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Union-closed families of sets

Piotr Wójcik*[,1]

Department of Discrete Mathematics, Adam Mickiewicz University, ul. Matejki 48/49, 60-769 Poznań, Poland

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Abstract

We use a lower bound on the number of small sets in an ideal to show that for each union-closed family of n sets there exists an element which belongs to at least

$$\frac{1+o(1)}{\log_2(\frac{4}{3})}\frac{n}{\log_2 n}$$

of them, provided n is large enough.

Keywords: Union-closed family; Frankl's conjecture

1. Introduction

A finite family $\mathscr{F}$ of sets is *union-closed* if for every $A, B \in \mathscr{F}$ we have also $A \cup B \in \mathscr{F}$. In this note we present some results concerning the following celebrated conjecture, stated by Frankl in 1979 [9,14].

Union-closed sets conjecture. For any finite union-closed family $\mathscr{F}$ of sets, in which at least one set is non-empty, there exists an element $x \in \bigcup \mathscr{F}$ which belongs to at least half of the sets of $\mathscr{F}$.

Despite its elementary statement, Frankl's conjecture is considered to be one of the most challenging problems in extremal set theory; there are few results in this direction (some of them are described shortly in the following section) but we still seem to be far from solving the problem. Knill observed in [2] that, if $\mathscr{F}$ is a finite union-closed family of finite sets and Y is a minimal subset of $\bigcup \mathscr{F}$ such that each non-empty set from $\mathscr{F}$ has non-empty intersection with Y, then $\{A \cap Y \mid A \in \mathscr{F} \setminus \{\emptyset\}\} = 2^Y \setminus \{\emptyset\}$.

* E-mail: wojcik@math.amu.edu.pl.

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This implies that $|Y| \leqslant \log_2(|\mathscr{F}\setminus\{\emptyset\}|+1)$, and so there exists an element in Y which belongs to at least $(|\mathscr{F}|-1)/\log_2|\mathscr{F}|$ sets of $\mathscr{F}$. In this note we show that for large n this estimate can be slightly strengthened, by a factor of $1/\log_2(\frac{4}{3}) \approx 2.40942$.

The structure of the paper is the following. In the first part of the note we state three equivalent versions of the union-closed sets conjecture (Theorem 2.1), one of which will be used for the main proof in the end. Then, in the next section, we prove a certain extremal set result about ideals (Theorem 3.1). Finally, we prove the strengthening of Knill's result mentioned above (Theorem 4.1).

2. Union-closed sets conjecture and normalized families of sets

In order to state our results we need to introduce some notation and a few definitions. Let $\mathbb{N}=\{1,2,\ldots\}$ and let $[\mathbb{N}]^{<\infty}$ be the family of all finite subsets of $\mathbb{N}$. Throughout this note, all sets and families of sets, except for $\mathbb{N}$ and $[\mathbb{N}]^{<\infty}$, are finite. Let $F=\bigcup\mathscr{F}$. For a family $\mathscr{F}$, a set $X \subseteq F$, an element $x \in F$ and an integer number k we define

$$\mathscr{F}_{\ni x}=\mathscr{F}\setminus\mathscr{F}_{\not\ni x}=\{A\in\mathscr{F} \mid x\in A\},$$

$$\mathscr{F}_{\supseteq X}=\mathscr{F}\setminus\mathscr{F}_{\not\supseteq X}=\{A\in\mathscr{F} \mid A\supseteq X\},$$

$$\mathscr{F}_{\subseteq X}=\mathscr{F}\setminus\mathscr{F}_{\not\subseteq X}=\{A\in\mathscr{F} \mid A\subseteq X\},$$

$$\mathscr{F}_{X}=\mathscr{F}_{\not\subseteq F\setminus X}=\{A\in\mathscr{F} \mid A\cap X\neq\emptyset\},$$

