S4 - 17B. Devoir 1

ETU003235: ANDERSON Soamiavaka Vanille
ETU003247: ANDRIANAJA Onja Fanilo
ETU003286: RABETOKOTANY Yvan Noah
ETU003298: RAJAONARIVONY Tandrifiniaina Dylan
ETU003305: RAKOTOARIVONY Loïc Dylan
ETU003331: RANAIVOSON Miora Randie
ETU003335: RANDRIAMAHEFA Liantsoa Alicia
ETU003348: RANDRIANIRINA Niriela Andraina
ETU003363: RATSITO Oelirivo Mitia
ETU003378: RAZAKANDISA Sariaka Niaina

Exercice 1:

1. (E): $y' = \sin(y)$

$$\frac{dy}{dx} = \sin(y)$$
$$\frac{dy}{sin(y)} = dx$$
$$\int \frac{dy}{sin(y)} = \int dx$$

<u>Calcul de $\int \frac{1}{\sin(x)} dy$</u>

$$\text{Posons } t = \tan(\frac{y}{2})$$
$$\text{donc } dt = \frac{1}{2}(1 + \tan^2(\frac{y}{2}))dy$$
$$dt = \frac{1}{2}(1 + t^2)dy$$
$$dy = \frac{2dt}{1 + t^2}$$

$$\sin(y) = \frac{2t}{1 + t^2}$$
$$\int \frac{1}{\sin(y)} dy = \int (\frac{1 + t^2}{2t})(\frac{2dt}{1 + t^2})$$
$$\int \frac{1}{\sin(y)} dy = \int \frac{1}{t}(dt)$$
$$\int \frac{1}{\sin(y)} dy = ln|t| + C, \quad C \in \mathbb{R}$$
$$\text{Or } t = \tan(\frac{y}{2}) \implies \int \frac{1}{\sin(y)} dy = ln|\tan(\frac{y}{2})| + C$$

On a:

$$
\begin{aligned}
ln|\tan(\frac{y}{2})| &= x + C, \quad C \in \mathbb{R} \\
|\tan(\frac{y}{2})| &= e^x \cdot e^C \\
\tan(\frac{y}{2}) &= ke^x, \quad k = e^C \\
\frac{y}{2} &= \tan^{-1}(ke^x) \\
y &= 2\tan^{-1}(ke^x)
\end{aligned}
$$

$$\boxed{y = 2\tan^{-1}(ke^x), \quad k \in \mathbb{R}}$$

---

2. (E): $x^2y' = e^y$

$$
\begin{aligned}
x^2y' &= e^y \\
x^2\frac{dy}{dx} &= e^y \\
\frac{dy}{e^y} &= \frac{dx}{x^2} \\
\int \frac{1}{e^y}\,dy &= \int \frac{1}{x^2}\,dx \\
\int e^{-y}\,dy &= \int x^{-2}\,dx \\
-e^{-y} &= -\frac{1}{x} + C, \quad C \in \mathbb{R} \\
\ln(e^{-y}) &= \ln|\frac{1}{x} + C| \\
-y &= \ln|\frac{1}{x} + C| \\
y &= -\ln|\frac{1}{x} + C|
\end{aligned}
$$

$$\boxed{y = -\ln|\frac{1}{x} + C|, \quad C \in \mathbb{R}}$$

3. (E): $(x^2+1)y' + 3xy = x^2$

   Etape 1: ESSM

$$\begin{aligned}(x^2+1)y' + 3xy &= 0\\ (x^2+1)y' &= -3xy\\ y'y^{-1} &= \frac{-3x}{x^2+1}\\ \frac{dy}{y} &= \frac{-3x}{x^2+1}dx\\ \int \frac{1}{y}\,dy &= \int -\frac{3x}{x^2+1}\,dx\\ ln|y| &= -\frac{3}{2}ln|x^2+1|+C, \quad C \in \mathbb{R}\\ y &= \lambda(x^2+1)^{-3/2}, \quad \lambda = e^C \quad (1)\end{aligned}$$

   Etape 2: EASM MVC

$$\begin{aligned}y' &= \lambda'(x^2+1)^{-\frac{3}{2}} + (-\frac{3}{2})\lambda(2x)(x^2+1)^{-\frac{3}{2}-1}\\ y' &= \lambda'(x^2+1)^{-\frac{3}{2}} - \frac{3}{2}\lambda(2x)(x^2+1)^{-\frac{5}{2}} \quad (2)\end{aligned}$$

