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ORIGINAL ARTICLE

Strategic asset allocation for insurers under Solvency II

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Abstract An important question in asset management is how solvency requirements impact the investment strategies of institutional investors. In this paper, we derive the optimal asset allocation of an insurer that minimizes its capital requirement for market risk determined with the Solvency II standard formula, subject to a target return on own funds. Solvency II is the risk-based framework for setting capital requirements of European insurance companies, in force since 2016. The solvency capital requirement is set such that the insurer's own funds can absorb losses over a 1-year horizon with a probability of at least 99.5%. Within this framework, we analyze the properties of an optimal asset allocation for a representative European life insurance company.

Keywords Investment · Insurance · Asset-liability management · Solvency II · Capital requirements

Introduction

One of the lasting consequences of the global financial crisis is that banks, insurance companies and pension funds are nowadays subject to stricter capital requirements, after the introduction of new regulatory frameworks. An important question for both academics and practitioners in asset management is how these capital requirements influence the investment strategies of financial institutions. In this paper, we derive and analyze the optimal strategic asset allocation for insurance firms that are subject to a capital requirement set by the Solvency II standard formula for market risk. Solvency II is the new regulatory framework for insurance firms in the European Economic Association, in force since 2016. The *solvency capital requirement* (SCR) of Solvency II is based on the 1-year 99.5% Value-at-Risk, such that an insurance company can meet its obligations over the next 12 months with a probability of at least 99.5%.

The Solvency II standard formula determines the SCR by applying a set of extreme shocks (losses) to the main balance sheet components of the insurer, both the assets and the liabilities. The resulting losses are then aggregated with a prescribed correlation matrix, to account for diversification benefits. Most large insurance companies have spent years to develop their own sophisticated internal models and risk measurement systems to determine the SCR, which can replace the standard formula, subject to approval by the regulator. But many smaller insurance firms, for whom developing an internal model is too costly, rely on the standard formula to calculate their SCR. It is therefore important to understand how this standard formula may influence their asset allocation and investment strategies.

In this paper, we derive the optimal asset allocation followed by an insurer that maximizes its expected return on own funds, subject to a capital constraint determined by the Solvency II standard formula for market risk. Equivalently, this model also shows how an insurer would invest when just aiming to minimize its required solvency capital, subject to a target expected return on its own funds (e.g., of 10%). Assessing the soundness of these investment strategies is relevant, as some insurance companies in

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practice may try to minimize their required capital and therefore focus on the 1-year 99.5% VaR given by the Solvency II standard formula as a yardstick for measuring risk.

The main contribution of this paper is that we analytically solve the investment problem where risk is determined by the Solvency II standard formula, and we show that the corresponding optimal investment strategies are sound in principle. For example, hedging the interest rate risk of the liabilities is a key component of the optimal strategy. In addition, the insurer invests in an efficient "asset-only" portfolio and in cash, a classical three fund separation result. One caveat is that the standard formula makes little distinction between individual assets *within* its market risk categories (interest, equity, property and credit risk), and therefore is only appropriate when applied to well-diversified portfolios. We also provide examples of the optimal asset allocation for a typical European life insurance company. The results show a large improvement in the return on solvency capital and better liability hedging, but the limitations of the standard formula as a simplified risk model also lead to relatively underdiversified portfolios with large concentrations of government bonds.

The next section reviews the related literature, and then the "Solvency II standard formula for market risk" section briefly introduces the Solvency II standard formulas for market risk. The section "Optimal asset allocation" derives the optimal strategic asset allocation analytically. The "Asset Allocation Examples" section provides numerical examples for a representative European life insurance company, and the "Conclusions" section concludes the paper.

Related literature on Solvency II and asset allocation

Solvency II, the regulatory framework for insurance companies in the European Economic Area (EEA), came into full force in 31 European countries as of January 1, 2016.[1] The Solvency II rules were under development since 2000, involving a long consultation process with national regulators and the European insurance industry. The Solvency II framework prescribes risk-based capital requirements, in combination with market-based valuation of the assets and the insurance liabilities. The aim of the solvency capital requirement is that the insurance company has sufficient funds to cover losses over a 1-year horizon with 99.5% probability. Both academicians and consultants have investigated the potential asset allocation implications of the regulatory framework, which we now briefly review.[2]

Amenc et al. (2006) were among the first to analyze the potential impact of the Solvency II framework on the asset-liability management of insurance companies. Amenc et al. (2006) recommend a core-satellite approach, with a dedicated core portfolio to fully hedge the interest rate sensitivity of the liabilities and any embedded options in the insurance liabilities.[3] Once the market value of the liabilities is fully hedged with the core asset portfolio, Amenc et al. argue that the remaining surplus assets can be managed with traditional asset allocation techniques (see also van Bragt and Kort 2011). Van Bragt et al. (2010) also find that a typical European life insurer can greatly reduce its solvency capital requirement by matching the interest rate sensitivity of the assets and the liabilities, showing that the new rules stimulate better asset-liability management (ALM). Solvency capital can also be reduced by limiting exposure to risky assets such as stocks and real estate, but this comes at the expense of a low expected return that can deteriorate the long-term viability of the insurer over a 10-year horizon. Van Bragt et al. (2010) conclude that insurers should set asset allocation policies that consider both the 1-year Solvency II capital requirement and the long-run risk-return profile of the asset allocation.

Several studies on the impact of the Solvency II rules on asset allocation focus on the question whether insurance companies will reduce exposure to risky assets such as stocks, corporate bonds and hedge funds (Rudschuck et al. 2010; Mittnik 2011; Fischer and Schlütter 2015; Braun et al. 2017, 2018). Investments in developed market equities are subject to a 39% capital charge under the latest Solvency II provisions, while emerging equity, private equity and hedge funds face a 49% charge. Consequently, when the new capital requirement rules are considered in isolation, they may lead to a large reduction in risky asset exposures (see, e.g., Rudschuck et al. 2010). Further, Braun et al. (2017, 2018) show that the Solvency II standard formula for market risk may inadvertently lead to poorly diversified investment portfolios. On the other hand, Höring (2013) points out that insurance companies are also subject to tests by ratings agencies such as S&P and Fitch,

[1] The EEA consists of the 28 European Union (EU) members, plus Norway, Lichtenstein and Iceland.

[2] Another line of studies criticizes the VaR methodology of Solvency II, raising serious questions about whether the specifications of the standard model will truly lead to a ruin probability below 0.5%: see Mittnik (2011) and Braun et al. (2015), for example. As Solvency II has come into force in 2016, in this paper we take the framework as given and focus on analyzing optimal asset allocation strategies.

[3] Amenc et al. (2006) point out that a potential obstacle to liability hedging in practice are the IFRS accounting standards, which require insurers to recognize profits and losses on derivatives used for liability hedging annually, while changes in the market value of the liabilities are not immediately recognized, leading to volatile net income.

which are typically even stricter than the Solvency II requirements when companies want to maintain a credit rating of A or higher.

Overall, the literature so far has considered the potential impact of Solvency II on the asset allocation of insurers using numerical techniques such as simulation and one-off examples. The contribution of this paper is that we will *analytically* derive the optimal strategic asset allocation subject to a capital requirement determined with the Solvency II standard formula. To provide insights about the properties of the optimal investment strategies, in this paper we focus on analytical solutions for a 1-period portfolio choice model. We refer to Duarte et al. (2017) and Escobar et al. (2018) for multi-period models with Solvency II constraints, solved using numerical and iterative techniques.