$$\mathscr{F}_{\leqslant k}=\{A\in\mathscr{F} \mid |A|\leqslant k\}.$$

We say that a set A *separates* elements x and y if $x\in A$ and $y\notin A$, or $x\notin A$ and $y\in A$. A family $\mathscr{F}$ of sets is *normalized* if $\emptyset\in\mathscr{F}$, $|F|=|\mathscr{F}|-1$ and for every two distinct elements $x,y\in F$ there is a set in $\mathscr{F}$ which separates x and y. A set G of a union-closed family $\mathscr{F}$ is a *generator* of $\mathscr{F}$ if G is not a union of two sets of $\mathscr{F}$ different from G. The family of all generators of $\mathscr{F}$ is denoted by $J(\mathscr{F})$. Finally, we say that a family $\mathscr{I}$ of sets is an *ideal* if for any set A from $\mathscr{I}$, the family $\mathscr{I}$ contains also all subsets of A.

The following result gives, in the case when $k=n/2$, the equivalent versions of the union-closed sets conjecture.

Theorem 2.1. *Let $n\geqslant 2$ and $k\geqslant 0$. The following assertions are equivalent:*

(i) *For every union-closed family $\mathscr{F}$ with $|\mathscr{F}|=n$ there exists an element $x\in F$ such that $|\mathscr{F}_{\ni x}|\geqslant k$.*

(ii) *For every union-closed family $\mathscr{F}$ with $|\mathscr{F}|=n$ and $\emptyset\in\mathscr{F}$ there exists a generator G of $\mathscr{F}$ such that $|\mathscr{F}_{\not\supseteq G}|\geqslant k$.*

(iii) *For every normalized union-closed family $\mathscr{F}$ with $|\mathscr{F}|=n$ there exists a generator G of $\mathscr{F}$ such that $|\mathscr{F}_{\not\supseteq G}|\geqslant k$.*

(iv) *For every normalized union-closed family* $\mathscr{F}$ *with* $|\mathscr{F}|=n$ *there exists a generator* G *of* $\mathscr{F}$ *such that* $|G|\geqslant k$.

The analogous equivalent formulations of Frankl's conjecture are given by Salzborn [11] who considered intersection-closed families of sets. However, for the completeness of the note, we present the proof of Theorem 2.1 below. Other equivalent versions of the union-closed sets conjecture were obtained by Poonen in [7] and by Zagaglia Salvi in [16]. The assertion (ii) of Theorem 2.1 corresponds to a statement on lattices which is known to be equivalent to Frankl's conjecture (see [2,7,14]). Knill proved in [2] that the assertion (ii) holds for every n and $k=n/2$ if $\mathscr{F}$ is a union-closed family generated by sets of size at most two. Poonen showed in [7] that assertion (i) of Theorem 2.1 holds for $n\leqslant 28$ and $k=n/2$. Slightly better result was obtained by Lo Faro [4] who proved that the assertion (i) holds for $n\leqslant 36$ and $k=n/2$. Roberts [10] has independently proved that (i) holds for $n\leqslant 40$ and $k=n/2$. The last two authors showed also that (i) holds if $|F|\leqslant 8$ and $k=n/2$.

In the proof of Theorem 2.1 we shall use several facts on union-closed families. For a family $\mathscr{G}$ of sets we define

$$\mathscr{G}^*=\left\{\mathscr{G}_B \mid B\subseteq \bigcup\mathscr{G}\right\}.$$

Lemma 2.2. *Let* $\mathscr{F}$ *be a union-closed family with* $\emptyset\in\mathscr{F}$ *and let* $\mathscr{G}$ *be a subfamily of* $\mathscr{F}$ *such that every set in* $\mathscr{F}$ *is a union of sets from* $\mathscr{G}$. *Then*

(i) $\mathscr{G}^*$ *is union-closed,*
(ii) $\emptyset\in\mathscr{G}^*$,
(iii) $\bigcup\mathscr{G}^*=\mathscr{G}\backslash\{\emptyset\}$,
(iv) $\mathscr{G}^*=\{\mathscr{G}_{\not\subseteq X}\mid X\in\mathscr{F}\}$,
(v) $|\mathscr{G}^*|=|\mathscr{F}|$,
(vi) *for every* $X,Y\in\bigcup\mathscr{G}^*$, $X\neq Y$, *there exists a set in* $\mathscr{G}^*$ *which separates* X *and* Y.
(vii) $J(\mathscr{G}^*)\subseteq\{\mathscr{G}_{\ni x}\mid x\in\bigcup\mathscr{G}\}\cup\{\emptyset\}$.

