$ Supplement Note 19: Magnetic Field of a Solenoid

The magnetic field $B(\mathbf{r})$ produced by a steady circular current loop is given by the *Biot-Savart law*:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{r}' \times \mathbf{R}}{||\mathbf{R}||^3} \,. \tag{1}$$

where $\mathbf{R} \equiv \mathbf{r} - \mathbf{r}'$ is the vector from the current source segment $I\, d\mathbf{r}'$ to the field point $\mathbf{r}$. The value of μ_0 in SI units is equal to $\frac{\mu_0}{4\pi} = 10^{-7}$.

$1 Magnetic Field of a Circular Loop

As a first example, let us look at the magnetic field produced by a planar circular current loop of radius $a = 0.5$ on the $x\!-\!y$ plane(Figure 1). We have

$$\begin{aligned}
\mathbf{r} =& (x, y, z) \\
\mathbf{r}' =& (a\cos\theta, a\sin\theta, 0) \quad \Rightarrow \quad d\mathbf{r}' = \ a\, d\theta(-\sin\theta, \cos\theta, 0) \\
\mathbf{R} =& \mathbf{r} - \mathbf{r}' = (x - a\cos\theta, y - a\sin\theta, z) \\
d\mathbf{r}' \times \mathbf{R} =& a\, d\theta\Big(z\ \cos\theta,\ z\ \sin\theta,\ a - y\sin\theta - x\cos\theta\Big) \\
R^2 =& x^2 + y^2 + a^2 + z^2 - 2a(x\sin\theta + y\cos\theta)
\end{aligned}$$

The magnetic field $\mathbf{B} \equiv (B_x, B_y, B_z)$ may then be explicitly written from (1) as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a d\theta}{R^3}\Big(z\ \cos\theta,\ z\ \sin\theta,\ a - y\sin\theta - x\cos\theta\Big) \tag{2}$$

The results for the field cannot be expressed analytically except on the axis of the loop where $x = y = 0$. In this case, Eq.(2) yields

$$\mathbf{B}(0, 0, z) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{a d\theta}{(a^2 + z^2)^{3/2}}(\cos\theta\, z,\ \sin\theta\, z,\ a) = \left(0, 0, \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}}\right) \tag{3}$$

Listed in *CircularLoopMag.py* is an example of *Python* code implementing the numerical calculation of the Biot-Savart Law (1). The calculated results on the z-axis together with the analytic results (3) are shown on the right half of Figure

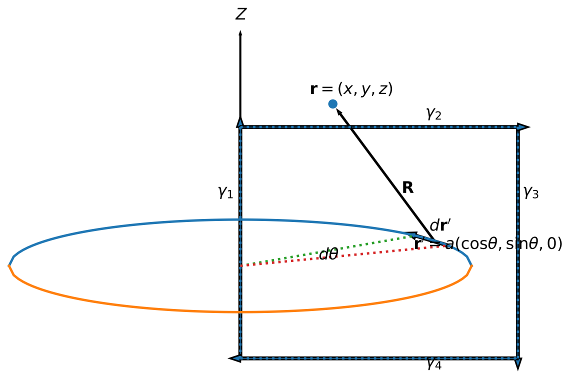

Figure 1: Geometry of the current loop problem

2. The results agree. The calculated results of the magnetic field on the $x-z$ plane are shown on the left half of Figure 2.

```
# CircularLoopMag.py: Calculate magnetic field from a circular loop
import numpy as np
from scipy.integrate import quad
import matplotlib.pyplot as plt
PI = np.pi
def dB(theta, r,a,i):
    x,y,z = r
    c,s = np.cos(theta), np.sin(theta)
    rp = np.array([ a*c, a*s,0])
    drp = np.array([-a*s, a*c, 0])
    R = r - rp
    db= np.cross(drp, R)  # cross product : dr' \times R
    R3pi4 = 1/(4*PI*np.linalg.norm(R)**3)
    DB = R3pi4*db
    return DB[i]
ZZ = np.linspace(0,1)
```