Solvency II standard formula for market risk

In this section, we summarize the Solvency II standard formula for determining the solvency capital requirement for market risk. The standard formula aims to approximate the 0.5% worst-case loss to the insurer's own funds (assets minus liabilities) over a horizon of one year, namely the 1-year 99.5% VaR. We first briefly define the notation for the insurer's balance sheet. Let A denote the market value of the assets of the insurer, separated in I different asset classes: $A = \sum_{i=1}^{I} A_i$. We distinguish the following asset classes:

$A_1 = A_{gov,1} =$ sovereign debt issued by EEA countries
$A_2 = A_{gov,2} =$ sovereign debt issued by other countries
$A_3 = A_{corp} =$ corporate debt
$A_4 = A_{eq} =$ equity
$A_5 = A_{prop} =$ property
$A_6 = A_{other} =$ other non-market assets
$A_7 = A_{cash} =$ cash

Equity includes listed stocks in developed and emerging markets, as well as private equity and hedge funds. Examples of other non-market assets are mortgages and reinsurance assets. The asset classes above are aligned with the way the Solvency II standard formula calculates risk capital: investments in debt, equity and property are charged in the market risk module, while cash and other assets are charged separately in the counter-party risk module.

Similarly, let $L = \sum_{n=1}^{N} L_N$ denote the market value of the liabilities of the insurer, consisting of $N = 2$ different sub-categories:

$L_1 = L_{tprov} =$ Technical provisions
$L_2 = L_{other} =$ Other liabilities

Finally, the amount of own funds of the insurer, denoted by F, is the difference between the assets and the liabilities: $F = A - L$.

Market risk sub-modules in the standard formula

The charge for market risk is first determined separately in six sub-modules, consisting of: I. interest rate risk, II. equity risk, III. property risk, IV. credit spread risk, V. currency risk, and VI. concentration risk. Let $SCR_{Mkt,k}$ denote the solvency capital requirement for market risk type k, for $k = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VI}$. For example, the capital requirement for property risk is set by applying a shock of $\Delta_{prop} = 25\%$ to the total value of the property investments (A_{prop})[4]:

$$SCR_{Mkt,III} = \Delta_{prop} A_{prop} \tag{1}$$

For equity risk, the standard formula distinguishes two different types of investments that are charges separately: (1) equity listed in developed markets, and (2) other equity, which includes emerging market stocks, private equity and hedge funds. A shock of $\Delta_{eq,1} = 39\%$ is applied to the value of developed equity market investments ($A_{eq,1}$), and a larger shock of $\Delta_{eq,2} = 49\%$ is applied to the value of all other equity investments ($A_{eq,2}$): $SCR_{eq,1} = \Delta_{eq,1} A_{eq,1}$, and $SCR_{eq,2} = \Delta_{eq,2} A_{eq,2}$.[5] The overall capital requirement for equity risk is set by aggregating the charges for developed equity ($SCR_{eq,1}$) and the other equity ($SCR_{eq,2}$) with the square root formula below, using a correlation parameter of $\rho_{eq} = 0.75$:

$$SCR_{Mkt,II} = \sqrt{\left(SCR_{eq,1}\right)^2 + \left(SCR_{eq,2}\right)^2 + 2\rho_{eq} SCR_{eq,1} SCR_{eq,2}} \tag{2}$$

Hence, some diversification benefits between developed and other equity are recognized.

In the spread risk module, each fixed income asset with credit risk is separately shocked with a percentage Δ_l that depends on the asset's duration and credit rating. The individual amounts are aggregated trough simple summation, without considering diversification benefits. Assets charged

[4] The property investments include listed real estate (e.g., REITS), direct property investments (unlisted), and the value of office buildings owned by the insurance company for its own use.

[5] The parameter values are adjusted slightly each month with a mechanism called the "symmetric adjustment," which reduces capital charges for equity in bear markets, and increases it in bull markets, to alleviate the concern that the capital requirements put pressure on insurers to sell their equity investments directly after a market crash.

in the credit spread risk sub-module include corporate bonds, loans, asset backed securities, credit derivatives and non-EEA government bonds. Government bonds from EEA countries are exempt from capital charges in the spread risk module (i.e., the regulator treats their spread risk as zero). Let $\Delta_{\text{gov},2}$ denote the weighted average shock for the portfolio of non-EEA sovereign bonds and let Δ_{corp} denote the weighted average shock for the portfolio of all corporate debt investments, including securitizations and credit derivatives. Then, the overall charge for credit spread risk is:

$$\text{SCR}_{\text{Mkt,IV}} = \Delta_{\text{gov},2} A_{\text{gov},2} + \Delta_{\text{corp}} A_{\text{corp}} \tag{3}$$

In sub-module for interest rate risk, the capital requirement is determined as the maximum loss of own funds resulting from a prescribed upward shock to the term structure of risk-free interest rates, and a given downward shock. The impact of the term-structure shocks is calculated separately for each asset and each liability, and then combined to see the impact on the insurer's own funds. For ease of exposition, we will assume that interest rate risk can be summarized with a simple duration-based calculation, following Höring (2013).[6] Let $D_{\text{A},i}$ denote the duration of asset i, and $D_{\text{L},n}$ the duration of liability n. Further, the parameter Δ_{rd} is the parallel downward shock to the interest rates, and Δ_{ru} is the upward shock. The capital requirement for interest rate risk then is:

$$\begin{aligned}\text{SCR}_{\text{Mkt,I}} = \max\Big\{&\Delta_{\text{rd}}\Big(\big(D_{\text{L},1}L_1 + D_{\text{L},2}L_2\big) - \big(D_{\text{A},1}A_1 + D_{\text{A},2}A_2 + D_{\text{A},3}A_3\big)\Big),\\ &\Delta_{\text{ru}}\Big(\big(D_{\text{A},1}A_1 + D_{\text{A},2}A_2 + D_{\text{A},3}A_3\big) - \big(D_{\text{L},1}L_1 + D_{\text{L},2}L_2\big)\Big)\Big\}\end{aligned} \tag{4}$$

Finally, the capital requirement for currency risk is set by applying a shock of $\Delta_{\text{cur}} = 25\%$ to the value of all investments denoted in a foreign currency. Let f_i denote the fraction of investment A_i invested in foreign currency, and then the capital requirement is:

$$\text{SCR}_{\text{Mkt,V}} = \Delta_{\text{cur}} \sum_{i=1}^{I} f_i A_i \tag{5}$$

The sixth market risk sub-module determines a capital charge for concentration risk, when the insurer's combined investments in a single company or bond issuer exceeds 1.5% to 15% of the insurer's total asset value, depending on the credit rating of the issuer. In the remainder of this paper, we assume that the insurer invests in well-diversified portfolios of stocks and bonds, such that the charge for concentration risk is zero.

[6] This simplification can be made without loss of generality. The crucial assumption for the overall framework is that the value of each SCR component increases 1-on-1 with the amount of total assets (or liabilities).