Proof. Clearly, (i)–(iii) hold. We have $\bigcup\mathscr{G}=F$. For each $C\subseteq F$ and $G\in\mathscr{G}$, $G\subseteq C$ if and only if $G\subseteq\bigcup\mathscr{G}_{\subseteq C}$. Hence,

$$\mathscr{G}^*=\{\mathscr{G}_{\not\subseteq C}\mid C\subseteq F\}=\{\mathscr{G}_{\not\subseteq\bigcup\mathscr{G}_{\subseteq C}}\mid C\subseteq F\}.$$

Since $\mathscr{F}$ is union-closed,

$$\{\bigcup\mathscr{G}_{\subseteq C}\mid C\subseteq F\}=\mathscr{F}.$$

Thus (iv) holds. Assertion (v) follows from (iv) and the fact that for every $X,Y\in\mathscr{F}$, if $X\neq Y$, then $\mathscr{G}_{\not\subseteq X}\neq\mathscr{G}_{\not\subseteq Y}$.

Let $X,Y\in\bigcup\mathscr{G}^*=\mathscr{G}\backslash\{\emptyset\}$, $X\neq Y$. Assume that $X\not\subseteq Y$ (the case $Y\not\subseteq X$ is symmetric). Then $X\in\mathscr{G}_{\not\subseteq Y}$ and $Y\notin\mathscr{G}_{\not\subseteq Y}$. Hence $\mathscr{G}_{\not\subseteq Y}$ separates X and Y. Since $\mathscr{G}\subseteq\mathscr{F}$, by (iv), $\mathscr{G}_{\not\subseteq Y}\in\mathscr{G}^*$. Therefore (vi) holds. For each $B\subseteq\bigcup\mathscr{G}$, $\mathscr{G}_B=\bigcup_{x\in B}\mathscr{G}_{\ni x}$. This implies (vii). $\square$

Lemma 2.3. *If $\mathscr{F}$ is a union-closed family of sets with $\emptyset \in \mathscr{F}$, then $\mathscr{F}^*$ is a normalized union-closed family and $|\mathscr{F}^*| = |\mathscr{F}|$.*

Proof. The result follows from Lemma 2.2 for $\mathscr{G} = \mathscr{F}$ (we have $\emptyset \in \mathscr{F}^*$ and $|\bigcup \mathscr{F}^*| = |\mathscr{F}| - 1 = |\mathscr{F}^*| - 1$). □

It can be easily verified that a normalized union-closed family of sets has a particularly 'regular' structure. For the proof of Theorem 2.1 we shall use only the following simple property of such a family.

Lemma 2.4. *If $\mathscr{F}$ is a normalized union-closed family of sets, then for every $X \in \mathscr{F}$,*

$$|\mathscr{F}_{\not\supseteq X}| = |X|.$$

Proof. For $x \in F$, let $T_{\mathscr{F}}(x) = \bigcup \mathscr{F}_{\not\ni x}$. Let $X \in \mathscr{F}$ and $x \in F$. Then, clearly,

(i) $x \in X$ if and only if $T_{\mathscr{F}}(x) \not\supseteq X$.

We shall show that

(ii) $T_{\mathscr{F}}$ is a bijection from F to $\mathscr{F} \setminus \{F\}$,

(iii) $T_{\mathscr{F}}(X) = \mathscr{F}_{\not\supseteq X}$.

