Consistent Constitutions

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Abstract

This article presents a general possibility theorem for *constitutions*, social aggregators which always generate a non-empty "top set" of alternatives that are ranked at least as highly as any other. We relax the requirement of collective rationality by defining two increasingly demanding consistency properties of aggregators, which aim only at avoiding cycles involving elements of the top set. We establish that if a social aggregator satisfies unanimity, preference reversal, and weak consistency (implied by negative transitivity), then the family of blocking coalitions is a filter. If the social aggregator satisfies strong consistency (implied by transitivity), then the family of blocking coalitions is an ultrafilter. We thus generalize many classic results, including the oligarchy theorem of Mas Colell and Sonnenschein, and the dictatorship theorems of Arrow, Gibbard and Satterthwaite, and Eliaz.

1 Introduction

Given a set of preferences, a natural question, the problem of social choice, is whether these preferences can be aggregated to arrive at a collective decision which satisfies some notion of group rationality. Since the seminal theorem of Arrow (1950), many scholars have shown that, for a broad class of preference aggregation mechanisms, the answer is typically *no*: the only rational aggregator is a dictatorship, or at best an oligarchy (Mas-Colell and Sonnenschein (1972)). The problem of social choice has a deep game-theoretic interpretation, exemplified by the dictatorship theorem of Gibbard and Satterthwaite (Gibbard 1973; Satterthwaite 1975); this connection is formalized by Eliaz (2004), who establishes a single "meta-theorem" for transitive social aggregators. This article present a general possibility theorem that generalizes all the aforementioned results, and several others.[1]

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1. In a different framework, Barberà (2003) establishes a deep theorem for finite populations which implies the theorems of Arrow (1950) and Gibbard (1973), as well as the liberal

Crucially, the result in this article relaxes the collective rationality assumption to a less stringent notion of consistency. While formal definition are left to the next section, consistency can be loosely described as a restriction of transitivity (or negative transitivity) aimed at avoiding cycles containing only certain alternatives, which are *best* in the sense that they are ranked at least as highly as any other alternative by the aggregator. These alternatives are particularly salient from a collective choice perspective, since they are not "dominated" by any other, and thus consistency can be understood as a minimal collective rationality requirement.

Mathematically, this article uses the concepts of filters and ultrafilters on sets, which have a long history in social choice theory (e.g. Kirman and Sondermann 1972). These concepts, borrowed from general topology, allow to shed light on the basic logical and set-theoretic properties of the problem of social choice. Furthermore, they allow to state general possibility theorems to arbitrarily large, and possibly infinite, populations of voters. One may be skeptical of the relevance of possibility results for infinite populations, since any real-world population of voters is finite. There are, however, a few reasons to study infinite sets of voters. For example, this could be deemed the appropriate modeling assumption for social choice problems involving future populations of unknown size. Furthermore, it is a common simplifying analytical assumption in the formal political economy literature to model electorates as continua of voters; a social-choice-theoretic result on infinite populations is thus of theoretical relevance.

The next section introduces the mathematical preliminaries and definitions. The main result is presented and proved in Section 3. Section 4 is dedicated to a detailed analysis of the connections between our main theorem and related results in the literature. The final section concludes.

2 Definitions and framework

2.1 The aggregation of preferences

Let A be a finite set of three or more alternatives and N a set of three or more individuals or voters. Δ will denote the set of all pairs of distinct alternatives, i.e. $\Delta \equiv A^2 \setminus \{(a,a) \mid a \in A\}$. Let $\mathscr{B}$ denote the set of all binary relations on A and $\mathscr{P}$ the set of all linear orders (antisymmetric, complete, transitive binary relations) on A; an element of $\mathscr{P}$ will be called a preference ordering.

A preference profile $\succ: N \to \mathscr{P}$ assigns to each individual $i \in N$ one preference ordering $\succ_i$. The set of all preference profiles will be denoted Π.

