$ Supplement Note 8: Oscillatory Motion and Euler-Cromer Method

Equation of Simple Harmonic Oscillation and Its Solution

Consider a particle of mass m connected to a spring with a spring constant equal to k. According to *Hooke's law* the force acting on the particle is given by

$$F = -kx \tag{1}$$

where x is the displacement of the particle from the equilibrium position. The Newton's second law of motion tells us that

$$F = ma = m\frac{d^2x}{dt^2} = -kx \tag{2}$$

Thus the equation of motion for a *simple harmonic oscillator* is given by

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \equiv -\omega^2 x \quad \Leftrightarrow \quad \frac{d^2x}{dt^2} + \omega^2 x = 0\,. \tag{3}$$

where

$$k = m\omega^2 \quad \Leftrightarrow \quad \omega \equiv \sqrt{\frac{k}{m}} \tag{4}$$

Eq.(3) is a *homogeneous second-order linear differential equation with constant coefficients*. It is called *second-order* since the highest order of differentiation for x is two. It is of the general form

$$a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0\,x = 0\,. \tag{5}$$

where the coefficients a_i are *constants*. It is *homogeneous* if the right hand side of (5) is zero. Notice that if x_1 and x_2 satisfy (3), ($\frac{d^2x_1}{dt^2} + \omega^2 x_1 = \frac{d^2x_2}{dt^2} + \omega^2 x_2 = 0$), *its linear superposition (i.e.* $C_1x_1 + C_2x_2$, C_1, C_2 *constants), also satisfies* (3) :

$$\frac{d^2}{dt^2}(C_1x_1 + C_2x_2) + \omega^2(C_1x_1 + C_2x_2) = 0 \tag{6}$$

This is true in general for homogeneous linear differential equations with constant coefficients. (3) is solved by rewriting it as a set of two simultaneous first order differential equations for $v \equiv \frac{dix}{dt}$ and x and then employing Euler's method to solve it numerically

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 x \quad \Rightarrow \quad v_{i+1} = v_i - \omega^2 x_i \, \delta t \tag{7a}$$

$$\frac{dx}{dt} = v \quad \Rightarrow \quad x_{i+1} = x_i + v_i \, \delta t \tag{7b}$$

The solution is determined by the initial condition: i.e. the values of x_0 and v_0. By using similar reasoning, the following conclusion may be obtained in general: *a general solution of an n–th order differential equation has n arbitrary constants* (c.f p. 10.3 of Lecture note).

The functions $x_1 \equiv \cos\omega t, x_2 \equiv \sin\omega t$ are solutions of (3). This may be easily proved by direct differentiation using chain rule with $\phi = \omega t$:

$$\frac{dx_1}{dt} = \frac{d}{dt}\cos\omega t = \frac{d\cos\phi}{d\phi}\frac{d\phi}{dt} = -\omega\sin\omega t$$

$$\frac{d^2x_1}{dt^2} = \frac{d^2}{dt^2}\cos\omega t = -\frac{d}{dt}\omega\sin\omega t = -\,\omega^2\cos\omega t = -\,\omega^2 x_1 \tag{8a}$$

$$\frac{dx_2}{dt} = \frac{d}{dt}\sin\omega t = \frac{d\sin\phi}{d\phi}\frac{d\phi}{dt} = \omega\cos\omega t$$

$$\frac{d^2x_2}{dt^2} = \frac{d^2}{dt^2}\sin\omega t = \frac{d}{dt}\omega\cos\omega t = -\,\omega^2\sin\omega t = -\,\omega^2 x_2 \tag{8b}$$

As discussed above, a second order differential equation contains two arbitrary constants. Hence *the general form of a solution to the simple harmonic oscillation* (3) is given by

$$x = Ax_1 + Bx_2 = A\cos\omega t + B\sin\omega t; \quad A, B \text{ are constants .} \tag{9}$$

(9) may also be written as

$$x = C\sin(\omega t + \phi_0) = C\sin\omega t\cos\phi_0 + C\cos\omega t\sin\phi_0 \tag{10}$$

where C, and ϕ_0 are constants. This leads to

$$C\cos\phi_0 = B,\ C\sin\phi_0 = A \quad \Rightarrow \quad C = \sqrt{A^2 + B^2}\,,$$

and

$$\frac{\sin\phi_0}{\cos\phi_0} \equiv \tan\phi_0 = \frac{A}{B} \quad \Rightarrow \quad \phi_0 = \tan^{-1}(\frac{A}{B})$$

Similarly, (9) may also be written as

$$x = C\cos(\omega t + \phi_0) = C\cos\omega t\cos\phi_0 - C\sin\omega t\sin\phi_0 \tag{11}$$

