Strategic Asset Allocation and Risk Budgeting for Insurers under Solvency II

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Abstract
Solvency II is a new risk-based framework for setting the capital requirements of European insurance companies, in force since January 2016. The *solvency capital requirement* (SCR) is set such that the insurer can meet its obligations over the next 12 months with a probability of at least 99.5%. In this paper we derive expressions for an investment's marginal contribution to the solvency capital requirement, to provide insight in the risk allocation and the trade-off between expected return and marginal risk. In addition we derive the optimal strategic asset allocation for an insurer that maximizes the expected return on its own funds, subject to a limit on the SCR for market risk determined with the Solvency II standard formula. We then provide numerical examples to illustrate how the new framework for asset allocation and risk budgeting under Solvency II can be applied at a representative European life insurance company.

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1. Introduction

This research analyzes the optimal asset allocation policy for European insurance companies under the new Solvency II regulatory framework. Solvency II is a risk-based system for setting capital requirements in the insurance industry, which came into force in the European Economic Association on 1 January 2016. The *solvency capital requirement* (SCR) is set such that the insurance company can meet its obligations over the next 12 months with a probability of at least 99.5%. The insurer needs to have own funds greater than the solvency capital requirement. The SCR can be calculated with a standard formula that applies a set of instantaneous extreme shocks (losses) to the main balance sheet components of the insurer, both the assets and the liabilities. The resulting losses are then aggregated with a given correlation matrix to derive the total SCR, taking into account diversification benefits and the correlation between the assets and the liabilities.

In this research project we will analytically derive the optimal strategic asset allocation for an insurer that maximizes the expected return on its own funds, subject to a limit on its solvency capital requirement determined with the Solvency II standard formulas for market risk and risk aggregation. We will assume that the insurance liabilities are fixed (given) in the short-run, but the interest-rate sensitivity of the liabilities will be explicitly taken into account. Given a set of expected asset class returns, we can then construct an efficient frontier showing the trade-off between the insurer's expected return on own funds versus the required amount of capital.

The aim of this paper is not simply to derive the "optimal" allocation, as it depends strongly on the assumptions made about the expected returns. Rather, when deriving the analytical solution we also obtain expressions for useful concepts such as an investment's *marginal contribution* to the capital requirement (*mcSCR*). These marginal contributions to risk provide important insights about how each asset class in the portfolio contributes to the total solvency capital requirement, while accounting for diversification benefits and the correlation with the insurance liabilities (hedging). Especially each investment's ratio of expected excess return to marginal risk gives useful information that can be used to improve the strategic asset allocation.

The marginal contributions to risk and return of the current asset allocation can provide the management team of the insurer important insights about the risk and return profile of the company, even when they do not intend to implement an optimal asset allocation. The framework can also deliver a set of implied expected returns that would make the current asset allocation optimal under a given SCR constraint, which allows the management team to check whether the current allocation is consistent with its own asset return expectations. In other words, we provide *a risk budgeting framewor*k for European insurers under Solvency II, aiming to provide more insights about the insurer's risk profile. Our work extends existing research on asset allocation and risk budgeting for pension funds by Berkelaar, Kobor and Kouwenberg (2006) to the case of insurers under the new Solvency II regulatory framework.

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1.1. Research objectives

- To derive an analytical framework for asset allocation and risk budgeting for a European life insurance company subject to the Solvency II regulations. The framework will be based on the Solvency II standard formulas for market risk and risk aggregation, while assuming that the insurance liability is fixed (given) in the short-run. The interest-rate sensitivity of the liabilities and its contribution to risk will be explicitly taken into account.

- To provide extensive numerical examples to illustrate how the new analytical framework for asset allocation and risk budgeting can be applied at a representative, but fictitious, European life insurance company.

- To make sure that the long-term return on assets is also considered as part of the asset allocation and risk budgeting framework, to avoid myopic investment strategies that focus only on the one-year Solvency II horizon and minimizing the capital requirement.

- To indicate how the asset allocation and risk budgeting framework can be applied when the insurance company uses an internal risk model, or credit rating agency model, applying numerical methods.

2. Related Literature

2.1. Literature on Solvency II and asset allocation for insurers

Solvency II, the new regulatory framework for insurance companies in the European Economic Area (EEA), has come into full force in 31 European countries as of January 1, 2016.
[1] The Solvency II rules have been under development since 2000, involving a long consultation process with national regulators and the European insurance industry. The new framework, prescribes risk-based capital requirements, in combination with market-based valuation of the assets and the insurance liabilities. The aim of the solvency capital requirement is that the insurance company has sufficient funds to cover losses over a 1-year horizon with 99.5% probability. The new rules have been tested in five Quantitative Impact Studies (QIS) during which insurers were asked to determine the value of their liabilities and the solvency capital requirement according to the latest technical specifications of Solvency II. During the 15-year development period of Solvency II both academicians and consultants have done research about the potential asset allocation implications of the new rules, which we now summarize briefly.

Amenc, Martellini, Foulquier and Sender (2006) were among the first who analyzed the potential impact of the Solvency II framework on the asset-liability management of insurance companies. Amenc et al. (2006) recommend a core-satellite approach, with a dedicated core portfolio to fully hedge the interest rate sensitivity of the liabilities, and any embedded options in the insurance liabilities. Once the market-value of the liabilities is fully hedged with the core asset portfolio, the

[1] The EEA consists of 31 countries: the 28 European Union (EU) members, plus Norway, Lichtenstein and Iceland. Switzerland has its own regulatory framework for insurers, similar to Solvency II.

remaining surplus assets can be managed with traditional asset allocation techniques. Similarly, Bragt and Kort (2011) discuss full hedging strategies for insurance liabilities, which would enable insurers to allocate surplus assets with an asset-only approach. Amenc et al. (2006) point out that a major obstacle to full liability hedging in practice are the IFRS accounting standards, which require insurers to recognize profits and losses on derivatives used for liability hedging annually, while changes in the market value of the liabilities are not immediately recognized. As a result, an insurance company following a liability hedging strategy with derivatives in practice will have to report excessively volatile accounting profits (net income).

Van Bragt, Steehouwer and Waalwijk (2010) investigate the impact of the Solvency II rules on the tradeoff between risk and return of a typical European life insurer. The main finding is that the life insurer can greatly reduce its solvency capital requirement by matching the interest-rate sensitivity of the assets and the liabilities, showing that the new rules stimulate better asset-liability management (ALM). Solvency capital can also be reduced by limiting exposure to risky assets such as stocks and real estate, but this comes at the expense of a low expected return that can deteriorate the long-term viability of the insurer over a 10-year horizon. Van Bragt et al. (2010) conclude that insurers should set asset allocation policies that consider both the 1-year Solvency II capital requirement and the long-run risk-return profile of the asset allocation.

A host of other studies consider the impact of the Solvency II rules on asset allocation (Rudschuck et al. 2010; Mittnik, 2011; Braun, Schmeiser and Schreiber, 2015b; Fischer and Schlütter, 2015), mainly focusing on the question whether insurance companies will reduce exposure to risky assets such as stocks, corporate bonds and hedge funds after the introduction of the new rules. Investments in developed market equities are subject to a 39% capital charge under the latest Solvency II provisions, while emerging equity, private equity and hedge funds face a 49% capital charge. Hence, ignoring diversification benefits, an insurance company investing 100 euro in hedge funds would have to hold 49 euro of additional own funds as a buffer against potential losses. Not surprisingly, many studies find that when the new capital requirement rules are considered in isolation, they may lead to a large reduction in risky asset exposures (e.g., Rudschuck et al. 2010). On the other hand, insurance companies are also subject to tests by ratings agencies such as S&P and Fitch, which appear to be even stricter than the Solvency II requirements when companies want to maintain rating levels of A or higher (Höring, 2013). Further, most insurance companies will not consider the solvency requirements in isolation, but also consider the long-term risk-return profile of the business.

Another line of studies criticizes the one-year 99.5% VaR methodology of Solvency II and raises serious questions about whether the specifications of the standard model will truly lead to a ruin probability below 0.5% (1 in 200 years); see Mittnik (2011) and Braun, Schmeiser and Schreiber (2015a), for example. Most criticisms of the Solvency II framework for setting capital requirements are valid and justified in our opinion. But as Solvency II has come into force in 2016, in this project we take the framework as given and we focus on deriving optimal asset allocation policies given the Solvency II standard model for market risk.

Overall, the literature so far has considered the potential impact of Solvency II on the asset allocation of insurers using numerical tools such as simulation and one-off examples. The contribution of this project is that we will *analytically* derive the optimal strategic asset allocation for an insurer that maximizes the expected return on its portfolio of assets, subject to a limit on its solvency capital requirement determined with the Solvency II standard formula for market risk. Most importantly, the analytical solution allows us to obtain expressions for useful risk management concepts such as an investment's *marginal contribution to the solvency capital requirement* (*mcSCR*). Marginal contributions to risk provide important insights about how each asset class in the portfolio contributes to the capital requirement, while taking into account diversification benefits and the correlation with the insurance liabilities. Further, we will derive an expression for each investment's *marginal return on solvency capital* (*mRoC*) and the *ratio of expected return to marginal solvency capital*, to facilitate a trade-off between expected return and risk when evaluating changes to the current asset allocation. Finally, we can derive the *implied expected returns* that would make the current asset allocation optimal within the framework, which the management team can then compare to its latest expectations about asset returns.

In sum, we aim to enhance both the asset allocation and risk budgeting process for insurers under Solvency II by deriving new metrics that provide insights about the risk and return of the asset allocation. Although insurers and consultants in practice may already apply measures of marginal risk and return, our analytical framework will provide a solid foundation for these measures that supports their application and interpretation in a Solvency II context.

2.2. Literature on risk budgeting

The project aims to extends the existing research on asset allocation and risk budgeting for pension funds by Berkelaar, Kobor and Kouwenberg (2006) to the case of insurers under the new Solvency II framework. Berkelaar et al. (2006) show how measures like marginal contribution to tracking error and marginal information ratio can provide better understanding of the risk and returns embedded in the active risk allocation of a pension fund. Berkelaar et al. (2006) derive these measures analytically, starting from a simple mean-variance framework for the optimal active risk allocation. We aim to derive similar marginal risk and return measures in a simple framework for an insurance company that maximizes the expected return on its portfolio of assets, subject to a limit on its solvency capital requirement, which is determined with the Solvency II standard model. Leblanc (2011) has already derived marginal contribution to risk expressions for the Solvency II standard formula for market risk, showing our intended approach is feasible. However, Leblanc (2011) does not derive an optimal asset allocation and does not consider the impact of the asset allocation on the expected return. Our framework will consider both risk and return, and will include the correlation of the assets with the insurance liabilities.

Risk budgeting is an established approach in asset management that aims to allocate a given risk budget efficiently over a set of asset classes, or a set of active investments (see, e.g., Litterman, 1996, Blitz and Hottinga, 2001, Lee and Lam, 2001, Chow and Kritzman, 2001, Sharpe, 2002, Molenkamp, 2004, Berkelaar, Kobor and Tsumagari, 2006, and Berkelaar, Kouwenberg and Kobor, 2006). Berkelaar, Kobor and Tsumagari (2006) explain that a risk budgeting process involves *risk measurement* ("What is our total risk today?"), *risk attribution* ("Which assets generate the total risk?"), and *risk allocation* ("How to better allocate risk in the future?"). We aim to extend the risk budgeting process to insurance companies under Solvency II.

3. Solvency II Standard Formula for the Capital Requirement

In this section we describe the Solvency II standard formula (SF) that insurers can use for determining their solvency capital requirement. We focus in particular on the market risk module, as our aim is to analyze the strategic asset allocation of the insurer.

Before introducing the standard formula, we first define the notation for the insurer's balance sheet. Let A denote the market value of the total assets of the insurer, separated in I different asset classes: $A = \sum_{i=1}^{I} A_i$. Similarly, let L denote the market value of the liabilities of the insurer, consisting of N different sub-categories: $L = \sum_{n=1}^{N} L_N$. The amount of own funds of the insurer, denoted by F, is equal to the difference between the assets and the liabilities: $F = A - L$.

For the asset allocation we use the following asset classes:

$A_1 = A_{gov,1}$ = sovereign debt issued by EEA countries
$A_2 = A_{gov,2}$ = sovereign debt issued by other countries
$A_3 = A_{corp}$ = corporate debt
$A_4 = A_{eq}$ = equity (developed markets, emerging, private equity and hedge funds)
$A_5 = A_{prop}$ = property
$A_6 = A_{other}$ = other non-market assets (e.g., mortgages, reinsurance assets)
$A_7 = A_{cash}$ = cash

The asset classes above are aligned with the way the Solvency II standard formula charges capital. Investments in debt (A_1, A_2, A_3), equity (A_4) and property (A_5) are charged in the market risk module, while cash (A_7) and other assets (A_6) are charged in the counter-party risk module.

The total liabilities consist of the following sub-categories:

$L_1 = L_{tprov}$ = Technical provisions
$L_2 = L_{other}$ = Other liabilities

3.1. The Solvency II standard formula for market risk

The Solvency II standard formula for market risk determines the capital requirement for six types of market risk: I. Interest rate risk, II. Equity risk, III. Property risk, IV. Credit spread risk, V. Currency risk, and VI. Concentration risk. Let $SCR_{Mkt,k}$ denote the solvency capital requirement for market risk type k, for $k = I, II, \dots, VI$, before taking into account diversification effects. We will now specify how the solvency capital requirement for each type of market risk is determined with the Solvency II standard formulas.

3.1.1. Interest rate risk

The capital requirement for interest rate risk is determined as the maximum loss of own funds resulting from a prescribed upward shock to the term structure of risk-free interest rates, and a given downward shock. The impact of the term-structure shock is determined separately for each asset and each liability, and then combined to see the impact on the insurer's own funds ($A - L$).

Whether the upward shock scenario or the downward shock gives the largest loss of own funds depends on which side of the balance sheet has the highest interest rate sensitivity. For example, when the liabilities have a longer duration than the assets, then the downward shock to the curve will determine the capital requirement for interest rate risk.

For ease of exposition, we will assume that the effect of a shock to the term structure of interest rates can be summarized with a simple duration-based calculation, following Höring (2013).[2] Let $D_{A,i}$ denote the duration of asset *i*, and $D_{L,n}$ the duration of liability *n*. Further, the parameter Δ_{rd} is the parallel downward shock to the interest rates, and Δ_{ru} is the upward shock. The capital requirement for interest rate risk then is:

$$SCR_{Mkt,I} = \max\{\Delta_{rd}((D_{L,1}L_1 + D_{L,2}L_2) - (D_{A,1}A_1 + D_{A,2}A_2 + D_{A,3}A_3)), \\ \Delta_{ru}\big(D_{A,1}A_1 + D_{A,2}A_2 + D_{A,3}A_3\big) - (D_{L,1}L_1 + D_{L,2}L_2))\} \tag{1}$$

3.1.2. Equity risk

The Solvency II standard formula distinguishes two types of equity investments:

$A_{eq,1}$ = developed: equity listed in developed markets,
$A_{eq,2}$ = all other: equity listed in emerging markets, private equity and hedge funds.

We assume that of the total equity investment $A_{eq}(= A_4)$, a fraction $w_{eq,1}$ is invested in developed equity and the remainder (1−$w_{eq,1}$) in other equity: $A_{eq,1} = w_{eq,1}A_{eq}$ and $A_{eq,2} = (1 - w_{eq,1})A_{eq}$. To determine the capital requirement for equity, a shock of $\Delta_{eq,1}$ is applied to the value of the equity investments in developed markets ($A_{eq,1}$). Similarly, a larger shock of $\Delta_{eq,2}(> \Delta_{eq,1})$ is applied to the value of the other equity investments ($A_{eq,2}$).

$$SCR_{eq,1} = \Delta_{eq,1}A_{eq,1} \tag{2}$$

$$SCR_{eq,2} = \Delta_{eq,2}A_{eq,2}$$

Typically, $\Delta_{eq,1}$ = 39% and $\Delta_{eq,2}$ = 49%, but the exact parameter values are adjusted monthly depending on the market's recent development with a mechanism called the "symmetric adjustment". The symmetric adjustment reduces capital charges for equity in bear markets, and increases it in bull markets. The purpose is to alleviate the concern that the capital requirements put pressure on insurers to sell their equity investments directly after a market crash.