We will be concerned with functions of the form

$$\phi : \Pi \to \mathscr{B}$$

paradoxes (Sen 1970; Kelsey 1985). Barberà does not assume unanimity, and his "locality" restriction is independent from preference reversal; his technique does not involve filters, but rather relies on the local property of pivotality.

which we will call **social aggregators** (Eliaz 2004).[2] Social aggregators map each preference profile to a *social relation* on the set of alternatives.

Fix social aggregator ϕ and profile $\succ$. We will use the following notation throughout. For any $(a,b) \in A^2$, we will use $aR(\succ)b$ to mean $(a,b) \in \phi(\succ)$, that is, the social aggregator ϕ relates a to b when the profile is $\succ$. Conversely, we will use $aX(\succ)b$ to mean that $(a,b) \notin \phi(\succ)$, and $aP(\succ)b$ to mean that $aR(\succ)b$ and $bX(\succ)a$.

2.2 Collective rationality

The class of social aggregators is very large, spanning a rich set of potential aggregation rules. In what follows, we will focus on aggregators satisfying some notion of "collective rationality", defined as a set of restrictions on the social relations they may produce.

Central to our analysis will be the concept of the *top set*, or set of best alternatives: the set of alternatives which are related to all others by the aggregator. In particular we will focus on aggregators which always generate a non-empty top set; we we will call these aggregators *constitutions*. As we will discuss in greater detail in Section 4, the class of constitutions includes many classic aggregation mechanisms studied in the literature, including social decision functions and social choice functions.

Definition 1. *Let ϕ be a social aggregator. The* ***top set*** *of $\phi(\succ)$, denoted $T_\phi(\succ)$, is the set of alternatives which the social aggregator relates to all others when the profile is $\succ$, that is:*

$$T_\phi(\succ) \equiv \{a \in A \mid \{a\} \times A \setminus \{a\} \subseteq \phi(\succ)\}.$$

If $T_\phi(\succ)$ is non-empty for all $\succ \in \Pi$, then ϕ is a ***constitution****.*

A second key range restriction will be to ensure the absence of cycles involving elements of the top set; we call this property *consistency*, which we formalize in two levels. Strong consistency requires that no alternative b outside the top set relate to an element a of the top set. Weak consistency relaxes this by allowing b to relate to a, as long as b relates to all alternatives c which do not relate to a.

Definition 2. *A constitution ϕ satisfies:*

1. ***Weak consistency*** *if for all profiles $\succ$ and for all $a \in T_\phi(\succ), b, c \in A$: $bR(\succ)a \implies [b \in T_\phi(\succ) \vee bX(\succ)c \implies cR(\succ)a]$;*
2. ***Strong consistency*** *if for all $a \in T_\phi(\succ), b \in A : bR(\succ)a \implies b \in T_\phi(\succ)$.*

2. The assumption that preferences are strict imposes additional structure on the aggregation problem; therefore, since our general theorem is essentially an impossibility result, focusing on aggregators with a smaller domain allows us to prove a stronger theorem.

Consistency generalizes the more familiar properties of transitivity and negative transitivity by limiting their reach outside the top set.[3] Ruling out all cycles involving alternatives outside the top set may seem like an excessively demanding property for a social relation, since certain considerations regarding alternatives outside the top set could perhaps be safely ignored. In our main result, we will show that even this weaker collective rationality requirement will produce what is essentially an impossibility result.

Proposition 1. *If ϕ is negatively transitive, it is weakly consistent; if it is transitive, it is strongly consistent.*

Proof. Let $a \in T_\phi(\succ), b \in A, bR(\succ)a$.

Suppose that ϕ is negatively transitive and for some profile $\succ, b \notin T_\phi(\succ)$: then for some $c \in A, bX(\succ)c$. If $cX(\succ)a$, then negative transitivity yields $bX(\succ)a$, a contradiction.