   (1) et (2) dans (E) ($(x^2+1)y' + 3xy = x^2$):

$$\begin{aligned}(x^2+1)(\lambda'(x^2+1)^{-\frac{3}{2}} - \frac{3}{2}\lambda(2x)(x^2+1)^{-\frac{5}{2}}) + 3x(\lambda(x^2+1)^{-3/2} &= x^2\\ \lambda'(x^2+1)^{\frac{1}{2}} &= x^2\\ \lambda' &= x^2(x^2+1)^{\frac{1}{2}}\\ d\lambda &= x^2(x^2+1)^{\frac{1}{2}}dx\\ \int d\lambda &= \int x^2(x^2+1)^{\frac{1}{2}}dx\end{aligned}$$

$$
\begin{aligned}
x^2 &= \sinh^2(t) \\
x &= \sinh(t) \\
dx &= \cosh(t)dt \\
x^2 + 1 &= \sinh^2(t) + 1 = \cosh^2(t) \\
\sqrt{x^2 + 1} &= \sqrt{\cosh^2(t)} = \cosh(t), \quad \cosh(t) > 0 \\
\int x^2\sqrt{x^2 + 1}\, dx &= \int \sinh^2(t) \cosh(t) \cosh(t)\, dt \\
\int x^2\sqrt{x^2 + 1}\, dx &= \int \sinh^2(t) \cosh^2(t)\, dt
\end{aligned}
$$

$$\sinh^2(t) = \frac{\cosh(2t) - 1}{2} \quad , \ \cosh^2(t) = \frac{\cosh(2t) + 1}{2}$$

$$\sinh^2(t)\cosh^2(t) = (\frac{\cosh(2t) - 1}{2})(\frac{\cosh(2t) + 1}{2})$$

$$\sinh^2(t)\cosh^2(t) = \frac{1}{4}(\cosh(2t) - 1)(\cosh(2t) + 1)$$

$$\sinh^2(t)\cosh^2(t) = \frac{1}{4}(\cosh^2(2t) - 1)$$

$$\sinh^2(t)\cosh^2(t) = \frac{1}{4}\sinh^2(2t)$$

Donc

$$\int \sinh^2(t)\cosh^2(t)\,dt = \int \frac{1}{4}\sinh^2(2t)\,dt$$

$$\int \sinh^2(t)\cosh^2(t)\,dt = \frac{1}{4}\int \sinh^2(2t)\,dt$$

$$\text{or } \sinh^2(2t)\,dt = \int \frac{\cosh(4t) - 1}{2}\,dt$$

$$\text{car } \sinh^2(t) = \frac{\cosh(2t) - 1}{2}$$

$$\frac{1}{2}\int \cosh(4t) - \frac{1}{2}\int 1dt$$

$$\cosh(4t)dt = \frac{1}{4}\sinh(4t)$$

$$\int \sinh^2(t)dt = \frac{1}{2}.\frac{1}{4}\sinh(4t) - \frac{1}{2}t$$

$$\int \sinh^2(t)dt = \frac{1}{8}\sinh(4t) - \frac{1}{2}t + C, \quad C \int \mathbb{R}$$

$$\int \sinh^2(t)dt = \frac{1}{4}(\frac{1}{8}\sinh(4t) - \frac{1}{2}t + C)$$

$$\int \sinh^2(t)dt = \frac{1}{32}\sinh(4t) - \frac{1}{8}t + C$$

$$x = \sinh(t), t = arcsinh(x) = \ln(x + \sqrt{x^2 + 1})$$

$$\sinh(4t) = 2\sin(2t)\cosh(2t)$$

$$\sinh(2t) = 2\sinh(t)\cosh(t) = 2x\sqrt{1 + x^2}$$

$$\cosh(2t) = \sqrt{1+\sinh^2(2t)} = \sqrt{1+(2x\sqrt{x^2+1})^2}$$
$$\cosh(2t) = \sqrt{1+4x^2(x^2+1)} = \sqrt{1+4x^4+4x^2}$$
$$\cosh(2t) = \sqrt{(2x^2+1)^2} = 2x^2+1, \quad x^2 >= 0$$

$$\sinh(4t) = 2(2x\sqrt{1+x^2})(2x^2+1)$$
$$\sinh(4t) = 4x(2x^2+1)\sqrt{1+x^2}$$

$$\frac{1}{32}\sinh(4t) - \frac{1}{8}t = \frac{1}{32}4x(2x^2+1)\sqrt{1+x^2} - \frac{1}{8}t + C$$
$$\int x^2\sqrt{x^2+1}dx = \frac{1}{8}x(2x^2+1)\sqrt{x^2+1} - \frac{1}{8}ln|x+\sqrt{x^2+1}|+C$$

$$\lambda = \frac{1}{8}x(2x^2+1)\sqrt{x^2-1} - \frac{1}{8}\ln|x+\sqrt{x^2+1}|+C$$
$$y = (\frac{1}{8}x(2x^2+1)\sqrt{x^2-1} - \frac{1}{8}\ln|x+\sqrt{x^2+1}|+C)(x^2+1)^{-\frac{3}{2}}$$

$$\boxed{y = (\frac{1}{8}x(2x^2+1)\sqrt{x^2-1} - \frac{1}{8}\ln|x+\sqrt{x^2+1}|+C)(x^2+1)^{-\frac{3}{2}}, \quad C \in \mathbb{R}}$$