The two separate capital requirements for developed market equity ($SCR_{eq,1}$) and other equity ($SCR_{eq,2}$) are aggregated with the square root formula, with a correlation parameter of ρ_{eq}=0.75:

[2] This simplification can be made without loss of generality. The crucial assumption for the overall framework is that the value of each SCR component increases 1-on-1 with the amount of total assets (or liabilities).

$$SCR_{Mkt,II} = \sqrt{(SCR_{eq,1})^2 + (SCR_{eq,2})^2 + 2\rho_{eq}SCR_{eq,1}SCR_{eq,2}} \quad (3)$$

Hence, some degree of diversification between developed equity and other equity is allowed for.

3.1.3. Property risk

The capital requirement for property risk is set by applying a shock of Δ_{prop}=25% to the value of the property investments (A_{prop}). The property investments include listed real estate (e.g., REITS), direct property investments (unlisted), and the value of office buildings owned by the insurance company for its own use.

$$SCR_{Mkt,III} = \Delta_{prop}A_{prop} \quad (4)$$

3.1.4. Credit spread risk

This module determines the capital requirement for tradable investments in corporate bonds, loans, securitizations (asset backed securities) and credit derivatives. The value of each investment *l* in this group is shocked with a percentage Δ_l that depends on its duration and credit rating. The individual capital requirements are aggregated trough simple summation, thus without considering diversification effects. Government bonds from EEA countries are currently exempt from capital requirements in the credit spread risk module (i.e., the spread risk for these bonds is set to zero).

Let $\Delta_{gov,2}$ denote the weighted average shock for the portfolio of non-EEA sovereign bonds and let Δ_{corp} denote the weighted average shock for the portfolio of all corporate debt investments, including securitizations and credit derivatives. Then the overall charge for credit spread risk is:

$$SCR_{Mkt,IV} = \Delta_{gov,2}A_{gov,2} + \Delta_{corp}A_{corp} \quad (5)$$

3.1.5. Currency risk

The capital requirement for currency risk is set by applying a shock of Δ_{cur}=25% to the value of all investments denoted in a foreign currency. Let f_i denote the fraction of investment A_i invested in foreign currency, then the capital requirement is:

$$SCR_{Mkt,V} = \Delta_{cur}\sum_{i=1}^{I} f_i A_i \quad (6)$$

3.1.6. Concentration risk

A capital requirement for concentration risk applies when the insurer's combined investments in a single name (e.g., a specific company, or bond issuer) exceeds 1.5% to 15% of the insurer's total asset value, depending on the credit rating of the issuer. The investments included in the concentration risk calculation include all forms of market risk: bonds, other debt, equity and property. Government bonds issued by EU member states in their domestic currency are exempted from concentration risk. Let *m* denote a specific issuer (e.g., Volkswagen), and $A_{conc,m}$ the insurer's total investments in this issuer. The charge $\Delta_{conc.m}$ for concentration risk applies if $A_{conc,m}$ exceeds the threshold percentage $T_{conc.m}$ of the total assets (A):

$$SCR_{conc,m} = \Delta_{conc,m}\max\{0, A_{conc,m} - T_{conc,m}A\} \quad (7)$$

The threshold $T_{conc,m}$ and the shock $\Delta_{conc,m}$ depend on the specific type of investments (e.g., exposure to a single property, or a covered bond, etc.) and the weighted credit rating of the issuer's assets. Finally, the single name exposures are aggregated as if the default risks have zero correlation:

$$SCR_{Mkt,VI} = \sqrt{\sum_{m=1}^{M}\left(SCR_{conc,m}\right)^2} \quad (8)$$

Insurance companies can reduce the capital requirement for concentration risk to zero by limiting the exposure to any single issuer below the threshold $T_{conc,m}$, ranging from 1.5% to 15%, depending on the asset type and the issuer's credit rating.

3.1.7. Aggregation of market risk

The capital charges for the six risk types are aggregated into a total capital requirement for market risk, SCR_{Market}, with the square root formula below:

$$\begin{aligned} SCR_{Market} &= \sqrt{\sum_{k=1}^{K}(SCR_{Mkt,k})^2 + \sum_{k=1}^{K}\sum_{\substack{j=1 \\ j\neq k}}^{K}\rho_{kj}SCR_{Mkt,k}SCR_{Mkt,j}} \\ &= \sqrt{\sum_{k=1}^{K}\sum_{j=1}^{K}\rho_{kj}SCR_{Mkt,k}SCR_{Mkt,j}} = (\mathbf{s}'\mathbf{R}\mathbf{s})^{½} \end{aligned} \quad (9)$$

where $\rho_{kj} = \rho_{Mkt,kj}$ is the correlation between market risk types k and j, prescribed by the regulator. Above we also use vector-matrix notation: $\mathbf{s} = (SCR_{Mkt,I}, SCR_{Mkt,II}, \dots, SCR_{Mkt,K})'$ is a $Kx1$ vector holding the SCR's for the market risk types, and $\mathbf{R}$ is a KxK matrix containing the correlation coefficients ρ_{kj}.

The correlation coefficients prescribed by the Solvency II regulation for the standard formula are shown in Table 1. Please note that the correlations of interest rate risk with equity, property and spread risk depend on whether the downward shock scenario for interest rate risk gives the largest loss of own funds (Panel A), or the upward shock (Panel B).

Table 1 Correlations for aggregation of market risks

Panel A: Decrease of the term-structure shock determines the interest rate risk

	Interest	Equity	Property	Spread	Currency	Concent.
Interest rate risk	1	0.5	0.5	0.5	0.25	0
Equity risk	0.5	1	0.75	0.75	0.25	0
Property risk	0.5	0.75	1	0.5	0.25	0
Spread risk	0.5	0.75	0.5	1	0.25	0
Currency risk	0.25	0.25	0.25	0.25	1	0
Concentration risk	0	0	0	0	0	1

Panel B: Increase of the term-structure shock determines the interest rate risk

	Interest	Equity	Property	Spread	Currency	Concent.
Interest rate risk	1	0	0	0	0.25	0
Equity risk	0	1	0.75	0.75	0.25	0
Property risk	0	0.75	1	0.5	0.25	0
Spread risk	0	0.75	0.5	1	0.25	0
Currency risk	0.25	0.25	0.25	0.25	1	0
Concentration risk	0	0	0	0	0	1

Note: The table shows the correlations used to aggregate capital requirements for market risk in the Solvency II standard formula, in Equation (9). The correlations in Panel A apply when the downward shock to the term-structure of interest rates determines the capital requirement (gives the biggest SCR). Panel B applies otherwise.

3.2. The total solvency capital requirement

Apart from market risk, the insurer also needs to hold capital for four types of non-market risk: non-life insurance underwriting risk, life insurance underwriting risk, health insurance underwriting risk, and counter-party default risk. Let $SCR_{Agg,h}$ denote the aggregated capital requirements for the following risk types: market risk (*h* = 1), non-life (*h* = 2), life (*h* = 3), health (*h* = 4), and counter-party risk (*h* = 5). Thus, $SCR_{Agg,1} = SCR_{Market}$. The capital requirements for the different risk types are then aggregated as follows to determine the total capital requirement:

$$SCR_{Total} = \sqrt{\sum_{h=1}^{5} (SCR_{Agg,h})^2 + \sum_{h=1}^{5} \sum_{\substack{j=1 \\ j \neq k}}^{5} \rho_{Agg,hj} SCR_{Agg,h} SCR_{Agg,j}} \qquad (10)$$

where $\rho_{Agg,hj}$ is the correlation between risk type *h* and *j*, prescribed by the regulator. As of March 2016, the prescribed correlations between market risk and the non-market risks are all 0.25. Table 2 shows the full correlation matrix.

Table 2 Correlations for aggregation of the total SCR

	Market	Non-life	Life	Health	Default
Market risk	1	0.25	0.25	0.25	0.25
Non-life risk	0.25	1	0	0	0.5
Life risk	0.25	0	1	0.25	0.25
Health risk	0.25	0	0.25	1	0.25
Default risk	0.25	0.5	0.25	0.25	1

Note: The table shows the correlations used to aggregate capital requirements for market risk and non-market risks with the Solvency II standard formula, in Equation (10) above, to determine the total solvency capital requirement.

The insurance company's own funds need to be larger than the total solvency capital requirement, $F = A - L \geq SCR_{Total}$. If the amount of own funds drop below the total SCR, the national regulator can take actions to force the insurer to improve its capital position. Hence, the solvency ratio, f, needs to be larger than one: $f = (A - L)/SCR_{Total} \geq 1$.

In practice life insurance companies set considerably higher targets for the solvency ratio, for example targets ranging from 150% to more than 200%. The stock market can strongly discount, and thus punish, listed insurance companies that do not meet the solvency target deemed appropriate for the firm (depending on the size of the firm and its diversification opportunities). For example, the shares of the Dutch life insurer Delta Lloyd dropped by 70% in the year 2015, after it announced that its solvency ratio fell below its target range of 140 to 180 percent. In the year 2016 Delta Lloyd raised capital from shareholders to increase its own funds. Eventually, the company was taken over by a large competitor with a stronger solvency ratio.

4. Risk Budgeting Measures for Solvency II

We will now derive expressions for marginal risk that can be used to assess how much the SCR increases when the allocation to a particular asset (or liability) is increased by a small amount. In addition, the related measure marginal *contribution* to risk shows how much of the total SCR a particular asset contributes (in %), after accounting for diversification benefits. For ease of exposition, below we start with the SCR for market risk, before eventually deriving marginal risk measures for the total SCR.

4.1. Marginal SCR for market risk types

The marginal SCR of market risk type *k*, denoted by $mSCR_k^{Mkt}$, gives the approximate increase in solvency capital for market risk when the SCR of risk type *k* increases by 1 unit (e.g., 1 euro):

$$mSCR_k^{Mkt} = \frac{\partial SCR_{Market}}{\partial SCR_{Mkt,k}} = \frac{\left(SCR_{Mkt,k} + \sum_{\substack{j=1 \\ j\neq k}}^{K} \rho_{kj} SCR_{Mkt,j}\right)}{SCR_{Market}} = \frac{\{\mathbf{Rs}\}_k}{SCR_{Market}} \tag{11}$$

We next define the marginal *contribution* to market SCR of risk type *k*, denoted by $mcSCR_k^{Mkt}$:

$$mcSCR_k^{Mkt} = \frac{SCR_{Mkt,k} \times mSCR_k^{Mkt}}{SCR^{Market}} = \frac{\left(SCR_{Mkt,k}^2 + \sum_{\substack{j=1 \\ j\neq k}}^{K} \rho_{kj} SCR_{Mkt,k} SCR_{Mkt,j}\right)}{(SCR_{Market})^2} = \frac{\{\mathbf{s}\}_k \times \{\mathbf{Rs}\}_k}{(SCR_{Market})^2} \tag{12}$$

The marginal contributions to risk sum up to one: $\sum_{k=1}^{K} mcSCR_k^{Mkt} = 1$. Thus, we can interpret $mcSCR_k^{Mkt}$ as the percentage of the capital requirement for market risk attributed to risk type *k*, after taking into account diversification effects.

4.2. Marginal risk measures for the asset allocation

For risk attribution and improving the asset allocation it is important to have information about the marginal risk of each asset class individually (A_1, A_2, …, A_I). Below we derive these marginal risk measures. For ease of exposition, we assume that the duration of the liabilities is greater than or equal to the duration of the assets, and that the insurer's SCR for concentration risk is zero.

Let $mSCR_{A,i}^{Mkt}$ denote the marginal SCR with respect to investment A_i in asset class *i*. This marginal risk measure gives the approximate increase in the SCR for market risk when the investment A_i in asset class *i* increases by 1 unit (e.g., 1 million euro). For example, the marginal SCR for the investment in government bonds from EEA countries ($A_1 = A_{gov,1}$) is:

$$mSCR_{A,1}^{Mkt} = \frac{\partial SCR_{Market}}{\partial A_1} = \frac{\partial SCR_{Market}}{\partial SCR_{Mkt,I}} \frac{\partial SCR_{Mkt,I}}{\partial A_1} + \frac{\partial SCR_{Market}}{\partial SCR_{Mkt,V}} \frac{\partial SCR_{Mkt,V}}{\partial A_1} \qquad (13)$$
$$= -mSCR_I^{Mkt} D_{A,1} \Delta_{rd} + mSCR_V^{Mkt} \Delta_{cur} f_1$$

The expression above shows that EEA government bonds provide a hedge against the liabilities through their interest rate sensitivity (on the left side), but they can potentially also involve foreign currency risk (the right side of the expression). The marginal SCR for non-EEA government bonds and corporate debt additionally involves credit spread risk:

$$mSCR_{A,2}^{Mkt} = mSCR_I^{Mkt} \frac{\partial SCR_{Mkt,I}}{\partial A_2} + mSCR_{IV}^{Mkt} \frac{\partial SCR_{Mkt,IV}}{\partial A_2} + mSCR_V^{Mkt} \frac{\partial SCR_{Mkt,V}}{\partial A_2} \qquad (14)$$
$$= -mSCR_I^{Mkt} D_{A,2} \Delta_{rd} + mSCR_{IV}^{Mkt} \Delta_{gov,2} + mSCR_V^{Mkt} \Delta_{cur} f_2$$

$$mSCR_{A,3}^{Mkt} = -mSCR_I^{Mkt} D_{A,3} \Delta_{rd} + mSCR_{IV}^{Mkt} \Delta_{corp} + mSCR_V^{Mkt} \Delta_{cur} f_3 \qquad (15)$$

The marginal SCR for investments in equity ($A_4 = A_{eq}$) is special, because of the distinction between developed market equity ($A_{eq,1}$) and other equity ($A_{eq,2}$), and the aggregation step in the standard formula that takes into account the correlation between them:

$$\begin{aligned} mSCR_{A,4}^{Mkt} = {} & mSCR_{II}^{Mkt} \frac{\left(SCR_{eq,1} + \rho_{eq} SCR_{eq,2}\right)}{SCR_{Mkt,II}} \Delta_{eq,1} \mathrm{w}_{eq,1} \\ & + mSCR_{II}^{Mkt} \frac{\left(SCR_{eq,2} + \rho_{eq} SCR_{eq,1}\right)}{SCR_{Mkt,II}} \Delta_{eq,2} \left(1 - \mathrm{w}_{eq,1}\right) \qquad (16) \\ & + mSCR_V^{Mkt} \Delta_{cur} f_{eq,1} \mathrm{w}_{eq,1} + mSCR_V^{Mkt} \Delta_{cur} f_{eq,2} \left(1 - \mathrm{w}_{eq,1}\right) \end{aligned}$$

Finally, the marginal SCR for investments in property ($A_5 = A_{prop}$) is:

$$mSCR_{A,5}^{Mkt} = mSCR_{III}^{Mkt} \Delta_{prop} + mSCR_V^{Mkt} \Delta_{cur} f_5 \qquad (17)$$

Please note that the marginal SCR for non-market assets (A_{other}) and cash (A_{cash}) is zero when considering the SCR for market risk, as these assets are charged in the counter-party risk module.

