A note on the union-closed sets conjecture

IAN ROBERTS

School of Engineering and Information Technology
Charles Darwin University
Darwin, Northern Territory 0909
Australia
Tan Roberts@cdu.edu.au

Jamie Simpson

Department of Mathematics and Statistics
Curtin University of Technology
GPO Box U1987, Perth
Western Australia 6845
Australia
simpson@maths.curtin.edu.au

Abstract

A collection \mathcal{A} of finite sets is closed under union if $A, B \in \mathcal{A}$ implies that $A \cup B \in \mathcal{A}$. The Union-Closed Sets Conjecture states that if \mathcal{A} is a union-closed collection of sets, containing at least one non-empty set, then there is an element which belongs to at least half of the sets in \mathcal{A} . We show that if q is the minimum cardinality of $\cup \mathcal{A}$ taken over all counterexamples \mathcal{A} , then any counterexample \mathcal{A} has cardinality at least 4q-1.

A collection \mathcal{A} of finite sets is closed under union if $A, B \in \mathcal{A}$ implies that $A \cup B \in \mathcal{A}$. The Union-Closed Sets Conjecture (also called Frankl's Conjecture) states that if \mathcal{A} is a union-closed collection of sets, containing at least one non-empty set, then there is an element which belongs to at least half of the sets in \mathcal{A} . The conjecture dates from 1979 and is generally attributed to Peter Frankl [2]. Some authors place the further condition on \mathcal{A} that it does not contain the empty set. The results of this note apply to either version of the conjecture. A number of necessary conditions for a counterexample have been established, including that if \mathcal{A} is a counterexample then $|\mathcal{A}| \geq 37$ [4] and that $|\cup \mathcal{A}| \geq 12$ [1] where $\cup \mathcal{A} = \cup_{A \in \mathcal{A}} A$. See [1] and [3] and their bibliographies for more background. In this note we show that if q is the minimum cardinality of $\cup \mathcal{A}$ taken over all counterexamples then for any counterexample \mathcal{A} we have $|\mathcal{A}| \geq 4q - 1$. It is known that $q \geq 12$ [1] so this implies that a counterexample has cardinality at least 47.

We say that A is a basis set in A if it is not the union of other sets in A. Suppose that A is a counterexample with |A| = 2n + 2 for some integer n. Then no element occurs in more than n of its sets. If we remove a basis set from A, then we get a smaller collection which is also a counterexample. A counterexample of minimum size must therefore contain an odd number of sets.

Let q be the minimum cardinality of $\cup \mathcal{A}$ taken over all counterexamples \mathcal{A} . We will obtain a lower bound for $|\mathcal{A}|$ in terms of q. Henceforth \mathcal{A} will be a minimal counterexample to the conjecture with $|\mathcal{A}| = 2n + 1$. Set $S = \cup \mathcal{A}$ and note that $|S| \geq q$. We define

$$\begin{array}{lll} \mathcal{A}_x &=& \{A \in \mathcal{A} : x \in A\} \\ \mathcal{A}_{\overline{x}} &=& \{A \in \mathcal{A} : x \not\in A\} \\ C_x &=& \cup \mathcal{A}_{\overline{x}} \\ \mathcal{C} &=& \{C_x : x \in S\} \\ \mathcal{D} &=& \mathcal{A} \setminus \{S\} \setminus \mathcal{C} \\ \mathcal{D}_x &=& \{A \in \mathcal{D} : x \in A\}. \end{array}$$

Clearly, C_x is an element of \mathcal{A} for each element x of S. Since \mathcal{A} is a minimal counterexample there will be elements which occur in exactly n of the sets in \mathcal{A} (otherwise we could remove a basis set to give a smaller counterexample). We set $H = \{x : |\mathcal{A}_x| = n\}$. Let \mathcal{B} be the collection of basis sets in \mathcal{A} . If a and b are elements of \mathcal{A} , then we say that a dominates b if whenever b occurs in a set A in \mathcal{A} then a also occurs in that set. This is equivalent to saying $b \notin C_a$. We will assume that \mathcal{A} contains no mutually dominating pair of elements, since if such a pair existed then we could replace it with a single element without altering the size of \mathcal{A} . Note that by this assumption the sets C_a are distinct.

