$ Supplement Note 18: Further Topics on the Periods of Simple Pendulum

Some of the Contents here have also been discussed in "CLASSICAL MECHANICS: A Modern Perspective" by Vernon Barger and Martin Olsson, McGraw-Hill. Some of the derivations may be difficult in first reading.

As discussed in Supplement Note 17, the periods T of a simple pendulum deviate from the value $T_0 \equiv 2\pi\sqrt{\frac{l}{g}}$ of a simple harmonic oscillator and increase as the magnitudes of the initial amplitudes θ_0 are increased. Numerical values of the amplitudes T versus the amplitudes θ_0 may also be obtained by modifying the code *Pendulum.py*. This is left as an exercise. Here let us seek a formula for the period T directly by using the *energy conservation law* for the simple pendulum:

$$\frac{1}{2}ml\left(\frac{d\theta}{dt}\right)^2 + mgl - mgl\cos\theta = T + V = E \tag{1}$$

This law may be derived from the equation of motion for the simple pendulum

$$\frac{d}{dt}\left(ml\,\frac{d\theta}{dt}\right) = -mg\sin\theta \tag{2}$$

by multiplying both sides of the equation with $\frac{d\theta}{dt}$:

$$ml\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} = ml\frac{d}{dt}\left(\frac{1}{2}(\frac{d\theta}{dt})^2\right) = -mg\sin\theta\frac{d\theta}{dt} = \frac{d}{dt}(mg\cos\theta)$$

The energy conservation law (1) may then be obtained by moving all the terms to the left hand side and performing an integration with respect to t:

$$\int dt\frac{d}{dt}\left\{\frac{1}{2}ml\left(\frac{d\theta}{dt}\right)^2 - mgl\cos\theta\right\} = \frac{1}{2}ml\left(\frac{d\theta}{dt}\right)^2 - mgl\cos\theta = \text{constant} = E - mgl$$

Hence the energy conservation law is also referred to as *the first integral* of Newton's equation of motion. By using the initial condition $\dot\theta = 0; \theta = \theta_0$, the conservation of energy (1) leads to

$$\frac{1}{2}ml\left(\frac{d\theta}{dt}\right)^2 + mgl - mgl\cos\theta = mgl - mgl\cos\theta_0$$

or

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0) \quad \Rightarrow \quad \frac{d\theta}{dt} = \pm\sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_0} \tag{3}$$

The sign of $\frac{d\theta}{dt}$ changes alternatively as the pendulum swings back and forth between $-\theta_0$ and θ_0. To find the period T, we look for the *inverse function* of $\theta(t)$, i.e. the relation $t(\theta)$ of time t as a function of θ:

$$t(\theta) \equiv \int_{-\theta_0}^{\theta} d\theta' \frac{dt(\theta')}{d\theta'} \tag{4}$$

The derivative $\frac{dt}{d\theta}$ may be obtained from (3) (the derivative of its inverse function). Explicitly, this is given by (The derivatives are positive for $-\theta_0 \leq \theta \leq \theta_0$):

$$\frac{dt}{d\theta} = \frac{1}{\frac{dt}{d\theta}} = \sqrt{\frac{l}{2g}}\frac{1}{\sqrt{\cos\theta - \cos\theta_0}} = \frac{T_0}{2\pi}\frac{1}{\sqrt{2}\sqrt{\cos\theta - \cos\theta_0}} \tag{5}$$

which leads to

$$t(\theta) = \frac{T_0}{2\pi}\int_{-\theta_0}^{\theta}\frac{d\theta'}{\sqrt{2}\sqrt{\cos\theta' - \cos\theta_0}} = \frac{T_0}{4\pi}\int_{-\theta_0}^{\theta}\frac{d\theta'}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta'}{2}}} \tag{6}$$

where the integrand was modified by using the identity

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2} \quad \Rightarrow \quad \cos\theta = 1 - 2\sin^2\frac{\theta}{2} \tag{7}$$

Hence the period T is given by (Note that the integrand is an even function of θ.)

$$T = 2t(\theta_0) = \frac{T_0}{2\pi}\int_{-\theta_0}^{\theta_0}\frac{d\theta'}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta'}{2}}} = \frac{T_0}{\pi}\int_0^{\theta_0}\frac{d\theta}{\sin\frac{\theta_0}{2}\sqrt{1 - \frac{\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta_0}{2}}}} \tag{8}$$

