{i_k} \right] dt + \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \sum_{k=0}^p c_{i_k} t^{i_k} dt \\ &\quad + \frac{1}{\Gamma(s)} \int_1^{+\infty} t^{s-1} \Theta(t) dt \\ &= \frac{1}{\Gamma(s)} \sum_{k=0}^p \frac{c_{i_k}}{s+i_k} + \frac{1}{\Gamma(s)} F(s),\end{aligned}$$

where p is such that all the singular terms have been incorporated into the finite sum and F is a function analytic for $\operatorname{Re}(s) > 0$. The above is valid for $\operatorname{Re}(s) > \mu$ and, by analytic continuation, also for $0 < \operatorname{Re}(s) < \mu$, with the exception of the poles at the points $s = -i_k$. Dividing now both sides by $\Gamma(1-s)$ and using Euler's reflection formula we obtain

$$\frac{\zeta_{\mathfrak{T},M}(s)}{\Gamma(1-s)} = \frac{\sin(\pi s)}{\pi} \sum_{k=0}^p \frac{c_{i_k}}{s+i_k} + \frac{\sin(\pi s)}{\pi} F(s).$$

As was pointed out in [34], the function $\zeta_{\mathfrak{T},M}$ has poles at the positive values of $-i_k$ and, whenever this equals an integer m , we obtain

$$\lim_{s \rightarrow m} \frac{\zeta_{\mathfrak{T},M}(s)}{\Gamma(1-s)} = (-1)^m c_{-m},$$

yielding the desired formula.

3. Application to other examples

The spectra of the Laplacian on the n -dimensional hemisphere \mathbb{S}_+^n with Dirichlet boundary conditions, and on the n -dimensional real projective space \mathbb{RP}^n are well known [2, 4] and quite similar, for geometrical reasons, to the spectra of spheres. This enables us to replicate the previous steps for these examples and obtain the recursions, and explicit formulae for the zeta functions and determinants which we indicate below. Since the proofs are analogous to those for the sphere given above, except in the case of hemispheres for which the results are new, we present them without proof. In addition, we also revisit the case of the quantum harmonic oscillator, already studied in [14], but for which we now present an explicit expression for the corresponding zeta function.

3.1. Hemispheres. The spectral determinant of hemispheres with Dirichlet boundary conditions has never, to the best of our knowledge, been analyzed in the literature before. We shall now show how the method described in the previous section for the case of spheres may be applied in a similar way to this case. Except for the zero eigenvalue, the spectrum of the Laplacian on the n -dimensional hemisphere with Dirichlet boundary conditions is, in fact, the same as that on the n -dimensional sphere, namely, of the form $k(k+n-1)$, with k a positive integer. The difference between the two lies in the corresponding multiplicities which are now given by

$$m_k^n = \binom{n+k-2}{k-1},$$

where $k \in \mathbb{N}$ - see [2]. As in the sphere case, we apply a shift to the spectrum of λ_n and the corresponding shifted zeta function is thus defined as

$$\zeta_{\mathbb{S}^n_+}(s, \lambda_n) = \sum_{k=1}^{\infty} \frac{m_k^n}{\left(k + \frac{n-1}{2}\right)^{2s}}$$

with the series being absolutely convergent for $\operatorname{Re}(s) > n$. From this it is possible to derive a recurrence relation similar to Lemma 2.3 which now reads as

$$\zeta_{\mathbb{S}^{n+2}_+}(s, \lambda_{n+2}) = \frac{\zeta_{\mathbb{S}^n_+}(s-1, \lambda_n) - \zeta_{\mathbb{S}^n_+}(s-\frac{1}{2}, \lambda_n) - \left(\frac{n^2-1}{4}\right) \zeta_{\mathbb{S}^n_+}(s, \lambda_n)}{n(n+1)} \quad (3.1)$$

valid for all positive integers n and complex numbers s in the domain of the functions involved. Furthermore,

$$\begin{aligned} \zeta_{\mathbb{S}^1_+}(s, \lambda_1) &= \zeta(2s) \\ \zeta_{\mathbb{S}^2_+}(s, \lambda_2) &= (2^{2s-1} - 1)\zeta(2s-1) - (2^{2s-1} - \frac{1}{2})\zeta(2s) \end{aligned}$$

so that $\zeta_{\mathbb{S}^n_+}(s, \lambda_n)$ may be determined for all n by successive applications of the recursion formula.

Theorem 3.1. *Let $n \in \mathbb{N}$. The zeta function $\zeta_{\mathbb{S}^n_+}(s, \lambda_n)$ satisfies the following identities*

$$\begin{aligned} \zeta_{\mathbb{S}^{2n-1}_+}(s, \lambda_{2n-1}) &= \frac{1}{2} \sum_{i=1}^n \bar{u}(n-1, i-1) \left[\zeta(2s-2i+2) - (n-1)\zeta(2s-2i+3) \right] \\ \zeta_{\mathbb{S}^{2n}_+}(s, \lambda_{2n}) &= \sum_{i=1}^n \bar{v}(n, i) \left[\left(2^{2s-1} - 2^{2i-2}\right) \zeta(2s-2i+1) - (2n-1) \left(2^{2s-1} - 2^{2i-3}\right) \zeta(2s-2i+2) \right] \end{aligned}$$

Proof.

- Odd dimensional case:

♦ Induction on n .

Induction step:

Following the recursion in (3.1) for $\zeta_{\mathbb{S}^{2n+1}}(s, \lambda_{2n+1})$ we obtain

$$\begin{aligned} \zeta_{\mathbb{S}^{2n+1}_+}(s, \lambda_{2n+1}) &= \frac{\zeta_{\mathbb{S}^{2n-1}_+}(s-1, \lambda_{2n-1}) - \zeta_{\mathbb{S}^{2n-1}_+}(s-\frac{1}{2}, \lambda_{2n-1}) - (n(n-1)) \zeta_{\mathbb{S}^{2n-1}_+}(s, \lambda_{2n-1})}{2n(2n-1)} \\ &= \frac{1}{2n(2n-1)} \left[\sum_{i=1}^n \bar{u}(n-1, i-1) (\zeta(2s-2i) - n\zeta(2s-2i+1)) \right. \\ &\quad \left. - (n-1)^2 \sum_{i=1}^n \bar{u}(n-1, i-1) (\zeta(2s-2i+2) - n\zeta(2s-2i+3)) \right] \\ &= \frac{1}{2} \sum_{i=1}^n \bar{u}(n-1, i-1) \left[\zeta(2s-2i+2) - (n-1)\zeta(2s-2i+3) \right] \end{aligned}$$

where we use the Induction Hypothesis and Proposition A.1 (iv), (v) and A.3 (ii) in that order. The even dimensional case can be shown in exactly the same way by means of Proposition A.5. \square

For completeness, and since the results for hemispheres are new, we will provide some insight into the proof in spite of its similarities with the case of spheres. We begin by obtaining the recursion (3.1) and its solution $\zeta_{\mathbb{S}_+^n}(s, \lambda_n)$, in terms of the Riemann zeta function as presented in Theorem 3.2 and in total analogy with Sections 2.3 and 2.4. Then we leverage such expressions together with Voros method, namely Corollary 2.2, to obtain the associated determinant. Namely, we obtain that $\det(\mathbb{S}_+^n)$ satisfies the following identity.

