European Journal of Combinatorics 43 (2015) 210–219

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

The graph formulation of the union-closed sets conjecture

Henning Bruhn [a], Pierre Charbit [b], Oliver Schaudt [c], Jan Arne Telle [d,1]

[a] Universität Ulm, Germany
[b] LIAFA, Université Paris Diderot, France
[c] Universität zu Köln, Germany
[d] Department of Informatics, University of Bergen, Norway

ARTICLE INFO

Article history:
Received 27 June 2013
Accepted 23 August 2014
Available online 29 September 2014

ABSTRACT

The union-closed sets conjecture asserts that in a finite non-trivial union-closed family of sets there has to be an element that belongs to at least half the sets. We show that this is equivalent to the conjecture that in a finite non-trivial graph there are two adjacent vertices each belonging to at most half of the maximal stable sets. In this graph formulation other special cases become natural. The conjecture is trivially true for non-bipartite graphs and we show that it holds also for the classes of chordal bipartite graphs, subcubic bipartite graphs, bipartite series-parallel graphs and bipartitioned circular interval graphs. We derive that the union-closed sets conjecture holds for all union-closed families being the union-closure of sets of size at most three.

1. Introduction

A family $\mathcal{F}$ of sets is *union-closed* if $X, Y \in \mathcal{F}$ implies $X \cup Y \in \mathcal{F}$. The following conjecture is usually attributed to Peter Frankl who dates its formulation to 1979 [7].

E-mail address: schaudto@uni-koeln.de (O. Schaudt).

1 Part of this research done while visiting LIAFA in 2011.

http://dx.doi.org/10.1016/j.ejc.2014.08.030
0195-6698/

Union-closed Sets Conjecture. *Let $\mathcal{F}$ be a finite union-closed family of sets with $\mathcal{F} \neq \{\emptyset\}$. Then there exists $x \in \bigcup_{X\in\mathcal{F}} X$ that lies in at least half of the members of $\mathcal{F}$.*

In spite of a great number of papers, see e.g. the good bibliography of Marković [16] for papers up to 2007, this conjecture is still wide open. Several special cases are known to hold, for example when $|\bigcup_{X\in\mathcal{F}} X|$ is bounded from above, the currently best bound being 11 by Bošnjak and Marković [1]. Similarly, the conjecture holds when $|\mathcal{F}|$ is bounded, the currently best bound being 46. This follows from a lemma by Lo Faro [15], and independently by Roberts and Simpson [22]. The conjecture also holds when certain sets are present in $\mathcal{F}$, such as a set of size 2. Possibly as a reflection of its general difficulty, Gowers [9] suggested that work on this conjecture could fruitfully be done as a collaborative Polymath project. See [2] for a survey of the literature on the union-closed sets conjecture.

Various equivalent formulations have been discovered. We mention in particular Poonen [18] who translates the conjecture into the language of lattice theory. Several subsequent results together with their proofs belong to lattice theory, for example Reinhold [21] who proves this conjecture for lower semimodular lattices.

In this paper we give a formulation of the conjecture in the language of graph theory. A set of vertices in a graph is *stable* (or *independent*) if no two vertices of the set are adjacent. A stable set is *maximal* if it is maximal under inclusion, that is, every vertex outside has a neighbour in the stable set.

Conjecture 1. *Let G be a finite graph with at least one edge. Then there will be two adjacent vertices each belonging to at most half of the maximal stable sets.*

Note that Conjecture 1 is true for non-bipartite graphs. Indeed, if vertices u and v are adjacent there is no stable set containing them both and so one of them must belong to at most half of the maximal stable sets. An odd cycle will therefore imply the existence of two adjacent vertices each belonging to at most half of the maximal stable sets. The conjecture is for this reason open only for bipartite graphs. Moreover, in a connected bipartite graph, for any two vertices u and v in different bipartition classes we have a path from u to v containing an odd number of edges, so that if u and v each belongs to at most half the maximal stable sets there will be two adjacent vertices each belonging to at most half the maximal stable sets. Conjecture 1 is therefore equivalent to the following.

Conjecture 2. *Let G be a finite bipartite graph with at least one edge. Then each of the two bipartition classes contains a vertex belonging to at most half of the maximal stable sets.*

We will show that Conjectures 1 and 2 are equivalent to the union-closed sets conjecture. The merit of this graph formulation is that other special cases become natural, in particular subclasses of bipartite graphs. As an illustration we verify the conjecture for the classes of chordal bipartite graphs and bipartitioned circular interval graphs, and for series–parallel and subcubic bipartite graphs. Translating the last result back to the set formulation, we obtain the following result which seems new.

Theorem 3. *Let $\mathcal{F}$ be a finite union-closed family of sets with $\mathcal{F} \neq \{\emptyset\}$. If $\mathcal{F}$ is the union-closure of sets of size at most three then it satisfies the union-closed sets conjecture.*

Another advantage of our reformulation is that it allows to test Frankl's conjecture in a probabilistic setting: in [3] it is shown that almost every random bipartite graph in the standard $G(m, n; p)$ model satisfies Conjecture 2 up to any given $\delta > 0$. That is, almost every such graph contains in each bipartition class a vertex for which the number of maximal stable sets containing it is at most $\frac{1}{2} + \delta$ times the total number of maximal stable sets. Note that no restriction is put on the sizes of the two sides, m and n, of the random bipartite graph. This allows us to deal with relatively sparse union-closed families, too.

