S4 - 17B. Devoir 1

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Exercice 1:

1. (E): $y' = \sin(y)$

$$\begin{aligned}
\frac{dy}{dx} &= \sin(y)\\
\frac{dy}{sin(y)} &= dx\\
\int \frac{dy}{sin(y)} &= \int dx
\end{aligned}$$

Calcul de $\int \frac{1}{\sin(x)} dy$

$$\begin{aligned}
\text{Posons } t &= \tan(\frac{y}{2})\\
\text{donc } dt &= \frac{1}{2}(1+\tan^2(\frac{y}{2}))dy\\
dt &= \frac{1}{2}(1+t^2)dy\\
dy &= \frac{2dt}{1+t^2}
\end{aligned}$$

$$\begin{aligned}
\sin(y) &= \frac{2t}{1+t^2}\\
\int \frac{1}{\sin(y)} dy &= \int (\frac{1+t^2}{2t})(\frac{2dt}{1+t^2})\\
\int \frac{1}{\sin(y)} dy &= \int \frac{1}{t}(dt)\\
\int \frac{1}{\sin(y)} dy &= ln|t| + C, \quad C \in \mathbb{R}\\
\text{Or } t = \tan(\frac{y}{2}) \implies \int \frac{1}{\sin(y)} dy &= ln|\tan(\frac{y}{2})| + C
\end{aligned}$$

On a:

$$\begin{aligned}
ln|\tan(\frac{y}{2})| &= x + C \\
|\tan(\frac{y}{2})| &= e^x \cdot e^C \\
\tan(\frac{y}{2}) &= ke^x, \quad k \in \mathbb{R} \\
\frac{y}{2} &= \tan^{-1}(ke^x) \\
y &= 2\tan^{-1}(ke^x)
\end{aligned}$$

$$\boxed{y = 2\tan^{-1}(ke^x), \quad k \in \mathbb{R}}$$

2. (E): $x^2y' = e^y$

$$\begin{aligned}
x^2y' &= e^y \\
x^2\frac{dy}{dx} &= e^y \\
\frac{dy}{e^y} &= \frac{dx}{x^2} \\
\int \frac{1}{e^y}\,dy &= \int \frac{1}{x^2}\,dx \\
\int e^{-y}\,dy &= \int x^{-2}\,dx \\
-e^{-y} &= -\frac{1}{x} + C, \quad C \in \mathbb{R} \\
\ln(e^{-y}) &= \ln|\frac{1}{x} + C| \\
-y &= \ln|\frac{1}{x} + C| \\
y &= -\ln|\frac{1}{x} + C|
\end{aligned}$$

$$\boxed{y = -\ln|\frac{1}{x} + C|}$$

3. (E): $(x^2+1)y' + 3xy = x^2$

Etape 1: ESSM

$$
\begin{aligned}
(x^2+1)y' + 3xy &= 0 \\
(x^2+1)y' &= -3xy \\
y'y^{-1} &= \frac{-3x}{x^2+1} \\
\frac{dy}{y} &= \frac{-3x}{x^2+1}dx \\
\int \frac{1}{y}\,dy &= \int -\frac{3x}{x^2+1}\,dx \\
ln|y| &= -\frac{3}{2}ln|x^2+1|{+}C, \quad C \in \mathbb{R} \\
y &= \lambda(x^2+1)^{-3/2}, \quad \lambda = e^C \quad (1)
\end{aligned}
$$

Etape 2: EASM MVC

$$
\begin{aligned}
y' &= \lambda'(x^2+1)^{-\frac{3}{2}} + (-\frac{3}{2})\lambda(2x)(x^2+1)^{-\frac{3}{2}-1} \\
y' &= \lambda'(x^2+1)^{-\frac{3}{2}} - \frac{3}{2}\lambda(2x)(x^2+1)^{-\frac{5}{2}} \quad (2)
\end{aligned}
$$