Suppose that ϕ is transitive: then for all a, b, c, if $aR(\succ)b$ and $bR(\succ)a$, then $aR(\succ)c$ if and only if $bR(\succ)c$. Since $bR(\succ)a$ and a is in the top set, it immediately follows that b must be in the top set, too. □

2.3 Filters and blocking coalitions

Since we assume all preferences to be strict, a constitution can be further characterized by the power it gives certain groups of voters, or coalitions, to have their preferences determine parts of the social relation. In particular, we will focus on the notion of *blocking coalitions*, sets of voters with the property that whenever everyone in the set prefers a to b, the social aggregator does not relate b to a. (Since we assume preferences to be antisymmetric, a and b must be distinct.) We will say that these coalitions "block b against a".

Definition 3. *Fix some social aggregator. A coalition $C \subseteq N$* ***blocks*** *b against a, denoted $C \in \mathcal{B}_{ab}$, if for every profile $\succ \in \Pi$, $[a \succ_i b \,\forall i \in C] \implies bX(\succ)a$. The family of all* ***blocking coalitions*** *$\mathcal{B}$ is the collection of all coalitions which block some alternative against another:*

$$\begin{aligned} \mathcal{B} &\equiv \bigcup_{(a,b)\in\Delta} \mathcal{B}_{ab} \\ &= \bigcup_{(a,b)\in\Delta} \{C \subseteq N \mid \forall \succ \in \Pi : [\forall i \in C : a \succ_i b] \implies bX(\succ)a\}. \end{aligned}$$

In the previous section, we defined some collective rationality properties of aggregators; the further restrictions that we will place will involve the composition of the set of blocking coalition. The first is *unanimity* (also known in the literature as Pareto efficiency). Unanimity requires the social aggregator to be minimally responsive to individuals' preferences.

3. Recall that a relation R is negatively transitive if and only if its complement is transitive, i.e. if for all $a, b, c \in A : (a, b), (b, c) \notin R \implies (a, c) \notin R$.

Definition 4. *A social aggregator satisfies* ***unanimity*** *for all* $(a, b) \in \Delta, \mathcal{B}_{ab}$ *is non-empty, or equivalently,* $N \in \mathcal{B}_{ab}$*.*

We have established $\mathcal{B}$ must be non-empty: we now formalize a procedure to determine its members beyond N. The strategy is to define a property which is *inter-profile*: given two profiles $\succ, \succ'$, as long as they have some property in common, learning some subset of $\phi(\succ)$ can imply a subset of $\phi(\succ')$. In particular, we wish to define some property that will allow us to infer from $\phi(\succ)$ whether some coalition is blocking or not. We define two increasingly demanding such properties: the first is due to Eliaz (2004); the second is classic.

Definition 5. *A social aggregator* ϕ *satisfies* ***preference reversal*** *if for all* (a, b)*, if there is some profile* $\succ$ *such that* $[a \succ_i b \iff i \in C]$ *and* $aP(\succ)b$*, then* $C \in \mathcal{B}_{ab}$*.* ϕ *satisfies* ***positive responsiveness*** *if for all* (a, b)*, if there is some profile* $\succ$ *such that* $[a \succ_i b \iff i \in C]$ *and* $bX(\succ)a$*, then* $C \in \mathcal{B}_{ab}$*.*

Finally, to make our discussion of constitutions' coalitional structure mathematically precise, we need to introduce the notion of filters on sets.

Definition 6. *A* ***filter*** $\mathcal{F}$ *on a set* X *is a non-empty family of subsets of* X *satisfying:*

1. ***Non-triviality****:* $\varnothing \notin \mathcal{F}$*;*
2. ***Upward closure****:* $S \in \mathcal{F}, S \subseteq S' \subseteq X \implies S' \in \mathcal{F}$*.*
3. *Closure under* ***finite intersection****:* $S_1, ..., S_k \in \mathcal{F} \implies \bigcap_{j=1}^{k} S_j \in \mathcal{F}$*.*

An ***ultrafilter*** *is a filter satisfying* ***maximality****:* $S \notin \mathcal{F} \implies X \setminus S \in \mathcal{F}$*, or equivalently, there is no filter* $\mathcal{F}'$ *on* X *such that* $\mathcal{F} \neq \mathcal{F}'$ *and* $\mathcal{F} \subseteq \mathcal{F}'$*.*

The collection $\mathcal{F}$ of subsets of X containing the elements of some $S \subseteq X$ is a special kind of filter, which we call a *principal filter*; S is the *base* of $\mathcal{F}$. When S is a singleton, $\mathcal{F}$ is an ultrafilter. It is a classic result that all filters on a finite set are principal, and all ultrafilters on a finite set have a singleton base.

