S4 - 17B. Devoir 1

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Exercice 1:

1. (E): $y' = \sin(y)$

$$\frac{dy}{dx} = \sin(y)$$
$$\frac{dy}{\sin(y)} = dx$$
$$\int \frac{dy}{\sin(y)} = \int dx$$

Calcul de $\int \frac{1}{\sin(x)} dy$

$$\text{Posons } t = \tan(\frac{y}{2})$$
$$\text{donc } dt = \frac{1}{2}(1 + \tan^2(\frac{y}{2}))dy$$
$$dt = \frac{1}{2}(1 + t^2)dy$$
$$dy = \frac{2dt}{1 + t^2}$$

$$\sin(y) = \frac{2t}{1 + t^2}$$
$$\int \frac{1}{\sin(y)} dy = \int (\frac{1 + t^2}{2t})(\frac{2dt}{1 + t^2})$$
$$\int \frac{1}{\sin(y)} dy = \int \frac{1}{t}(dt)$$
$$\int \frac{1}{\sin(y)} dy = ln|t| + C, \quad C \in \mathbb{R}$$
$$\text{Or } t = \tan(\frac{y}{2}) \implies \int \frac{1}{\sin(y)} dy = ln|\tan(\frac{y}{2})| + C$$

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On a:

$$ln|\tan(\frac{y}{2})| = x + C, \quad C \in \mathbb{R}$$

$$|\tan(\frac{y}{2})| = e^x \cdot e^C$$

$$\tan(\frac{y}{2}) = ke^x, \quad k = e^C$$

$$\frac{y}{2} = \tan^{-1}(ke^x)$$

$$y = 2\tan^{-1}(ke^x)$$

$$\boxed{y = 2\tan^{-1}(ke^x), \quad k \in \mathbb{R}}$$

2. (E): $x^2 y' = e^y$

$$x^2 y' = e^y$$

$$x^2 \frac{dy}{dx} = e^y$$

$$\frac{dy}{e^y} = \frac{dx}{x^2}$$

$$\int \frac{1}{e^y}\, dy = \int \frac{1}{x^2}\, dx$$

$$\int e^{-y}\, dy = \int x^{-2}\, dx$$

$$-e^{-y} = -\frac{1}{x} + C, \quad C \in \mathbb{R}$$

$$\ln(e^{-y}) = \ln|\frac{1}{x} + C|$$

$$-y = \ln|\frac{1}{x} + C|$$

$$y = -\ln|\frac{1}{x} + C|$$

$$\boxed{y = -\ln|\frac{1}{x} + C|}$$

3. (E): $(x^2 + 1)y' + 3xy = x^2$

Etape 1: ESSM

$$(x^2 + 1)y' + 3xy = 0$$
$$(x^2 + 1)y' = -3xy$$
$$y'y^{-1} = \frac{-3x}{x^2 + 1}$$
$$\frac{dy}{y} = \frac{-3x}{x^2 + 1}dx$$
$$\int \frac{1}{y}\,dy = \int -\frac{3x}{x^2 + 1}\,dx$$
$$ln|y| = -\frac{3}{2}ln|x^2 + 1| + C, \quad C \in \mathbb{R}$$
$$y = \lambda(x^2 + 1)^{-3/2}, \quad \lambda = e^C \quad (1)$$

Etape 2: EASM MVC

$$y' = \lambda'(x^2 + 1)^{-\frac{3}{2}} + (-\frac{3}{2})\lambda(2x)(x^2 + 1)^{-\frac{3}{2}-1}$$
$$y' = \lambda'(x^2 + 1)^{-\frac{3}{2}} - \frac{3}{2}\lambda(2x)(x^2 + 1)^{-\frac{5}{2}} \quad (2)$$