(1) et (2) dans (E) ($(x^2+1)y' + 3xy = x^2$):

$$
\begin{aligned}
(x^2+1)(\lambda'(x^2+1)^{-\frac{3}{2}} - \frac{3}{2}\lambda(2x)(x^2+1)^{-\frac{5}{2}}) + 3x(\lambda(x^2+1)^{-3/2} &= x^2 \\
\lambda'(x^2+1)^{\frac{1}{2}} &= x^2 \\
\lambda' &= x^2(x^2+1)^{\frac{1}{2}} \\
d\lambda &= x^2(x^2+1)^{\frac{1}{2}}dx \\
\int d\lambda &= \int x^2(x^2+1)^{\frac{1}{2}}dx
\end{aligned}
$$

$$
\begin{aligned}
x^2 &= \sinh^2(t) \\
x &= \sinh(t) \\
dx &= \cosh(t)dt \\
x^2 + 1 &= \sinh^2(t) + 1 = \cosh^2(t) \\
\sqrt{x^2+1} &= \sqrt{\cosh^2(t)} = \cosh(t), \quad \cosh(t) > 0 \\
\int x^2\sqrt{x^2+1}\,dx &= \int \sinh^2(t)\cosh(t)\cosh(t)\,dt \\
\int x^2\sqrt{x^2+1}\,dx &= \int \sinh^2(t)\cosh^2(t)\,dt
\end{aligned}
$$

$$\sinh^2(t) = \frac{\cosh(2t)-1}{2} \quad , \ \cosh^2(t) = \frac{\cosh(2t)+1}{2}$$

$$\sinh^2(t)\cosh^2(t) = (\frac{\cosh(2t)-1}{2})(\frac{\cosh(2t)+1}{2})$$

$$\sinh^2(t)\cosh^2(t) = \frac{1}{4}(\cosh(2t)-1)(\cosh(2t)+1)$$

$$\sinh^2(t)\cosh^2(t) = \frac{1}{4}(\cosh^2(2t)-1)$$

$$\sinh^2(t)\cosh^2(t) = \frac{1}{4}\sinh^2(2t)$$

Donc

$$\int \sinh^2(t)\cosh^2(t)\,dt = \int \frac{1}{4}\sinh^2(2t)\,dt$$

$$\int \sinh^2(t)\cosh^2(t)\,dt = \frac{1}{4}\int \sinh^2(2t)\,dt$$

$$\text{or } \ \sinh^2(2t)\,dt = \int \frac{\cosh(4t)-1}{2}\,dt$$

$$\text{car } \ \sinh^2(t) = \frac{\cosh(2t)-1}{2}$$

$$\frac{1}{2}\int \cosh(4t) - \frac{1}{2}\int 1dt$$

$$\cosh(4t)dt = \frac{1}{4}\sinh(4t)$$

$$\int \sinh^2(t)dt = \frac{1}{2}.\frac{1}{4}\sinh(4t) - \frac{1}{2}t$$

$$\int \sinh^2(t)dt = \frac{1}{8}\sinh(4t) - \frac{1}{2}t + C, \quad C \int \mathbb{R}$$

$$\int \sinh^2(t)dt = \frac{1}{4}(\frac{1}{8}\sinh(4t) - \frac{1}{2}t + C)$$

$$\int \sinh^2(t)dt = \frac{1}{32}\sinh(4t) - \frac{1}{8}t + C$$

$$x = \sinh(t), t = arcsinh(x) = \ln(x + \sqrt{x^2+1})$$

$$\sinh(4t) = 2\sin(2t)\cosh(2t)$$

$$\sinh(2t) = 2\sinh(t)\cosh(t) = 2x\sqrt{1+x^2}$$

$$\cosh(2t) = \sqrt{1+\sinh^2(2t)} = \sqrt{1+(2x\sqrt{x^2+1})^2}$$
$$\cosh(2t) = \sqrt{1+4x^2(x^2+1)} = \sqrt{1+4x^4+4x^2}$$
$$\cosh(2t) = \sqrt{(2x^2+1)^2} = 2x^2+1, \quad x^2 >= 0$$

$$\sinh(4t) = 2(2x\sqrt{1+x^2})(2x^2+1)$$
$$\sinh(4t) = 4x(2x^2+1)\sqrt{1+x^2}$$

$$\frac{1}{32}\sinh(4t) - \frac{1}{8}t = \frac{1}{32}4x(2x^2+1)\sqrt{1+x^2} - \frac{1}{8}t + C$$
$$\int x^2\sqrt{x^2+1}dx = \frac{1}{8}x(2x^2+1)\sqrt{x^2+1} - \frac{1}{8}ln|x+\sqrt{x^2+1}|+C$$

