

S4 GROUPE n

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Exercice 1:

1. (E): $y' = \sin(y)$

2. (E): $x^2 y' = e^y$

$$x^2 y' = e^y$$

$$x^2 \frac{dy}{dx} = e^y$$

$$\frac{dy}{e^y} = \frac{dx}{x^2}$$

$$\int \frac{1}{e^y} dy = \int \frac{1}{x^2} dx$$

$$\int e^{-y} dy = \int x^{-2} dx$$

$$-e^{-y} = -\frac{1}{x} + C, \quad C \in \mathbb{R}$$

$$\ln(e^{-y}) = \ln\left|\frac{1}{x} + C\right|$$

$$-y = \ln\left|\frac{1}{x} + C\right|$$

$$y = -\ln\left|\frac{1}{x} + C\right|$$

$$y = -\ln\left|\frac{1}{x} + C\right|$$

3. (E): $(x^2 + 1)y' + 3xy = x^2$

Etape 1: ESSM

$$\begin{aligned}
(x^2 + 1)y' + 3xy &= 0 \\
(x^2 + 1)y' &= -3xy \\
y'y^{-1} &= \frac{-3x}{x^2 + 1} \\
\frac{dy}{y} &= \frac{-3x}{x^2 + 1} dx \\
\int \frac{1}{y} dy &= \int -\frac{3x}{x^2 + 1} dx \\
\ln|y| &= -\frac{3}{2}\ln|x^2 + 1| + C, \quad C \in \mathbb{R} \\
y &= \lambda(x^2 + 1)^{-3/2}, \quad \lambda = e^C \quad (1)
\end{aligned}$$

Etape 2: EASM MVC

$$\begin{aligned}
y' &= \lambda'(x^2 + 1)^{-\frac{3}{2}} + \left(-\frac{3}{2}\right)\lambda(2x)(x^2 + 1)^{-\frac{3}{2}-1} \\
y' &= \lambda'(x^2 + 1)^{-\frac{3}{2}} - \frac{3}{2}\lambda(2x)(x^2 + 1)^{-\frac{5}{2}} \quad (2)
\end{aligned}$$

(1) et (2) dans (E) $((x^2 + 1)y' + 3xy = x^2)$:

$$\begin{aligned}
(x^2 + 1)(\lambda'(x^2 + 1)^{-\frac{3}{2}} - \frac{3}{2}\lambda(2x)(x^2 + 1)^{-\frac{5}{2}}) + 3x(\lambda(x^2 + 1)^{-3/2}) &= x^2 \\
\lambda'(x^2 + 1)^{\frac{1}{2}} &= x^2 \\
\lambda' &= x^2(x^2 + 1)^{\frac{1}{2}} \\
d\lambda &= x^2(x^2 + 1)^{\frac{1}{2}} dx \\
\int d\lambda &= \int x^2(x^2 + 1)^{\frac{1}{2}} dx
\end{aligned}$$

$$x^2 = \sinh^2(t)$$

$$x = \sinh(t)$$

$$dx = \cosh(t)dt$$

$$x^2 + 1 = \sinh^2(t) + 1 = \cosh^2(t)$$

$$\sqrt{x^2 + 1} = \sqrt{\cosh^2(t)} = \cosh(t), \quad \cosh(t) > 0$$

$$\int x^2 \sqrt{x^2 + 1} dx = \int \sinh^2(t) \cosh(t) \cosh(t) dt$$

$$\int x^2 \sqrt{x^2 + 1} dx = \int \sinh^2(t) \cosh^2(t) dt$$

$$\begin{aligned}
\sinh^2(t) &= \frac{\cosh(2t) - 1}{2} \quad , \quad \cosh^2(t) = \frac{\cosh(2t) + 1}{2} \\
\sinh^2(t) \cosh^2(t) &= \left(\frac{\cosh(2t) - 1}{2}\right) \left(\frac{\cosh(2t) + 1}{2}\right) \\
\sinh^2(t) \cosh^2(t) &= \frac{1}{4}(\cosh(2t) - 1)(\cosh(2t) + 1) \\
\sinh^2(t) \cosh^2(t) &= \frac{1}{4}(\cosh^2(2t) - 1) \\
\sinh^2(t) \cosh^2(t) &= \frac{1}{4}\sinh^2(2t)
\end{aligned}$$

