Union-closed families of sets

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A R T I C L E I N F O

A B S T R A C T

A family of sets is union-closed if it contains the union of any two of its elements. Reimer (2003) [16] and Czédli (2009) [2] investigated the average size of an element of a union-closed family consisting of m subsets of a ground set with n elements. We determine the minimum average size precisely, verifying a conjecture of Czédli, Maróti and Schmidt (2009) [3]. As a consequence, the union-closed conjecture holds if $m \geqslant \frac{2}{3}.2^n$ — in this case some element of $[n]$ is in at least half the sets of the family.

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1. Introduction

Let $\mathcal{A}$ be a finite family of finite sets; as often, we shall assume that $\mathcal{A}$ is non-empty and consists of subsets of $[n] = \{1, \ldots, n\}$, i.e.

$$\emptyset \neq \mathcal{A} \subset \mathcal{P}(n) = \mathcal{P}([n]).$$

The *degree of $x \in [n]$ in $\mathcal{A}$* is

$$d_{\mathcal{A}}(x) = \big|\{A \in \mathcal{A}: x \in A\}\big|.$$

Recall that $\mathcal{A}$ is an *up-set* (in $\mathcal{P}(n)$) if $B \in \mathcal{A}$ whenever $A \subset B$, $A \in \mathcal{A}$ and $B \in \mathcal{P}(n)$. A similar, but weaker property of $\mathcal{A}$ is that it is *union-closed*: $A \cup B \in \mathcal{A}$ whenever $A, B \in \mathcal{A}$. As usual, we shall assume that the groundset $[n]$ is as small as possible: $\bigcup_{A \in \mathcal{A}} A = [n]$. Trivially, if $\mathcal{A}$ is an up-set

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then $d_\mathcal{A}(x) \geqslant |\mathcal{A}|/2$ for every $x \in [n]$: indeed, $A \mapsto A \cup \{x\}$ is an injection from $\{A \in \mathcal{A}: x \notin A\}$ into $\{A \in \mathcal{A}: x \in A\}$. Clearly, not every union-closed family has this property; e.g. if $\mathcal{A} = \{[k]: 0 \leqslant k \leqslant n\}$ then $\mathcal{A}$ is union-closed and $d_\mathcal{A}(n) = 1 = |\mathcal{A}|/(n+1)$. However, it seems that it may be true that

$$d_\mathcal{A}(x) \geqslant |\mathcal{A}|/2 \quad \text{for some } x \in [n]. \tag{1}$$

Indeed, the *union-closed degree conjecture*, or simply *union-closed conjecture*, usually attributed to Frankl, says that this is the case: *if $\mathcal{A}$ is a union-closed family then some element is contained in at least half of the sets in $\mathcal{A}$.* (In fact, the first such attribution appeared in 1985 [4], but the conjecture was well known by the mid-1970s as a "folklore conjecture".)

Over the years, much work has been done on this conjecture, and yet it has only been proved when the size of the groundset [n] or the system $\mathcal{A}$ is very restricted. Bošnjak and Marković [1] settled it for $n \leqslant 11$, improving on results of Marković [12], Morris [13], Gao and Yu [6], Poonen [15] and Lo Faro [10]. Roberts and Simpson [17] show that if q is the smallest n for which there is a counterexample, then any counterexample $\mathcal{A}$ has $|\mathcal{A}| \geqslant 4q - 1$. Since $q \geqslant 12$, this implies the conjecture for $|\mathcal{A}| \leqslant 46$, improving bounds given by Lo Faro [11], Poonen [15] and Sarvate and Renaud [18,19]. Vaughan [20–22] settled the conjecture for families which contain any of a number of small subfamilies. There are various other results on the conjecture, including [5,7,14].

In 2003 Reimer [16] considered the *union-closed size problem*, a problem closely related to the union-closed degree conjecture: what is

$$f(m) = \min \|\mathcal{A}\|,$$

where the minimum is over all union-closed families $\mathcal{A}$ of finite sets consisting of m sets, and $\|\mathcal{A}\|$ is the *total size of the sets in $\mathcal{A}$*,

$$\|\mathcal{A}\| = \sum_{A \in \mathcal{A}} |A|.$$

Reimer [16] proved that

$$f(m) \geqslant \frac{m}{2} \log_2 m. \tag{2}$$

Clearly, equality holds whenever $m = 2^k$, since we may take $\mathcal{A} = \mathcal{P}(k)$. In fact, (2) is strict in every other case.

Recently, Czédli [2] studied a refinement of the function $f(m)$. For $0 \leqslant m \leqslant 2^n$, let

$$f(n, m) = \min \|\mathcal{A}\|,$$

where the minimum is over all union-closed families $\mathcal{A}$ of subsets of [n] with $|\mathcal{A}| = m$. (The inequality $m \leqslant 2^n$ is not really a restriction, since if $\mathcal{A}$ is a family of subsets of [n] then $m \leqslant 2^n$.)

Clearly, if $\|\mathcal{A}\| \geqslant mn/2$ and $\mathcal{A} \subset \mathcal{P}(n)$ then (1) holds. Hence, setting

$$m_0 = m_0(n) = \min\{m^*: f(n, m) \geqslant mn/2 \text{ if } m \geqslant m^*\},$$

the union-closed conjecture holds whenever $|\mathcal{A}| \geqslant m_0$. *A priori* it is not clear that m_0 is smaller than 2^n (after all, it could be that Reimer's inequality (2) is close to best possible), but a moment's thought tells us that this is the case. In fact, Czédli [2] has proved that m_0 is not too close to 2^n: $2^n - m_0 \geqslant \lfloor 2^{n/2} \rfloor$. From the other direction, Czédli, Maróti and Schmidt [3] showed that $m_0 \geqslant 2\lceil 2^n/3 \rceil$. Our main aim in this paper is to determine $f(n, m)$ precisely and give the extremal families for all n and m, and so settle the union-closed size problem completely. In particular, our result implies that $m_0 = 2\lceil 2^n/3 \rceil$; *a fortiori*, the union-closed conjecture holds for families in $\mathcal{P}(n)$ with at least $\frac{2}{3}.2^n$ elements. This result was conjectured by Czédli, Maróti and Schmidt [3]; they also proved that it does hold provided the union-closed conjecture holds.

The rest of this paper is organised as follows. In Section 2, we shall discuss the initial segments of the colex order, enabling us to state our main theorem, giving the value of $f(n, m)$. Section 3 consists of the proof of the main theorem, and in Section 4 we shall determine the structure of all extremal

families. In Section 5 we shall determine some quantitative bounds on $f(n,m)$: our main goal shall be to determine m_0. In Section 6 we prove bounds for the sizes of a restricted class of union-closed family. In Section 7 we prove a slightly stronger result in the case when we have a hypothetical counterexample to the union-closed conjecture. Finally, in Section 8, we conjecture a relationship between the total size of union-closed family and the minimum degree of an element in it.

2. The main result

Before stating our main theorem, let us recall the definition of the colex order on finite sets of positive integers, and in particular the Kruskal–Katona theorem on the cardinalities of shadows. The *colex order* on $\mathbb{N}^{(<\infty)}$, the collection of finite sets of positive integers, is the linear order $<$ on $\mathbb{N}^{(<\infty)}$ in which $A < B$ if and only if $\max(A \triangle B) \in B$. E.g., writing 124 for the set $\{1, 2, 4\}$, the colex order on $\mathbb{N}^{(<\infty)}$ starts as follows: $\emptyset < 1 < 2 < 12 < 3 < 13 < 23 < 123 < 4 < 14 < 124$. It is immediate that $<$ is indeed a linear order. We write $\mathcal{I}(m)$ for the initial segment of length m of $\mathbb{N}^{(<\infty)}$ in the colex order, and $\mathcal{I}_k(m)$ for the initial segment of length m of $\mathbb{N}^{(k)}$ in the colex order. The fundamental theorem of Kruskal [9] and Katona [8] states that if $\mathcal{A} \subset \mathbb{N}^{(k)}$ with $|\mathcal{A}| = m$, then the lower shadow of $\mathcal{A}$ is at least as large as the lower shadow of $\mathcal{I}_k(m)$; consequently, $|\partial^i(\mathcal{A})| \geqslant |\partial^i(\mathcal{I}_k(m))|$ for all $\mathcal{A} \subset \mathbb{N}^{(k)}$ with $|\mathcal{A}| = m$ and $0 \leqslant i \leqslant k$. Let us state an easy consequence of this Kruskal–Katona theorem as a lemma, since this is the fact that we shall exploit.