Note that the maximal stable sets in a graph are exactly the independent dominating sets. A stable set of a graph is a clique of the complement graph and the graph formulation of the conjecture can also be stated in terms of maximal cliques, instead of maximal stable sets. The family of all maximal stable sets of a bipartite graph, or rather maximal complete bipartite cliques (bicliques) of the bipartite complement graph, was studied by Prisner [19] who gave upper bounds on the size of this family, also when excluding certain subgraphs. More recently, Duffus, Frankl and Rödl [5] and Ilinca and

Kahn [11] investigated the number of maximal stable sets in certain regular and biregular bipartite graphs. In work related to the graph parameter Boolean-width, Rabinovich, Vatshelle and Telle [20] study balanced cuts of a graph that bound the number of maximal stable sets. However, we have not found in the graph theory literature any previous work focusing on the number of maximal stable sets that vertices belong to.

2. Equivalence of the conjectures

For a subset S of vertices of a graph we denote by $N(S)$ the set of vertices adjacent to a vertex in S. All our graphs will be finite, and whenever we consider a union-closed family $\mathcal{F}$ of sets, it will be a finite family, all of whose member-sets will be finite as well. As Poonen [18] observed the latter assumption does not restrict generality, while the conjecture becomes false if $\mathcal{F}$ is allowed to have infinitely many sets.

We need two easy auxiliary statements. The proof of the first is trivial so we omit it.

Observation 4. *Let G be a bipartite graph with bipartition U, W, and let S be a maximal stable set. Then $S = (U \cap S) \cup (W \setminus N(U \cap S))$.*

Lemma 5. *Let G be a bipartite graph with bipartition U, W, and let S and T be maximal stable sets. Then $(U \cap S \cap T) \cup (W \setminus N(S \cap T))$ is a maximal stable set.*

Proof. Clearly, $R = (U \cap S \cap T) \cup (W \setminus N(S \cap T))$ is stable. Moreover, any vertex of $W \setminus R$ has a neighbour in R. Pick any vertex $u \in U \setminus R$. By the choice of R it holds that $u \notin S$ or $\notin T$, let us say that $u \notin T$. As T is maximal, u has a neighbour $w \in W \cap T$. This neighbour w cannot be adjacent to any vertex in $U \cap S \cap T$ as T is stable. So, w belongs to R as well, which shows that R is a maximal stable set. □

For a fixed graph G let us denote by $\mathcal{A}$ the family of all maximal stable sets, and for any vertex v let us write $\mathcal{A}_v$ for the sets of $\mathcal{A}$ that contain v and $\mathcal{A}_{\overline{v}}$ for the sets of $\mathcal{A}$ that do not contain v. Note that $\mathcal{A}$ depends on G. Let us call a vertex v *rare* if $|\mathcal{A}_v| \leq \frac{1}{2}|\mathcal{A}|$.

Theorem 6. *Conjecture 2 is equivalent to the union-closed sets conjecture.*

Proof. Let us consider first a union-closed family $\mathcal{F} \neq \{\emptyset\}$, which, without loss of generality, we may assume to include $\emptyset$ as a member. We put $U = \bigcup_{X \in \mathcal{F}} X$ and we define a bipartite graph G with partition classes U and $\mathcal{F}$, where we make $X \in \mathcal{F}$ adjacent with all $u \in X$.

Consider the mapping $\tau : \mathcal{A} \to \mathcal{F}$ given by $\tau : S \mapsto U \setminus S$. We claim that τ is a bijection. First note that indeed $\tau(S) \in \mathcal{F}$ for every maximal stable set: set $A = U \cap S$ and $\mathcal{B} = \mathcal{F} \cap S$. If $U \subseteq S$ then $U \setminus S = \emptyset \in \mathcal{F}$, by assumption. So, assume $U \nsubseteq S$, which implies $\mathcal{B} \neq \emptyset$. As S is a maximal stable set, it follows that $U \setminus S = U \setminus A = N(\mathcal{B})$. On the other hand, $N(\mathcal{B})$ is just the union of the $X \in S \cap \mathcal{F} = \mathcal{B}$, which is by the union-closed property equal to a set X' in $\mathcal{F}$.

To see that τ is injective note that, by Observation 4, S is determined by $U \cap S$, which in turn determines $U \setminus S$. For surjectivity, consider $X \in \mathcal{F}$. We set $A = U \setminus N(X)$ and observe that $S = A \cup (\mathcal{F} \setminus N(A))$ is a stable set. Moreover, as $X \in \mathcal{F} \setminus N(A)$ every vertex in $U \setminus A$ is a neighbour of $X \in S$, which means that S is maximal.

