$ Supplement Note 5: Euler's Formula and Properties of the Trigonometric Functions Sine and Cosine

By substituting $z = i\theta$ into the series expansion of the exponential function

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

and separating the series into real parts and imaginary parts by using the relation, $i^2 = -1 \Rightarrow\ i^3 = -i,$ and $i^4 = 1$, we obtain the following expression:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right). \end{aligned} \tag{1}$$

It turns out that **the real parts** and **the imaginary parts** in (1) are precisely **the cosine function** $\cos(\theta)$ and **the sine function** $\sin(\theta)$ of the trigonometric functions, respectively. Equation (1) is a *proof* for the famous *Euler's formula*:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2}$$

Explicitly the trigonometric functions are given by

$$\cos\theta \equiv 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots = \sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n}}{(2n)!} \tag{3a}$$

$$\sin\theta \equiv \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots = \sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \tag{3b}$$

It follows from (3) that the function $\cos\theta$, which contains only terms with even powers of θ, is an *even function of* θ, while $\sin\theta$ is an *odd function of* θ. This is in agreement with the fact that the sign reversing of θ is equivalent to the interchanging of the 1st and the 4th quadrants (or the 2nd and the 3rd quadrants) with the effect that $x \leftrightarrow x; y \leftrightarrow -y$(c.f. Figure 1):

$$\cos(-\theta) = \frac{x}{r} = \cos\theta; \tag{4a}$$

$$\sin(-\theta) = \frac{-y}{r} = -\sin\theta \tag{4b}$$

A replacement of θ by $-\theta$ in (1) and combining with (4) leads to the relation:

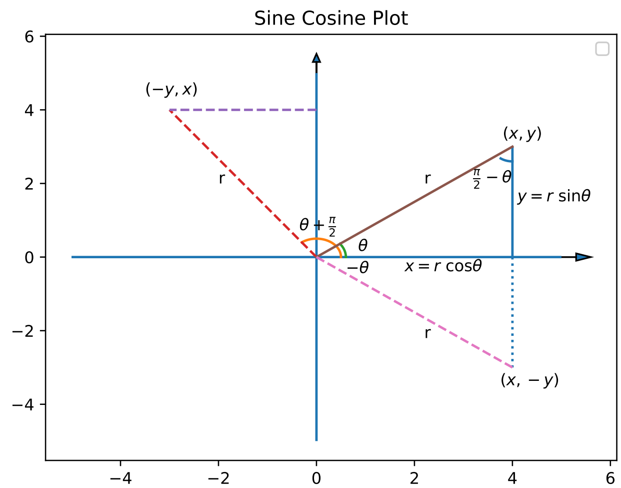

Figure 1: Sine, Cosine and Rectangular Triangle

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta \tag{5}$$

The following identities for the trigonometric functions $\cos\theta$ and $\sin\theta$, may be obtained by adding and subtracting (5) from (1):

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{6a}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{6b}$$

A simple manipulation of (6) yields the following identity Eq.(2.4) in **Lecture Note**, which is a direct consequence of **Pythagorean theorem**:

$$\cos^2\theta + \sin^2\theta = \frac{e^{2i\theta} + 2 + e^{-2i\theta}}{4} + \frac{e^{i\theta} - 2 + e^{-2i\theta}}{-4} = 1 \tag{7}$$

The key in the proof for the product identity, $e^{x+y} = e^x\, e^y$, of the exponential function is the binomial expansion formula

$$(x+y)^n = \sum_{m=0}^{n} \binom{n}{m} x^m y^{n-m}\,. \tag{8}$$

Notice that *the binomial expansion formula* (8) *is valid* as long as the *distributive law* and the *commutative law* for the arithmetic *addition and multiplication* operations are satisfied. In addition to the real numbers system $\mathcal{R}$, these laws are also satisfied by the complex numbers system C. Hence the product formula of the exponential function can be extended to *complex numbers system* C.

$$e^{i(x+y)} = e^{ix+iy} = e^{ix}\, e^{iy} \tag{9}$$

By using Euler's formula (2) and the identity $i^2 = -1$, this yields:

$$\begin{aligned} \cos(x+y) + i\sin(x+y) &= (\cos x + i\sin x)\,(\cos y + i\sin y) \\ &= (\cos x\cos y - \sin x\sin y) + i\,(\cos x\sin y + \sin x\cos y) \end{aligned} \tag{10}$$

