

S4 - 17B. Devoir 1

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Exercice 1:

Resolution des equations differentielles du 1er ordre suivants:

1. (E): $y' = \sin(y)$

$$\begin{aligned}\frac{dy}{dx} &= \sin(y) \\ \frac{dy}{\sin(y)} &= dx \\ \int \frac{dy}{\sin(y)} &= \int dx\end{aligned}$$

Calcul de $\int \frac{1}{\sin(x)} dy$

Posons $t = \tan(\frac{y}{2})$

donc $dt = \frac{1}{2}(1 + \tan^2(\frac{y}{2}))dy$

$$dt = \frac{1}{2}(1 + t^2)dy$$

$$dy = \frac{2dt}{1 + t^2}$$

$$\sin(y) = \frac{2t}{1 + t^2}$$

$$\int \frac{1}{\sin(y)} dy = \int \left(\frac{1 + t^2}{2t}\right) \left(\frac{2dt}{1 + t^2}\right)$$

$$\int \frac{1}{\sin(y)} dy = \int \frac{1}{t} (dt)$$

$$\int \frac{1}{\sin(y)} dy = \ln|t| + C, \quad C \in \mathbb{R}$$

$$\text{Or } t = \tan\left(\frac{y}{2}\right) \implies \int \frac{1}{\sin(y)} dy = \ln\left|\tan\left(\frac{y}{2}\right)\right| + C$$

On a:

$$\ln|\tan(\frac{y}{2})| = x + C, \quad C \in \mathbb{R}$$

$$|\tan(\frac{y}{2})| = e^x \cdot e^C$$

$$\tan(\frac{y}{2}) = ke^x, \quad k = e^C$$

$$\frac{y}{2} = \tan^{-1}(ke^x)$$

$$y = 2 \tan^{-1}(ke^x)$$

$y = 2 \tan^{-1}(ke^x), \quad k \in \mathbb{R}$

2. (E): $x^2 y' = e^y$

$$x^2 y' = e^y$$

$$x^2 \frac{dy}{dx} = e^y$$

$$\frac{dy}{e^y} = \frac{dx}{x^2}$$

$$\int \frac{1}{e^y} dy = \int \frac{1}{x^2} dx$$

$$\int e^{-y} dy = \int x^{-2} dx$$

$$-e^{-y} = -\frac{1}{x} + C, \quad C \in \mathbb{R}$$

$$\ln(e^{-y}) = \ln|\frac{1}{x} + C|$$

$$-y = \ln|\frac{1}{x} + C|$$

$$y = -\ln|\frac{1}{x} + C|$$

$y = -\ln|\frac{1}{x} + C|, \quad C \in \mathbb{R}$

3. (E): $(x^2 + 1)y' + 3xy = x^2$

Etape 1: ESSM

$$\begin{aligned}
 (x^2 + 1)y' + 3xy &= 0 \\
 (x^2 + 1)y' &= -3xy \\
 y'y^{-1} &= \frac{-3x}{x^2 + 1} \\
 \frac{dy}{y} &= \frac{-3x}{x^2 + 1} dx \\
 \int \frac{1}{y} dy &= \int -\frac{3x}{x^2 + 1} dx \\
 \ln|y| &= -\frac{3}{2}\ln|x^2 + 1| + C, \quad C \in \mathbb{R} \\
 y &= \lambda(x^2 + 1)^{-3/2}, \quad \lambda = e^C \quad (1)
 \end{aligned}$$

Etape 2: EASM MVC

$$\begin{aligned}
 y' &= \lambda'(x^2 + 1)^{-3/2} + \left(-\frac{3}{2}\right)\lambda(2x)(x^2 + 1)^{-3/2-1} \\
 y' &= \lambda'(x^2 + 1)^{-3/2} - \frac{3}{2}\lambda(2x)(x^2 + 1)^{-5/2} \quad (2)
 \end{aligned}$$

(1) et (2) dans (E) $((x^2 + 1)y' + 3xy = x^2)$:

$$\begin{aligned}
 (x^2 + 1)(\lambda'(x^2 + 1)^{-3/2} - \frac{3}{2}\lambda(2x)(x^2 + 1)^{-5/2}) + 3x(\lambda(x^2 + 1)^{-3/2}) &= x^2 \\
 \lambda'(x^2 + 1)^{\frac{1}{2}} &= x^2 \\
 \lambda' &= x^2(x^2 + 1)^{\frac{1}{2}} \\
 d\lambda &= x^2(x^2 + 1)^{\frac{1}{2}} dx \\
 \int d\lambda &= \int x^2(x^2 + 1)^{\frac{1}{2}} dx
 \end{aligned}$$