Potential Energy and Energy Conservation

For the simple harmonic oscillator problem, the *potential energy* V_{pot} of the spring is defined as

$$V_{\text{pot}}(x) \equiv \frac{1}{2}kx^2 \tag{12}$$

so that *the force* (1) *may be defined as the negative of the derivative of the potential with respect to* x:

$$F = -\frac{d}{dx}V_{\text{pot}} = -kx \tag{13}$$

As discussed in Supplement Note7, the *rate of change of* E_{kin} (the *power* acting on the particle) is equal to *the product of force acting on it times its velocity*.

$$P \equiv \frac{dE_{\text{kin}}}{dt} = \frac{d}{dv}(\frac{1}{2}mv^2)\,\frac{dv}{dt} = mv\,\frac{dv}{dt} = Fv \tag{14}$$

By integrating (14) with respect to t and making a substitution of integration to x, we obtain

$$\int_{t_1}^{t_2} \frac{dE_{\text{kin}}}{dt} = E_{\text{kin}}(t_2) - E_{\text{kin}}(t_1) = \int_{t_1}^{t_2} F\frac{dx}{dt}dt = \int_{x_1}^{x_2} F dx \equiv W \tag{15}$$

where the integration

$$W \equiv \int_{x_1}^{x_2} F dx$$

is referred to as *the work done on the particle* by the force and Eq.(15) states that: *the total work done on the particle is equal to the increment of the total kinetic energy*. With the connection of force to the potential energy (13) this yields

$$E_{\text{kin}}(t_2) - E_{\text{kin}}(t_1) = \int_{x_1}^{x_2} F dx = \int_{x_1}^{x_2} (-\frac{dV_{\text{pot}}}{dx})dx = -\left(V_{\text{pot}}(x_2) - V_{\text{pot}}(x_1)\right)\,. \tag{16}$$

Thus *the increase of the kinetic energy is compensated by the loss of the potential energy*. It follows from (16) that

$$E_{\rm kin}(t_1) + V_{\rm pot}(x_1) = E_{\rm kin}(t_2) + V_{\rm pot}(x_2) \tag{17}$$

By defining the *total energy E* of a particle as *the sum of the kinetic energy and the potential energy* $E \equiv E_{\rm kin} + V_{\rm pot}$, Eq.(17) then expresses the *law of conservation of energy*, i.e. the total energy E of a particle is a constant. This is one of the most important implications of Newton's second law of motion.

Using (4) and the identities $\sin^2 x = \frac{1}{2}(1-\cos 2x), \cos^2 x = \frac{1}{2}(1+\cos 2x)$ we obtain from the general solution (10) for the simple harmonic oscillator,

$$V_{\rm pot} = \frac{1}{2}kx^2 = \frac{1}{2}kC^2 \sin^2(\omega t + \phi_0) = \frac{1}{4}kC^2\,(1 - \cos 2(\omega t + \phi_0)) \tag{18}$$

$$E_{\rm kin} = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 C^2 \cos^2(\omega t + \phi_0) = \frac{1}{4}kC^2\,(1 + \cos 2(\omega t + \phi_0)) \tag{19}$$

The following implications can be immediately obtained:

1. The total energy E of a simple harmonic oscillator is a constant.

$$E \equiv E_{\rm kin} + V_{\rm pot} = \frac{1}{2}kC^2\ . \tag{20}$$

2. The average value of the oscillator term $\cos 2(\omega t + \phi_0)$ is zero. Thus *the average values of the kinetic energy E_{kin} and the potential energy V_{pot} are equal to each other*:

$$\overline{E_{\rm kin}} = \overline{V_{\rm pot}} = \frac{1}{4}kC^2\ . \tag{21}$$

   Each one of them is equal to half of the total energy, $E = \frac{1}{2}kC^2$.

Euler-Cromer Method

It is found that the Euler method fails for oscillatory motions. An example with $\omega = 9.284$ is coded in *SimpleHarmonic.py*. Although the motion is basically oscillatory, the amplitude of the oscillation *grows* with time, in contrary to the exact solution, (9). This behavior can not be remedied by decreasing the time

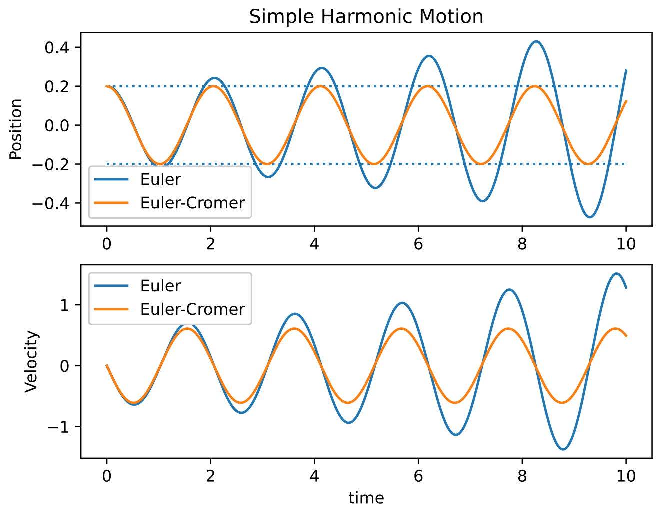

Figure 1: Simple Harmonic Oscillator

step δt. This is related to the fact that in the Euler method(7), the *energy is not conserved*.

