$ Supplement Note 7: Aerodynamic Drag

1. Energy and Power

The *kinetic energy* of a particle is defined as

$$E_{\rm kin} \equiv \frac{1}{2}\, m\, v^2 = \frac{1}{2}\, m\, \mathbf{v}\cdot\mathbf{v}\ . \tag{1}$$

The unit of energy is *joul* in MKS system and *erg* in cgs system.

$$1\ \text{joul} = 1\ \text{kg m}^2/\text{s}^2,\ 1\ \text{erg} = 1\ \text{g cm}^2/\text{s}^2 \quad \Rightarrow \quad 1\ \text{joul} = 10^7\ \text{erg}$$

The *time rate of change of the kinetic energy* of a body $\frac{dE_{\rm kin}}{dt} \equiv P$ is called *power*. Its unit in MKS system is *watt*

$$1\text{watt} = 1\ \text{joul/s}.$$

The motion of a particle is governed by *Newton's Second law*:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\,\frac{d\mathbf{v}}{dt} = m\,\mathbf{a} \tag{2}$$

where $\mathbf{p} \equiv m\,\mathbf{v}$ is the *momentum* and $\mathbf{a} \equiv \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$ is the *acceleration* of the particle. A direct differentiation of (1) with respect to t leads to

$$P \equiv \frac{dE_{\rm kin}}{dt} = \frac{d}{dv}\Big(\frac{1}{2}mv^2\Big)\,\frac{dv}{dt} = mv\,\frac{dv}{dt} = Fv \tag{3}$$

Notice that when the directions of $\mathbf{v}$ and $\mathbf{F}$ differ, Eq.(3) should be replaced by

$$P \equiv \frac{dE_{\rm kin}}{dt} = m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} = \mathbf{v}\cdot\mathbf{F} \tag{4}$$

Eq.(4) states that the *rate of change of E_{kin}* (the *power* acting on the particle) is equal to *the dot product of the force acting on it and its velocity*. By integrating Eq.(4) with respect to t, we obtain

$$\int_{t_1}^{t_2} \frac{dE_{\rm kin}}{dt} = E_{\rm kin}(t_2) - E_{\rm kin}(t_1) = \int_{t_1}^{t_2} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}\cdot d\mathbf{r} \tag{5}$$

The dot product $\mathbf{F}\cdot d\mathbf{r}$ between the force acting on the particle and the displacement of the particle is known as the *work* done on the particle. Eq.(5) states that *the total work done on the particle is equal to the increment of the total kinetic energy*. This is an important implication of Newton's second law of motion.

2. The Effect of Air Drag

As a body moving through air, it experiences a drag force in the direction opposite to its motion. In general, this force is given by

$$F_{\text{drag}} \approx -B_1 v - B_2 v^2 \tag{6}$$

At extremely low velocities the first term dominates, and its coefficient B_1 can be calculated for objects with simple shapes. This is known as *Stokes' law*. In particular, the drag force for a sphere with radius R moving through a viscous flow with a dynamic *viscosity* of η is given by

$$F_{\text{drag}} = 6\pi\eta R v \tag{7}$$

At any reasonable velocity the second term in (6) dominates for most objects. The exact value of the coefficient B_2 is a difficult problem. However, an approximate estimate can be made as follows. In a duration of time Δt, an object with a frontal area A and velocity v moves through the atmosphere must push an amount of air with a volume equal to $V = A \cdot v\Delta t$ out of the way with velocities close to v. If the density of air is ρ, the total mass m is equal to

$$m = \rho V = \rho A v \Delta t$$

The rate of energy gain of the air is then given by

$$Fv = \frac{dE_{\text{air}}}{dt} \approx \frac{\frac{1}{2}mv^2}{\Delta t} = \frac{1}{2}\rho A\ v\ v^2 \tag{8}$$

From (3), this implies that the force acting on the air (*by the object*) is equal to

$$F = \frac{1}{2}\rho A\ v^2$$

The reaction force from Newton's third law of motion then yields the drag force on the object is given by

$$F_{\text{drag}} = -\frac{1}{2}\, C\rho A\, v^2 \tag{9}$$

where C,which is known as the drag coefficient, is a dimensionless factor related to the geometry of the body.