$$x^2 = \sinh^2(t)$$

$$x = \sinh(t)$$

$$dx = \cosh(t)dt$$

$$x^2 + 1 = \sinh^2(t) + 1 = \cosh^2(t)$$

$$\sqrt{x^2 + 1} = \sqrt{\cosh^2(t)} = \cosh(t), \quad \cosh(t) > 0$$

$$\int x^2 \sqrt{x^2 + 1} dx = \int \sinh^2(t) \cosh(t) \cosh(t) dt$$

$$\int x^2 \sqrt{x^2 + 1} dx = \int \sinh^2(t) \cosh^2(t) dt$$

$$\begin{aligned}
\sinh^2(t) &= \frac{\cosh(2t) - 1}{2} \quad , \quad \cosh^2(t) = \frac{\cosh(2t) + 1}{2} \\
\sinh^2(t) \cosh^2(t) &= \left(\frac{\cosh(2t) - 1}{2}\right) \left(\frac{\cosh(2t) + 1}{2}\right) \\
\sinh^2(t) \cosh^2(t) &= \frac{1}{4}(\cosh(2t) - 1)(\cosh(2t) + 1) \\
\sinh^2(t) \cosh^2(t) &= \frac{1}{4}(\cosh^2(2t) - 1) \\
\sinh^2(t) \cosh^2(t) &= \frac{1}{4}\sinh^2(2t)
\end{aligned}$$

Donc

$$\begin{aligned}
\int \sinh^2(t) \cosh^2(t) dt &= \int \frac{1}{4} \sinh^2(2t) dt \\
\int \sinh^2(t) \cosh^2(t) dt &= \frac{1}{4} \int \sinh^2(2t) dt \\
\text{or } \sinh^2(2t) dt &= \int \frac{\cosh(4t) - 1}{2} dt \\
\text{car } \sinh^2(t) &= \frac{\cosh(2t) - 1}{2} \\
\frac{1}{2} \int \cosh(4t) - \frac{1}{2} \int 1 dt \\
\cosh(4t) dt &= \frac{1}{4} \sinh(4t) \\
\int \sinh^2(t) dt &= \frac{1}{2} \cdot \frac{1}{4} \sinh(4t) - \frac{1}{2} t \\
\int \sinh^2(t) dt &= \frac{1}{8} \sinh(4t) - \frac{1}{2} t + C, \quad C \in \mathbb{R} \\
\int \sinh^2(t) dt &= \frac{1}{4} \left(\frac{1}{8} \sinh(4t) - \frac{1}{2} t + C \right) \\
\int \sinh^2(t) dt &= \frac{1}{32} \sinh(4t) - \frac{1}{8} t + C
\end{aligned}$$

$$x = \sinh(t), t = \operatorname{arcsinh}(x) = \ln(x + \sqrt{x^2 + 1})$$

$$\sinh(4t) = 2 \sinh(2t) \cosh(2t)$$

$$\sinh(2t) = 2 \sinh(t) \cosh(t) = 2x\sqrt{1+x^2}$$

$$\begin{aligned}\cosh(2t) &= \sqrt{1 + \sinh^2(2t)} = \sqrt{1 + (2x\sqrt{x^2 + 1})^2} \\ \cosh(2t) &= \sqrt{1 + 4x^2(x^2 + 1)} = \sqrt{1 + 4x^4 + 4x^2} \\ \cosh(2t) &= \sqrt{(2x^2 + 1)^2} = 2x^2 + 1, \quad x^2 \geq 0\end{aligned}$$

$$\begin{aligned}\sinh(4t) &= 2(2x\sqrt{1 + x^2})(2x^2 + 1) \\ \sinh(4t) &= 4x(2x^2 + 1)\sqrt{1 + x^2}\end{aligned}$$

$$\begin{aligned}\frac{1}{32} \sinh(4t) - \frac{1}{8}t &= \frac{1}{32}4x(2x^2 + 1)\sqrt{1 + x^2} - \frac{1}{8}t + C \\ \int x^2\sqrt{x^2 + 1}dx &= \frac{1}{8}x(2x^2 + 1)\sqrt{x^2 + 1} - \frac{1}{8}\ln|x + \sqrt{x^2 + 1}| + C\end{aligned}$$

$$\begin{aligned}\lambda &= \frac{1}{8}x(2x^2 + 1)\sqrt{x^2 - 1} - \frac{1}{8}\ln|x + \sqrt{x^2 + 1}| + C \\ y &= \left(\frac{1}{8}x(2x^2 + 1)\sqrt{x^2 - 1} - \frac{1}{8}\ln|x + \sqrt{x^2 + 1}| + C\right)(x^2 + 1)^{-\frac{3}{2}}\end{aligned}$$

$$\boxed{y = \left(\frac{1}{8}x(2x^2 + 1)\sqrt{x^2 - 1} - \frac{1}{8}\ln|x + \sqrt{x^2 + 1}| + C\right)(x^2 + 1)^{-\frac{3}{2}}, \quad C \in \mathbb{R}}$$

