$ Supplement Note 14: Eigenvalues and Eigenvectors

$1 Definition

If the *direction* of a vector $\mathbf{x}$ is preserved after being operated by an operator A,i.e.

$$A\mathbf{x} = \lambda\mathbf{x} \,, \tag{1}$$

where λ is a scalar (a complex number!), then the vector $\mathbf{x}$ is called an *eigenvector* of the operator A and the number λ is an *eigenvalue* of the operator.

Example 1. The vectors $\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$ are eigenvectors of *the Pauli matrix* $\sigma_z \equiv \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ with eigenvalues equal to 1 and -1 respectively:

$$\sigma_z\mathbf{e}_1 \equiv \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = \mathbf{e}_1, \quad \sigma_z\mathbf{e}_2 \equiv \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = -\mathbf{e}_2 \tag{2}$$

Example 2. The vectors $\mathbf{e}'_1 = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{e}'_2 = \begin{pmatrix}1\\-1\end{pmatrix}$ are eigenvectors of *the Pauli matrix* $\sigma_x \equiv \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ with eigenvalues equal to 1 and -1 respectively:

$$\sigma_x\mathbf{e}'_1 \equiv \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = \mathbf{e}'_1, \quad \sigma_x\mathbf{e}'_2 \equiv \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = -\mathbf{e}'_2 \tag{3}$$

Example 3. The vectors $\mathbf{e}''_1 = \begin{pmatrix}1\\i\end{pmatrix}$ and $\mathbf{e}''_2 = \begin{pmatrix}1\\-i\end{pmatrix}$ are eigenvectors of *the Pauli matrix* $\sigma_y \equiv \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$ with eigenvalues equal to 1 and -1 respectively:

$$\sigma_y\mathbf{e}''_1 \equiv \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1\\i\end{pmatrix} = \begin{pmatrix}1\\i\end{pmatrix} = \mathbf{e}''_1, \quad \sigma_y\mathbf{e}''_2 \equiv \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix} = \begin{pmatrix}-1\\i\end{pmatrix} = -\mathbf{e}''_2 \tag{4}$$

$2 Homogeneous Linear Differential Equations With Constant Coefficients

The differential operator $D \equiv \frac{d}{dx}$ is a *linear operator* on the *vector space* of differentiable functions, since the following *linearity condition* is satisfied:

$$D\left(\alpha f(x) + \beta g(x)\right) = \alpha\, D\, f(x) + \beta\, D\, g(x) : \quad \alpha,\ \beta \text{ are constants.} \tag{5}$$

The first order *homogeneous linear differential equation with constant coefficient*

$$\frac{d}{dx}\, y - \lambda\, y \equiv D\, y - \lambda\, y = 0\ . \tag{6}$$

may be seen as an *eigenvalue equation* for the operator D. Its solution, the *eigenvector* (*eigenfunction*), is given by the exponential function $y = e^{\lambda x}$

$$D\, y \equiv \frac{d}{dx}\, e^{\lambda x} = \lambda\, e^{\lambda x} = \lambda\, y\ , \tag{7}$$

Repeated application of the operator D on the eigenvector $e^{\lambda x}$ leads to

$$D^n\, y \equiv \Big(\frac{d}{dx}\Big)^n\, e^{\lambda x} = \lambda^n\, e^{\lambda x} = \lambda^n\, y \tag{8}$$

The series expansion for the eigenfunction of D, $y(x) = f(x) = e^{\lambda x}$, may be obtained by combining the boundary condition $y(0) = 1$ with Eq.(8) and the Taylor-MacLaughlin series expansion (**Section 8.4** of Lecture Note),

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} (\frac{d}{dx})^n f(0) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \lambda^n f(0) = \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!} \tag{9}$$

The eigenvalue equation (8) also provides general solutions to *homogeneous linear differential equations with constant coefficients*. We limit our discussion to second order. Generalization to higher order is straightforward. Such an equation may be seen as a linear equation for the linear operator $\mathcal{L} \equiv a_2 D^2 + a_1 D + a_0$:

$$\mathcal{L}y \equiv a_2\, D^2 y + a_1\, Dy + a_0\, y \equiv a_2\, y'' + a_1\, y' + a_0\, y = 0 \tag{10}$$

where a_0, a_1, a_2 are constants. By using (8), equation (10) leads to

$$(a_2\, \lambda^2 + a_1\, \lambda + a_0)\, e^{\lambda x} = 0 \tag{11}$$

As described in **Section** 10.2.1(p.10-5), in general, the quadratic equation (11) has two roots, λ_1, λ_2 (This is **Theorem** 14.6, the *Fundamental Theorem of Algebra*). If the two roots are different, then the linear combinations of $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ yield the desired general solutions to the linear equation (10) $\mathcal{L}y = 0$:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \tag{12}$$

since in general *a second order differential equation contains two arbitrary constants*. In the cases with double roots, (12) may be recast in the following form

$$y = \left(c_1' - \frac{c_2'}{\lambda_2 - \lambda_1}\right) e^{\lambda_1 x} + \frac{c_2'}{\lambda_2 - \lambda_1} e^{\lambda_2 x} = c_1' e^{\lambda_1 x} + c_2' \frac{e^{\lambda_2 x} - e^{\lambda_1 x}}{\lambda_2 - \lambda_1} \tag{13}$$

When the parameters a_1, a_2, a_3 of the differential equation (10) are varied so that the two roots converge to the same value $\lambda_2 \to \lambda_1$, the last term in (13) changes smoothly to (**Comment 10.2** in Lecture Note)

$$\lim_{\lambda_2 \to \lambda_1} \frac{e^{\lambda_2 x} - e^{\lambda_1 x}}{\lambda_2 - \lambda_1} = \left. \frac{de^{\lambda x}}{d\lambda} \right|_{\lambda = \lambda_1} = x e^{\lambda_1 x}$$