Aggregation of market risk capital

The capital charges for the six market risk types are aggregated into a total capital requirement for market risk, $\text{SCR}_{\text{Market}}$, with the square root formula below:

$$\begin{aligned}\text{SCR}_{\text{Market}} &= \sqrt{\sum_{k=1}^{K}(\text{SCR}_{\text{Mkt},k})^2 + \sum_{k=1}^{K}\sum_{j=1;j\neq k}^{K} \rho_{kj}\text{SCR}_{\text{Mkt},k}\text{SCR}_{\text{Mkt},j}}\\ &= \sqrt{\sum_{k=1}^{K}\sum_{j=1}^{K} \rho_{kj}\text{SCR}_{\text{Mkt},k}\text{SCR}_{\text{Mkt},j}} = (\mathbf{s}'\mathbf{R}\mathbf{s})^{\frac{1}{2}}\end{aligned} \tag{6}$$

where $\rho_{kj} = \rho_{\text{Mkt},kj}$ is the correlation between market risk types k and j, prescribed by the regulator. Above we also use vector–matrix notation: $\mathbf{s}$ is a $K \times 1$ vector holding the SCR's for the market risk types, and $\mathbf{R}$ is a $K \times K$ matrix containing the correlation coefficients ρ_{kj}.

Table 1 shows the correlation coefficients prescribed by the Solvency II standard formula. The correlations for interest rate risk depend on whether the downward shock scenario for interest rate risk gives the largest loss of own funds (Panel A of Table 1), or the upward shock (Panel B of Table 1). It is important to recognize that the regulator calibrated the correlation matrix to reflect a 0.5% worst-case scenario, and therefore the values are relatively high compared to correlation estimates based on historical data.

Apart from market risk, insurance companies also need to hold capital for four types of non-market risks: non-life insurance underwriting risk, life insurance underwriting risk, health insurance underwriting risk, and counter-party default risk. The total capital requirement for the insurer is determined by aggregating the capital requirements for the five different risk types, including market risk, with a square root formula. In this paper on strategic asset allocation, we assume that the underwriting risks of the insurance company are fixed in the short run, and we therefore limit our attention to solvency capital for market risk only.

Optimal asset allocation

We now derive and analyze the optimal asset allocation for an insurer that maximizes the expected return on its own funds, subject to an upper limit on the solvency capital for market risk. The purpose is to understand how the solvency requirement influences the investment strategies of insurers who rely on the standard formula as a risk model, rather than using their own internal model (e.g., mean–variance). The framework is equivalent to a model where the insurer minimizes its required solvency capital, which is an estimate of the 1-year 99.5% VaR, subject to a target expected

Table 1 Correlations for aggregation of market risks

	Interest	Equity	Property	Spread	Currency	Concent.
Panel A: Decrease in the term-structure shock determines the interest rate risk						
Interest rate risk	1	0.5	0.5	0.5	0.25	0
Equity risk	0.5	1	0.75	0.75	0.25	0
Property risk	0.5	0.75	1	0.5	0.25	0
Spread risk	0.5	0.75	0.5	1	0.25	0
Currency risk	0.25	0.25	0.25	0.25	1	0
Concentration risk	0	0	0	0	0	1
Panel B: Increase in the term-structure shock determines the interest rate risk						
Interest rate risk	1	0	0	0	0.25	0
Equity risk	0	1	0.75	0.75	0.25	0
Property risk	0	0.75	1	0.5	0.25	0
Spread risk	0	0.75	0.5	1	0.25	0
Currency risk	0.25	0.25	0.25	0.25	1	0
Concentration risk	0	0	0	0	0	1

The table shows the correlations used to aggregate the capital requirements for the six types of market risks in the Solvency II standard formula, in Eq. (6). The correlations in Panel A apply when the downward shock to the term-structure of interest rates determines the capital requirement (gives the biggest loss in own funds). Panel B applies otherwise, when the upward shock to the term-structure leads to the biggest loss of own funds

return on own funds. Hence, the model also shows what kind of investment strategy an insurer would follow if it just wants to minimize its capital requirement, thereby maximizing its solvency ratio, while meeting an expected return target. The framework is not normative (prescribing the "best" way to invest), but rather behavioral, showing how insurers would invest if they are pre-occupied with managing their Solvency II capital requirement given by the standard formula.

In the analysis, we assume that the insurer can take a short position in EEA Treasury bills, with duration close to zero. In the Solvency II standard formula, these short-term government bonds are effectively riskless, receiving a negligible charge for interest rate risk and no charge for credit risk. In practice, insurers can hedge the interest rate risk of the liabilities by entering swap contracts where the insurance company pays a floating rate and receives a fixed rate. Such a swap position is similar to shorting Treasury bills and investing the proceeds in long-dated government bonds. Thus, below we assume that the insurer invests an amount A_0 in a riskless asset with expected return r_f, such that the budget constraint is: $A = A_0 + \sum_{i=1}^{I} A_i$. Further, the riskless asset can also be shorted ($A_0 < 0$).

The objective and constraints

Let $\mu_{A,i}$ denote the expected return on asset i and $\mu_{L,n}$ the expected growth rate of liability n, both measured over a period of 1 year, the horizon of Solvency II. Then the expected increase in own funds ΔF is: $E[\Delta F] = E[\Delta A - \Delta L] = r_f A_0 + \sum_{i=1}^{I} \mu_{A,i} A_i - \sum_{n=1}^{N} \mu_{L,n} L_n$. The insurer's objective is to maximize the expected increase in own funds ΔF subject to an upper limit of $\mathrm{SCR}_{\mathrm{Market}}^{\mathrm{Max}}$ on the capital charge for market risk $\mathrm{SCR}_{\mathrm{Market}}(\boldsymbol{a}; L_1, L_2)$:

$$\max_{\boldsymbol{a}} E[\Delta F(\boldsymbol{a})] = r_f A + \sum_{i=1}^{I} \left(\mu_{A,i} - r_f\right) A_i - \sum_{n=1}^{N} \mu_{L,n} L_n \\ \text{s.t.} \quad \mathrm{SCR}_{\mathrm{Market}}(\boldsymbol{a}; L_1, L_2) \le \mathrm{SCR}_{\mathrm{Market}}^{\mathrm{Max}} \tag{7}$$

where $\boldsymbol{a} = (A_1, A_2, \ldots, A_I)'$ is a column vector of length I containing the risky asset amounts.

We note that the optimization problem (7) effectively maximizes the *return on risk-adjusted capital* (RoRAC), when solvency capital for market risk is at the upper limit. RoRAC is defined as the expected increase in own funds divided by the required capital for market risk.