---

4. (E): $y' + y = 2e^x + 4\sin(x)$

   Etape 1: ESSM

$$y' + y = 0$$
$$y' = -y$$
$$\frac{dy}{y} = -1$$
$$\int \frac{1}{y}dy = -\int 1dx$$
$$ln|y| = -x + C, \quad C \in \mathbb{R}$$
$$y = -e^{-x}.e^C$$
$$y = \lambda e^{-x} \quad (1)$$

<u>Etape 2</u>: EASM MVC

$$y' = -\lambda e^{-x} + \lambda' e^{-x}$$
$$y' = e^{-x}(\lambda + \lambda')(2)$$

$$\begin{aligned}
&(1) \text{ et } (2) \text{ dans (E)}:\\
(-\lambda + \lambda)e^{-x} + \lambda e^{-x} &= 2e^{x} + 4\sin(x)\\
\lambda' &= 2e^{x} + 4sin(x)\\
\frac{d\lambda}{dx}e^{-x} &= 2e^{x} + 4sin(x)\\
\int d\lambda &= \int (2e^{2x} + 4e^{x}\sin(x))\,dx\\
\int d\lambda &= \int 2e^{2x}\,dx + 4\int e^{x}\sin(x))\,dx\\
\int d\lambda &= e^{2x} + C + 4[-\cos(x)e^{x} + \int e^{x}\cos(x)\,dx]
\end{aligned}$$

Soit $I = \int e^{x}\cos(x)$

$$\begin{cases} u = e^{x} \implies u' = e^{x} \\ v = -\cos(x) \implies v' = \sin(x) \end{cases}$$

$-\cos(x)e^{x} + \int e^{x}\cos(x)\,dx$

$$\begin{cases} u = e^{x} \implies v' = \cos(x) \\ u' = e^{x} \implies v' = \sin(x) \end{cases}$$

$$-e^{x}cos(x) + -e^{x}sin(x) - \int e^{x}\sin(x))\,dx$$
$$-e^{x}cos(x) + -e^{x}sin(x) - I$$
$$\begin{aligned}
I &= e^{x}(sin(x) - cos(x)) - I\\
2I &= e^{x}(sin(x) - cos(x))\\
I &= \frac{e^{x}}{2}(sin(x) - cos(x))
\end{aligned}$$

$$\int d\lambda = e^{2x} + c + 2e^{x}(sin(x) - cos(x)) + k$$
$$\lambda x = e^{2x} + e^{2x}(\sin(x) - \cos(x)) + C$$
$$\lambda x = e^{x}(e^{x} + 2(\sin(x) - \cos(x)) + C$$

$$y = e^{x}.e^{-x}(e^{x} + 2(\sin(x) - \cos(x)) + Ce^{-x}$$
$$y = e^{x} + 2\sin(x) - 2\cos(x) + Ce^{-x}$$

$$\boxed{y = e^{x} + 2\sin(x) - 2\cos(x) + Ce^{-x}, \quad C \in \mathbb{R}}$$

---

5. (E): $y' - 2y = 2x^3 + x$ ou $y(0) = 1$

   Etape 1: ESSM

$$y' - 2y = 0$$
$$y' = 2y$$
$$\frac{dy}{dx} = 2y$$
$$\int \frac{dy}{y} = \int 2dx$$
$$\ln|y| = 2x + c, \quad c \in \mathbb{R}$$
$$y = e^{2x+c}$$
$$y = e^{2x} \cdot e^{c}$$
$$\text{On pose } \lambda = e^{c}$$
$$\underline{y = \lambda e^{2x}}$$

Etape 2: EASM MVC

$y' - 2y = 2x^3 + x$ avec $y = \lambda e^{2x}$

$$y' = \lambda e^{2x} + 2e^{2x}\lambda \quad (2)$$

(1) et (2) dans (E):

$$\begin{aligned}
\lambda e^{2x} + 2e^{2x}\lambda - 2(\lambda e^{2x}) &= 2x^3 + x \\
\lambda e^{2x} &= 2x^3 + x \\
\frac{d\lambda}{dx} &= \frac{2x^3 + x}{e^{2x}} \\
d\lambda &= \frac{2x^3 + x}{e^{2x}} dx \\
\int d\lambda &= \int \frac{2x^3 + x}{e^{2x}} dx \\
\lambda &= \int \frac{2x^3}{e^{2x}} dx + \int \frac{x}{e^{2x}} dx
\end{aligned}$$