Lemma 1. *If $\mathcal{D}$ is a down-set, then $\|\mathcal{D}\| \leqslant \|\mathcal{I}(|\mathcal{D}|)\|$.*

A consequence of this is a simple relationship between the total sizes of the initial segments of the colex order.

Lemma 2. *For all $m_1, m_2 \geqslant 1$ we have*

$$\left\|\mathcal{I}(m_1 + m_2)\right\| \geqslant \left\|\mathcal{I}(m_1)\right\| + \left\|\mathcal{I}(m_2)\right\| + \min(m_1, m_2). \tag{3}$$

This follows (for $m_1 \geqslant m_2$) by considering the down-set $\mathcal{I}(m_1) \cup \{A \cup \{0\}\colon A \in \mathcal{I}(m_2)\}$, the total size of which is precisely the right hand side of (3).

Our main theorem, Theorem 3, giving the smallest average size of a set in a union-closed family, was proved by Czédli, Maróti and Schmidt [3] *under the assumption* that the union-closed conjecture holds; they also conjectured that it holds unconditionally. As a by-product of our theorem, we find that although the function $f(n,m)$ is a refinement of $f(m)$ it is, in fact, independent of n.

We need one more important ingredient before we can state our main result, for which we give two definitions. First, for $m < 2^n$ write m in the following form: $m = 2^{n-1} + 2^{n-2} + \cdots + 2^k + m'$, where k and m' are integers, with $m' \leqslant 2^{k-1}$. Then define a family $\mathcal{M}(n,m)$ of subsets of $\mathcal{P}(n)$ by $\mathcal{M}(n,m) = \{B \in \mathcal{P}(n)\colon B \cap [k+1, n] \neq \emptyset\} \cup \{B + k\colon B \in \mathcal{I}(m')\}$. Here and elsewhere we use the shorthand $B + k$ for $B \cup \{k\}$, and $B - k$ for $B \setminus \{k\}$. Furthermore, for $m = 2^n$ we define $\mathcal{M}(n,m) = \mathcal{P}(n)$.

Alternatively, we can define $\mathcal{M}(n,m)$ through another order on $\mathcal{P}(n)$. For $A, B \in \mathcal{P}(n)$, let the *mixed order* to be a total order $<_M$ given by

$$A <_M B \quad \Longleftrightarrow \quad \begin{cases} \max(A) > \max(B) & \text{or} \\ \max(A) = \max(B) & \text{and} \quad \max(A \triangle B) \in B. \end{cases}$$

Then $\mathcal{M}(n,m)$ is the initial segment of size m of the mixed order on $\mathcal{P}(n)$. (On the set of all finite subsets of $\mathbb{N}$ the mixed order does not have a smallest element, so does not have finite initial segments.)

It can easily be seen that for each n with $2^n \geqslant m$, the families $\mathcal{P}(n) \setminus \mathcal{M}(n, 2^n - m)$ are identical. We define $\mathcal{F}(m)$ to be $\mathcal{P}(n) \setminus \mathcal{M}(n, 2^n - m)$ for any n with $2^n \geqslant m$. If $k = \lceil \log_2(m) \rceil$, we can write $\mathcal{F}(m) = \mathcal{P}(k-1) \cup \{A + k\colon A \in \mathcal{U}\}$, where $\mathcal{U} = \mathcal{P}(k-1) \setminus \mathcal{I}(2^k - m)$ is an up-set contained in $\mathcal{P}(k-1)$. Then we shall prove the following result.

Theorem 3. *For $n, m \geqslant 1$ and $2^n \geqslant m$ we have*

$$f(n, m) = f(m) = \| \mathcal{F}(m) \|, \tag{4}$$

i.e. if $\mathcal{F} \subset \mathcal{P}(n)$ is a union-closed family and $|\mathcal{F}| = m$ then $\| \mathcal{F} \| \geqslant \| \mathcal{F}(m) \|$. Equality holds when $\mathcal{F} = \mathcal{F}(m)$.

3. Proof of Theorem 3

As in [16], we consider applying up-compressions to a union-closed family. To perform an up-compression in direction i, we add i to any set A of $\mathcal{A}$ for which $A + i$ is not already in $\mathcal{A}$. Formally, given a family $\mathcal{A} \subset \mathcal{P}(n)$, an element $i \in [n]$, and a set $A \in \mathcal{A}$, define

$$u_{(\mathcal{A}, i)}(A) = \begin{cases} A: & A + i \in \mathcal{A}, \\ A + i: & A + i \notin \mathcal{A}. \end{cases}$$

Now, define the *up-compression of $\mathcal{A}$ in direction i* by

$$u_i(\mathcal{A}) = \{ u_{(\mathcal{A}, i)}(A) : A \in \mathcal{A} \}.$$

Also, define $\mathcal{A}_i$ by $\mathcal{A}_i = u_i \ldots u_1(\mathcal{A})$; in particular, $\mathcal{A}_0 = \mathcal{A}$. We then define $u(\mathcal{A}) = \mathcal{A}_n$, and for $A \in \mathcal{A}$ define $A_i = u_{(\mathcal{A}_{i-1}, i)} \ldots u_{(\mathcal{A}_1, 2)} u_{(\mathcal{A}, 1)}(A)$: this is the "image of A" in $\mathcal{A}_i$, so that $\mathcal{A}_i = \{ A_i : A \in \mathcal{A} \}$. Also, occasionally we write $u_{\mathcal{A}}(A)$ for A_n.

We shall make use of a result of Reimer [16]; for the sake of completeness, we prove it here.

Theorem 4. *Let $\mathcal{A} \subset \mathcal{P}(n)$. Then*

(i) *$u(\mathcal{A})$ is an up-set;*
(ii) *if $\mathcal{A}$ is union-closed, the family $\mathcal{A}_i$ is union-closed for each $0 \leqslant i \leqslant n$;*
(iii) *if $\mathcal{A}$ is union-closed, the cubes $[A, u_{\mathcal{A}}(A)]$ and $[B, u_{\mathcal{A}}(B)]$ are disjoint for A and B distinct sets in $\mathcal{A}$.*

Proof. (i) This is well known and easily checked for any family $\mathcal{A}$. In fact, all that has to be shown is that if $1 \leqslant i < j \leqslant n$ then $u_i(\mathcal{A}_j) = \mathcal{A}_j$.

(ii) It suffices to show that $\mathcal{A}_1 = u_1(\mathcal{A})$ is union-closed; then by induction each $\mathcal{A}_i$ is union-closed also. Now, suppose A and B are sets in $\mathcal{A}_1$ — our task to show $A \cup B$ is also in $\mathcal{A}_1$. If $1 \in A$ then either A or $A - 1$ is in $\mathcal{A}$, and either B or $B - 1$ is also in $\mathcal{A}$. Hence either $A \cup B$ or $A \cup B - 1$ is in $\mathcal{A}$, as this family is union-closed. In either case we have $A \cup B \in u_1(\mathcal{A}) = \mathcal{A}_1$. If, on the other hand, $1 \notin A \cup B$, the sets A, B, $A + 1$ and $B + 1$ must all appear in $\mathcal{A}$ and $\mathcal{A}_1$. Hence in $\mathcal{A}$ we have the sets $A \cup B$ and $A \cup B + 1$, since $\mathcal{A}$ is union-closed. But then these sets are in $\mathcal{A}_1$ also, so $A \cup B = u_{(\mathcal{A}, 1)}(A \cup B) \in \mathcal{A}_1$.

