

S4 - 17B. Devoir 1

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Exercice 1:

1. (E): $y' = \sin(y)$

$$\begin{aligned}\frac{dy}{dx} &= \sin(y) \\ \frac{dy}{\sin(y)} &= dx \\ \int \frac{dy}{\sin(y)} &= \int dx\end{aligned}$$

Calcul de $\int \frac{1}{\sin(x)} dy$

Posons $t = \tan(\frac{y}{2})$

$$\text{donc } dt = \frac{1}{2}(1 + \tan^2(\frac{y}{2}))dy$$

$$dt = \frac{1}{2}(1 + t^2)dy$$

$$dy = \frac{2dt}{1 + t^2}$$

$$\sin(y) = \frac{2t}{1 + t^2}$$

$$\int \frac{1}{\sin(y)} dy = \int \left(\frac{1 + t^2}{2t}\right) \left(\frac{2dt}{1 + t^2}\right)$$

$$\int \frac{1}{\sin(y)} dy = \int \frac{1}{t} (dt)$$

$$\int \frac{1}{\sin(y)} dy = \ln|t| + C, \quad C \in \mathbb{R}$$

$$\text{Or } t = \tan(\frac{y}{2}) \implies \int \frac{1}{\sin(y)} dy = \ln|\tan(\frac{y}{2})| + C$$

On a:

$$\ln|\tan(\frac{y}{2})| = x + C, \quad C \in \mathbb{R}$$

$$|\tan(\frac{y}{2})| = e^x \cdot e^C$$

$$\tan(\frac{y}{2}) = ke^x, \quad k = e^C$$

$$\frac{y}{2} = \tan^{-1}(ke^x)$$

$$y = 2 \tan^{-1}(ke^x)$$

$$\boxed{y = 2 \tan^{-1}(ke^x), \quad k \in \mathbb{R}}$$

2. (E): $x^2 y' = e^y$

$$x^2 y' = e^y$$

$$x^2 \frac{dy}{dx} = e^y$$

$$\frac{dy}{e^y} = \frac{dx}{x^2}$$

$$\int \frac{1}{e^y} dy = \int \frac{1}{x^2} dx$$

$$\int e^{-y} dy = \int x^{-2} dx$$

$$-e^{-y} = -\frac{1}{x} + C, \quad C \in \mathbb{R}$$

$$\ln(e^{-y}) = \ln|\frac{1}{x} + C|$$

$$-y = \ln|\frac{1}{x} + C|$$

$$y = -\ln|\frac{1}{x} + C|$$

$$\boxed{y = -\ln|\frac{1}{x} + C|, \quad C \in \mathbb{R}}$$

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3. (E): $(x^2 + 1)y' + 3xy = x^2$
Etape 1: ESSM

$$\begin{aligned}
(x^2 + 1)y' + 3xy &= 0 \\
(x^2 + 1)y' &= -3xy \\
y'y^{-1} &= \frac{-3x}{x^2 + 1} \\
\frac{dy}{y} &= \frac{-3x}{x^2 + 1} dx \\
\int \frac{1}{y} dy &= \int -\frac{3x}{x^2 + 1} dx \\
\ln|y| &= -\frac{3}{2}\ln|x^2 + 1| + C, \quad C \in \mathbb{R} \\
y &= \lambda(x^2 + 1)^{-3/2}, \quad \lambda = e^C \quad (1)
\end{aligned}$$

Etape 2: EASM MVC

$$\begin{aligned}
y' &= \lambda'(x^2 + 1)^{-\frac{3}{2}} + \left(-\frac{3}{2}\right)\lambda(2x)(x^2 + 1)^{-\frac{3}{2}-1} \\
y' &= \lambda'(x^2 + 1)^{-\frac{3}{2}} - \frac{3}{2}\lambda(2x)(x^2 + 1)^{-\frac{5}{2}} \quad (2)
\end{aligned}$$

