$ Supplement Note 3: Real Part and Imaginary Part of e^{ix}

Let us substitute $z = ix$ in the series expansion of the exponential function e^z. Using the relation $i^2 = -1 \quad \Rightarrow \quad i^3 = -i, \quad$ and $\quad i^4 = 1$, we separate the terms into real part $C(x)$ and imaginary part $S(x)$:

$$
\begin{aligned}
e^{ix} \equiv & C(x) + i\,S(x) \\
= & \; 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \ldots \\
= & \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)
\end{aligned} \tag{1}
$$

Explicitly they are given by

$$C(x) \equiv 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{2a}$$

$$S(x) \equiv x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{2b}$$

It turns out that **the functions $S(x)$ and $C(x)$ are precisely the sine function** $\sin(x)$ **and the cosine** $\cos(x)$ **of the trigonometric functions**, respectively.

$$S(x) \equiv \sin x, \quad \text{and} \quad C(x) \equiv \cos x \tag{3}$$

This fact will be demonstrated step by step below. They have the same values of the trigonometric functions at $x = 0$:

$$C(0) = 1 = \cos 0 \quad \text{and} \quad S(0) = 0 = \sin 0\,, \tag{4}$$

$C(x)$ is an even function of x, while $S(x)$ is an odd function. This is *in agreement with the corresponding properties of the trigonometric functions*: $\cos(-x) = \cos(x),\ \sin(-x) = -sin(x)$.

$$C(-x) = C(x); \quad S(-x) = -S(x) \tag{5}$$

A replacement of x with $-x$ in (1) leads to the relation

$$e^{-ix} = C(x) - iS(x) \tag{6}$$

The following identities *which are used frequently for the trigonometric functions* $\cos x$ and $\sin x$, may be obtained by adding and subtracting (6) from (1):

$$C(x) = \frac{e^{ix} + e^{-ix}}{2} \tag{7a}$$

$$S(x) = \frac{e^{ix} - e^{-ix}}{2i} \tag{7b}$$

A direct manipulation of (7) yields the following result

$$C^2(x) + S^2(x) = \frac{e^{2ix} + 2 + e^{-2ix}}{4} + \frac{e^{ix} - 2 + e^{-2ix}}{-4} = 1 \tag{8}$$

Identifying $C(x)$ with $\cos(x)$ and $S(x)$ with $\sin(x)$, this is precisely the identity Eq.(2.4) in **Lecture Note**, a direct consequence of **Pythagorean theorem**:

$$\cos^2 x + \sin^2 x = 1 \tag{2.4}$$

In accord with basic properties of cosine and sine functions, an immediate implication of (8) is

$$|C| \leq 1, \quad \text{and} \quad |S| \leq 1 \tag{9}$$

The product identity (1.8) of the exponential function in **Lecture Note** can be extended to *complex numbers system C* since the arithmetic operations addition and multiplication in C satisfy the distributive law and the commutative law, *all you need to prove the identity!*. This leads to

$$e^{i(x+y)} = e^{ix+iy} = e^{ix}\, e^{iy} \tag{10}$$

From (7a), and (7b), we obtain the following sum relations for $C(x)$ and $S(x)$:

$$\begin{aligned}
C(x+y) &= \frac{e^{i(x+y)} + e^{-i(x+y)}}{2} = \frac{e^{ix}\, e^{iy} + e^{-ix}\, e^{-iy}}{2} \\
&= \frac{(e^{ix} + e^{-ix})\,(e^{iy} + e^{-iy}) + (e^{ix} - e^{-ix})\,(e^{iy} - e^{-iy})}{4} \\
&= \frac{(e^{ix} + e^{-ix})}{2}\frac{(e^{iy} + e^{-iy})}{2} - \frac{(e^{ix} - e^{-ix})}{2i}\frac{(e^{iy} - e^{-iy})}{2i} \\
&= C(x)C(y) - S(x)S(y) \qquad \text{(11a)} \\
S(x+y) &= \frac{e^{i(x+y)} - e^{-i(x+y)}}{2i} = \frac{e^{ix}\, e^{iy} - e^{-ix}\, e^{-iy}}{2i} \\
&= \frac{(e^{ix} + e^{-ix})\,(e^{iy} - e^{-iy}) + (e^{ix} - e^{-ix})\,(e^{iy} + e^{-iy})}{4i} \\
&= \frac{(e^{ix} + e^{-ix})}{2}\frac{(e^{iy} - e^{-iy})}{2i} + \frac{(e^{ix} - e^{-ix})}{2i}\frac{(e^{iy} + e^{-iy})}{2} \\
&= C(x)S(y) + S(x)C(y) \qquad \text{(11b)}
\end{aligned}$$