To solve investment problem (7) analytically, we make two simplifying assumptions. First, we assume that the insurance liabilities have longer duration than the assets, such that the downward shock to the term-structure determines the capital requirement for interest rate risk:

$$\mathrm{SCR}_{\mathrm{Mkt,I}} = \Delta_{\mathrm{rd}} \left(\left(D_{\mathrm{L},1} L_1 + D_{\mathrm{L},2} L_2 \right) - \left(D_{\mathrm{A},1} A_1 + D_{\mathrm{A},2} A_2 + D_{\mathrm{A},3} A_3 \right) \right) \tag{8}$$

Second, we assume that the weights of developed and other equity within the insurer's equity portfolio are fixed at $w_{\mathrm{eq},1}$ and $1 - w_{\mathrm{eq},1}$. We then treat equity as a single asset class with (variable) invested amount A_{eq}, and solvency shock Δ_{eq}.[7]

Under these two assumptions, the capital requirements for the market risk types, $\mathbf{s} = (\mathrm{SCR}_{\mathrm{Mkt,I}}, \mathrm{SCR}_{\mathrm{Mkt,II}}, \ldots, \mathrm{SCR}_{\mathrm{Mkt},K})'$, are a linear function of the asset amounts in the vector $\boldsymbol{a}$:

$$\mathbf{s} = \mathbf{V}\boldsymbol{a} + \boldsymbol{c_L} \tag{9}$$

where $\mathbf{s}$ is a $K \times 1$ vector holding the SCR's for the market risk types, and $\mathbf{V}$ is a $K \times I$ matrix containing the asset shock parameters, and $\boldsymbol{c_L}$ is a $K \times 1$ vector with the liability shock amounts defined as: $\boldsymbol{c_L} = (\Delta_{rd}\left(D_{L,1}L_1 + D_{L,2}L_2\right), 0, \ldots, 0)'$. Let $\mathbf{v}(k)$ denote row k of the $K \times I$ coefficient matrix $\mathbf{V}$, then its elements are:

$$\begin{aligned}
\mathbf{v}(1) &= \left(-D_{\mathrm{A},1}\Delta_{\mathrm{rd}}, -D_{\mathrm{A},2}\Delta_{\mathrm{rd}}, -D_{\mathrm{A},3}\Delta_{\mathrm{rd}}, 0, 0\right)\\
\mathbf{v}(2) &= \left(0, 0, 0, \Delta_{\mathrm{eq}}, 0\right)\\
\mathbf{v}(3) &= \left(0, 0, 0, 0, \Delta_{\mathrm{prop}}\right)\\
\mathbf{v}(4) &= \left(0, \Delta_{\mathrm{gov},2}, \Delta_{\mathrm{corp}}, 0, 0\right)\\
\mathbf{v}(5) &= (\Delta_{\mathrm{cur}}f_1, \Delta_{\mathrm{cur}}f_2, \Delta_{\mathrm{cur}}f_3, \Delta_{\mathrm{cur}}f_4, \Delta_{\mathrm{cur}}f_5).
\end{aligned} \tag{10}$$

The optimal asset allocation

We now solve for the optimal allocation, first writing the asset allocation problem as follows:

$$\begin{aligned}
&\max_{\boldsymbol{a}} E[\Delta F(\boldsymbol{a})] = r_f A + \boldsymbol{\mu}_{\mathbf{A}}'\boldsymbol{a} - \mu_{\mathrm{L},1}L_1 - \mu_{\mathrm{L},2}L_2\\
&\text{s.t.}\quad \left((\mathbf{V}\boldsymbol{a} + \boldsymbol{c}_{\mathbf{L}})'\mathbf{R}(\mathbf{V}\boldsymbol{a} + \boldsymbol{c}_{\mathbf{L}})\right)^{1/2} \le \mathrm{SCR}^{\mathrm{Max}}_{\mathrm{Market}}
\end{aligned} \tag{11}$$

where $\boldsymbol{\mu}_{\mathbf{A}} = \left(\mu_{\mathrm{A},1} - r_f, \mu_{\mathrm{A},2} - r_f, \ldots, \mu_{\mathrm{A,I}} - r_f\right)'$ is a column vector of length I with the expected *excess* asset returns, relative to the risk-free rate.

The first-order condition for optimality is:

$$\boldsymbol{\mu}_{\mathbf{A}} - \lambda\frac{\mathbf{V}'\mathbf{R}(\mathbf{V}\boldsymbol{a} + \boldsymbol{c}_{\mathbf{L}})}{\mathrm{SCR}_{\mathrm{Market}}} = 0, \quad \text{for } \lambda \ge 0 \tag{12}$$

We solve for the optimal asset allocation $\boldsymbol{a}^*$, assuming that the matrix $\mathbf{V}'\mathbf{RV}$ is invertible:

$$\begin{aligned}
\boldsymbol{a}^* &= \frac{\mathrm{SCR}^{\mathrm{Max}}_{\mathrm{Market}}}{\lambda}(\mathbf{V}'\mathbf{RV})^{-1}\boldsymbol{\mu}_{\mathbf{A}} - (\mathbf{V}'\mathbf{RV})^{-1}\mathbf{V}'\mathbf{R}\boldsymbol{c}_{\mathbf{L}}\\
&= \frac{\mathrm{SCR}^{\mathrm{Max}}_{\mathrm{Market}}}{\lambda}(\mathbf{V}'\mathbf{RV})^{-1}\boldsymbol{\mu}_{\mathbf{A}} - \mathbf{V}^{-1}\boldsymbol{c}_{\mathbf{L}}
\end{aligned} \tag{13}$$

using

$$(\mathbf{V}'\mathbf{RV})^{-1}\mathbf{V}'\mathbf{R} = \left(\mathbf{V}^{-1}(\mathbf{V}'\mathbf{R})^{-1}\right)\mathbf{V}'\mathbf{R} = \mathbf{V}^{-1}.$$

We then solve for lambda, assuming that the solvency constraint is binding:

$$\begin{aligned}
\mathrm{SCR}^{\mathrm{Max}\ 2}_{\mathrm{Market}} &= (\mathbf{a}^{*\prime}\mathbf{V}' + \boldsymbol{c}_{\mathbf{L}}')\mathbf{R}(\mathbf{V}\mathbf{a}^* + \boldsymbol{c}_{\mathbf{L}})\\
&= \left(\frac{\mathrm{SCR}^{\mathrm{Max}}_{\mathrm{Market}}}{\lambda}\boldsymbol{\mu}_{\mathbf{A}}'(\mathbf{V}'\mathbf{RV})^{-1}\mathbf{V}' - \boldsymbol{c}_{\mathbf{L}}'(\mathbf{V}^{-1})'\mathbf{V}' + \boldsymbol{c}_{\mathbf{L}}'\right)\\
&\quad\times \mathbf{R}\left(\frac{\mathrm{SCR}^{\mathrm{Max}}_{\mathrm{Market}}}{\lambda}\mathbf{V}(\mathbf{V}'\mathbf{RV})^{-1}\boldsymbol{\mu}_{\mathbf{A}} - \boldsymbol{c}_{\mathbf{L}} + \boldsymbol{c}_{\mathbf{L}}\right)\\
&= \frac{\mathrm{SCR}^{\mathrm{Max}\ 2}_{\mathrm{Market}}}{\lambda^2}\left(\boldsymbol{\mu}_{\mathbf{A}}'(\mathbf{V}'\mathbf{RV})^{-1}\mathbf{V}'\mathbf{R}\right)\left(\mathbf{V}(\mathbf{V}'\mathbf{RV})^{-1}\boldsymbol{\mu}_{\mathbf{A}}\right)\\
&= \frac{\mathrm{SCR}^{\mathrm{Max}\ 2}_{\mathrm{Market}}}{\lambda^2}\boldsymbol{\mu}_{\mathbf{A}}'(\mathbf{V}'\mathbf{RV})^{-1}\boldsymbol{\mu}_{\mathbf{A}}
\end{aligned}$$

Thus, $\lambda = \sqrt{\boldsymbol{\mu}_{\mathbf{A}}'\left(\mathbf{V}_{\mathbf{A}}'\mathbf{R}\mathbf{V}_{\mathbf{A}}\right)^{-1}\boldsymbol{\mu}_{\mathbf{A}}}$. It is easy to prove that this expression is equal to the return on solvency capital of an optimal allocation for the "assets-only" case, $\mathrm{RoRAC}^*_{\mathrm{NoLiab}} = \sqrt{\boldsymbol{\mu}_{\mathbf{A}}'\left(\mathbf{V}_{\mathbf{A}}'\mathbf{R}\mathbf{V}_{\mathbf{A}}\right)^{-1}\boldsymbol{\mu}_{\mathbf{A}}}$, when the insurance liabilities are equal to zero.