To check (ii), let $y, z \in F$ and $y \neq z$. Since $\mathscr{F}$ is normalized, there exists a set $A \in \mathscr{F}$ which separates y and z. We may assume that $y \in A$ and $z \notin A$. Then, by (i), $T_{\mathscr{F}}(y) \not\supseteq A$ and $T_{\mathscr{F}}(z) \supseteq A$, so $T_{\mathscr{F}}(y) \neq T_{\mathscr{F}}(z)$. Hence, $T_{\mathscr{F}}$ is an injection. This implies that $|T_{\mathscr{F}}(F)| = |F|$, and since $\mathscr{F}$ is normalized, $|T_{\mathscr{F}}(F)| = |\mathscr{F} \setminus \{F\}|$. Thus, since $T_{\mathscr{F}}(F) \subseteq \mathscr{F} \setminus \{F\}$, we have $T_{\mathscr{F}}(F) = \mathscr{F} \setminus \{F\}$, and (ii) follows.

By (i), $x \in X$ if and only if $T_{\mathscr{F}}(x) \in \mathscr{F}_{\not\supseteq X}$. Consequently,

$$T_{\mathscr{F}}(X) \subseteq \mathscr{F}_{\not\supseteq X}$$

and, since $\mathscr{F}_{\not\supseteq X} \subseteq \mathscr{F} \setminus \{F\}$, by (ii), $T_{\mathscr{F}}^{-1}(\mathscr{F}_{\not\supseteq X}) \subseteq X$. The last inclusion yields $\mathscr{F}_{\not\supseteq X} \subseteq T_{\mathscr{F}}(X)$. Therefore, (iii) holds. Clearly, (iii) and (ii) imply Lemma 2.4. □

Proof of Theorem 2.1. Let us fix n and k. Clearly, assertion (ii) implies (iii). The equivalence of (iii) and (iv) follows immediately from Lemma 2.4. Thus, it is enough to prove that, say, (ii) follows from (i) and (iv) implies (i).

(i) $\Rightarrow$ (ii): Suppose that (i) holds. Let $\mathscr{F}$ be a union-closed family such that $|\mathscr{F}| = n$ and $\emptyset \in \mathscr{F}$. Set $\mathscr{G} = J(\mathscr{F})$. By Lemma 2.2 (i) and (v), $\mathscr{G}^*$ satisfies assumptions of (i). From (i), there exists $G \in \bigcup \mathscr{G}^*$ such that $|(\mathscr{G}^*)_{\ni G}| \geqslant k$. Clearly, by Lemma 2.2 (iii), $G \in \mathscr{G}$. Hence, Lemma 2.2 (iv) gives

$$(\mathscr{G}^*)_{\ni G} = \{\mathscr{G}_{\not\supseteq X} \mid X \in \mathscr{F} \text{ and } G \in \mathscr{G}_{\not\supseteq X}\} = \{\mathscr{G}_{\not\supseteq X} \mid X \in \mathscr{F}_{\not\supseteq G}\}.$$

Thus, $|(\mathscr{G}^*)_{\ni G}| = |\mathscr{F}_{\not\supseteq G}|$, which implies that $|\mathscr{F}_{\not\supseteq G}| \geqslant k$, so (ii) holds.

(iv) $\Rightarrow$ (i): Suppose now that (iv) holds. Let $\mathscr{F}$ be a union-closed family with $|\mathscr{F}| = n$. We may assume that $\emptyset \in \mathscr{F}$ (otherwise it is enough to show that (i) holds for the family obtained from $\mathscr{F}$ by replacing some generator by the empty set). By

Lemma 2.3, $\mathscr{F}^*$ satisfies assumptions of (iv). From (iv) and Lemma 2.2 (vii) it follows that there exists $x \in \bigcup\mathscr{F}$ such that $\mathscr{F}_{\ni x} \in J(\mathscr{F}^*)$ and $|\mathscr{F}_{\ni x}| \geqslant k$. Thus (i) holds and the proof of Theorem 2.1 is complete. □

Remark. If $\mathscr{F}$ is a union-closed family and $\emptyset \in \mathscr{F}$, we can define a union-preserving bijection of $\mathscr{F}$ onto a normalized union-closed family by mapping each $X \in \mathscr{F}$ to $\mathscr{F}_{\not\supseteq X}$. This can be used to prove the implication (iii) $\Rightarrow$ (ii) more directly.