$$\begin{aligned} -\log [\det(\mathbb{S}_+^n)] &= \zeta'_{\mathbb{S}_+^n}(0) = \zeta'_{\mathbb{S}_+^n}(0, \lambda_n) + \sum_{m=1}^{\lfloor \mu \rfloor} FP[\zeta_{\mathbb{S}_+^n}(m, \lambda_n)] \frac{(\lambda_n)^m}{m} + \sum_{m=2}^{\lfloor \mu \rfloor} c_{-m} H_{m-1} \frac{(\lambda_n)^m}{m!} \\ &\quad + \sum_{m=\lfloor \mu \rfloor + 1}^{\infty} \zeta_{\mathbb{S}_+^n}(m, \lambda_n) \frac{(\lambda_n)^m}{m} \end{aligned} \quad (3.2)$$

3.1.1. The case of \mathbb{S}_+^{2n-1} . We observe that $\zeta_{\mathbb{S}_+^{2n-1}}(s, \lambda_{2n-1})$ as given in Theorem 3.1 is convergent in the half-plane $\operatorname{Re}(s) > n - 1 = \mu$, and $\lambda_{2n-1} = (n-1)^2$. The first part of equation (3.2) is given by the following expression.

$$\zeta'_{\mathbb{S}_+^{2n-1}}(0, \lambda_{2n-1}) = \sum_{i=1}^n \bar{u}(n-1, i-1) [\zeta'(-2i+2) - (n-1)\zeta'(-2i+3)] \quad (3.3)$$

Now consider the coefficients c_{-m}^{2n-1} as defined by (2.5) which, in this case, translates into

$$\begin{aligned} c_{-m}^{2n-1} &= \operatorname{Res}\left(\zeta_{\mathbb{S}_+^{2n-1}}(s, \lambda_{2n-1}), m\right) \Gamma(m) \\ &= -\bar{u}(n-1, m)(m-1)! \frac{(n-1)}{4}, \end{aligned}$$

with the third term in (3.2) then becoming

$$\sum_{m=2}^{n-1} c_{-m}^{2n-1} H_{m-1} \frac{(\lambda_{2n-1})^m}{m!} = - \sum_{m=2}^n \bar{u}(n-1, m) \frac{(n-1)^{2m+1}}{4m} H_{m-1}. \quad (3.4)$$

Next, we evaluate the finite part of $\zeta_{\mathbb{S}_+^{2n-1}}(m, \lambda_{2n-1})$ as indicated in (2.4). This gives

$$\begin{aligned} FP[\zeta_{\mathbb{S}_+^{2n-1}}(m, \lambda_{2n-1})] &= \frac{1}{2} \sum_{i=1}^n \bar{u}(n-1, i-1) FP[\zeta(2m-2i+2) - (n-1)\zeta(2m-2i+3)] \\ &= \frac{1}{2} \sum_{i=1}^n \bar{u}(n-1, i-1) \zeta(2m-2i+2) \\ &\quad - (n-1) \frac{1}{2} \sum_{\substack{i=1 \\ i \neq m+1}}^n \bar{u}(n-1, i-1) \zeta(2m-2i+3) \\ &\quad - \bar{u}(n-1, m) \frac{(n-1)}{2} \gamma \end{aligned}$$

We proceed with the evaluation of the second and fourth terms in expression (3.2), making use of Proposition 2.5 (ii) and Proposition A.3 (iii) and (iv) in the following way:

$$\begin{aligned}
& \sum_{m=n}^{\infty} \zeta_{\mathbb{S}_+^{2n-1}}(m, \lambda_{2n-1}) \frac{(\lambda_{2n})^m}{m} + \sum_{m=1}^{n-1} FP[\zeta_{\mathbb{S}_+^{2n-1}}(m, \lambda_{2n-1})] \frac{(\lambda_{2n-1})^m}{m} \\
&= \sum_{m=1}^{\infty} \frac{(n-1)^{2m}}{2m} \left[\sum_{i=1}^n \bar{u}(n-1, i-1) \zeta(2m-2i+2) - (n-1) \sum_{\substack{i=1 \\ i \neq m+1}}^n \bar{u}(n-1, i-1) \zeta(2m-2i+3) \right] \\
&\quad - \sum_{i=1}^{n-1} \bar{u}(n-1, i) \frac{(n-1)^{2i+1}}{2i} \gamma \\
&= \frac{1}{2} \sum_{i=1}^n \bar{u}(n-1, i-1) \left[\sum_{m=2-i}^{\infty} \frac{\zeta(2m, n)}{m+i-1} (n-1)^{2m+2i-2} - (n-1) \sum_{\substack{m=2-i \\ m \neq 0}}^{\infty} \frac{\zeta(2m+1, n)}{m+i-1} (n-1)^{2m+2i-2} \right] \\
&\quad - \sum_{i=1}^{n-1} \bar{u}(n-1, i) \frac{(n-1)^{2i+1}}{2i} (\gamma - 2H_{n-1})
\end{aligned}$$

Once again, to simplify the two infinite sums inside the brackets, consider Proposition 2.6 (i) and (ii) ($a = n$ and $t = n-1$) to obtain

$$\begin{aligned}
& \frac{1}{2} \sum_{i=1}^n \bar{u}(n-1, i-1) \left[\sum_{k=0}^{2i-2} \binom{2i-2}{k} [\zeta'(-k) + (-1)^k \zeta'(-k, 2n-1)] (n-1)^{2i-k-2} \right. \\
&\quad - \sum_{k=0}^{2i-3} \binom{2i-3}{k} [\zeta'(-k) + (-1)^{k+1} \zeta'(-k, 2n-1)] (n-1)^{2i-k-2} \\
&\quad + \frac{(n-1)^{2i-1}}{i-1} [\phi(2i-2) - \phi(n) + \gamma] \\
&\quad \left. - 2\zeta'(-2i+2, n) + 2(n-1)\zeta'(-2i+3, n) \right] \\
&\quad - \sum_{i=1}^{n-1} \bar{u}(n-1, i) \frac{(n-1)^{2i+1}}{2i} (\gamma - 2H_{n-1}) \\
&= \sum_{i=1}^n \bar{u}(n-1, i-1) \left[\sum_{k=0}^{i-1} \binom{2i-2}{2k} \zeta'(-2k) (n-1)^{2i-2k-2} - \sum_{k=0}^{i-1} \binom{2i-3}{2k-1} \zeta'(-2k+1) (n-1)^{2i-2k-1} \right. \\
&\quad \left. - \zeta'(-2i+2) + (n-1)\zeta'(-2i+3) \right] + \sum_{i=1}^{n-1} \bar{u}(n-1, i) \frac{(n-1)^{2i+1}}{2i} H_{2i-1} \\
&= \frac{1}{(2n-2)!} \sum_{k=0}^{2n-1} \zeta'(-k) [s(2n-1, k+1) + (-1)^k s(2n-2, k)] \\
&\quad - \sum_{i=1}^n \bar{u}(n-1, i-1) [\zeta'(-2i+2) - (n-1)\zeta'(-2i+3)] + \sum_{i=1}^{n-1} \bar{u}(n-1, i) \frac{(n-1)^{2i+1}}{2i} H_{2i-1}.
\end{aligned}$$

Combining this with (3.3) and (3.4) we are able to compute (3.2) and obtain $\det(\mathbb{S}_+^{2n-1})$ as given in Theorem 3.2.