(1) et (2) dans (E) $((x^2 + 1)y' + 3xy = x^2)$:

$$\begin{aligned}
(x^2 + 1)(\lambda'(x^2 + 1)^{-\frac{3}{2}} - \frac{3}{2}\lambda(2x)(x^2 + 1)^{-\frac{5}{2}}) + 3x(\lambda(x^2 + 1)^{-3/2}) &= x^2 \\
\lambda'(x^2 + 1)^{\frac{1}{2}} &= x^2 \\
\lambda' &= x^2(x^2 + 1)^{\frac{1}{2}} \\
d\lambda &= x^2(x^2 + 1)^{\frac{1}{2}} dx \\
\int d\lambda &= \int x^2(x^2 + 1)^{\frac{1}{2}} dx
\end{aligned}$$

$$x^2 = \sinh^2(t)$$

$$x = \sinh(t)$$

$$dx = \cosh(t)dt$$

$$x^2 + 1 = \sinh^2(t) + 1 = \cosh^2(t)$$

$$\sqrt{x^2 + 1} = \sqrt{\cosh^2(t)} = \cosh(t), \quad \cosh(t) > 0$$

$$\int x^2 \sqrt{x^2 + 1} dx = \int \sinh^2(t) \cosh(t) \cosh(t) dt$$

$$\int x^2 \sqrt{x^2 + 1} dx = \int \sinh^2(t) \cosh^2(t) dt$$

$$\begin{aligned}
\sinh^2(t) &= \frac{\cosh(2t) - 1}{2} \quad , \quad \cosh^2(t) = \frac{\cosh(2t) + 1}{2} \\
\sinh^2(t) \cosh^2(t) &= \left(\frac{\cosh(2t) - 1}{2}\right) \left(\frac{\cosh(2t) + 1}{2}\right) \\
\sinh^2(t) \cosh^2(t) &= \frac{1}{4}(\cosh(2t) - 1)(\cosh(2t) + 1) \\
\sinh^2(t) \cosh^2(t) &= \frac{1}{4}(\cosh^2(2t) - 1) \\
\sinh^2(t) \cosh^2(t) &= \frac{1}{4}\sinh^2(2t)
\end{aligned}$$

Donc

$$\begin{aligned}
\int \sinh^2(t) \cosh^2(t) dt &= \int \frac{1}{4} \sinh^2(2t) dt \\
\int \sinh^2(t) \cosh^2(t) dt &= \frac{1}{4} \int \sinh^2(2t) dt \\
\text{or } \sinh^2(2t) dt &= \int \frac{\cosh(4t) - 1}{2} dt \\
\text{car } \sinh^2(t) &= \frac{\cosh(2t) - 1}{2} \\
\frac{1}{2} \int \cosh(4t) - \frac{1}{2} \int 1 dt \\
\cosh(4t) dt &= \frac{1}{4} \sinh(4t) \\
\int \sinh^2(t) dt &= \frac{1}{2} \cdot \frac{1}{4} \sinh(4t) - \frac{1}{2} t \\
\int \sinh^2(t) dt &= \frac{1}{8} \sinh(4t) - \frac{1}{2} t + C, \quad C \in \mathbb{R} \\
\int \sinh^2(t) dt &= \frac{1}{4} \left(\frac{1}{8} \sinh(4t) - \frac{1}{2} t + C \right) \\
\int \sinh^2(t) dt &= \frac{1}{32} \sinh(4t) - \frac{1}{8} t + C
\end{aligned}$$

$$x = \sinh(t), t = \operatorname{arcsinh}(x) = \ln(x + \sqrt{x^2 + 1})$$

$$\sinh(4t) = 2 \sinh(2t) \cosh(2t)$$

$$\sinh(2t) = 2 \sinh(t) \cosh(t) = 2x\sqrt{1+x^2}$$

$$\begin{aligned}\cosh(2t) &= \sqrt{1 + \sinh^2(2t)} = \sqrt{1 + (2x\sqrt{x^2 + 1})^2} \\ \cosh(2t) &= \sqrt{1 + 4x^2(x^2 + 1)} = \sqrt{1 + 4x^4 + 4x^2} \\ \cosh(2t) &= \sqrt{(2x^2 + 1)^2} = 2x^2 + 1, \quad x^2 \geq 0\end{aligned}$$