Now, assuming that Conjecture 2 is true, there is a rare $u \in U$, that is, $|\mathcal{A}_u| \leq \frac{1}{2}|\mathcal{A}|$. Clearly $\mathcal{A}$ is the disjoint union of $\mathcal{A}_u$ and of $\mathcal{A}_{\overline{u}}$, so that

$$|\tau(\mathcal{A}_{\overline{u}})| = |\mathcal{A}_{\overline{u}}| \geq \frac{1}{2}|\mathcal{A}| = \frac{1}{2}|\mathcal{F}|.$$

As $u \in \tau(S) \in \mathcal{F}$ for every $S \in \mathcal{A}_{\overline{u}}$, the union-closed sets conjecture follows.

Consider a bipartite graph G with bipartition U, W and at least one edge. Define $\mathcal{F} := \{U \setminus S : S \in \mathcal{A}\}$, and note that $\mathcal{F} \neq \{\emptyset\}$ as G has at least two distinct maximal stable sets. By Observation 4, there is a bijection between $\mathcal{F}$ and $\mathcal{A}$. Moreover, it is a direct consequence of Lemma 5 that $\mathcal{F}$ is union-closed. Analogously we may consider the other partition class of G. From this, it is straightforward that Conjecture 2 follows from the union-closed sets conjecture. □

3. Application to four graph classes

We now prove Conjecture 1 in four graph classes. The techniques we will use are not new. We still think this is a useful exercise as it gives us the opportunity to illustrate how some of the known methods can be adapted from the set formulation to the new setting. Moreover, our last application gives, in our opinion, a nice example of possible interactions between the set and the graph formulation.

3.1. Generalizing the singleton observation

For a set X of vertices we define $\mathcal{A}_X$ to be the set of maximal stable sets containing all of X. As before, we abbreviate $\mathcal{A}_{\{x\}}$ to $\mathcal{A}_x$.

Lemma 7. *Let x be a vertex of a bipartite graph G. Then there is an injection $\mathcal{A}_{N(x)} \to \mathcal{A}_x$.*

Proof. We define

$$\begin{aligned} i : \mathcal{A}_{N(x)} &\to \mathcal{A}_x, \\ S &\mapsto (S \setminus L_1) \cup \{x\} \cup (L_2 \setminus N(S \cap L_3)), \end{aligned}$$

where L_i denotes the set of vertices at distance i to x. That $i(S)$ is stable and maximal is a direct consequence of the definition. Moreover, $i(S) = i(T)$ for $S, T \in \mathcal{A}_{N(x)}$ implies that S and T are identical outside $L_1 \cup L_2$. Moreover, S and T are also identical on $L_1 \cup L_2$: first, $L_1 = N(x)$ shows that L_1 lies in both S and T. Second, since every vertex in L_2 is a neighbour of one in $L_1 \subseteq S \cap T$, no vertex of L_2 can lie in either of S or T. Thus, $S = T$, and we see that i is an injection. □

We denote by $N^2(x) = N(N(x))$ the second neighbourhood of a vertex x. The following lemma gives a sufficient condition for a vertex to be rare.

Lemma 8. *Let x, y be two adjacent vertices in a bipartite graph G with $N^2(x) \subseteq N(y)$. Then y is rare.*

Proof. From $N^2(x) \subseteq N(y)$ it follows that every maximal stable set containing y must contain all of $N(x)$. Thus, $\mathcal{A}_y \subseteq \mathcal{A}_{N(x)}$, which means by Lemma 7 that $|\mathcal{A}_y| \le |\mathcal{A}_x|$ and as $|\mathcal{A}_y| + |\mathcal{A}_x| \le |\mathcal{A}|$ the lemma is proved. □

As a special case the lemma implies that the neighbour of any vertex of degree 1 is always rare. In the set formulation this corresponds to the following: if a union-closed set family contains a singleton, then the family satisfies the union-closed sets conjecture. This fact, already observed in 1989 by Sarvate and Renaud [24], is not only one of the oldest known facts about the conjecture but also one of the simplest.

We now apply the lemma to the class of *chordal bipartite* graphs. This is the class of bipartite graphs in which every cycle with length at least six has a chord.

This graph class was originally defined in 1978 by Golumbic and Gross [8]. It is also known as the class of bipartite weakly chordal graphs.

A vertex v in a bipartite graph is *weakly simplicial* if the neighbourhoods of its neighbours form a chain under inclusion. Hammer, Maffray and Preissmann [10], and also Pelsmajer, Tokaz and West [17] prove the following.

Theorem 9 (*Hammer, Maffray and Preissmann [10], Pelsmajer, Tokaz and West [17]*)**.** *A bipartite graph with at least one edge is chordal bipartite if and only if every induced subgraph has a weakly simplicial vertex. Moreover, such a vertex can be found in each of the two bipartition classes.*

Let us say that a bipartite graph *satisfies Frankl's conjecture* if each of its bipartition classes contains a rare vertex. In order to avoid repeating the trivial condition that the graph has to contain at least one edge, we will also consider edgeless graphs to satisfy Frankl's conjecture.