4.3. Marginal risk measures for the liabilities and contributions to risk

We can similarly derive the marginal risk of the liabilities:

$$mSCR_{L,1}^{Mkt} = \frac{\partial SCR_{Market}}{\partial L_{tprov}} = \frac{\partial SCR_{Market}}{\partial SCR_{Mkt,I}} \frac{\partial SCR_{Mkt,I}}{\partial L_{tprov}} = mSCR_I^{Mkt} D_{L,1} \Delta_{rd} \qquad (18)$$

$$mSCR_{L,2}^{Mkt} = \frac{\partial SCR_{Market}}{\partial L_{other}} = mSCR_{I}^{Mkt} D_{L,2} \Delta_{rd} \tag{19}$$

We next define the marginal *contribution* of asset i to the SCR for market risk, denoted by $mcSCR_{A,i}^{Mkt}$, and the marginal *contribution* of liability *n*, denoted by $mcSCR_{L,n}^{Mkt}$, as

$$mcSCR_{A,i}^{Mkt} = \frac{A_i \times mSCR_{A,i}^{Mkt}}{SCR_{Market}} \tag{20}$$

$$mcSCR_{L,n}^{Mkt} = \frac{L_n \times mSCR_{L,n}^{Mkt}}{SCR_{Market}} \tag{21}$$

The marginal contributions sum up to one: $\sum_{i=1}^{I} mcSCR_{A,i}^{Mkt} + \sum_{n=1}^{N} mcSCR_{L,n}^{Mkt} = 1$. Hence, we can interpret $mcSCR_{A,i}^{Mkt}$ as the percentage of the capital requirement for market risk that can be attributed to investment *i*, after taking into account diversification effects. Similarly, $mc_n^L SCR$ is the percentage of the market risk SCR attributed to liability *n*. Please note that all assets *and* all liability types charged for market risk in the Solvency II standard formula need to be included in the summation for the weights to add up to one.

4.4. Expected returns and return on solvency capital

When analyzing the strategic asset allocation under Solvency II, insurers need to consider the expected return generated per unit of solvency capital required to cover market risk. We now define the *return on solvency capital* (*RoC*) to facilitate this trade-off. Let A and L denote the initial value of the assets and the liabilities, and $F(= A - L)$ the insurer's own funds. Further, let $\mu_{A,i}$ denote the expected return on asset *i* and $\mu_{L,n}$ the expected growth rate of liability *n*, both measured over a period of one year, the horizon of Solvency II. Then the expected increase in own funds ΔF is:

$$E[\Delta F] = E[\Delta A - \Delta L] = \sum_{i=1}^{I} \mu_{A,i} A_i - \sum_{n=1}^{N} \mu_{L,n} L_n \tag{22}$$

We then divide the expected increase in own funds by the required capital for market risk:

$$RoC_{Mkt} = \frac{E[\Delta F]}{SCR_{Market}} = \left(\sum_{i=1}^{I} \mu_{A,i} A_i - \sum_{n=1}^{N} \mu_{L,n} L_n\right) / SCR_{Market} \tag{23}$$

The return on capital, RoC_{Mkt}, measures the expected increase in own funds per unit of solvency capital for market risk SCR_{Market} required. The measure above can also be labelled the return on solvency capital, or the return on risk-adjusted capital (RoRAC), but we prefer RoC to keep the notation short. It is a risk-adjusted return measure, similar to the Sharpe ratio and information ratio.

To provide more insight into the risk-adjusted expected returns of the asset allocation, we define the marginal RoC of each asset class and liability type individually:

$$mRoC_{A,i}^{Mkt} = \frac{\partial RoC_{Mkt}}{\partial A_i} = \frac{\mu_{A,i} SCR_{Market} - \left(\sum_{i=1}^{I} \mu_{A,i} A_i - \sum_{n=1}^{N} \mu_{L,n} L_n\right) mSCR_{A,i}^{Mkt}}{(SCR_{Market})^2} = \frac{\mu_{A,i} - RoC_{Mkt} \times mSCR_{A,i}^{Mkt}}{SCR_{Market}} \tag{24}$$

$$mRoC_{L,n}^{Mkt} = \frac{\partial RoC_{Mkt}}{\partial L_n} = \frac{-\mu_{L,n} - RoC_{Mkt} \times mSCR_{L,n}^{Mkt}}{SCR_{Market}} \tag{25}$$

The marginal returns on capital sum up to zero when they are weighted by the asset and liability amounts: $\sum_{i=1}^{I}(A_i \times mRoC_{A,i}^{Mkt}) + \sum_{n=1}^{N}(L_n \times mRoC_{L,n}^{Mkt}) = 0$. To improve the overall return on solvency capital, the insurer should increase the allocation to assets with relatively high marginal return on solvency capital ($mRoC$), and decrease the amount invested in assets with low $mRoC$.

4.5. Marginal risk measures for total SCR

The marginal risk measures so far have been derived for the solvency capital for market risk, SCR_{Market}. The insurer's solvency requirement is determined by *total* SCR, which also includes capital charges for non-market risks. In the Solvency II standard formula, charges for market and non-market risks are aggregated into a total SCR using the square root formula in (10) and the correlation matrix in Table 2. The following measure estimates how the *total* solvency capital requirement (SCR_{Total}) increases after a 1 unit increase in the SCR for market risk (SCR_{Market}):

$$mSCR_{Mkt}^{Total} = \frac{\partial SCR_{Total}}{\partial SCR_{Market}} = \frac{\left(SCR_{Market} + \sum_{j=2}^{5} \rho_{Agg,1j} SCR_{Agg,j}\right)}{SCR_{Total}} \tag{26}$$

Similarly, we can calculate the marginal SCR for the non-market risk types: $mSCR_{Agg,j}^{Total}$, for *j* = 2, 3, 4, 5, referring to non-life underwriting risk, life risk, health risk and counter-party default risk.

We next define the marginal *contribution* to total SCR of risk type *j*, denoted by $mcSCR_j^{Total}$:

$$mcSCR_j^{Total} = \frac{SCR_{Agg,j} \times mSCR_{Agg,j}^{Total}}{SCR^{Total}} = \frac{\left(SCR_{Agg,j}^2 + \sum_{\substack{h=1 \\ h \neq j}}^{5} \rho_{Agg,hj} SCR_{Agg,h} SCR_{Agg,j}\right)}{(SCR_{Total})^2} \tag{27}$$

where we use the convention $SCR_{Agg,1} = SCR_{Market}$.

As before, the marginal contributions to total SCR sum up to one: $\sum_{j=1}^{J} mcSCR_j^{Total} = 1$. Hence, we can interpret $mcSCR_j^{Total}$ as the percentage of total SCR attributed to risk type *j*, after taking into account diversification effects, providing useful insights about the composition of the SCR.

Similarly, we can also assess how the *total* SCR changes when the amount invested in an asset class, A_i , increases by 1 unit (e.g., 1 million euro):

$$mSCR_{A,i}^{Total} = \frac{\partial SCR_{Total}}{\partial SCR_{Market}} \times \frac{\partial SCR_{Market}}{\partial A_i} = mSCR_{Mkt}^{Total} \times mSCR_{A,i}^{Mkt} \tag{28}$$

The $mSCR_{A,i}^{Total}$ measures the marginal change in total SCR for asset class *i*, after taking into account diversification effects with other market risks *and* non-market risks. In Appendix A we also define and derive the (marginal) return on capital for the total SCR.

4.6. Numerical marginals for internal models

Many insurance companies have developed an internal risk model to determine the Solvency II capital requirement, replacing the standard formula. Internal risk models need to be approved by the national regulator, after a thorough model validation process. One advantage of an internal risk model is that it can be tailor-made to reflect the risks taken by a specific insurance company. For example, if an insurer has large investments in infrastructure or mortgages, the risk of these investments can be modelled in great detail in an internal model. In addition, the development of an internal model tends to foster an increased understanding of risk within the organization.

A further advantage of using an internal model is that more advanced and more refined techniques for risk modelling can be applied. For example, in the standard formula different market risks are aggregated with a correlation matrix, see Equation (9), often referred to as the square root formula. This square root formula only correctly aggregates risks if the following assumptions both hold (see, Dhaene et al., 2005, Devineau and Loisel, 2009): (i) The distribution of the risk types is in the elliptical class. ii) The value of the insurer's own funds is a linear function of the risks.

Regarding assumption (i), the elliptical family of distributions includes the multi-variate normal distribution and the multi-variate Student's *t* distribution. Assumption (i) basically implies that the marginal distributions of the different risk types all have the same shape (e.g., Student's *t* with one particular value for the degrees of freedom). Assumption (ii) means that changes in the insurer's assets and liabilities need to be a linear function of the risk drivers, which is strongly violated in practice (Devineau and Loisel 2009, Kousaris, 2011a). For example, the market value of the liabilities is a complex non-linear function of interest rates, spreads and survival probabilities. Further, the asset side of the insurer's balance sheet can be a non-linear function of the risk drivers as well, especially when derivatives are used for hedging.

An alternative for the use of the standard formula is to fit different marginal distributions for each market risk driver (e.g., equity returns, property returns, credit spreads), reflecting its unique tail risk. The marginal distributions are then aggregated into a multi-variate distribution of market risk drivers with a copula, a function that links the distributions. Copula functions that allow for tail dependence are especially suited for Solvency II, e.g. the Student *t* copula or the Clayton copula, as they can model the tendency of negative extremes of risk drivers to occur *jointly* in worst case scenarios. The SCR for market risk typically cannot be determined analytically for an internal model consisting of different marginal distributions joined by a copula function. In those cases, the SCR is determined by Monte Carlo simulation, drawing thousands of multivariate returns from the statistical model for the risk drivers and determining the impact on the insurer's own funds in each scenario. For example, suppose a simulation has generated 10,000 scenarios for changes in own funds due to market risk, ranked ascending from losses to gains. The 50th loss scenario then gives the SCR for market risk, occurring with 0.5% probability (50/10,000).

Marginal risk measures such as *mSCR* can still be computed when using an internal model for market risk. If the internal model results in an analytical expression for the SCR (0.5% VaR), then the *mSCR* is simply the first derivative of this expression with respect to the amount invested in an asset, similar to Equation (13). If the internal model uses numerical techniques such as Monte Carlo simulation, then the *mSCR* can be calculated as a numerical derivative:

$$mSCR_{A,i}^{Mkt} = \frac{SCR_{Market}(\boldsymbol{a} + \Delta_A \boldsymbol{e_i}; L_1, L_2) - SCR_{Market}(\boldsymbol{a} - \Delta_A \boldsymbol{e_i}; L_1, L_2)}{2\Delta_A} \tag{29}$$

where Δ_A is a relatively small increase in the amount invested (e.g., 1 million EUR), $\boldsymbol{e_i}$ is a unit column vector of length I, equal to 1 in row *i* and 0 elsewhere. The expression above evaluates the market SCR after a small increase in the amount invested in asset i, $A_i + \Delta_A$, and after a small decrease, $A_i - \Delta_A$. The difference in the two resulting SCR's divided by $2\Delta_A$ is the marginal SCR of asset *i*.[3]

[3] The numerical calculations for the *mSCR*'s do not have to be time-consuming, if they are done at the same time as the Monte Carlo simulation for determining the total SCR. The simulation software needs to keep track of the distribution of the insurer's own funds, not just for the current asset allocation $\boldsymbol{a}$, but also for a set of "disturbed" allocations where the amounts invested in the asset classes are shifted up and down by Δ_A.

5. Optimal Asset Allocation

As part of the risk budgeting process, we can derive the "optimal" asset allocation given the insurer's expected asset returns, the insurer's liabilities and the standard formula for market risk capital under Solvency II. We define an asset allocation as optimal when it maximizes the expected return on the insurer's own funds, subject to a given upper limit on the solvency capital requirement for market risk.

We will assume that insurers can take a short position in EEA Treasury bills, with duration close to zero. In the Solvency II standard formula these short-term government bonds are effectively riskless, receiving a negligible charge for interest rate risk and no charge for credit risk. In practice insurers can hedge the interest rate risk of the liabilities well by entering swap contracts where the insurance company pays a floating rate and receives a fixed rate. Such a swap position is similar to shorting Treasury bills and investing the proceeds in long-dated government bonds.

Thus, in the analysis below we assume that the insurer invests an amount by A_0 in a riskless asset with expected return r_f, such that the budget constraint is: $A = A_0 + \sum_{i=1}^{I} A_i$. Further, the riskless asset can also be shorted ($A_0 < 0$).

5.1. The objective and constraints

The objective function is to maximize the expected change in own funds ΔF subject to an upper limit of SCR_{Market}^{Max} on the solvency capital for market risk $SCR_{Market}(\boldsymbol{a}; L_1, L_2)$:

$$\begin{aligned} \max_{\boldsymbol{a}} E[\Delta F(\boldsymbol{a})] &= r_f A_0 + \sum\nolimits_{i=1}^{I} \mu_{A,i} A_i - \sum\nolimits_{n=1}^{N} \mu_{L,n} L_n \\ &= r_f A + \sum\nolimits_{i=1}^{I} (\mu_{A,i} - r_f) A_i - \sum\nolimits_{n=1}^{N} \mu_{L,n} L_n \end{aligned} \quad (30)$$

$$\text{s.t.} \qquad SCR_{Market}(\boldsymbol{a}; L_1, L_2) \leq SCR_{Market}^{Max}$$

where $\boldsymbol{a} = (A_1, A_2, \cdots, A_I)'$ is a column vector of length I containing the risky asset amounts.

We assume that the insurance liabilities have longer duration than the assets, such that the downward shock to the term-structure determines the capital requirement for interest rate risk:

$$SCR_{Mkt,I} = \Delta_{rd}\big((D_{L,1}L_1 + D_{L,2}L_2) - (D_{A,1}A_1 + D_{A,2}A_2 + D_{A,3}A_3)\big) \quad (31)$$

For the insurer's equity investments, we assume that the weights of developed equity and other equity are fixed at $\mathrm{w}_{eq,1}$ and $1 - \mathrm{w}_{eq,1}$. We then treat equity as a single asset class with invested amount A_4, and solvency shock: $\Delta_{eq} = \mathrm{w}_{eq,1}\Delta_{eq,1} + (1 - \mathrm{w}_{eq,1})\Delta_{eq,2}$. Under these assumptions, the capital requirements for the market risk types, $\boldsymbol{s} = (SCR_{Mkt,I}, SCR_{Mkt,II}, \dots, SCR_{Mkt,K})'$, are a linear function of the asset allocation amounts in the vector $\boldsymbol{a}$:

$$\mathbf{s} = \mathbf{V}\boldsymbol{a} + \boldsymbol{c_L} \tag{32}$$

where $\mathbf{s} = (SCR_{Mkt,I}, SCR_{Mkt,II}, \dots, SCR_{Mkt,K})'$ is a $Kx1$ vector holding the SCR's for the market risk types, and $\mathbf{V}$ is a KxI matrix containing the asset shock parameters, and $\boldsymbol{c_L}$ is a $Kx1$ vector with the liability shock amounts defined as: $\boldsymbol{c_L} = (\Delta_{rd}(D_{L,1}L_1 + D_{L,2}L_2), 0, \dots, 0)'$.

Let $\mathbf{v}(k)$ denote row k of the coefficient matrix $\mathbf{V}$, then the expressions below define the contents of the KxI matrix $\mathbf{V}$.

$$\begin{aligned}
\mathbf{v}_{\boldsymbol{A}}(1) &= (-D_{A,1}\Delta_{rd}, -D_{A,2}\Delta_{rd}, -D_{A,3}\Delta_{rd}, 0, 0) \\
\mathbf{v}_{\boldsymbol{A}}(2) &= (0, 0, 0, \Delta_{eq}, 0) \\
\mathbf{v}_{\boldsymbol{A}}(3) &= (0, 0,0, 0, \Delta_{prop}) \\
\mathbf{v}_{\boldsymbol{A}}(4) &= (0, \Delta_{gov,2}, \Delta_{corp}, 0, 0) \\
\mathbf{v}_{\boldsymbol{A}}(5) &= (\Delta_{cur}f_1, \Delta_{cur}f_2, \Delta_{cur}f_3, \Delta_{cur}f_4, \Delta_{cur}f_5)
\end{aligned} \tag{33}$$

Above we assume that the insurer's investments track broadly diversified benchmarks for stocks, corporate bonds and listed property, such that exposure to a single issuer remains below the threshold and the charge for concentration risk is zero ($SCR_{Mkt,VI}$= 0).