Posons $A = \int \frac{2x^3}{e^{2x}} dx$ et $B = \int \frac{x}{e^{2x}} dx$

(1) <u>Calculons A</u>:

$$\begin{aligned}
B &= \int \frac{x}{e^{2x}} dx \\
B &= \int e^{-2x} x dx \\
B &= -\frac{1}{2} \int -2e^{-2x} x dx
\end{aligned}$$

Posons $K = \int -2e^{-2x} x dx$

Rappelons l'integration par partie

$\int u'v = uv - \int v'u$

$$\begin{aligned}
&\text{avec } u' = -2e^{-2x} && v = x \\
&\text{alors } u = e^{-}2x && v' = 1
\end{aligned}$$

$$\begin{aligned}
K &= e^{-2x} x - \int 1e^{-2x} \\
K &= e^{-2x} x + \frac{1}{2} \int -2e^{-2x} \\
K &= e^{-2x} x + \frac{1}{2} e^{-2x}
\end{aligned}$$

$$\underline{K = e^{-2x} x + \frac{1}{2} e^{-2x}}$$

$$\text{Donc } B = -\frac{1}{2}(e^{-2x}x + \frac{1}{2}e^{-2x})$$
$$B = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$

$\underline{B = -\frac{1}{2}e^{-2x}x - \frac{1}{4}e^{-2x}}$

(2) Calculons maintenant A:

$$A = \int 2x^3 \cdot e^{-2x}dx$$
$$A = -\int -2x^3 \cdot e^{-2x}dx$$

Posons $M = \int 2x^3 \cdot e^{-2x}dx$
Rappelons l'integration par partie
$\int u'v = uv - \int v'u$

avec $u' = -2e^{-2x}$ $\qquad$ $u = e^{-2x}$
alors $v = x^3$ $\qquad$ $v' = 3x^2$

$$M = e^{-2x}x^3 - \int 3x^2e^{-2x}dx$$
$$M = e^{-2x}x^3 + \frac{3}{2}\int -2x^2e^{-2x}dx$$

Posons $N = \int -2x^2e^{-2x}dx$

avec $u' = -2e^{-2x}$ $\qquad$ $v = x^2$
alors $u = e^{-2x}$ $\qquad$ $v' = 2x$

$$N = e^{-2x}x^2 - \int +2xe^{-2x}dx$$
$$N = e^{-2x}x^2 + \int -2xe^{-2x}dx$$

D'apres le calcul de K, on peut avoir:
$\underline{N = e^{-2x}x^2 + e^{-2x}x + \frac{1}{2}e^{-2x}}$

On peut aussi alors ecrire:
$M = e^{-2x}x^3 + \frac{3}{2}e^{-2x}x^2 + \frac{3}{2}e^{-2x} + \frac{3}{4}e^{-2x}$
Et on peut aussi avoir:
$A = -x^3e^{-2x} - \frac{3}{2}x^2e^{-2x} - \frac{3}{2}xe^{-2x} - \frac{3}{4}e^{-2x}$
On a alors trouve $\lambda = A + B$

$$\lambda = -x^3e^{-2x} - \frac{3}{2}x^2e^{-2x} - \frac{3}{2}xe^{-2x} - \frac{3}{4}e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$
$$\lambda = -e^{-2x}(x^3 + \frac{3}{2}x^2 + 2x + 1)$$

Et comme $y = \lambda e^{2x}$
$$\boxed{y = -x^3 - \frac{3}{2}x^2 - 2x - 1}$$

---

6. (E): $y' + 2xy = e^{x-x^2}$
   Etape 1: ESSM

$$\begin{aligned} y' - 2xy &= 0 \\ y' &= 2xy \\ \frac{dy}{dx} &= 2xy \\ \int \frac{dy}{y} &= \int 2xdx \\ \ln|y| &= x^2 + k, \quad k \in \mathbb{R} \\ y_0 &= e^{x^2} + e^k \\ y_0 &= e^{x^2} \cdot c, \quad c = e^k \end{aligned}$$