Proposition 10. *Chordal bipartite graphs satisfy Frankl's conjecture.*

Proof. For a given bipartition class, let x be a weakly simplicial vertex in it. Among the neighbours of x denote by y the one whose neighbourhood includes the neighbourhoods of all other neighbours of x. Then y is rare, by Lemma 8. □

We now reformulate Proposition 10 in the set-theoretic setting. For this, we need the following notions. Let $\mathcal{S}$ be a set family and $X = \bigcup_{S\in\mathcal{S}} S$. We call X the *ground set* of $\mathcal{S}$. A *β-cycle* in $\mathcal{S}$ is a sequence $(S_1, x_1, S_2, x_2, \ldots, S_k, x_k)$ with $k \geq 3$ such that the S_i are distinct sets of $\mathcal{S}$, the x_i are distinct elements of X, and the following holds: for all $i \in \{1, 2, \ldots, k\}$, $x_i \in S_i \cap S_{i+1}$, and for all $j \notin \{i, i+1\}$, $x_i \notin S_j$ (where we identify S_{k+1} and S_1). A set family is called *β-acyclic* if it does not contain a β-cycle.

The *incidence graph* of $\mathcal{S}$, denoted $I(\mathcal{S})$, is the bipartite graph on the vertex set $\mathcal{S} \cup X$ and edge set

$$E(I(\mathcal{S})) = \{xS : x \in X, S \in \mathcal{S}, x \in S\}.$$

Obviously, every bipartite graph is isomorphic to the incidence graph of some set family. As the following result shows, there is a close connection between chordal bipartite graphs and β-acyclic set families.

Theorem 11 (*Tarjan and Yannakakis [25]*)**.** *Let $\mathcal{S}$ be a set family. Then $\mathcal{S}$ is β-acyclic if and only if $I(\mathcal{S})$ is chordal bipartite.*

This implies the following.

Corollary 12. *Let $\mathcal{S}$ be a β-acyclic set family and $\mathcal{F}$ be its union-closure. Then $\mathcal{F}$ satisfies the union-closed sets conjecture.*

Proof. By Theorem 11, $I(\mathcal{S})$ is chordal bipartite. Proposition 10 implies that there is a vertex $x \in X = \bigcup_{S\in\mathcal{S}} S$ contained in at most half of the maximal stable sets of $I(\mathcal{S})$. Following the proof of Theorem 6, this means that x is contained in at least half of the sets of the union-closure of the family $\{N_{I(\mathcal{S})}(S) : S \in \mathcal{S}\}$. But this is exactly $\mathcal{F}$, and the proof is complete. □

Let us stress the fact that Corollary 12 is equivalent to Proposition 10.

3.2. Generalizing the pairs observation

Going beyond chordal bipartite graphs, we quickly encounter graphs that cannot be handled anymore by Lemma 8. For example, no vertex in an even cycle of length at least six can be proved to be rare by applying Lemma 8. We will, therefore, strengthen the lemma.

For this, let us extend our notation a bit. For two vertices u, v let us denote by $\mathcal{A}_{uv}$ the subfamily of $\mathcal{A}$, every member of which contains both u and v. Similarly, by $\mathcal{A}_{u\overline{v}}$ we denote the subfamily of $\mathcal{A}$ containing the sets that contain u but not v, and by $\mathcal{A}_{\overline{uv}}$ the subfamily of $\mathcal{A}$ of sets containing neither of u and v.

Lemma 13. *Let G be a bipartite graph. Let y and z be two neighbours of a vertex x so that $N^2(x) \subseteq N(y) \cup N(z)$. Then one of y and z is rare.*

Proof. We may assume that $|\mathcal{A}_{y\overline{z}}| \leq |\mathcal{A}_{\overline{y}z}|$. Now, from $N^2(x) \subseteq N(y) \cup N(z)$ we deduce that $\mathcal{A}_{yz} = \mathcal{A}_{N(x)}$. Thus, by Lemma 7, we obtain $|\mathcal{A}_{yz}| \leq |\mathcal{A}_x|$. Since $\mathcal{A}_x \subseteq \mathcal{A}_{\overline{yz}}$ it follows that $|\mathcal{A}_y| = |\mathcal{A}_{y\overline{z}}| + |\mathcal{A}_{yz}| \leq |\mathcal{A}_{\overline{y}z}| + |\mathcal{A}_{\overline{yz}}| = |\mathcal{A}_{\overline{y}}|$. As $|\mathcal{A}| = |\mathcal{A}_y| + |\mathcal{A}_{\overline{y}}|$, we see that y is rare. □

In particular, at least one of the neighbours of any vertex of degree 2 is always rare—translated to the set formulation of the union-closed sets conjecture this gives an easy fact that is well known: if one of the sets in the union-closed family $\mathcal{F}$ is a pair, then one of the members of this pair lies in at least half of the members of $\mathcal{F}$; see Sarvate and Renaud [24].