3.1.2. The case of \mathbb{S}_+^{2n} . We observe that $\zeta_{\mathbb{S}_+^{2n}}(s, \lambda_{2n})$ as given in Theorem 3.1 is convergent in the section $\Re(s) > n = \mu$ and $\lambda_{2n-1} = (\frac{2n-1}{2})^2$. The first part of equation (3.2) is given by the following expression.

$$\begin{aligned} \zeta'_{\mathbb{S}_+^{2n}}(0, \lambda_{2n}) &= \sum_{i=1}^n \bar{v}(n, i) \left[\log(2) (\zeta'(-2i+1) - (2n-1)\zeta'(-2i+2)) \right. \\ &\quad \left. + (1 - 2^{2i-1}) \zeta'(-2i+1) - (2n-1)(1 - 2^{2i-2}) \zeta'(-2i+2) \right]. \end{aligned}$$

Now consider the coefficients c_{-m}^{2n-1} as defined by (2.5) which, in this case, are given by

$$\begin{aligned} c_{-m}^{2n} &= \text{Res} \left(\zeta_{\mathbb{S}_+^{2n}}(s, \lambda_{2n}), m \right) \Gamma(m) \\ &= 2^{2m-3} \bar{v}(n, m) (m-1)! \end{aligned}$$

and the third part of (3.2) is then

$$\sum_{m=2}^{n-1} c_{-m}^{2n} H_{m-1} \frac{(\lambda_{2n})^m}{m!} = \sum_{m=2}^n \bar{v}(n, m) \frac{(2n-1)^{2m}}{8m} H_{m-1} \quad (3.5)$$

Next we evaluate the finite part of $\zeta_{\mathbb{S}_+^{2n}}(m, \lambda_{2n})$. Since

$$\begin{aligned} FP[\zeta_{\mathbb{S}_+^{2n}}(m, \lambda_{2n})] &= \sum_{\substack{i=1 \\ i \neq m}}^n \bar{v}(n, i) (2^{2m-1} - 2^{2i-2}) \zeta(2m-2i+1) \\ &\quad - (2n-1) \sum_{i=1}^n \bar{v}(n, i) (2^{2m-1} - 2^{2i-3}) \zeta(2m-2i+2) \\ &\quad + \bar{v}(n, m) 2^{2m-2} (\gamma + \log(2)) \end{aligned}$$

we may proceed by evaluating the second and fourth parts of expression (3.2) using Proposition 2.5 (ii) to convert the Riemann zeta functions into Hurwitz zeta functions, and Proposition A.5 (iii) and (iv) to simplify the resulting extra terms in a similar manner to the even sphere case.

$$\begin{aligned} &\sum_{m=n}^{\infty} \zeta_{\mathbb{S}_+^{2n}}(m, \lambda_{2n}) \frac{(\lambda_{2n})^m}{m} + \sum_{m=1}^{n-1} FP[\zeta_{\mathbb{S}_+^{2n}}(m, \lambda_{2n})] \frac{(\lambda_{2n})^m}{m} = \\ &= \sum_{m=1}^{\infty} \frac{\left(\frac{2n-1}{2}\right)^{2m}}{m} \left[\sum_{\substack{i=1 \\ i \neq m}}^n \bar{v}(n, i) (2^{2m-1} - 2^{2i-2}) \zeta(2m-2i+1) \right. \\ &\quad \left. - (2n-1) \sum_{i=1}^n \bar{v}(n, i) (2^{2m-1} - 2^{2i-3}) \zeta(2m-2i+2) \right] \\ &\quad + \sum_{i=1}^n \bar{v}(n, i) \frac{(2n-1)^{2i}}{4i} (\gamma + \log(4)) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n \frac{\bar{v}(n, i)}{2} \left[\sum_{\substack{m=1-i \\ m \neq 0}}^{\infty} \frac{\zeta(2m+1, 2n)}{m+i} (2n-1)^{2m+2i} - 2^{2i-1} \sum_{\substack{m=1-i \\ m \neq 0}}^{\infty} \frac{\zeta(2m+1, n)}{m+i} \left(\frac{2n-1}{2}\right)^{2m+2i} \right. \\
&\quad \left. - (2n-1) \sum_{\substack{m=2-i \\ m \neq 0}}^{\infty} \frac{\zeta(2m, 2n)}{m+i-1} (2n-1)^{2m+2i-2} - (2n-1)2^{2i-2} \sum_{\substack{m=2-i \\ m \neq 0}}^{\infty} \frac{\zeta(2m, n)}{m+i-1} \left(\frac{2n-1}{2}\right)^{2m+2i-2} \right] \\
&\quad + \sum_{i=1}^n \bar{v}(n, i) \frac{(2n-1)^{2i}}{4i} \left(\gamma + \log(4) + H_{n-1} - 2H_{2n} \right)
\end{aligned}$$

By substituting $a = 2n$ and $t = 2n-1$ and $a = n$ and $t = \frac{2n-1}{2}$ into Proposition 2.6 (ii), we can expand and simplify the first and second infinite sums within the brackets. Likewise, by substituting into Proposition 2.6 (i), we can expand and simplify the third and fourth infinite sums.