By replacing y with $-y$ and using (5), we also obtain

$$C(x-y) = C(x)C(y) + S(x)S(y) \qquad \text{(12a)}$$

$$S(x-y) = C(x)S(y) - S(x)C(y) \qquad \text{(12b)}$$

(11) and (12) are identical with the important *sum formulas for the trigonometric functions Eqs. (2.20), (2.23), (2.24) and (2.25)*. Further, we have

$$\begin{aligned}
C(2x) &= \frac{e^{2ix} + e^{-2ix}}{2} = \frac{(e^{ix} + e^{-ix})^2 - 2}{2} \\
&= 2C^2(x) - 1 = C^2(x) - S^2(x) = 1 - 2S^2(x) \qquad \text{(13a)} \\
S(2x) &= \frac{e^{2ix} - e^{-2ix}}{2i} = \frac{(e^{ix} - e^{-ix})(e^{ix} + e^{-ix})}{2i} \\
&= 2S(x)C(x) \qquad \text{(13b)}
\end{aligned}$$

$$\begin{aligned}
C(3x) &= \frac{e^{3ix} + e^{-3ix}}{2} = \frac{(e^{ix} + e^{-ix})(e^{2ix} - 1 + e^{-2ix})}{2} = \frac{(e^{ix} + e^{-ix})[(e^{ix} + e^{-ix})^2 - 3]}{2} \\
&= 4C^3(x) - 3C(x) \qquad \text{(14a)} \\
S(3x) &= \frac{e^{3ix} - e^{-3ix}}{2i} = \frac{(e^{ix} - e^{-ix})(e^{2ix} + 1 + e^{-2ix})}{2i} = \frac{(e^{ix} - e^{-ix})[(e^{ix} - e^{-ix})^2 + 3]}{2i} \\
&= -4S^3(x) + 3S(x) \qquad \text{(14b)}
\end{aligned}$$

where (8) was used in (13a). Eq.(13) is identical with the **double-angle formulas** and Eq.(14) with the **triple-angle formulas** of the trigonometric functions,

Let us now calculate the values of the functions $C(x)$ and $S(x)$ for $x = \frac{\pi}{2}$ where $\pi = 3.141592653589793\ldots$ is the *ratio of the circumference of a circle to its diameter*. To avoid numerical cancellation, every two terms in the series with alternating signs are grouped together:

$$\begin{aligned} C(x) =& \left(1 - \frac{x^2}{1\cdot 2}\right) + \frac{x^4}{4!}\left(1 - \frac{x^2}{5\cdot 6}\right) + \ldots \\ =& \sum_{m=0}^{\infty} \frac{x^{4m}}{(4m)!}\left(1 - \frac{x^2}{(4m+1)(4m+2)}\right) \qquad (15a) \\ S(x) =& x\left(1 - \frac{x^3}{2\cdot 3}\right) + \frac{x^5}{5!}\left(1 - \frac{x^2}{6\cdot 7}\right) + \ldots \\ =& \sum_{m=0}^{\infty} \frac{x^{4m+1}}{(4m+1)!}\left(1 - \frac{x^2}{(4m+2)(4m+3)}\right) \qquad (15b) \end{aligned}$$

An explicit calculation is given in python code *t3.py*. The convergence of the series is extremely rapid and both results again agree with the corresponding values of the trigonometric functions:

$$C\left(\frac{\pi}{2}\right) = \ 0 = \ \cos\left(\frac{\pi}{2}\right), \quad \text{and} \quad S\left(\frac{\pi}{2}\right) = \ 1 = \ \sin\left(\frac{\pi}{2}\right) \qquad (16)$$

Similar calculation are performed for 20 points evenly distributed in the interval $[0, \frac{\pi}{2}]$ and the graph of the function $C(x)$ and $S(x)$ are plotted together with the trigonometric sine function and cosine functions. The two sets of functions can not be distinguished in the plot because their values are identical. Notice that in the interval $[0, \frac{\pi}{2}]$, $S(x)$ is an increasing function while $C(x)$ a decreasing function:

$$0 < x < \frac{\pi}{2} \quad \Rightarrow \quad 0 < C(x),\ S(x) < 1 \qquad (17)$$

The following relations are direct consequences of (16) (12) and (11):

$$C\left(\frac{\pi}{2}-y\right)=C\left(\frac{\pi}{2}\right)C(y)+S\left(\frac{\pi}{2}\right)S(y)=S(y) \tag{18a}$$

$$S\left(\frac{\pi}{2}-y\right)=C\left(\frac{\pi}{2}\right)S(y)+S\left(\frac{\pi}{2}\right)C(y)=C(y) \tag{18b}$$

$$C\left(\frac{\pi}{2}+y\right)=C\left(\frac{\pi}{2}\right)C(y)-S\left(\frac{\pi}{2}\right)S(y)=-S(y) \tag{18c}$$

$$S\left(\frac{\pi}{2}-y\right)=\;-C\left(\frac{\pi}{2}\right)S(y)+S\left(\frac{\pi}{2}\right)C(y)=C(y) \tag{18d}$$