$$\begin{aligned}\sinh(4t) &= 2(2x\sqrt{1 + x^2})(2x^2 + 1) \\ \sinh(4t) &= 4x(2x^2 + 1)\sqrt{1 + x^2}\end{aligned}$$

$$\begin{aligned}\frac{1}{32} \sinh(4t) - \frac{1}{8}t &= \frac{1}{32} 4x(2x^2 + 1)\sqrt{1 + x^2} - \frac{1}{8}t + C \\ \int x^2 \sqrt{x^2 + 1} dx &= \frac{1}{8}x(2x^2 + 1)\sqrt{x^2 + 1} - \frac{1}{8} \ln|x + \sqrt{x^2 + 1}| + C\end{aligned}$$

$$\begin{aligned}\lambda &= \frac{1}{8}x(2x^2 + 1)\sqrt{x^2 - 1} - \frac{1}{8} \ln|x + \sqrt{x^2 + 1}| + C \\ y &= \left(\frac{1}{8}x(2x^2 + 1)\sqrt{x^2 - 1} - \frac{1}{8} \ln|x + \sqrt{x^2 + 1}| + C\right)(x^2 + 1)^{-\frac{3}{2}}\end{aligned}$$

$$\boxed{y = \left(\frac{1}{8}x(2x^2 + 1)\sqrt{x^2 - 1} - \frac{1}{8} \ln|x + \sqrt{x^2 + 1}| + C\right)(x^2 + 1)^{-\frac{3}{2}}, \quad C \in \mathbb{R}}$$

4. (E): $y' + y = 2e^x + 4 \sin(x)$

Etape 1: ESSM

$$\begin{aligned}y' + y &= 0 \\ y' &= -y \\ \frac{dy}{y} &= -1 \\ \int \frac{1}{y} dy &= - \int 1 dx \\ \ln|y| &= -x + C, \quad C \in \mathbb{R} \\ y &= -e^{-x} \cdot e^C \\ y &= \lambda e^{-x} \quad (1)\end{aligned}$$

Etape 2: EASM MVC

$$\begin{aligned}y' &= -\lambda e^{-x} + \lambda' e^{-x} \\y' &= e^{-x}(\lambda + \lambda')(2)\end{aligned}$$

(1) et (2) dans (E):

$$(-\lambda + \lambda)e^{-x} + \lambda e^{-x} = 2e^x + 4 \sin(x)$$

$$\lambda' = 2e^x + 4 \sin(x)$$

$$\frac{d\lambda}{dx} e^{-x} = 2e^x + 4 \sin(x)$$

$$\int d\lambda = \int (2e^{2x} + 4e^x \sin(x)) dx$$

$$\int d\lambda = \int 2e^{2x} dx + 4 \int e^x \sin(x) dx$$

$$\int d\lambda = e^{2x} + C + 4[-\cos(x)e^x + \int e^x \cos(x) dx]$$

Soit $I = \int e^x \cos(x)$

$$\begin{cases} u = e^x \implies u' = e^x \\ v = -\cos(x) \implies v' = \sin(x) \end{cases}$$

$$-\cos(x)e^x + \int e^x \cos(x) dx$$

$$\begin{cases} u = e^x \implies v' = \cos(x) \\ u' = e^x \implies v' = \sin(x) \end{cases}$$

$$-e^x \cos(x) + -e^x \sin(x) - \int e^x \sin(x) dx$$

$$-e^x \cos(x) + -e^x \sin(x) - I$$

$$I = e^x(\sin(x) - \cos(x)) - I$$

$$2I = e^x(\sin(x) - \cos(x))$$

$$I = \frac{e^x}{2}(\sin(x) - \cos(x))$$

$$\begin{aligned}\int d\lambda &= e^{2x} + c + 2e^x(\sin(x) - \cos(x)) + k \\ \lambda x &= e^{2x} + e^{2x}(\sin(x) - \cos(x)) + C \\ \lambda x &= e^x(e^x + 2(\sin(x) - \cos(x)) + C\end{aligned}$$