By introducing a new variable z by

$$\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\ \sin z \Rightarrow \frac{1}{2}\cos\frac{\theta}{2}\,d\theta = \sin\frac{\theta_0}{2}\ \cos z\,dz; \quad \theta = \theta_0 \Rightarrow z = \frac{\pi}{2}$$

Eq.(8) becomes (Note: $\cos\frac{\theta}{2} = \sqrt{1 - \sin^2\frac{\theta}{2}} = \sqrt{1 - \sin^2\frac{\theta_0}{2}\sin^2 z}$)

$$\frac{T}{T_0} = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{\cos z\,dz}{\frac{1}{2}\cos\frac{\theta}{2}\sqrt{1 - \sin^2 z}} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{dz}{\sqrt{1 - \sin^2\frac{\theta_0}{2}\sin^2 z}} \tag{9}$$

θ_0 (in degree)	θ_0	$\frac{T-T_0}{T_0}$	from(11)	from(10)
0°	0.0000	0.0000000	0.0000000	0.0000000
5°	0.0873	0.0004762	0.0004760	0.0004757
10°	0.1745	0.0019072	0.0019039	0.0018990
15°	0.2618	0.0043006	0.0042837	0.0042593
20°	0.3491	0.0076690	0.0076154	0.0075384
25°	0.4363	0.0120306	0.0118991	0.0117115
30°	0.5236	0.0174088	0.0171347	0.0167468

Table 1: Deviation of the periods of simple pendulums from simple harmonic oscillators

The integral (9) is known as the *complete elliptic integral of the first kind*. It can not be evaluated in closed form. For small values of angular displacement, $\theta_0 \ll 1$, the integrand in (9) may be expanded in a power series of the small quantity $\sin^2 \frac{\theta_0}{2}$ and integrated term by term:

$$\begin{aligned}
\frac{T}{T_0} =& \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{dz}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2 z}} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} dz\left\{1+\frac{1}{2}\sin^2\frac{\theta_0}{2}\sin^2 z+\cdots\right\} \\
=& \frac{2}{\pi}\int_0^{\frac{\pi}{2}} dz\left\{1+\frac{1}{2}\sin^2\frac{\theta_0}{2}\frac{1-\cos 2z}{2}+\cdots\right\} \\
=& \frac{2}{\pi}\left[1+\frac{1}{4}\sin^2\frac{\theta_0}{2}\left(z-\frac{\sin 2z}{2}\right)+\cdots\right]_0^{\frac{\pi}{2}} \\
=& 1+\frac{1}{4}\sin^2\frac{\theta_0}{2}+\cdots \qquad (10) \\
=& 1+\frac{\theta_0^2}{16}+\cdots \qquad (11)
\end{aligned}$$

Listed in *TT0.py* are codes for numerical evaluation of the complete elliptic integral of the first kind (9) using the sub-package *scipy.integrate.quad*. Calculated results for the deviation $\frac{T-T_0}{T_0}$ together with the approximation from (11) and (10) are given in Table 1. The fact that the results from (11) which are further approximations for the sine function in (10) yield better results. This is due to the fact that the inequality $0 < \sin\theta_0 < \theta_0$ compensates somewhat the further corrections from the next neglected terms in the expansion which are positive.

```
# TT0.py
# numerical integration of complete elliptic integral
import numpy as np
from scipy.integrate import quad
import matplotlib.pyplot as plt
Pi = np.pi
pi6, pi2= Pi/6, Pi/2
def ft(z,theta0):
    st,sz = np.sin(theta0/2), np.sin(z)
    f = np.sqrt(1- (st*sz)**2)
    return 1/f
thetad = np.linspace(0,30,7)
print("degree  radians    exact       approx      approx1")
for d in thetad:         # in degree
    theta0 = d *Pi/180   # d in gradient
    sint = np.sin(theta0/2)
    I = quad(ft, 0, pi2, args = (theta0,))
    tt0 = I[0]/pi2
    tdif, tt, tts  = tt0 - 1,theta0**2/16 , sint**2/4
    print(r"{:4.1f}    {:6.4f}    {:9.7f}    {:9.7f}" \
          r"   {:9.7f}’.format(d, theta0, tdif, tt, tts))
```