Etape 2: EASM
$y' - 2xy = e^{x-x^2} \quad (E)$
MVC:

$$y = ce^{x^2} \quad (1)$$
$$y' = c'e^{x^2} + c2xe^{x^2} \quad (2)$$

(1) et (2) dans (E):

$$\begin{aligned}
c'e^{x^2} + c2xe^{x^2} - c2xe^{x^2} &= e^{x-x^2} \\
c'e^{x^2} &= e^{x-x^2} \\
\frac{dc}{dx} \cdot e^{x^2} &= e^{x-x^2} \\
\int dc &= \int \frac{e^{x-x^2}}{e^{x^2}}\,dx \\
c &= \int \frac{e^x}{e^{x^2} \cdot e^{x^2}} \\
c &= \int \frac{e^x}{e^{2x^2}}\,dx \\
c &= \int \frac{1}{e^{2x^2}} + \int e^x\,dx \\
c &= \int e^{-2x} + \int e^x\,dx \\
c &= -\frac{-e^{-2x^2}}{4x} + e^x + \lambda, \quad \lambda \in \mathbb{R}
\end{aligned}$$

Or $y = c \cdot e^{x^2}$

$$\begin{aligned}
y &= (-\frac{e^{-2x^2} \cdot e^{x^2}}{4x} + e^x \cdot e^{x^2}) + \lambda e^{x^2} \\
&= (-\frac{e^{x^2-2x^2}}{4x} + e^x \cdot e^{x^2}) + \lambda e^{x^2} \\
y &= y_p + y_0
\end{aligned}$$

$$\boxed{y = (-\frac{e^{-x^2}}{4x} + e^{x^2+x} + \lambda e^{x^2}), \quad \lambda \in \mathbb{R}}$$

---

7. (E): $y' \cos(2y) - \sin(y) = 0$

$$\begin{aligned} y'cos(2y) - sin(y) &= 0, \\ y'cos(2y) &= sin(y) \\ y'\frac{cos(2y)}{cos(2y)} &= \frac{sin(y)}{cos(2y)}, \\ y' &= \frac{sin(y)}{cos(2y)} or cos(2y) = 1 - 2cos(y), \\ y' &= \frac{sin(y)}{1 - 2cos^2(y)} \end{aligned}$$

Posons $u = cos(y)$
Alors $\frac{du}{dy} = -sin(y) \Rightarrow dy = \frac{du}{-sin(y)}$,
On a alors $: y = \int \frac{sin(y)}{1-2u^2} \cdot \frac{du}{-sin(y)}$
$y = \int \frac{1}{2u^2-1}$
Posons $w = \sqrt{2}u \Rightarrow u = \frac{w}{\sqrt{2}}$
Alors $\frac{dw}{du} = \sqrt{2} \Rightarrow w = \frac{dw}{\sqrt{2}}$

$$\begin{aligned} y &= \int \frac{1}{w^2 - 1} \cdot \frac{dw}{\sqrt{2}} \\ y &= \frac{1}{\sqrt{2}} \int \frac{1}{w^2 - 1} dw \\ y &= \frac{1}{\sqrt{2}} \int \frac{1}{(w-1)(w+1)} dw \end{aligned}$$

<u>Simplifions dabord la primitive</u>
$\frac{1}{(w-1)(w+1)} = \frac{A}{w-1} + \frac{B}{w+1} (E), \quad A, B \in \mathbb{R},$
On multiplie chaque terme par (x-1)
$\frac{1}{(w-1)} = A + \frac{B(w-1)}{(w+1)}$
Pour u = 1,
$\frac{1}{2} = A \quad (1)$
On multiplie chaque terme par (x+1)

$\frac{1}{(w+1)} = \frac{A(w+1)}{(w-1)} + B$
Pour u = -1,
$-\frac{1}{2} = B \quad (2)$

(1) et (2) dans (E):

$$\frac{1}{(w-1)(w+1)} = \frac{\frac{1}{2}}{(w-1)} + \frac{-\frac{1}{2}}{(w+1)}$$
$$\frac{1}{(w-1)(w+1)} = \frac{1}{2(w-1)} - \frac{1}{2(w+1)}$$
$$\text{On a alors: } y = \frac{1}{\sqrt{2}}[\int \frac{1}{2(w-1)} - \int \frac{1}{2(w+1)}]$$

$$y = \frac{1}{\sqrt{2}}[\frac{1}{2}\int \frac{1}{(w-1)} - \frac{1}{2}\int \frac{1}{(w+1)}]$$
$$y = \frac{1}{2\sqrt{2}}[\int \frac{1}{(w-1)} - \int \frac{1}{(w+1)}]$$
$$y = \frac{\sqrt{2}}{4}[ln|w-1| + C - ln|w+1| + K], \quad C, K \in \mathbb{R}$$
$$y = \frac{\sqrt{2}}{4}[ln|w-1| - ln|w+1| + T], T \in \mathbb{R}$$