$$\begin{aligned}
E_{i+1} = \frac{1}{2}mv_{i+1}^2 + \frac{1}{2}kx_{i+1}^2 &= \frac{1}{2}m(v_i - \omega^2 x_i\,\delta t)^2 + \frac{1}{2}k(x_i + v_i\,\delta t)^2 \\
&= \frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 + v_i x_i \delta t(-m\omega^2 + k) + \frac{1}{2}(\delta t)^2\left(m\omega^4 x_i^2 + kv_i^2\right) \\
&= E_i + \frac{1}{2}(\delta t)^2\omega^2\left(kx_i^2 + mv_i^2\right) = E_i + (\delta t)^2\omega^2\,E_i
\end{aligned} \tag{22}$$

Thus although the first order δt of the error $E_{i+1}-E_i$ vanishes due to (4), a tiny error $(\delta t)^2\omega^2\,E_i$ keeps adding, so that the energy is slowly increasing, demonstrated by the growth of the amplitude. This defect may be fixed by a simple modification which is known as the *Euler-Cromer method*. In this method, Eq.(7) are replaced by

$$\begin{aligned}
v_{i+1} &= v_i - \omega^2 x_i\,\delta t \\
x_{i+1} &= x_i + v_{i+1}\,\delta t\,.
\end{aligned} \tag{23}$$

With the Euler method, the *previous* values of both v and x are used to calculate the new values of both v and x. However, with the Euler-Cromer method, the *previous* values of both v and x are used to calculate the new value of v, but the *new value of v is used to calculate the new value of x.* This minor change in the algorithm in the Euler-Cromer method *conserves energy exactly* over each complete period of the motion and thus changes the whole picture.

$$
\begin{aligned}
E_{i+1} =& \frac{1}{2}mv_{i+1}^2 + \frac{1}{2}kx_{i+1}^2 = \frac{1}{2}m(v_i - \omega^2 x_i\,\delta t)^2 + \frac{1}{2}k\left(x_i + (v_i - \ \omega^2 x_i\,\delta t)\,\delta t\right)^2 \\
=&\frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 + v_i x_i \delta t(-m\omega^2 + k) + \frac{1}{2}(\delta t)^2\left(m\omega^4 x_i^2 + kv_i^2 - 2k\omega^2 x_i^2\right) + \ldots \\
=&E_i + \frac{1}{2}(\delta t)^2\omega^2\left(kx_i^2 - mv_i^2\right) = E_i + (\delta t)^2\omega^2\left(E_{\mathrm{kin}}(t_1) - V_{\mathrm{pot}}(x_1)\right)
\end{aligned}
\tag{24}
$$

where the neglected terms are of higher orders of δt. The relation (21) guarantees that the leading errors ($\propto (\delta t)^2$) *cancel each other over each period.* In particular, the oscillation amplitude remains unchanged for results obtained with the Euler-Cromer algorithm, as shown in *SimpleHarmonic.py*.

```
# SimpleHarmonic.py
# simulation of Simple Harmonic Oscillation
import numpy as np
import matplotlib.pyplot as plt
t , x , v= 0 , 0.2 , 0
x1 , v1= 0.2 , 0
tf , dt = 10, 0.02 #tf = final time
omega = 9.284
size = tf // dt
T, y1, y2, V1, V2 = [], [], [], [], []
while t < tf:
    T.append(t)
    y1.append(x)
    V1.append(v)
    y2.append(x1)
    V2.append(v1)
    xold = x
    x  = x + v* dt                #Euler
    v  = v  - omega * xold * dt #Euler
```

```
    v1 = v1 - omega * x1* dt    #Euler-Cromer
    x1 = x1 + v1* dt            #Euler-Cromer: v1 is the updated value
    t += dt
plt.figure()
plt.subplot(2,1,1)
plt.title('Simple Harmonic Motion')
plt.ylabel('Position')
plt.plot(T,y1,label='Euler')
plt.plot(T,y2,label='Euler-Cromer')
plt.hlines(0.2,0,tf, linestyles='dotted')
plt.hlines(-0.2,0,tf, linestyles='dotted')
plt.legend()
plt.subplot(2,1,2)
#plt.title('Simple Harmonic Motion Velocity')
plt.plot(T,V1,label='Euler')
plt.ylabel('Velocity')
plt.plot(T,V2,label='Euler-Cromer')
#plt.hlines(0.2,0,tf, linestyles='dotted')
#plt.hlines(-0.2,0,tf, linestyles='dotted')
plt.xlabel('time')
plt.legend()
fig = plt.gcf()
fig.savefig('SimpleHarmonic.eps', format='eps')
plt.show()
```