5.2. The optimal asset allocation with liabilities

We now derive the optimal asset allocation, while explicitly taking into account the liabilities of the insurer (L_1, L_2) and their impact on the solvency capital for market risk, $SCR_{Market}(\boldsymbol{a}; L_1, L_2)$. We can write the optimal asset allocation problem as follows:

$$\begin{aligned}
&\max_{\boldsymbol{a}} E[\Delta F(\boldsymbol{a})] = r_f A + \boldsymbol{\mu_A}'\boldsymbol{a} - \mu_{L,1}L_1 - \mu_{L,2}L_2 \\
&\text{s.t.} \qquad ((\mathbf{V}\boldsymbol{a} + \boldsymbol{c_L})'\mathbf{R}(\mathbf{V}\boldsymbol{a} + \boldsymbol{c_L}))^{½} \leq SCR_{Market}^{Max}
\end{aligned} \tag{34}$$

where $\boldsymbol{\mu_A} = (\mu_{A,1} - r_f, \mu_{A,2} - r_f, \cdots, \mu_{A,I} - r_f)'$ is a column vector of length I with the expected *excess* asset returns, relative to the risk-free rate. The first order condition for optimality is:

$$\boldsymbol{\mu_A} - \lambda \frac{\mathbf{V}'\mathbf{R}(\mathbf{Va} + \boldsymbol{c_L})}{SCR_{Market}} = \mathbf{0}, \qquad \text{for } \lambda \geq 0 \tag{35}$$

We solve for the optimal asset allocation $\boldsymbol{a}^*$, assuming that the matrix $\mathbf{V}'\mathbf{R}\mathbf{V}$ is invertible:

$$\begin{aligned}
\boldsymbol{a}^* &= \frac{SCR_{Market}^{Max}}{\lambda}(\mathbf{V}'\mathbf{R}\mathbf{V})^{-1}\boldsymbol{\mu_A} - (\mathbf{V}'\mathbf{R}\mathbf{V})^{-1}\mathbf{V}'\mathbf{R}\boldsymbol{c_L} \\
&= \frac{SCR_{Market}^{Max}}{\lambda}(\mathbf{V}'\mathbf{R}\mathbf{V})^{-1}\boldsymbol{\mu_A} - \mathbf{V}^{-1}\boldsymbol{c_L}
\end{aligned} \tag{36}$$

using $(\mathbf{V}'\mathbf{R}\mathbf{V})^{-1}\mathbf{V}'\mathbf{R} = (\mathbf{V}^{-1}(\mathbf{V}'\mathbf{R})^{-1})\mathbf{V}'\mathbf{R} = \mathbf{V}^{-1}$.

We then solve for lambda, assuming that the solvency constraint is binding:

$$
\begin{aligned}
{SCR_{Market}^{Max}}^{2} &= (\mathbf{a}^{*\prime}\mathbf{V}' + \boldsymbol{c}_L')\mathbf{R}(\mathbf{V}\mathbf{a}^* + \boldsymbol{c}_L) = \\
&\left(\frac{SCR_{Market}^{Max}}{\lambda}\boldsymbol{\mu_A}'(\mathbf{V}'\mathbf{R}\mathbf{V})^{-\mathbf{1}}\mathbf{V}' - \boldsymbol{c}_L'(\mathbf{V}^{-\mathbf{1}})'\mathbf{V}' + \boldsymbol{c_L}'\right)\mathbf{R}\left(\frac{SCR_{Market}^{Max}}{\lambda}\mathbf{V}(\mathbf{V}'\mathbf{R}\mathbf{V})^{-\mathbf{1}}\boldsymbol{\mu_A} - \boldsymbol{c_L} + \boldsymbol{c_L}\right) \\
&= \left(\frac{SCR_{Market}^{Max}}{\lambda}\boldsymbol{\mu_A}'(\mathbf{V}'\mathbf{R}\mathbf{V})^{-\mathbf{1}}\mathbf{V}'\mathbf{R}\right)\left(\frac{SCR_{Market}^{Max}}{\lambda}\mathbf{V}(\mathbf{V}'\mathbf{R}\mathbf{V})^{-\mathbf{1}}\boldsymbol{\mu_A}\right) \\
&= \frac{{SCR_{Market}^{Max}}^{2}}{\lambda^2}\boldsymbol{\mu_A}'(\mathbf{V}'\mathbf{R}\mathbf{V})^{-\mathbf{1}}\boldsymbol{\mu_A}
\end{aligned}
$$

Thus, $\lambda = \sqrt{\boldsymbol{\mu_A}'(\mathbf{V_A}'\mathbf{R}\mathbf{V_A})^{-\mathbf{1}}\boldsymbol{\mu_A}}$. In Appendix B we show that this expression is equal to the return on capital of an optimal allocation for the "assets-only" case, $RoC_{NoLiab}^* = \sqrt{\boldsymbol{\mu_A}'(\mathbf{V_A}'\mathbf{R}\mathbf{V_A})^{-\mathbf{1}}\boldsymbol{\mu_A}}$, when the insurance liabilities are equal to zero.

We can now write the optimal asset allocation as the sum of two portfolios:

$$
\boldsymbol{a}^* = \left(\frac{SCR_{Market}^{Max}}{RoC_{NoLiab}^*}\right)(\mathbf{V}'\mathbf{R}\mathbf{V})^{-\mathbf{1}}\boldsymbol{\mu_A} - \mathbf{V}^{-\mathbf{1}}\boldsymbol{c_L} = \boldsymbol{a}_{\boldsymbol{NoLiab}}^* + \boldsymbol{a}_{\boldsymbol{Hedge}}^* \qquad (37)
$$

The first component, $\boldsymbol{a}_{\boldsymbol{NoLiab}}^*$, is the optimal portfolio for the "asset-only" investment problem without liabilities ($L_1 = 0, L_2 = 0$) and with market risk limit SCR_{Market}^{Max}, see Appendix B for the derivation. The relative weights of the risky assets in this portfolio are fixed; only the amounts invested in the riskless asset and the risky portfolio depend on the risk target SCR_{Market}^{Max}. The second component, $\boldsymbol{a}_{\boldsymbol{Hedge}}^* = -\mathbf{V}^{-\mathbf{1}}\boldsymbol{c_L}$ is a liability hedge portfolio that exactly offsets the capital charge for interest risk that arises from the liabilities. Hence, the optimal investment strategy is to hedge the interest rate risk of the liabilities (with $\boldsymbol{a}_{\boldsymbol{Hedge}}^*$) and then to invest in an efficient asset-only portfolio ($\boldsymbol{a}_{\boldsymbol{NoLiab}}^*$). Hence, we have a three fund separation result: all insurers invest in the riskless asset, the optimal portfolio of risky assets $\boldsymbol{a}_{\boldsymbol{NoLiab}}^*$ and a liability hedge portfolio $\boldsymbol{a}_{\boldsymbol{Hedge}}^*$.

We can see the role of the liability hedge portfolio more clearly by inspecting the capital charges for the K market risk types, before risk aggregation, denoted by $\mathbf{s}^*$:

$$
\mathbf{s}^* = \mathbf{V}\mathbf{a}^* + \boldsymbol{c_L} = \mathbf{V}\boldsymbol{a}_{\boldsymbol{NoLiab}}^* + \mathbf{V}\boldsymbol{a}_{\boldsymbol{Hedge}}^* + \boldsymbol{c_L} = \mathbf{V}\boldsymbol{a}_{\boldsymbol{NoLiab}}^* - \mathbf{V}\mathbf{V}^{-\mathbf{1}}\boldsymbol{c_L} + \boldsymbol{c_L} = \mathbf{V}\boldsymbol{a}_{\boldsymbol{NoLiab}}^* = \mathbf{s}_{\boldsymbol{NoLiab}}^*
$$

where $\boldsymbol{s}_{\boldsymbol{NoLiab}}^* = \mathbf{V}\boldsymbol{a}_{\boldsymbol{NoLiab}}^* = ({SCR_{Market}^{Max}}/{RoC_{NoLiab}^*})\mathbf{V}(\mathbf{V}'\mathbf{R}\mathbf{V})^{-\mathbf{1}}\boldsymbol{\mu_A}$ is the vector of capital charges for the market risk types of the optimal asset-only allocation, when the liabilities are zero.

The equation above shows that the effect of the interest rate shocks on the liabilities, captured in the vector $\boldsymbol{c_L}$, is exactly offset by the effect of these shocks on the liability hedge portfolio $\boldsymbol{a}_{\boldsymbol{Hedge}}^*$. Hence, the impact of the liabilities on the solvency capital requirement is completely neutralized by the liability hedge portfolio. In practice, this means that the liability hedge portfolio reduces the duration gap between the assets and the liabilities to zero. On top of that an optimal asset-only portfolio $\boldsymbol{a}_{NoLiab}^{*}$ is held, which may carry some interest-rate risk exposure itself, but only if this is an efficient way to generate a higher expected return on the assets.

The first-order conditions for the optimal asset allocation above can be rewritten as follows:

$$(\mu_{A,i} - r_f) - RoC_{NoLiab}^{*} \times mSCR_{A,i}^{Mkt} = 0, \text{ for } i = 1, 2, \dots, I \tag{38}$$

where RoC_{NoLiab}^{*} is the return on capital of an optimal portfolio for the "asset-only" investment problem without insurance liabilities ($L_1 = 0, L_2 = 0$). We note that the first-order condition is similar to requiring the *marginal RoC*'s of the assets to be equal to zero, but using the return on capital of the optimal asset-only portfolio (RoC_{NoLiab}^{*}) in the expression.

As alternative interpretation of the first-order condition is that the ratio of expected excess return to marginal risk (*mSCR*) of all the risky assets is equal, to the constant RoC_{NoLiab}^{*}:

$$\frac{(\mu_{i,A} - r_f)}{mSCR_{A,i}^{Mkt}} = \lambda = RoC_{NoLiab}^{*}, \qquad \text{for } i = 1, 2, \dots, I \tag{39}$$

5.3. Lessons about redundant assets and diversification

For the derivation of the optimal asset allocation we had to assume that the IxI matrix $\mathbf{V'RV}$ is invertible (non-singular). What does this assumption imply? Recall that i-th column of the KxI matrix $\mathbf{V}$ contains the charges for the K risk market types when a one dollar is invested in asset i. $\mathbf{R}$ is the KxK matrix with correlations between the K market risk types that is used to aggregate the capital requirements with the square root formula in Equation (9). The correlation matrix $\mathbf{R}$ as specified by the Solvency II standard formula, given in Table 1, is invertible. It follows that $\mathbf{V'RV}$ is invertible (positive definite) if and only if the KxI matrix $\mathbf{V}$ is of full rank.

Practically speaking the full rank condition for $\mathbf{V}$ means:
1. There can be no more than K risky assets in the optimization: $I \leq K$.
2. For each type of market risk $k = 1, 2, \dots, K$ in the standard formula, there should be at least one risky assets that has exposure to risk type k.
3. The exposures of the risky assets to the market risk types contained in $\mathbf{V}$ should not be linearly dependent (i.e., the columns of $\mathbf{V}$ should be independent).

We now provide some examples where these conditions are violated to gain useful insights. The standard formula for market has five sources of risk (ignoring concentration risk): interest rate risk, equity risk, property risk, credit spread risk and currency risk. Hence, the optimization can only have a maximum of I = 5 asset classes. If there would be more than five assets in the optimization, say I = 6, at least one asset would have exactly the same risk exposure as a linear combination of the other assets. This means that one of the six assets is redundant. Suppose this redundant asset has a relatively low expected return relative to the other assets. In a setting with no-short selling restrictions this would create an arbitrage opportunity, in the sense that the portfolio expected return can be increased unlimited without increasing the solvency capital for market risk.[4] In a setting where short-selling is not allowed, no money would be invested in the redundant asset.

Now suppose that there are only $I = 5$ asset classes, but none of these assets has currency risk, because the insurer hedges all foreign currency exposure. In this case there are only four sources of market risk that determine the solvency requirement, and hence there can only be four or less assets classes in the portfolio choice problem without creating a redundant asset.

It is also important to note that the Solvency II standard formula for market risk is a simplified model that does not distinguish the risk of assets *within* the same market risk type k well. For example, suppose that the insurer can choose between two portfolios of developed equity:

1. The MSCI World: a well-diversified portfolio of 1,637 stocks spread over 23 developed markets. Suppose the expected return is 7% per annum. All currency risk is hedged.

2. The MSCI Belgium: a portfolio of 10 stocks from one developed country, Belgium. Suppose the expected return is 8% per annum. Any currency risk is hedged.

Judged from the perspective of the Solvency II standard formula for market risk, the two portfolios have exactly the same risk exposure and risk charge: 39% of the amount invested. Hence, seen in the context of optimization problem (30), the MSCI Belgium portfolio is superior, offering a higher expected return for the same level of risk, while the MSCI World is an inferior (redundant) asset. In an optimal asset allocation with short-selling constraints no money would be invested in the MSCI World, and all investments in developed market equity would end up in the MSCI Belgium. When short-selling is allowed, an "arbitrage opportunity" would arise. Clearly, the Solvency II standard formula is of no use when considering which developed stock markets to invest in, as it ignores differences in risk between these two equity portfolios.[5]

Hence, the conclusion is that the insurer should not consider the Solvency II standard formula for market risk when selecting specific assets *within* each risk type, such as particular stocks or bonds. Basically, the standard formula ignores the potential for the diversification of assets *within* market risk types. Insurers need to develop an internal risk model to better distinguish the risk of individual assets *within* each risk type, and to consider diversification potential *within* each risk type. In the absence of such a model, only well-diversified portfolios should be considered for getting exposure to the market risk types of the standard formula. For example, the MSCI World for developed equity, a broadly diversified corporate bond portfolio for spread risk, etc. After that, the standard formula can be applied for risk aggregation and risk allocation *between* the risk types.

5.4. Extensions

Several simplifying assumptions had to be made for the analytical solution of the asset-liability management problem in Section 5.2. Modelling an actual insurance company with all its details and investment constraints would require numerical solution techniques. Such numerical solutions

[4] Short the redundant asset and long the replicating asset with the same risk exposure but higher expected return.
[5] In the 10-year period Oct-2006 to Sept-2016, the return standard deviation of MSCI World was 16.5%, versus 24.4% for MSCI Belgium. The standard deviation estimates are based on monthly price returns (source MSCI).

typically do not provide intuition or insights about why a particular asset allocation is optimal, and for this reason we believe it is worthwhile to first study the main principles of asset liability management under some simplifying assumptions. Below and in the appendices we indicate how our framework can be extended when some of these assumptions are dropped.

For example, our asset allocation framework imposes a limit on the SCR for *market* risk. The rationale is that exposures to non-market risks, such as life insurance underwriting risk, are typically fixed, or quite difficult to change in the short run. However, some insurers may want to explicitly impose an upper limit SCR_{Total}^{Max} on the *total* SCR, while optimizing their asset allocation. In Appendix D we discuss this extension.

In Section 5.2 we assumed that the duration of the assets is longer than the liabilities, such that the "curve down" scenarios determines the capital requirement for interest rate risk. Further, we did not rule out negative values for the asset amounts $\boldsymbol{a}$ and the capital charges $\boldsymbol{s}$. In Appendix C we provide a formulation of the optimization problem that includes non-negativity constraints on the amounts invested ($\boldsymbol{a} \geq 0$) and the charges for the risk types ($\boldsymbol{s} \geq 0$), as well as two inequality constraints to make sure that the SCR for interest rate risk is the maximum of the "curve up" and "curve down" scenarios. The resulting quadratic optimization problem with linear inequality constraints cannot be solved analytically, but a numerical solution is easily found with modern solution algorithms (e.g., interior point methods). Below we provide some intuition regarding the trade-off between return and marginal risk when such constraints are imposed.

5.4.1. First-order conditions with no borrowing and no short-selling constraints

In the asset allocation examples we have assumed that insurers can take a short position in a risk-less asset (e.g., a EEA Treasury bill). In practice this can also be implemented with a swap contract where the insurance company pays floating and receives a fixed rate, useful to hedge the high interest rate sensitivity of the insurance liabilities. Some insurers, however, completely avoid the use of derivatives and leverage for accounting reasons. Instead, such insurers typically buy a long-only asset portfolio that closely matches the cash flows of the liabilities.[6]

A cash flow matching approach can be mimicked by pre-selecting a portfolio of long-term EEA government bonds that has longer duration (i.e., interest rate sensitivity) than the liabilities. The insurance liabilities can then be fully hedged by investing a fraction (< 1) of the total market risk assets in this liability hedge portfolio. The remainder of the assets are then invested in asset classes with higher expected returns, such as corporate bonds and equity.