Lemma 1 If $a \notin H$, then $H \subseteq C_a$ and if $x \in H$, then $H \setminus \{x\} \subseteq C_x$.

Proof: Let $y \in H$. We show that $y \in C_a$ where $a \notin H$. Since y occurs in n sets and a occurs in fewer than n sets, there exists A in A such that $y \in A$ and $a \notin A$ which implies $y \in C_a$. It follows that $H \subseteq C_a$.

For the second part let $y \in H$, $y \neq x$. We are assuming that x and y do not dominate each other. So there exists $A \in \mathcal{A}$ such that $y \in A$, $x \notin A$ so $A \subseteq C_x$ and therefore $y \in C_x$.

Lemma 2 No set C_a is a basis set.

Proof: Suppose that C_a is a basis set. If $a \notin H$ then by Lemma 1 $\mathcal{A}\setminus\{C_a\}$ is a unionclosed collection in which no element occurs more than n-1 times, contradicting the minimality of \mathcal{A} . If, on the other hand, $a \in H$, then choose a basis set B which contains a. Then by Lemma 1 $\mathcal{A}\setminus\{B,C_a\}$ is a union-closed collection of 2n-1 sets in which no element occurs more than n-1 times, again contradicting the minimality of \mathcal{A} . **Theorem 3** If $x \in H$ and $a \notin H$, then $|\mathcal{D}_x| = n - q$ and $|\mathcal{D}_a| < n - q$.

Proof: Let $x \in H$. By Lemma 1, x occurs in all sets C_b with $b \neq x$ as well as in S, so x occurs in $|\mathcal{A}_x| - q = n - q$ sets in \mathcal{D} .

Now assume, without loss of generality, that $|\mathcal{D}_a|$ is maximal for $a \in S \setminus H$.

If there exists $b \neq a$ such that $a \notin C_b$, then b dominates a and there exists a basis set which contains b but not a. By Lemma 2 this belongs to \mathcal{D} so b occurs more often in \mathcal{D} than a which is a contradiction.

If, on the other hand, $a \in C_b$ for all $b \neq a$ then a occurs in q sets of $A \setminus D$. If a occurs in n - q or more sets in D then it occurs in at least n sets in A which means $a \in H$.

Theorem 4 If A is a minimal counterexample, then $|A| \ge 4q - 1$.

Proof: Consider \mathcal{A}_x and $\mathcal{A}_{\overline{x}}$ for some $x \in H$. Then $|\mathcal{A}_x| = n$, $|\mathcal{A}_{\overline{x}}| = n+1$ and $|\mathcal{D}_x| = n-q$ by Theorem 3. Since $\mathcal{A}_{\overline{x}}$ is union-closed, there exists an element a in at least half its sets. By Theorem 3 a occurs in at most n-q sets in \mathcal{D} . By Lemma 1, each set in $\mathcal{A} \setminus \mathcal{D}$ except C_x contains x so $\mathcal{A}_{\overline{x}} \subseteq \mathcal{D} \cup \{C_x\}$. Thus a is in at most n-q+1 sets in $\mathcal{A}_{\overline{x}}$. Hence $|\mathcal{A}_{\overline{x}}| \leq 2(n-q+1)$ which implies $n \geq 2q-1$. Thus $|\mathcal{A}| = 2n+1 \geq 4q-1$.

Corollary 5 If A is a minimal counterexample, then $|A| \ge 47$.

Proof: Bošnjak and Marković [1] have recently shown that a minimal counterexample to the conjecture has q > 12.

Acknowledgement We thank the referee for many suggestions.

References

- I. Bošnjak and P. Marković, The 11-element case of Fankl's conjecture, Electr. J. Combin. 15 (2008), #R88.
- [2] P. Frankl, Extremal set systems, in *The Handbook of Combinatorics*, MIT Press, North-Holland, 1995 (Chap. 24).
- [3] P. Marković, An attempt at Frankl's Conjecture, Publ. Inst. Math. (Beograd) (N.S.) 81 (95) (2007), 29–43.
- [4] R. Morris, FC-families and improved bounds for Frankl's conjecture, *Europ. J. Combin.* 27 (2004), 269–282.