$$\begin{aligned}
& \sum_{i=1}^n \bar{v}(n, i) \left\{ \sum_{k=0}^{2i-1} \binom{2i-1}{k} [\zeta'(-k) + (-1)^{k+1} \zeta'(-k, 4n-1)] (2n-1)^{2i-k-1} \right. \\
& \quad - \sum_{k=0}^{2i-2} \binom{2i-2}{k} [\zeta'(-k) + (-1)^k \zeta'(-k, 4n-1)] (n-1)^{2i-k-1} \\
& \quad - 2^{2i-1} \sum_{k=0}^{2i-1} \binom{2i-1}{k} [\zeta'(-k, \frac{1}{2}) + (-1)^{k+1} \zeta'(-k, 2n-\frac{1}{2})] \left(\frac{2n-1}{2} \right)^{2i-k-1} \\
& \quad + 2^{2i-2} (2n-1) \sum_{k=0}^{2i-1} \binom{2i-1}{k} [\zeta'(-k, \frac{1}{2}) + (-1)^{k+1} \zeta'(-k, 2n-\frac{1}{2})] \left(\frac{2n-1}{2} \right)^{2i-k-2} \\
& \quad - \frac{(2n-1)^{2i}}{i} [\phi(2i) - \phi(2n) + \gamma] - 2\zeta'(-2i+1, 2n) \\
& \quad + 2^{2i-1} \left[\frac{\left(\frac{2n-1}{2} \right)^{2i}}{i} [\phi(2i) - \phi(n) + \gamma] + 2\zeta'(-2i+1, n) \right] \\
& \quad \left. + (2n-1) [2\zeta'(-2i+2, 2n) - 2^{2i-1} \zeta'(-2i+2, n)] \right\} \\
& \quad + \sum_{i=1}^n \bar{v}(n, i) \frac{(2n-1)^{2i}}{4i} (\gamma + \log(4) + H_{n-1} - 2H_{2n}) \\
& = \sum_{i=1}^n \bar{v}(n, i) \left\{ \sum_{k=1}^i \left[2^{2k-1} \binom{2i-1}{2k-1} \zeta'(-2k+1) (2n-1)^{2i-2k} - 2^{2k-2} \binom{2i-2}{2k-2} \zeta'(-2k+2) (2n-1)^{2i-2k+1} \right] \right. \\
& \quad - \log(2) \left(\sum_{k=0}^{2i-1} (-1)^{k+1} \binom{2i-1}{k} (2n-1)^{2i-k-1} \zeta(-k) + (2n-1) \sum_{j=1}^{n-1} (2j)^{2i-2} - \frac{(2n-1)^{2i}}{2i} \right) \\
& \quad \left. + (2^{2i-1} - 1) \zeta'(-2i+1) - (2n-1) (2^{2i+2} - 1) \zeta'(-2i+2) \right\} \\
& \quad - \sum_{i=1}^n \bar{v}(n, i) \frac{(2n-1)^{2i}}{4i} H_{2i-1} \\
& = \frac{1}{(2n-1)!} \sum_{k=0}^n \zeta'(-k) [s(2n, k+1) + (-1)^{k+1} s(2n-1, k)] - \sum_{i=1}^n \bar{v}(n, i) \frac{(2n-1)^{2i}}{4i} H_{2i-1} \\
& \quad - \sum_{i=1}^n \bar{v}(n, i) \left\{ \log(2) \left(\sum_{k=0}^{2i-1} (-1)^{k+1} \binom{2i-1}{k} (2n-1)^{2i-k-1} \zeta(-k) + (2n-1) \sum_{j=1}^{n-1} (2j)^{2i-2} - \frac{(2n-1)^{2i}}{2i} \right) \right. \\
& \quad \left. - (2^{2i-1} - 1) \zeta'(-2i+1) + (2n-1) (2^{2i+2} - 1) \zeta'(-2i+2) \right\}
\end{aligned}$$

By using the previous equation along with equations (3.5) and (3.1.2) in (3.2), we can derive the expression for $\det(\mathbb{S}_+^{2n})$ as presented in Theorem 3.2. We note that the terms multiplied by $\log(2)$ cancel in a manner identical to that in the sphere case, using Proposition 2.5 (vi) and (vii).

Theorem 3.2. *Let $n \in \mathbb{N}$. The determinant of the Laplacian on odd - \mathbb{S}_+^{2n-1} - and even - \mathbb{S}_+^{2n} - dimensional hemispheres satisfies the following identity.*

$$\begin{aligned} -\log [\det (\mathbb{S}_+^{2n-1})] &= \zeta'_{\mathbb{S}_+^{2n-1}}(0) \\ &= \frac{1}{(2n-2)!} \sum_{k=0}^{2n-1} \zeta'(-k) [s(2n-1, k+1) + (-1)^k s(2n-2, k)] \\ &\quad - \sum_{i=1}^{n-1} \bar{u}(n-1, i) \frac{(n-1)^{2i+1}}{2i} \left(\frac{H_{i-1}}{2} - H_{2i-1} \right) \\ -\log [\det (\mathbb{S}_+^{2n})] &= \zeta'_{\mathbb{S}_+^{2n}}(0) \\ &= \frac{1}{(2n-1)!} \sum_{k=0}^{2n-1} \zeta'(-k) [s(2n, k+1) + (-1)^{k+1} s(2n-1, k)] \\ &\quad + \sum_{i=1}^n \bar{v}(n, i) \frac{(2n-1)^{2i}}{4i} \left(\frac{H_{i-1}}{2} - H_{2i-1} \right) \end{aligned}$$

As in the case of spheres, we give the first 100 numerical values in Appendix B, and show the graphs of the first 10 000 values in Figure 2. Note that now the behavior for even and odd dimensions is even more striking, in that the corresponding limits as the dimension grows are different.

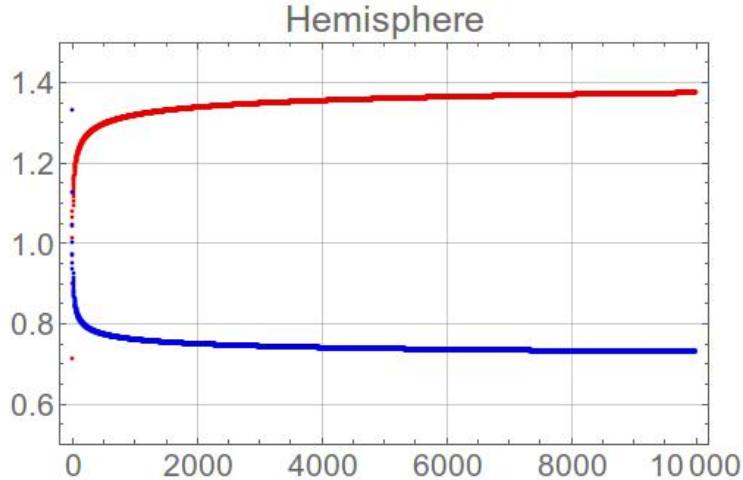


FIGURE 2. The first 10 000 values for $\det (\mathbb{S}_+^n)$ (odd dimensions: blue; even dimensions: red).

3.2. Real projective space. The case of the real projective space has recently been considered by Hartmann and Spreafico in [18]. Since our approach is different and, above all, the results obtained are given in a different form, we now indicate the main steps in the application of our method for this case, leaving out the main part of the proofs, as they are similar to those done above.

The spectrum of the Laplacian is given by integers of the form $2k(2k + n - 1)$ with the associated multiplicities

$$m_k^n = \binom{n+2k}{2k} - \binom{n+2k-2}{2k-2}$$

where $k \in \mathbb{N}_0$ - see [4]. As in the sphere case, we apply a shift on the spectrum of λ_n and the corresponding shifted zeta function is thus defined as

$$\zeta_{\mathbb{RP}^n}(s, \lambda_n) := \sum_{k=1}^{\infty} \frac{m_k^n}{\left(2k + \frac{n-1}{2}\right)^{2s}}$$

with the series being absolutely convergent for $\operatorname{Re}(s) > n$ - see Theorem 3.3. From this it is possible to derive a recurrence relation similar to that given in Lemma 2.3, namely,

$$\zeta_{\mathbb{RP}^{n+4}}(s, \lambda_{n+4}) = \frac{\zeta_{\mathbb{RP}^n}(s-2, \lambda_n) - \left(\frac{n^2+1}{2}\right) \zeta_{\mathbb{RP}^n}(s-1, \lambda_n) + \left(\frac{n^2-1}{4}\right)^2 \zeta_{\mathbb{RP}^n}(s, \lambda_n)}{n(n+1)(n+2)(n+3)} - \left(\frac{2}{n+1}\right)^{2s}$$

valid for all positive integers n and complex numbers s in the domain of the functions involved. Furthermore,

$$\begin{aligned} \zeta_{\mathbb{RP}^1}(s, \lambda_1) &= 2^{1-2s} \zeta(2s) \\ \zeta_{\mathbb{RP}^2}(s, \lambda_2) &= 2^{2-2s} \zeta\left(2s-1, \frac{5}{4}\right) \\ \zeta_{\mathbb{RP}^3}(s, \lambda_3) &= (1 - 2^{2-2s}) \zeta(2(s-1)) - 1 \\ \zeta_{\mathbb{RP}^4}(s, \lambda_4) &= \frac{2^{-2s-1}}{3} \left[16 \zeta\left(2s-3, \frac{7}{4}\right) - \zeta\left(2s-1, \frac{7}{4}\right) \right] \end{aligned}$$

so that $\zeta_{\mathbb{RP}^n}(s, \lambda_n)$ may be determined for all n by successive applications of the recursion formula. Remarkably, even though the recursion highlights the need to treat four independent cases, in the end we are able to recover the dimensional dichotomy similar to the previous spaces.