$$\begin{aligned}y &= e^x \cdot e^{-x}(e^x + 2(\sin(x) - \cos(x)) + Ce^{-x}) \\ y &= e^x + 2\sin(x) - 2\cos(x) + Ce^{-x}\end{aligned}$$

$$\boxed{y = e^x + 2\sin(x) - 2\cos(x) + Ce^{-x}, \quad C \in \mathbb{R}}$$

5. (E): $y' - 2y = 2x^3 + x$ ou $y(3) = 1$
Etape 1: ESSM

$$\begin{aligned}y &= \lambda e^{f(-\frac{-2}{1})} \\ y &= \lambda e^{2x} \quad (1)\end{aligned}$$

Etape 2: EASM
MVC

$$y' = \lambda e^{2x} + 2e^{2x}\lambda \quad (2)$$

(1) et (2) dans (E):

$$\begin{aligned}\lambda e^{2x} + 2e^{2x}\lambda - 2(\lambda e^{2x}) &= 2x^3 + x \\ \lambda e^{2x} &= 2x^3 + x \\ \frac{d\lambda}{dx} &= \frac{2x^3 + x}{e^{2x}} \\ d\lambda &= \frac{2x^3 + x}{e^{2x}} dx \\ \int d\lambda &= \int \frac{2x^3 + x}{e^{2x}} dx\end{aligned}$$

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6. (E): $y' + 2xy = e^{x-x^2}$
Etape 1: ESSM

$$\begin{aligned}
 y' - 2xy &= 0 \\
 y' &= 2xy \\
 \frac{dy}{dx} &= 2xy \\
 \int \frac{dy}{y} &= \int 2x dx \\
 \ln |y| &= x^2 + k, \quad k \in \mathbb{R} \\
 y_0 &= e^{x^2} + e^k \\
 y_0 &= e^{x^2} \cdot c, \quad c = e^k
 \end{aligned}$$

Etape 2: EASM
 $y' - 2xy = e^{x-x^2} \quad (E)$
MVC:

$$\begin{aligned}
 y &= ce^{x^2} \quad (1) \\
 y' &= c'e^{x^2} + c2xe^{x^2} \quad (2)
 \end{aligned}$$

(1) et (2) dans (E):

$$c'e^{x^2} + c2xe^{x^2} - c2xe^{x^2} = e^{x-x^2}$$

$$c'e^{x^2} = e^{x-x^2}$$

$$\frac{dc}{dx} \cdot e^{x^2} = e^{x-x^2}$$

$$\int dc = \int \frac{e^{x-x^2}}{e^{x^2}} dx$$

$$c = \int \frac{e^x}{e^{x^2} \cdot e^{x^2}}$$

$$c = \int \frac{e^x}{e^{2x^2}} dx$$

$$c = \int \frac{1}{e^{2x^2}} + \int e^x dx$$

$$c = \int e^{-2x} + \int e^x dx$$

$$c = -\frac{e^{-2x^2}}{4x} + e^x + \lambda, \quad \lambda \in \mathbb{R}$$

Or $y = c \cdot e^{x^2}$

$$y = \left(-\frac{e^{-2x^2} \cdot e^{x^2}}{4x} + e^x \cdot e^{x^2}\right) + \lambda e^{x^2}$$

$$= \left(-\frac{e^{x^2-2x^2}}{4x} + e^x \cdot e^{x^2}\right) + \lambda e^{x^2}$$

$$y = y_p + y_0$$

$$y = \left(-\frac{e^{-x^2}}{4x} + e^{x^2+x} + \lambda e^{x^2}\right), \quad \lambda \in \mathbb{R}$$

7. (E): $y' \cos(2y) - \sin(y) = 0$

Exercice 2:

1. (E): $xy' = y + 3xy^2$

$$\begin{aligned}\frac{x}{x}y' &= \frac{1}{x}y + \frac{3xy^2}{x} \\ y' &= \frac{1}{x}y + 3y^2\end{aligned}$$