$$\lambda = \frac{1}{8}x(2x^2+1)\sqrt{x^2-1} - \frac{1}{8}\ln|x+\sqrt{x^2+1}|+C$$
$$y = (\frac{1}{8}x(2x^2+1)\sqrt{x^2-1} - \frac{1}{8}\ln|x+\sqrt{x^2+1}|+C)(x^2+1)^{-\frac{3}{2}}$$

$$\boxed{y = (\frac{1}{8}x(2x^2+1)\sqrt{x^2-1} - \frac{1}{8}\ln|x+\sqrt{x^2+1}|+C)(x^2+1)^{-\frac{3}{2}}}$$

4. (E): $y' + y = 2e^x + 4\sin(x)$
Etape 1: ESSM

$$y' + y = 0$$
$$y' = -y$$
$$\frac{dy}{y} = -1$$
$$\int \frac{1}{y}dy = -\int 1dx$$
$$ln|y| = -x + C, \quad C \in \mathbb{R}$$
$$y = -e^{-x}.e^C$$
$$y = \lambda e^{-x} \quad (1)$$

Etape 2: EASM MVC

$$y' = -\lambda e^{-x} + \lambda' e^{-x}$$
$$y' = e^{-x}(\lambda + \lambda')(2)$$

(1) et (2) dans (E):

$$\begin{aligned}
(-\lambda + \lambda)e^{-x} + \lambda e^{-x} &= 2e^x + 4\sin(x) \\
\lambda' &= 2e^x + 4sin(x) \\
\frac{d\lambda}{dx}e^{-x} &= 2e^x + 4sin(x) \\
\int d\lambda &= \int (2e^{2x} + 4e^x \sin(x))\, dx \\
\int d\lambda &= \int 2e^{2x}\, dx + 4\int e^x \sin(x))\, dx \\
\int d\lambda &= e^{2x} + C + 4[-\cos(x)e^x + \int e^x \cos(x)\, dx]
\end{aligned}$$

Soit $I = \int e^x \cos(x)$

$$\begin{cases} u = e^x \implies u' = e^x \\ v = -\cos(x) \implies v' = \sin(x) \end{cases}$$

$-\cos(x)e^x + \int e * x \cos(x)\, dx$

$$\begin{cases} u = e^x \implies v' = \cos(x) \\ u' = e^x \implies v' = \sin(x) \end{cases}$$

$$-e^x cos(x) + -e^x sin(x) - \int e^x \sin(x))\, dx$$
$$-e^x cos(x) + -e^x sin(x) - I$$
$$\begin{aligned}
I &= e^x(sin(x) - cos(x)) - I \\
2I &= e^x(sin(x) - cos(x)) \\
I &= \frac{e^x}{2}(sin(x) - cos(x))
\end{aligned}$$

$$\int d\lambda = e^{2x} + c + 2e^{x}(sin(x) - cos(x)) + k$$
$$\lambda x = e^{2x} + e^{2x}(\sin(x) - \cos(x)) + C$$
$$\lambda x = e^{x}(e^{x} + 2(\sin(x) - \cos(x)) + C$$

$$y = e^{x}.e^{-x}(e^{x} + 2(\sin(x) - \cos(x)) + Ce^{-x}$$
$$y = e^{x} + 2\sin(x) - 2\cos(x) + Ce^{-x}$$

$\boxed{y = e^{x} + 2\sin(x) - 2\cos(x) + Ce^{-x}}$

5. (E): $y' - 2y = 2x^3 + x$ ou $y(3) = 1$
   Etape 1: ESSM

$$y = \lambda e^{\int(-\frac{-2}{1})}$$

6. (E): $y' + 2xy = e^{x-x^2}$

7. (E): $y' \cos(2y) - \sin(y) = 0$

Exercice 2:

1. (E): $xy' = y + 3xy^2$

$$\frac{x}{x}y' = \frac{1}{x}y + \frac{3xy^2}{x}$$
$$y' = \frac{1}{x}y + 3y^2$$

On pose $z = y^{-1} = \frac{1}{y} \implies z' = -\frac{y'}{y^2}$

$$\begin{aligned} -\frac{y'}{y^2} &= -\frac{1}{xy} - 3 \\ \text{d'ou } z' &= -\frac{1}{x}z - 3 \\ z' + \frac{1}{x}z &= -3 \quad (E) \end{aligned}$$

<u>ESSM</u>
$z' + \frac{1}{x}z = 0$

$$\begin{aligned} z_H &= \lambda e^{-\int \frac{1}{x}}, \quad \lambda \in \mathbb{R} \\ &= \lambda e^{-ln|x|} \\ z_H &= \lambda \frac{1}{x}, \quad \lambda \in \mathbb{R} \end{aligned}$$

<u>EASM</u>

MVC:

$$\begin{aligned} t &= g(x)\frac{1}{x} \quad (1) \\ t' &= \frac{g'(x)x - g(x)}{x^2} \quad (2) \end{aligned}$$

(1) et (2) dans (E)

$$\begin{aligned} \frac{g'(x)x - g(x)}{x^2} + \frac{1}{x}g(x)\frac{1}{x} &= -3 \\ \frac{g'(x)x}{x^2} - \frac{g(x)}{x^2} + \frac{g(x)}{x^2} &= -3 \\ g'(x) &= -3x \\ g(x) &= -\frac{3}{2}x^2 + C, \quad C \in \mathbb{R} \end{aligned}$$

$$
\begin{aligned}
z_p &= (-\frac{3}{2}x^2 + C)\frac{1}{x} + \lambda(\frac{1}{x}) \\
z_p &= (-\frac{3}{2}x + \psi(\frac{1}{x} + \frac{1}{x})), \quad \psi = \frac{C}{x} \\
z_p &= -\frac{3}{2}x + \psi\frac{2}{x} \\
z_p &= -\frac{3x^2 + 4\psi}{2x}
\end{aligned}
$$

$z = z_H + z_P$ et $y = y_H + y_P$
or $z = \frac{1}{y}$
Alors $y = \frac{1}{z_H} + \frac{1}{z_P}$ $\boxed{y = \dfrac{x}{\lambda} + \dfrac{2x}{3x^2 + 4\psi}, \quad \lambda, \psi \in \mathbb{R}}$

2. (E): $y' + \frac{y}{x+1} = \frac{1}{2}(x+1)^3 y^3$
   C'est une equation de Bernoulli

$$
\begin{aligned}
\frac{y'}{y^3} + \frac{y}{(x+1)y^3} &= \frac{1}{2}(x+3)^3 \\
\frac{y'}{y^3} + \frac{(x+1)^{-1}}{y^2} &= \frac{1}{2}(x+3)^3 \quad (E)
\end{aligned}
$$

Changement de variable

$$
\begin{aligned}
u = \frac{1}{y^2} &= y^{-2} \\
\frac{du}{dx} &= -2y'y^{-3} \\
-\frac{1}{2}u' &= \frac{y'}{y^3} \quad (A)
\end{aligned}
$$

D'ou
$(E1):\quad \frac{1}{-2}u' + (x+1)^{-1}u = \frac{1}{2}(x+1)^3$

Etape 1: ESSM

$$
\begin{aligned}
-\frac{1}{2}u' + (x+1)^{-1}u &= 0 && (1)\\
\frac{1}{2}u' &= (x+1)^{-1}u && (2)\\
\frac{du}{dx} &= 2(x+1)^{-1}u && (3)\\
\frac{du}{u} &= 2(x+1)^{-1}dx && (4)\\
\ln|u| &= 2\ln|x+1| + C, \quad C \in \mathbb{R} && (5)\\
\ln|u| &= \ln|(x+1)^2| + C && (6)\\
u &= e^{\ln|(x+1)^2|} \times e^{C} && (7)\\
u &= (x+1)^2 \times \lambda, \quad \lambda = e^{C} \quad (E1_1) && (8)
\end{aligned}
$$