3 A general theorem

We can now state and prove our general theorem on consistent constitutions.

Theorem 1. *Let ϕ be a constitution satisfying unanimity and preference reversal. (i) If ϕ is weakly consistent, then $\mathcal{B}$ is a filter; (ii) If ϕ is strongly consistent, then $\mathcal{B}$ is an ultrafilter.*

$\mathcal{B}$ is non-empty due to unanimity, and closed upward by definition. We proceed by way of simple lemmas.

Lemma 1 (Non-triviality)**.** *If ϕ is a constitution satisfying unanimity, then $\mathcal{B}$ is non-trivial.*

Proof. Consider any profile $\succ$ where some alternative a is preferred to all other alternatives by all individuals, i.e. $\forall i \in N, b \in A \setminus \{b\} : a \succ_i b$. Then $bX(\succ)a$ and $\{a\} = T_\phi(\succ)$; it follows that $aP(\succ)b$. Observe that the set of voters who strictly prefer b to a is empty, thus $\varnothing \notin \mathcal{B}_{ba}$ and $\varnothing \notin \mathcal{B}$ in general. $\square$

We can now tackle the core of the proof, by adding preference reversal and the consistency requirements.

Our first main goal is to show that $\mathcal{B}$ is closed under finite intersection. A key step to prove this is to show that if some coalition blocks one alternative over another, then it blocks *all* alternatives over any other; this step is commonly known as a "contraction" lemma.

Lemma 2 (Contraction)**.** *If ϕ is a weakly consistent constitution satisfying unanimity and preference reversal, then $\mathcal{B} = \bigcap_{(a,b)\in\Delta} \mathcal{B}_{ab}$.*

Proof. It is immediate that $\bigcap_{(a,b)\in\Delta} \mathcal{B}_{ab} \subseteq \mathcal{B}$; we now prove the converse statement in two steps.

Step 1. We show that if C blocks b over a, then it blocks any alternative x over a: that is, for any $(a,b) \in \Delta, \mathcal{B}_{ab} \subseteq \bigcap_{x\in A\setminus\{a\}} \mathcal{B}_{ax}$. Suppose $C \in \mathcal{B}_{ab}$ and let $\succ$ be such that:

$$a \succ_i b \iff i \in C \quad \text{and} \quad \forall c \in A \setminus \{a,b\}, \forall j \in N : b \succ_j c.$$

We have $bX(\succ)a$ and by unanimity, $cX(\succ)b$. Therefore $\{a\} = T_\phi(\succ)$: weak consistency yields $cX(\succ)a$, so $aP(\succ)c$. By preference reversal, $C \in \mathcal{B}_{ac}$ and thus $C \in \bigcap_{x\in A\setminus\{a\}} \mathcal{B}_{ax}$.

Step 2. Now we show that if C blocks b over a, then it blocks any alternative y over b; that is, $\mathcal{B}_{ab} \subseteq \bigcap_{y\in A\setminus\{b\}} \mathcal{B}_{by}$. To see this, take some $C' \in \mathcal{B}_{ba}$ and consider a profile $\succ'$ such that for all every alternatives $x \in A \setminus \{a,b,c\}$ (when these exist), $b \succ'_k x$ for all voters k, and furthermore:

$$\forall i \in C : a \succ'_i c \quad \text{and} \quad \forall j \in C' : b \succ'_j a.$$

Because preferences are transitive, $b \succ'_i c$ for all $i \in C \cap C'$. Suppose that for all $j \notin C \cap C' : c \succ_j b$. Because $C' \in \mathcal{B}_{ba}$ and $C \in \mathcal{B}_{ac}$ (by Step 1), we have

