PWM Simulation and Characterization

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1 PWM Defined as a Function of Duty Cycle

We can define a PWM signal with amplitudes of -1 to 1 as a function of duty cycle like so:

$$x(t) = \begin{cases} 1 & 0 \leq t \leq \alpha T_0 \\ -1 & \alpha T_0 \leq t \leq T_0(1-\alpha) \end{cases} \tag{1}$$

Where:

T_0 = Fundamental Period
α = Duty Cycle

1.1 PWM Example

Figure 1 is an example of a PWM square wave signal at 179 kHz, which is my student ID number rounded up to 3 significant digits, at 25 percent duty cycle. For the following calculations, we will use this example at different duty cycles.

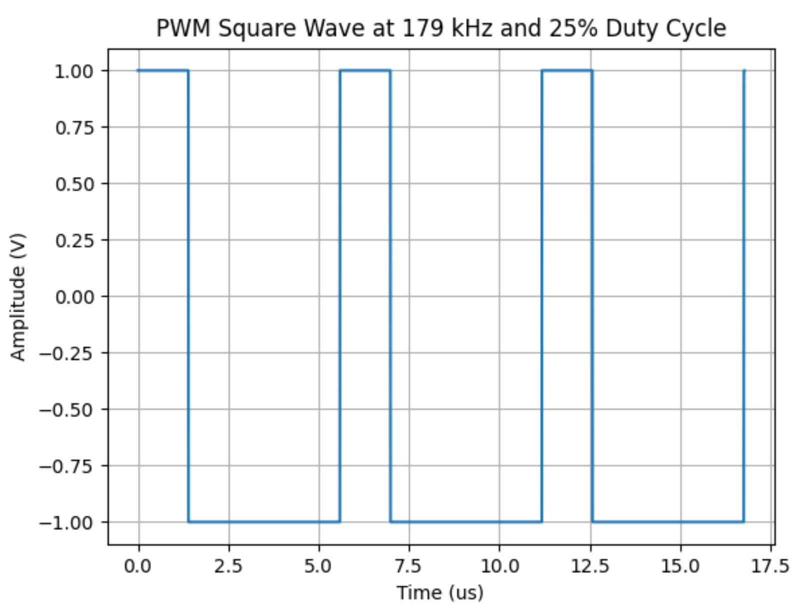

Figure 1: PWM Example

1.2 Derivation of Frequency Spectra

We can derive the math defintion using a Fourier Transform to get the signal in the frequency spectra. Since this is a PWM square wave, we will need to setup an improper integral defining the amplitudes at their specified time intervals.

$$\int_{a}^{b} x(t)e^{-j\omega t}\,\mathrm{d}t = \int_{a_1}^{b_1} 1e^{-j\omega t}\,\mathrm{d}t + \int_{a_2}^{b_2} -1e^{-j\omega t}\,\mathrm{d}t$$

$$= \int_{a_1}^{b_1} e^{-j\omega t}\,\mathrm{d}t - \int_{a_2}^{b_2} e^{-j\omega t}\,\mathrm{d}t$$

We can pull out $\frac{1}{\omega j}$ from the resulting Riemann sums.

$$= \frac{1}{\omega j}[[-e^{-\omega j b_1} + e^{-\omega j a_1}] + [e^{-\omega j b_2} + e^{-\omega j a_2}]]$$

Then we can combine both Riemann sums into one whole Riemann sum and use Euler's formula on the terms.

$$= \frac{1}{\omega j}(-\cos\omega b_1 - j\sin\omega b_1 + \cos\omega a_1 + j\sin\omega a_1 + \cos\omega b_2 + j\sin\omega b_2 + \cos\omega a_2 + j\sin\omega a_2)$$

Since $b_1 = a_2$ and $a_1 = 0$, we can cancel out some terms.

$$= \frac{1}{\omega j}(1 + \cos\omega b_2 + j\sin\omega b_2)$$

Using the following (adjusted) Euler's identity we can derive the complete Fourier transform.

$$\sin\omega = \frac{1}{2j}(\cos\omega + j\sin\omega - \cos\omega + j\sin\omega)$$

We can balance out the above equation with our equation we got in the previous integration.

$$\sin\omega b_2 = \frac{1}{2j}(\cos\omega b_2 + j\sin\omega b_2 - \cos\omega b_2 + j\sin\omega b_2)$$

First is to balance the $\frac{1}{2j}$ coefficient with the our coefficient

$$\frac{2\sin\omega b_2}{\omega} = \frac{1}{\omega j}(\cos\omega b_2 + j\sin\omega b_2 - \cos\omega b_2 + j\sin\omega b_2)$$

Next, we expand the term on the right and add $\frac{1}{\omega j}$

$$\frac{2\sin\omega b_2}{\omega} + \frac{1}{\omega j} = \frac{1}{\omega j} + \frac{\cos\omega b_2}{\omega j} + \frac{j\sin\omega b_2}{\omega j} - \frac{\cos\omega b_2}{\omega j} - \frac{j\sin\omega b_2}{\omega j}$$

We can then add $\frac{\cos\omega b_2}{\omega j} + \frac{j\sin\omega b_2}{\omega j}$ to both sides,

cancel out some j's, and combine like-terms to get the resulting Fourier transform:

$$X(t) = \frac{3\sin\omega b_2}{\omega} - j\frac{1 + \cos\omega b_2}{\omega}$$

1.3 Further Calculations and Explainations

From our equation $X(t) = \frac{3\sin\omega b_2}{\omega} - j\frac{1+\cos\omega b_2}{\omega}$, we can understand a few things. First, since $b_2 = T_0(1-\alpha)$, we can directly get the frequency value for each point by using the equation $\frac{1}{T}$, which is the equation for converting time to frequency. Since b_2 is a point in time from 0 to T_0, we can replace T with b_2 to get the resulting equation $\frac{1}{b_2}$ to get our x-values in the frequency domain.