We can now write the optimal asset allocation as the sum of two portfolios:

$$\begin{aligned}
\boldsymbol{a}^* &= \left(\frac{\mathrm{SCR}^{\mathrm{Max}}_{\mathrm{Market}}}{\mathrm{RoRAC}^*_{\mathrm{NoLiab}}}\right)(\mathbf{V}'\mathbf{RV})^{-1}\boldsymbol{\mu}_{\mathbf{A}} - \mathbf{V}^{-1}\boldsymbol{c}_{\mathbf{L}}\\
&= \boldsymbol{a}^*_{\mathbf{NoLiab}} + \boldsymbol{a}^*_{\mathbf{Hedge}}
\end{aligned} \tag{14}$$

The first component, $\boldsymbol{a}^*_{\mathbf{NoLiab}}$, is the optimal portfolio for the "asset-only" investment problem without liabilities ($L_1 = 0$, $L_2 = 0$) and with market risk limit $\mathrm{SCR}^{\mathrm{Max}}_{\mathrm{Market}}$. The relative weights of the risky assets in this portfolio are fixed; only the amounts invested in the riskless asset and the risky portfolio depend on the risk target $\mathrm{SCR}^{\mathrm{Max}}_{\mathrm{Market}}$. The second component $\boldsymbol{a}^*_{\mathbf{Hedge}} = -\mathbf{V}^{-1}\boldsymbol{c}_{\mathbf{L}}$ is a liability hedge portfolio that exactly offsets the capital charge for interest risk that arises from the liabilities. Hence, the optimal investment strategy is to hedge the interest rate risk of the liabilities (with $\boldsymbol{a}^*_{\mathbf{Hedge}}$) and then to invest in an efficient asset-only portfolio ($\boldsymbol{a}^*_{\mathbf{NoLiab}}$).[8] We have a three fund separation result: all insurers invest in the riskless asset, the optimal portfolio of risky assets $\boldsymbol{a}^*_{\mathbf{NoLiab}}$ and a liability hedge portfolio $\boldsymbol{a}^*_{\mathbf{Hedge}}$.

It is easy to prove that the strategy in (14) is also optimal for an alternative formulation of the model where the insurer minimizes its required solvency capital (1-year

[7] $\Delta_{\mathrm{eq}} = \sqrt{\left(w_{\mathrm{eq},1}\Delta_{\mathrm{eq},1}\right)^2 + \left(\left(1 - w_{\mathrm{eq},1}\right)\Delta_{\mathrm{eq},2}\right)^2 + 2\rho_{\mathrm{eq}}w_{\mathrm{eq},1}\left(1 - w_{\mathrm{eq},1}\right)\Delta_{\mathrm{eq},1}\Delta_{\mathrm{eq},2}}$, with $\rho_{\mathrm{eq}} = 0.75$.

[8] We note that the liability hedge portfolio reduces the duration gap between the assets and the liabilities to zero. On top of that an optimal asset-only portfolio $\boldsymbol{a}^*_{\mathbf{NoLiab}}$ is held, which may carry some interest rate risk exposure itself, but only if this is an efficient way to generate a higher expected return on the assets.

99.5% VaR), subject to a target expected return on its own funds. In other words, the three fund separation result also applies to an insurer that aims to minimize its capital requirement for market risk (i.e., to maximize its solvency ratio), while meeting a target expected return (e.g., 10%).

Implications about redundant assets and diversification

For the derivation of the optimal asset allocation, we had to assume that the *IxI* matrix $\mathbf{V}'\mathbf{R}\mathbf{V}$ is invertible (non-singular). The correlation matrix $\mathbf{R}$ as specified by the Solvency II standard formula, given in Table 1, is invertible. It follows that $\mathbf{V}'\mathbf{R}\mathbf{V}$ is invertible (positive definite) if and only if the $K \times I$ matrix $\mathbf{V}$ is of full rank. Practically, the full rank condition for $\mathbf{V}$ means:

1. There can be no more than K risky assets in the optimization: $I \leq K$.
2. For each type of market risk $k = 1, 2, \ldots, K$ in the standard formula, there should be at least one risky asset that has exposure to risk type k.
3. The exposures of the risky assets to the market risk types contained in $\mathbf{V}$ should not be linearly dependent (i.e., the columns of $\mathbf{V}$ should be independent).

We now provide some examples where these conditions are violated to gain insight. The standard formula for market has five main sources of risk: interest rate risk, equity risk, property risk, credit spread risk and currency risk. Hence, the optimization can only have a maximum of $I = 5$ asset classes. If there are more than five assets in the optimization, say $I = 6$, at least one asset has exactly the same risk exposure as a linear combination of the other assets and that asset would therefore be redundant. Suppose this redundant asset has a relatively low expected return relative to the other assets. In a setting with no-short selling restrictions, this would create an arbitrage opportunity, in the sense that the portfolio expected return can be increased unlimited without increasing the solvency capital for market risk.[9] In a setting where short-selling is not allowed, the redundant asset would get a zero weight.

It is also important to recognize that the Solvency II standard formula for market risk is a simplified model that does not distinguish the risk of assets *within* the same market risk type k well. For example, suppose that the insurer can choose between two equity portfolios:

1. The MSCI World: a well-diversified portfolio of 1637 stocks spread over 23 developed markets. Suppose the expected return is 7% per annum. All currency risks are hedged.
2. The MSCI Belgium: a portfolio of 10 stocks from one developed country, Belgium. Suppose the expected return is 8% per annum. Any currency risk is hedged.

Judging from the perspective of the Solvency II standard formula for market risk, the two portfolios have exactly the same risk exposure and risk charge: 39% of the amount invested. Hence, seen in the context of optimization problem (7), the MSCI Belgium portfolio is superior, offering a higher expected return for the same level of risk, while the MSCI World is an inferior (redundant) asset. In an optimal asset allocation with short-selling constraints, no money would be invested in the MSCI World, and all investments in developed market equity would end up in the MSCI Belgium. When short-selling is allowed, an "arbitrage opportunity" would arise.

Clearly, the Solvency II standard formula is of no use when considering which developed stock markets to invest in, as it ignores differences in risk between these two equity portfolios.[10] In general, the standard formula ignores the diversification benefits of investing in different assets *within* the same market risk type. Insurers need to develop an internal risk model to better distinguish risks and diversification potential *within* each risk type. In the absence of such a model, only well-diversified portfolios (e.g., the MSCI World Index for equity) should be considered for getting exposure to the risk types of the standard formula.