On pose $z = y^{-1} = \frac{1}{y} \implies z' = -\frac{y'}{y^2}$

$$\begin{aligned}-\frac{y'}{y^2} &= -\frac{1}{xy} - 3 \\ \text{d'où } z' &= -\frac{1}{x}z - 3 \\ z' + \frac{1}{x}z &= -3 \quad (E)\end{aligned}$$

ESSM
 $z' + \frac{1}{x}z = 0$

$$\begin{aligned}z_H &= \lambda e^{-\int \frac{1}{x}}, \quad \lambda \in \mathbb{R} \\ &= \lambda e^{-\ln|x|} \\ z_H &= \lambda \frac{1}{x}, \quad \lambda \in \mathbb{R}\end{aligned}$$

EASM

MVC:

$$\begin{aligned}t &= g(x) \frac{1}{x} \quad (1) \\ t' &= \frac{g'(x)x - g(x)}{x^2} \quad (2)\end{aligned}$$

(1) et (2) dans (E)

$$\begin{aligned}\frac{g'(x)x - g(x)}{x^2} + \frac{1}{x}g(x)\frac{1}{x} &= -3 \\ \frac{g'(x)x}{x^2} - \frac{g(x)}{x^2} + \frac{g(x)}{x^2} &= -3 \\ g'(x) &= -3x \\ g(x) &= -\frac{3}{2}x^2 + C, \quad C \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}z_p &= \left(-\frac{3}{2}x^2 + C\right)\frac{1}{x} + \lambda\left(\frac{1}{x}\right) \\ z_p &= \left(-\frac{3}{2}x + \psi\left(\frac{1}{x} + \frac{1}{x}\right)\right), \quad \psi = \frac{C}{x} \\ z_p &= -\frac{3}{2}x + \psi\frac{2}{x} \\ z_p &= -\frac{3x^2 + 4\psi}{2x}\end{aligned}$$

$$z = z_H + z_P \text{ et } y = y_H + y_P$$

or $z = \frac{1}{y}$

$$\text{Alors } y = \frac{1}{z_H} + \frac{1}{z_P} \quad \boxed{y = \frac{x}{\lambda} + \frac{2x}{3x^2 + 4\psi}, \quad \lambda, \psi \in \mathbb{R}}$$

2. (E): $y' + \frac{y}{x+1} = \frac{1}{2}(x+1)^3y^3$
C'est une equation de Bernoulli

$$\begin{aligned}\frac{y'}{y^3} + \frac{y}{(x+1)y^3} &= \frac{1}{2}(x+1)^3 \\ \frac{y'}{y^3} + \frac{(x+1)^{-1}}{y^2} &= \frac{1}{2}(x+1)^3 \quad (E)\end{aligned}$$

Changement de variable

$$\begin{aligned}u &= \frac{1}{y^2} = y^{-2} \\ \frac{du}{dx} &= -2y'y^{-3} \\ -\frac{1}{2}u' &= \frac{y'}{y^3} \quad (A)\end{aligned}$$