For magnitude: $||\vec{X}(t)|| = \sqrt{(\frac{3\sin\omega b_2}{\omega})^2 + (\frac{1+\cos\omega b_2}{\omega})^2}$

For phase: $\theta = \tan^{-1}\frac{\frac{1+\cos\omega b_2}{\omega}}{\frac{3\sin\omega b_2}{\omega}}$

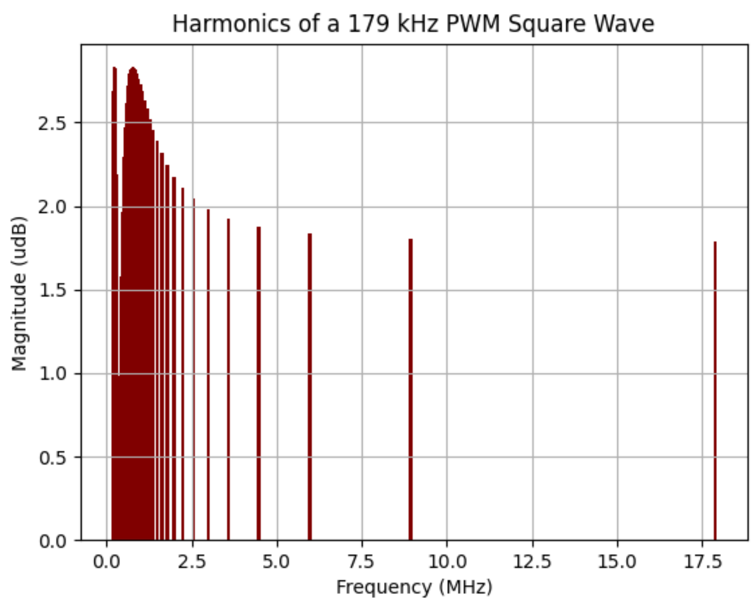

Figure 2: Harmonics of the PWM Square Wave

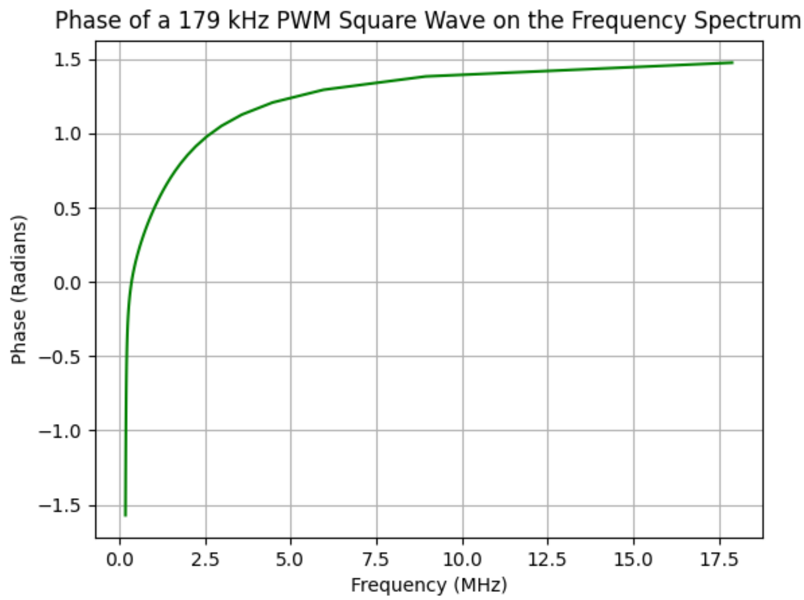

Figure 3: Phase of the PWM Square Wave

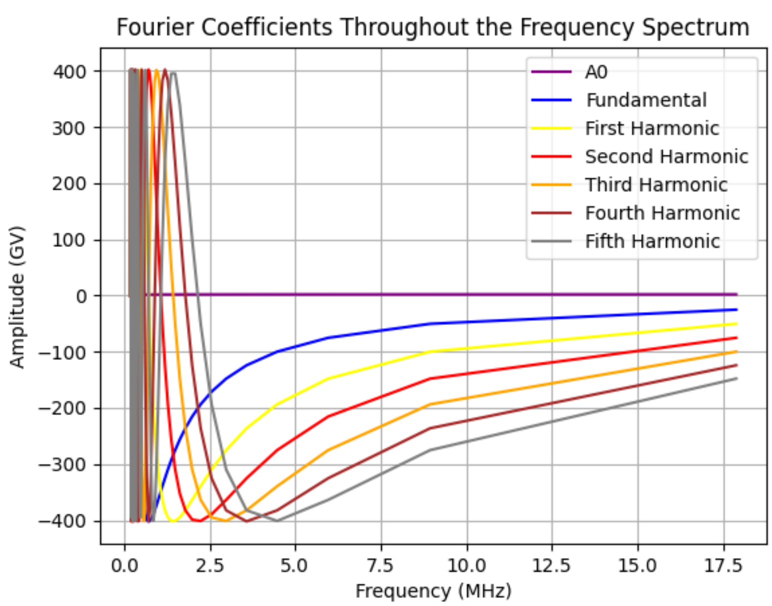

Figure 4: Coefficients Throughout the Frequency Spectrum