Retour au changement de variable

$$w = \sqrt{2}u$$
$$\text{Alors } y = \frac{\sqrt{2}}{4}(ln|\sqrt{2}u - 1| - ln|\sqrt{2}u + 1| + T), T \in \mathbb{R}$$
$$u = cos(y)$$
$$\text{Alors } y = \frac{\sqrt{2}}{4}(ln|\sqrt{2}cos(y) - 1| - ln|\sqrt{2}cos(y) + 1| + T), T \in \mathbb{R}$$

$$\boxed{y = \frac{\sqrt{2}}{4}(ln|\sqrt{2}cos(y) - 1| - ln|\sqrt{2}cos(y) + 1| + T), T \in \mathbb{R}}$$

Exercice 2:

Resolution des équations différentielles non linéaires du 1er ordre

1. (E): $xy' = y + 3xy^2$

$$\frac{x}{x}y' = \frac{1}{x}y + \frac{3xy^2}{x}$$
$$y' = \frac{1}{x}y + 3y^2$$

On pose $z = y^{-1} = \frac{1}{y} \implies z' = -\frac{y'}{y^2}$

$$-\frac{y'}{y^2} = -\frac{1}{xy} - 3$$
$$\text{d'ou } z' = -\frac{1}{x}z - 3$$
$$z' + \frac{1}{x}z = -3 \quad (E)$$

ESSM
$z' + \frac{1}{x}z = 0$

$$z_H = \lambda e^{-\int \frac{1}{x}}, \quad \lambda \in \mathbb{R}$$
$$= \lambda e^{-ln|x|}$$
$$z_H = \lambda \frac{1}{x}, \quad \lambda \in \mathbb{R}$$

EASM

MVC:

$$t = g(x)\frac{1}{x} \quad (1)$$
$$t' = \frac{g'(x)x - g(x)}{x^2} \quad (2)$$

(1) et (2) dans (E)

$$\begin{aligned}
\frac{g'(x)x - g(x)}{x^2} + \frac{1}{x}g(x)\frac{1}{x} &= -3 \\
\frac{g'(x)x}{x^2} - \frac{g(x)}{x^2} + \frac{g(x)}{x^2} &= -3 \\
g'(x) &= -3x \\
g(x) &= -\frac{3}{2}x^2 + C, \quad C \in \mathbb{R}
\end{aligned}$$

$$\begin{aligned}
z_p &= (-\frac{3}{2}x^2 + C)\frac{1}{x} + \lambda(\frac{1}{x}) \\
z_p &= (-\frac{3}{2}x + \psi(\frac{1}{x} + \frac{1}{x})), \quad \psi = \frac{C}{x} \\
z_p &= -\frac{3}{2}x + \psi\frac{2}{x} \\
z_p &= -\frac{3x^2 + 4\psi}{2x}
\end{aligned}$$

$z = z_H + z_P$ et $y = y_H + y_P$
or $z = \frac{1}{y}$
Alors $y = \frac{1}{z_H} + \frac{1}{z_P}$ $\boxed{y = \frac{x}{\lambda} + \frac{2x}{3x^2 + 4\psi}, \quad \lambda, \psi \in \mathbb{R}}$

---

2. (E): $y' + \frac{y}{x+1} = \frac{1}{2}(x+1)^3 y^3$
   <u>C'est une equation de Bernoulli</u>

$$\begin{aligned}
\frac{y'}{y^3} + \frac{y}{(x+1)y^3} &= \frac{1}{2}(x+3)^3 \\
\frac{y'}{y^3} + \frac{(x+1)^{-1}}{y^2} &= \frac{1}{2}(x+3)^3 \quad (E)
\end{aligned}$$

Changement de variable

$$
\begin{aligned}
u = \frac{1}{y^2} = y^{-2} \\
\frac{du}{dx} = -2y'y^{-3} \\
-\frac{1}{2}u' = \frac{y'}{y^3} \quad (A)
\end{aligned}
$$

D'ou
$(E1):\quad \frac{1}{-2}u' + (x+1)^{-1}u = \frac{1}{2}(x+1)^3$

<u>Etape 1: ESSM</u>

$$
\begin{aligned}
-\frac{1}{2}u' + (x+1)^{-1}u = 0 && (1)\\
\frac{1}{2}u' = (x+1)^{-1}u && (2)\\
\frac{du}{dx} = 2(x+1)^{-1}u && (3)\\
\frac{du}{u} = 2(x+1)^{-1}dx && (4)\\
\ln|u| = 2\ln|x+1| + C, \quad C \in \mathbb{R} && (5)\\
\ln|u| = \ln|(x+1)^2| + C && (6)\\
u = e^{\ln|(x+1)^2|} \times e^C && (7)\\
u = (x+1)^2 \times \lambda, \quad \lambda = e^C \quad (E1_1) && (8)
\end{aligned}
$$

