$ Supplement Note 4: Calculation of π via Archimedes method

The ratio of the circumference S of a circle is related to its radius R via

$$S = 2\pi R \quad \Rightarrow \quad \pi = \frac{S}{2R} \tag{1}$$

S may be obtained from the limit of the circumferences of the inscribed regular polygons with sides N:

$$S = \lim_{N\to\infty} S_N \; ; \quad S_N = N\, D_N$$

We start with $N = 6$ and $R = 1$, we have

$$D_6 = 1 \quad \Rightarrow \quad S_6 = 6 \quad \Rightarrow \quad \pi_{\text{approx}} = \frac{S_6}{2R} = 3$$

The corresponding results for the case with doubling number N of sides: may be obtain by using Pythagorean theorem, (c.f Figure 1):

$$a = \sqrt{1 - \left(\frac{D_N}{2}\right)^2} \quad ; \quad b = 1 - a$$
$$D_{2N} = \sqrt{b^2 + \left(\frac{D_N}{2}\right)^2}$$

This is coded in *Archimedes-pi.py* and some of the results are listed in Table 1. Notice that, we could also start with an inscribed square with $N = 4$.

```
# Archimedes-pi.py
# Calculation of pi via Archimedes method
import numpy as np
#initialize
Pi= np.pi
d, nn= 1 , 6
print('pi =', Pi)
```

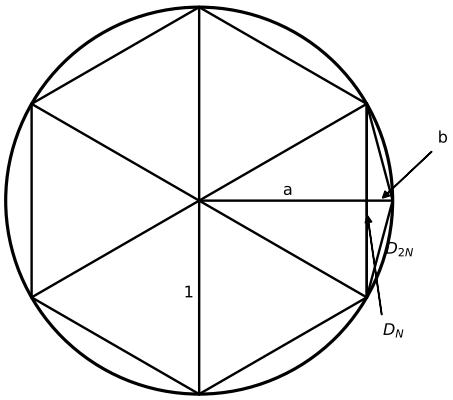

Figure 1: inscribed polygon of side N

```
print('Calculation starts with n = 6')
with open('pi.out','w') as fo:
    while nn <= 45000000:
        p = nn * d /2.0
        pe = p-Pi
        ss = "n= {0:8d}, pi_n = {1:21.16f}, error= {2:.2e}".format(nn,p,pe)
        print(ss)
        fo.write(ss+'\n')
        nn *=2
        ah2 = (d/2)**2
        a = np.sqrt(1 - ah2)
        d = np.sqrt((1-a)**2 + ah2)
###
print('Calculation starts with n = 4')
d, nn= np.sqrt(2) , 4
print('pi =', Pi)
while nn <= 45000000:
    p = nn * d /2.0
    pe = p-Pi
```

```
ss = "n= {0:8d}, pi_n = {1:21.16f}, error= {2:.2e}".format(nn,p,pe)
print(ss)
nn *=2
ah2 = (d/2)**2
a = np.sqrt(1 - ah2)
d = np.sqrt((1-a)**2 + ah2)
```

n	π_n	error
6	3.0000000000000000	-1.42e-01
12	3.1058285412302489	-3.58e-02
24	3.1326286132812378	-8.96e-03
48	3.1393502030468667	-2.24e-03
96	3.1410319508905098	-5.61e-04
192	3.1414524722854624	-1.40e-04
384	3.1415576079118579	-3.50e-05
768	3.1415838921483186	-8.76e-06
1536	3.1415904632280505	-2.19e-06
3072	3.1415921059992717	-5.48e-07
6144	3.1415925166921577	-1.37e-07
12288	3.1415926193653840	-3.42e-08
24576	3.1415926450336911	-8.56e-09
49152	3.1415926514507682	-2.14e-09
98304	3.1415926530550373	-5.35e-10
196608	3.1415926534561045	-1.34e-10
393216	3.1415926535563719	-3.34e-11
786432	3.1415926535814380	-8.36e-12
1572864	3.1415926535877046	-2.09e-12
3145728	3.1415926535892713	-5.22e-13
6291456	3.1415926535896630	-1.30e-13
12582912	3.1415926535897611	-3.20e-14
25165824	3.1415926535897856	-7.55e-15

Table 1: calculated value of π