$aX(\succ')b$ and $cX(\succ')a$; by unanimity, $xX(\succ')b$. It follows that $\{b\} = T_\phi(\succ')$. Again, weak consistency yields $cX(\succ')b$, and thus $bP(\succ')c$. Preference reversal implies $C \cap C' \in \mathcal{B}_{bc}$, and therefore $C \in \mathcal{B}_{bc}$ by upward closure. By Step 1, then $C \in \bigcap_{y \in A \setminus \{b\}} \mathcal{B}_{by}$. $\square$

Lemma 3 (Intersection)**.** *If ϕ is a weakly consistent constitution satisfying unanimity and preference reversal, then $\mathcal{B}$ is closed under finite intersection.*

Proof. Take $C_1, C_2 \in \mathcal{B}$ and let $\succ$ now be such that:

$$\forall i \in C_1, x \in A \setminus \{a,b,c\} : a \succ_i b \succ_i x \quad \text{and} \quad \forall j \in C_2 : b \succ_j c.$$

As above, the transitivity of individual preferences implies $a \succ_i c$ for all $i \in C_1 \cap C_2$: suppose then that $c \succ_k a$ for all $k \notin C_1 \cap C_2$. Because C_1 blocks b over a and x over a (by Lemma 2), $bX(\succ)a$ and $xX(\succ)a$. Applying the same logic to C_2 gives $cX(\succ)b$. Then $\{a\} = T_\phi(\succ)$, and thus $aP(\succ)b$: weak consistency yields $cX(\succ)a$, and therefore $aP(\succ)c$. This and preference reversal imply that $C_1 \cap C_2 \in \mathcal{B}_{ac}$, as desired. $\square$

We have now proved part (i) of the theorem, as well as much of part (ii); all is left to do is focus on strongly consistent constitutions.

Lemma 4 (Maximality)**.** *If ϕ is a strongly consistent constitution satisfying unanimity and preference reversal, then $\mathcal{B}$ is maximal.*

Proof. Suppose towards a contradiction that $C \notin \mathcal{B}$ and $N \setminus C \notin \mathcal{B}$. Consider profile $\succ$ such that for all voters, the three most-preferred alternatives are a, b, c (which implies $T_\phi(\succ) \subseteq \{a,b,c\}$) and:

$$a \succ_i b \iff i \in C \quad \text{and} \quad a \succ_i c \iff i \in C \quad \text{and} \quad \forall j \in N : b \succ_j c.$$

By unanimity, $cX(\succ)b$ and thus $T_\phi(\succ) \subseteq \{a,b\}$. By preference reversal, it cannot be that $aP(\succ)b$, $aP(\succ)c$, or $bP(\succ)a$: else $C \in \mathcal{B}_{ab}$, $C \in \mathcal{B}_{ac}$, or $N \setminus C \in \mathcal{B}_{ba}$ respectively. It follows that $aR(\succ)b$ and $bR(\succ)a$, so strong consistency implies $T_\phi(\succ) = \{a,b\}$ and $cX(\succ)a$. Thus $aP(\succ)c$, a contradiction. $\square$

This concludes the proof of the theorem. $\blacksquare$

4 Related results

4.1 Oligarchies and dictatorships

This section connects Theorem 1 with well-known possibility results, and gives a straightforward social-choice interpretation to the concept of filters on a population of voters.

Definition 7. *A social aggregator ϕ is an **oligarchy** of C if $C \in \mathcal{B}_{ab}$ for all $(a,b) \in \Delta$. ϕ is a **dictatorship** of i if it is an oligarchy of $\{i\}$.*

When ϕ is an oligarchy of C, $\mathcal{B}$ is a principal filter with base C; when ϕ is dictatorial, the base of $\mathcal{B}$ is a single voter, the dictator. When ϕ is a constitution and i is the dictator, for any profile $\succ$ the top set $T_\phi(\succ)$ will be the alternative that maximizes $\succ_i$.

This gives us the first immediate corollary of Theorem 1.