We conclude this section with a result that lets us associate an ideal to a union-closed family of sets.

Lemma 2.5. *For every union-closed family $\mathscr{F}$ with $\emptyset \in \mathscr{F}$ there exists an ideal $\mathscr{I} \subseteq 2^{J(\mathscr{F})\setminus\{\emptyset\}}$ such that the map sending $\mathscr{G} \in \mathscr{I}$ to $\bigcup\mathscr{G}$ is a bijection from $\mathscr{I}$ to $\mathscr{F}$.*

Proof. Let $J' = J(\mathscr{F})\setminus\{\emptyset\}$ and let σ be any bijection from J' to $\{1,\dots,|J'|\}$. For $\mathscr{G} \subseteq J'$ define the weight of $\mathscr{G}$ setting

$$w(\mathscr{G}) = \sum_{G\in\mathscr{G}} 2^{\sigma(G)-1}. \tag{2.6}$$

For any $A \in \mathscr{F}$, let $\phi(A)$ be the subfamily of J' with minimal weight, such that $\bigcup\phi(A) = A$. Finally, define

$$\mathscr{I} = \{\phi(A) \mid A \in \mathscr{F}\}.$$

Note that ϕ is a bijection from $\mathscr{F}$ to $\mathscr{I}$. Moreover, for each $\mathscr{G} \in \mathscr{I}$, $\phi^{-1}(\mathscr{G}) = \bigcup\mathscr{G}$. Thus, the map sending $\mathscr{G}$ to $\bigcup\mathscr{G}$ is a bijection from $\mathscr{I}$ to $\mathscr{F}$.

We shall show that $\mathscr{I}$ is an ideal. Let $\mathscr{A} \in \mathscr{I}$ and $\mathscr{B} \subseteq \mathscr{A}$. Suppose to the contrary that $\mathscr{B} \notin \mathscr{I}$. Since $\mathscr{B} \subseteq J'$ and $\mathscr{F}$ is union-closed, $\bigcup\mathscr{B} \in \mathscr{F}$. Let $\mathscr{B}' = \phi(\bigcup\mathscr{B})$. Then $\mathscr{B}'$ is the subfamily of J' with the minimal weight such that $\bigcup\mathscr{B}' = \bigcup\mathscr{B}$. Hence, $w(\mathscr{B}') < w(\mathscr{B})$ because $\mathscr{B}' \in \mathscr{I}$, $\mathscr{B} \notin \mathscr{I}$ and w is an injection. Let

$$\mathscr{A}' = (\mathscr{A}\setminus\mathscr{B}) \cup \mathscr{B}'.$$

Since $\bigcup\mathscr{B}' = \bigcup\mathscr{B}$ and $\mathscr{B} \subseteq \mathscr{A}$, we have $\bigcup\mathscr{A}' = \bigcup\mathscr{A}$. Thus, by the assumption that $\mathscr{A} \in \mathscr{I}$, $w(\mathscr{A}') \geqslant w(\mathscr{A})$. But from (2.6),

$$w(\mathscr{A}') \leqslant w(\mathscr{A}\setminus\mathscr{B}) + w(\mathscr{B}') = w(\mathscr{A}) - w(\mathscr{B}) + w(\mathscr{B}') < w(\mathscr{A})$$

which leads to a contradiction, so $\mathscr{I}$ is an ideal. □

Remark. Note that each subfamily $\mathscr{G}$ of generators which belongs to the ideal $\mathscr{I}$ satisfying Lemma 2.5 forms a minimal cover of the set $\bigcup\mathscr{G}$, i.e. for every $G \in \mathscr{G}$, $\bigcup(\mathscr{G}\setminus\{G\}) \neq \bigcup\mathscr{G}$. This observation follows from the injectivity of the map sending $\mathscr{G}$ to $\bigcup\mathscr{G}$ and from the fact that $\mathscr{G}\setminus\{G\} \in \mathscr{I}$ (since $\mathscr{I}$ is an ideal).