Next we give an application of Lemma 13 to a class of graphs derived from *circular interval graphs*. The class of circular interval graphs plays a fundamental role in the structure theorem of claw-free graphs of Chudnovsky and Seymour [4]. Circular interval graphs are defined as follows: let a finite subset of a circle be the vertex set, and for a given set of subintervals of the circle consider two vertices

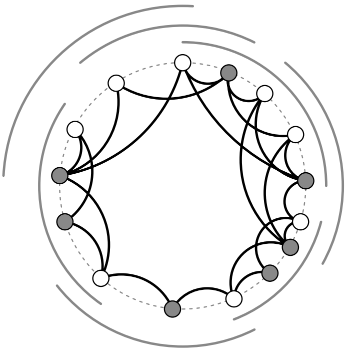

Fig. 1. A bipartitioned circular interval graph.

to be adjacent if there is an interval containing them both. This class is equivalent to what is known as the proper circular arc graphs.

Circular interval graphs are not normally bipartite. The only exceptions are even cycles and disjoint unions of paths. Nevertheless, we may obtain a non-trivial class of bipartite graphs from circular interval graphs: for any circular interval graph, partition its vertex set into two parts and delete every edge with both its end vertices in the same class. We call any graph arising in this manner a *bipartitioned circular interval graph* (see Fig. 1).

Proposition 14. *Bipartitioned circular interval graphs satisfy Frankl's conjecture.*

Proof. Consider a bipartitioned circular interval graph defined by intervals $\mathcal{I}$, and let x be a non-isolated vertex of the graph.

For every neighbour u of x we choose an interval $I_u \in \mathcal{I}$ containing both x and u. If $\bigcup_{v\in N(x)} I_v$ covers the whole circle, then there are already two such intervals I_y and I_z that cover the circle. Clearly, every vertex not in the same bipartition class as y and z is adjacent to at least one of them. In particular, $N^2(x) \subseteq N(y) \cup N(z)$.

So, let us assume that there is a point p on the circle that is not covered by any I_v, $v \in N(x)$. We choose y as the first neighbour of x from p in the clockwise order, and z as the first neighbour of x from p in the anticlockwise order. Then y, v, z appear in clockwise order for every $v \in N(x)$. Moreover, $v' \in I_y \cup I_z$ for every vertex v' such that y, v', z appear in clockwise order.

Let us show that again $N^2(x) \subseteq N(y) \cup N(z)$. For this consider a vertex $u \in N^2(x)$, and a neighbour w of x that is adjacent to u. Thus, there is an interval $J \in \mathcal{I}$ containing both u and w. If y, u, z appear in clockwise order, then $u \in I_y \cup I_z$, which implies $u \in N(y) \cup N(z)$. If not, then J meets one of y or z as y, w, z appear in clockwise order. Thus, by virtue of J, the vertex u is adjacent to at least one of y and z.

In both cases, we apply Lemma 13 in order to see that one of y and z is rare. As the choice of x was arbitrary, we find rare vertices in both bipartition classes. □

3.3. *Bipartite series–parallel graphs*

Recall that a graph is called *series–parallel* if it does not contain K_4 as a minor. Equivalently, a graph is series–parallel if and only if it is of treewidth at most two. Reusing some of the tools presented above, we can settle Frankl's conjecture for bipartite series–parallel graphs.

Theorem 15. *Bipartite series–parallel graphs satisfy Frankl's conjecture.*

It is an easy observation, probably first described by Poonen [18], that in the union-closed sets conjecture we may always suppress one of two elements that appear in exactly the same sets. We adapt here this observation to the graph setting.

For this, let us call a graph G *reduced* if there is no vertex v whose neighbourhood is equal to the union of neighbourhoods of some other vertices. In particular, reduced graphs are *twin-free*, that is, no two vertices have identical neighbourhoods.

Lemma 16. *For any bipartite graph G there is a reduced induced subgraph G' so that G satisfies Frankl's conjecture if G' satisfies it.*

Proof. Assume there are pairwise distinct vertices $u, v_1, v_2, \ldots, v_k$ such that $N(u) = \bigcup_{i=1}^{k} N(v_i)$. Then $\mathcal{A}_u = \mathcal{A}_{\{v_1, v_2, \ldots, v_k\}}$. Thus, if A is a maximal stable set of G, then $A-u$ is one of $G-u$, and conversely, any maximal stable set A' of $G-u$ is already maximally stable in G if $\{v_1, v_2, \ldots, v_k\} \not\subseteq A'$; otherwise $A'+u$ is a maximal stable set of G. Hence, a rare vertex of $G-u$ is also rare in G. The assertion is now obtained by iteratively deleting vertices such as u from G. $\square$

The following lemma gives us enough information on the local structure of a series–parallel graph to prove the theorem with Lemmas 8 and 13.