4. (E): $y' + y = 2e^x + 4\sin(x)$

Etape 1: ESSM

$$\begin{aligned}y' + y &= 0 \\ y' &= -y \\ \frac{dy}{y} &= -1 \\ \int \frac{1}{y}dy &= - \int 1dx \\ \ln|y| &= -x + C, \quad C \in \mathbb{R} \\ y &= -e^{-x} \cdot e^C \\ y &= \lambda e^{-x} \quad (1)\end{aligned}$$

Etape 2: EASM MVC

$$\begin{aligned}y' &= -\lambda e^{-x} + \lambda' e^{-x} \\y' &= e^{-x}(\lambda + \lambda')(2)\end{aligned}$$

(1) et (2) dans (E):

$$(-\lambda + \lambda')e^{-x} + \lambda e^{-x} = 2e^x + 4\sin(x)$$

$$\lambda' = 2e^x + 4\sin(x)$$

$$\frac{d\lambda}{dx}e^{-x} = 2e^x + 4\sin(x)$$

$$\int d\lambda = \int (2e^{2x} + 4e^x \sin(x)) dx$$

$$\int d\lambda = \int 2e^{2x} dx + 4 \int e^x \sin(x) dx$$

$$\int d\lambda = e^{2x} + C + 4[-\cos(x)e^x + \int e^x \cos(x) dx]$$

Soit $I = \int e^x \cos(x)$

$$\begin{cases} u = e^x \implies u' = e^x \\ v = -\cos(x) \implies v' = \sin(x) \end{cases}$$

$$-\cos(x)e^x + \int e^x \cos(x) dx$$

$$\begin{cases} u = e^x \implies v' = \cos(x) \\ u' = e^x \implies v' = \sin(x) \end{cases}$$

$$-e^x \cos(x) + -e^x \sin(x) - \int e^x \sin(x) dx$$

$$-e^x \cos(x) + -e^x \sin(x) - I$$

$$I = e^x(\sin(x) - \cos(x)) - I$$

$$2I = e^x(\sin(x) - \cos(x))$$

$$I = \frac{e^x}{2}(\sin(x) - \cos(x))$$

$$\int d\lambda = e^{2x} + c + 2e^x(\sin(x) - \cos(x)) + k$$

$$\lambda x = e^{2x} + e^{2x}(\sin(x) - \cos(x)) + C$$

$$\lambda x = e^x(e^x + 2(\sin(x) - \cos(x))) + C$$

$$y = e^x \cdot e^{-x}(e^x + 2(\sin(x) - \cos(x))) + Ce^{-x}$$

$$y = e^x + 2\sin(x) - 2\cos(x) + Ce^{-x}$$

$$\boxed{y = e^x + 2\sin(x) - 2\cos(x) + Ce^{-x}, \quad C \in \mathbb{R}}$$

5. (E): $y' - 2y = 2x^3 + x$ ou $y(0) = 1$

Etape 1: ESSM

$$y' - 2y = 0$$

$$y' = 2y$$

$$\frac{dy}{dx} = 2y$$

$$\int \frac{dy}{y} = \int 2dx$$

$$\ln |y| = 2x + c, \quad c \in \mathbb{R}$$

$$y = e^{2x+c}$$

$$y = e^{2x} \cdot e^c$$

On pose $\lambda = e^c$

$$\underline{y = \lambda e^{2x}}$$

Etape 2: EASM MVC

$$y' - 2y = 2x^3 + x \text{ avec } y = \lambda e^{2x}$$

$$y' = \lambda e^{2x} + 2e^{2x}\lambda \quad (2)$$

(1) et (2) dans (E):

$$\lambda e^{2x} + 2e^{2x}\lambda - 2(\lambda e^{2x}) = 2x^3 + x$$

$$\lambda e^{2x} = 2x^3 + x$$

$$\frac{d\lambda}{dx} = \frac{2x^3 + x}{e^{2x}}$$

$$d\lambda = \frac{2x^3 + x}{e^{2x}} dx$$

$$\int d\lambda = \int \frac{2x^3 + x}{e^{2x}} dx$$

$$\lambda = \int \frac{2x^3}{e^{2x}} dx + \int \frac{x}{e^{2x}} dx$$

Posons $A = \int \frac{2x^3}{e^{2x}} dx$ et $B = \int \frac{x}{e^{2x}} dx$

(1) Calculons A:

$$B = \int \frac{x}{e^{2x}} dx$$

$$B = \int e^{-2x} x dx$$

$$B = -\frac{1}{2} \int -2e^{-2x} x dx$$

Posons $K = \int -2e^{-2x} x dx$

Rappelons l'intégration par partie

$$\int u'v = uv - \int v'u$$

$$\text{avec } u' = -2e^{-2x}$$

$$v = x$$

$$\text{alors } u = e^{-2x}$$

$$v' = 1$$

$$K = e^{-2x} x - \int 1e^{-2x}$$

$$K = e^{-2x} x + \frac{1}{2} \int -2e^{-2x}$$

$$K = e^{-2x} x + \frac{1}{2} e^{-2x}$$

$$\underline{K = e^{-2x} x + \frac{1}{2} e^{-2x}}$$

$$\text{Donc } B = -\frac{1}{2}(e^{-2x}x + \frac{1}{2}e^{-2x})$$

$$B = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$

$$\underline{B = -\frac{1}{2}e^{-2x}x - \frac{1}{4}e^{-2x}}$$

(2) Calculons maintenant A:

$$A = \int 2x^3 \cdot e^{-2x} dx$$

$$A = - \int -2x^3 \cdot e^{-2x} dx$$

Posons $M = \int 2x^3 \cdot e^{-2x} dx$

Rappelons l'intégration par partie

$$\int u'v = uv - \int v'u$$

$$\begin{array}{ll} \text{avec } u' = -2e^{-2x} & u = e^{-2x} \\ \text{alors } v = x^3 & v' = 3x^2 \end{array}$$

$$M = e^{-2x}x^3 - \int 3x^2e^{-2x}dx$$

$$M = e^{-2x}x^3 + \frac{3}{2} \int -2x^2e^{-2x}dx$$

Posons $N = \int -2x^2e^{-2x}dx$

$$\begin{array}{ll} \text{avec } u' = -2e^{-2x} & v = x^2 \\ \text{alors } u = e^{-2x} & v' = 2x \end{array}$$

$$N = e^{-2x}x^2 - \int +2xe^{-2x}dx$$

$$N = e^{-2x}x^2 + \int -2xe^{-2x}dx$$

D'après le calcul de K, on peut avoir:

$$\underline{N = e^{-2x}x^2 + e^{-2x}x + \frac{1}{2}e^{-2x}}$$

On peut aussi alors écrire:

$$M = e^{-2x}x^3 + \frac{3}{2}e^{-2x}x^2 + \frac{3}{2}e^{-2x}x + \frac{3}{4}e^{-2x}$$

Et on peut aussi avoir:

$$A = -x^3e^{-2x} - \frac{3}{2}x^2e^{-2x} - \frac{3}{2}xe^{-2x} - \frac{3}{4}e^{-2x}$$

On a alors trouvé $\lambda = A + B$

$$\begin{aligned}\lambda &= -x^3e^{-2x} - \frac{3}{2}x^2e^{-2x} - \frac{3}{2}xe^{-2x} - \frac{3}{4}e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} \\ \lambda &= -e^{-2x}\left(x^3 + \frac{3}{2}x^2 + 2x + 1\right)\end{aligned}$$

Et comme $y = \lambda e^{2x}$

$$y = -x^3 - \frac{3}{2}x^2 - 2x - 1$$

6. (E): $y' + 2xy = e^{x-x^2}$

Etape 1: ESSM

$$\begin{aligned}y' - 2xy &= 0 \\ y' &= 2xy \\ \frac{dy}{dx} &= 2xy \\ \int \frac{dy}{y} &= \int 2x dx \\ \ln |y| &= x^2 + k, \quad k \in \mathbb{R} \\ y_0 &= e^{x^2} + e^k \\ y_0 &= e^{x^2} \cdot c, \quad c = e^k\end{aligned}$$