In our framework the use of leverage can be ruled out by imposing the constraint that the portfolio weights of the risky assets add up to one: $\sum_{i=1}^{I} A_i = \boldsymbol{a}'\boldsymbol{\iota} = A$, where A is the initial amount of market risk assets. Insurers that want to impose a no-borrowing constraint most likely also want to ensure that all risky asset amounts are non-negative, $\boldsymbol{a} \geq \boldsymbol{0}$. The first order conditions for optimality of the asset allocation then are as follows:

[6] One motivation is that derivatives can lead to volatility of the reported profits under IFRS accounting standards. Even though hedging the liabilities with derivatives helps to reduce economic risks, the effect of the hedge is not recognized in IFRS accounts. Another motivation is that the use of derivatives and leverage exposes a firm to additional operational risk: an incorrect or inaccurate implementation of the hedge.

$$\frac{(\mu_{i,A} - \kappa)}{mSCR_{A,i}^{Mkt}} = \lambda, \quad \text{for all } i \text{ with } A_i^* > 0 \tag{40}$$

$$\frac{(\mu_{i,A} - \kappa)}{mSCR_{A,i}^{Mkt}} \leq \lambda, \quad \text{for all } i \text{ with } A_i^* = 0 \tag{41}$$

where κ is a Lagrange multiplier for the equality constraint $\boldsymbol{a}'\boldsymbol{\iota} = A$, and the vector of expected returns is now defined as $\boldsymbol{\mu_A} = (\mu_{A,1}, \mu_{A,2}, \cdots, \mu_{A,I})'$.

Please note that the expected asset returns in (40) are no longer measured in "excess" of the risk-free rate r_f, but rather in excess of κ, a Lagrange multiplier for the constraint $\boldsymbol{a}'\boldsymbol{\iota} = A$. Hence, the trade-off between expected return and marginal risk becomes different, and more difficult to assess when the exact value of κ is not known (i.e., not explicitly solved for). We refer to Herold (2005) for an excellent exposition about how κ and λ can be set, and how implied expected returns can be calculated, in the presence of a constraint that the risky asset weights sum up to one. In Herold (2005) the values of κ and λ are selected by prespecifying the risk premia for two asset classes.

The second condition for optimality, the inequality (41), shows that asset classes with a zero weight in the optimal solution ($A_i^* = 0$) tend to offer a trade-off between return and marginal risk that is inferior compared to the other asset classes. Later on in Section 6.5 we will see this property at work in the numerical examples for a representative life insurer: non-EEA government bonds and covered bonds then receive zero weights in the optimal asset allocation, because they offer return to maginal risk ratios that are worse than the other asset classes.

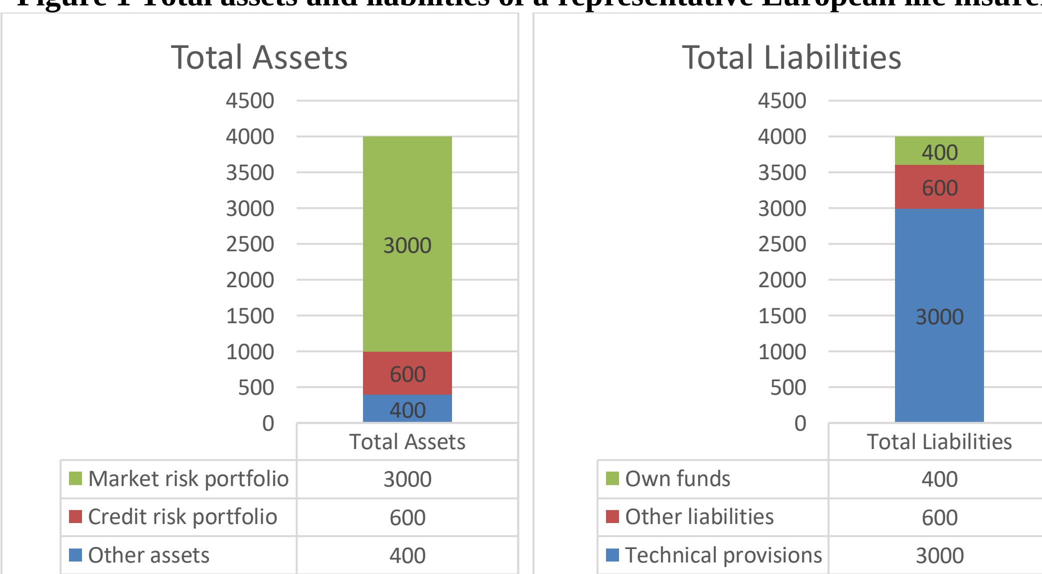
6. Asset Allocation Examples

We will now illustrate how the analytical framework for asset allocation and risk budgeting can be applied for a representative, but fictitious, European life insurance company, using numerical examples inspired by real life insurance companies. Höring (2013) has created the balance sheet and investment portfolio of a typical European life insurer using annual reports of individual insurance companies, the results of the fifth quantitative impact study for Solvency II (QIS5), and information from regulators and credit rating agencies. We will use Höring's representative life insurer for our numerical illustrations of the risk budgeting framework in this section.

6.1. Balance sheet and initial asset allocation

The total liabilities of the insurer, shown in Figure 1 below in the right panel, mainly consist of the technical provisions of EUR 3.0 billion, representing the value of the insurance liabilities. Under Solvency II the liability cash flows have to be discounted with interest rates from the swap market, plus certain spreads. As a result, fluctuations in the value of the liabilities due to changes in swap interest rates are a major source of risk. In addition, the insurer has EUR 600 million in other liabilities, such as short-term debt and deferred tax liabilities. Finally, the insurer has EUR 400 million of own funds. The own funds are crucial for determining the insurer's ability to absorb losses and to assess its solvency.

Figure 1 Total assets and liabilities of a representative European life insurer



Note: the figure shows the balance sheet of a representative European life insurer (source: Höring, 2013). All amounts are in millions of euro.

The insurer has EUR 4.0 billion of total assets, as shown in the left panel of Figure 1. The assets consist of a portfolio of market risk assets worth 3.0 billion, an illiquid credit risk portfolio of 0.6 billion and 0.4 billion in other assets. Market risk assets, such as equity, bonds and property, are charged in the market risk module of Solvency II. The credit risk portfolio of EUR 600 million contains all assets charged in the counter-party risk module of Solvency II, such as mortgages, policy loans, reinsurance assets and cash held at banks. Finally, EUR 400 million is in 'other assets', such as intangibles, goodwill and deferred tax assets.

The insurer's initial asset allocation of the market risk assets is shown in Figure 2 and Table 3 below (for further details, see Höring 2013). Sovereign debt (40%), corporate debt (29%) and covered bonds (13%) together make up 82% of the portfolio. Apart from fixed income securities, there are small fractions invested in equity (7%) and real estate (11%).

Figure 2 Allocation of market risk assets

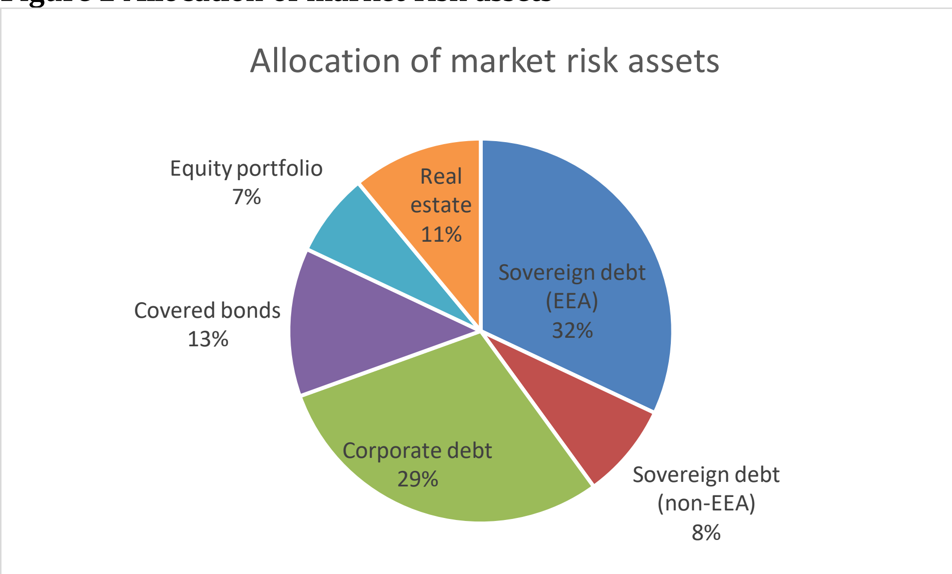


Table 3 Market risk assets portfolio

	Weight	Value	Duration	DV01	E[r]
Market risk assets					
Sovereign debt (EEA)	32.0%	960	6.9	0.66	1.50%
Sovereign debt (non-EEA)	8.0%	240	6.9	0.17	1.75%
Corporate debt	29.5%	885	5.4	0.48	2.40%
Covered bonds	12.5%	375	6.2	0.23	1.75%
Global equities	4.5%	135	0	0	4.50%
Other equities	2.5%	75	0	0	5.50%
Real estate	11.0%	330	0	0	3.50%
Total	100.0%	3,000	5.1	1.54	2.27%

Notes: Value of the assets and DV01 are measured in millions of euro. DV01 is the dollar duration, the expected change in value when the yield increases by one basis point (0.01%). E[r] denotes the expected return.

The last column of Table 3 shows the expected returns for the market risk assets, to be used in the asset allocation examples. These expected returns reflect a world in which interest rates are positive, but the expected returns of risky assets are relatively low. In addition, we use a short-term EEA government bond as the risk-free asset (e.g., a 3-month German Treasury bill), with an annualized expected return of 0.25%.

Table 4 summarizes the insurer's balance sheet, including the interest rate sensitivity of the assets and liabilities, measured with duration (in years). The assets have a duration of 4.6 years, implying that a downward shift of the yield curve with one basis point would lead to an increase in the value of the assets by EUR 1.83 million (DV01). The liabilities have a duration of 5.1 years, meaning that the liabilities increase in value by EUR 2.67 million for a one basis point downward shift of the yield curve. Overall the insurer has a duration gap between the liabilities and the assets of 2.1 years. As a result, a one basis point downward shift of the curve leads to an expected loss of EUR 0.84 million in own funds.

For the insurance liabilities we assume a growth rate of 3% per year. For the other assets and other liabilities we assume that they grow at the risk-free rate of 0.25%, for ease of exposition. Given these expected returns and the initial asset allocation in Table 3, the own funds of the representative European life insurer are expected to decrease by EUR 1.3 million in one year time. Cleary, the current asset allocation is inadequate, as the expected return on own funds is negative (-0.34%), while the insurer is exposed to high interest rate risk.

Table 4 Total assets and liabilities

	Weight	Value	Duration	DV01	E[*r*]
Total assets					
Total market risk portfolio	75.0%	3,000	5.1	1.54	2.27%
Credit risk portfolio	15.0%	600	4.9	0.29	3.50%
Other assets	10.0%	400	0	0	0.25%
Total	100.0%	4,000	4.6	1.83	2.25%
Total liabilities					
Technical provisions	75.0%	3,000	8.9	2.67	3.00%
Other liabilities	15.0%	600	0	0	0.25%
Own funds	10.0%	400	0	0	-0.34%
Total	100.0%	4,000	6.7	2.67	2.25%

Notes: Value of the assets and DV01 are measured in millions of euro. DV01 is the dollar duration, the expected change in value when the yield increases by one basis point (0.01%). E[*r*] denotes the expected return.

6.2. Initial SCR and marginal analysis

Table 5 shows the capital requirement for market risk (SCR_{Market}) based on the Solvency II standard formula, equal to EUR 297 million after taking into account diversification benefits of EUR 64 million. We refer to Höring (2013) for more details about the calculation of SCR for this example.[7] The last two columns of Table 5 show the marginal risk (*mSCR*) and the marginal contributions to risk (*mcSCR*) of the five main market risk types. For example, an increase of the SCR for interest rate risk by 1 million euro raises the aggregate SCR for market risk by only EUR 0.8 million, illustrating the effect of diversification among the risk types.

Equity risk has relatively low diversification benefits with a marginal SCR of 0.87, reflecting the relatively high correlation of equity with other risks in the standard formula (see Table 1). Currency risk has relatively high diversification benefits, with a marginal SCR of only 0.3, but it is questionable if currency risk can earn a positive expected return. The marginal *contributions* to risk (*mcSCR*) in the last column show that interest rate risk accounts for the biggest share of the SCR for market risk at 30%, followed by spread risk (28%), property risk (22%) and finally equity risk (19%). It is interesting to note that equity has a weight of only 7% in the asset allocation of the insurer, but it accounts for 19% of the capital requirement for market risk.

Table 6 show an analysis of the initial asset allocation and solvency position of the representative European life insurer using marginals. The column denoted *mSCR* shows the increase in SCR for market risk when the investment in an asset is increased by EUR 1 million, funded by risk-free assets (short T-bills). Equity has the highest marginal SCR (0.27), followed by real estate (0.20) and corporate debt (0.02). EEA government bonds have a negative marginal SCR of -0.07, implying that increasing the allocation to bonds will *reduce* risk. This is the result of the duration gap between the liabilities and the assets: increasing the duration of the assets by investing in government bonds helps to reduce the mismatch. For the same reason, non-EEA government bonds and covered bonds also have negative marginal SCR's.

The column denoted "adj. *mcSCR*" shows the percentage contribution to the SCR of each asset class, after netting out all interest rate risk and assigning it to the insurance liabilities. If we apply the original *mcSCR* formulas form Section 4.2 and 4.3, the technical provisions have a *mcSCR* of 96%, while government bonds have a *mcSCR* of -24%. Because assets with positive interest rate sensitivity (duration) help to reduce the SCR, their contribution to risk can become negative. To make the *mcSCR* numbers more meaningful, in the column "Adj. *mcSCR*" in Table 6 we assign the entire net interest rate risk exposure to the liabilities (30%), while for the assets we show the risk contributions for other sources of risk (i.e., excluding interest rate risk).

An analysis of the adjusted *mcSCR* figures reveals that the net interest rate risk exposure contributes 30% to the SCR of EUR 297.4 million, followed by real estate risk (22%), the credit spread exposure of corporate bonds (22%), equity risk (19%), and the credit spread risk of covered bonds (4%) and non-EEA government bonds (2%). In the next section we eliminate the biggest source of risk, namely interest rate risk, with a liability hedge.

[7] For developed equity and other equity we use shocks of 30% and 40%, respectively, to make the results directly comparable Höring (2013), who used these parameter values from QIS5. In 2016 these shocks are 39% and 49%.

Table 5 Solvency capital requirement for market risk

Market risk category	SCR	Sub-SCR	Shock	Value	*mSCR*	*mcSCR*
1. Interest rate risk	112.2				0.80	30%
Market risk assets		-205.5	-6.8%	3,000		
Credit risk portfolio		-38.9	-6.5%	600		
Technical provisions		356.6	11.9%	3,000		
2. Equity risk	66.1				0.87	19%
Developed equity		40.5	30.0%	135		
Other equity		30.0	40.0%	75		
Gross equity risk		70.5				
Diversification benefits		-4.4				
3. Property risk	82.5		25.0%	330	0.80	22%
4. Credit spread risk	100.9				0.83	28%
Gov. debt (non-EEA)		6.0	2.5%	240		
Corporate debt		79.2	9.0%	885		
Covered bonds		15.7	4.2%	375		
5. Currency risk	0.0		25.0%	0	0.30	0%
Gross SCR for market risk	361.7					
Diversification benefits	-64.3					
Total SCR for market risk	297.4					

Notes: SCR, Value and *mSCR* are denoted in millions of euro. SCR is the solvency capital requirement for market risk, determined with the Solvency II standard formula. The column "Shock" shows the percentage decrease in the value of the investment ("Value") prescribed by the Solvency II formula to calculate the SCR. For developed equity and other equity we use shocks of 30% and 40%, respectively, to make the results comparable to Höring (2013). The column "*mSCR*" shows the marginal SCR: the increase in the SCR for market risk when the SCR for a particular risk type increases by 1 unit (1 million euro). The column "*mcSCR*" shows the marginal *contribution* to the SCR for market risk in %, which add up to 100% when summed over all market risk types.