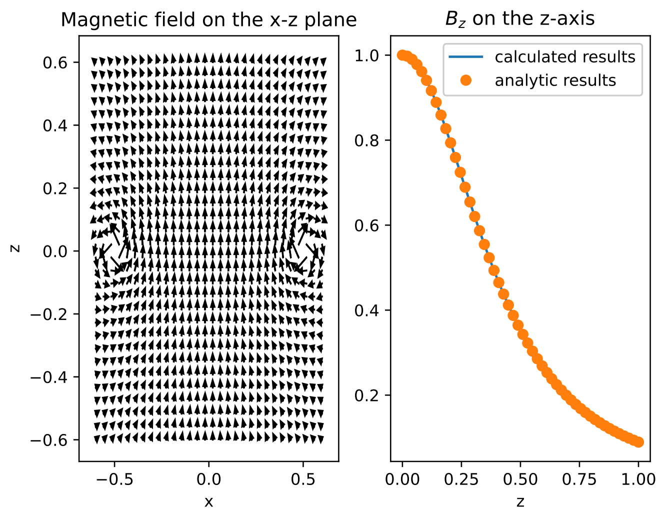

Figure 2: Magnetic field from a circular loop

```
a = 0.5
BZ,bz=[],[]
for z in ZZ:
    r= np.array([0,0,z])
    za = np.sqrt(a**2+r[2]**2)
    Bz = a**2/(2* za**3)
    BZ.append(Bz)
    I = quad(dB, 0,2*PI, args = (r,a,2))
    bz.append(I[0])
plt.figure()
plt.subplot(1,2,2)
plt.plot(ZZ,bz, label='calculated results')
plt.plot(ZZ,BZ,'o', label='analytic results')
plt.title(r'$B_z$ on the z-axis')
plt.xlabel('z')
plt.legend()
plt.subplot(1,2,1)
plt.title('Magnetic field on the x-z plane')
ndim=30
```

```
Z1 = np.linspace(-0.6,0.6,ndim)
X1 = np.linspace(-0.6,0.6,ndim)
for z in Z1:     # y = 0
    for x in X1:
        r= np.array([x,0,z])
        E = []
        for i in range(3):
            I = quad(dB, 0,2*PI, args = (r,a,i))
            E.append(I[0])
        plt.arrow(x,z,0.01*E[0],0.01*E[2],head_width=0.02,
                 head_length=0.02,fc='k', ec='k')
plt.ylabel('z')
plt.xlabel('x')
fig = plt.gcf()
fig.savefig('CircularLoopMag.eps', format='eps')
plt.show()
```

Supplement Problem 1 *Calculate the line integral along the closed loop γ*

$$\oint_{\gamma\equiv\gamma_1+\gamma_2+\gamma_3+\gamma_4} \mathbf{B}(\mathbf{r})\cdot d\mathbf{r} \tag{4}$$

where $\gamma_1 \equiv \{(0,0,z) : -1 \le z \le 2\}, \gamma_2 \equiv \{(x,0,2) : 0 \le x \le 3\}, \gamma_3 \equiv \{(3,0,z) : 2 \ge z \ge -1\}, \gamma_4 \equiv \{(x,0,-1) : 3 \ge x \ge 0\}$*(see Fig. 1). Verify the validity of the Ampère's law*

$$\oint \mathbf{B}\cdot d\mathbf{r} = \mu_0 I \tag{5}$$

You may check (5) *by using* (3) *and carrying out the line integral on the z-axis:*

$$\begin{aligned}\int_{-\infty}^{\infty} \mathbf{B}(0,0,z)\cdot d\mathbf{r} = \int_{-\infty}^{\infty} B_z(0,0,z)dz = \int_{-\infty}^{\infty} \frac{\mu_0 I a^2 dz}{2(a^2+z^2)^{3/2}}\\ =\mu_0 I a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a\sec^2\theta d\theta}{2(a^2+a^2\tan^2\theta)^{3/2}} = \mu_0 I a^2\end{aligned}$$

where a change of variable $z = \tan\theta$ was used for the integration.

Supplement Problem 2 *Calculate the magnetic field from a solenoid.* (The only modification is by replacing the loop with a helical coil!)