(1) et (2) dans (E) $((x^2 + 1)y' + 3xy = x^2)$:

$$(x^2 + 1)(\lambda'(x^2 + 1)^{-\frac{3}{2}} - \frac{3}{2}\lambda(2x)(x^2 + 1)^{-\frac{5}{2}}) + 3x(\lambda(x^2 + 1)^{-3/2} = x^2$$
$$\lambda'(x^2 + 1)^{\frac{1}{2}} = x^2$$
$$\lambda' = x^2(x^2 + 1)^{\frac{1}{2}}$$
$$d\lambda = x^2(x^2 + 1)^{\frac{1}{2}}dx$$
$$\int d\lambda = \int x^2(x^2 + 1)^{\frac{1}{2}}dx$$

$$x^2 = \sinh^2(t)$$
$$x = \sinh(t)$$
$$dx = \cosh(t)dt$$
$$x^2 + 1 = \sinh^2(t) + 1 = \cosh^2(t)$$
$$\sqrt{x^2 + 1} = \sqrt{\cosh^2(t)} = \cosh(t), \quad \cosh(t) > 0$$
$$\int x^2 \sqrt{x^2 + 1}\, dx = \int \sinh^2(t) \cosh(t) \cosh(t)\, dt$$
$$\int x^2 \sqrt{x^2 + 1}\, dx = \int \sinh^2(t) \cosh^2(t)\, dt$$

$$\sinh^2(t) = \frac{\cosh(2t) - 1}{2} \quad , \quad \cosh^2(t) = \frac{\cosh(2t) + 1}{2}$$

$$\sinh^2(t)\cosh^2(t) = (\frac{\cosh(2t) - 1}{2})(\frac{\cosh(2t) + 1}{2})$$

$$\sinh^2(t)\cosh^2(t) = \frac{1}{4}(\cosh(2t) - 1)(\cosh(2t) + 1)$$

$$\sinh^2(t)\cosh^2(t) = \frac{1}{4}(\cosh^2(2t) - 1)$$

$$\sinh^2(t)\cosh^2(t) = \frac{1}{4}\sinh^2(2t)$$

Donc

$$\int \sinh^2(t)\cosh^2(t)\, dt = \int \frac{1}{4}\sinh^2(2t)\, dt$$

$$\int \sinh^2(t)\cosh^2(t)\, dt = \frac{1}{4}\int \sinh^2(2t)\, dt$$

$$\text{or } \sinh^2(2t)\, dt = \int \frac{\cosh(4t) - 1}{2}\, dt$$

$$\text{car } \sinh^2(t) = \frac{\cosh(2t) - 1}{2}$$

$$\frac{1}{2}\int \cosh(4t) - \frac{1}{2}\int 1 dt$$

$$\cosh(4t) dt = \frac{1}{4}\sinh(4t)$$

$$\int \sinh^2(t) dt = \frac{1}{2}.\frac{1}{4}\sinh(4t) - \frac{1}{2}t$$

$$\int \sinh^2(t) dt = \frac{1}{8}\sinh(4t) - \frac{1}{2}t + C, \quad C \int \mathbb{R}$$

$$\int \sinh^2(t) dt = \frac{1}{4}(\frac{1}{8}\sinh(4t) - \frac{1}{2}t + C)$$

$$\int \sinh^2(t) dt = \frac{1}{32}\sinh(4t) - \frac{1}{8}t + C$$

$$x = \sinh(t), t = arcsinh(x) = \ln(x + \sqrt{x^2 + 1})$$

$$\sinh(4t) = 2\sin(2t)\cosh(2t)$$

$$\sinh(2t) = 2\sinh(t)\cosh(t) = 2x\sqrt{1 + x^2}$$

$$\cosh(2t) = \sqrt{1 + \sinh^2(2t)} = \sqrt{1 + (2x\sqrt{x^2+1})^2}$$
$$\cosh(2t) = \sqrt{1 + 4x^2(x^2+1)} = \sqrt{1 + 4x^4 + 4x^2}$$
$$\cosh(2t) = \sqrt{(2x^2+1)^2} = 2x^2 + 1, \quad x^2 >= 0$$