Lemma 17 (*Juvan, Mohar and Thomas [12]*). *Every non-empty series–parallel graph G has one of the following.*

(a) *a vertex of degree at most one,*
(b) *two twins of degree two,*
(c) *two distinct vertices u, v and two not necessarily distinct vertices $w, z \in V(G)\setminus\{u, v\}$ such that $N(v) = \{u, w\}$ and $N(u) \subseteq \{v, w, z\}$, or*
(d) *five distinct vertices v_1, v_2, u_1, u_2, w such that $N(w) = \{u_1, u_2, v_1, v_2\}$ and $N(v_i) = \{w, u_i\}$ for $i = 1, 2$.*

Proof of Theorem 15. Let G be a non-empty bipartite series–parallel graph, say with bipartition classes (U, W), and we may assume that G does not contain any isolated vertex. Our argument is symmetric, so it suffices to show that there is a rare vertex among the vertices in U. The class of series–parallel graphs is closed under induced subgraphs, and thus by Lemma 16 we may assume that G is reduced.

Let L be the set of *leaves* of G, that is, the set of degree 1 vertices. If there is a leaf in W, we obtain with Lemma 8 a rare vertex in U. So we may assume that $L \subseteq U$. Let $G' = G - L$ be the graph obtained by deleting all leaves. Since $L \subseteq U$, every vertex in $U \cap V(G')$ is of degree at least 2. In particular, G' is not empty.

We claim that in G' there is some vertex $x \in W$ of degree at most 2. If the claim is true then Lemma 13 yields that some $y \in N_{G'}(x) \subseteq U$ is rare in G, since every neighbour of x in $G - G'$ is a leaf.

So it remains to prove the claim. Lemma 17 yields that G' contains one of the configurations in (a), (b), (c), or (d). Clearly, (d) is not possible since G' is bipartite and thus triangle-free.

In case (a), there is a leaf in G', which then needs to be contained in W because every vertex in $U \cap V(G')$ has degree at least 2. In case (b), let u, v be the two twins of degree 2. If $u, v \in U$ then u and v are twins in G as well, which is impossible as G is reduced. Consequently, $u, v \in W$ and the claim is again verified. In the last case (c), there are two distinct vertices u, v and two not necessarily distinct vertices $w, z \in V(G) \setminus \{u, v\}$ such that $N(v) = \{u, w\}$ and $N(u) \subseteq \{v, w, z\}$. But G' is bipartite and so $uw \notin E(G')$. In particular, both u and v are of degree at most two. Since u and v are adjacent, one of them is contained in W. This completes the proof. $\square$

3.4. Subcubic bipartite graphs

Let us now turn to *subcubic* bipartite graphs: bipartite graphs in which no vertex has a degree greater than 3.

Theorem 18. *Subcubic bipartite graphs satisfy Frankl's conjecture.*

Unlike the previous classes, subcubic graphs do not have an easily exploitable local structure. In particular, Lemmas 8 and 13 will have only limited use. Nevertheless, we can verify Frankl's conjecture by adapting two results on the set formulation of the union-closed sets conjecture into the graph setting. Both results, one of Vaughan and the other of Knill, have surprisingly involved proofs. For a union-closed family $\mathcal{F}$, we say that an element of $\bigcup \mathcal{F}$ is *abundant* if the element appears in at least half of the member-sets of $\mathcal{F}$.

Theorem 19 (*Vaughan [26]*). *Let $\mathcal{F}$ be a union-closed family containing three distinct sets of size* 3 *such that there is an element contained in all three sets. Then there is an abundant element in the union of the three sets.*

While Vaughan's theorem gives a local condition, not unlike Lemmas 8 and 13, when a particular union-closed family satisfies the conjecture, the following result of Knill treats a special class of union-closed sets, which he calls *graph-generated families*. In this context, we view edges of a graph H as subsets of $V(H)$ of size two.

Theorem 20 (*Knill [13]*). *Given a graph H with at least one edge, let $\mathcal{B} = \{\bigcup F : F \subseteq E(H)\}$. Then there is an edge $e \in E(H)$ such that $|\{S \in \mathcal{B} : e \subseteq S\}| \leq \frac{|\mathcal{B}|}{2}$.*

This result was restated later as a conjecture by El-Zahar [6], who was probably unaware that it had already been proven by Knill. Finally, El-Zahar's conjecture was solved again by Llano, Montellano-Ballesteros, Rivera-Campo and Strausz [14].

We first translate Knill's theorem to the bipartite graph setting.

Lemma 21. *Let G be a twin-free bipartite graph with bipartition U, W, where every vertex in U is of degree* 2*. Then there is a rare vertex in U.*

Proof. Again, let $\mathcal{A}$ be the set of maximal stable sets of G. Observe that any two distinct vertices x, y of H are adjacent if and only if they have a common neighbour $u \in U$ in G. That is, G is a subdivision of the graph H. As G is twin-free and every vertex of U is of degree 2, every edge $e = xy$ of H corresponds to a unique vertex $u_e \in U$ with $N(u_e) = \{x, y\}$.