D'où

$$(E1) : \quad \frac{1}{-2}u' + (x+1)^{-1}u = \frac{1}{2}(x+1)^3$$

Etape 1: ESSM

$$-\frac{1}{2}u' + (x+1)^{-1}u = 0 \quad (1)$$

$$\frac{1}{2}u' = (x+1)^{-1}u \quad (2)$$

$$\frac{du}{dx} = 2(x+1)^{-1}u \quad (3)$$

$$\frac{du}{u} = 2(x+1)^{-1}dx \quad (4)$$

$$\ln|u| = 2\ln|x+1| + C, \quad C \in \mathbb{R} \quad (5)$$

$$\ln|u| = \ln|(x+1)^2| + C \quad (6)$$

$$u = e^{\ln|(x+1)^2|} \times e^C \quad (7)$$

$$u = (x+1)^2 \times \lambda, \quad \lambda = e^C \quad (E1_1) \quad (8)$$

Etape 2: EASM

$$u' = \lambda'(x+1)^2 + 2(x+1)\lambda \quad (E1_2)$$

$(E1_1)$ et $(E1_2)$ dans $(E1)$

$$\begin{aligned}
-\frac{1}{2}[\lambda'(x+1)^2 + 2(x+1)\lambda] + (x+1)^{-1}(x+1)^2\lambda &= \frac{1}{2}(x+1)^3 \\
-\frac{\lambda'}{2}(x+1)^2 - (x+1)\lambda + (x+1)\lambda &= \frac{1}{2}(x+1)^3 \\
-\frac{\lambda'}{2}(x+1)^2 &= \frac{1}{2}(x+1)^3 \\
\lambda' &= -(x+1) \\
\frac{d\lambda}{dx} &= -(x+1) \\
d\lambda &= -(x+1)dx \\
\int d\lambda &= -\int (x+1)dx \\
\lambda &= -\left(\frac{x^2}{2} + x\right) + K, \quad K \in \mathbb{R}
\end{aligned}$$

La solution generale de (E1) est:

$$\begin{aligned}
u &= \left(-\left(\frac{x^2}{2} + x\right) + K \right) (x+1)^2 \\
u &= \left(-\frac{x^2}{2} - x + K \right) (x+1)^2 \\
u' &= \left(-\frac{2}{2}x - 1 \right) (x+1)^2 + 2(x+1) \left(-\frac{x^2}{2} - x + K \right) \\
u' &= (-x - 1) (x+1)^2 + 2(x+1) \left(-\frac{x^2}{2} - x + K \right) \\
u' &= -(x+1)^3 + 2(x+1) \left(-\frac{x^2}{2} - x + K \right) \\
u' &= (x+1) \left[-(x+1)^2 + 2 \left(-\frac{x^2}{2} - x + 2K \right) \right] \\
u' &= (x+1) \left[-(x+1)^2 - x^2 - 2x + 2K \right] \\
u' &= (x+1) \left[-x^2 - 2x - 1 - x^2 - 2x + 2K \right] \\
u' &= (x+1) \left[-2x^2 - 4x - 1 + 2K \right] \quad (B)
\end{aligned}$$

En reprenant (A)

$$\begin{aligned}
\frac{y'}{y^3} &= -\frac{1}{2}u' \\
\frac{dy}{dx} &= -\frac{1}{2}u'y^3 \\
\frac{dy}{y^3} &= -\frac{1}{2}u'dx \\
\int \frac{dy}{y^3} &= -\frac{1}{2} \int u'dx \\
\int y^{-3}dy &= -\frac{1}{2} \int u'dx \\
\frac{y^{-2}}{-2} &= -\frac{1}{2} \int u'dx \\
y^{-2} &= \int u'dx \quad (A')
\end{aligned}$$

(B) dans (A')

$$\begin{aligned}
y^{-2} &= \int (x+1)[-2x^2 - 4x - 1 + 2K] dx \\
&= \int (x+1)(-2x^2) + (x+1)(-4x) + (x+1)(-1) + (x+1)(2K) dx \\
&= \int (-2x^3 - 2x^2 - 4x^2 - 4x - x - 1 + 2Kx + 2K) dx \\
&= \int (-2x^3 - 6x^2 - 5x - 1 + 2Kx + 2K) dx \\
&= -\frac{2}{4}x^4 - \frac{6}{3}x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C \\
y^{-2} &= -\frac{1}{2}x^4 - 2x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C \\
\Rightarrow y^2 &= (-\frac{1}{2}x^4 - 2x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C)^{-1} \\
\Rightarrow y &= \pm(-\frac{1}{2}x^4 - 2x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C)^{-\frac{1}{2}}
\end{aligned}$$

$y = \pm(-\frac{1}{2}x^4 - 2x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C)^{-\frac{1}{2}}, \quad K, C \in \mathbb{R}$

3. (E): $y'(1 - \sin(x) \cos(x)) + y^2 \cos(x) - y' + \sin(x) = 0$
si $y = \cos(x)$ est une solution particulière
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4. (E): $y = xy' + (y')^3$