The real parts and the imaginary parts of (10) then yield the first two sets of the *sum formulas for the trigonometric functions*:

$$\cos(x+y) = \cos x\ \cos y - \sin x\ \sin y \tag{11a}$$

$$\sin(x+y) = \cos x\ \sin y + \sin x\ \cos y \tag{11b}$$

The other two sets can be obtained by replacing y with $-y$ and using (4):

$$\cos(x-y) = \cos x\ \cos y + \sin x\ \sin y \tag{12a}$$

$$\sin(x-y) = \cos x\ \sin y - \sin x\ \cos y \tag{12b}$$

The **double-angle formulas** of the trigonometric functions are obtained by setting $y = x$ in Equation (11) and utilizing (7)

$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1 \tag{13a}$$

$$\sin(2x) = 2\sin x\cos x \tag{13b}$$

The **half-angle formulas** follow directly from (13a)

$$\cos^2 x = \frac{1+\cos(2x)}{2} \tag{14a}$$

$$\sin^2 x = \frac{1-\cos(2x)}{2} \tag{14b}$$

Finally, the **triple-angle formulas** of the trigonometric functions may be obtained from Euler's formula (2):

$$\begin{aligned}
\cos(3x) &= \frac{e^{3ix}+e^{-3ix}}{2} = \frac{(e^{ix}+e^{-ix})(e^{2ix}-1+e^{-2ix})}{2} \\
&= \frac{(e^{ix}+e^{-ix})[(e^{ix}+e^{-ix})^2-3]}{2} \\
&= 4\cos^3 x - 3\cos x \qquad (15a) \\
\sin(3x) &= \frac{e^{3ix}-e^{-3ix}}{2i} = \frac{(e^{ix}-e^{-ix})(e^{2ix}+1+e^{-2ix})}{2i} \\
&= \frac{(e^{ix}-e^{-ix})[(e^{ix}-e^{-ix})^2+3]}{2i} \\
&= -4\sin^3 x + 3\sin x \qquad (15b)
\end{aligned}$$

Let us now calculate from the series expansion (3) the values of the trigonometric functions at $\theta = 0$ and at $\theta = \frac{\pi}{2}$, where $\pi = 3.141592653589793\ldots$ is the *ratio of the circumference of a circle to its diameter*. From (3) it is easily seen that :

$$\cos 0 = 1; \quad \sin 0 = 0\,, \qquad (16)$$

in consistent with the fact that as $\theta \to 0 \Rightarrow y \to 0, x \to r$. During the actual computation, in order to reduce numerical cancellation, every two terms in the series with alternating signs are grouped together:

$$\begin{aligned}
\cos x &= \left(1-\frac{x^2}{1\cdot 2}\right)+\frac{x^4}{4!}\left(1-\frac{x^2}{5\cdot 6}\right)+\ldots \\
&= \sum_{m=0}^{\infty}\frac{x^{4m}}{(4m)!}\left(1-\frac{x^2}{(4m+1)(4m+2)}\right) \qquad (17a) \\
\sin x &= x\left(1-\frac{x^3}{2\cdot 3}\right)+\frac{x^5}{5!}\left(1-\frac{x^2}{6\cdot 7}\right)+\ldots \\
&= \sum_{m=0}^{\infty}\frac{x^{4m+1}}{(4m+1)!}\left(1-\frac{x^2}{(4m+2)(4m+3)}\right) \qquad (17b)
\end{aligned}$$

An explicit calculation for the values of $\cos(\frac{\pi}{2})$ and $\sin(\frac{\pi}{2})$ is demonstrated in the python code *CosSinSeries.py*. The convergence of the series is extremely rapid. Sums of the leading six pairs already yield results with error $\leq 10^{-16}$. The results are

$$\cos\left(\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \sin\left(\frac{\pi}{2}\right) = 1\,, \qquad (18)$$

in consistent with the fact that as $\theta \to \frac{\pi}{2} \Rightarrow x \to 0,\ y \to r$. Insertion of (18) into the double-angle formulas (13) leads to the results:

$$\cos\pi = \cos^2\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\pi}{2}\right) = -1 \tag{19a}$$

$$\sin\pi = 2\sin\left(\frac{\pi}{2}\right)\,\cos\left(\frac{\pi}{2}\right) = 0 \tag{19b}$$