Corollary 1 (Eliaz). *If the set of voters is finite, then every transitive constitution satisfying unanimity and preference reversal is a dictatorship.*[4]

To see the connection with our general result, it is enough to recall that (by Proposition 1), transitive constitutions satisfy strong consistency. The fact that all ultrafilters on finite sets have a singleton base yields dictatorship.

4.2 Independence and Arrow's theorem

As shown by Eliaz (2004), the class of constitutions spans a large set of commonly studied aggregation mechanisms. First we consider the class of **social decision functions**, i.e. social aggregators which always produce complete and acyclic relations. These are constitutions because they generate a choice function.[5] In fact, the term "constitution" has been previously used in the social choice literature to refer to social welfare functions, i.e. transitive social decision functions (e.g. Arrow 1977); it is used in a strictly more general sense in the present paper.

Our main goal now is to show that preference reversal is implied by the axioms postulated by Arrow (1950). In particular, Arrow introduced the property of *independence of irrelevant alternatives*, or independence for short.

Definition 8. *A social aggregator ϕ satisfies **independence of irrelevant alternatives** if for all alternatives (a,b) and for all profiles $\succ, \succ'$:*

$$(a,b) \in \phi(\succ) \wedge (a,b) \notin \phi(\succ') \implies \{i \in N \mid a \succ_i b\} \neq \{j \in N \mid a \succ'_j b\}.$$

4. It should be noted that Ninjbat (2015) proposes to further weaken preference reversal with a property called *preference change*: if $aP(\succ)b$ and $bR(\succ')a$, then $\{i \mid a \succ_i b\} \neq \{i \mid a \succ'_i b\}$. In particular, Ninjbat (in his Theorem 8) argues that Eliaz's method of proof can be used to establish a stronger result where preference reversal is replaced with preference change.

5. Given a relation R on a set A, a choice function C maps every non-empty subset S of A to a subset of itself such that for all $x \in C(S), y \in S : xRy$. Clearly, if $\phi(\succ)$ generates a choice function, $T_\phi(\succ)$ is non-empty. R is acyclic and complete if and only if it generates a choice function; see Sen (2018) for a proof.

Since we ruled out individual indifference, for some social aggregators independence implies preference reversal. Eliaz (2004) shows that this is the case for transitive and unanimous constitutions. Below we prove a stronger statement.

We say that an aggregator ϕ is **total** if $\phi(\succ)$ is total for all profiles $\succ$, that is, for all $(a,b) \in \Delta : (a,b) \notin \phi(\succ) \implies (b,a) \in \phi(\succ)$.

Proposition 2. *Let ϕ be a negatively transitive or transitive constitution satisfying unanimity. Then the following are equivalent:*

1. *ϕ satisfies positive responsiveness;*

2. *ϕ is total and satisfies independence of irrelevant alternatives.*

Proof. For necessity, suppose ϕ satisfies positive responsiveness. Take any distinct a, b and some profile $\succ$ where $a \succ_N x$ for all $x \in A \setminus \{a,b\}$. By unanimity, $T_\phi(\succ) \subseteq \{a,b\}$ and thus $aR(\succ)b$ or $bR(\succ)a$. Suppose without loss of generality that $aR(\succ)b$: then for any profile $\succ'$ where $C \equiv \{i \in N \mid a \succ_i b\} = \{i \in N \mid a \succ'_i b\}$, it must be that $aR(\succ')b$. Because C is arbitrary, ϕ is total and satisfies independence.

For sufficiency, suppose that ϕ is total and satisfies independence, and suppose towards a contradiction that it not positively responsive. Then there exist $a, b \in A, \succ, \succ' \in \Pi$, such that $bX(\succ)a$ (which implies $aR(\succ)b$ by totality), $bR(\succ')a$, and yet $C \equiv \{i \mid a \succ_j b\} \subseteq C' \equiv \{i \mid a \succ'_i b\}$.