$$\sinh(4t) = 2(2x\sqrt{1+x^2})(2x^2+1)$$
$$\sinh(4t) = 4x(2x^2+1)\sqrt{1+x^2}$$

$$\frac{1}{32}\sinh(4t) - \frac{1}{8}t = \frac{1}{32}4x(2x^2+1)\sqrt{1+x^2} - \frac{1}{8}t + C$$
$$\int x^2\sqrt{x^2+1}dx = \frac{1}{8}x(2x^2+1)\sqrt{x^2+1} - \frac{1}{8}ln|x + \sqrt{x^2+1}| + C$$

$$\lambda = \frac{1}{8}x(2x^2+1)\sqrt{x^2-1} - \frac{1}{8}\ln|x + \sqrt{x^2+1}| + C$$
$$y = (\frac{1}{8}x(2x^2+1)\sqrt{x^2-1} - \frac{1}{8}\ln|x + \sqrt{x^2+1}| + C)(x^2+1)^{-\frac{3}{2}}$$

$$\boxed{y = (\frac{1}{8}x(2x^2+1)\sqrt{x^2-1} - \frac{1}{8}\ln|x + \sqrt{x^2+1}| + C)(x^2+1)^{-\frac{3}{2}}}$$

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4. (E): $y' + y = 2e^x + 4\sin(x)$
   Etape 1: ESSM

$$y' + y = 0$$
$$y' = -y$$
$$\frac{dy}{y} = -1$$
$$\int \frac{1}{y}dy = -\int 1dx$$
$$ln|y| = -x + C, \quad C \in \mathbb{R}$$
$$y = -e^{-x}.e^C$$
$$y = \lambda e^{-x} \quad (1)$$

Etape 2: EASM MVC

$$y' = -\lambda e^{-x} + \lambda' e^{-x}$$
$$y' = e^{-x}(\lambda + \lambda')(2)$$

(1) et (2) dans (E):
$$(-\lambda + \lambda)e^{-x} + \lambda e^{-x} = 2e^x + 4\sin(x)$$
$$\lambda' = 2e^x + 4sin(x)$$
$$\frac{d\lambda}{dx}e^{-x} = 2e^x + 4sin(x)$$
$$\int d\lambda = \int (2e^{2x} + 4e^x \sin(x))\,dx$$
$$\int d\lambda = \int 2e^{2x}\,dx + 4\int e^x \sin(x))\,dx$$
$$\int d\lambda = e^{2x} + C + 4[-\cos(x)e^x + \int e^x \cos(x)\,dx]$$

Soit $I = \int e^x \cos(x)$

$$\begin{cases} u = e^x \implies u' = e^x \\ v = -\cos(x) \implies v' = \sin(x) \end{cases}$$

$-\cos(x)e^x + \int e * x \cos(x)\,dx$

$$\begin{cases} u = e^x \implies v' = \cos(x) \\ u' = e^x \implies v' = \sin(x) \end{cases}$$

$-e^x cos(x) + -e^x sin(x) - \int e^x \sin(x))\,dx$
$$-e^x cos(x) + -e^x sin(x) - I$$
$$I = e^x(sin(x) - cos(x)) - I$$
$$2I = e^x(sin(x) - cos(x))$$
$$I = \frac{e^x}{2}(sin(x) - cos(x))$$

$$\int d\lambda = e^{2x} + c + 2e^x(sin(x) - cos(x)) + k$$
$$\lambda x = e^{2x} + e^{2x}(\sin(x) - \cos(x)) + C$$
$$\lambda x = e^x(e^x + 2(\sin(x) - \cos(x)) + C$$

$$y = e^x . e^{-x}(e^x + 2(\sin(x) - \cos(x)) + Ce^{-x}$$
$$y = e^x + 2\sin(x) - 2\cos(x) + Ce^{-x}$$

$$\boxed{y = e^x + 2\sin(x) - 2\cos(x) + Ce^{-x}}$$

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5. (E): $y' - 2y = 2x^3 + x$ ou $y(3) = 1$
   <u>Etape 1: ESSM</u>

$$y = \lambda e^{\int(-\frac{-2}{1})}$$

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6. (E): $y' + 2xy = e^{x-x^2}$