Etape 2: EASM

$$y' - 2xy = e^{x-x^2} \quad (E)$$

MVC:

$$y = ce^{x^2} \quad (1)$$

$$y' = c'e^{x^2} + c2xe^{x^2} \quad (2)$$

(1) et (2) dans (E):

$$c'e^{x^2} + c2xe^{x^2} - c2xe^{x^2} = e^{x-x^2}$$

$$c'e^{x^2} = e^{x-x^2}$$

$$\frac{dc}{dx} \cdot e^{x^2} = e^{x-x^2}$$

$$\int dc = \int \frac{e^{x-x^2}}{e^{x^2}} dx$$

$$c = \int \frac{e^x}{e^{x^2} \cdot e^{x^2}}$$

$$c = \int \frac{e^x}{e^{2x^2}} dx$$

$$c = \int \frac{1}{e^{2x^2}} + \int e^x dx$$

$$c = \int e^{-2x} + \int e^x dx$$

$$c = -\frac{e^{-2x^2}}{4x} + e^x + \lambda, \quad \lambda \in \mathbb{R}$$

Or $y = c \cdot e^{x^2}$

$$y = \left(-\frac{e^{-2x^2} \cdot e^{x^2}}{4x} + e^x \cdot e^{x^2}\right) + \lambda e^{x^2}$$

$$= \left(-\frac{e^{x^2-2x^2}}{4x} + e^x \cdot e^{x^2}\right) + \lambda e^{x^2}$$

$$y = y_p + y_0$$

$$y = \left(-\frac{e^{-x^2}}{4x} + e^{x^2+x} + \lambda e^{x^2}\right), \quad \lambda \in \mathbb{R}$$

7. (E): $y' \cos(2y) - \sin(y) = 0$

$$\begin{aligned}
 y' \cos(2y) - \sin(y) &= 0, \\
 y' \cos(2y) &= \sin(y) \\
 y' \frac{\cos(2y)}{\cos(2y)} &= \frac{\sin(y)}{\cos(2y)}, \\
 y' &= \frac{\sin(y)}{\cos(2y)} \text{ or } \cos(2y) = 1 - 2\cos(y), \\
 y' &= \frac{\sin(y)}{1 - 2\cos^2(y)}
 \end{aligned}$$

Posons $u = \cos(y)$

Alors $\frac{du}{dy} = -\sin(y) \Rightarrow dy = \frac{du}{-\sin(y)}$,

On a alors : $y = \int \frac{\sin(y)}{1-2u^2} \cdot \frac{du}{-\sin(y)}$

$$y = \int \frac{1}{2u^2-1}$$

Posons $w = \sqrt{2}u \Rightarrow u = \frac{w}{\sqrt{2}}$

Alors $\frac{dw}{du} = \sqrt{2} \Rightarrow w = \frac{dw}{\sqrt{2}}$

$$\begin{aligned}
 y &= \int \frac{1}{w^2-1} \cdot \frac{dw}{\sqrt{2}} \\
 y &= \frac{1}{\sqrt{2}} \int \frac{1}{w^2-1} dw \\
 y &= \frac{1}{\sqrt{2}} \int \frac{1}{(w-1)(w+1)} dw
 \end{aligned}$$

Simplifions d'abord la primitive

$$\frac{1}{(w-1)(w+1)} = \frac{A}{w-1} + \frac{B}{w+1} (E), \quad A, B \in \mathbb{R},$$

On multiplie chaque terme par $(x-1)$

$$\frac{1}{(w-1)} = A + \frac{B(w-1)}{(w+1)}$$

Pour $u = 1$,

$$\frac{1}{2} = A \quad (1)$$

On multiplie chaque terme par $(x+1)$

$$\frac{1}{(w+1)} = \frac{A(w+1)}{(w-1)} + B$$

Pour $u = -1$,

$$-\frac{1}{2} = B \quad (2)$$

(1) et (2) dans (E):

$$\frac{1}{(w-1)(w+1)} = \frac{\frac{1}{2}}{(w-1)} + \frac{-\frac{1}{2}}{(w+1)}$$

$$\frac{1}{(w-1)(w+1)} = \frac{1}{2(w-1)} - \frac{1}{2(w+1)}$$

On a alors: $y = \frac{1}{\sqrt{2}} \left[\int \frac{1}{2(w-1)} - \int \frac{1}{2(w+1)} \right]$

$$y = \frac{1}{\sqrt{2}} \left[\frac{1}{2} \int \frac{1}{(w-1)} - \frac{1}{2} \int \frac{1}{(w+1)} \right]$$

$$y = \frac{1}{2\sqrt{2}} \left[\int \frac{1}{(w-1)} - \int \frac{1}{(w+1)} \right]$$

$$y = \frac{\sqrt{2}}{4} [\ln|w-1| + C - \ln|w+1| + K], \quad C, K \in \mathbb{R}$$

$$y = \frac{\sqrt{2}}{4} [\ln|w-1| - \ln|w+1| + T], T \in \mathbb{R}$$

Retour au changement de variable

$$w = \sqrt{2}u$$

$$\text{Alors } y = \frac{\sqrt{2}}{4} (\ln|\sqrt{2}u-1| - \ln|\sqrt{2}u+1| + T), T \in \mathbb{R}$$

$$u = \cos(y)$$

$$\text{Alors } y = \frac{\sqrt{2}}{4} (\ln|\sqrt{2}\cos(y)-1| - \ln|\sqrt{2}\cos(y)+1| + T), T \in \mathbb{R}$$