(iii) In proving this assertion we may assume by symmetry that $B \not\subset A$. Now, we claim that for all $0 \leqslant i \leqslant n$, $A_i \cup B \in \mathcal{A}_i$. For $i = 0$ this is immediate from $\mathcal{A}$ being union-closed. Now, let $1 \leqslant k \leqslant n$, and assume that $A_{k-1} \cup B \in \mathcal{A}_{k-1}$. To prove our claim, we consider three cases.

If $k \in A_{k-1}$ then $A_k = A_{k-1}$, and $k \in A_{k-1} \cup B$. Hence $A_k \cup B = A_{k-1} \cup B \in \mathcal{A}_k$, as required.

If $k \notin A_{k-1}$ but $k \in A_k$, then $A_{k-1} \cup B \in \mathcal{A}_{k-1}$ is enough to guarantee $A_k \cup B = A_{k-1} \cup B \cup \{k\} \in \mathcal{A}_k$.

Finally, if $k \notin A_k$ then we have both A_k and $A_k + k$ in $\mathcal{A}_{k-1}$. Hence by the induction hypothesis both $A_k \cup B$ and $A_k \cup B + k$ are in $\mathcal{A}_{k-1}$ (although they may be the same set). This is enough to guarantee $A_k \cup B \in \mathcal{A}_k$, as required.

Consequently, if $B \not\subset A$ in $\mathcal{A}$, then $B \not\subset A_i$ for any i. Indeed, suppose $B \subset A_i$, and take i minimal such that this holds. Then we must have $A_{i-1} = A_i - i$, $i \in B$ and $A_i \notin \mathcal{A}_{i-1}$. But then by the above we have $A_{i-1} \cup B = A_i \in \mathcal{A}_{i-1}$, a contradiction.

But if the cubes $[A, u_{\mathcal{A}}(A)]$ and $[B, u_{\mathcal{A}}(B)]$ intersect, then we have $B \subset A_n$. Since we have shown this cannot happen, these cubes are disjoint. $\quad\square$

Using this result, we can give the following sufficient condition for a union-closed family to have the smallest possible total size. To state this result, for $A \in \mathcal{A}$ we set $r_{\mathcal{A}}(A) = |u_{\mathcal{A}}(A) \setminus A|$: this is the total number of elements added to A by our compressions.

Lemma 5. *Suppose that $\mathcal{A} \subset \mathcal{P}(n)$ is a union-closed family of m elements such that the following hold*:

(i) *there exists an integer k so that for each $A \in \mathcal{A}$, $k \leqslant r_{\mathcal{A}}(A) \leqslant k+1$;*
(ii) $\bigcup_{A \in \mathcal{A}} [A, u_{\mathcal{A}}(A)] = \mathcal{P}(n)$;
(iii) $\|u(\mathcal{A})\|$ *is the minimal total size of an up-set with m elements in $\mathcal{P}(n)$.*

Then $f(n, m) = \|\mathcal{A}\|$; that is, $\mathcal{A}$ has the smallest possible total size for a union-closed family of m sets in $\mathcal{P}(n)$.

Proof. Let $\mathcal{A}'$ be another union-closed family of m elements in $\mathcal{P}(n)$ — our task is to show that $\|\mathcal{A}'\| \geqslant \|\mathcal{A}\|$. Note that from the second condition of the lemma, together with Theorem 4, we have that $\sum_{A \in \mathcal{A}'} 2^{r_{\mathcal{A}'}(A)} \leqslant \sum_{A \in \mathcal{A}} 2^{r_{\mathcal{A}}(A)} = 2^n$. Together with the first condition of the lemma, this gives $\sum_{A \in \mathcal{A}'} r_{\mathcal{A}'}(A) \leqslant \sum_{A \in \mathcal{A}} r_{\mathcal{A}}(A)$. Also, the third condition of the lemma implies $\|u(\mathcal{A}')\| \geqslant \|u(\mathcal{A})\|$. Hence we have

$$\|\mathcal{A}'\| = \|u(\mathcal{A}')\| - \sum_{A \in \mathcal{A}'} r_{\mathcal{A}'}(A) \geqslant \|u(\mathcal{A})\| - \sum_{A \in \mathcal{A}} r_{\mathcal{A}}(A) = \|\mathcal{A}\|,$$

as required. $\quad\square$

Lemma 6. *Let $\mathcal{A} = \mathcal{F}(m)$, viewed as a subset of $\mathcal{P}(n)$. Then $\mathcal{A}$ satisfies all the conditions of Lemma 5.*

Proof. Let $k = \lceil \log_2(m) \rceil$, and write $\mathcal{F}(m) = \mathcal{P}(k-1) \cup \{A + k : A \in \mathcal{U}\}$, where $\mathcal{U}$ is an up-set contained in $\mathcal{P}(k-1)$. For a set $S \in \mathcal{A}$, it is easy to see that

$$u_{\mathcal{A}}(S) = \begin{cases} S \cup [k, n]: & k \notin S, \ S \notin \mathcal{U}, \\ S \cup [k+1, n]: & \text{otherwise.} \end{cases}$$

Hence for each $A \in \mathcal{A}$ we have $n - k \leqslant r_{\mathcal{A}}(A) \leqslant n - k + 1$, and so the first condition of Lemma 5 is satisfied.

For the second condition, consider $S \in \mathcal{P}(n)$. First, suppose $k \notin S$. Then let $S = S_1 \cup S_2$, with $S_1 \subset [1, k-1]$, $S_2 \subset [k+1, n]$. Now, $S_1 \in \mathcal{A}$, and $u_{\mathcal{A}}(S_1)$ is either $S_1 \cup [k+1, n]$ or $S_1 \cup [k, n]$. In either case, $S \in [S_1, u_{\mathcal{A}}(S_1)]$.

Next suppose $k \in S$. Let $S = S_1 \cup S_2 \cup \{k\}$, again with $S_1 \subset [1, k-1]$, $S_2 \subset [k+1, n]$. If $S_1 \cup \{k\} \in \mathcal{A}$ then $u_{\mathcal{A}}(S_1 \cup \{k\}) = S_1 \cup [k+1, n]$, and so $S \subset [S_1 \cup \{k\}, u_{\mathcal{A}}(S_1 \cup \{k\})]$. If $S_1 \cup \{k\} \notin \mathcal{A}$ then $S_1 \in \mathcal{A}$ and $u_{\mathcal{A}}(S_1) = S_1 \cup [k, n]$. In this case we have $S \in [S_1, u_{\mathcal{A}}(S_1)]$.

For the third condition, note that $u(\mathcal{A}) = \{S: [k, n] \subset S\} \cup \{S \cup [k+1, n]: S \in \mathcal{U}\}$. Recalling that $\mathcal{U}$ is the complement of an initial segment of colex in $\mathcal{P}(k-1)$, let $D_{\max}$ be the maximal set in this initial segment of colex. Then

$$\mathcal{P}(n) \setminus u(\mathcal{A}) = \{B \in \mathbb{N}^{(<\infty)}: B < D_{\max} \cup [k+1, n]\} = \mathcal{I}(2^n - m).$$

So $\|\mathcal{P}(n) \setminus u(\mathcal{A})\|$ is maximal over down-sets with $2^n - m$ elements, and hence $\|u(\mathcal{A})\|$ is minimal over up-sets of $\mathcal{P}(n)$ with m elements. $\quad\square$

Putting together Lemmas 5 and 6, we find that $f(n, m) = \|\mathcal{F}(m)\|$, proving Theorem 3.

4. Extremal examples for Theorem 3

In this section, we examine the proof of Theorem 3 to determine the extremal families for $f(m)$. The following lemma shows that in fact the conditions of Lemma 5 are not only sufficient but also necessary for a union-closed family $\mathcal{A}$ of m sets to have $\|\mathcal{A}\| = f(m)$.