Table 6 Marginal analysis of the initial asset allocation

Market risk portfolio	Value	Weight	E[*r*]	*mSCR*	Adj. *mcSCR*	*mRoC* x 1%	*mRoC** x 1%	E[r_e] /*mSCR*
Equity portfolio	210	7.0%	4.86%	0.27	19%	0.64%	0.01%	0.168
Real estate	330	11.0%	3.50%	0.20	22%	0.45%	-0.01%	0.162
Gov bonds EEA	960	32.0%	1.50%	-0.07	0%	0.16%	0.33%	-0.170
Gov bonds non-EEA	240	8.0%	1.75%	-0.05	2%	0.20%	0.32%	-0.285
Corporate debt	885	29.5%	2.40%	0.02	22%	0.29%	0.25%	1.256
Covered bonds	375	12.5%	1.75%	-0.03	4%	0.20%	0.27%	-0.484
Treasury bills EEA	0	0.0%	0.25%	0.00	0%	0.00%	0.00%	---
Total portfolio	3,000	100.0%	2.27%	0.01	70%	0.27%	0.24%	1.370

Assets	Value	Weight	E[*r*]	*mSCR*	Adj. *mcSCR*	*mRoC* x 1%	*mRoC** x 1%	E[r_e] /*mSCR*
Market risk portfolio	3,000	75.0%	2.27%	0.01	70%	0.27%	0.24%	1.370
Credit risk portfolio	600	15.0%	3.50%	-0.05	0%	0.43%	0.55%	-0.628
Other assets	400	10.0%	0.25%	0.00	0%	0.00%	0.00%	---
Total assets	4,000	100.0%	2.25%	0.00	70%	0.27%	0.26%	6.052

Liabilities	Value	Weight	E[*r*]	*mSCR*	Adj. *mcSCR*	*mRoC* x 1%	*mRoC** x 1%	E[r_e] /*mSCR*
Technical provisions	3,000	75.0%	3.00%	0.09	30%	-0.36%	-0.61%	0.290
Other liabilities	600	15.0%	0.25%	0.00	0%	0.00%	-0.03%	---
Own funds	400	10.0%	-0.34%	0.00	70%	0.27%	0.26%	6.052
Total liabilities	4,000	100.0%	2.25%	0.07	100%	-0.25%	-0.44%	0.281

Return/risk trade-off	
E[Increase own funds]	-1.3
SCR market risk	297.4
Solvency ratio f_M	135%
Return on SCR (*RoC*)	-0.5%

Dollar duration (DV01)	
Assets	1.83
Liabilities	2.67
Gap	0.84

Leverage of assets	
Long	4,000
Short	0
Leverage	1.0

Notes: "SCR market risk" is the solvency capital requirement for market risk, determined with the standard formula. The solvency ratio f_M is the ratio of the insurer's own funds over the SCR for market risk. The column *mSCR* shows the *marginal* SCR: the increase in the SCR for market risk when the value of a particular asset or liability increases by 1 unit (1 million euro). The column "Adj. *mcSCR*" shows the marginal *contribution* (in %) to the SCR for market risk of an asset or liability, after netting out the *mcSCR* for interest rate risk and assigning it to the technical provisions. The "Return on SCR" (*RoC*) is the expected increase in the insurer's own funds divided by the SCR for market risk. The column "*mRoC* x 1%" shows the *marginal RoC*: the expected change in the return on solvency capital when the weight of a particular asset or liability is increased by 1%. The column "*mRoC** x 1%" shows the marginal *RoC*, but evaluated at the RoC^*_{NoLiab} of an optimal *asset-only* portfolio (i.e., without liabilities). "E[r_e]/*mSCR*" denotes the ratio of expected return to marginal risk: the expected return in excess of the risk-free rate divided by the marginal SCR.

6.3. Liability hedge

As first step for improving the current asset allocation, the insurer should hedge the interest rate risk of the insurance liabilities. In practice, insurers can hedge the interest rate risk effectively with interest rate swap contracts, as Solvency II specifies that the swap curve needs to be used to calculate the present value of the insurance liabilities. Another hedging approach often applied in practice is to match the future cash flows of the liabilities and the assets as closely as possible, using long-term government bonds. Here we abstract from these implementation issues, and we use an investment in EEA government bonds as a simple duration hedge.

We start by calculating the investment in EEA government bonds that would reduce the duration gap of the balance sheet to zero. The insurance liabilities have dollar duration of EUR 2.67 million. As partial hedge, the insurer has investments in traded bonds and the illiquid credit risk portfolio (e.g., mortgages), with a dollar duration of 1.83. Thus a liability hedge portfolio with a dollar duration EUR 0.84 million (=2.67 – 1.83) reduces the duration gap to zero. Given that EEA government bonds have a dollar duration of 0.00069 (duration of 6.9 years times 0.01%), this implies an additional investment in EEA government bonds of: 0.84 / 0.00069 = 1,217 million EUR. This investment will be financed with a short position in EEA T-bills, the risk-free asset.[8]

Table 7 shows that the new asset allocation with the liability hedge included: it has a SCR for market risk of EUR 219 million, a reduction of 26% compared to the initial allocation. The insurer's ratio of own funds to market SCR has increased from 135% to 183%. In addition, the expected return on the insurer's own funds has increased to 3.5%, from -0.3% initially. The return on the total asset portfolio is leveraged by a factor 1.3 (= 5,217 / 4,000), due to the short position of EUR 1.2 billion in EEA Treasury bills. This leverage also helps to increase the expected return on the insurer's own funds. Leverage is not necessarily a source of additional risk in this case, but simply one component of a hedge that reduces interest rate risk.

After implementing the hedge, the SCR for interest rate risk is exactly equal to zero.[9] The column "Adj. *mSCR*" in Table 7 shows the adjusted marginal contributions to risk (in %), after netting out all the interest rate risk.[10] The credit spread risk of corporate bonds contributes 32% to the SCR of EUR 219 million, followed by real estate risk (31%), equity risk (28%), and the credit spread risk of covered bonds (6%) and non-EEA government bonds (2%). In the next section we look for trades that can increase the expected return of the asset allocation, while keeping the SCR for market risk constant.

[8] Treasury bills have a duration of approximately zero due to their maturity of only a few months.

[9] The SCR for interest rate risk is now on a knife-edge: after a slight increase in bond investments the duration of the assets is higher than the liabilities (negative duration gap), and the "curve shift up" shock scenario determines interest rate risk; after a slight decrease in bonds investments the duration gap is positive and the "curve shift down" applies. In Table 7 we show the marginal risk estimates for a slight increase of the amount invested in bonds.

[10] Without adjustment, the *mcSCR* of the liabilities is -93%, the *mcSCR* of the market risk assets is 183% and the *mcSCR* of the credit risk portfolio (non-market) is 10%. In Table 7, when reporting Adj. *mcSCR*, we exclude the interest rate risk exposures because the *mcSCR* for interest rate risk adds up to zero.

Table 7 Marginal analysis of asset allocation with liability hedge

Market risk portfolio	Value	Weight	E[r]	*mSCR*	Adj. *mcSCR*	*mRoC* x 1%	*mRoC** x 1%	E[r_e] /*mSCR*
Equity portfolio	210	7.0%	4.86%	0.29	28%	0.50%	-0.04%	0.157
Real estate	330	11.0%	3.50%	0.21	31%	0.35%	-0.04%	0.156
Gov bonds EEA	2,177	72.6%	1.50%	0.05	0%	0.17%	0.07%	0.238
Gov bonds non-EEA	240	8.0%	1.75%	0.07	2%	0.19%	0.05%	0.202
Corporate debt	885	29.5%	2.40%	0.12	32%	0.25%	0.03%	0.180
Covered bonds	375	12.5%	1.75%	0.08	6%	0.18%	0.02%	0.179
Treasury bills EEA	-1,217	-40.6%	0.25%	0.00	0%	0.00%	0.00%	---
Total portfolio	3,000	100.0%	2.78%	0.13	100%	0.31%	0.06%	0.190

Assets	Value	Weight	E[r]	*mSCR*	Adj. *mcSCR*	*mRoC* x 1%	*mRoC** x 1%	E[r_e] /*mSCR*
Market risk portfolio	3,000	75.0%	2.78%	0.13	100%	0.31%	0.06%	0.190
Credit risk portfolio	600	15.0%	3.50%	0.04	0%	0.55%	0.48%	0.878
Other assets	400	10.0%	0.25%	0.00	0%	0.00%	0.00%	---
Total assets	4,000	100.0%	2.63%	0.11	100%	0.31%	0.12%	0.226

Liabilities	Value	Weight	E[r]	*mSCR*	Adj. *mcSCR*	*mRoC* x 1%	*mRoC** x 1%	E[r_e] /*mSCR*
Technical provisions	3,000	75.0%	3.00%	-0.07	0%	-0.42%	-0.34%	-0.406
Other liabilities	600	15.0%	0.25%	0.00	0%	0.00%	-0.05%	---
Own funds	400	10.0%	3.47%	0.11	100%	0.31%	0.12%	0.226
Total liabilities	4,000	100.0%	2.63%	-0.04	100%	-0.29%	-0.25%	-0.592

Return/risk trade-off	
E[Increase own funds]	13.9
SCR market risk	218.8
Solvency ratio f_M	183%
Return on SCR (*RoC*)	6.3%

Dollar duration (DV01)	
Assets	2.67
Liabilities	2.67
Gap	0

Leverage of assets	
Long	5,217
Short	-1,217
Leverage	1.3

Notes: "SCR market risk" is the solvency capital requirement for market risk, determined with the standard formula. The solvency ratio f_M is the ratio of the insurer's own funds over the SCR for market risk. The column *mSCR* shows the *marginal* SCR: the increase in the SCR for market risk when the value of a particular asset or liability increases by 1 unit (1 million euro). The column "Adj. *mcSCR*" shows the marginal *contribution* (in %) to the SCR for market risk of an asset or liability, after netting out the *mcSCR* for interest rate risk and assigning it to the technical provisions. The "Return on SCR" (*RoC*) is the expected increase in the insurer's own funds divided by the SCR for market risk. The column "*mRoC* x 1%" shows the *marginal RoC*: the expected change in the return on solvency capital when the weight of a particular asset or liability is increased by 1%. The column "*mRoC** x 1%" shows the marginal *RoC*, but evaluated at the RoC^*_{NoLiab} of an optimal *asset-only* portfolio (i.e., without liabilities). "E[r_e]/*mSCR*" denotes the ratio of expected return to marginal risk: the expected return in excess of the risk-free rate divided by the marginal SCR.

6.4. Stepwise improvement of the asset allocation

We will now explore ways to improve the asset allocation with liability hedge in Table 7. Suppose the insurance company would like to increase the expected return on own funds, while allowing the SCR for market risk to increase up to the initial value of EUR 297 million. The column "*mRoC* x 1%" in Table 7 shows the expected increase in the return on solvency capital, if the weight of an asset class is increased by 1% (of EUR 4,000 million). A good start might be to increase the weight of the asset with the highest marginal return on capital (*mRoC*) stepwise, until eventually the SCR target is breached. Then reduce the weight of the asset with the lowest *mRoC*, etc. However, a drawback of this approach is that the *mRoC* values are quite sensitive to the return on capital of the current portfolio. Ideally, we would like to evaluate the *mRoC*'s using the return on capital of an optimal "asset-only" portfolio (RoC^*_{NoLiab}), as in the condition for optimality in Equation (38). However, in practice we typically do not know that particular *RoC* value in advance, unless we use an optimizer.

The insurer's initial asset allocation in Table 6 is a good example of this problem: the *mRoC* of equity is highest at 0.64%, while the *mRoC* of EEA government bonds is lowest among the risky assets at 0.16%, suggesting that increasing the equity allocation at the expense of EEA bonds is a good idea. However, this is actually a poor suggestion, given that selling EEA bonds would further increase the large duration gap of the initial balance sheet. The problem is that the *mRoC*'s are calculated using the current allocation's negative return on solvency capital of -0.5%, a sub-par risk-adjusted return. Starting from *RoC* = -0.5%, a large increase in risk (SCR) brings the return on capital closer to zero, thus increasing the *RoC*. If instead the *mRoC*'s would be calculated using RoC^*_{NoLiab} = 16.5%, which is the return on capital of an optimal asset-only portfolio with the same SCR (=297), then EEA government bonds have the highest *mRoC* at 0.33%, and equity one of the lowest *mRoC* at 0.01%. These figures are shown in the column "$mRoC^*$ x 1%" in Table 6, and they are quite different. In sum, marginal returns on capital are sensitive to the *RoC* of the current asset allocation and do not always provide good trade ideas.

Another measure sometimes used to improve asset allocations is the implied expected return:

$$\mu_{A,i}^{Imp} = r_f + \lambda \times mSCR_{A,i}^{Mkt}, \text{ for } i = 1, 2, \ldots, I \qquad (42)$$

The implied expected return, $\mu_{A,i}^{Imp}$, is the expected return that would make the *current* asset allocation optimal, following from the optimality condition in Equation (38). The parameter λ is a scaling parameter, which can be freely chosen. Apart from providing information about the consistency between the insurer's return expectations and the current asset allocation, implied returns can also be used to improve the asset allocation stepwise. If the implied expected return of an asset class is *lower* than the actual expected return ($\mu_{A,i}^{Imp} < \mu_{A,i}$) then its weight can be *increased*, and vice versa. However, implied expected returns are quite sensitive to the value of the scaling parameter λ and as a result they do not always give good trade ideas, similar to marginal returns. A way out of this problem is to use $\lambda = RoC^*_{NoLiab}$, the optimal return on capital of an asset-only portfolio, but we typically do not know this value in advance. We refer Herold (2005) for an in-depth discussion of these issues and suggestions about how to set a reasonable value for λ.

A better alternative is to focus on the ratio of expected excess return to marginal risk, denoted by "E[r_e]/*mSCR*" in Table 6 and Table 7. Assets offering the highest return to marginal risk ratio are attractive to invest more in, until eventually the risk budget for SCR is exceeded. Then risk can be reduced by lowering the allocation to the asset with the lowest return to marginal risk ratio. Step by step, this way we can approach an efficient asset allocation. One caveat is that assets that offer a positive excess return and *negative* marginal risk, such as government bonds in the initial allocation in Table 6, should always be the first priority to invest in, as they *increase* the expected return while also *lowering* risk. Such "free lunches" will typically only occur when the interest rate risk of the insurance liabilities is not properly hedged yet, as is the case in Table 6.

We have applied this stepwise improvement process based on "E[r_e]/*mSCR*" to the asset allocation in Table 7, targeting for an SCR of EUR 297 million. In the first step the allocation to EEA government bonds was increased by 10%, as it has the highest return to marginal risk ratio in Table 7. This step was repeated until the weight of EEA government bonds reached 123%, and the SCR limit was breached. Then the allocation of covered bonds, having the lowest return to marginal risk ratio, was reduced by 10%. As the SCR then fell below the risk target again, another 10% increase in EEA bonds followed. In the next step the covered bond weight was reduced to zero, hitting the lower bound under no short-selling. This process was repeated, with smaller step sizes as the marginal risk ratios of the assets (with non-zero weights) became more similar. The final result was a portfolio consisting of 12% equity, 7% property, 146% EEA government bonds and 21.5% corporate debt. This asset allocation results in an SCR of 296 million euro, an expected increase in own funds of EUR 30 million (7.5%), and a return on solvency capital of 10%. In the next sub-section we will compare this asset allocation to an optimal portfolio.