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7. (E): $y' \cos(2y) - \sin(y) = 0$

Exercice 2:

1. (E): $xy' = y + 3xy^2$

$$\frac{x}{x}y' = \frac{1}{x}y + \frac{3xy^2}{x}$$
$$y' = \frac{1}{x}y + 3y^2$$

On pose $z = y^{-1} = \frac{1}{y} \implies z' = -\frac{y'}{y^2}$

$$-\frac{y'}{y^2} = -\frac{1}{xy} - 3$$
$$\text{d'ou } z' = -\frac{1}{x}z - 3$$
$$z' + \frac{1}{x}z = -3 \quad (E)$$

<u>ESSM</u>
$z' + \frac{1}{x}z = 0$

$$z_H = \lambda e^{-\int \frac{1}{x}}, \quad \lambda \in \mathbb{R}$$
$$= \lambda e^{-ln|x|}$$
$$z_H = \lambda \frac{1}{x}, \quad \lambda \in \mathbb{R}$$

<u>EASM</u>

MVC:

$$t = g(x)\frac{1}{x} \quad (1)$$
$$t' = \frac{g'(x)x - g(x)}{x^2} \quad (2)$$

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(1) et (2) dans (E)

$$\frac{g'(x)x - g(x)}{x^2} + \frac{1}{x}g(x)\frac{1}{x} = -3$$

$$\frac{g'(x)x}{x^2} - \frac{g(x)}{x^2} + \frac{g(x)}{x^2} = -3$$

$$g'(x) = -3x$$

$$g(x) = -\frac{3}{2}x^2 + C, \quad C \in \mathbb{R}$$

$$z_p = (-\frac{3}{2}x^2 + C)\frac{1}{x} + \lambda(\frac{1}{x})$$

$$z_p = (-\frac{3}{2}x + \psi(\frac{1}{x} + \frac{1}{x})), \quad \psi = \frac{C}{x}$$

$$z_p = -\frac{3}{2}x + \psi\frac{2}{x}$$

$$z_p = -\frac{3x^2 + 4\psi}{2x}$$

$z = z_H + z_P$ et $y = y_H + y_P$

or $z = \frac{1}{y}$

Alors $y = \frac{1}{z_H} + \frac{1}{z_P}$ $\boxed{y = \frac{x}{\lambda} + \frac{2x}{3x^2 + 4\psi}, \quad \lambda, \psi \in \mathbb{R}}$

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2. (E): $y' + \frac{y}{x+1} = \frac{1}{2}(x+1)^3 y^3$

   C'est une equation de Bernoulli

$$\frac{y'}{y^3} + \frac{y}{(x+1)y^3} = \frac{1}{2}(x+3)^3$$

$$\frac{y'}{y^3} + \frac{(x+1)^{-1}}{y^2} = \frac{1}{2}(x+3)^3 \quad (E)$$

Changement de variable

$$u = \frac{1}{y^2} = y^{-2}$$

$$\frac{du}{dx} = -2y'y^{-3}$$

$$-\frac{1}{2}u' = \frac{y'}{y^3} \quad (A)$$

D'ou
$(E1): \quad \frac{1}{-2}u' + (x+1)^{-1}u = \frac{1}{2}(x+1)^3$

Etape 1: ESSM

$$-\frac{1}{2}u' + (x+1)^{-1}u = 0 \tag{1}$$

$$\frac{1}{2}u' = (x+1)^{-1}u \tag{2}$$

$$\frac{du}{dx} = 2(x+1)^{-1}u \tag{3}$$

$$\frac{du}{u} = 2(x+1)^{-1}dx \tag{4}$$

$$\ln|u| = 2\ln|x+1|+C, \quad C \in \mathbb{R} \tag{5}$$

$$\ln|u| = \ln|(x+1)^2|+C \tag{6}$$

$$u = e^{\ln|(x+1)^2|} \times e^C \tag{7}$$

$$u = (x+1)^2 \times \lambda, \quad \lambda = e^C \quad (E1_1) \tag{8}$$

Etape 2: EASM

$$u' = \lambda'(x+1)^2 + 2(x+1)\lambda \quad (E1_2)$$

$(E1_1)$ et $(E1_2)$ dans (E1)