<u>Etape 2: EASM</u>

$$u' = \lambda'(x+1)^2 + 2(x+1)\lambda \quad (E1_2)$$

$(E1_1)$ et $(E1_2)$ dans (E1)

$$
\begin{aligned}
-\frac{1}{2}[\lambda'(x+1)^2+2(x+1)\lambda]+(x+1)^{-1}(x+1)^2\lambda &= \frac{1}{2}(x+1)^3 \\
-\frac{\lambda'}{2}(x+1)^2-(x+1)\lambda+(x+1)\lambda &= \frac{1}{2}(x+1)^3 \\
-\frac{\lambda'}{2}(x+1)^2 &= \frac{1}{2}(x+1)^3 \\
\lambda' &= -(x+1) \\
\frac{d\lambda}{dx} &= -(x+1) \\
d\lambda &= -(x+1)dx \\
\int d\lambda &= -\int(x+1)\,dx \\
\lambda &= -(\frac{x^2}{2}+x)+K, \quad K\in\mathbb{R}
\end{aligned}
$$

La solution generale de (E1) est:

$$
\underline{u=\left(-\left(\frac{x^2}{2}+x\right)+K\right)(x+1)^2}
$$

$$
\begin{aligned}
u &= (-\frac{x^2}{2}-x+K)(x+1)^2 \\
u' &= (-\frac{2}{2}x-1)(x+1)^2+2(x+1)(-\frac{x^2}{2}-x+K) \\
u' &= (-x-1)(x+1)^2+2(x+1)(-\frac{x^2}{2}-x+K) \\
u' &= -(x+1)^3+2(x+1)(-\frac{x^2}{2}-x+K) \\
u' &= (x+1)[-(x+1)^2+2(-\frac{x^2}{2}-x+2K)] \\
u' &= (x+1)[-(x+1)^2-x^2-2x+2K] \\
u' &= (x+1)[-x^2-2x-1-x^2-2x+2K] \\
u' &= (x+1)[-2x^2-4x-1+2K] \quad (B)
\end{aligned}
$$

En reprenant (A)

$$\begin{aligned}
\frac{y'}{y^3} &= -\frac{1}{2}u' \\
\frac{dy}{dx} &= -\frac{1}{2}u'y^3 \\
\frac{dy}{y^3} &= -\frac{1}{2}u'dx \\
\int \frac{dy}{y^3} &= -\frac{1}{2}\int u'dx \\
\int y^{-3}dy &= -\frac{1}{2}\int u'dx \\
\frac{y^{-2}}{-2} &= -\frac{1}{2}\int u'dx \\
y^{-2} &= \int u'dx \quad (A')
\end{aligned}$$

(B) dans (A')

$$\begin{aligned}
y^{-2} &= \int (x+1)[-2x^2-4x-1+2K]\,dx \\
&= \int (x+1)(-2x^2)+(x+1)(-4x)+(x+1)(-1)+(x+1)(2K)\,dx \\
&= \int \big(-2x^3-2x^2-4x^2-4x-x-1+2Kx+2K\big)\,dx \\
&= \int \big(-2x^3-6x^2-5x-1+2Kx+2K\big)\,dx \\
&= -\frac{2}{4}x^4-\frac{6}{3}x^3-\frac{5}{2}x^2-x+Kx^2+2Kx+C \\
y^{-2} &= -\frac{1}{2}x^4-2x^3-\frac{5}{2}x^2-x+Kx^2+2Kx+C \\
\implies y^2 &= (-\frac{1}{2}x^4-2x^3-\frac{5}{2}x^2-x+Kx^2+2Kx+C)^{-1} \\
\implies y &= \pm(-\frac{1}{2}x^4-2x^3-\frac{5}{2}x^2-x+Kx^2+2Kx+C)^{-\frac{1}{2}}
\end{aligned}$$

$$\boxed{y = \pm(-\frac{1}{2}x^4-2x^3-\frac{5}{2}x^2-x+Kx^2+2Kx+C)^{-\frac{1}{2}}, \quad K, C \in \mathbb{R}}$$

3. (E): $y'(1-\sin(s)\cos(x))+y^2\cos(x)-y'+\sin(x)=0$
si $y=\cos(x)$ est une solution particuliere

$$y'(1-\sin x\cos x)+y^2\cos x-y+\sin x=0$$
$$(1-\sin x\cos x)y'-y+\cos xy^2=-\sin x \quad (E)$$

On pose $y=\frac{1}{z}+\cos x$, alors $y'=-\frac{z'}{z^2}$.
(1) dans (E) :

$$(1-\sin x\cos x)\left(-\frac{z'}{z^2}\right)-\left(\frac{1}{z}+\cos x\right)+\cos x\left(\frac{1}{z}+\cos x\right)^2=-\sin x.$$