By independence, there is some j such that $b \succ_j a$ and $a \succ'_j b$. By unanimity, there is some $k \in C$, and similarly, there is some $l \in N \setminus C'$. We will assume that $\succ$ is such that $\succ_l$ is the preference of all voters who are not j or k. Again by independence, we can suppose without loss of generality that there is some third alternative c such that $\succ'$ is as follows: $a \succ'_j b \succ'_j c$ and $a \succ'_k c \succ'_k b$ and $b \succ'_l a \succ'_l c$. Unanimity and totality imply $aP(\succ')c$. Recall that $bR(\succ')a$: then both transitivity and negative transitivity yield $bR(\succ')c$.

Consider finally a third profile $\succ''$ such that $b \succ''_j c \succ''_j a$ and $c \succ''_k a \succ''_k b$ and $b \succ''_l c \succ''_l a$. Unanimity and totality yield $cP(\succ'')a$. By independence, $bR(\succ')c \implies bR(\succ'')c$, and since $a \succ_i b \iff a \succ''_i b$ for all i, $aP(\succ'')b$. Negative transitivity and transitivity both yield $bR(\succ'')a$, a contradiction. $\square$

As we noted previously, positive responsiveness implies preference reversal. We can now state Arrow's theorem and appreciate that it is a specific case of Theorem 1.

Corollary 2 (**Arrow**). *If the set of voters is finite, then every social welfare function satisfying unanimity and independence of irrelevant alternatives is a dictatorship.*

The first to establish the existence of nondictatorial aggregation mechanisms in the infinite case was Fishburn (1970). It was then proved independently by Kirman and Sondermann (1972) and Hansson (1976) that for transitive social aggregators, the set of blocking coalitions is an ultrafilter – which yields dictatorship in the finite case.[6] Recall that since social welfare functions are transitive, they are strongly consistent, and therefore Theorem 1 strictly generalizes the results due to Kirman and Sondermann and Hansson.

4.3 Social choice functions and non-manipulability

Another aggregation mechanism belonging to the class of constitutions is a **social choice function**, i.e. a function of the form $F : \Pi \to A$. Social choice functions map preference profiles into a single social choice. A fundamental result on social choice functions is due to Gibbard (1973) and Satterthwaite (1975).

Definition 9. *A social choice function F is **non-manipulable** if for all $\succ, \succ_i'$: $F(\succ) \succ_i F(\succ_{-i}, \succ_i)$.*

Like in the context of constitutions, we may say that a social choice function F is dictatorial if there is some voter i such that F always selects the alternative that maximizes $\succ_i$. Indeed, while the output of F is not a social relation, F is equivalent to a constitution ϕ_F defined as follows:

$$\phi_F : \Pi \to \mathscr{B} \qquad \text{where} \qquad \phi_F(\succ) = \{x\} \times A \setminus \{x\} \iff F(\succ) = x.$$

Observe that ϕ_F is transitive (and negatively transitive), and thus strongly consistent.

It has been long known that there are strong connections between these two result. The original proof by Gibbard (1973) crucially relied on Arrow's theorem, building on a connection on which Vickrey (1960) had remarked. The parallels between the two were made even clearer by Reny (2001), who proves them side by side using the same procedure due to Geanakoplos (1996). The formal connection was finally drawn by Eliaz (2004), who shows that if F is onto and strategy-proof, then ϕ_F satisfies unanimity and preference reversal.[7] The Gibbard-Satterthwaite theorem follows immediately:

Corollary 3 (**Gibbard-Satterthwaite**). *If the set of voters is finite, then every onto and non-manipulable social choice function is dictatorial.*

4.4 Weakening collective rationality

Arrow's assumption of transitivity of the social relation has been weakened in many important papers. Here we focus on a fundamental result by Mas-Colell and Sonnenschein (1972),[8] who study social decision functions satisfying **quasitransitivity**. Recall that a relation R is quasitransitive if its asymmetric

6. For an overview of fundamental filter and ultrafilter results in social choice theory, see for instance Austen-Smith and Banks (1999) and Sen (2018).

7. The argument relies on the connection between non-manipulability and monotonicity (Muller and Satterthwaite 1977).

8. An equivalent result was independently proved, but never published, by Gibbard (1969).

subset is transitive. Quasitransitivity is weaker than transitivity and stronger than acylicity. Observe that if R is complete, then it is quasitransitive if and only if it is negatively transitive. Therefore quasitransitive social decision functions are weakly consistent constitutions.