$$-\frac{1}{2}[\lambda'(x+1)^2 + 2(x+1)\lambda] + (x+1)^{-1}(x+1)^2\lambda = \frac{1}{2}(x+1)^3$$

$$-\frac{\lambda'}{2}(x+1)^2 - (x+1)\lambda + (x+1)\lambda = \frac{1}{2}(x+1)^3$$

$$-\frac{\lambda'}{2}(x+1)^2 = \frac{1}{2}(x+1)^3$$

$$\lambda' = -(x+1)$$

$$\frac{d\lambda}{dx} = -(x+1)$$

$$d\lambda = -(x+1)dx$$

$$\int d\lambda = -\int (x+1)\,dx$$

$$\lambda = -(\frac{x^2}{2} + x) + K, \quad K \in \mathbb{R}$$

La solution generale de (E1) est:

$$\underline{u = \left(-\left(\frac{x^2}{2} + x\right) + K\right)(x+1)^2}$$

$$u = (-\frac{x^2}{2} - x + K)(x+1)^2$$

$$u' = (-\frac{2}{2}x - 1)(x+1)^2 + 2(x+1)(-\frac{x^2}{2} - x + K)$$

$$u' = (-x - 1)(x+1)^2 + 2(x+1)(-\frac{x^2}{2} - x + K)$$

$$u' = -(x+1)^3 + 2(x+1)(-\frac{x^2}{2} - x + K)$$

$$u' = (x+1)[-(x+1)^2 + 2(-\frac{x^2}{2} - x + 2K)]$$

$$u' = (x+1)[-(x+1)^2 - x^2 - 2x + 2K]$$

$$u' = (x+1)[-x^2 - 2x - 1 - x^2 - 2x + 2K]$$

$$u' = (x+1)[-2x^2 - 4x - 1 + 2K] \quad (B)$$

En reprenant (A)

$$\frac{y'}{y^3} = -\frac{1}{2}u'$$

$$\frac{dy}{dx} = -\frac{1}{2}u'y^3$$

$$\frac{dy}{y^3} = -\frac{1}{2}u'dx$$

$$\int \frac{dy}{y^3} = -\frac{1}{2}\int u'dx$$

$$\int y^{-3}dy = -\frac{1}{2}\int u'dx$$

$$\frac{y^{-2}}{-2} = -\frac{1}{2}\int u'dx$$

$$y^{-2} = \int u'dx \quad (A')$$

(B) dans (A')

$$y^{-2} = \int (x+1)[-2x^2 - 4x - 1 + 2K]\,dx$$

$$= \int (x+1)(-2x^2) + (x+1)(-4x) + (x+1)(-1) + (x+1)(2K)\,dx$$

$$= \int \left(-2x^3 - 2x^2 - 4x^2 - 4x - x - 1 + 2Kx + 2K\right)dx$$

$$= \int \left(-2x^3 - 6x^2 - 5x - 1 + 2Kx + 2K\right)dx$$

$$= -\frac{2}{4}x^4 - \frac{6}{3}x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C$$

$$y^{-2} = -\frac{1}{2}x^4 - 2x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C$$

$$\implies y^2 = (-\frac{1}{2}x^4 - 2x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C)^{-1}$$

$$\implies y = \pm(-\frac{1}{2}x^4 - 2x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C)^{-\frac{1}{2}}$$

$$\boxed{y = \pm(-\frac{1}{2}x^4 - 2x^3 - \frac{5}{2}x^2 - x + Kx^2 + 2Kx + C)^{-\frac{1}{2}}}$$

3. (E): $y'(1 - \sin(s)\cos(x)) + y^2 \cos(x) - y' + \sin(x) = 0$
   si $y = \cos(x)$ est une solution particuliere

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4. (E): $y = xy' + (y')^3$