$ Supplement Note 7: Energy, Power and Aerodynamic Drag

The motion of a particle is governed by *Newton's Second law*, which states that *the time rate of change of momentum p of a body is equal to the force F acting on the particle*.

$$F = \frac{dp}{dt} , \tag{1}$$

where the *momentum* p is defined as the product of mass m and the velocity v of the particle:

$$p = mv . \tag{2}$$

Thus Newton's Second Law (1) may also be written as

$$F = \frac{d}{dt}(mv) = m\,\frac{dv}{dt} = m\,\frac{d^2x}{dt} = ma , \tag{3}$$

where $a \equiv \frac{dv}{dt} = \frac{d^2x}{dt}$ is the *acceleration* of the particle.

The *kinetic energy* of a particle is defined as

$$E_{\rm kin} \equiv \frac{1}{2}mv^2 . \tag{4}$$

The unit of energy is *joul* in MKS system and *erg* in cgs system.

$$1 \text{ joul} = 1 \text{ kgm}^2/\text{s}^2,\ 1 \text{ erg} = 1 \text{ gcm}^2/\text{s}^2 \quad \Rightarrow \quad 1 \text{ joul} = 10^7 \text{ erg}$$

Another commonly used unit for energy is *electron volt*(eV), it is equal to the amount of energy needed to bring a particle with a charge equal to that of an electron to a voltage difference of 1 volt(the magnitude of the charge of an electron e is equal to 1.602×10^{-19} coul):

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ jouls}$$

The *time rate of change of the kinetic energy* of a body P is called *power*. Its unit is *watt* (joul/second). For historic reason, the unit *horse power* is also frequently used. 1 horse power = 746 watts. By a direct differentiation on (4), and using *the chain rule*(**Theorem 3.3**), we obtain

$$P \equiv \frac{dE_{\text{kin}}}{dt} = \frac{d}{dv}(\frac{1}{2}mv^2)\ \frac{dv}{dt} = mv\ \frac{dv}{dt} = Fv \tag{5}$$

where Newton's second law (3) has been used. Eq. (5) states that the *rate of change of E_{kin}* (the *power* acting on the particle) is equal to *the product of force acting on it times its velocity*. By integrating (5) with respect to t and employing a substitution of integration to x, we obtain

$$\int_{t_1}^{t_2} \frac{dE_{\text{kin}}}{dt} = E_{\text{kin}}(t_2) - E_{\text{kin}}(t_1) = \int_{t_1}^{t_2} F\frac{dx}{dt}dt = \int_{x_1}^{x_2} F dx \tag{6}$$

The product $F\ dx$ (force$\times$displacement) is known as the *work* done on the particle. Eq.(6) states that *the total work done on the particle is equal to the increment of the total kinetic energy*. This is an important implication of Newton's second law of motion.

The Effect of Air Drag

As a body moving through air, it experiences a drag force in the direction opposite to its motion. In general, this force is given by

$$F_{\text{drag}} \approx -B_1 v - B_2 v^2 \tag{7}$$

At extremely low velocities the first term dominates, and its coefficient B_1 can be calculated for objects with simple shapes. This is known as *Stokes' law*. In particular, the drag force for a sphere with radius R moving through a viscous flow with a dynamic *viscosity* of η is given by

$$F_{\text{drag}} = 6\pi\eta R v \tag{8}$$

At any reasonable velocity the second term in (7) dominates for most objects. The exact value of the coefficient B_2 is is a difficult problem. However, an approximate estimate can be made as follows. In a duration of time dt, an object with a frontal area A and velocity v moves through the atmosphere must push an amount of air with a volume equal to $V = A \cdot vdt$ out of the way. If the density of air is ρ, the total mass m is equal to

$$m = \rho V = \rho A v dt$$

The rate of energy gain of the air is then given by

$$\frac{dE_{\text{air}}}{dt} = \frac{d}{dt}(\frac{1}{2}mv^2) = \frac{1}{2}\rho A\ v\ v^2 \tag{9}$$

From (5), this implies that the force acting on the air (*by the object*) is equal to

$$F = \frac{1}{2}\rho A\ v^2$$

The reaction force from Newton's third law of motion then yields the drag force on the object is given by

$$F_{\text{drag}} = -\frac{1}{2}\ C\rho A\ v^2 \tag{10}$$

C is a dimensionless factor related to the geometry of the body.