3. Colexicographic order and small sets in ideals

In this section we present a precise lower estimate for the number of sets of size bounded from above in an ideal of a given size. We show that this estimate is achieved by the ideal which consists of the first finite subsets of natural numbers in the colexicographic order. Define a bijection $\mathrm{clx} : [\mathbb{N}]^{<\infty} \to \mathbb{N} \cup \{0\}$ such that for each $A \in [\mathbb{N}]^{<\infty}$,

$$\mathrm{clx}(A) = \sum_{x \in A} 2^{x-1}.$$

The *colexicographic order* on $[\mathbb{N}]^{<\infty}$ is defined in such a way that for $A, B \in [\mathbb{N}]^{<\infty}$, we have $A < B$ whenever $\mathrm{clx}(A) < \mathrm{clx}(B)$. For integers $n \geqslant 0$ and k, let $\mu(n,k)$ be the number of sets of size at most k in the ideal which consists of the first n finite subsets of natural numbers in the colexicographic order, i.e.

$$\mu(n,k) = |\{A \in [\mathbb{N}]^{<\infty} \mid \mathrm{clx}(A) < n \text{ and } |A| \leqslant k\}|.$$

Our aim is to prove the following theorem.

Theorem 3.1. *If $\mathscr{I}$ is an ideal and $k \geqslant 0$, then*

$$|\mathscr{I}_{\leqslant k}| \geqslant \mu(|\mathscr{I}|, k).$$

Note that for every $i \geqslant 0$, the number of ones in the binary expansion of i is equal to the size of the set $A \in [\mathbb{N}]^{<\infty}$ such that $\mathrm{clx}(A) = i$. It follows that $\mu(n,k)$ is the number of non-negative integers less than n which have at most k ones in the binary expansion. We use this fact to show some properties of $\mu(n,k)$ we shall employ later in the proof of Theorem 3.1.

Lemma 3.2. *Let n, m, k, r be integers such that $0 \leqslant n \leqslant m$ and $r \in \{0, 1\}$. Then*
(i) $\mu(n, k-1) + \mu(n+r, k) = \mu(2n+r, k)$,
(ii) $\mu(n, k-1) + \mu(m, k) \geqslant \mu(n+m, k)$.

Proof. For integers $j \geqslant 0$ and k, let

$$h(j,k) = \begin{cases} 1 & \text{if } j \text{ has at most } k \text{ ones in its binary expansion,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu(n,k) = \sum_{j=0}^{n-1} h(j,k)$. Observe that for every $j \geqslant 0$ and k, $h(2j,k) = h(j,k)$ and $h(2j+1,k) = h(j,k-1)$. Hence,

$$\begin{aligned} \mu(n,k-1) + \mu(n+r,k) &= \sum_{j=0}^{n-1} h(j,k-1) + \sum_{j=0}^{n+r-1} h(j,k) \\ &= \sum_{j=0}^{n-1} h(2j+1,k) + \sum_{j=0}^{n+r-1} h(2j,k) \\ &= \sum_{j=0}^{2n+r-1} h(j,k) = \mu(2n+r,k) \end{aligned}$$

and so (i) follows.