6.5. An optimal allocation

We now derive the optimal asset allocation that maximizes the expected return on own funds, given a budget for the SCR for market risk equal to the initial value of EUR 297 million. In Appendix C we provide a formulation of the optimization problem that includes non-negativity constraints for the amounts invested in the risky assets ($\boldsymbol{a} \geq 0$) and the charges for the risk types ($\boldsymbol{s} \geq 0$), as well as two inequality constraints to make sure that the SCR for interest rate risk is the maximum of the "curve up" and "curve down" scenarios. The resulting quadratic optimization problem with linear inequality constraints cannot be solved analytically, but a numerical solution is easily found with modern solution algorithms (e.g., interior point methods). Table 8 shows the optimal solution for a market risk SCR target of EUR 297 million.

The optimal allocation in Table 8 has an expected return on own funds of 7.6% per year, while the SCR for market risk is equal to target of EUR 297 million. The expected return has improved substantially compared to the insurer's initial allocation in Table 6, which had an expected return on own funds of only -0.3% (EUR -1.3 million). For the optimal asset allocation the return on solvency capital is positive at *RoC* = 10.2%, compared to -0.5% initially. Remarkably, the asset allocation found in the previous section using the stepwise process, based on the ratios of return to marginal risk, is very close to the optimal solution in Table 8.

The optimal asset allocation has a 146% weight in EEA government bonds, partially financed with a short position in riskless EEA Treasury bills of -86%. A weight of 98% in EEA government bonds is sufficient to completely hedge the interest rate risk of the insurance liabilities, all else equal, so the remaining weight of 48% (= 146% – 98%) is an active allocation to earn a risk premium. In addition, the allocation invests in three 'risky' asset classes: 12% in equity, 7% in real estate and 21% in corporate bonds. Non-EEA government bonds and covered bonds have zero weights, as their ratio of return to marginal risk (E[r_e]/*mSCR*) is unattractive compared to the other assets. Hence, these two asset classes are redundant. The fact that the optimal allocation contains only four asset classes was to be expected, given that there are only four risk factors (equity, property, credit spread and interest rate risk) that determine the SCR for market risk: see the discussion following the analytical solution in Section 5.3.

Inspecting the marginal contributions to risk (adj. *mcSCR*), they are quite evenly distributed: 36% of the total risk budget of EUR 297 million is allocated to EEA government bonds, 34% to equity risk, 15% to spread risk through corporate debt and 14% to real estate.[11]

The marginal returns on capital (*mRoC*) of the optimal allocation seem to suggest that the return on solvency capital can be improved by increasing some portfolio weights further, especially for equity (*mRoC* = 0.27%) and property (*mRoC*=0.20%). The return on capital indeed can be increased, but the SCR for market risk will increase as well and exceed the limit of EUR 297.4 million. The marginal returns on capital are more useful when they are calculated according to the first-order condition of the optimization problem in Equation (38): evaluated at the RoC of an optimal asset-only allocation, namely RoC^*_{NoLiab} = 0.165. Table 8 shows that these marginal returns are all equal to zero for the market risk assets with positive weights. Further, they are negative for the two redundant assets, non-EEA government bonds and covered bonds.

As marginal returns on capital are quite sensitive to the exact method of calculation, our preferred marginal measure is E[r_e]/*mSCR*, the ratio of expected excess return to marginal risk. Table 8 shows that the return to marginal risk ratios are equal to 0.165 for the risky assets with positive weights (equity, property, EEA bonds and corporate bonds), and less than 0.165 for the redundant assets with zero weight (covered bonds, non-EEA bonds). This is in line with the first-order conditions for optimality in (40) and (41). Not coincidentally, the value 0.165 is also the return on capital of an optimal asset-only allocation with SCR = 297: RoC^*_{NoLiab} = 0.165.

We finally compare the optimal allocation in Table 8 to the analytical solution derived previously in Section 5.2. An analytical solution can only be derived without short-selling constraints, thus short positions ($\boldsymbol{a} < 0$) are possible and capital charges can become negative ($\boldsymbol{s} < 0$). However, we can always check if the analytical solution is feasible (i.e., meets the constraints) for the current numerical example. To apply the analytical solution, we first have to reduce the number of asset classes to four, equal to the number of relevant market risk types (interest rate, equity, property and spread risk).[12] The four relevant assets are equity, property, EEA government bonds and corporate bonds, while non-EEA government bonds and covered bonds are redundant. Setting the interest rate shock equal to the "curve up" scenario, the analytical solution for SCR = 297 is

[11] The adjusted contributions to risk were calculated by assigning the entire interest rate risk exposure, net of liability risk, to EEA government bonds. Without this adjustment, government bonds have a *mcSCR* of 111%, but most of this interest rate risk is cancelled by the negative *mcSCR* of the insurance liabilities which is -98%. Corporate bonds and the credit risk portfolio also have interest rate risk exposure, amounting to 24%.

[12] With more than four assets the matrix *V* would not be of full rank, and *V'RV* not invertible.

identical to the numerical solution in Table 8. Hence, all non-negativity constraints are satisfied in this case, and therefore the analytical and numerical solution coincide exactly.

Table 8 Marginal analysis of optimal allocation with SCR of 297 million

Market risk portfolio	Value	Weight	E[*r*]	*mSCR*	Adj. *mcSCR*	*mRoC* x 1%	*mRoC** x 1%	E[r_e] /*mSCR*
Equity portfolio	367	12.2%	4.86%	0.28	34%	0.24%	0.00%	0.165
Real estate	213	7.1%	3.50%	0.20	14%	0.17%	0.00%	0.165
Gov bonds EEA	4,377	145.9%	1.50%	0.08	36%	0.06%	0.00%	0.165
Gov bonds non-EEA	0	0.0%	1.75%	0.10	0%	0.07%	-0.01%	0.157
Corporate debt	634	21.1%	2.40%	0.13	15%	0.11%	0.00%	0.165
Covered bonds	0	0.0%	1.75%	0.10	0%	0.06%	-0.02%	0.148
Treasury bills EEA	-2,592	-86.4%	0.25%	0.00	0%	0.00%	0.00%	---
Total portfolio	3,000	100.0%	3.32%	0.19	100%	0.16%	0.00%	0.165

Assets	Value	Weight	E[*r*]	*mSCR*	Adj. *mcSCR*	*mRoC* x 1%	*mRoC** x 1%	E[r_e] /*mSCR*
Market risk portfolio	3,000	75.0%	3.32%	0.19	100%	0.16%	0.00%	0.165
Credit risk portfolio	600	15.0%	3.50%	0.05	0%	0.36%	0.32%	0.610
Other assets	400	10.0%	0.25%	0.00	0%	0.00%	0.00%	---
Total assets	4,000	100.0%	3.04%	0.15	100%	0.17%	0.05%	0.189

Liabilities	Value	Weight	E[*r*]	*mSCR*	Adj. *mcSCR*	*mRoC* x 1%	*mRoC** x 1%	E[r_e] /*mSCR*
Technical provisions	3,000	75.0%	3.00%	-0.10	0%	-0.24%	-0.19%	-0.282
Other liabilities	600	15.0%	0.25%	0.00	0%	0.00%	-0.03%	---
Own funds	400	10.0%	7.55%	0.15	100%	0.17%	0.05%	0.189
Total liabilities	4,000	100.0%	3.04%	-0.06	100%	-0.16%	-0.14%	-0.478

Return/risk trade-off	
E[Increase own funds]	30.2
SCR market risk	297.4
Solvency ratio f_M	135%
Return on SCR (*RoC*)	10.2%

Dollar duration (DV01)	
Assets	3.65
Liabilities	2.67
Gap	-0.98

Leverage of assets	
Long	6,592
Short	-2,592
Leverage	1.6

Notes: "SCR market risk" is the solvency capital requirement for market risk, determined with the standard formula. The solvency ratio f_M is the ratio of the insurer's own funds over the SCR for market risk. The column *mSCR* shows the *marginal* SCR: the increase in the SCR for market risk when the value of a particular asset or liability increases by 1 unit (1 million euro). The column "Adj. *mcSCR*" shows the marginal *contribution* (in %) to the SCR for market risk of an asset or liability, after netting out the *mcSCR* for interest rate risk and assigning it to the investment in EEA government bonds. The "Return on SCR" (*RoC*) is the expected increase in the insurer's own funds divided by the SCR for market risk. The column "*mRoC* x 1%" shows the *marginal RoC*: the expected change in the return on solvency capital when the weight of a particular asset or liability is increased by 1%. The column "*mRoC** x 1%" shows the marginal *RoC*, but evaluated at the RoC^*_{NoLiab} of an optimal *asset-only* portfolio (i.e., without liabilities). "E[r_e]/*mSCR*" denotes the ratio of expected return to marginal risk: the expected return in excess of the risk-free rate divided by the marginal SCR.

6.6. Total returns on assets versus minimization of SCR

A concern often raised about the Solvency II regulatory framework is that it may lead insurers to excessively reduce risk in an attempt to lower their solvency capital requirement. Life insurance companies can achieve the biggest reductions in solvency capital by hedging the interest rate risk and longevity risks of their insurance liabilities. Arguably, liability hedging has clear economic benefits, if the costs and operational risks of implementing the hedge are within limits. Beyond that, solvency capital can be reduced further by limiting exposure to risky assets such as hedge funds, private equity and stocks, which are subject to relatively high capital charges in the standard formula. However, a policy of de-risking the asset allocation potentially comes at the expense of a reduced expected return that can deteriorate the long-term viability of the insurer over a 10-year or 20-year horizon (van Bragt et al., 2010). Low expected returns on capital can jeopardize dividend payouts to shareholders. In addition, if a life insurer earns a return on equity that is consistently lower than other firms with a similar risk profile, investors may be reluctant to participate in new equity issues (an important lifeline in times of stress).

An important question therefore is whether our proposed asset allocation framework in Section 5 inadvertently leads to low expected returns on assets. The answer is: no. The proposed framework trades off the expected return on own funds versus the solvency capital requirement. The expected return on own funds is equal to the expected return on assets minus the expected return on the liabilities. Assuming that the liabilities are fixed in the short term, the asset allocation framework thus effectively trades off the expected return on assets versus SCR.

For example, the life insurer's initial balance sheet in Table 6 has an expected return on assets of 2.3% per year and a return on own funds of -0.3%. The "optimal" asset allocation in Table 8 has the same SCR as the initial allocation, but an expected return on assets of 3.0% per year and a return on own funds of 7.6%. The return on own funds is enhanced by a short position in EEA Treasury bills (i.e., leverage) that helps to hedge the liabilities and provides some additional interest rate risk exposure to earn a term premium. The asset classes equity and real estate are also present in the optimal portfolio, with a combined weight of 19%. Clearly, no de-risking of the asset allocation took place.

As a further example, in Table 9 we show the optimal asset allocation for a more conservative SCR target of 200 million euro. Even with this lower risk target, the optimal asset allocation still contains 8% equity, 5% property and 14% corporate debt, as well as 136% in EEA government bonds. Again, the portfolio is not completely de-risked. This allocation has an expected return on assets of 2.6% per year and an expected return on own funds of 3.5%. Figure 3 show the entire efficient frontier of expected return on own funds versus SCR for market risk.

The bottom line is that the proposed asset allocation framework does not necessarily drive out risky assets, if they offer a reasonable risk/return trade-off. Attempts to de-risk the asset allocation in practice probably occur when an insurer has a solvency ratio below a minimum target level deemed necessary by its shareholders (e.g., 130%). This may result in an attempt to minimize the SCR at all costs. If the liabilities are already fully hedged, other options for reducing the SCR are raising more equity from investors or de-risking the asset allocation. But, as shown by Bragt et al. (2010), a fully de-risked asset allocation with low expected returns can lead to more solvency problems in the long run, unless the underlying insurance business has a sufficiently high level of operational profitability.

Table 9 Marginal analysis of optimal allocation with SCR of 200 million

Market risk portfolio	Value	Weight	E[r]	*mSCR*	Adj. *mcSCR*	*mRoC* x 1%	*mRoC** x 1%	E[r_e] /*mSCR*
Equity portfolio	247	8.2%	4.86%	0.28	34%	0.53%	0.00%	0.165
Real estate	144	4.8%	3.50%	0.20	14%	0.37%	0.00%	0.165
Gov bonds EEA	4,072	135.7%	1.50%	0.08	36%	0.14%	0.00%	0.165
Gov bonds non-EEA	0	0.0%	1.75%	0.10	0%	0.17%	-0.02%	0.157
Corporate debt	426	14.2%	2.40%	0.13	15%	0.25%	0.00%	0.165
Covered bonds	0	0.0%	1.75%	0.10	0%	0.16%	-0.03%	0.148
Treasury bills EEA	-1,889	-63.0%	0.25%	0.00	0%	0.00%	0.00%	---
Total portfolio	3,000	100.0%	2.79%	0.15	100%	0.29%	0.00%	0.165

Assets	Value	Weight	E[r]	*mSCR*	Adj. *mcSCR*	*mRoC* x 1%	*mRoC** x 1%	E[r_e] /*mSCR*
Market risk portfolio	3,000	75.0%	2.79%	0.15	100%	0.29%	0.00%	0.165
Credit risk portfolio	600	15.0%	3.50%	0.05	0%	0.57%	0.47%	0.610
Other assets	400	10.0%	0.25%	0.00	0%	0.00%	0.00%	---
Total assets	4,000	100.0%	2.64%	0.12	100%	0.30%	0.07%	0.194

Liabilities	Value	Weight	E[r]	*mSCR*	Adj. *mcSCR*	*mRoC* x 1%	*mRoC** x 1%	E[r_e] /*mSCR*
Technical provisions	3,000	75.0%	3.00%	-0.10	0%	-0.41%	-0.28%	-0.282
Other liabilities	600	15.0%	0.25%	0.00	0%	0.00%	-0.05%	---
Own funds	400	10.0%	3.53%	0.12	100%	0.30%	0.07%	0.194
Total liabilities	4,000	100.0%	2.64%	-0.06	100%	-0.28%	-0.21%	-0.393

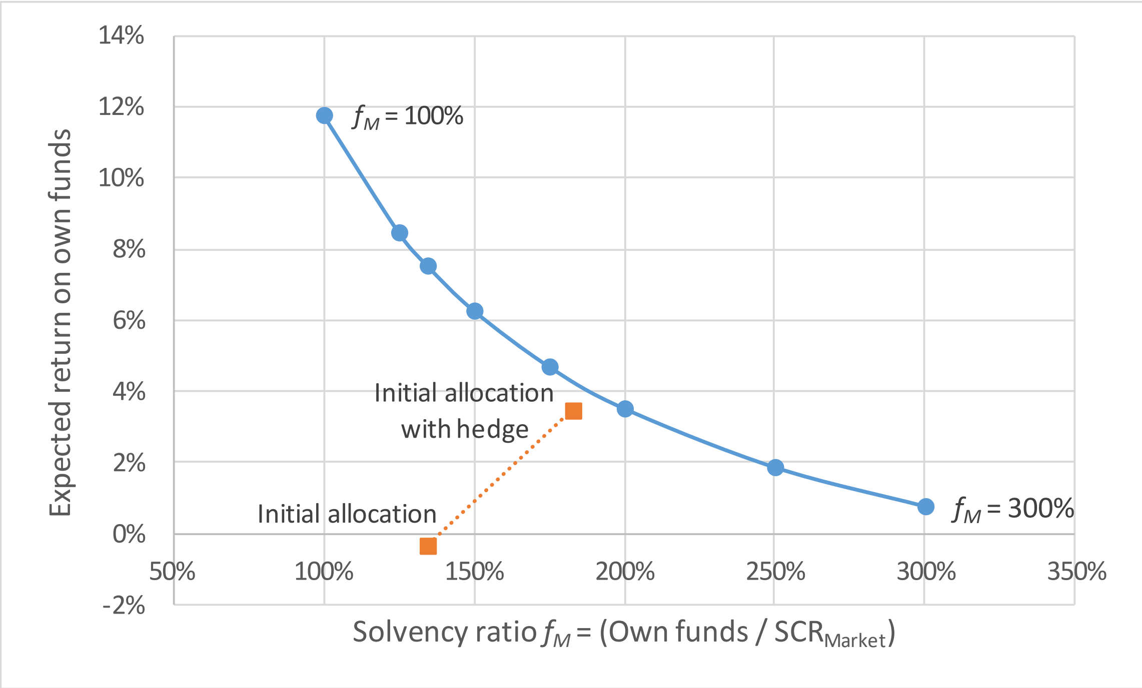
Return/risk trade-off	
E[Increase own funds]	14.1
SCR market risk	200.0
Solvency ratio f_M	200%
Return on SCR (*RoC*)	7.1%

Dollar duration (DV01)	
Assets	3.33
Liabilities	2.67
Gap	-0.66

Leverage of assets	
Long	5,889
Short	-1,889
Leverage	1.5

Notes: "SCR market risk" is the solvency capital requirement for market risk, determined with the standard formula. The solvency ratio f_M is the ratio of the insurer's own funds over the SCR for market risk. The column *mSCR* shows the *marginal* SCR: the increase in the SCR for market risk when the value of a particular asset or liability increases by 1 unit (1 million euro). The column "Adj. *mcSCR*" shows the marginal *contribution* (in %) to the SCR for market risk of an asset or liability, after netting out the *mcSCR* for interest rate risk and assigning it to the investment in EEA government bonds. The "Return on SCR" (*RoC*) is the expected increase in the insurer's own funds divided by the SCR for market risk. The column "*mRoC* x 1%" shows the *marginal RoC*: the expected change in the return on solvency capital when the weight of a particular asset or liability is increased by 1%. The column "*mRoC** x 1%" shows the marginal *RoC*, but evaluated at the RoC^*_{NoLiab} of an optimal *asset-only* portfolio (i.e., without liabilities). "E[r_e]/*mSCR*" denotes the ratio of expected return to marginal risk: the expected return in excess of the risk-free rate divided by the marginal SCR.