Lemma 7. *Suppose $\mathcal{A} \subset \mathcal{P}(n)$ is a union-closed family with $|\mathcal{A}| = m$ and $\|\mathcal{A}\| = f(m)$. Then $\mathcal{A}$ satisfies the conditions of Lemma 5; that is*

(i) *there exists an integer k so that for each $A \in \mathcal{A}$, $k \leqslant |u_{\mathcal{A}}(A) \setminus A| \leqslant k + 1$;*
(ii) $\bigcup_{A \in \mathcal{A}} [A, u_{\mathcal{A}}(A)] = \mathcal{P}(n)$;
(iii) *$\|u(\mathcal{A})\|$ is the minimal total size of an up-set with m elements in $\mathcal{P}(n)$.*

Proof. Just as in the proof of Theorem 3, if we set $\mathcal{A}' = \mathcal{F}(m)$ we have $\sum_{A \in \mathcal{A}} r_{\mathcal{A}}(A) \leqslant \sum_{A \in \mathcal{A}'} r_{\mathcal{A}'}(A)$, and $\|u(\mathcal{A})\| \geqslant \|u(\mathcal{A}')\|$. If either of the first two conditions of the lemma fail to hold for $\mathcal{A}$, then the first of these inequalities is strict. If the third fails to hold, then the second is strict. In either case, $\|\mathcal{A}'\| > \|\mathcal{A}\|$. $\quad \square$

For a union-closed family $\mathcal{A}$, we define $g(\mathcal{A})$ to be the groundset of $\mathcal{A}$; that is $g(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$. Then, making use of Lemma 7, we can put the following conditions on a union-closed family of extremal size:

Lemma 8. *Suppose that $\mathcal{A}$ is union-closed, with $|\mathcal{A}| = m$ and $\|\mathcal{A}\| = f(m)$. Let $n_0 = \lceil \log_2 m \rceil$. Then $|g(\mathcal{A})| = n_0$, and there is a subset $R \subset g(\mathcal{A})$ with $|R| = n_0 - 1$ such that $\mathcal{P}(R) \subset \mathcal{A}$. Let $g(\mathcal{A}) \setminus R = \{x\}$. Then $\{A \in \mathcal{P}(R): A + x \in \mathcal{A}\}$ is an up-set in $\mathcal{P}(R)$ of size $m - 2^{k-1}$, having the minimal possible total size for such an up-set.*

Proof. Suppose $|g(\mathcal{A})| = n$; then we may assume $g(\mathcal{A}) = [n]$. View $\mathcal{A}$ as a subset of $\mathcal{P}(n)$. Since $\mathcal{A}$ satisfies the first condition of Lemma 7, there must be some k such that for each $A \in \mathcal{A}$ we have $k \leqslant |u_{\mathcal{A}}(A) \setminus A| \leqslant k + 1$. Also, we must have $\bigcup_{A \in \mathcal{A}} [A, u(A)] = \mathcal{P}(n)$; and hence we must have $2^k |\mathcal{A}| \leqslant 2^n \leqslant 2^{k+1} |\mathcal{A}|$. Thus $k = n - n_0$. However, $[n] \in \mathcal{A}$, and $u_{\mathcal{A}}([n]) = [n]$, and so $0 \in \{k, k + 1\}$, giving $n = n_0$ or $n = n_0 - 1$. The latter is impossible, as $m > 2^{n_0 - 1}$, so $n = n_0$ and $k = 0$.

For the existence of the subset R, by the union-closed property it is enough to show that $\mathcal{A}$ contains the empty set and all but at most 1 singleton. If $\emptyset \notin \mathcal{A}$, then $\emptyset \notin [A, u_{\mathcal{A}}(A)]$ for any $A \in \mathcal{A}$, contradicting the second condition of Lemma 7. Now, suppose some singleton $\{i\}$ is not in $\mathcal{A}$. Then from the second condition of Lemma 7, there exists $A \in \mathcal{A}$ with $\{i\} \in [A, u_{\mathcal{A}}(A)]$. Thus $A \subset \{i\}$, and so $A = \emptyset$. If there are two such singletons i_1 and i_2, then $\{i_1, i_2\} \subset u_{\mathcal{A}}(\emptyset)$, and so $|u_{\mathcal{A}}(\emptyset) \setminus \emptyset| \geqslant 2$. This contradicts the first condition of Lemma 7, since $k = 0$.

Now, taking $[n] \setminus R = \{x\}$, let $S = \{A \subset \mathcal{P}(R): A + x \in \mathcal{A}\}$. If $R_1 \in S$ and $R_1 \subset R_2$ with $R_2 \in \mathcal{P}(R)$, then $R_2 \setminus R_1 \in \mathcal{P}(R)$, and hence $R_2 \setminus R_1 \in \mathcal{A}$. Thus $(R_1 + x) \cup (R_2 \setminus R_1) = R_2 + x \in \mathcal{A}$, and so R_2 is also in S. Hence S is an up-set in $\mathcal{P}(R)$. If S does not have minimal total size for an up-set of $|S|$ sets in $\mathcal{P}(R)$, then let S' be another up-set of $|S|$ sets in $\mathcal{P}(R)$ with $\|S'\| < \|S\|$. Let $\mathcal{A}' = \mathcal{P}(R) \cup \{A + x: A \in S'\}$. Then $\mathcal{A}'$ is union-closed and $|\mathcal{A}'| = |\mathcal{A}|$, but $\|\mathcal{A}'\| < \|\mathcal{A}\|$, which is a contradiction. $\quad \square$

In fact, this is enough to show that the extremal families $\mathcal{F}(m)$ are unique up to isomorphism, since it is a consequence of the Kruskal–Katona theorem that the up-set in $\mathcal{P}(n)$ of m sets with smallest total size is unique.

5. Quantitative bounds

An interesting implication of Theorem 3 is that $\|\mathcal{A}\| \geqslant |\mathcal{A}| n/2$ if $|\mathcal{A}|$ is large enough. Czédli, Maróti and Schmidt [3] proved that $\|\mathcal{I}(m)\| \leqslant m(n/2 - 1)$ exactly when $m < 2^n/3$. Since for $m \leqslant 2^{n-1}$ we have $\|\mathcal{M}(n, m)\| = \|\mathcal{I}(m)\| + m$, this will yield the union-closed conjecture for families $\mathcal{A}$ with at least $\frac{2}{3} 2^n$ elements. We are able to give a somewhat simpler proof of this result, making use of the following simple upper and lower bounds on $\|\mathcal{I}(m)\|$.

Lemma 9. *For all positive integers m,*

$$m(\log_2 m - 1)/2 < \left\| \mathcal{I}(m) \right\| \leqslant m(\log_2 m)/2. \tag{5}$$

Equality in the upper bound of (5) is only achieved for $m = 2^k$.

Proof. We proceed by induction on m. For $m = 1$ we have equality in the upper bound. For $m \geqslant 2$, write $m = k + \ell$, where $k = 2^r$ is a power of 2 and $0 \leqslant \ell < k$. If $\ell = 0$, then $\mathcal{I}(m) = \mathcal{P}(r)$, and we get equality in the upper bound of (5), so we shall assume $\ell > 0$. We note that $\mathcal{I}(m)$ consists of $\mathcal{P}(r)$, together with $\{A \cup \{r+1\}: A \in \mathcal{I}(\ell)\}$. From this, we have

$$\|\mathcal{I}(m)\| = k(\log_2 k)/2 + \|\mathcal{I}(\ell)\| + \ell.$$

Let us start with the upper bound of (5). By the induction hypothesis we have that

$$\|\mathcal{I}(m)\| \leqslant k(\log_2 k)/2 + \ell\big((\log_2 \ell)/2 + 1\big),$$

so it is enough to show that, for $0 < \ell < k$,

$$k(\log_2 k)/2 + \ell\big((\log_2 \ell)/2 + 1\big) < (k + \ell)\big(\log_2(k + \ell)\big)/2.$$

This is equivalent to

$$2^{2\ell} k^k \ell^\ell < (k + \ell)^{k+\ell},$$

i.e.

$$2^{2\ell} \ell^\ell < \left(\frac{k + \ell}{k}\right)^k (k + \ell)^\ell. \tag{6}$$

For fixed $\ell > 0$, the right hand side of (6) is increasing for $k \geqslant \ell$, and we have equality when $k = \ell$, so the upper bound of (5) holds with strict inequality for all $k > \ell$.