$y = \frac{\sqrt{2}}{4} (\ln \sqrt{2}\cos(y)-1 - \ln \sqrt{2}\cos(y)+1 + T), T \in \mathbb{R}$
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Exercice 2:

Resolution des equations differentielles non lineaires du 1er ordre suivants:

1. (E): $xy' = y + 3xy^2$

$$\begin{aligned}\frac{x}{x}y' &= \frac{1}{x}y + \frac{3xy^2}{x} \\ y' &= \frac{1}{x}y + 3y^2\end{aligned}$$

On pose $z = y^{-1} = \frac{1}{y} \implies z' = -\frac{y'}{y^2}$

$$\begin{aligned}-\frac{y'}{y^2} &= -\frac{1}{xy} - 3 \\ \text{d'où } z' &= -\frac{1}{x}z - 3 \\ z' + \frac{1}{x}z &= -3 \quad (E)\end{aligned}$$

ESSM

$$z' + \frac{1}{x}z = 0$$

$$\begin{aligned}z_H &= \lambda e^{-\int \frac{1}{x}}, \quad \lambda \in \mathbb{R} \\ &= \lambda e^{-\ln|x|}\end{aligned}$$

$$z_H = \lambda \frac{1}{x}, \quad \lambda \in \mathbb{R}$$

EASM

MVC:

$$t = g(x) \frac{1}{x} \quad (1)$$

$$t' = \frac{g'(x)x - g(x)}{x^2} \quad (2)$$

(1) et (2) dans (E)

$$\begin{aligned}\frac{g'(x)x - g(x)}{x^2} + \frac{1}{x}g(x)\frac{1}{x} &= -3 \\ \frac{g'(x)x}{x^2} - \frac{g(x)}{x^2} + \frac{g(x)}{x^2} &= -3 \\ g'(x) &= -3x \\ g(x) &= -\frac{3}{2}x^2 + C, \quad C \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}z_p &= \left(-\frac{3}{2}x^2 + C\right)\frac{1}{x} + \lambda\left(\frac{1}{x}\right) \\ z_p &= \left(-\frac{3}{2}x + \psi\left(\frac{1}{x} + \frac{1}{x}\right)\right), \quad \psi = \frac{C}{x} \\ z_p &= -\frac{3}{2}x + \psi\frac{2}{x} \\ z_p &= -\frac{3x^2 + 4\psi}{2x}\end{aligned}$$

$$z = z_H + z_P \text{ et } y = y_H + y_P$$

or $z = \frac{1}{y}$

$$\text{Alors } y = \frac{1}{z_H} + \frac{1}{z_P} \quad \boxed{y = \frac{x}{\lambda} + \frac{2x}{3x^2 + 4\psi}, \quad \lambda, \psi \in \mathbb{R}}$$

2. (E): $y' + \frac{y}{x+1} = \frac{1}{2}(x+1)^3y^3$
C'est une equation de Bernoulli

$$\begin{aligned}\frac{y'}{y^3} + \frac{y}{(x+1)y^3} &= \frac{1}{2}(x+1)^3 \\ \frac{y'}{y^3} + \frac{(x+1)^{-1}}{y^2} &= \frac{1}{2}(x+1)^3 \quad (E)\end{aligned}$$

Changement de variable

$$\begin{aligned} u &= \frac{1}{y^2} = y^{-2} \\ \frac{du}{dx} &= -2y'y^{-3} \\ -\frac{1}{2}u' &= \frac{y'}{y^3} \quad (A) \end{aligned}$$