Theorem 3.3. *For each $n \in \mathbb{N}$ the zeta function $\zeta_{\mathbb{RP}^n}(s, \lambda_n)$ satisfies the following identities:*

$$\begin{aligned} \zeta_{\mathbb{RP}^{2n-1}}(s, \lambda_{2n-1}) &= \sum_{i=1}^n \bar{u}(n-1, i-1) (\tau_n - (-1)^n 2^{2i-2s-2}) \zeta(2s-2i+2) - (n-1)^{-2s} \\ \zeta_{\mathbb{RP}^{2n}}(s, \lambda_{2n}) &= \sum_{i=1}^n \bar{v}(n, i) 2^{4i-2s-2} \zeta(2s-2i+1, \frac{5}{4} + \frac{\tau_n}{2}) - \gamma_n \left[\frac{2}{(2n-1)} \right]^{2s} \end{aligned}$$

where

$$\gamma_n = \begin{cases} 0, & \text{if } n = 1, 2 \\ 1, & \text{otherwise} \end{cases} \quad \tau_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

The determinant of the Laplacian on odd- and even-dimensional real projective spaces, \mathbb{RP}^{2n-1} and \mathbb{RP}^{2n} , respectively, satisfies

$$\begin{aligned}\zeta'_{\mathbb{RP}^{2n-1}}(0) &= -\log \left[\det(\mathbb{RP}^{2n-1}) \right] \\ &= \frac{1}{(2n-2)!} \sum_{k=0}^{n-1} 2^{2k+1} \zeta'(-2k) [s(2n-1, 2k+1) + s(2n-2, 2k)] + \log(4(n-1)) \\ \zeta'_{\mathbb{RP}^{2n}}(0) &= -\log \left[\det(\mathbb{RP}^{2n}) \right] \\ &= \frac{1}{(2n-1)!} \sum_{k=0}^{2n-1} (1 - \tau_k 2^{k+1}) \zeta'(-k) [s(2n, k+1) + s(2n-1, k)] \\ &\quad + \log(4n-2) + \sum_{i=1}^n \bar{v}(n, i) \frac{(2n-1)^{2i}}{4i} \left(\frac{H_{i-1}}{2} - H_{2i-1} \right)\end{aligned}$$

The first 10 000 values are displayed in Figure 3.

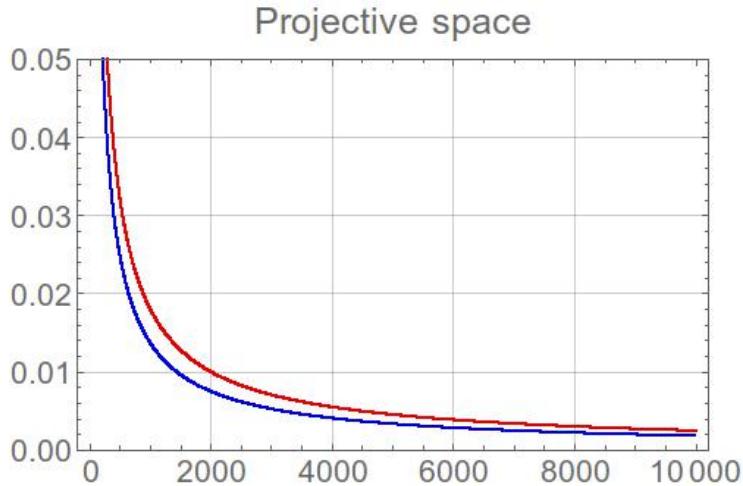


FIGURE 3. The first 10 000 values for $\det(\mathbb{RP}^n)$ (odd dimensions: blue; even dimensions: red).

3.3. The quantum harmonic oscillator revisited. In [14] the first author studied the determinant of the quantum harmonic oscillator in \mathbb{R}^n , whose spectrum is now given by integers of the form $2k+n$ with the associated multiplicities

$$m_k^n = \binom{n+k-1}{k},$$

where $k \in \mathbb{N}_0$. The corresponding zeta function is thus defined as

$$\zeta_{H^n}(s) := \sum_{k=0}^{\infty} \frac{m_k}{(2k+n)^s},$$

with the series being absolutely convergent for $\operatorname{Re}(s) > n$ – see [14] for the details. From this it was possible to derive the recurrence relation [14, Theorem A]

$$\zeta_{H^{n+2}}(s) = \frac{1}{4n(n+1)} \zeta_{H^n}(s-2) - \frac{n}{4(n+1)} \zeta_{H^n}(s),$$

for all positive integers n and complex numbers s in the domain of the functions involved. Furthermore,

$$\begin{aligned} \zeta_{H^1}(s) &= (1 - 2^{-s}) \zeta(s) \\ \zeta_{H^2}(s) &= 2^{-s} \zeta(s-1), \end{aligned}$$

so that $\zeta_{H^n}(s)$ may be determined for all n by successive applications of the recursion formula above. Most of the study in [14] then followed along lines closer to a more classical complex analytic line of approach to the study of zeta functions, without making use of these formulae. However, the similarities with the identities given in Lemma 2.3 and the other cases considered in this paper are striking, and, indeed, a similar approach is possible, allowing us to determine an explicit formula for $\zeta_{H^n}(s)$ – note that the study carried out in [14] was mostly concerned with finding, in a rigorous way, the asymptotic behavior of the determinant as n became large.

Theorem 3.4. *Let $n \in \mathbb{N}$. The zeta function $\zeta_{H^n}(s)$ is given by*

$$\begin{aligned} \zeta_{H^{2n}}(s) &= \frac{2^{-s}}{(2n-1)!} \sum_{i=1}^n u(n, i) \zeta(s-2i+1) \\ \zeta_{H^{2n-1}}(s) &= \frac{1}{4^n (2n-2)!} \sum_{i=1}^n v(n, i) \left(4 - 2^{-s+2i}\right) \zeta(s-2i+2) \end{aligned}$$

while the corresponding determinants satisfy

$$-\log [\det(H^{2n})] = \frac{1}{(2n-1)!} \sum_{i=1}^n u(n, i) [\zeta'(-2i+1) - \log(2)\zeta(-2i+1)]$$

and

$$-\log [\det(H^{2n-1})] = \frac{1}{4^n (2n-2)!} \sum_{i=2}^n \left(4 - 4^i\right) v(n, i) \zeta'(-2i+2) + \frac{(-1)^n 8 \log(2)(2n-2)!}{16^n ((n-1)!)^2}.$$

Appendix A. Central factorial numbers and relevant properties

The Stirling numbers of the first kind, $s(n, k)$ can be defined as the coefficients of the expansion of the falling factorial polynomial of degree n , denoted $[x]_n$

$$[x]_0 = 1, \quad [x]_n = x(x-1)\dots(x-n+1) =: \sum_{k=0}^n s(n, k)x^k. \quad (\text{A.1})$$

They have received a lot of attention and are at the core of the study of permutations, as $s(n, k)$ counts the number of permutations of n elements with k cycles. We shall only present here their properties which are relevant for our purposes, and refer to [11, Chapter 5] for a more complete treatment of these numbers.

