Graphs and Combinatorics (2000) 16:115–116

Graphs and Combinatorics

Frankl's Conjecture Is True for Lower Semimodular Lattices

Jürgen Reinhold

Universität Hannover, Institut für Mathematik, Welfengarten 1, 30167 Hannover, Germany. e-mail: reinhold@math.uni-hannover.de

Abstract. It is shown that every finite lower semimodular lattice L with $|L| \geq 2$ contains a join-irreducible element x such that at most $|L|/2$ elements $y \in L$ satisfy $y \geq x$.

An element a of a lattice L is $\vee$-*irreducible* if $a = b \vee c$ implies $a = b$ or $a = c$. The lattice-theoretical version of the famous *union-closed sets conjecture* alias *Frankl's conjecture* is the following

Conjecture. *In every finite lattice L with $|L| \geq 2$ there is a $\vee$-irreducible element a such that $|\{x \in L : a \leq x\}| \leq |L|/2$.*

Let a, b be elements of a *poset* (= partially ordered set) P. We call a a *lower cover* of b and write $a \prec b$ if $a < b$ and the *interval* $[a, b] = \{c \in P : a \leq c \leq b\}$ has only the two elements a, b. The *principal filter* $\{c \in P : a \leq c\}$ will be denoted by $[a)$, the *principal ideal* $\{c \in P : c \leq a\}$ by $(a]$. Every finite lattice has a smallest element $\perp$ and a greatest element $\top$. A lower cover of $\top$ is called a *coatom*. A mapping $\varphi : P \to P'$ between posets is *order preserving* if $a \leq b \Rightarrow \varphi(a) \leq \varphi(b)$ holds for all $a, b \in P$, and φ is an *order embedding* if this implication becomes an equivalence for all $a, b \in P$.

Recall that a lattice L is *lower semimodular* if the implication $b \prec b \vee a \Rightarrow b \wedge a \prec a$ holds for all $a, b \in L$. It seems that even most lattice-theorists are unaware of the following description of semimodularity although a similar characterization of modularity (see e.g. [1,3.5]) is widely used.

Lemma 1. *A lattice L is lower semimodular if and only if for all $a, b \in L$ with $b \prec b \vee a$ the mapping $\varphi : [a, b \vee a] \to [b \wedge a, b]$, $x \mapsto b \wedge x$ is an order embedding.*

Proof. Let L be lower semimodular and $b \prec b \vee a$. Clearly, φ is order preserving. Let $x, y \in [a, b \vee a]$ such that $x \not\leq y$. Assume that $\varphi(x) \leq \varphi(y)$. Then

$$b \wedge x = \varphi(x) = \varphi(x) \wedge \varphi(y) = b \wedge x \wedge y \leq x \wedge y < x. \qquad (1)$$

Since $x \in [a, b \vee a]$, we have $b \prec b \vee a = b \vee x$, and since L is lower semimodular, it follows that $b \wedge x \prec x$. Thus in (1), $b \wedge x = x \wedge y$. Then $a \leq x \wedge y = b \wedge x \leq b$, which contradicts $b \prec b \vee a$.

For the converse, suppose that L is not lower semimodular. Then there exist $a, b, c \in L$ such that $b \prec b \vee c$ and $b \wedge c < a < c$. Then $b \vee a = b \vee c$, so $b \prec b \vee a$. But $\varphi(a) = \varphi(c)$, so φ is not an order embedding. □

Now, we improve the main results from [2].

Theorem 2. *In every finite lower semimodular lattice L with $|L| \geq 2$ there is a $\vee$-irreducible element a such that $|[a)| \leq |L|/2$.*

Proof. If $\top$ is $\vee$-irreducible, we are done. Otherwise, there is a coatom b and a $\vee$-irreducible element a with $a \nleq b$. Then $b \prec \top = b \vee a$. By Lemma 1, $\varphi : [a) \to [b \wedge a, b]$, $x \mapsto b \wedge x$ is an order embedding. Since $[b \wedge a, b] \subseteq L \backslash [a)$, it follows that $|[a)| \leq |L|/2$. □

Theorem 3. *If $|[a)| = |L|/2$ for every $\vee$-irreducible element $a \neq \perp$ of a lower semimodular finite lattice L, then L is Boolean.*

Proof. Now, the order embedding φ in the proof of Theorem 2 must be an isomorphism between the sublattices $[a)$ and $L \backslash [a) = (b]$ of L. For $x \in [a)$, one has $a \vee (b \wedge x) \leq x$. On the other hand, $\varphi(a \vee (b \wedge x)) = b \wedge (a \vee (b \wedge x)) \geq b \wedge x = \varphi(x)$, and since φ is an order embedding, it follows that $a \vee (b \wedge x) = x$, i.e. the inverse map is $\varphi^{-1} : y \mapsto a \vee y$. Hence the $\vee$-irreducibles in L different from a are exactly the $\vee$-irreducibles of $(b]$, and each of them is dominated by $|(b]|/2$ elements of $(b]$. Now, a simple induction proves the claim. □

References

1. Crawley P., Dilworth, R.P.: Algebraic Theory of Lattices, Prentice-Hall, Englewood Cliffs, New York, 1973
2. Abe, T., Nakano, B.: Frankl's Conjecture is True for Modular Lattices, Graph. Comb. **14**, 305–311 (1998)

Received: April 19, 1999
Revised: August 16, 1999