Let $\mathcal{B} = \{\bigcup F : F \subseteq E(H)\}$, and note that $\mathcal{B} = \{N_G(U') : U' \subseteq U\}$. We will establish a bijection between $\mathcal{B}$ and $\mathcal{A}$. For this, denote by $\mathcal{A}_{\cap W}$ the intersections of maximal stable sets of G with W. We define a mapping $\mathcal{B} \to \mathcal{A}_{\cap W}$ by $N_G(U') \mapsto W \backslash N_G(U')$, for $U' \subseteq U$. As $(W \backslash N_G(U')) \cup (U \backslash N_G(W \backslash N_G(U')))$ is a maximal stable set, this mapping is a bijection. Recall that Observation 4 asserts that every maximal stable set is determined by its intersection with one of the bipartition classes. Thus, the bijection $\mathcal{B} \to \mathcal{A}_{\cap W}$ extends to a bijection $\mathcal{B} \to \mathcal{A}$. In particular, $|\mathcal{A}| = |\mathcal{B}|$.

Now, for any $S \in \mathcal{B}$ there exists $U' \subseteq U$ so that $N_G(U') = S$. Any edge $e \in E(H)$ between vertices $x, y \in W$ is contained in S if and only if $x, y \notin W \setminus N_G(U')$, which means that the unique maximal stable set $A \in \mathcal{A}$ with $A \cap W = W \setminus N_G(U')$ needs to contain u_e, the vertex in U with neighbours x, y. Therefore, the number of $S \in \mathcal{B}$ with $e \subseteq S$ is equal to the number of maximal stable sets containing u_e.

Applying Theorem 20 we obtain an edge $e = xy \in E(H)$ such that $|\{S \in \mathcal{B} : \{x, y\} \subseteq S\}| \leq \frac{|\mathcal{B}|}{2}$. This then implies that u_e lies in at most $\frac{|\mathcal{B}|}{2} = \frac{|\mathcal{A}|}{2}$ maximal stable sets, which completes the proof. $\square$

Proof of Theorem 18. Let G be a subcubic bipartite graph with bipartition $U \cup W$, and let $\mathcal{A}$ be the set of maximal stable sets of G. By Lemma 16, we may assume that G is reduced and, in particular, twin-free.

Let us prove that there is a rare vertex in U. Then, by symmetry, we know that there must be a rare vertex in W too. If W contains a vertex of degree 1 or 2, we are done by Lemma 13. So, let us assume that every vertex in W has degree 3.

First assume that there is a vertex $u \in U$ of degree 1. Let $x \in W$ be its unique neighbour, and let $y, z \in U$ be the other two neighbours of x. By Lemma 13, y or z is rare and we are done.

Now assume that there is a vertex $u \in U$ of degree 3, say $N(u) = \{x, y, z\}$. Consider the set $\mathcal{B} = \{U \backslash S : S \in \mathcal{A}\}$, which is union-closed by Lemma 5. Then $N(x), N(y), N(z) \in \mathcal{B}$, and $u \in N(x) \cap$

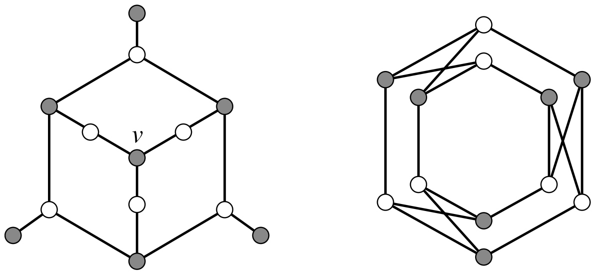

Fig. 2. Left: No neighbour of v is rare. Right: Lemma 8 or 13 not applicable.

$N(y) \cap N(z)$. Note that $N(x)$, $N(y)$, $N(z)$ are three distinct sets as G is twin-free. From Theorem 19 we know that there is an abundant element of $\mathcal{B}$ in $N(x) \cup N(y) \cup N(z)$, and hence this is a rare vertex in U.

The remaining case, when every vertex in U is of degree 2 is taken care of by Lemma 21. $\square$

As a by-product we may now deduce Theorem 3, that the union-closure of any collection $\mathcal{S} \neq \{\emptyset\}$ of sets of size at most 3 satisfies the union-closed sets conjecture. First we note that by Theorem 19 we may assume that no element appears in more than three sets of $\mathcal{S}$. Then the graph $I(\mathcal{S})$ is subcubic, and thus contains rare vertices in both bipartition classes. Translating this back to the set formulation as in the proof of Corollary 12, we obtain an abundant element.

We point out that, while Theorem 3 draws heavily on Vaughan's theorem, it is not a straightforward consequence of the latter result. Rather Vaughan's theorem has to be used in conjunction with Knill's theorem, while our graph reformulation serves as glue.

4. Discussion

Recall that Lemmas 8 and 13 do not apply to the subcubic case. This is because none of the neighbours of a vertex of degree at least 3 have to be rare. An example is given in Fig. 2 on the left, where no neighbour of the vertex v is rare. Note that both graphs in Fig. 2 are subcubic.

This is not new, in the sense that it corresponds directly to an observation of Sarvate and Renaud [23] in the set formulation: a set of size three need not contain any element appearing in at least half of the member sets of the union-closed family.