For the lower bound of (5), we proceed similarly. We again write $m = k + \ell$, with k a power of 2 and $0 \leqslant \ell < k$. If $\ell = 0$, the result is trivial, so assume $\ell > 0$. By the bound from the induction hypothesis, the lower bound in (5) follows if

$$k(\log_2 k)/2 + \ell\big((\log_2 \ell)/2 + 1/2\big) > (k + \ell)\big((\log_2(k + \ell))/2 - 1/2\big).$$

Writing $\ell = rk$, this simplifies to

$$r \log_2 r + 1 + 2r > (1 + r)\log_2(1 + r). \tag{7}$$

It is easy to verify that (7) holds for $0 < r \leqslant 1$, implying the lower bound in (5). $\quad\square$

Lemma 10. *For $m, n \geqslant 1$, the inequality $\|\mathcal{I}(m)\| \leqslant m(n/2 - 1)$ holds exactly when $m \leqslant \lfloor 2^n/3 \rfloor$.*

Proof. We proceed by induction on n. Since the assertion is trivial for $n \leqslant 2$, in proving the induction step, we may assume that $n \geqslant 3$. First, if $m \geqslant 2^{n-1}$ then from Lemma 9 we have $\|\mathcal{I}(m)\| > m((\log_2 m)/2 - 1/2) \geqslant m(n/2 - 1)$. If $m \leqslant 2^{n-3}$, then again from Lemma 9 we have $\|\mathcal{I}(m)\| \leqslant m((\log_2 m)/2) \leqslant m(n/2 - 1)$. Hence, we may assume that $m = 2^{n-2} + m'$ for some $0 < m' < 2^{n-2}$, and so

$$\|\mathcal{I}(m)\| = 2^{n-2}(n-2)/2 + \|\mathcal{I}(m')\| + m'.$$

From this we see that $\|\mathcal{I}(m)\| \leqslant m(n/2 - 1)$ if and only if $\|\mathcal{I}(m')\| \leqslant m'(n/2 - 2)$. By the induction hypothesis, this is true if and only if $m' \leqslant \lfloor 2^{n-2}/3 \rfloor$, which is in turn true if and only if $m \leqslant \lfloor 2^n/3 \rfloor$. $\quad\square$

Corollary 11. *If $\mathcal{A} \subset \mathcal{P}(n)$ is a union-closed family with $|\mathcal{A}| > \frac{2}{3} 2^n$, then $\|\mathcal{A}\| \geqslant n|\mathcal{A}|/2$, and in particular the union-closed conjecture holds for $\mathcal{A}$.*

Proof. This follows from Theorem 3 and Lemma 10. Indeed, from Theorem 3 we have

$$\|\mathcal{A}\| \leqslant \|\mathcal{F}(m)\|$$
$$= \|\mathcal{P}(n) \setminus \mathcal{M}(n, 2^n - m)\|,$$

so it suffices to show that $\|\mathcal{M}(n, 2^n - m)\| \leqslant n(2^n - m)/2$. This follows from Lemma 10, since $\|\mathcal{M}(n, 2^n - m)\| = \|\mathcal{I}(2^n - m)\| + (2^n - m)$, and $2^n - m < \frac{1}{3} 2^n$. $\quad\square$

6. Restricted union-closed families

In this section we prove results about the minimal total size of a union-closed family in $\mathcal{P}(n)$ with certain restrictions, which are motivated by the fact that the family $\mathcal{F}(m)$ is not "truly" in $\mathcal{P}(n)$ for $n > \lceil \log_2(m) \rceil$, in the sense that no element of $\mathcal{F}(m)$ contains n.

Theorem 12. *Let $f_r(n, m)$ be defined by*

$$f_r(n, m) = \min(\|\mathcal{A}\|),$$

where the minimum is taken over all union-closed families in $\mathcal{P}(n)$ which consist of m sets and contain $[n]$. Then

$$f_r(n, m) = \begin{cases} f(m): & m > 2^{n-1}, \\ f(m-1) + n: & m \leqslant 2^{n-1}. \end{cases}$$

The extremal examples are $\mathcal{F}(m)$ when $m > 2^{n-1}$ and $\mathcal{F}(m-1) \cup \{[n]\}$ when $m \leqslant 2^{n-1}$.

We shall prove this later, after having proved a result about families satisfying a different restriction, that of point-separation.

We call a family $\mathcal{A}$ *point-separating* if for all i and j in $[n]$ there is a set $A \in \mathcal{A}$ with $|A \cap \{i, j\}| = 1$. In studying UC-families, we may restrict our attention to families $\mathcal{F}$ in $\mathcal{P}(n)$ which are point-separating and for which $d_{\mathcal{F}}(i) \geqslant 1$ for each $1 \leqslant i \leqslant n$; we call such families *normal*. Note that if $\mathcal{A} \subset \mathcal{P}(n)$ is a normal union-closed family, $[n] = \bigcup_{A \in \mathcal{A}} A$ is a set of $\mathcal{A}$.

Theorem 13. *Let $f_p(n, m)$ be defined by*

$$f_p(n, m) = \min(\|\mathcal{A}\|),$$

where the minimum is taken over all normal union-closed families in $\mathcal{P}(n)$ which consist of m sets. Let $\mathcal{F}_p(n, k)$ be given by

$$\mathcal{F}_p(n, k) = \mathcal{F}(k) \cup \{[i]: 1 \leqslant i \leqslant n\}.$$

Then for any integers m and n with $n + 1 \leqslant m \leqslant 2^n$ there exists k such that $|\mathcal{F}_p(n, k)| = m$, and then we have

$$f_p(n, m) = \|\mathcal{F}_p(n, k)\|.$$

In fact, we shall use Theorem 13 to prove Theorem 12. To prove Theorem 13, we shall again consider applying up-compressions to our family; in the case where $\mathcal{A}$ is normal we can give some additional restrictions on the process.

Lemma 14. *Suppose that $\mathcal{A}$ is a normal union-closed family in $\mathcal{P}(n)$. Then there are distinct sets A_0, A_1, …, A_n in $\mathcal{A}$ such that $|r_{\mathcal{A}}(A_0)| = 0$, and for $1 \leqslant i \leqslant n$ we have $r_{\mathcal{A}}(A_i) \leqslant i - 1$.*

Proof. Without loss of generality, we may assume that $d_{\mathcal{A}}(i) \leqslant d_{\mathcal{A}}(j)$ if $i \leqslant j$. Now, for $1 \leqslant i \leqslant n$ let $A_i = \bigcup_{i \notin A \in \mathcal{A}} A$. Since $\mathcal{A}$ is union-closed, this is a set of $\mathcal{A}$. If $j > i$, then $d_{\mathcal{A}}(j) \geqslant d_{\mathcal{A}}(i)$. Since $\mathcal{A}$ is point-separating, i and j cannot be in precisely the same sets, and so $j \in A_i$. Hence $|A_i| \geqslant n - i$. Also, $|u_{\mathcal{A}}(A_i)| \leqslant n - 1$, since $[n] \in \mathcal{A}$. Thus $r_{\mathcal{A}}(A_i) \leqslant (n - 1) - (n - i) = i - 1$. To show that the A_i are distinct, if $i < j \in [n]$ then $j \in A_i$ but $j \notin A_j$. Setting $A_0 = [n]$, we have $A_0, A_1, \ldots, A_n$ as required. □

Lemma 15. *Let $\mathcal{A}$ be a normal union-closed family in $\mathcal{P}(n)$, and suppose $d_{\mathcal{A}}(i) \leqslant d_{\mathcal{A}}(j)$ for $i \leqslant j$. Then $u(\mathcal{A})$ contains all sets in $\mathcal{P}(n)$ of size $n - 1$.*

Proof. As in the proof of Lemma 14, we have sets A_i for $1 \leqslant i \leqslant n$ so that $i \notin A_i$, but $j \in A_i$ for all $j > i$. Now, after applying $u_{i-1} \ldots u_1$ to $\mathcal{A}$ to reach $\mathcal{A}_{i-1}$ we have $[n] - i = A_i \cup [1, i-1] \in \mathcal{A}_{i-1}$. The remaining compressions $u_n \ldots u_i$ do not affect this set, as $[n] \in \mathcal{A}$, and so $[n] - i \in u(\mathcal{A})$. □

Now, analogously to Lemma 5 we can give conditions under which a union-closed point-separating family in $\mathcal{P}(n)$ with m sets is extremal. For a set S and an integer r we define $S^{(r)}$ to be $\{T \subset S: |T| = r\}$.