The *Euler's identity* follows immediately from (19) and (2):

$$e^{i\pi} + 1 = (\cos\pi + i\ \sin\pi) + 1 = 0 \tag{20}$$

A further insertion of (19) into the double-angle formulas (13) leads to

$$\cos(2\pi) = \cos^2\pi - \sin^2\pi = 1 \tag{21a}$$

$$\sin(2\pi) = 2\sin\pi\ \cos\pi = 0 \tag{21b}$$

From (21) and Euler's formula (2) we arrive at the following important identity:

$$e^{2\pi i} = \cos(2\pi) + i\ \sin(2\pi) = 1 \tag{22}$$

Using (22) and the product formula (9) we see that *the exponential function* $e^{i\theta}$ *together with its real and imaginary parts* $\cos(\theta),\ \sin(\theta)$ *are periodic functions with a period of* 2π:

$$e^{i(\theta+2\pi)} = e^{i\theta}e^{2\pi i} = e^{i\theta} \tag{23a}$$

$$\cos(\theta + 2\pi) = \cos\theta \tag{23b}$$

$$\sin(\theta + 2\pi) = \sin\theta \tag{23c}$$

The code *CosSinSeries.py* also contains similar calculation of the series for the trigonometric functions $\cos\theta$ and $\sin\theta$ for 20 other values of θ in the range $[0, \frac{\pi}{2}]$. The results have been checked with the sine function and cosine function in the numpy package and the graph of the functions are plotted in Figure 2. As θ increases from 0 to $\frac{\pi}{2}$, the value of $\sin\theta$ increases from 0 to 1 while that of the $\cos\theta$ decreases from 1 to 0.

$$0 < \theta < \frac{\pi}{2} \quad \Rightarrow \quad 0 < \cos\theta,\ \sin\theta < 1 \tag{24}$$

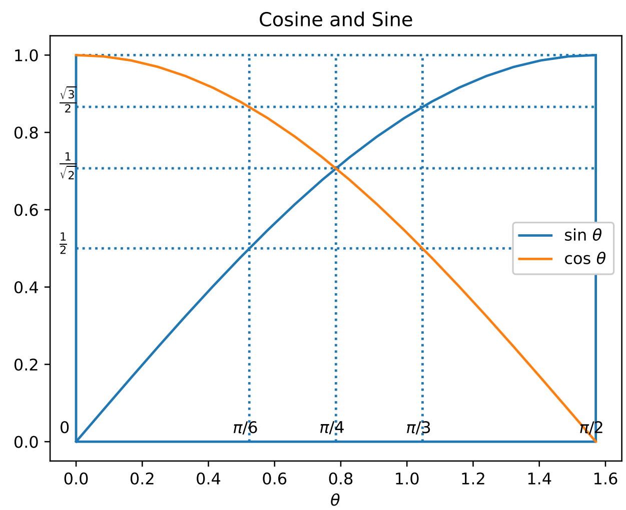

Figure 2: Cosine and Sine in the range $[0, \frac{\pi}{2}]$

The following relations are direct consequences of (18) (12) and (11):

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2}\right)\cos\theta + \sin\left(\frac{\pi}{2}\right)\sin\theta = \sin\theta \tag{25a}$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2}\right)\cos\theta + \cos\left(\frac{\pi}{2}\right)\sin\theta = \cos\theta \tag{25b}$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = \cos\left(\frac{\pi}{2}\right)\cos\theta - \sin\left(\frac{\pi}{2}\right)\sin\theta = -\sin\theta \tag{25c}$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \sin\left(\frac{\pi}{2}\right)\cos\theta + \cos\left(\frac{\pi}{2}\right)\sin\theta = \cos\theta \tag{25d}$$

Equations (25a), (25b) are consistent with the fact that interchanging of the two acute angles $\theta \leftrightarrow \left(\frac{\pi}{2} - \theta\right)$ is equivalent to the interchanging $x \leftrightarrow y$. Similarly, Equations (25c) (25d) are consequences of the fact that a *counter clockwise rotation* of an angle of $\frac{\pi}{2}$ (i.e. $\theta \rightarrow \frac{\pi}{2} + \theta$) is equivalent to the interchanging of $(x, y) \leftrightarrow (-y, x)$ (Figure.1). Notice that Equations (25) are derived from the product rule, *they are valid for all values of* θ.