As chordal bipartite graphs are exactly the $(C_6, C_8, C_{10}, \ldots)$-free graphs one may be tempted to generalize Proposition 10 by allowing one more even cycle, the 6-cycle, as an induced subgraph. While Lemma 8 is no longer strong enough even for the graph C_6, Lemma 13 easily takes care of any graph with a degree 2 vertex in each bipartition class. In general, however, Lemma 13 turns out to be too weak to prove the conjecture for $(C_8, C_{10}, C_{12}, \ldots)$-free graphs: the graph on the right in Fig. 2 is of that form but has no vertices covered by Lemma 13.

With the results in the previous section we aim to substantiate the usefulness of the graph formulation of the union-closed sets conjecture. Moreover, we believe that a good number of other graph classes should be within reach. Does Frankl's conjecture hold for planar graphs, regular graphs or for graphs of treewidth 3?

Acknowledgements

We thank both referees for their careful comments. The second author was partially supported by the French Agence Nationale de la Recherche under reference anr-10-jcjc-Heredia, and by INRIA-Project Team GANG.

References

[1] I. Bošnjak, P. Marković, The 11-element case of Frankl's conjecture, Electron. J. Combin. 15 (2008) R88.
[2] H. Bruhn, O. Schaudt, The journey of the union-closed sets conjecture, Graphs Combin. (2013) in press.

[3] H. Bruhn, O. Schaudt, The union-closed sets conjecture almost holds for almost all random bipartite graphs, 2013. arXiv:1302.7141 [math.CO].
[4] M. Chudnovsky, P.D. Seymour, Claw-free graphs. III. Circular interval graphs, J. Combin. Theory Ser. B 98 (4) (2008) 812–834.
[5] D. Duffus, P. Frankl, V. Rödl, Maximal independent sets in bipartite graphs obtained from Boolean lattices, European J. Combin. 32 (1) (2011) 1–9.
[6] M. El-Zahar, A graph-theoretic version of the union-closed sets conjecture, J. Graph Theory 26 (1997) 155–163.
[7] P. Frankl, Handbook of Combinatorics, Vol. 2, MIT Press, Cambridge, MA, USA, 1995, pp. 1293–1329.
[8] M.C. Golumbic, C.F. Goss, Perfect elimination and chordal bipartite graphs, J. Graph Theory 2 (1978) 155–163.
[9] T. Gowers, Gowers's weblog: possible future PolyMath projects, 2009. http://gowers.wordpress.com/2009/09/16/possible-future-polymath-projects.
[10] P.L. Hammer, F. Maffray, M. Preissmann, A characterization of chordal bipartite graphs, Rutcor Research Report, Rutgers University, New Brunswick, NJ, 1989.
[11] L. Ilinca, J. Kahn, Counting maximal antichains and independent sets, Order 30 (2013) 427–435.
[12] M. Juvan, B. Mohar, R. Thomas, List edge-colorings of series–parallel graphs, Electron. J. Combin. 6 (1999) 1077–8926.
[13] E. Knill, Graph generated union-closed families of sets, 1994. arXiv:math/9409215 [math.CO].
[14] B. Llano, J.J. Montellano-Ballesteros, E. Rivera-Campo, R. Strausz, On conjectures of Frankl and El-Zahar, J. Graph Theory 57 (2008) 344–352.
[15] G. Lo Faro, Union-closed sets conjecture: improved bounds, J. Combin. Math. Combin. Comput. 16 (1994) 97–102.
[16] P. Marković, An attempt at Frankl's conjecture, Publ. Inst. Math. (Beograd) (N.S.) 81 (95) (2007) 29–43.
[17] M.J. Pelsmajer, J. Tokaz, D.B. West, New proofs for strongly chordal graphs and chordal bipartite graphs, Preprint, 2004.
[18] B. Poonen, Union-closed families, J. Combin. Theory Ser. A 59 (1992) 253–268.
[19] E. Prisner, Bicliques in graphs I: bounds on their number, Combinatorica 20 (1) (2000) 109–117.
[20] Yuri Rabinovich, Jan Arne Telle, Martin Vatshelle, Upper bounds on boolean-width with applications to exact algorithms, in: Gregory Gutin, Stefan Szeider (Eds.), IPEC, in: Lecture Notes in Computer Science, vol. 8246, Springer, 2013, pp. 308–320.
[21] J. Reinhold, Frankl's conjecture is true for lower semimodular lattices, Graphs Combin. 16 (1) (2000) 115–116.
[22] I. Roberts, J. Simpson, A note on the union-closed sets conjecture, Australas. J. Combin. 47 (2010) 265–269.
[23] D.G. Sarvate, J.-C. Renaud, Improved bounds for the union-closed sets conjecture, Ars Combin. 29 (1989) 181–185.
[24] D.G. Sarvate, J.-C. Renaud, On the union-closed sets conjecture, Ars Combin. 27 (1989) 149–154.
[25] R.E. Tarjan, M. Yannakakis, Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs, SIAM J. Comput. 13 (1984) 566–579.
[26] T.P. Vaughan, Three-sets in a union-closed family, J. Combin. Math. Combin. Comput. 49 (2004) 73–84.