Lemma 16. *Suppose that $\mathcal{A}$ is a normal union-closed family of m sets in $\mathcal{P}(n)$, and the following hold*:

(i) *Let $n_i = |\{A \in \mathcal{A}: r_\mathcal{A}(A) = i\}|$. Then there exist $k, s \geqslant 0$ and $r > 0$ such that*

$$n_i = \begin{cases} 2: & i = 0, \\ 1: & 1 \leqslant i \leqslant k - 1, \\ r: & i = k, \\ s: & i = k + 1, \\ 0: & i > k + 1; \end{cases}$$

(ii) $\bigcup_{A \in \mathcal{A}} [A, u_\mathcal{A}(A)] = \mathcal{P}(n)$;

(iii) $\|u(\mathcal{A})\|$ *is the smallest possible total size of an up-set of m sets in $\mathcal{P}(n)$ containing $[n]^{(n-1)}$*.

Then $f_p(n, m) = \|\mathcal{A}\|$; that is, $\mathcal{A}$ has the smallest possible total size for a normal union-closed family in $\mathcal{P}(n)$.

Proof. If $\mathcal{A}'$ is another normal union-closed family with m sets in $\mathcal{P}(n)$, our task is to show that $\|\mathcal{A}'\| \geqslant \|\mathcal{A}\|$. We may assume that $d_{\mathcal{A}'}(i) \leqslant d_{\mathcal{A}'}(j)$ for $i \leqslant j$. Let $n_i' = |\{A \in \mathcal{A}': r_{\mathcal{A}'}(A) = i\}|$. Now, applying Lemma 14 to $\mathcal{A}'$, we have that for $0 \leqslant j \leqslant n$ there are at least $j + 2$ sets A of $\mathcal{A}'$ with $r(A) \leqslant j$, and so $\sum_{i=0}^{j} n_i' \geqslant j + 2$. Also, from the second condition of the lemma together with Theorem 4, we have that $\sum_{A \in \mathcal{A}'} 2^{r_{\mathcal{A}'}(A)} \leqslant \sum_{A \in \mathcal{A}} 2^{r_\mathcal{A}(A)} = 2^n$. Combining these conditions, we must have $\sum_{A \in \mathcal{A}'} r_{\mathcal{A}'}(A) \leqslant \sum_{A \in \mathcal{A}} r_\mathcal{A}(A)$. Also, applying Lemma 15, $u(\mathcal{A}')$ is an up-set containing $[n]^{(n-1)}$. Hence from the third condition of the lemma we have $\|u(\mathcal{A}')\| \geqslant \|u(\mathcal{A})\|$. Combining these inequalities, we have

$$\|\mathcal{A}'\| = \|u(\mathcal{A}')\| - \sum_{A \in \mathcal{A}'} r_{\mathcal{A}'}(A) \geqslant \|u(\mathcal{A})\| - \sum_{A \in \mathcal{A}} r_\mathcal{A}(A) = \|\mathcal{A}\|,$$

as required. □

To finish the proof of Theorem 13, we now need to show that the family $\mathcal{F}_p(n, m)$ satisfies these conditions.

Lemma 17. *For all integers m and n with $n + 1 \leqslant m \leqslant 2^n$, the family $\mathcal{F}_p(n, m)$ satisfies the conditions of Lemma 16.*

Proof. For some integer k, write $m = 2^{k-1} + m'$ with $0 < m' \leqslant 2^{k-1}$. Then we can write $\mathcal{F}_p(n, m)$ as the disjoint union

$$\mathcal{F}_p(n, m) = \mathcal{P}(k - 1) \cup \{A + k: A \in \mathcal{U}\} \cup \{[i]: k + 1 \leqslant i \leqslant n\},$$

where $\mathcal{U} = \mathcal{P}(k - 1) \setminus \mathcal{I}(2^{k-1} - m')$ is an up-set of m' elements in $\mathcal{P}(k - 1)$, which minimises $\|\mathcal{U}\|$ amongst such up-sets. For convenience, we define $\mathcal{A} = \mathcal{F}_p(n, m)$. Then it can be shown that the following describes $u(\mathcal{A})$:

$$u_{\mathcal{A}}(A) = \begin{cases} [n]: & A = [n], \\ [n] - (i+1): & A = [i], \ k \leqslant i \leqslant n-1, \\ A \cup [k+1,n]: & A \in \{B+k \colon B \in \mathcal{U}\} \setminus \{[k]\}, \\ A \cup [k+1,n]: & A \in \mathcal{U}, \\ A \cup [k,n]: & A \in \mathcal{P}(k-1) \setminus \mathcal{U}. \end{cases}$$

Hence considering $r_{\mathcal{A}}(A)$ for the various sets $A \in \mathcal{A}$ we have:

$$r_{\mathcal{A}}(A) = \begin{cases} 0: & A = [n], \\ (n-1) - i: & A = [i], \ k \leqslant i \leqslant n-1, \\ n-k: & A \in \{B+k \colon B \in \mathcal{U}\} \setminus \{[k]\}, \\ n-k: & A \in \mathcal{U}, \\ n-k+1: & A \in \mathcal{P}(k-1) \setminus \mathcal{U}, \end{cases}$$

and so

$$n_i = \begin{cases} 2: & i = 0, \\ 1: & 1 \leqslant i < n-k, \\ 2|\mathcal{U}|: & i = n-k, \\ 2^{n-1} - |\mathcal{U}|: & i = n-k+1, \\ 0: & i > n-k+1. \end{cases}$$

This sequence satisfies Condition 1 of Lemma 16. For Condition 2, let $S \in \mathcal{P}(n)$; we wish to find $A \in \mathcal{A}$ with $S \in [A, u_{\mathcal{A}}(A)]$. We break this down into three cases, depending on the intersection of S with $[1,k]$. Firstly, if $[1,k] \subset S$ then let i be minimal with $i \notin S$. Then $u_{\mathcal{A}}([i-1]) = [n] - i$, and so $S \in [[i-1], u_{\mathcal{A}}([i-1])]$, as required. Secondly, if $k \notin S$, then write $S = S_1 \cup S_2$, with $S_1 \subset [1, k-1]$ and $S_2 \subset [k+1, n]$. Then $u_{\mathcal{A}}(S_1) = S_1 \cup [k+1, n]$, and so $S \in [S_1, u_{\mathcal{A}}(S_1)]$. Finally, if $[1,k] \not\subset S$ and $k \in S$, write $S = S_1 \cup \{k\} \cup S_2$, with $S_1 \subset [1, k-1]$ and $S_2 \subset [k+1, n]$. If $S_1 \in \mathcal{U}$, then $u_{\mathcal{A}}(S_1 + k) = (S_1 + k) \cup [k+1, n]$, and $S \in [S_1 + k, u_{\mathcal{A}}(S_1 + k)]$. If $S_1 \notin \mathcal{U}$, then $u(S_1) = S_1 \cup [k, n]$, and $S \in [S_1, u_{\mathcal{A}}(S_1)]$.