Inserting $\theta = \frac{\pi}{4}$ into (25a) lead to the identity $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right)$. Combining this with (7), $\cos^2(\frac{\pi}{4}) + \sin^2(\frac{\pi}{4}) = 1$, we obtain (c.f. Figure 3.)

$$2\cos^2\left(\frac{\pi}{4}\right) = 1 \quad \Rightarrow \quad \cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \tag{26}$$

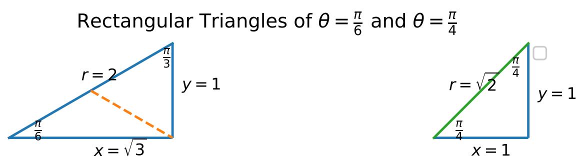

Figure 3: Rectangular Triangle For $\theta = \frac{\pi}{6}$ and $\theta = \frac{\pi}{4}$

Similarly, by inserting $x = \frac{\pi}{6}$ in (15) and using (18) we obtain

$$\cos\left(\frac{\pi}{2}\right) = 0 = 4\cos^3\left(\frac{\pi}{6}\right) - 3\cos\left(\frac{\pi}{6}\right) \quad \Rightarrow \quad \cos\left(\frac{\pi}{6}\right)\left\{4\cos^2\left(\frac{\pi}{6}\right) - 3\right\} = 0$$
$$\sin\left(\frac{\pi}{2}\right) = 1 = -4\sin^3\left(\frac{\pi}{6}\right) + 3\sin\left(\frac{\pi}{6}\right) \quad \Rightarrow \quad \left\{\sin\left(\frac{\pi}{6}\right) + 1\right\}\left\{2\sin\left(\frac{\pi}{6}\right) - 1\right\}^2 = 0$$

This leads to the results(Figure 3)

$$\cos\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \quad \text{and} \quad \sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \tag{27}$$

The behavior of the trigonometric functions in other quadrants can be derived by using sum formulas (11) similar to the derivation in (25). We have

$$\cos(\pi + \theta) = \cos\pi\cos\theta - \sin\pi\sin\theta = -\cos\theta \tag{28a}$$
$$\sin\pi + \theta) = \sin\pi\cos\theta + \cos\pi\sin\theta = -\sin\theta \tag{28b}$$

A plot of the trigonometric functions $\cos\theta$ and $\sin\theta$ in the range $[-2\pi, 2\pi]$ is shown in Figure 4.

```
# CosSinSeries.py:  Real part and Imaginary part of e^{ix}
import numpy as np
```