Extensions

Modeling an actual insurance company with all its details and investment constraints requires numerical solution techniques, but they typically do not provide insights about why a particular asset allocation is optimal. For this reason, we studied the main principles of asset-liability management analytically in a stylized model. For example, in "The optimal asset allocation" section we assumed that the duration of the assets is longer than the liabilities and we did not rule out negative values for the asset amounts $\boldsymbol{a}$ and the capital charges $\boldsymbol{s}$. In the Appendix, we provide a formulation of the optimization problem that does include nonnegativity constraints ($\boldsymbol{a} \geq 0$, $\boldsymbol{s} \geq 0$), as well as two inequality constraints to make sure that the SCR for interest rate risk is the maximum of the "curve up" and "curve down" scenarios. The resulting quadratic optimization problem with linear inequality constraints cannot be solved

[9] Short the redundant asset and long the replicating asset with the same risk exposure but higher expected return.

[10] In the 10-year period Oct 2006 to Sept 2016, the return standard deviation of MSCI World was 16.5%, versus 24.4% for MSCI Belgium. The standard deviation estimates are based on monthly price returns (source MSCI).

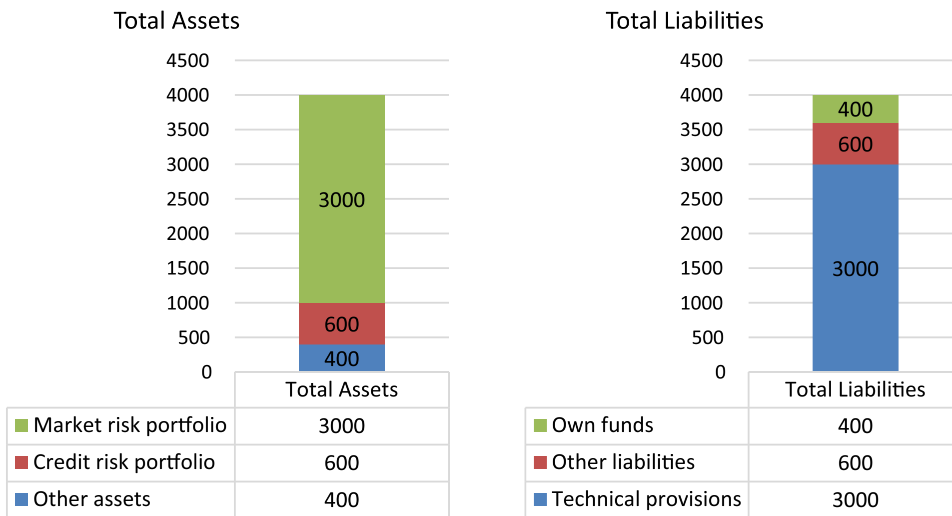

Fig. 1 Balance sheet of a representative European life insurer (*Source*: Höring 2013). All amounts are in millions of euro

analytically, but a numerical solution is easily found with modern solution algorithms (e.g., interior point methods).

Asset allocation examples

To gain more insights about how the solvency requirements could influence the investment strategy of insurers that rely on the Solvency II standard formula, we now calculate optimal asset allocations for a representative European life insurance company. Höring (2013) has created the balance sheet and investment portfolio of a typical European life insurer before the introduction of Solvency II, using annual reports of individual insurance companies, as well as information from regulators and credit rating agencies. We will use Höring's representative life insurer for our numerical illustrations in this section.

Balance sheet and initial asset allocation

The total liabilities of the insurer, shown in the right panel of Fig. 1, mainly consist of the technical provisions of EUR 3.0 billion, representing the value of the insurance liabilities. Under Solvency II, the liability cash flows have to be discounted with interest rates from the swap market, plus certain spreads. As a result, fluctuations in the value of the liabilities due to changes in swap interest rates are a major source of risk. In addition, the insurer has EUR 600 million in other liabilities, such as short-term debt and deferred tax liabilities. Finally, the insurer has EUR 400 million of own funds. The own funds are crucial for determining the insurer's ability to absorb losses and to assess its solvency.

The insurer has EUR 4.0 billion of total assets, as shown in the left panel of Fig. 1. The assets consist of a portfolio of market risk assets worth 3.0 billion, an illiquid credit risk portfolio of 0.6 billion and 0.4 billion in other assets. Market risk assets, such as equity, bonds and property, are charged in the market risk module of Solvency II. The credit risk portfolio of EUR 600 million contains all assets charged in the counter-party risk module of Solvency II, such as mortgages, policy loans, reinsurance assets and cash held at banks. Finally, EUR 400 million is in "other assets," such as intangibles, goodwill and deferred tax assets.

The insurer's initial asset allocation of the market risk assets is shown in Table 2 (for further details, see Höring 2013). Sovereign debt (40%), corporate debt (29%) and covered bonds (13%) together make up more than 80% of the portfolio, while only small fractions are invested in equity (7%) and real estate (11%). Table 2 also shows the expected returns for the market risk assets, used in the asset allocation examples. These expected returns reflect a world in which interest rates are positive, but the expected returns of risky assets are relatively low. In addition, we use a short-term EEA government bond as the risk-free asset (e.g., a 3-month German Treasury bill), with an annualized expected return of 0.25%.

Table 2 also summarizes the representative life insurer's balance sheet, including the interest rate sensitivity of the assets and liabilities. The assets have a duration of 4.6 years, while the liabilities have a longer duration of 6.7 years, creating a duration gap of 2.1 years. As a result, a one basis point downward shift of the curve leads to an expected loss of EUR 0.84 million in own funds. For the insurance liabilities, we assume a growth rate of 3% per year. For the other assets and other liabilities, we assume that they grow at the risk-free rate of 0.25%, for ease of exposition.

Table 2 Initial asset allocation and balance sheet

	E[r] (%)	Duration	Value	Weight (%)
Market risk portfolio				
Developed equity	4.50	0	135	4.5
Other equity	5.50	0	75	2.5
Real estate	3.50	0	330	11.0
Govt. bonds EEA	1.50	6.9	960	32.0
Govt. bonds non-EEA	1.75	6.9	240	8.0
Corporate debt	2.40	5.4	885	29.5
Covered bonds	1.75	6.2	375	12.5
Treasury bills EEA	0.25	0	0	0
Total portfolio	*2.27*	*5.1*	*3000*	*100.0*
Assets				
Market risk portfolio	2.27	5.1	3000	75.0
Credit risk portfolio	3.50	4.9	600	15.0
Other assets	0.25	0	400	10.0
Total assets	*2.25*	*4.6*	*4000*	*100.0*
Liabilities				
Technical provisions	3.00	8.9	3000	75.0
Other liabilities	0.25	0	600	15.0
Own funds	− 0.34	0	400	10.0
Total liabilities	*2.25*	*6.7*	*4000*	*100.0*

Return/risk trade-off		Dollar duration (DV01)		Leverage of assets	
E[Increase own funds]	− 1.3	Assets	1.83	Long	4000
SCR market risk	297.4	Liabilities	2.67	Short	0
Solvency ratio f_M	135%	Gap	0.84	Leverage	1.0
RoRAC	− 0.5%				

