S4 - 17B. Devoir 2

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Exercice 2:

Resolution des equations differentielles non lineraires de 2nd ordre

5. (E): $y' = xy' + \sqrt{1 + (y')^2}$

   Forme de Lagrange

   Une equation differentielle du premier ordre de Lagrange est donnee par :

$$y = x\varphi(y') + \psi(y'). \tag{1}$$

Avec $p = y'$, on a :

$$\begin{aligned} p &= xp + \sqrt{1+p^2} \\ p - xp &= \sqrt{1+p^2} \\ p(1-x) &= \sqrt{1+p^2} \\ p^2(1-x)^2 &= 1+p^2, \\ p^2 - 2xp^2 + x^2p^2 &= 1 + p^2, \\ p^2(x^2-2x) &= 1, \\ p^2 &= \frac{1}{x^2-2x}, \end{aligned}$$

Avec

x	$-\infty$	0	2	$+\infty$
x	$-$	0	$+$	$+$
$x-2$	$-$	$-$	0	$+$
$x(x-2)$	$+$	$-$	$-$	$+$

Pour que $p^2 > 0$ alors $x^2 - 2x = x(x-2) > 0$

On a

$$x^2 - 2x = \frac{1}{p^2}$$

$$x^2 - 2x - \frac{1}{p^2} = 0$$

$$
\begin{aligned}
\Delta &= (-2)^2 - 4(\frac{1}{p^2}) \\
&= 4 + \frac{4}{p^2} \\
&= \frac{4p^2+4}{p^2} \\
\Delta &= 4\frac{p^2+1}{p^2} \\
\sqrt{\Delta} &= \frac{2}{p}\sqrt{p^2+1}
\end{aligned}
$$

$$
\begin{aligned}
x &= \frac{2 \pm \frac{2}{p}\sqrt{p^2+1}}{2} \\
x &= \frac{2p \pm 2\sqrt{p^2+1}}{2p} \\
x &= \frac{2p}{2p} \pm \frac{2\sqrt{p^2+1}}{2p} \\
x &= 1 \pm 2\sqrt{\frac{p^2}{p^2} + \frac{1}{p^2}} \\
x &= 1 \pm 2\sqrt{1 + \frac{1}{p^2}}
\end{aligned}
$$

Alors $x = 1 - 2\sqrt{1+\frac{1}{p^2}} < 0$ OU $x = 1 + 2\sqrt{1+\frac{1}{p^2}} > 0$
sattisfaissent $x \in]-\infty; 0[U]0; +\infty[$

Si $x < 0,$ alors il n'y a aucune solution pour (E)
Si $p = y' = 0,$ alors y: constante alors$y = Cx + K \quad C, K \in \mathbb{R}$
Si $x > 0$

$$
\begin{aligned}
y &= xy' + \sqrt{a + (y')^2} \\
y &= 1 \pm 2(\sqrt{1 + \frac{1}{p^2}})(p) + \sqrt{1 + p^2} \\
y &= 1 \pm 2(\sqrt{p^2 + \frac{p^2}{p^2}}) + \sqrt{1 + p^2} \\
y &= 1 \pm 2(\sqrt{p^2 + 1}) + \sqrt{1 + p^2}
\end{aligned}
$$

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6. (E): $x = yy' + (y')^3$

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7. (E): $2xy' = y + y'$

$$
\begin{aligned}
2xy' &= y + y' \\
-y' + 2xy' &= y \\
y'(-1 + 2x) &= y \\
y'\frac{(-1 + 2x)}{(-1 + 2x)} &= \frac{1}{(-1 + 2x)} \cdot y \\
y' &= \frac{1}{(-1 + 2x)} \cdot y
\end{aligned}
$$

Calcul de la solution generale :

$$
\begin{aligned}
y &= \lambda e^{-\int \frac{1}{\frac{1}{(-1+2x)}}}, \quad \lambda \in \mathbb{R} \\
&= \lambda e^{-\int (-1+2x)}, \quad \lambda \in \mathbb{R}
\end{aligned}
$$

Calculons d'abord l'integrale :

$$
\begin{aligned}
-\int(-1+2x) &= -(-x+x^2) \\
&= x - x^2
\end{aligned}
$$

Alors :

$$y = \lambda e^{x-x^2}, \quad \lambda \in \mathbb{R}$$

$\boxed{y = \lambda e^{x-x^2}, \quad \lambda \in \mathbb{R}}$

Exercice 3:

Resolution des equations differentielles du second ordre:

1. (E): $y'' + y = xe^x + 2e^{-x}$ ou $y(0) = y'(0) = 1$

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2. (E): $y'' - 4y' + y = xe^{2x}$
   L'equation caracteristique (K): $r^2 - 4r + 1 = 0$

$$
\begin{aligned}
\Delta &= (-4)^2 - 4(1)(1) \\
&= 16 - 4 \\
&= 12 > 0 \\
\sqrt{\Delta} &= 2\sqrt{3}
\end{aligned}
$$

   $r_1 = 2 + \sqrt{3}$, $r_2 = 2 - \sqrt{3}$ alors la solution y_h de l'equation homogene est:
   $Ae^{(2+\sqrt{3})x} + Be^{(2-\sqrt{3})x}$ / A, B $\in \mathbb{R}$

$f(x) = xe^{2x}$ de la forme $P_n(x)e^{\delta x}$

$$\delta = 2$$
$$n = 1$$

Donc

$$y_p = H_n(x)e^{\delta x}$$
$$y_p = (ax+b)e^{2x}$$
$$, \quad a, b \in \mathbb{R}$$