Example 1 *A* 70*kg sky diver free-falls from an altitude of* 1400 *m high. For the air drag, assume that* $C \approx 0.5$*,* $A \approx 1\ m^2$ *and the air density is* $\rho \approx 1\ kg/m^3$.

In this case, Newton's second law (2) becomes

$$\frac{dv}{dt} = -\,g + \frac{C\rho A}{2m}\, v^2 \tag{10}$$

The Euler's approximation now takes the form

$$\begin{aligned} y_{i+1} &= y_i + v_i\, \delta t \\ v_{i+1} &= v_i + \left(-g + \frac{C\rho A}{2m}\, v_i^2\right) \delta t \end{aligned}$$

This is coded in *Skydiver.py* where the case without friction is also included. With the effect of airdrag, the velocity of the skydiver reaches a *terminal velocity* which may be obtained by setting $\frac{dv}{dt} = 0$ in Eq.(10):

$$v_{terminal} = -\sqrt{\frac{2gm}{C\rho A}} \approx -\,52 \text{ m/sec}$$

As can be seen in the figure, the sky diver approaches this limit in about 11 minutes.

```
# Skydiver.py
# simulation of freely falling skydiver using Euler's method
import numpy as np
import matplotlib.pyplot as plt
t , y2  , v2 = 0 , 1400, 0 #initial condition
C, rho, A, m = 0.5, 1, 1, 70
Cdrag = 0.5 * C * rho * A/m
```

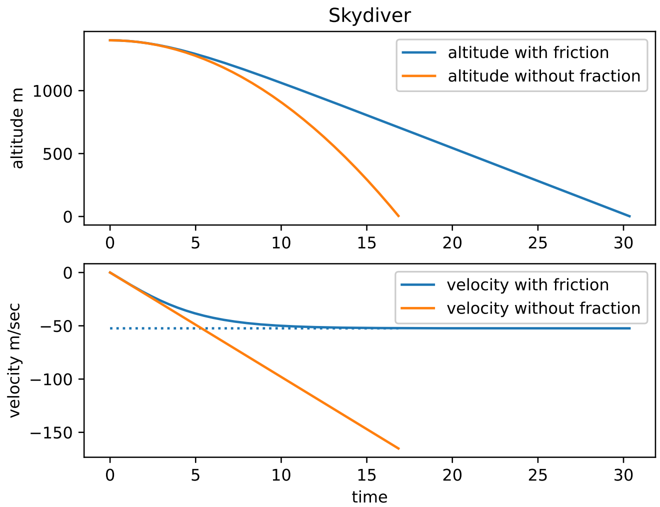

Figure 1: Skydiver

```
print(Cdrag)
dt = 0.05  # time step used
g = 9.8    # m/s^2 acceleration due to gravity
T2, Y2, V2 = [],[],[]
while y2 > 0 :
    T2.append(t)
    Y2.append(y2)
    V2.append(v2)
    v2 = v2 - g*dt + Cdrag*v2*v2*dt
    y2 = y2 + v2*dt
    t = t + dt
# neglect air friction
t , y  , v = 0 , 1400, 0 #initial condition
T1, Y1, V1 = [],[],[]
#print(V2)
while y > 0 :
    T1.append(t)
    Y1.append(y)
    V1.append(v)
```

```
    v = v - g*dt
    y = y + v*dt
    t = t + dt
tmax = T1[-1]
vlim = - np.sqrt(g/Cdrag)
plt.figure()
plt.subplot(2,1,1)
plt.title('Skydiver')
plt.ylabel('altitude m')
plt.plot(T2,Y2, label= 'altitude with friction')
plt.plot(T1,Y1, label= 'altitude without fraction')
plt.legend()
plt.subplot(2,1,2)
plt.xlabel('time')
plt.ylabel('velocity m/sec')
plt.hlines(vlim,0,tmax, linestyles='dotted')
plt.plot(T2, V2, label = 'velocity with friction')
plt.plot(T1, V1, label = 'velocity without fraction')
plt.legend()
fig = plt.gcf()
fig.savefig('Skydiver.eps', format='eps')
plt.show()
```