Etape 2: EASM

$$u' = \lambda'(x+1)^2 + 2(x+1)\lambda \quad (E1_2)$$

$(E1_1)$ et $(E1_2)$ dans (E1)

$$
\begin{aligned}
-\frac{1}{2}[\lambda'(x+1)^2 + 2(x+1)\lambda] + (x+1)^{-1}(x+1)^2\lambda &= \frac{1}{2}(x+1)^3\\
-\frac{\lambda'}{2}(x+1)^2 - (x+1)\lambda + (x+1)\lambda &= \frac{1}{2}(x+1)^3\\
-\frac{\lambda'}{2}(x+1)^2 &= \frac{1}{2}(x+1)^3\\
\lambda' &= -(x+1)\\
\frac{d\lambda}{dx} &= -(x+1)\\
d\lambda &= -(x+1)dx\\
\int d\lambda &= -\int (x+1)\,dx\\
\lambda &= -(\frac{x^2}{2} + x) + K, \quad K \in \mathbb{R}
\end{aligned}
$$

La solution generale de (E1) est:

$$\underline{u = \left(-\left(\frac{x^2}{2} + x\right) + K\right)(x+1)^2}$$

$$
\begin{aligned}
u &= (-\frac{x^2}{2} - x + K)(x+1)^2 \\
u' &= (-\frac{2}{2}x - 1)(x+1)^2 + 2(x+1)(-\frac{x^2}{2} - x + K) \\
u' &= (-x-1)(x+1)^2 + 2(x+1)(-\frac{x^2}{2} - x + K) \\
u' &= -(x+1)^3 + 2(x+1)(-\frac{x^2}{2} - x + K) \\
u' &= (x+1)[-(x+1)^2 + 2(-\frac{x^2}{2} - x + 2K)] \\
u' &= (x+1)[-(x+1)^2 - x^2 - 2x + 2K] \\
u' &= (x+1)[-x^2 - 2x - 1 - x^2 - 2x + 2K] \\
u' &= (x+1)[-2x^2 - 4x - 1 + 2K] \quad (B)
\end{aligned}
$$

En reprenant (A)

$$
\begin{aligned}
\frac{y'}{y^3} &= -\frac{1}{2}u' \\
\frac{dy}{dx} &= -\frac{1}{2}u'y^3 \\
\frac{dy}{y^3} &= -\frac{1}{2}u'dx \\
\int \frac{dy}{y^3} &= -\frac{1}{2}\int u'dx \\
\int y^{-3}dy &= -\frac{1}{2}\int u'dx \\
\frac{y^{-2}}{-2} &= -\frac{1}{2}\int u'dx \\
y^{-2} &= \int u'dx \quad (A')
\end{aligned}
$$

(B) dans (A')

$$
\begin{aligned}
y^{-2} &= \int (x+1)[-2x^2-4x-1+2K]\,dx \\
&= \int (x+1)(-2x^2)+(x+1)(-4x)+(x+1)(-1)+(x+1)(2K)\,dx \\
&= \int \big(-2x^3-2x^2-4x^2-4x-x-1+2Kx+2K\big)\,dx \\
&= \int \big(-2x^3-6x^2-5x-1+2Kx+2K\big)\,dx \\
&= -\frac{2}{4}x^4-\frac{6}{3}x^3-\frac{5}{2}x^2-x+Kx^2+2Kx+C \\
y^{-2} &= -\frac{1}{2}x^4-2x^3-\frac{5}{2}x^2-x+Kx^2+2Kx+C \\
\implies y^2 &= (-\frac{1}{2}x^4-2x^3-\frac{5}{2}x^2-x+Kx^2+2Kx+C)^{-1} \\
\implies y &= \pm(-\frac{1}{2}x^4-2x^3-\frac{5}{2}x^2-x+Kx^2+2Kx+C)^{-\frac{1}{2}}
\end{aligned}
$$

$$\boxed{y = \pm(-\frac{1}{2}x^4-2x^3-\frac{5}{2}x^2-x+Kx^2+2Kx+C)^{-\frac{1}{2}}}$$

3. (E): $y'(1-\sin(s)\cos(x))+y^2\cos(x)-y'+\sin(x)=0$
   si $y=\cos(x)$ est une solution particuliere

4. (E): $y = xy'+(y')^3$