We shall, however, make extensive use of central factorial numbers of the first kind $t(n, k)$, which have received less attention in the literature. They can be defined as the coefficients of the expansion of the central factorial polynomial of degree n , denoted by $x^{[n]}$.

$$x^{[0]} = 1, \quad x^{[n]} = x \left(x + \frac{n}{2} - 1 \right) \dots \left(x - \frac{n}{2} + 1 \right) =: \sum_{k=0}^n t(n, k) x^k \quad (\text{A.2})$$

A short introductory reference to central factorial numbers may be found in [29, pp. 213-217], and we refer to [5] for a more systematic treatment, including their main properties and a variety of applications. Although more closely related to the Stirling numbers of the first kind, the central factorial numbers are also related to the Euler and Bernoulli numbers (see [6, 22]). In this appendix, we shall summarize and introduce several properties which were used throughout the paper.

The identities in the following proposition follow directly from the aforementioned definitions. For more details we refer to [5, pp. 428, Proposition 2.1] with the exception of entry (vii) which is given in [5, pp. 480, Proposition 7.3]. We remark the similarities between the recursion in entry (i) and the recursion given for $\zeta_{S^n}(s, \lambda_n)$ in Lemma 2.3.

Proposition A.1. *Let $n \in \mathbb{N}$. The following identities hold.*

- (i) $t(n, k) = t(n-2, k-2) - \left(\frac{n-2}{2}\right)^2 t(n-2, k) \quad 2 \leq k \leq n$
- (ii) $s(n, k) = s(n-1, k-1) - (n-1) s(n-1, k) \quad 2 \leq k \leq n$
- (iii) $t(n, n) = s(n, n) = 1 \quad n \geq 0$
- (iv) $t(n, k) = s(n, k) = 0 \quad n < k$
- (v) $t(n, 0) = s(n, 0) = \delta_{n,0} \quad n \geq 0$
- (vi) $t(2n, 2k+1) = t(2n+1, 2k) = 0 \quad n, k \geq 0$
- (vii) $s(n, k) = (-1)^{n+k} \sum_{i=1}^n \binom{i-1}{k-1} \left(\frac{n}{2}\right)^{i-k} t(n, k) \quad n, k \geq 1$

It will be useful to consider the central factorial numbers with even and odd indices separately, which is also justified by Proposition A.1 (vi). These are defined as

$$u(n, k) := t(2n, 2k) \quad (\text{A.3})$$

$$v(n, k) := 4^{n-k} t(2n-1, 2k-1) \quad (\text{A.4})$$

Both of these integer sequences are present in the On-line Encyclopedia of Integer Sequences (OEIS) under A008955 for $u(n, k)$ and under A008956 for $v(n, k)$. In Table 1 their values for $n, k = 0, \dots, 6$ are given. We remark that $\text{sgn}(s(n, k)) = \text{sgn}(v(n, k)) = \text{sgn}(u(n, k)) = (-1)^{n-k}$ for $n, k \geq 1$ which follows straight from their respective definitions.

TABLE 1. Central factorial numbers
(Riordan [29, pp.217, Table 6.1])

$u(n, k)$							$v(n, k)$								
$k \setminus n$	0	1	2	3	4	5	6	$k \setminus n$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0
1		1	-1	4	-36	576	-14400	1		1	-1	9	-225	11025	-893025
2			1	-5	49	-820	21076	2			1	-10	259	-12916	1057221
3				1	-14	273	-7645	3				1	-35	1974	-172810
4					1	-30	1023	4					1	-84	8778
5						1	-55	5						1	-165
6							1	6							1

A.1. Central factorial numbers with even indices.

A detailed treatment of the central factorial numbers with even indices $u(n, k)$ may be found in Shiha [31], where a recurrence relation, generating function, distribution and general formula are presented, together with several combinatorial identities and applications. For our purposes we shall only need the following lemma which is also a direct consequence of definitions (A.2) and (A.3).

Lemma A.2 (Shiha [31, pp. 6]). *For $n \in \mathbb{N}$, then*

$$\sum_{k=1}^n u(n, k)x^k = \prod_{i=0}^{n-1} (x - i^2)$$

for all $x \in \mathbb{R}$.

Consider now the following associated sequence of numbers

$$\bar{u}(n, k) := \frac{2}{(2n)!} u(n, k) \quad (\text{A.5})$$

These are key to understanding the functional determinant of the Laplacian on odd-dimensional spheres and, in particular, they are intrinsic to the associated spectral zeta function as we shall see in Theorem 2.4. A few of their properties are summarized in the following proposition and shall be used several times.

Proposition A.3. *Let $n \in \mathbb{N}$. The following identities hold true:*

- (i) $u(n, k) = u(n-1, k-1) - (n-1)^2 u(n-1, k) \quad 2 \leq k \leq n$
- (ii) $\bar{u}(n, k) = \frac{\bar{u}(n-1, k-1) - (n-1)^2 \bar{u}(n-1, k)}{2n(2n-1)} \quad 1 \leq k \leq n$
- (iii) $\sum_{k=1}^n \bar{u}(n, k)j^{2k} = 0 \quad 0 \leq j < n$
- (iv) $\sum_{k=1}^n \bar{u}(n, k)n^{2k} = 1$
- (v) $\sum_{i=k}^n \binom{2i}{2k} n^{2i-2k} \bar{u}(n, i) = \frac{s(2n, 2k) + s(2n+1, 2k+1)}{(2n)!} \quad 1 \leq k \leq n$

Proof.

- (i) Follows from the definition of $u(n, i)$ in (A.3) applied to Proposition A.1 (i).
- (ii) Follows from (i) and the definition of $\bar{u}(n, i)$ in (A.5).
- (iii) Considering Lemma A.2 with $x = j^2$ for $0 \leq j < n$ and diving both sides by $\frac{(2n)!}{2}$ we obtain

$$\sum_{k=1}^n \bar{u}(n, k) j^{2k} = \frac{2}{(2n)!} \prod_{i=0}^{n-1} (j^2 - i^2),$$

and since $1 \leq i, j < n$, the RHS of the previous equation is 0.