The table shows the asset allocation and balance sheet of a representative European life insurance company, based on data from Höring (2013). "E[r]" denotes the expected asset returns, and the growth rate of the liabilities. "Value" is the amount of assets or liabilities measured in millions of Euro. "SCR market risk" is the solvency capital requirement for market risk, determined with the Solvency II standard formula. The solvency ratio f_M is the ratio of the insurer's own funds over the SCR for market risk. The "RoRAC" is the expected increase in the insurer's own funds divided by the SCR for market risk. Dollar duration (DV01) is the expected decrease in value (in millions of Euro), when the yield curve rises by one basis point (0.01%)

Given the balance sheet and asset allocation in Table 2, the solvency capital requirement for market risk of the representative life insurer is EUR 297 million. We refer to Höring (2013) for more details about the SCR calculation.[11] Hence, initially the life insurer's own funds of 400 million euro are 135% of the SCR for market risk, a relatively low ratio, while the expected return on risk-adjusted capital (RoRAC) is only − 0.5%. The initial asset allocation appears inadequate, as the expected return on own funds is negative (− 0.34%), while the insurer is exposed to high interest rate risk and a relatively high solvency capital requirement.

[11] For developed equity and other equity, we use shocks of 30% and 40%, respectively, to make the results directly comparable Höring (2013), who used these parameter values from QIS5. In 2016, these shocks are 39% and 49%.

Example of an optimal strategic allocation

As an example, we now analyze the optimal asset allocation that maximizes the expected return on own funds, given a limit on the SCR for market risk equal to the initial value of EUR 297 million. The SCR limit of EUR 297 million is just an assumption in this example, and in fact, the insurer can choose any combination of expected return and SCR on the efficient frontier shown in Fig. 2, depending on its risk preferences. We solve the extended version of the asset allocation problem in the Appendix that includes nonnegativity constraints ($\boldsymbol{a} \geq 0$, $\boldsymbol{s} \geq 0$), and inequality constraints to make sure that the SCR for interest rate risk is the maximum of the "curve up" and "curve down" scenarios. Table 3 shows the optimal solution.

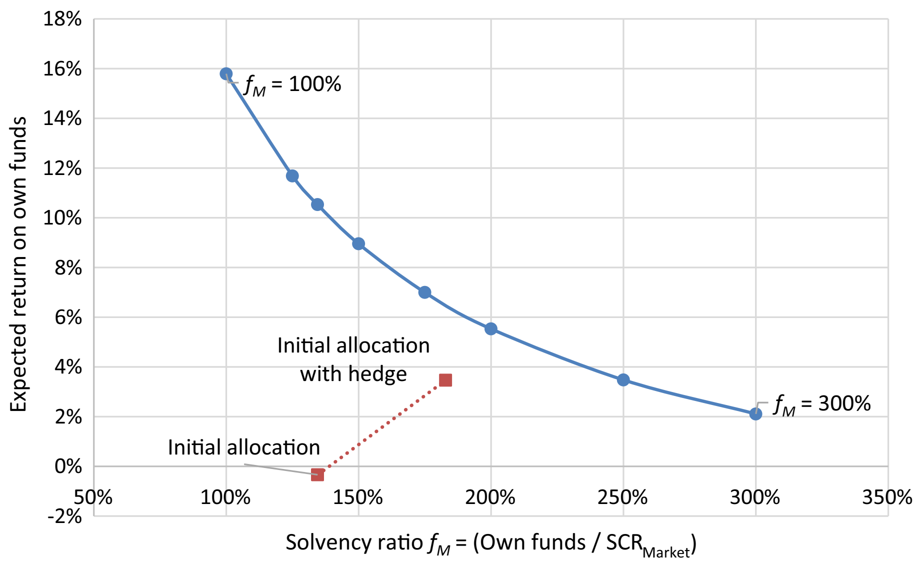

Fig. 2 Efficient frontier of expected return on own funds versus solvency ratio. The solvency ratio f_M is calculated using the SCR for market risk in the denominator: f_M = own funds/SCR_{Market}

Table 3 Optimal asset allocation with SCR limit of 297 million

	E[r] (%)	Value	Weight (%)
Market risk portfolio			
Equity portfolio	4.86	295	9.8
Real estate	3.50	313	10.4
Govt. bonds EEA	1.50	4872	162.4
Govt. bonds non-EEA	1.75	0	0.0
Corporate debt	2.40	904	30.1
Covered bonds	1.75	0	0.0
Treasury bills EEA	0.25	− 3384	− 112.8
Total portfolio	*3.72*	*3000*	*100.0*
Assets			
Market risk portfolio	3.72	3000	75.0
Credit risk portfolio	3.50	600	15.0
Other assets	0.25	400	10.0
Total assets	*3.34*	*4000*	*100.0*
Liabilities			
Technical provisions	3.00	3000	75.0
Other liabilities	0.25	600	15.0
Own funds	10.53	400	10.0
Total liabilities	*3.34*	*4000*	*100.0*

Return/risk trade-off		Dollar duration (DV01)		Leverage of assets	
E[Increase own funds]	42.1	Assets	4.14	Long	7384
SCR market risk	297.4	Liabilities	2.67	Short	− 3384
Solvency ratio f_M	135%	Gap	− 1.47	Leverage	1.8
RoRAC	14.2%				

The table shows the optimal asset allocation and balance sheet of the representative European life insurance, after optimization of the expected return on own funds subject to a limit on SCR for market risk of 297.4 million Euro. "E[r]" denotes the expected asset returns, and the growth rate of the liabilities. "Value" is the amount of assets or liabilities measured in millions of Euro. "SCR market risk" is the solvency capital requirement for market risk, determined with the Solvency II standard formula. The solvency ratio f_M is the ratio of the insurer's own funds over the SCR for market risk. The "RoRAC" is the expected increase in the insurer's own funds divided by the SCR for market risk. Dollar duration (DV01) is the expected decrease in value (in millions of Euro), when the yield curve rises by one basis point (0.01%)

The optimal allocation in Table 3 has an expected return on own funds of 10.5% per year, a substantial improvement compared to the initial allocation's return of only − 0.3% in Table 2. The optimal asset allocation has a 162% weight in EEA government bonds, partially financed with a short position in riskless EEA Treasury bills of − 113%. As a weight of 98% in EEA government bonds is sufficient to hedge the interest rate risk of the insurance liabilities, the remaining 64% is an active allocation to earn a risk premium (the term premium). The attractiveness of Eurozone government bonds may partially be explained by the fact they do not receive a capital charge for credit spread risk in the Solvency II standard formula.

In addition, the optimal allocation invests in three risky asset classes: 9.8% in equity, 10.4% in real estate and 30.1% in corporate bonds. Non-EEA government bonds and covered bonds have zero weights in the optimal allocation, as their ratios of expected return to marginal risk are relatively unattractive. Hence, these two asset classes are redundant. The fact that the optimal allocation contains only four asset classes was to be expected, given that there are only four risk factors (equity, property, credit spread and interest rate risk) that determine the SCR for market risk: see the discussion following the analytical solution in "Implications about redundant assets and diversification" section. These results illustrate that strategic asset allocation with the Solvency standard formula for market risk can result in relatively underdiversified and concentrated portfolios, which is also demonstrated by extensive simulation studies in Braun et al. (2017, 2018).

