$ Supplement Note 7.1: Oscillatory Motion and Euler-Cromer Method

Equation of Simple Harmonic Oscillation and Its Solution

Consider a particle of mass m connected to a spring with a spring constant equal to k. According to *Hooke's law* the force acting on the particle is given by

$$F = -\,kx \tag{1}$$

where x is the displacement of the particle from the equilibrium position. From Newton's second law, the equation of motion for a *simple harmonic oscillator* is given by

$$F = ma = m\,\frac{d^2x}{dt^2} = -\,kx \quad \Leftrightarrow \quad \frac{d^2x}{dt^2} + \,\omega^2 x = 0\,. \tag{2}$$

where

$$k = m\omega^2 \quad \Leftrightarrow \quad \omega \equiv \sqrt{\frac{k}{m}} \tag{3}$$

Eq.(2) is a *homogeneous second-order linear differential equation with constant coefficients*. It may be viewed as a *homogeneous linear equation* for the *linear* differential *operator* $\mathcal{L}$, which operates on the function $x(t)$ (considered as a *vector* in the *vector space* of differentiable functions):

$$\mathcal{L}(x) \equiv \frac{d^2x}{dt^2} - \,\omega^2 x = 0\,. \tag{4}$$

The operator $\mathcal{L}$ satisfies the *linearity conditions*:

$$\begin{aligned}
\mathcal{L}(x_1 + x_2) \;&\equiv \frac{d^2(x_1+x_2)}{dt^2} - \,\omega^2(x_1+x_2) = \Big(\frac{d^2x_1}{dt^2} - \,\omega^2 x_1\Big) + \Big(\frac{d^2x_2}{dt^2} - \,\omega^2 x_2\Big)\\
&= \mathcal{L}(x_1) + \mathcal{L}(x_2) && \text{(5a)}\\
\mathcal{L}(C\,x) \;&\equiv \frac{d^2(C\,x)}{dt^2} - \,\omega^2(C\,x) = \;C\,\Big(\frac{d^2x}{dt^2} - \,\omega^2 x\Big)\\
&= C\;\mathcal{L}(x) && \text{(5b)}
\end{aligned}$$

where x, x_1, x_2 are functions(*vectors*) and C is an arbitrary constant. The *linearity conditions* Eq.(5) may be combined into a single equation:

$$\mathcal{L}(C_1\ x_1 + C_2\ x_2)\ = C_1\ \mathcal{L}(x_1)\ + C_2\ \mathcal{L}(x_2) \tag{6}$$

A direct differentiation will show that the functions $x_1 \equiv \cos\omega t$ and $x_2 \equiv \sin\omega t$ are solutions of Eq.(2). It follows from Eq.(4) and the linearity property (5) that *the linear superpositions* $x = C_1x_1 + C_2x_2$ also satisfy Eq.(2). In general a second order differential equation contains two arbitrary constants. Hence *the general form of a solution to the simple harmonic oscillation Eq.* (2) is given by

$$x = Ax_1 + Bx_2 = A\cos\omega t + B\sin\omega t = C\sin(\omega t + \phi_0) \tag{7}$$

where A, B or C, ϕ_0 are constants which may be determined by the initial consitions $x(t_0)$ and $\frac{dx(t_0)}{dt}$.

Potential Energy and Energy Conservation

As discussed in Supplement Note7 the *power* acting on the particle, which is equal to the *rate of change of the kinetic energy* $E_{\text{kin}} \equiv \frac{1}{2}mv^2$ is given by *the product of force acting on it times its velocity*:

$$P \equiv \frac{dE_{\text{kin}}}{dt} = \frac{d}{dt}\Big(\frac{1}{2}mv^2\Big) = mv\ \frac{dv}{dt} = Fv \tag{8}$$

The *potential energy* for the simple harmonic oscillator, V_{pot}, is defined as

$$V_{\text{pot}}(x) \equiv \frac{1}{2}kx^2 \tag{9}$$

so that *the force* (1) *is equal to the negative of the derivative of the potential with respect to* x:

$$F = -\ \frac{d\ V_{\text{pot}}(x)}{dx} = -kx \tag{10}$$

Using Eq.(10) and Eq. (8) we see that the time rate change of the *total energy* $E \equiv E_{\text{kin}} + V_{\text{pot}}$ vanishes:

$$\frac{dE}{dt} = \frac{dE_{\text{kin}}}{dt} + \frac{dV_{\text{pot}}}{dt} = F\ v + \frac{d\ V_{\text{pot}}(x)}{dx}\frac{dx}{dt} = 0 \tag{11}$$

Eq.(11) expresses the *law of conservation of energy*, i.e. the total energy E of a particle is a constant. This is one of the most important implications of Newton's second law of motion.