For Condition 3, we have

$$u(\mathcal{A}) = \{A \cup [k,n] \colon A \in \mathcal{P}(n-1)\} \cup \{A \cup [k+1,n] \colon A \in \mathcal{U}\} \cup [n]^{(n-1)}.$$

The first two terms in the union form an up-set in $\mathcal{P}(n)$ of m' sets of minimal total size. It is an easy consequence of the Kruskal–Katona theorem that the union of this set with $[n]^{(n-1)}$ gives an up-set of minimal total size containing $[n]^{(n-1)}$. $\quad\square$

Putting together Lemmas 16 and 17, we have proved that every family $\mathcal{F}_p(n,k)$ is extremal − that is, if $|\mathcal{F}_p(n,k)| = m$ then $\|\mathcal{F}_p(n,k)\| = f_p(n,m)$. To complete the proof of Theorem 13, it remains only to show that for $n+1 \leqslant m \leqslant 2^n$ there is some choice of k with $|\mathcal{F}_p(n,k)| = m$. However, this is immediate, since $|\mathcal{F}_p(n,k)| - |\mathcal{F}_p(n,k-1)| \leqslant 1$ for $1 \leqslant k \leqslant 2^n$, $|\mathcal{F}_p(n,0)| = n$, and $|\mathcal{F}_p(n,2^n)| = 2^n$.

6.1. Proof of Theorem 12

We now use Theorem 13 to prove Theorem 12. Let $\mathcal{A}$ be a union-closed family in $\mathcal{P}(n)$, containing $[n]$. If $|\mathcal{A}| > 2^{n-1}$, then $\|\mathcal{A}\| \geqslant f(m)$, so we are done. If $|\mathcal{A}| \leqslant 2^{n-1}$, we form an equivalence relation on $[n]$ by setting

$$i \equiv j \iff \{A \in \mathcal{A} \colon i \in A\} = \{A \in \mathcal{A} \colon j \in A\}.$$

Let the equivalence classes of this relation be $C_1, \ldots, C_k$ for some $k \leqslant n$. Define a family $\mathcal{A}'$ in $\mathcal{P}(k)$ by

$$A \in \mathcal{A}' \iff \bigcup_{i \in A} C_i \in \mathcal{A}.$$

$\mathcal{A}'$ is a union-closed family of m sets in $\mathcal{P}(k)$, and it is point-separating. We now split into two cases. If $m \leqslant 2^{k-1}$ then for some $m' < 2^{k-1}$ we have $|\mathcal{F}_p(k, m')| = m$, and then

$$\|\mathcal{A}'\| \geqslant \|\mathcal{F}_p(k, m')\| \geqslant f(m-1) + k.$$

However, since every element of $[n]$ appears in a set of $\mathcal{A}$, $\|\mathcal{A}\| \geqslant \|\mathcal{A}'\| + (n - k) \geqslant f(m-1) + n$, as required.

On the other hand, if $m > 2^{k-1}$ we write $m = 2^{k-1} + m'$. Then every element of $[k]$ appears at least m' times in $\mathcal{A}'$, and so every element of $[n]$ appears at least m' times in $\mathcal{A}$. Hence we have

$$
\begin{aligned}
\|\mathcal{A}\| &\geqslant \|\mathcal{A}'\| + (n-k)m' \\
&\geqslant f(m) + (n-k)m'.
\end{aligned}
$$

Also, let $\mathcal{F}(m) \setminus \mathcal{F}(m-1) = \{S\}$. Then since $m = 2^{k-1} + m'$, the set S is in $\{k + A \colon A \in \mathcal{U}\}$, for $\mathcal{U}$ an up-set of $\mathcal{P}(k-1)$ with $|\mathcal{U}| = m'$. Hence we have $|S| \geqslant k - \log_2(m')$, and so

$$
\begin{aligned}
\|\mathcal{A}\| &\geqslant f(m) + (n-k)m' \\
&= f(m-1) + |S| + (n-k)m' \\
&\geqslant f(m-1) + k - \log_2(m') + (n-k)m' \\
&= f(m-1) + n + (n-k)(m'-1) - \log_2(m') \\
&\geqslant f(m-1) + n.
\end{aligned}
$$

The last line follows since $(n-k)(m'-1) - \log_2(m') \geqslant 0$ for all $m' \geqslant 1$ and $(n-k) \geqslant 1$. This completes the proof of Theorem 12. $\quad\square$

Theorem 12 is enough to verify a conjecture of Wójcik [23], which states that the smallest average size for a union-closed family containing an n-set is achieved by the family $\mathcal{P}(k) \cup \{[n]\}$, for either $k = \lfloor \log_2(n) \rfloor$ or $k = \lceil \log_2(n) \rceil$. Given Theorem 12, to prove this result one only needs to minimise $f_r(n, m)/m$ over m. To do so is a simple but tedious calculation, which we shall omit.

7. Union-closed and rooted families

We call a family $\mathcal{B} \subset \mathcal{P}(n)$ *rooted* if for each $B \in \mathcal{B}$ there is an $i \in B$ such that $\{B' \colon i \in B' \subset B, |B'| = |B| - 1\} \subset \mathcal{B}$. Also, we call $\mathcal{B} \subset \mathcal{P}(n)$ *simply rooted* if for each $\varnothing \neq B \in \mathcal{B}$ we have $[\{i\}, B] \subset \mathcal{B}$ for some $i \in B$. These definitions are motivated by the following simple observation.

Lemma 18. *For a family $\mathcal{A} \subset \mathcal{P}(n)$, $\mathcal{A}$ is union-closed if and only if $\mathcal{B} = \mathcal{P}(n) \setminus \mathcal{A}$ is simply rooted.*

Proof. The family $\mathcal{A}$ is union-closed if and only if $\bigcup_{B' \subset B, B' \in \mathcal{A}} B' \neq B$ whenever $B \in \mathcal{B}$. This is true precisely when for each $B \in \mathcal{B}$ there is an $i \in B$ with $[\{i\}, B] \subset \mathcal{B}$. $\quad\square$

Also, from the proof of Theorem 3 we can read out a slight strengthening for union-closed families $\mathcal{A}$ with $|\mathcal{A}| \geqslant 2^{n-1}$.

Theorem 19. *Let $\mathcal{A} \subset \mathcal{P}(n)$ be a union-closed family, and $\mathcal{B} = \mathcal{P}(n) \setminus \mathcal{A}$ with $|\mathcal{B}| = m$. Suppose the largest down-set contained in $\mathcal{B}$ is $\mathcal{D}$. Then $\|\mathcal{A}\| \geqslant \|\mathcal{P}(n)\| - \|\mathcal{I}(m)\| - m + |\mathcal{D}|$.*

Proof. Here, with u defined as at the beginning of Section 3, we have $\|u(\mathcal{A})\| \geqslant \|\mathcal{P}(n)\| - \|\mathcal{I}(m)\|$, since $u(\mathcal{A})$ is an up-set on $2^n - m$ sets. Also, for every $A \in \mathcal{A}$ we have $[A, u_{\mathcal{A}}(A)] \setminus A \subset \mathcal{B} \setminus \mathcal{D}$. Hence $\sum_{A \in \mathcal{A}} |u_{\mathcal{A}}(A) \setminus A| \leqslant |\mathcal{P}(n) \setminus (\mathcal{A} \cup \mathcal{D})| = m - |\mathcal{D}|$. Combining these inequalities,

$$
\begin{aligned}
\|\mathcal{A}\| &= \|u(\mathcal{A})\| - \sum_{A \in \mathcal{A}} |u_{\mathcal{A}}(A) \setminus A| \\
&\geqslant \|\mathcal{P}(n)\| - \|\mathcal{I}(m)\| - m + |\mathcal{D}|,
\end{aligned}
$$

as required. $\quad\square$

We use this to prove a slightly better bound on the total size of a hypothetical counterexample to the union-closed conjecture. The following is a corollary of Theorem 3.