- (iv) Similarly, considering Lemma A.2 with $x = n^2$ and diving both sides by $\frac{(2n)!}{2}$ we obtain

$$\sum_{k=1}^n \bar{u}(n, k) n^{2k} = \frac{2}{(2n)!} \prod_{i=0}^{n-1} (n^2 - i^2) = 1.$$

- (v) Consider the following

$$\begin{aligned} \sum_{i=k}^n \binom{2i}{2k} n^{2i-2k} \bar{u}(n, i) &= \frac{2}{(2n)!} \sum_{i=k}^n \binom{2i}{2k} n^{2i-2k} t(2n, 2i) \\ &= \frac{2}{(2n)!} \sum_{i=2k}^{2n} \binom{i}{2k} n^{i-2k} t(2n, i) && \text{Proposition A.1 (vi)} \\ &= \frac{2}{(2n)!} \sum_{i=2k}^{2n} \left[\binom{i-1}{2k-1} + \binom{i-1}{2k} \right] n^{i-2k} t(2n, i) \\ &= \frac{2}{(2n)!} [s(2n, 2k) - n s(2n, 2k+1)] && \text{Proposition A.1 (vii)} \\ &= \frac{1}{(2n)!} [s(2n, 2k) + s(2n+1, 2k+1)] && \text{Proposition A.1 (ii)} \end{aligned}$$

□

A.2. Central factorial numbers with odd indices.

A comprehensive treatment of the central factorial numbers with odd indices $v(n, k)$ may be found in [35] and is very similar to the treatment given for $u(n, k)$ with the exception of a general formula. One notes the unsurprising similarities in what follows. The following lemma is a direct consequence of definitions (A.2) and (A.4).

Lemma A.4 (Zaid, Shiha and El-Desouki [35, pp. 62]). *For $n \in \mathbb{N}$, then*

$$\sum_{k=1}^n v(n, k) x^k = x \prod_{i=1}^{n-1} (x - (2i-1)^2)$$

for all $x \in \mathbb{R}$.

Consider now the following associated sequence of numbers

$$\bar{v}(n, k) := \frac{4^{-(n-1)}}{(2n-1)!} v(n, k) \quad (\text{A.6})$$

As before, these are key to understanding the functional determinant of the Laplacian on even-dimensional spheres and, in particular, they are intrinsic to the associated spectral zeta function

as we shall see in Theorem 2.4. A few of their properties are presented below in the following proposition which is the counterpart of Proposition A.3 for even indices, and will also be used throughout the paper.

Proposition A.5. *Let $n \in \mathbb{N}$. The following identities hold true.*

- (i) $v(n, k) = v(n - 1, k - 1) - (2n - 3)^2 v(n - 1, k)$ $2 \leq k \leq n$
- (ii) $\bar{v}(n, k) = \frac{\bar{v}(n - 1, k - 1) - (2n - 3)^2 \bar{v}(n - 1, k)}{4(2n - 2)(2n - 1)}$ $1 \leq k \leq n$
- (iii) $\sum_{k=1}^n \bar{v}(n, k)(2j - 1)^{2k-1} = 0$ $0 \leq j < n$
- (iv) $\sum_{k=1}^n \bar{v}(n, k)(2n - 1)^{2k-1} = 1$
- (v) $\sum_{i=k}^n \binom{2i-1}{2k-1} (2n - 1)^{2i-2k} \bar{v}(n, i) = (s(2n, 2k) + s(2n - 1, 2k - 1)) \frac{2^{-2k+1}}{(2n - 1)!}$ $1 \leq k \leq n$

Proof.

- (i) Follows from the definition of $v(n, i)$ in (A.4) applied to Proposition A.1 (i).
- (ii) Follows from (i) and the definition of $\bar{v}(n, i)$ in (A.6).
- (iii) Considering Lemma A.4 with $x = (2j - 1)^2$ for $0 \leq j < n$ and multiplying both sides by $\frac{4^{-(n-1)}}{(2n-1)!(2j-1)}$ we obtain

$$\sum_{k=1}^n \bar{v}(n, k)(2j - 1)^{2k-1} = \frac{4^{-(n-1)}(2j - 1)}{(2n - 1)!} \prod_{i=1}^{n-1} ((2j - 1)^2 - (2i - 1)^2)$$

and since $1 \leq i, j < n$, the RHS of the previous equation is 0.

- (iv) Similarly, considering Lemma A.4 with $x = (2n - 1)^2$ and multiplying both sides by $\frac{4^{-(n-1)}}{(2n-1)!(2n-1)}$ we obtain

$$\sum_{k=1}^n \bar{v}(n, k)(2n - 1)^{2k-1} = \frac{4^{-(n-1)}}{(2n - 2)!} \prod_{i=1}^{n-1} ((2n - 1)^2 - (2i - 1)^2) = 1$$

(v) In a similar manner to the proof of Proposition A.3 (v), making use of the identities in Proposition A.1 consider the following

$$\begin{aligned}
\sum_{i=k}^n \binom{2i-1}{2k-1} (2n-1)^{2i-2k} \bar{v}(n, i) &= \frac{2^{2-2k}}{(2n-1)!} \sum_{i=k}^n \binom{2i-1}{2k-1} \left(\frac{2n-1}{2}\right)^{2i-2k} t(2n-1, 2i-1) \\
&= \frac{2^{2-2k}}{(2n-1)!} \sum_{i=2k}^{2n} \binom{i}{2k-1} \left(\frac{2n-1}{2}\right)^{i-2k+1} t(2n-1, i) \\
&= \frac{2^{2-2k}}{(2n-1)!} \sum_{i=2k-1}^{2n} \left(\binom{i-1}{2k-1} + \binom{i-1}{2k-2} \right) \\
&\quad \times \left(\frac{2n-1}{2}\right)^{i-2k+1} t(2n-1, i) \\
&= \frac{2^{2-2k}}{(2n-1)!} \left[s(2n-1, 2k-1) - \left(\frac{2n-1}{2}\right) s(2n-1, 2k) \right] \\
&= \frac{2^{1-2k}}{(2n-1)!} (s(2n-1, 2k-1) + s(2n, 2k))
\end{aligned}$$

□

From Proposition A.5 we are able to simplify some expressions which are presented below and will prove useful in this essay. They are quite simple but we provide a proof for clarity.

Proposition A.6. *Let $n \in \mathbb{N}$. The following identities hold true.*

$$\begin{aligned}
(i) \quad &\sum_{i=1}^n \bar{v}(n, i) \left(\sum_{j=1}^{2n-1} \log\left(\frac{2j-1}{2}\right) (2n-2j)^{2i-1} - \sum_{j=1}^{4n-2} \log(j)(2n-j-1)^{2i-1} \right) = \log(4n-2) \\
(ii) \quad &\sum_{i=1}^n \bar{v}(n, i) \left(\sum_{j=1}^{n-1} \log(j)(2j)^{2i-1} - \sum_{j=1}^{2n-1} \log(j)j^{2i-1} \right) = -\log(2) \sum_{i=1}^n \sum_{j=1}^{n-1} \bar{v}(n, i)(2j)^{2i-1} - \log(2n-1)
\end{aligned}$$

Proof. (i) Note that

$$\begin{aligned}
&\sum_{j=1}^{2n-1} \log\left(\frac{2j-1}{2}\right) (2n-2j)^{2i-1} - \sum_{j=1}^{4n-2} \log(j)(2n-j-1)^{2i-1} = \\
&= - \underbrace{\sum_{j=1}^{2n-1} \log(2)(2n-2j)^{2i-1}}_{=0} - \sum_{j=1}^{2n-1} \log(2j)(2n-2j-1)^{2i-1} \\
&= \log(4n) + \sum_{j=1}^{n-1} \log(2n+2j)(2j+1)^{2i-1} - \sum_{j=1}^{n-1} \log(2n-2j)(2j-1)^{2i-1} \\
&= \log(4n-2)(2n-1)^{2i-1} - \sum_{j=1}^{n-1} \log\left(\frac{2n-2j}{2n+2j-2}\right) (2j-1)^{2i-1}
\end{aligned}$$

where, in the first step, the second sum was separated in its even and odd parts. Now applying $\sum_{i=1}^n \bar{v}(n, i)$ on both sides together with Proposition A.5 (iii) and (iv) yields the result.