Developpement :

$$-\frac{z'}{z^2}+\frac{z'\sin x\cos x}{z^2}+\sin^2 x\cos x-\frac{1}{z}-\cos x+\cos x\left(\frac{1}{z^2}+\frac{2\cos x}{z}+\cos^2 x\right)=0.$$

Developpement de $\cos x\left(\frac{1}{z}+\cos x\right)^2$:

$$\cos x\left(\frac{1}{z^2}+\frac{2\cos x}{z}+\cos^2 x\right)=\frac{\cos x}{z^2}+\frac{2\cos^2 x}{z}+\cos^3 x.$$

Avec $\cos^3 x=(1-\sin^2 x)\cos x$:

$$\cos^3 x=\cos x-\sin^2 x\cos x.$$

Substitution dans (E) :

$$-\frac{z'}{z^2}+\frac{z'\sin x\cos x}{z^2}-\frac{1}{z}+\sin^2 x\cos x-\cos x+\frac{\cos x}{z^2}+\frac{2\cos^2 x}{z}+\cos x-\sin^2 x\cos x=0.$$

Rearrangement :

$$(\sin x\cos x-1)z'+(2\cos^2 x-1)z=-\cos x.$$

Hypothese $z_0=\lambda e^{\varphi(x)}$, où

$$\varphi(x)=\int\frac{-(2\cos^2 x-1)}{(\sin x\cos x-1)}dx.$$

Calcul de I :

$$I = \int \frac{2\cos^2 x - 1}{\sin x \cos x - 1} dx.$$

Avec $2\cos^2 x - 1 = \cos 2x$:

$$I = \int \frac{\cos 2x}{\sin 2x - 2} dx.$$

Changement de variable $u = \sin 2x - 2$, $du = 2\cos 2x dx$:

$$I = \int \frac{du}{u} = \ln|u| + C = \ln|\sin 2x - 2| + C.$$

Expression de z_0 :

$$z_0 = K(\sin 2x - 2)^{-1}.$$

(3) dans (E) :

$$(\sin x \cos x - 1)K'(\sin 2x - 2)^{-1} - 2K \cos 2x(\sin 2x - 2)^{-2} = -\cos x.$$

Resolution :

$$K' = -2\cos x, \quad K = -2\sin x + \varepsilon.$$

Solution finale : $\boxed{y = \dfrac{\sin 2x - 2}{-2\sin x + \varepsilon} + \cos x, \quad \varepsilon \in \mathbb{R}}$

---

4. (E): $y = xy' + (y')^3$
   Lagrange (E) : $y = x\varphi(y') + \psi(y)$

   1er cas: Si $\varphi(y') = y'$ c-a-d $y = xy' + \psi(y')$
   alors on pose $p = y'$
   donc on a $y' = xp + \varphi(p)$ et $y' = p + xp' + p'\psi'(p)$
   mais comme $y' = p$ alors on a $p = p + xp' + p'\psi(p)$
   ou encore $0 = p'(x + \psi'(p))$ c-a-d $p' = 0$ ou $x + \psi'(p) = 0$
   donc p: constante ou $x = -\psi'(p)$

Si p = y' est une constante alors $y = Cx + K, K \in \mathbb{Q}$ ou si $x = -\psi'(p)$ alors $y = -p\psi'(p) + \psi(p)$ (car $y = xp + \varphi(p)$)

$$\begin{aligned} (E) : y = xy' + (y')^3 \\ \text{On pose } p = y' \\ y = xp + p^3 \implies y' = x + xp' + 3p^2p' \\ y' = p \implies p = x + p'x + 3p'p^2 \end{aligned}$$

Or $p'(x + \psi'(p)) = 0 \quad p' = 0$ ou $x - 3p^2 = 0$

$$\begin{aligned} x &= -3p^2 \quad /y = xp + p^3 \\ 3p^2 &= -x \\ p^2 &= -\frac{x}{3} \end{aligned}$$

$$\begin{aligned} y &= xp + p(-\frac{x}{3}) \\ &= xp - \frac{x}{3}p \\ y &= \frac{2xp}{3} \\ p &= \frac{3y}{2x} \end{aligned}$$

$$\begin{aligned} \text{Nous rappelons que } p^2 &= -\frac{x}{3} \\ (\frac{3y}{2x})^2 &= -\frac{x}{3} \\ \frac{9y^2}{4x^2} &= -\frac{x}{3} \\ 27y^2 &= -4x^3 \\ y^2 &= -\frac{4}{27}x^3 \end{aligned}$$

Solution generale:
$y = Cx + C^3, \quad C \in \mathbb{R}$

$$\boxed{y^2 = -\frac{4}{27}x^3}$$