$ Supplement Note 6: Euler's Method

The *Euler method* is the simplest numerical approach to solve *linear differential equations* of the form

$$\frac{df(t)}{dt} = g(t)\ . \tag{1}$$

In this method, the values of the function f at increasing arguments t are obtained by repeated use of the formula

$$f(t + \Delta t) \approx f(t) + g(t)\,\Delta t\ . \tag{2}$$

(2) is identical with Eq.(3.2) in the **Lecture Note** which was obtained during the proof of **Theorem 3.1 Differentiability implies continuity** and is based directly on the definition of the *derivative*. Let us review the derivation here. The existence of the derivative

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{3}$$

may be explicitly expressed as

$$|\Delta t| \to 0\ \Rightarrow \epsilon \equiv \left|\frac{f(t + \Delta t) - f(t)}{\Delta t} - f'(t)\right| \to 0. \tag{4}$$

In short, the difference $|\epsilon|$ *becomes as small as required if* $|\Delta t|$ *is small enough*. The Euler's method (2) is obtained by multiplying (4) with Δt:

$$f(t_0 + \Delta t) = f(t_0) + f'(t_0) \cdot \Delta t + \epsilon \cdot \Delta t \tag{5}$$

Euler's approach can readily be generalized to numerically solving the linear differential equations of vectors:

$$\frac{d\mathbf{f}(t)}{dt} = \mathbf{g}(t) \tag{6}$$

The definition of derivative for vector

$$\frac{d\mathbf{f}(t)}{dt} \equiv \lim_{\Delta t \to 0} \frac{\mathbf{f}(t + \Delta t) - \mathbf{f}(t)}{\Delta t} \tag{7}$$

and its equivalence

$$|\Delta t| \to 0 \ \Rightarrow \epsilon \equiv \left|\frac{\mathbf{f}(t+\Delta t) - \mathbf{f}(t)}{\Delta t} - \frac{d\mathbf{f}(t)}{dt}\right| \to 0. \tag{8}$$

means that the relations hold for *every components*.

$$\mathbf{f}(t+\Delta t) = \mathbf{f}(t) + \frac{d\mathbf{f}(t)}{dt} \cdot \Delta t + \ldots \approx \mathbf{f}(t) + \mathbf{g}(t)\ \Delta t \tag{9}$$

Written explicitly, Euler's method for a set of simultaneous linear differential equations (6) is given by

$$f_i(t+\Delta t) = f_i(t) + g_i(t)\ \Delta t; \quad i = 1, 2, \ldots\ . \tag{10}$$

As a first application of Euler's method, let us consider the problem of a freely falling object near Earth's surface. The gravitation force on the object is given by $F = -mg$, where m is the mass of the object, and $g = 9.8\text{m/s}^2$. It follows from Newton's Law $F = ma$ that

$$\frac{dv}{dt} = a = -g \quad \Rightarrow \quad v(t+\Delta t) = v(t) - g \cdot \Delta t \tag{11}$$

$$\frac{dy}{dt} = v \quad \Rightarrow \quad y(t+\Delta t) = y(t) + v(t) \cdot \Delta t \tag{12}$$

In the code *FreeFall.py* three sets of increment $\Delta t = 0.5, 0.1, 0.05$ are used and the calculated values together with the exact results $v(t) = -gt, \quad y(t) = 100 - \frac{1}{2}gt^2$ are shown in the plot. Notice that all the curves except for $y(t)$ with $\Delta t = 0.5$ are almost identical! Euler approximation is exact for linear functions.

```
# FreeFall.py
# simulation of freely falling of object near Earth's surface
import numpy as np
import matplotlib.pyplot as plt
ddt = [0.5, 0.1, 0.05]
g = 9.8    # m/s^2 acceleration due to gravity
fg, (ax1,ax2)= plt.subplots(2,1,sharex = True)
ax1.set_xlabel('time')
```

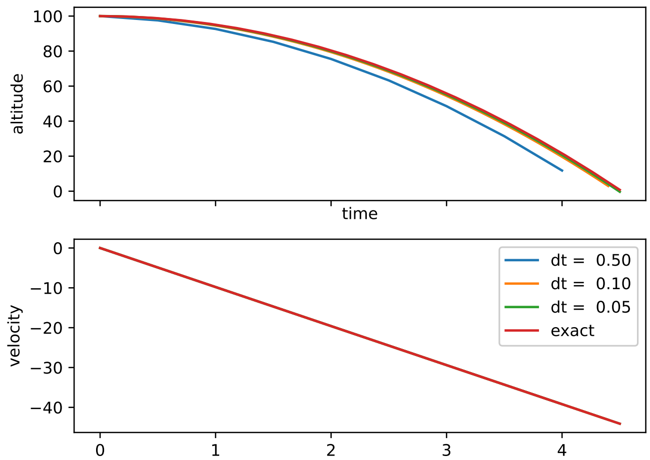

Figure 1: Free Fall

```
ax1.set_ylabel('altitude')
ax2.set_ylabel('velocity')
for dt in ddt:                  # calculation with dt = 0.5; 0.1; and 0.05
    t, y, v= 0 , 100, 10        #initial condition
    T1, y1, v1 = [],[],[]
    while y > -1.4:
        T1.append(t)
        y1.append(y)
        v1.append(v)
        v = v - g*dt
        y = y + v*dt
        t = t + dt
    ax1.plot(T1,y1, label= r'dt = {0:5.2f}'.format(dt))
    ax2.plot(T1,v1, label= r'dt = {0:5.2f}'.format(dt))
plt.legend()
fig = plt.gcf()
fig.savefig('FreeFall.eps', format='eps')
plt.show()
```