(ii) Note that

$$\begin{aligned}
 \sum_{j=1}^{n-1} \log(j)(2j)^{2i-1} - \sum_{j=1}^{2n-1} \log(j)j^{2i-1} &= - \sum_{j=1}^{n-1} \log(2)(2j)^{2i-1} - \sum_{j=1}^{n-1} \log(2j+1)(2j+1)^{2i-1} \\
 &= - \sum_{j=1}^{n-1} \log(2)(2j)^{2i-1} - \sum_{j=1}^{n-2} \log(2j+1)(2j+1)^{2i-1} \\
 &\quad - \log(2n-1)(2n-1)^{2i-1}
 \end{aligned}$$

and now applying $\sum_{i=1}^n \bar{v}(n, i)$ on both sides together with Proposition A.5 (iii) and (iv) yields the result. \square

Appendix B. Numerical values

TABLE 2. Numerical evaluation of the determinant of the Laplace operator on \mathbb{S}^n , \mathbb{S}_+^n and \mathbb{RP}^n for $n = 1, 2, \dots, 100$.

det(\mathbb{S}^n)			det(\mathbb{S}_+^n)			det(\mathbb{RP}^n)					
1	39.47842	2	3.195311	1	6.283185	2	0.713127	1	9.869604	2	2.240353
3	3.338851	4	1.736943	3	1.328388	4	0.896916	3	2.004050	4	1.312925
5	1.762919	6	1.290018	5	1.126034	6	0.969512	5	1.226325	6	0.985390
7	1.222521	8	1.050409	7	1.045956	8	1.011931	7	0.909314	8	0.804377
9	0.946733	10	0.896183	9	1.001319	10	1.040842	9	0.732550	10	0.686519
11	0.778048	12	0.786904	11	0.972171	12	1.062300	11	0.618221	12	0.602558
13	0.663546	14	0.704655	13	0.951307	14	1.079117	13	0.537518	14	0.539189
15	0.580375	16	0.640108	15	0.935449	16	1.092804	15	0.477157	16	0.489387
17	0.517020	18	0.587870	17	0.922876	18	1.104258	17	0.430114	18	0.449052
19	0.467028	20	0.544581	19	0.912592	20	1.114048	19	0.392302	20	0.415618
21	0.426495	22	0.508027	21	0.903976	22	1.122559	21	0.361172	22	0.387384
23	0.392915	24	0.476683	23	0.896618	24	1.130057	23	0.335047	24	0.363179
25	0.364604	26	0.449464	25	0.890236	26	1.136739	25	0.312775	26	0.342163
27	0.340384	28	0.425571	27	0.884630	28	1.142750	27	0.293538	28	0.323722
29	0.319410	30	0.404403	29	0.879653	30	1.148199	29	0.276737	30	0.307391
31	0.301055	32	0.385501	31	0.875192	32	1.153175	31	0.261923	32	0.292814
33	0.284847	34	0.368503	33	0.871164	34	1.157746	33	0.248752	34	0.279711
35	0.270420	36	0.353124	35	0.867501	36	1.161966	35	0.236958	36	0.267861
37	0.257491	38	0.339133	37	0.864149	38	1.165881	37	0.226330	38	0.257085
39	0.245830	40	0.326342	39	0.861067	40	1.169528	39	0.216696	40	0.247239
41	0.235257	42	0.314597	41	0.858219	42	1.172938	41	0.207921	42	0.238201
43	0.225623	44	0.303770	43	0.855575	44	1.176137	43	0.199889	44	0.229874
45	0.216803	46	0.293752	45	0.853113	46	1.179148	45	0.192509	46	0.222172
47	0.208698	48	0.284452	47	0.850811	48	1.181989	47	0.185701	48	0.215026
49	0.201221	50	0.275793	49	0.848653	50	1.184676	49	0.179400	50	0.208374
51	0.194300	52	0.267707	51	0.846623	52	1.187225	51	0.173549	52	0.202167
53	0.187874	54	0.260138	53	0.844708	54	1.189647	53	0.168100	54	0.196357
55	0.181891	56	0.253035	55	0.842899	56	1.191953	55	0.163012	56	0.190909
57	0.176305	58	0.246355	57	0.841185	58	1.194152	57	0.158249	58	0.185786
59	0.171076	60	0.240060	59	0.839558	60	1.196254	59	0.153780	60	0.180960
61	0.166171	62	0.234115	61	0.838011	62	1.198265	61	0.149578	62	0.176405
63	0.161561	64	0.228491	63	0.836536	64	1.200193	63	0.145618	64	0.172098
65	0.157217	66	0.223163	65	0.835129	66	1.202043	65	0.141881	66	0.168018
67	0.153118	68	0.218106	67	0.833784	68	1.203822	67	0.138346	68	0.164147
69	0.149243	70	0.213298	69	0.832496	70	1.205533	69	0.134999	70	0.160469
71	0.145573	72	0.208723	71	0.831261	72	1.207181	71	0.131822	72	0.156970
73	0.142093	74	0.204362	73	0.830077	74	1.208770	73	0.128805	74	0.153635
75	0.138786	76	0.200199	75	0.828938	76	1.210304	75	0.125934	76	0.150454
77	0.135642	78	0.196222	77	0.827842	78	1.211787	77	0.123198	78	0.147416
79	0.132647	80	0.192418	79	0.826787	80	1.213220	79	0.120589	80	0.144510
81	0.129791	82	0.188775	81	0.825770	82	1.214608	81	0.118097	82	0.141728
83	0.127064	84	0.185283	83	0.824788	84	1.215953	83	0.115714	84	0.139063
85	0.124457	86	0.181932	85	0.823840	86	1.217256	85	0.113434	86	0.136505
87	0.121963	88	0.178713	87	0.822923	88	1.218521	87	0.111249	88	0.134050
89	0.119575	90	0.175619	89	0.822035	90	1.219749	89	0.109153	90	0.131690
91	0.117284	92	0.172642	91	0.821176	92	1.220942	91	0.107142	92	0.129421
93	0.115086	94	0.169775	93	0.820343	94	1.222102	93	0.105209	94	0.127236
95	0.112975	96	0.167013	95	0.819536	96	1.223231	95	0.103350	96	0.125131
97	0.110945	98	0.164348	97	0.818752	98	1.224330	97	0.101562	98	0.123101
99	0.108992	100	0.161777	99	0.817991	100	1.225400	99	0.0998387	100	0.121143

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