Or $y = y_h + y_p$

$\boxed{y = Ae^{(2+\sqrt{3})x} + Be^{(2-\sqrt{3})x} + (ax+b)e^{2x}}$

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3. (E): $y'' + 2y' + 5y = cosx$ ou $y(0) = y'(0) = 0$
   Etape 1 : ESSM:

$$y'' + 2y^t + 5y = 0$$

Equation caracteristique:

$$r^2 + 2r + 5 = 0$$
$$\Delta = 2^2 - 4 \times 1 \times 5 = 4 - 20 = -16$$
$$r = \frac{-2 \pm \sqrt{-16}}{2 \times 1} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

D'ou on a:

$$y = e^{-x}(E\cos(2x) + F\sin(2x))$$

Etape 2 : EASM:

$y'' + 2y' + 5y = \cos(x) \quad (E)$

$$y_p = A\cos(x) + B\sin(x) \quad (1)$$
$$y_p' = -A\sin(x) + B\cos(x) \quad (2)$$
$$y_p'' = -A\cos(x) - B\sin(x) \quad (3)$$

(1), (2), (3) dans (E):

$$(-A\cos(x) - B\sin(x)) + 2(-A\sin(x) + B\cos(x)) + 5(A\cos(x) + B\sin(x)) = \cos(x)$$
$$(-A + 2B + 5A)\cos(x) + (-B - 2A + 5B)\sin(x) = \cos(x)$$
$$(4A + 2B)\cos(x) + (4B - 2A)\sin(x) = \cos(x)$$

Systeme:

$$4A + 2B = 1$$
$$-2A + 4B = 0$$

Resolution:

$$-2A + 4B = 0 \Rightarrow A = 2B$$
$$4(2B) + 2B = 1 \Rightarrow 8B + 2B = 1$$
$$10B = 1 \Rightarrow B = \frac{1}{10}$$
$$A = 2B = \frac{2}{10} = \frac{1}{5}$$

D'ou:

$$y = e^{-x}(E\cos(2x) + F\sin(2x)) + \frac{1}{5}\cos(x) + \frac{1}{10}\sin(x)$$
$$y(0) = e^{0}(E\cos(0) + F\sin(0)) + \frac{1}{5}\cos(0) + \frac{1}{10}\sin(0) = 0$$
$$E + \frac{1}{5} = 0 \Rightarrow E = -\frac{1}{5}$$

Derivee:

$$
\begin{aligned}
y'(0) &= 0 \\
y' &= e^{-x}(E\cos(2x) + F\sin(2x)) - e^{-x}(-2E\sin(2x) + 2F\cos(2x)) - \frac{1}{5}\sin(
\\
y'(0) &= e^{0}(E\cos(0) + F\sin(0)) - e^{0}(-2E\sin(0) + 2F\cos(0)) - \frac{1}{5}\sin(0) + \frac{1}{10}
\\
-E + 2F + \frac{1}{10} &= 10 \\
-(-\frac{1}{5}) + 2F + \frac{1}{10} &= 0 \implies \frac{1}{5} + 2F + \frac{1}{10} = 0 \\
2F &= -\frac{1}{5} - \frac{1}{10} \\
2F &= -\frac{3}{10} \\
F &= -\frac{3}{20}
\end{aligned}
$$

$$\boxed{y = e^{-x}(-\frac{1}{5}\cos(2x) - \frac{3}{20}\sin(2x)) + \frac{1}{5}\cos(x) + \frac{1}{10}\sin(x)}$$

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4. (E): $y'' - 4y + 13y = cos(3x)$

Etape 1 : ESSM

$y'' - 4y + 13y = 0$

Equation caracteristique :

$$
\begin{aligned}
r^2 - 4r + 13 &= 0 \\
\Delta &= (-4)^2 - 4 \times 1 \times 13 = 16 - 52 \\
r &= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i
\end{aligned}
$$

$\Delta < 0$, donc $y = e^{2x}(A\cos 3x + B\sin 3x)$

Etape 2 : EASM

$$
\begin{aligned}
y'' - 4y' + 13y &= \cos(3x) \\
f(x) &= \cos(3x) \\
y_p &= m\cos(3x) + n\sin(3x) \\
y_p' &= -3m\sin(3x) + 3n\cos(3x) \\
y_p'' &= -9m\cos(3x) - 9n\sin(3x)
\end{aligned}
$$

Substituons dans l'equation :

$$
\begin{aligned}
(-9m\cos(3x) - 9n\sin(3x)) - 4(-3m\sin(3x) + 3n\cos(3x)) + 13(m\cos(3x) + n\sin(3x)) &= \text{c} \\
(-9m + 12n + 13m)\cos(3x) + (-9n - 12m + 13n)\sin(3x) &= \text{c} \\
(4m + 12n)\cos(3x) + (4n - 12m)\sin(3x) &= \text{c}
\end{aligned}
$$

On obtient le systeme :

$$
\begin{aligned}
4m + 12n &= 1 \\
-12m + 4n &= 0
\end{aligned}
$$

Resolution :

$$
\begin{aligned}
-12m + 4n &= 0 \\
4n &= 12m \\
n &= 3m
\end{aligned}
$$

Substitution dans la premiere equation :

$$
\begin{aligned}
4m + 12(3m) &= 1 \\
4m + 36m &= 1 \\
40m &= 1 \\
m &= \frac{1}{40}
\end{aligned}
$$

Donc, on a solution :

$$
\boxed{y = e^{2x}(A\cos(3x) + B\sin(3x)) + \frac{1}{40}\cos(3x) + \frac{3}{40}\sin(3x)}
$$