Euler-Cromer Method

For a numerical approach using Euler's method, the second order differential equaiton Eq.(2) is rewritten as a set of two simultaneous first order differential equations for $v \equiv \frac{dx}{dt}$ and x:

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 x \quad \Rightarrow \quad v_{i+1} = v_i - \omega^2 x_i \, \delta t \tag{12a}$$

$$\frac{dx}{dt} = v \quad \Rightarrow \quad x_{i+1} = x_i + v_i \, \delta t \tag{12b}$$

An example with $\omega = 9.284$ is coded in *SimpleHarmonic.py*. It is found that *the Euler method fails for oscillatory motions*. As may be seen in Figure 1, although the motion is basically oscillatory, *the amplitudes* of the oscillations *grow* with time, in contrary to the exact solution (7). This behavior can not be remedied by decreasing the time step δt. This is related to the fact that in the Euler method(12), the *energy is not conserved*:

$$\begin{aligned}
E_{i+1} &= \frac{1}{2}mv_{i+1}^2 + \frac{1}{2}kx_{i+1}^2 = \frac{1}{2}m(v_i - \omega^2 x_i \, \delta t)^2 + \frac{1}{2}k(x_i + v_i \, \delta t)^2 \\
&= \frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 + v_i x_i \delta t(-m\omega^2 + k) + \frac{1}{2}(\delta t)^2 \left(m\omega^4 x_i^2 + kv_i^2\right) \\
&= E_i + \frac{1}{2}(\delta t)^2 \omega^2 \left(kx_i^2 + mv_i^2\right) = E_i + (\delta t)^2 \omega^2 \, E_i
\end{aligned} \tag{13}$$

Thus although the first order error δt of $E_{i+1} - E_i$ vanishes due to (3), a tiny error $(\delta t)^2\omega^2 \; E_i$ remains.The energy is slowly increasing, leading to the growth of the amplitude. This defect may be fixed by a simple modification which is known as the *Euler-Cromer method*. In this method, Eq.(12) are replaced by

$$\begin{aligned}
v_{i+1} &= v_i - \; \omega^2 x_i \, \delta t \\
x_{i+1} &= x_i + \; v_{i+1} \, \delta t \, .
\end{aligned} \tag{14}$$

With the Euler method, the *previous* values of both v and x are used to calculate the new values of both v and x. In the Euler-Cromer method, the *previous* values

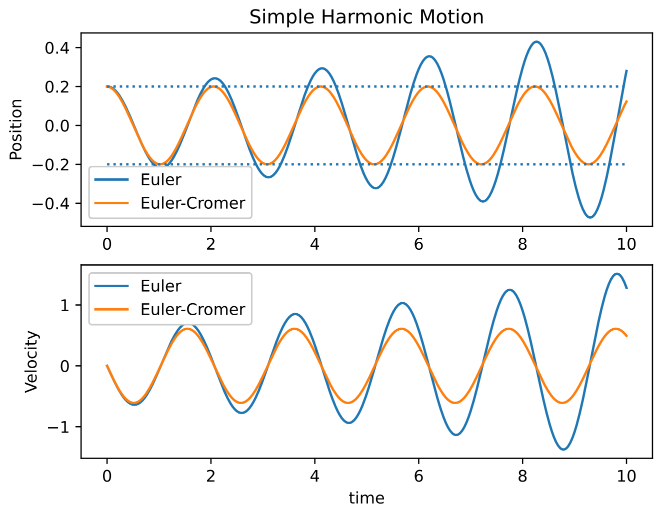

Figure 1: Simple Harmonic Oscillator

of both v and x are used to calculate the new value of v, the same as the Euler's method. But the *new value of v* (and the *previous value* of x) is used to calculate the new value of x. This minor change of the algorithm in the Euler-Cromer method *conserves energy over each cycle* of the periodic motion.