D'où

$$(E1) : \quad \frac{1}{-2}u' + (x+1)^{-1}u = \frac{1}{2}(x+1)^3$$

Etape 1: ESSM

$$-\frac{1}{2}u' + (x+1)^{-1}u = 0 \quad (1)$$

$$\frac{1}{2}u' = (x+1)^{-1}u \quad (2)$$

$$\frac{du}{dx} = 2(x+1)^{-1}u \quad (3)$$

$$\frac{du}{u} = 2(x+1)^{-1}dx \quad (4)$$

$$\ln|u| = 2\ln|x+1| + C, \quad C \in \mathbb{R} \quad (5)$$

$$\ln|u| = \ln|(x+1)^2| + C \quad (6)$$

$$u = e^{\ln|(x+1)^2|} \times e^C \quad (7)$$

$$u = (x+1)^2 \times \lambda, \quad \lambda = e^C \quad (E1_1) \quad (8)$$

Etape 2: EASM

$$u' = \lambda'(x+1)^2 + 2(x+1)\lambda \quad (E1_2)$$

$(E1_1)$ et $(E1_2)$ dans $(E1)$

$$\begin{aligned}
-\frac{1}{2}[\lambda'(x+1)^2 + 2(x+1)\lambda] + (x+1)^{-1}(x+1)^2\lambda &= \frac{1}{2}(x+1)^3 \\
-\frac{\lambda'}{2}(x+1)^2 - (x+1)\lambda + (x+1)\lambda &= \frac{1}{2}(x+1)^3 \\
-\frac{\lambda'}{2}(x+1)^2 &= \frac{1}{2}(x+1)^3 \\
\lambda' &= -(x+1) \\
\frac{d\lambda}{dx} &= -(x+1) \\
d\lambda &= -(x+1)dx \\
\int d\lambda &= -\int (x+1)dx \\
\lambda &= -\left(\frac{x^2}{2} + x\right) + K, \quad K \in \mathbb{R}
\end{aligned}$$

La solution generale de (E1) est:

$$u = \left(-\left(\frac{x^2}{2} + x\right) + K \right) (x+1)^2$$

$$\begin{aligned}
u &= \left(-\frac{x^2}{2} - x + K\right)(x+1)^2 \\
u' &= \left(-\frac{2}{2}x - 1\right)(x+1)^2 + 2(x+1)\left(-\frac{x^2}{2} - x + K\right) \\
u' &= (-x - 1)(x+1)^2 + 2(x+1)\left(-\frac{x^2}{2} - x + K\right) \\
u' &= -(x+1)^3 + 2(x+1)\left(-\frac{x^2}{2} - x + K\right) \\
u' &= (x+1)\left[-(x+1)^2 + 2\left(-\frac{x^2}{2} - x + 2K\right)\right] \\
u' &= (x+1)\left[-(x+1)^2 - x^2 - 2x + 2K\right] \\
u' &= (x+1)\left[-x^2 - 2x - 1 - x^2 - 2x + 2K\right] \\
u' &= (x+1)\left[-2x^2 - 4x - 1 + 2K\right] \quad (B)
\end{aligned}$$

En reprenant (A)

$$\begin{aligned}
\frac{y'}{y^3} &= -\frac{1}{2}u' \\
\frac{dy}{dx} &= -\frac{1}{2}u'y^3 \\
\frac{dy}{y^3} &= -\frac{1}{2}u'dx \\
\int \frac{dy}{y^3} &= -\frac{1}{2} \int u'dx \\
\int y^{-3}dy &= -\frac{1}{2} \int u'dx \\
\frac{y^{-2}}{-2} &= -\frac{1}{2} \int u'dx \\
y^{-2} &= \int u'dx \quad (A')
\end{aligned}$$

(B) dans (A')

$$\begin{aligned}
y^{-2} &= \int (x+1)[-2x^2 - 4x - 1 + 2K] dx \\
&= \int (x+1)(-2x^2) + (x+1)(-4x) + (x+1)(-1) + (x+1)(2K) dx \\
&= \int (-2x^3 - 2x^2 - 4x^2 - 4x - x - 1 + 2Kx + 2K) dx \\
&= \int (-2x^3 - 6x^2 - 5x - 1 + 2Kx + 2K) dx \\
&= -\frac{2}{4}x^4 - \frac{6}{3}x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C \\
y^{-2} &= -\frac{1}{2}x^4 - 2x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C \\
\Rightarrow y^2 &= (-\frac{1}{2}x^4 - 2x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C)^{-1} \\
\Rightarrow y &= \pm(-\frac{1}{2}x^4 - 2x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C)^{-\frac{1}{2}}
\end{aligned}$$

$$y = \pm(-\frac{1}{2}x^4 - 2x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C)^{-\frac{1}{2}}, \quad K, C \in \mathbb{R}$$

3. (E): $y'(1 - \sin(x) \cos(x)) + y^2 \cos(x) - y' + \sin(x) = 0$
 si $y = \cos(x)$ est une solution particuliere

$$\begin{aligned} y'(1 - \sin x \cos x) + y^2 \cos x - y' + \sin x &= 0 \\ (1 - \sin x \cos x)y' - y' + \cos x y^2 &= -\sin x \quad (E) \end{aligned}$$

On pose $y = \frac{1}{z} + \cos x$, alors $y' = -\frac{z'}{z^2}$.