Discussion

A concern often raised about the Solvency II regulatory framework is that it may lead insurers to excessively reduce risk in an attempt to lower their solvency capital requirement, by limiting exposure to risky assets such as stocks, private equity and hedge funds. Such a "de-risking" policy can reduce expected returns and the insurer's long-term viability (van Bragt et al. 2010).

We note that the structure of the optimal investment strategy in Eq. (14) is sound in principle, namely liability hedging combined with an optimal asset-only portfolio. The optimal allocation in Table 3 indeed has a substantially higher return on assets compared to the initial situation, as the model maximizes the expected return on assets subject to a limit on the capital requirement. Investments in equity and real estate are also present in the optimal allocation, with a combined weight of 20.2%, slightly more than the initial weight of 18.0%. We have also calculated the optimal asset allocation for a more conservative SCR target of 200 million euro, corresponding to a solvency ratio of 200%. Even with this stricter risk limit, the optimal asset allocation still contains 6.6% equity, 7.0% property and 20.3% corporate debt, and hence the portfolio is not completely de-risked. Figure 2 shows the entire efficient frontier, the trade-off between expected return on own funds versus the SCR for market risk.

One potential concern about the optimal portfolio in Table 3 is the large weight of EEA government bonds, while covered bonds and non-EEA bonds have zero weights. As demonstrated in section "Implications about redundant assets and diversification", one issue causing this lack of diversification is that the Solvency II standard formula is a simplified risk model with a limited number of risk drivers.[12] Further, the parameters were calibrated conservatively by CEIOPS to reflect a worst-case scenario that occurs once in 200 years. The main purpose of this simplified model is to set *minimum* capital requirements, not to guide the strategic trade-off between risk and expected return that determines the insurer's long-term profitability. When it comes to strategic asset allocation, insurers are therefore advised to rely on a more elaborate internal risk model.

Conclusions

In this paper, we analyze the asset allocation of insurance companies that maximize their expected return on own funds, subject to an upper limit on the capital requirement for market risk, determined with the Solvency II standard formula. The model solution also applies to insurers who minimize their capital requirement for market risk, subject to meeting a target expected return on their own funds. Our model illustrates how insurers could respond to the Solvency II capital constraint, if they use its standard formula as the only yardstick for risk (an estimate of the 1-year 99.5% VaR) and aim to maximize their solvency ratio. We derive the optimal asset allocation analytically and show that it consists of a liability hedge portfolio combined with an optimal asset-only portfolio and cash, a three fund separation result. The optimal investment strategy is sound in principle, as hedging the interest rate risk of the liabilities is a key component.

However, we also note that when the Solvency II standard formula is used as the yardstick for measuring risk, little-to-no distinction is made between individual assets *within* market risk categories (interest, equity, property and credit risk). From the perspective of the standard formula, an investment in the well-diversified MSCI World index

[12] The allocation also results in a balance sheet with a negative duration gap, so that the upward shock to the yield curve determines interest rate risk. In that case, the correlation matrix in Panel B of Table 1 applies, assigning zero correlations to interest rate risk, increasing the diversification benefits of fixed income investments.

and the undiversified MSCI Belgium index is equally risky, both receiving the same capital charge for developed equity. Insurers applying the standard formula are therefore advised to use only well-diversified portfolios to gain exposure to interest rate, equity, property and spread risk.

Further, our illustrative examples for a typical European life insurance company show that the Solvency II standard formula can give rise to a relatively underdiversified portfolio with large investments in European government bonds, which receive no charge for spread risk. When it comes to strategic asset allocation, insurers are advised to develop an internal risk model that better recognizes risk, as well as opportunities for diversification and earning risk premiums. The standard formula was calibrated to set *minimum* capital levels so that the company can survive a 0.5% worst-case scenario, and not necessarily for managing the long-term strategic trade-off between return and risk.

Appendix: Optimization with Nonnegativity Constraints

Below we provide a formulation of the optimal asset allocation problem that includes nonnegativity constraints on the amounts invested in the risky assets ($\boldsymbol{a} \geq 0$) and on the capital charges for the risk types ($\boldsymbol{s} \geq 0$). Further, the model includes two inequality constraints to make sure that the SCR for interest rate risk (s_1) is the maximum of the two charges in the "curve up" and "curve down" scenarios for the term-structure of interest rates.

To facilitate an efficient numerical solution of the optimization problem, the objective function is to maximize the expected change in own funds ΔF, while putting a penalty $\gamma > 0$ on the solvency capital requirement for market risk ($\mathrm{SCR}_{\mathrm{Market}} = \boldsymbol{s}'\mathbf{R}\boldsymbol{s}$). This optimization problem has a quadratic objective function and linear inequality constraints. A numerical solution can be found easily with modern solution algorithms, such as interior point methods. An efficient frontier of expected return on own funds versus the SCR for market risk can be derived by changing the risk aversion parameter γ.

$$\max_{\boldsymbol{a}} E[\Delta F(\boldsymbol{a})] = \left(r_f A + \boldsymbol{\mu}_{\mathrm{A}}' \boldsymbol{a} - \mu_{\mathrm{L},1} L_1 - \mu_{\mathrm{L},2} L_2 \right) - \gamma \boldsymbol{s}'\mathbf{R}\boldsymbol{s} \tag{15}$$

$$\begin{aligned} s_1 &\geq \Delta_{\mathrm{rd}} \left(\left(D_{\mathrm{L},1} L_1 + D_{\mathrm{L},2} L_2 \right) - \left(D_{\mathrm{A},1} A_1 + D_{\mathrm{A},2} A_2 + D_{\mathrm{A},3} A_3 \right) \right) \\ s_1 &\geq \Delta_{\mathrm{ru}} \left(\left(D_{\mathrm{A},1} A_1 + D_{\mathrm{A},2} A_2 + D_{\mathrm{A},3} A_3 \right) - \left(D_{\mathrm{L},1} L_1 + D_{\mathrm{L},2} L_2 \right) \right) \end{aligned} \tag{16}$$

$$\begin{aligned} s_2 &\geq \Delta_{\mathrm{eq}} A_{\mathrm{eq}} \\ &= \sqrt{\left(\mathrm{w}_{\mathrm{eq},1} \Delta_{\mathrm{eq},1} \right)^2 + \left(\left(1 - \mathrm{w}_{\mathrm{eq},1} \right) \Delta_{\mathrm{eq},2} \right)^2 + 2 \rho_{\mathrm{eq}} \mathrm{w}_{\mathrm{eq},1} \left(1 - \mathrm{w}_{\mathrm{eq},1} \right) \Delta_{\mathrm{eq},1} \Delta_{\mathrm{eq},2}} A_{\mathrm{eq}} \end{aligned} \tag{17}$$

$$s_3 \geq \Delta_{\mathrm{prop}} A_{\mathrm{prop}} \tag{18}$$

$$s_4 \geq \Delta_{\mathrm{gov},2} A_{\mathrm{gov},2} + \Delta_{\mathrm{corp}} A_{\mathrm{corp}} \tag{19}$$

$$s_5 \geq \Delta_{\mathrm{cur}} \sum_{i=1}^{I} f_i A_i \tag{20}$$

$$\boldsymbol{s} \geq 0, \boldsymbol{a} \geq 0 \tag{21}$$

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