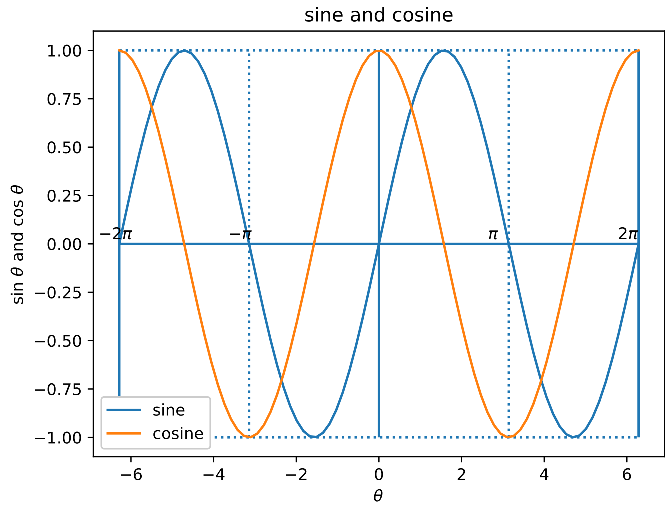

Figure 4: $\cos\theta$ and $\sin\theta$ in the interval $\theta \in [-2\pi, 2\pi]$

```
import matplotlib.pyplot as plt
PI = np.pi
print('pi, pi/2=', PI, 0.5* PI)
# calculation of cos(pi/2) and sin(pi/2) :
# The real part and the Imaginary part of e^{ix}, at x = pi/2
sp = 'C and S: the Real parts and the Imaginary parts of e^{i pi/2}'
print(sp)
x= 0.5* PI
C, S, tc, ts, x2  = 0, 0, 1, x, x*x
# tc, ts are leading errors of calculated C and S, respectively
print('           C               error                    S              error')
for m in range(6):
    n = 4*m
    fn1,fn2,fn3,fn4 , fn5 = n+1.0, n+2.0, n+3.0, n+4.0, n+5.0
    xn1 = x2/(fn1*fn2)
    C += tc*(1- xn1)
    sn1 = x2/(fn2*fn3)
    S += ts*(1- sn1)
    ss = "{0:21.16f},  {1:.2e}, {2:21.16f} , {3:.2e}\n".format(C,tc,S,ts)
```

```
    print(ss)
    xn2 = x2/(fn3*fn4) # prepare for the calculation of next pairs
    sn2 = x2/(fn4*fn5) # prepare for the calculation of next pairs
    tc *= xn1*xn2
    ts *= sn1*sn2
print('calculated value of cos(pi/2) =', C)
print('calculated value of sin(pi/2) =', S)
##########################
xx = np.linspace(0, 0.5* PI, 20)
C, S = [],[]
for x in xx:
    c, s, tc, ts, x2, sinx, cosx  = 0, 0, 1, x, x*x, np.sin(x), np.cos(x)
    print('x =', x, '   sin(x)=', sinx, 'cos(x)=', cosx)
    for m in range(5):
        n = 4*m
        fn1,fn2,fn3,fn4 , fn5 = n+1.0, n+2.0, n+3.0, n+4.0, n+5.0
        xn1 = x2/(fn1*fn2)
        xn2 = x2/(fn3*fn4)
        sn1 = x2/(fn2*fn3)
        sn2 = x2/(fn4*fn5) # prepare for next ts
        c += tc*(1- xn1)
        s += ts*(1- sn1)
        ss = "{0:21.16f},  {1:.2e}, {2:21.16f} , {3:.2e}\n".format(c,tc,s,ts)
        print(ss)
        tc *= xn1*xn2
        ts *= sn1*sn2
    print('c=', c, '  , C(x)-cos(x) =', "{:.2e}".format(c-cosx), '***')
    print('s=', s, '  , S(x)-sin(x) =', "{:.2e}".format(s-sinx), '***\n')
    C.append(c)
    S.append(s)
####
import matplotlib.pyplot as plt
xmin, xmax =0 , 0.5* PI
```

```
pi6, pi3 , pi4 = PI/6.0, PI/3.0, PI/4.0
sq2, sq3 = np.sqrt(2), np.sqrt(3)
plt.title('Cosine and Sine')
plt.plot(xx,S,label=r'$\sin\ \theta$')
plt.plot(xx,C,label=r'$\cos\ \theta$')
plt.hlines(0,xmin,xmax)
plt.hlines(1,xmin,xmax, linestyles='dotted')
plt.hlines(0.5,xmin,xmax, linestyles='dotted')
plt.hlines(1.0/sq2,xmin,xmax, linestyles='dotted')
plt.hlines(sq3/2.0,xmin,xmax, linestyles='dotted')
plt.vlines(pi6,0,1, linestyles='dotted')
plt.vlines(pi3,0,1, linestyles='dotted')
#plt.vlines(0,-1,1)
plt.vlines(pi4,0,1, linestyles='dotted')
plt.text(-0.05, 1.0/sq2,r'$\frac{1}{\sqrt{2}}$')
plt.text(-0.05, sq3/2.0,r'$\frac{\sqrt{3}}{2}$')
plt.text(-0.05, 1.0/2.0,r'$\frac{1}{2}$')
plt.text(pi4-0.05,0.02,r'$\pi/4$')
#plt.hlines(-1,xmin,xmax, linestyles='dotted')
plt.vlines(xmin,0,1)
plt.vlines(xmax,0,1)
plt.text(-0.05,0.02,r'$0$')
plt.text(xmax-0.05,0.02,r'$\pi/2$')
plt.text(pi6-0.05,0.02,r'$\pi/6$')
plt.text(pi3-0.05,0.02,r'$\pi/3$')
plt.xlabel(r'$\theta$')
plt.legend()
fig = plt.gcf()
fig.savefig('CosSin.eps', format='eps')
plt.show()
```