Example 1 *A* 70*kg sky diver free-falls from an altitude of* 1400 *m high. For the air drag, assume that* $C \approx 0.5$*,* $A \approx 1\ m^2$ *and the air density is* $\rho \approx 1\ kg/m^3$.

In this case, Newton's second law (3) becomes

$$\frac{dv}{dt} = -g + \frac{C\rho A}{2m}\ v^2 \tag{11}$$

The Euler's approximation now takes the form

$$\begin{aligned} y_{i+1} =& y_i + v_i \delta t \\ v_{i+1} =& v_i + (-g + \frac{C\rho A}{2m}\ v_i^2)\delta t \end{aligned}$$

This is coded in *Skydiver.py* where the case without friction is also included. With the effect of airdrag, the velocity of the skydiver reaches a *terminal velocity* which may be obtained by setting $\frac{dv}{dt} = 0$: (11)

$$v_{terminal} = -\sqrt{\frac{2gm}{C\rho A}} \approx -52\ \text{m/sec}$$

As can be seen in the figure, the sky diver approaches this limit in about 11 minutes.

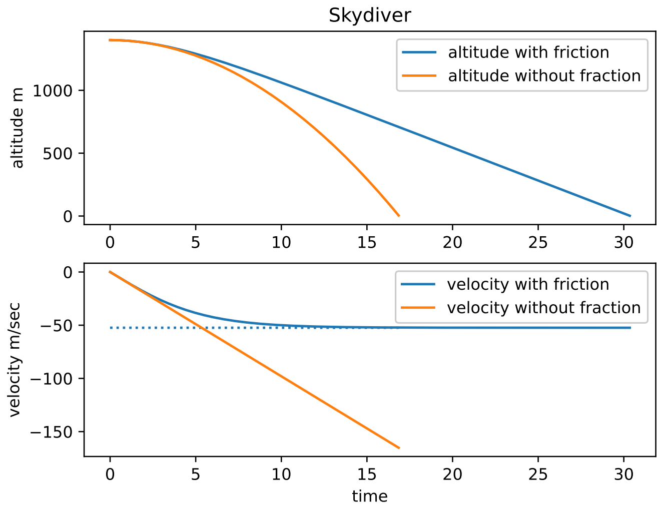

Figure 1: Skydiver

```
# Skydiver.py
# simulation of freely falling of object near Earth's surface
import numpy as np
import matplotlib.pyplot as plt
# calculation with dt = 0.5
t , y2  , v2 = 0 , 1400, 0 #initial condition
C, rho, A, m = 0.5, 1, 1, 70
Cdrag = 0.5 * C * rho * A/m
print(Cdrag)
dt = 0.05
g = 9.8    #m/s^2 acceleration due to gravity
T2, Y2, V2 = [],[],[]
while y2 > 0 :
    T2.append(t)
    Y2.append(y2)
    V2.append(v2)
    v2 = v2 - g*dt + Cdrag*v2*v2*dt
    y2 = y2 + v2*dt
    t = t + dt
```

```
t , y  , v = 0 , 1400, 0 #initial condition
T1, Y1, V1 = [],[],[]
#print(V2)
while y > 0 :
    T1.append(t)
    Y1.append(y)
    V1.append(v)
    v = v - g*dt
    y = y + v*dt
    t = t + dt
tmax = T1[-1]
vlim = - np.sqrt(g/Cdrag)
plt.figure()
plt.subplot(2,1,1)
plt.title('Skydiver')
plt.ylabel('altitude m')
plt.plot(T2,Y2, label= 'altitude with friction')
plt.plot(T1,Y1, label= 'altitude without fraction')
plt.legend()
plt.subplot(2,1,2)
plt.xlabel('time')
plt.ylabel('velocity m/sec')
plt.hlines(vlim,0,tmax, linestyles='dotted')
plt.plot(T2, V2, label = 'velocity with friction')
plt.plot(T1, V1, label = 'velocity without fraction')
plt.legend()
###
fig = plt.gcf()
fig.savefig('Skydiver.eps', format='eps')
plt.show()
```