Corollary 20. *Suppose that $\mathcal{A} \subset \mathcal{P}(n)$ is a union-closed family which is a counterexample to the union-closed conjecture; that is $d_{\mathcal{A}}(i) < |\mathcal{A}|/2$ for each $i \in [n]$. Let $\mathcal{B} = \mathcal{P}(n) \setminus \mathcal{A}$, and $m = |\mathcal{B}|$. Then $\|\mathcal{I}(m)\| > m(n/2 - 1)$.*

This follows because we must have $mn/2 < \|\mathcal{B}\| \leqslant \|\mathcal{M}(n, m)\| \leqslant \|\mathcal{I}(m)\| + m$. In fact, we can prove the following slight strengthening of Corollary 20.

Theorem 21. *Suppose that $\mathcal{A} \subset \mathcal{P}(n)$ is a union-closed family which is a counterexample to the union-closed conjecture. Let $\mathcal{B} = \mathcal{P}(n) \setminus \mathcal{A}$, and $m = |\mathcal{B}|$. Then $\|\mathcal{I}(m)\| > m(n/2 - 1 + 1/n)$.*

Proof. For each $i \in [n]$, let $\mathcal{B}_i = \{B \in \mathcal{B}: [i, B] \subset \mathcal{B}\}$, $m_{i,+} = |\{B \in \mathcal{B}: i \in B\}|$, and $m_{i,-} = m - m_{i,+} = |\{B \in \mathcal{B}: i \notin B\}|$. Also, let $\|\mathcal{B}\| = m(n/2 + r)$; since $\mathcal{A}$ is a counterexample to the union-closed conjecture, $r > 0$. We relate r and $m_{i,+}$ by

$$\sum_{i=1}^{n}(m_{i,+} - m_{i,-}) = \sum_{i=1}^{n}(2m_{i,+} - m)$$
$$= 2\|\mathcal{B}\| - nm$$
$$= 2mr.$$

Also, since by Lemma 18 the family $\mathcal{B}$ is simply rooted, $\sum_{i=1}^{n}|\mathcal{B}_i| \geqslant m$. So we have

$$\sum_{i=1}^{n}|\mathcal{B}_i| - (m_{i,+} - m_{i,-}) \geqslant m(1 - 2r).$$

In particular, there is some j with $|\mathcal{B}_j| - m_{j,+} + m_{j,-} \geqslant m(1 - 2r)/n$; without loss of generality, we may assume $j = n$. Now, we define two families of sets

$$\mathcal{B}_n^+ = \{B \subset \mathcal{P}(n - 1): B + n \in \mathcal{B}\},$$
$$\mathcal{B}_n^- = \{B \subset \mathcal{P}(n - 1): B \in \mathcal{B}\}.$$

Then, setting $\mathcal{D}_+$ to be the largest down-set contained in $\mathcal{B}_n^+$, we have $\{B - n: B \in B_n\} \subset \mathcal{D}_+$, and so $|\mathcal{D}_+| \geqslant |\mathcal{B}_n| \geqslant m_{n,+} - m_{n,-} + m(1 - 2r)/n$. This gives us

$$\|\mathcal{B}\| = \|\mathcal{B}_n^+\| + \|\mathcal{B}_n^-\| + m_{n,+}$$
$$\leqslant \|\mathcal{I}(m_{n,+})\| + m_{n,+} - |\mathcal{D}_+| + \|\mathcal{I}(m_{n,-})\| + m_{n,-} + m_{n,+}$$
$$\leqslant \|\mathcal{I}(m_{n,+})\| + \|\mathcal{I}(m_{n,-})\| + m_{n,-} - m(1 - 2r)/n + m$$
$$\leqslant \|\mathcal{I}(m)\| - m(1 - 2r)/n + m.$$

Where the last line follows from Lemma 2. Hence we have $\|\mathcal{I}(m)\| \geqslant (n/2 + r)m + m(1 - 2r)/n - m > m(n/2 - 1 + 1/n)$, as required. $\square$

Theorem 21 provides only a slight strengthening of our bound on the size of a counterexample to the union-closed conjecture − approximately, it improves $|\mathcal{B}| > 2^n/3$ to $|\mathcal{B}| > 2^n(1/3 + 2/9n)$ for $\mathcal{P}(n) \setminus \mathcal{B}$ such a counterexample.

8. Conjectures on rooted families

By Theorem 3 and Lemma 18, if $\mathcal{B}$ is a simply rooted family in $\mathcal{P}(n)$ then

$$\|\mathcal{B}\| \leqslant \|\mathcal{M}(n,m)\| \leqslant \|\mathcal{I}(m)\| + m. \tag{8}$$

We conjecture two strengthenings of this result. Firstly, we conjecture that (8) holds for a family which has only the weaker property of being rooted. Also, we note that the families $\mathcal{M}(n,m)$ for which equality is achieved in (8) are very asymmetric; for $m \leqslant 2^{n-1}$ every set in $\mathcal{B}$ contains n. This leads us to conjecture the following strengthening of (8) for simply rooted or rooted families:

Conjecture 22. *Suppose $\mathcal{B}$ is a rooted family in $\mathcal{P}(n)$. Then*

$$\|\mathcal{B}\| \leqslant \|\mathcal{I}(|\mathcal{B}|)\| + \max_{i \in [n]} d_{\mathcal{B}}(i).$$

We note that the truth of this conjecture for simply rooted families would show that if $\mathcal{P}(n) \setminus \mathcal{B}$ were a counterexample to the union-closed conjecture, then we would have $\|\mathcal{I}(|B|)\| \geqslant |\mathcal{B}|(n-1)/2$, and so $|\mathcal{B}| \geqslant (2/3)2^n$. In particular, Conjecture 22 implies that the union-closed conjecture holds for every family $\mathcal{A} \subset \mathcal{P}(n)$ with $|\mathcal{A}| > 2^n/3$.

If $[\{i\}, B] \subset \mathcal{B}$, we say that B is *simply rooted at i*. If all sets in $\mathcal{B}$ are simply rooted at the same i, then Conjecture 22 reduces to a weaker form of Theorem 3. We can also prove Conjecture 22 in the case where each set $B \in \mathcal{B}$ is simply rooted at one of two elements i and j in $[n]$. Indeed, in this case we let $\mathcal{B}_i$, $\mathcal{B}_j$ and $\mathcal{B}_{ij}$ be the families $\{B \in \mathcal{B}: \ j \notin B\}$, $\{B \in \mathcal{B}: \ i \notin B\}$ and $\{B \in \mathcal{B}: \ \{i,j\} \subset B\}$, with sizes m_i, m_j and m_{ij} respectively. Since $\{B - i: \ B \in \mathcal{B}_i\}$, $\{B - j: \ B \in \mathcal{B}_j\}$ and $\{B \setminus \{i,j\}: \ B \in \mathcal{B}_{ij}\}$ are all down-sets, we have

$$\begin{aligned}
\|\mathcal{B}\| &\leqslant \|\mathcal{I}(m_i)\| + \|\mathcal{I}(m_j)\| + \|\mathcal{I}(m_{ij})\| + m_i + m_j + 2m_{ij} \\
&\leqslant \|\mathcal{I}(m_i + m_j)\| + \max(m_i, m_j) + \|\mathcal{I}(m_{ij})\| + 2m_{ij} \\
&\leqslant \|\mathcal{I}(m)\| + \max(m_i, m_j) + m_{ij}.
\end{aligned}$$

The last inequality uses the fact that $m_{ij} \leqslant m_i + m_j$; indeed, we can remove either i or j from each set in $\mathcal{B}_{ij}$, while remaining in $\mathcal{B}$. This is exactly the statement of Conjecture 22 in this special case, as $\max(m_i, m_j) + m_{ij} = \max(d_{\mathcal{B}}(i), d_{\mathcal{B}}(j))$.

Acknowledgments

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