(18) are frequently used formulas for trigonometric functions. In particular, the following identity is obtained by inserting $y=\frac{\pi}{4}$ in (18a):

$$C\left(\frac{\pi}{4}\right)=S\left(\frac{\pi}{4}\right)$$

Combining this with (8), *the Pythagorean theorem*, $C^2(\frac{\pi}{4})+S^2(\frac{\pi}{4})=1$, we obtain

$$2C^2(\frac{\pi}{4})=1 \quad\Rightarrow\quad C(\frac{\pi}{4})=S(\frac{\pi}{4})=\frac{1}{\sqrt{2}} \tag{19}$$

Similarly, by inserting $x=\frac{\pi}{6}$ in (14) and using (16) we obtain

$$C\left(\frac{\pi}{2}\right)=0=4C^3\left(\frac{\pi}{6}\right)-3C\left(\frac{\pi}{6}\right) \quad\Rightarrow\quad C\left(\frac{\pi}{6}\right)\left\{4C^2\left(\frac{\pi}{6}\right)-3\right\}=0$$

$$S\left(\frac{\pi}{2}\right)=1=-4S^3\left(\frac{\pi}{6}\right)+3S\left(\frac{\pi}{6}\right) \quad\Rightarrow\quad \left\{S\left(\frac{\pi}{6}\right)+1\right\}\left\{2S\left(\frac{\pi}{6}\right)-1\right\}^2=0$$

This leads to the results

$$C\left(\frac{\pi}{6}\right)=\;S\left(\frac{\pi}{3}\right)=\;\frac{\sqrt{3}}{2},\quad\text{and}\quad S\left(\frac{\pi}{6}\right)=\;C\left(\frac{\pi}{3}\right)=\;\frac{1}{2} \tag{20}$$

The results (19) and (20) agree with *the corresponding values of the trigonometric functions*! The following supplement problem provides behaviors of $C(x)$ and $S(x)$ (or *trigonometric functions*!) by using the sum relations (11), etc. The sine and cosine functions, together with the exponential function e^x, are plotted at the end of *t3.py*. The exponential function e^x increases rapidly for positive x, while it approaches 0 as $x\to-\infty$ ($e^{-x}=\frac{1}{e^x}$).

Supplement Problem 2 *Show that*

$$C(\pi) = -1, \quad C\left(\frac{3\pi}{2}\right) = 0, \quad C(2\pi) = 1, \tag{21a}$$

$$S(\pi) = 0, \quad S\left(\frac{3\pi}{2}\right) = -1, \quad S(2\pi) = 0 \tag{21b}$$

$$0 > C(x) > -1; \quad 1 > S(x) > 0 \tag{21c}$$

$$\pi < x < \frac{3\pi}{2} \quad \Rightarrow \quad 0 > C(x), S(x) > -1 \tag{21d}$$

$$\frac{3\pi}{2} < x < 2\pi \quad \Rightarrow \quad 1 > C(x) > 0; \quad 0 > S(x) > -1 \tag{21e}$$