Corollary 4 (Mas Colell–Sonnenschein). *If the set of voters is finite, then every quasitransitive social decision function satisfying unanimity and preference reversal is an oligarchy.*

A general result involving infinite populations is due to Blair and Pollak (1982), who show that in the case of quasitransitive social decision functions, the family of blocking coalitions is a filter. Theorem 1 implies both.

While these results do not speak to the size of the coalition of voters defining the oligarchy, centralization of power is not fully avoided. To see this, consider the equivalent characterization of oligarchy in finite populations, for social aggregators satisfying positive responsiveness. If ϕ is an oligarchy of C, then every $i \in C$ is a *weak dictator* in the sense that $a \succ_i b$ always implies $aR(\succ)b$. Indeed, were that not the case, then positive responsiveness would yield $N \setminus \{i\} \in \mathcal{B}_{ab}$; then by closure under finite intersection, $\bigcap_{i \in N} N \setminus \{i\} = \varnothing \in \mathcal{B}$, a contradiction.[9]

Two other key results which investigating the consequences of relaxing transitivity are due to Blair et al. (1976) and Blau (1979). Blair and colleagues prove a version of Corollary 4 for collective choice rules mapping preference profiles to path-independent choice functions (which implies quasitransitivity).[10] The requirement that $\phi(\succ)$ always generate a choice function is more restrictive than the mere non-emptiness of the top set, as it requires completeness. Theorem 1 is independent from (and in fact stronger than) their result.[11] Blau (1979) shows that if unanimity and independence are satisfied, dictatorship arises even when preferences are aggregated into a semiorder, a condition which sits at the middle between transitivity and quasitransitivity, and (which allows for intransitive equivalence, as may be desirable to overcome difficulties such as the sorites paradox). Blau's result independent from our main theorem.

5 Conclusion

This article builds on the "meta-theorem" of Eliaz to present a single statement which implies many well-known results, including those of Arrow, Gibbard-Satterthwaite, and Mas Colell and Sonnenschein. As such, it makes several contributions.

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9. Observe that $\{N\}$ is a filter on N. Therefore a quasitransitive social decision function where $\mathcal{B} = \{N\}$, i.e. the Pareto-extension rule (Sen 1969), exists – although all disagreement among voters results in social indifference.

10. Recall the definition of choice functions from footnote 5. A choice function C is path-independent if for all non-empty $S_1, S_2 \subseteq A : C(S_1 \cup S_2) = C(C(S_1) \cup C(S_2))$.

11. For a general result concerning collective choice rules whose range is the class of choice functions, see Man and Takayama (2013).

First of all, it significantly relaxes the range restriction on the social aggregator, as it is shown that transitivity and quasitransitivity of the social relation are not necessary for a dictatorship and oligarchy result. Indeed, a much weaker notion of consistency aimed at avoiding cycles with elements of the top set is sufficient. Furthermore, we demonstrate that the set-theoretic notion of filters can be fruitfully used not only to study social decision functions satisfying independence of irrelevant alternatives, as in most classic applications, but also to the more general environment of constitutions satisfying preference reversal. Finally, we extend the general meta-theorem to possibly infinite populations.

The use of the abstract notion of filters sheds light on the set-theoretic foundations of the problem of social choice; this can help researchers see connections between results sharing a common logical structure. Consider for instance the emerging literature on judgment aggregation, which is concerned with the aggregation of sets of logically connected propositions (of which relations over a set of alternatives are an example). Results strictly related to Arrow's and Gibbard-Satterthwaite in this more general framework have been established by Dietrich and List (2007a, 2007b): their proofs make use of the property of maximality of ultrafilters. While the notion of a top set has no straightforward interpretation in the context of judgment aggregation, future research should aim to uncover whether there exists a single collective rationality requirement from which these theorems, as well as Theorem 1 presented here and its corollaries, can be derived.

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