Donc

$$\begin{aligned}
\int \sinh^2(t) \cosh^2(t) dt &= \int \frac{1}{4} \sinh^2(2t) dt \\
\int \sinh^2(t) \cosh^2(t) dt &= \frac{1}{4} \int \sinh^2(2t) dt \\
\text{or } \sinh^2(2t) dt &= \int \frac{\cosh(4t) - 1}{2} dt \\
\text{car } \sinh^2(t) &= \frac{\cosh(2t) - 1}{2} \\
\frac{1}{2} \int \cosh(4t) - \frac{1}{2} \int 1 dt \\
\cosh(4t) dt &= \frac{1}{4} \sinh(4t) \\
\int \sinh^2(t) dt &= \frac{1}{2} \cdot \frac{1}{4} \sinh(4t) - \frac{1}{2} t \\
\int \sinh^2(t) dt &= \frac{1}{8} \sinh(4t) - \frac{1}{2} t + C, \quad C \in \mathbb{R} \\
\int \sinh^2(t) dt &= \frac{1}{4} \left(\frac{1}{8} \sinh(4t) - \frac{1}{2} t + C \right) \\
\int \sinh^2(t) dt &= \frac{1}{32} \sinh(4t) - \frac{1}{8} t + C
\end{aligned}$$

$$x = \sinh(t), t = \operatorname{arcsinh}(x) = \ln(x + \sqrt{x^2 + 1})$$

$$\sinh(4t) = 2 \sinh(2t) \cosh(2t)$$

$$\sinh(2t) = 2 \sinh(t) \cosh(t) = 2x\sqrt{1+x^2}$$

$$\begin{aligned}\cosh(2t) &= \sqrt{1 + \sinh^2(2t)} = \sqrt{1 + (2x\sqrt{x^2 + 1})^2} \\ \cosh(2t) &= \sqrt{1 + 4x^2(x^2 + 1)} = \sqrt{1 + 4x^4 + 4x^2} \\ \cosh(2t) &= \sqrt{(2x^2 + 1)^2} = 2x^2 + 1, \quad x^2 \geq 0\end{aligned}$$

$$\begin{aligned}\sinh(4t) &= 2(2x\sqrt{1 + x^2})(2x^2 + 1) \\ \sinh(4t) &= 4x(2x^2 + 1)\sqrt{1 + x^2}\end{aligned}$$

$$\begin{aligned}\frac{1}{32} \sinh(4t) - \frac{1}{8}t &= \frac{1}{32}4x(2x^2 + 1)\sqrt{1 + x^2} - \frac{1}{8}t + C \\ \int x^2\sqrt{x^2 + 1}dx &= \frac{1}{8}x(2x^2 + 1)\sqrt{x^2 + 1} - \frac{1}{8}\ln|x + \sqrt{x^2 + 1}| + C\end{aligned}$$

$$\begin{aligned}\lambda &= \frac{1}{8}x(2x^2 + 1)\sqrt{x^2 - 1} - \frac{1}{8}\ln|x + \sqrt{x^2 + 1}| + C \\ y &= \left(\frac{1}{8}x(2x^2 + 1)\sqrt{x^2 - 1} - \frac{1}{8}\ln|x + \sqrt{x^2 + 1}| + C\right)(x^2 + 1)^{-\frac{3}{2}}\end{aligned}$$

$$\boxed{y = \left(\frac{1}{8}x(2x^2 + 1)\sqrt{x^2 - 1} - \frac{1}{8}\ln|x + \sqrt{x^2 + 1}| + C\right)(x^2 + 1)^{-\frac{3}{2}}}$$

4. (E): $y' + y = 2e^x + 4\sin(x)$
Etape 1: ESSM

$$\begin{aligned}y' + y &= 0 \\ y' &= -y \\ \frac{dy}{y} &= -1 \\ \int \frac{1}{y}dy &= - \int 1dx \\ \ln|y| &= -x + C, \quad C \in \mathbb{R} \\ y &= -e^{-x} \cdot e^C \\ y &= \lambda e^{-x} \quad (1)\end{aligned}$$

Etape 2: EASM MVC

$$\begin{aligned}y' &= -\lambda e^{-x} + \lambda' e^{-x} \\y' &= e^{-x}(\lambda + \lambda')(2)\end{aligned}$$