$$C(2\pi + x) = C(x); \qquad S(2\pi + x) = S(x) \tag{21f}$$

```
# t3.py:  Real part and Imaginary part of e^{ix}
import numpy as np
import matplotlib.pyplot as plt
Pi = np.pi
pi2 = 0.5* Pi
print('pi, pi/2=', Pi, pi2)
#calculation of C(pi/2) and S(pi/2) : the real part and the Imaginary part
#  of e^{ix}, for x = pi/2
sp = 'C(pi/2) and S(pi/2): the Real part and the Imaginary part of e^{i pi/2}'
print(sp)
x= pi2
c, s, tc, t, x2  = 0, 0, 1, x, x*x
# tc, t are estimated errors of calculated c and s, respectively
print('          C               error                S            error')
for m in range(5):
    n = 4*m
    fn1,fn2,fn3,fn4 , fn5 = n+1.0, n+2.0, n+3.0, n+4.0, n+5.0
    xn1 = x2/(fn1*fn2)
    xn2 = x2/(fn3*fn4)
    c += tc*(1- xn1)
    sn1 = x2/(fn2*fn3)
    s += t*(1- sn1)
    ss = "{0:21.16f},  {1:.2e}, {2:21.16f} , {3:.2e}\n".format(c,tc,s,t)
```

```
    print(ss)
    sn2 = x2/(fn4*fn5) # prepare for next t
    tc *= xn1*xn2
    t *= sn1*sn2
print('c=', c, '  , C(pi/2)-cos(pi/2) =', "{:.2e}".format(c), '***')
print('s=', s, '  , S(pi/2)-sin(pi/2) =', "{:.2e}".format(s-1), '***\n')
##########################
import matplotlib.pyplot as plt
xx = np.linspace(0, pi2,20)
C, S = [],[]
for x in xx:
    c, s, tc, t, x2, sinx, cosx  = 0, 0, 1, x, x*x, np.sin(x), np.cos(x)
    print('x =', x, '   sin(x)=', sinx, 'cos(x)=', cosx)
    for m in range(5):
        n = 4*m
        fn1,fn2,fn3,fn4 , fn5 = n+1.0, n+2.0, n+3.0, n+4.0, n+5.0
        xn1 = x2/(fn1*fn2)
        xn2 = x2/(fn3*fn4)
        sn1 = x2/(fn2*fn3)
        sn2 = x2/(fn4*fn5) # prepare for next t
        c += tc*(1- xn1)
        s += t*(1- sn1)
        ss = "{0:21.16f},  {1:.2e}, {2:21.16f} , {3:.2e}\n".format(c,tc,s,t)
        print(ss)
        tc *= xn1*xn2
        t *= sn1*sn2
    print('c=', c, '  , C(x)-cos(x) =', "{:.2e}".format(c-cosx), '***')
    print('s=', s, '  , S(x)-sin(x) =', "{:.2e}".format(s-sinx), '***\n')
    C.append(c)
    S.append(s)
####
xmin, xmax =0 , pi2
pi6, pi3 , pi4 = Pi/6.0, Pi/3.0, Pi/4.0
```

```
sq2, sq3 = np.sqrt(2), np.sqrt(3)
x = np.linspace(xmin,xmax,80)
CC = np.cos(x)
SS = np.sin(x)
plt.plot(x,SS,label='sine')
plt.plot(x,CC,label='cosine')
plt.plot(xx,S,label='S(x)')
plt.plot(xx,C,label='C(x)')
plt.hlines(0,xmin,xmax)
plt.hlines(1,xmin,xmax, linestyles='dotted')
plt.hlines(0.5,xmin,xmax, linestyles='dotted')
plt.hlines(1.0/sq2,xmin,xmax, linestyles='dotted')
plt.hlines(sq3/2.0,xmin,xmax, linestyles='dotted')
plt.vlines(pi6,0,1, linestyles='dotted')
plt.vlines(pi3,0,1, linestyles='dotted')
#plt.vlines(0,-1,1)
plt.vlines(pi4,0,1, linestyles='dotted')
plt.text(-0.05, 1.0/sq2,r'$\frac{1}{\sqrt{2}}$')
plt.text(-0.05, sq3/2.0,r'$\frac{\sqrt{3}}{2}$')
plt.text(-0.05, 1.0/2.0,r'$\frac{1}{2}$')
plt.text(pi4-0.05,0.02,r'$\pi/4$')
#plt.hlines(-1,xmin,xmax, linestyles='dotted')
plt.vlines(xmin,0,1)
plt.vlines(xmax,0,1)
plt.text(-0.05,0.02,r'$0$')
plt.text(xmax-0.05,0.02,r'$\pi/2$')
plt.text(pi6-0.05,0.02,r'$\pi/6$')
plt.text(pi3-0.05,0.02,r'$\pi/3$')
plt.xlabel('x')
plt.legend()
plt.show()
xmin, xmax =-2*np.pi, 2*np.pi
x = np.linspace(xmin,xmax,80)
```

```
C = np.cos(x)
S = np.sin(x)
EXP= np.exp(x)
plt.figure()
plt.subplot(2,1,1)
plt.plot(x,EXP,label='exponential')
plt.title('exponential function')
plt.legend()
plt.subplot(2,1,2)
##
plt.plot(x,S,label='sine')
plt.plot(x,C,label='cosine')
plt.hlines(0,xmin,xmax)
plt.hlines(1,xmin,xmax, linestyles='dotted')
plt.vlines(np.pi,-1,1, linestyles='dotted')
plt.vlines(0,-1,1)
plt.vlines(-np.pi,-1,1, linestyles='dotted')
plt.hlines(-1,xmin,xmax, linestyles='dotted')
plt.vlines(xmin,-1,1)
plt.vlines(xmax,-1,1)
plt.text(-2*np.pi-0.5,0.02,r'$-2\pi$')
plt.text(2*np.pi-0.5,0.02,r'$2\pi$')
plt.text(-np.pi-0.5,0.02,r'$-\pi$')
plt.text(np.pi-0.5,0.02,r'$\pi$')
plt.title('sine and cosine')
plt.legend()
plt.show()
```