Figure 3 **Efficient frontier of expected return on own funds versus solvency ratio**



Note: the figure show the efficient frontier of expected return on own funds (on the *y*-axis) versus the insurer's solvency ratio (on the *x*-axis). The solvency ratio f_M is calculated using the SCR for market risk (excluding non-market risks) in the denominator: f_M = own funds / SCR_{Market}.

7. Conclusions

In this paper we develop an asset allocation and risk budgeting framework for life insurers under Solvency II. In our framework the insurance company maximizes the expected return on own funds, subject to an upper limit on the solvency capital requirement for market risk. We derive the optimal asset allocation analytically and show that it consists of a liability hedge portfolio combined with an optimal asset-only portfolio, a three fund separation result. The liability hedge portfolio fully hedges the interest rate risk of the insurance liabilities. In practice life insurers can also (partially) hedge the longevity risk of the insurance liabilities, and hedge the risk of equity-linked insurance products using derivatives. The purpose of liability hedging is twofold: reduce economic risk exposures, and as a result also lower the capital requirement under Solvency II.

Once the liabilities have been hedged, the insurer can invest in efficient asset-only portfolio, providing an optimal tradeoff between expected return and risk. We note that when the Solvency II standard formula is used as the yardstick for measuring risk, little to no distinction is made between individual assets *within* market risk categories (interest, equity, property and credit risk). From the perspective of the standard formula, an investment in the broadly diversified MSCI World index and the undiversified MSCI Belgium index are equally risky, both receiving the same capital charge for developed equity. Therefore, insurers applying the standard formulas should first select well-diversified portfolios to gain exposure to interest rate, equity, property and credit risk. Then an asset allocation framework trading off the expected return on own funds versus the required solvency capital can be applied to find an efficient allocation to these broadly diversified portfolios, providing diversification *between* risk types.

We also derived several risk budgeting measures, and applied them to analyze and improve the asset allocation of a representative European life insurer. Measures such as marginal contribution to SCR can provide meaningful insights about the risk exposures on the insurer's balance sheet, while taking into account diversification effects. Further, based on the numerical examples and our analytical framework, we conclude that the ratio of expected return to marginal risk of asset classes is the most useful measure for improving the efficiency of an asset allocation step by step. Alternative measures, such as marginal returns on capital and implied expected returns, are rather sensitive to scaling parameters that are typically unknown in advance (unless an optimization problem is solved), and therefore less useful in our assessment.

Finally, we would like point out that any efficient asset allocation is typically sensitive to the return expectations formulated by the investor (Chopra and Ziemba, 1993). Small changes in expected returns can lead to large changes in the optimal allocation, but in practice expected returns are notoriously hard to estimate precisely. We therefore recommend life insurers to consider several sets of long-term expected return forecasts for the asset classes, obtained from different sources. A truly "optimal" asset allocation should be robust to changes in return forecasts, and perform well under several expected return scenarios, most likely further increasing the need for diversification. In the literature robust portfolio optimization methods and Bayesian techniques for parameter uncertainty are often been applied for this purpose (Fabozzi, 2007; Scherer 2007). An interesting avenue for further research is to extend the framework in this paper in this direction.

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Appendix A: Return on Total Solvency Capital

The return on capital in the main text is defined using the solvency capital requirement (SCR) for *market risk* in the denominator. In this appendix we define the return on capital with respect to the *total* SCR, which includes non-market risks:

$$RoC_{Total} = \frac{E[\Delta F]}{SCR_{Total}} = \left(\sum\nolimits_{i=1}^{I} \mu_{A,i} A_i - \sum\nolimits_{n=1}^{N} \mu_{L,n} L_n\right) / SCR_{Total} \tag{43}$$

The return RoC_{Total} shows the expected increase in own funds expressed as a percentage of the total solvency capital required under Solvency II.

RoC_{Total} is also equal to the expected return on own funds ($E[\Delta F/F]$) multiplied by the current solvency ratio f:

$$RoC_{Total} = \frac{E[\Delta F]}{F}\left(\frac{F}{SCR_{Total}}\right) = E\left[\frac{\Delta F}{F}\right] \times f$$

Finally, the marginal returns on capital for the assets and liabilities are:

$$mRoC_{A,i}^{Total} = \frac{\partial RoC_{Total}}{\partial A_i} = \frac{\mu_{A,i} - RoC_{Total} \times mSCR_{A,i}^{Total}}{SCR_{Total}} \tag{44}$$

$$mRoC_{L,n}^{Total} = \frac{\partial RoC_{Total}}{\partial L_n} = \frac{-\mu_{L,n} - RoC_{Total} \times mSCR_{L,n}^{Total}}{SCR_{Total}} \tag{45}$$

Appendix B: Optimal Asset-Only Allocation

In this appendix we derive the optimal "asset-only" solution for the asset allocation framework in Section 5.1, when the liabilities are zero ($L_1 = L_2 = 0$). The problem then simplifies as follows:

$$\max_{\boldsymbol{a}} E[\Delta F(\boldsymbol{a})] = r_f A + \boldsymbol{\mu_A}'\boldsymbol{a}$$

$$\text{s.t.} \qquad \mathbf{s'Rs} = (\boldsymbol{a}'\mathbf{V'RV}\boldsymbol{a})^{½} \leq SCR_{Market}^{Max} \tag{46}$$

where $\boldsymbol{\mu_A} = (\mu_{A,1} - r_f, \mu_{A,2} - r_f, \cdots, \mu_{A,I} - r_f)'$ is a column vector of length I that contains the expected excess returns of the assets. The first order condition for optimality is

$$\boldsymbol{\mu_A} - \lambda \frac{\mathbf{V'RV}\boldsymbol{a}}{SCR_{Market}} = \mathbf{0}, \qquad \text{for } \lambda \geq 0 \tag{47}$$

We solve for the optimal asset-only allocation $\boldsymbol{a}^*$ **,** assuming that $\mathbf{V'RV}$ is non-singular:

$$\boldsymbol{a}^* = \frac{SCR_{Market}^{Max}}{\lambda}(\mathbf{V'RV})^{-1}\boldsymbol{\mu_A} \tag{48}$$

We then solve for lambda, assuming the solvency constraint is binding:

$$\mathbf{s'Rs} = \boldsymbol{a}^{*\prime}\mathbf{V'RV}\boldsymbol{a}^* = \frac{{SCR_{Market}^{Max}}^2}{\lambda^2}\boldsymbol{\mu_A}'(\mathbf{V'RV})^{-1}\boldsymbol{\mu_A} = {SCR_{Market}^{Max}}^2 \tag{49}$$

Thus $\lambda = \sqrt{\boldsymbol{\mu_A}'(\mathbf{V'RV})^{-1}\boldsymbol{\mu_A}}$. We can now write the optimal asset allocation as follows, using the notation $\boldsymbol{a}_{\boldsymbol{NoLiab}}^*$ for the solution:

$$\boldsymbol{a}_{\boldsymbol{NoLiab}}^* = \left(\frac{SCR_{Market}^{Max}}{\sqrt{\boldsymbol{\mu_A}'(\mathbf{V'RV})^{-1}\boldsymbol{\mu_A}}}\right)(\mathbf{V'RV})^{-1}\boldsymbol{\mu_A} \tag{50}$$

We can prove that the return on capital of the optimal asset-only allocation is equal to λ:

$$RoC_{NoLiab}^* = \frac{\boldsymbol{\mu_A}'\boldsymbol{a}_{\boldsymbol{NoLiab}}^*}{SCR_{Market}^{Max}} = \frac{\boldsymbol{\mu_A}'(\mathbf{V'RV})^{-1}\boldsymbol{\mu_A}}{\sqrt{\boldsymbol{\mu_A}'(\mathbf{V'RV})^{-1}\boldsymbol{\mu_A}}} = \sqrt{\boldsymbol{\mu_A}'(\mathbf{V'RV})^{-1}\boldsymbol{\mu_A}} = \lambda \tag{51}$$

The first-order optimality condition for the individual assets can now be rewritten as:

$$(\mu_{A,i} - r_f) - RoC_{NoLiab}^* \times mSCR_{A,i}^{Mkt} = 0, \text{ for } i = 1, 2, \dots, I \tag{52}$$

This condition is equivalent to the marginal returns on solvency capital being equal to zero:

$$mRoC_{A,i}^{Mkt} = 0, \text{ for } i = 1, 2, \dots, I \tag{53}$$

Hence, an asset allocation is optimal when a marginal change in the amounts allocated to the assets does not help to improve the return on solvency capital further. Another way to express this condition is that the ratio of expected return to marginal risk (*mSCR*) is equal for all assets:

$$\frac{(\mu_{A,i} - r_f)}{mSCR_{A,i}^{Mkt}} = \lambda = RoC_{NoLiab}^*, \text{ for } i = 1, 2, \dots, I \tag{54}$$

Appendix C. Optimization with Non-Negativity Constraints

Below we present a formulation of the optimal asset allocation problem that includes non-negativity constraints for the amounts invested in the risky assets ($\boldsymbol{a} \geq 0$) and for the capital charges for the risk types ($\boldsymbol{s} \geq 0$). Further, it includes two inequality constraints to make sure that the SCR for interest rate risk (s_1) is the maximum of the charges in "curve up" and "curve down" scenarios for the term-structure of interest rates.

To facilitate an efficient numerical solution of the optimization problem, the objective function is to maximize the expected change in own funds ΔF, while putting a penalty $\gamma > 0$ on the solvency capital requirement for market risk ($SCR_{Market} = \boldsymbol{s}'\mathbf{R}\boldsymbol{s}$). This optimization problem has a quadratic objective function and linear inequality constraints. A numerical solution can be easily found with modern solution algorithms, such as interior point methods. An efficient frontier of expected return on own funds versus the SCR for market risk can be derived by changing the risk aversion parameter γ.

$$\max_{\boldsymbol{a}} E[\Delta F(\boldsymbol{a})] = (r_f A + \boldsymbol{\mu_A}'\boldsymbol{a} - \mu_{L,1}L_1 - \mu_{L,2}L_2) - \gamma \boldsymbol{s}'\mathbf{R}\boldsymbol{s} \qquad (55)$$

$$\begin{aligned} s_1 &\geq \Delta_{rd}\left(\left(D_{L,1}L_1 + D_{L,2}L_2\right) - \left(D_{A,1}A_1 + D_{A,2}A_2 + D_{A,3}A_3\right)\right) \\ s_1 &\geq \Delta_{ru}\left(\left(D_{A,1}A_1 + D_{A,2}A_2 + D_{A,3}A_3\right) - (D_{L,1}L_1 + D_{L,2}L_2)\right) \end{aligned} \qquad (56)$$

$$s_2 \geq (\Delta_{eq,1}w_{eq,1} + \Delta_{eq,2}(1 - w_{eq,1}))A_{eq} \qquad (57)$$

$$s_3 \geq \Delta_{prop}A_{prop} \qquad (58)$$

$$s_4 \geq \Delta_{gov,2}A_{gov,2} + \Delta_{corp}A_{corp} \qquad (59)$$

$$s_5 \geq \Delta_{cur}\sum_{i=1}^{I} f_i A_i \qquad (60)$$

$$\boldsymbol{s} \geq 0, \boldsymbol{a} \geq 0 \qquad (61)$$

Appendix D. Optimization with a Risk Limit on Total SCR

The asset allocation framework in Section 5 imposes a limit on the SCR for market risk. The rationale is that exposures to non-market risks, such as life insurance underwriting risk, are typically fixed, or quite difficult to change in the short run. However, some insurers may want to explicitly impose an upper limit SCR_{Total}^{Max} on *total* SCR, while optimizing their asset allocation:

$$\left(\sum_{h=1}^{5}(SCR_{Agg,h})^2 + \sum_{h=1}^{5}\sum_{\substack{j=1 \\ j\neq k}}^{5}\rho_{Agg,hj}SCR_{Agg,h}SCR_{Agg,j}\right) \leq (SCR_{Total}^{Max})^2 \qquad (62)$$

Here it is plausible to assume that the SCR's for non-market risks are fixed (given), while the SCR for market risk $SCR_{Agg,1} = SCR_{Market}(\boldsymbol{a})$ depends on the asset allocation $\boldsymbol{a}$.

The constraint on total SCR above is a relatively complex non-linear function, as one of its components, the capital requirement for market risk ($SCR_{Agg,1} = SCR_{Market}(\boldsymbol{a})$) is itself given by a square-root formula, and depends on the asset allocation $\boldsymbol{a}$. Hence, numerical solution techniques are required. The best way to tackle the numerical optimization is to incorporate the non-linear SCR expression into the objective function, as illustrated in Appendix C, while imposing only linear constraints, making it better suited for an efficient numerical solution.

An alternative approach we can linearize the constraint above, using the following first-order Taylor approximation for the total capital requirement:

$$\begin{aligned} SCR_{Total}(\boldsymbol{a}) = \overline{SCR}_{Total} + \overline{mSCR}_{Mkt}^{Total} \times \left(SCR_{Market}(\boldsymbol{a}) - \overline{SCR}_{Market}\right) \\ \leq SCR_{Total}^{Max} \end{aligned} \qquad (63)$$

where the overlined variables ($\overline{SCR}_{Total}, \overline{SCR}_{Market}, \overline{mSCR}_{Mkt}^{Total}$) are fixed at their current (initial) values. The linearized constraint can be reformulated as follows[13]:

$$SCR_{Market}(\boldsymbol{a}) \leq \overline{SCR}_{Market} + \left(\frac{SCR_{Total}^{Max} - \overline{SCR}_{Total}}{\overline{mSCR}_{Mkt}^{Total}}\right) \qquad (64)$$

Hence, the framework for asset allocation with a constraint on SCR for *market* risk in Section 5 can now be applied to impose a constraint on *total* SCR, using the following adjusted upper limit: $SCR_{Market}^{Max} = \overline{SCR}_{Market} + (SCR_{Total}^{Max} - \overline{SCR}_{Total})/\overline{mSCR}_{Mkt}^{Total}$. The linear approximation above will be accurate as long as the difference between the initial value of total SCR ($\overline{SCR}_{Total}$) and its upper limit (SCR_{Total}^{Max}) is small. Further, the approximation is exact in the special case where the initial SCR value is equal to the target ($\overline{SCR}_{Total} = SCR_{Total}^{Max}$).

[13] Assuming $\overline{mSCR}_{Mkt}^{Total} > 0$: the marginal SCR of market risk is positive.