(1) et (2) dans (E):

$$(-\lambda + \lambda)e^{-x} + \lambda e^{-x} = 2e^x + 4\sin(x)$$

$$\lambda' = 2e^x + 4\sin(x)$$

$$\frac{d\lambda}{dx}e^{-x} = 2e^x + 4\sin(x)$$

$$\int d\lambda = \int (2e^{2x} + 4e^x \sin(x)) dx$$

$$\int d\lambda = \int 2e^{2x} dx + 4 \int e^x \sin(x) dx$$

$$\int d\lambda = e^{2x} + C + 4[-\cos(x)e^x + \int e^x \cos(x) dx]$$

Soit $I = \int e^x \cos(x)$

$$\begin{cases} u = e^x \implies u' = e^x \\ v = -\cos(x) \implies v' = \sin(x) \end{cases}$$

$$-\cos(x)e^x + \int e^x \cos(x) dx$$

$$\begin{cases} u = e^x \implies v' = \cos(x) \\ u' = e^x \implies v' = \sin(x) \end{cases}$$

$$-e^x \cos(x) + -e^x \sin(x) - \int e^x \sin(x) dx$$

$$-e^x \cos(x) + -e^x \sin(x) - I$$

$$I = e^x(\sin(x) - \cos(x)) - I$$

$$2I = e^x(\sin(x) - \cos(x))$$

$$I = \frac{e^x}{2}(\sin(x) - \cos(x))$$

$$\int d\lambda = e^{2x} + c + 2e^x(\sin(x) - \cos(x)) + k$$

$$\lambda x = e^{2x} + e^{2x}(\sin(x) - \cos(x)) + C$$

$$\lambda x = e^x(e^x + 2(\sin(x) - \cos(x))) + C$$

$$y = e^x \cdot e^{-x}(e^x + 2(\sin(x) - \cos(x))) + Ce^{-x}$$

$$y = e^x + 2\sin(x) - 2\cos(x) + Ce^{-x}$$

$$\boxed{y = e^x + 2\sin(x) - 2\cos(x) + Ce^{-x}}$$

5. (E): $y' - 2y = 2x^3 + x$ ou $y(3) = 1$
6. (E): $y' + 2xy = e^{x-x^2}$
7. (E): $y' \cos(2y) - \sin(y) = 0$
8. (E): $y' - 2y = 2x^3 + x$ ou $y(3) = 1$

Exercice 2:

1. (E): $xy' = y + 3xy^2$
2. (E): $y' + \frac{y}{x+1} = \frac{1}{2}(x+1)^3 y^3$
C'est une equation de Bernoulli

$$\frac{y'}{y^3} + \frac{y}{(x+1)y^3} = \frac{1}{2}(x+1)^3$$

$$\frac{y'}{y^3} + \frac{(x+1)^{-1}}{y^2} = \frac{1}{2}(x+1)^3 \quad (E)$$

Changement de variable

$$u = \frac{1}{y^2} = y^{-2} \implies \frac{du}{dx} = -2y'y^{-3}$$

D'où

$$(E1) : \quad \frac{1}{2}u' + (x+1)^{-1}u = \frac{1}{2}(x+1)^3 \quad (A)$$

Etape 1: ESSM

$$-\frac{1}{2}u' + (x+1)^{-1}u = 0 \quad (1)$$

$$\frac{1}{2}u' = (x+1)^{-1}u \quad (2)$$

$$\frac{du}{dx} = 2(x+1)^{-1}u \quad (3)$$

$$\frac{du}{u} = 2(x+1)^{-1}dx \quad (4)$$

$$\ln|u| = 2\ln|x+1| + C, \quad C \in \mathbb{R} \quad (5)$$

$$\ln|u| = \ln|(x+1)^2| + C \quad (6)$$

$$u = e^{\ln|(x+1)^2|} \times e^C \quad (7)$$

$$u = (x+1)^2 \times \lambda, \quad \lambda = e^C \quad (E1_1) \quad (8)$$

Etape 2: EASM

$$u' = \lambda'(x+1)^2 + 2(x+1)\lambda \quad (E1_2)$$

(E1₁) et (E1₂) dans (E1)

$$-\frac{1}{2}[\lambda'(x+1)^2 + 2(x+1)\lambda] + (x+1)^{-1}(x+1)^2\lambda = \frac{1}{2}(x+1)^3$$

$$-\frac{\lambda'}{2}(x+1)^2 - (x+1)\lambda + (x+1)\lambda = \frac{1}{2}(x+1)^3$$

$$-\frac{\lambda'}{2}(x+1)^2 = \frac{1}{2}(x+1)^3$$

$$\lambda' = -(x+1)$$

$$\frac{d\lambda}{dx} = -(x+1)$$

$$d\lambda = -(x+1)dx$$

$$\int d\lambda = - \int (x+1) dx$$

$$\lambda = -\left(\frac{x^2}{2} + x\right) + K, \quad K \in \mathbb{R}$$

La solution generale de (E1) est: $u = -\left(\frac{x^2}{2} + x\right) + K)(x+1)^2$

3. (E): $y'(1 - \sin(s) \cos(x)) + y^2 \cos(x) - y' + \sin(x) = 0$
si $y = \cos(x)$ est une solution particuliere
4. (E): $y = xy' + (y')^3$