Hence the general solution to (10) with identical roots λ_1 is given by

$$y = (c_1' + c_2' \, x) e^{\lambda_1 x} \tag{14}$$

$3 Diagonalization of Symmetric Matrix

As was shown in **Theorem 6.1**, *for a symmetric operator A, the eigenvectors corresponding to different eigenvalues are mutually orthogonal*. Let $\{\mathbf{v}_i\}_{i=1,\ldots,n}$ be the *normalized eigenvectors*, i.e. $|\mathbf{v}_i| = 1$, of a symmetric operator A, with the corresponding eigenvalues given by λ_i:

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i, \quad i = 1, \ldots, n \tag{15}$$

We may use the set of the orthogonal unit vectors $\{\mathbf{v}_i\}_{i=1,\ldots,n}$ as the new set of basis vectors. From (15), the matrix representation of the operator A in this set is then in the *diagonal form*(The dimension of the space is taken to be $n = 3$):

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{16}$$

This process is called *diagonalization of symmetric matrix*. For a rigid body, the *moment of inertia tensor I*, Eq.(11.16) in the Lecture Note, is given by

$$I_{ij} = \sum m(r^2 \delta_{ij} - x_i x_j) = \int \rho(\mathbf{r})(r^2 \delta_{ij} - x_i x_j) d^3 r; \quad i, j = 1, 2, 3 \tag{17}$$

where $\rho(\mathbf{r})$ is the mass density. The inertia tensor I is symmetric. Its eigenvectors are orthogonal. The directions of these eigenvectors are called *the principal axes of inertia*, and their eigenvalues $I_i; i = 1, 2, 3$ are referred to as the *principal moments of inertia*. In terms of this new net of basis vectors, the inertial tensor can be reduced to diagonal form:

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{18}$$

Example 4. *Determine the principal moments of inertia for the following homogeneous body of mass M.*

1. *A thin rod of length l.*

   Taking the direction of the rod as the z-axis. Here we have $\rho l = M$

$$I_1 = I_2 = \int_{-\frac{l}{2}}^{\frac{l}{2}} \rho z^2 dz = \frac{2}{3}\rho\left(\frac{l}{2}\right)^3 = \frac{1}{12}\rho l^3 = \frac{1}{12}Ml^2 \quad ; \quad I_3 = 0 \tag{19}$$

2. *A rectangular parallelepiped of sides a, b, c.*

$$I_1 = \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{b}{2}}^{\frac{b}{2}} dy \int_{-\frac{c}{2}}^{\frac{c}{2}} dz \rho(y^2 + z^2) = \frac{\rho}{12}(acb^3 + abc^3) = \frac{1}{12}M(b^2 + c^2) \tag{20}$$

   where the identity $abc\rho = M$ have been used. Similar, we have

$$I_2 = \frac{1}{12}M(a^2 + c^2); \quad \text{and} \quad I_3 = \frac{1}{12}M(a^2 + b^2) \tag{21}$$

3. *A circular cylinder of radius R and height h.*

   It is convenient to use the *cylindrical coordinates* r, ϕ, z :

$$x = r\cos\phi, \; y = r\sin\phi$$

with the volume element $d^3r = \rho\ rdr\ d\phi\ dz \Rightarrow M = \pi R^2 h\rho$:

$$
\begin{aligned}
I_1 =& \iiint \rho d^3r\ (y^2 + z^2) \\
=&\rho \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \int_0^R rdr \int_0^{2\pi} d\phi (r^2 \sin^2\phi + z^2) = \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \int_0^R rdr (r^2\pi + 2\pi z^2) \\
=&\rho\pi \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \Big(\frac{R^4}{4} + R^2 z^2\Big) = \pi\rho R^2 \Big(\frac{R^2}{4} h + \frac{h^3}{12}\Big) \\
=&\frac{M}{4}\Big(R^2 + \frac{h^2}{3}\Big) \qquad (22) \\
I_2 =& \iiint \rho d^3r\ (x^2 + z^2) \\
=&\rho \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \int_0^R rdr \int_0^{2\pi} d\phi (r^2 \cos^2\phi + z^2) = \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \int_0^R rdr (r^2\pi + 2\pi z^2) \\
=&\frac{M}{4}\Big(R^2 + \frac{h^2}{3}\Big) = I_3 \qquad (23) \\
I_3 =& \iiint \rho d^3r\ (x^2 + y^2) \\
=&\rho \int_{-\frac{h}{2}}^{\frac{h}{2}} dz \int_0^R rdr \int_0^{2\pi} d\phi\ r^2 = h\rho \int_0^R 2\pi r^3 dr = h\rho\pi \frac{R^4}{2} \\
=&\frac{R^2}{2} M \qquad (24)
\end{aligned}
$$

The equality $I_1 = I_2$ also follows directly from symmetry!

4. *A sphere of radius R.*

Here we use the *spherical coordinates* r, θ, ϕ

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

with the volume element $d^3r = r^2 \sin\theta\ dr\ d\theta\ d\phi \Rightarrow \frac{4\pi R^3\rho}{3} = M$. The identity $I_1 = I_2 = I_3$ follows from symmetry.

$$
\begin{aligned}
I_1 =& I_2 = I_3 = \frac{1}{3}(I_1 + I_2 + I_3) = \iiint_{r^3 \le R} \frac{2}{3}(x^2 + y^2 + z^2)\rho d^3r \\
=&\frac{2\rho}{3} \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\ r^2 = \frac{2\rho}{3}\frac{R^5}{5}\ 4\pi \\
=&\frac{2}{5} R^2 M \qquad (25)
\end{aligned}
$$