(1) dans (E) :

$$(1 - \sin x \cos x) \left(-\frac{z'}{z^2} \right) - \left(\frac{1}{z} + \cos x \right) + \cos x \left(\frac{1}{z} + \cos x \right)^2 = -\sin x.$$

Developpement :

$$-\frac{z'}{z^2} + \frac{z' \sin x \cos x}{z^2} + \sin^2 x \cos x - \frac{1}{z} - \cos x + \cos x \left(\frac{1}{z^2} + \frac{2 \cos x}{z} + \cos^2 x \right) = 0.$$

Developpement de $\cos x \left(\frac{1}{z} + \cos x \right)^2$:

$$\cos x \left(\frac{1}{z^2} + \frac{2 \cos x}{z} + \cos^2 x \right) = \frac{\cos x}{z^2} + \frac{2 \cos^2 x}{z} + \cos^3 x.$$

Avec $\cos^3 x = (1 - \sin^2 x) \cos x$:

$$\cos^3 x = \cos x - \sin^2 x \cos x.$$

Substitution dans (E) :

$$-\frac{z'}{z^2} + \frac{z' \sin x \cos x}{z^2} - \frac{1}{z} + \sin^2 x \cos x - \cos x + \frac{\cos x}{z^2} + \frac{2 \cos^2 x}{z} + \cos x - \sin^2 x \cos x = 0.$$

Rearrangement :

$$(\sin x \cos x - 1)z' + (2 \cos^2 x - 1)z = -\cos x.$$

Hypothese $z_0 = \lambda e^{\varphi(x)}$, où

$$\varphi(x) = \int \frac{-(2 \cos^2 x - 1)}{(\sin x \cos x - 1)} dx.$$

Calcul de I :

$$I = \int \frac{2 \cos^2 x - 1}{\sin x \cos x - 1} dx.$$

Avec $2 \cos^2 x - 1 = \cos 2x$:

$$I = \int \frac{\cos 2x}{\sin 2x - 2} dx.$$

Changement de variable $u = \sin 2x - 2$, $du = 2 \cos 2x dx$:

$$I = \int \frac{du}{u} = \ln |u| + C = \ln |\sin 2x - 2| + C.$$

Expression de z_0 :

$$z_0 = K(\sin 2x - 2)^{-1}.$$

(3) dans (E) :

$$(\sin x \cos x - 1)K'(\sin 2x - 2)^{-1} - 2K \cos 2x(\sin 2x - 2)^{-2} = -\cos x.$$

Resolution :

$$K' = -2 \cos x, \quad K = -2 \sin x + \varepsilon.$$

Solution finale :

$$y = \frac{\sin 2x - 2}{-2 \sin x + \varepsilon} + \cos x, \quad \varepsilon \in \mathbb{R}$$

4. (E): $y = xy' + (y')^3$

Lagrange (E) : $y = x\varphi(y') + \psi(y)$

1er cas: Si $\varphi(y') = y'$ c-a-d $y = xy' + \psi(y')$

alors on pose $p = y'$

donc on a $y' = xp + \varphi(p)$ et $y' = p + xp' + p'\psi(p)$

mais comme $y' = p$ alors on a $p = p + xp' + p'\psi(p)$

ou encore $0 = p'(x + \psi(p))$ c-a-d $p' = 0$ ou $x + \psi(p) = 0$

donc p: constante ou $x = -\psi(p)$

Si $p = y'$ est une constante alors $y = Cx + K, K \in \mathbb{Q}$ ou si $x = -\psi'(p)$
alors $y = -p\psi'(p) + \psi(p)$ (car $y = xp + \varphi(p)$)

$$(E) : y = xy' + (y')^3$$

On pose $p = y'$

$$y = xp + p^3 \implies y' = x + xp' + 3p^2p'$$

$$y' = p \implies p = x + p'x + 3p'p^2$$

Or $p'(x + \psi'(p)) = 0$ $p' = 0$ ou $x - 3p^2 = 0$

$$\begin{aligned} x &= -3p^2 & / y &= xp + p^3 \\ 3p^2 &= -x \\ p^2 &= -\frac{x}{3} \end{aligned}$$

$$\begin{aligned} y &= xp + p\left(-\frac{x}{3}\right) \\ &= xp - \frac{x}{3}p \\ y &= \frac{2xp}{3} \\ p &= \frac{3y}{2x} \end{aligned}$$

Nous rappelons que $p^2 = -\frac{x}{3}$

$$\left(\frac{3y}{2x}\right)^2 = -\frac{x}{3}$$

$$\frac{9y^2}{4x^2} = -\frac{x}{3}$$

$$27y^2 = -4x^3$$

$$y^2 = -\frac{4}{27}x^3$$

Solution generale:

$$y = Cx + C^3, \quad C \in \mathbb{R}$$

$$y^2 = -\frac{4}{27}x^3$$