$$\begin{aligned}E_{i+1} &= \frac{1}{2}mv_{i+1}^2 + \frac{1}{2}kx_{i+1}^2 = \frac{1}{2}m(v_i - \omega^2 x_i\,\delta t)^2 + \frac{1}{2}k\left(x_i + (v_i - \ \omega^2 x_i\,\delta t)\,\delta t\right)^2 \\ &= \frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 + v_i x_i \delta t(-m\omega^2 + k) + \frac{1}{2}(\delta t)^2\left(m\omega^4 x_i^2 + kv_i^2 - 2k\omega^2 x_i^2\right) + \ldots \\ &= E_i + \frac{1}{2}(\delta t)^2\omega^2\left(kx_i^2 - mv_i^2\right) = E_i + (\delta t)^2\omega^2\,(E_{\mathrm{kin}}(t_1) - V_{\mathrm{pot}}(x_1)) \qquad (15)\end{aligned}$$

where the neglected terms are of errors of higher orders than $(\delta t)^2$. The cancellation of the kinetic energy E_{kin} and the potential energy V_{pot} may be proved in general by using the *action and angle variables* in the *Hamiltonian mechanics*. For the simple harmonic oscillation, this may be proved by using the general form of the solution (7) $x = C\sin(\omega t + \phi_0) \Rightarrow v = C\omega\cos(\omega t + \phi_0)$:

$$E_{\mathrm{kin}} = \frac{1}{2}mv^2 = \frac{mC^2\omega^2}{2}\cos^2(\omega t + \phi_0) = \frac{kC^2(1 + \cos 2(\omega t + \phi_0))}{4} \qquad (16a)$$

$$V_{\mathrm{pot}} = \frac{1}{2}kx^2 = \frac{kC^2}{2}\sin^2(\omega t + \phi_0) = \frac{kC^2(1 - \cos 2(\omega t + \phi_0))}{4} \qquad (16b)$$

The integration of the oscillation term $\cos 2(\omega t + \phi_0)$ vanishes over each cycle of the periodic motion. Hence with the Euler-Cromer algorithm, the leading errors ($\propto (\delta t)^2$) *cancel each other over each period*, and the amplitudes of the oscillation remain unchanged as was shown in Figure 1.

```
# SimpleHarmonic.py
# simulation of Simple Harmonic Oscillation
import numpy as np
import matplotlib.pyplot as plt
t , x , v= 0 , 0.2 , 0
x1 , v1= 0.2 , 0
tf , dt = 10, 0.02 #tf = final time
omega = 9.284
size = tf // dt
T, y1, y2, V1, V2 = [], [], [], [], []
while t < tf:
    T.append(t)
    y1.append(x)
    V1.append(v)
    y2.append(x1)
    V2.append(v1)
    xold = x
    x  = x + v* dt              #Euler
    v  = v  - omega * xold * dt #Euler
    v1 = v1 - omega * x1* dt    #Euler-Cromer
    x1 = x1 + v1* dt            #Euler-Cromer: v1 is the updated value
    t += dt
plt.figure()
plt.subplot(2,1,1)
plt.title('Simple Harmonic Motion')
plt.ylabel('Position')
plt.plot(T,y1,label='Euler')
plt.plot(T,y2,label='Euler-Cromer')
plt.hlines(0.2,0,tf, linestyles='dotted')
```

```
plt.hlines(-0.2,0,tf, linestyles='dotted')
plt.legend()
plt.subplot(2,1,2)
#plt.title('Simple Harmonic Motion Velocity')
plt.plot(T,V1,label='Euler')
plt.ylabel('Velocity')
plt.plot(T,V2,label='Euler-Cromer')
#plt.hlines(0.2,0,tf, linestyles='dotted')
#plt.hlines(-0.2,0,tf, linestyles='dotted')
plt.xlabel('time')
plt.legend()
fig = plt.gcf()
fig.savefig('SimpleHarmonic.eps', format='eps')
plt.show()
```