We now prove (ii) by induction on m. If $m \leqslant 1$, then (ii) is a particular case of (i). Assume thus that $m \geqslant 2$. Let $n = 2p + r_1$ and $m = 2q + r_2$, where p, q are integers and $r_1, r_2 \in \{0, 1\}$. Since $m \geqslant 2$ and $n \leqslant m$, we have $q \geqslant 1$, $p \leqslant q < m$ and $p + r_1 \leqslant q + r_2 < m$. Using (i) and the induction hypothesis, we get

$$\begin{aligned}\mu(n,k-1)+\mu(m,k) &= \mu(2p+r_1,k-1)+\mu(2q+r_2,k)\\ &= \mu(p,k-2)+\mu(p+r_1,k-1)\\ &\quad+\mu(q,k-1)+\mu(q+r_2,k)\\ &\geqslant \mu(p+q,k-1)+\mu(p+q+r_1+r_2,k).\end{aligned}$$

We need to show that the above sum is not smaller than $\mu(n+m,k)$. This fact follows easily from (i) whenever $r_1 + r_2 \leqslant 1$. Thus, it is enough to verify it when $r_1 + r_2 = 2$. For any $j \geqslant 0$ and k, if j has at most $k-1$ ones in its binary expansion, then $j+1$ has at most k ones. Hence $h(j, k-1) \leqslant h(j+1, k)$. Consequently, by (i), if $r_1 + r_2 = 2$, then

$$\begin{aligned}\mu(n,k-1)+\mu(m,k) &\geqslant \mu(p+q,k-1)+\mu(p+q+2,k)\\ &= \mu(p+q+1,k-1)-h(p+q,k-1)\\ &\quad+\mu(p+q+1,k)+h(p+q+1,k)\\ &\geqslant \mu(p+q+1,k-1)+\mu(p+q+1,k)\\ &= \mu(n+m,k),\end{aligned}$$

which completes the proof of Lemma 3.2. □

Proof of Theorem 3.1. We use induction on $|\mathscr{I}|$. If $|\mathscr{I}| \leqslant 1$, then clearly Theorem 3.1 holds. Assume that $|\mathscr{I}| > 1$. Let $x \in \bigcup \mathscr{I}$ and $\mathscr{I}'_x = \{A \setminus \{x\} \mid A \in \mathscr{I}_{\ni x}\}$. Then $\mathscr{I}'_x$ and $\mathscr{I}_{\not\ni x}$ are ideals such that $\mathscr{I}'_x \subseteq \mathscr{I}_{\not\ni x}$ and $|\mathscr{I}'_x| \leqslant |\mathscr{I}_{\not\ni x}| < |\mathscr{I}|$. Hence, we can use the induction hypothesis for $\mathscr{I}'_x$ and $\mathscr{I}_{\not\ni x}$, which together with Lemma 3.2 (ii) gives

$$\begin{aligned}|\mathscr{I}_{\leqslant k}| &= |(\mathscr{I}'_x)_{\leqslant k-1}| + |(\mathscr{I}_{\not\ni x})_{\leqslant k}|\\ &\geqslant \mu(|\mathscr{I}'_x|, k-1) + \mu(|\mathscr{I}_{\not\ni x}|, k)\\ &\geqslant \mu(|\mathscr{I}'_x| + |\mathscr{I}_{\not\ni x}|, k) = \mu(|\mathscr{I}|, k),\end{aligned}$$

as required. □

Lemma 3.3. *For each $n \geqslant 1$ and $k \geqslant 0$,*

$$\mu(n,k) \geqslant \sum_{i=0}^{k} \binom{\lfloor \log_2 n \rfloor}{i}.$$

Proof. Let $l = \lfloor \log_2 n \rfloor$. Clearly, $\mu(n,k) \geqslant \mu(2^l, k)$. Let $0 \leqslant i \leqslant l$. Since the binary expansion of $2^l - 1$ consists of l ones, the number of non-negative integers less than 2^l

which have i ones in the binary expansion is equal to $\binom{l}{i}$. Hence, $\mu(2^l,k)=\sum_{i=0}^{k}\binom{l}{i}$, and the result follows. $\square$

Lemma 3.4. *Let* $0<a<1$. *If* n *is sufficiently large, then*

$$\mu(n,\lfloor a\log_2 n\rfloor)>c(a)(\log_2 n)^{-1/2}\,n^{-a\log_2 a-(1-a)\log_2(1-a)},$$

where $c(a)$ *is a constant depending only on* a.

Proof. Let $l=\lfloor\log_2 n\rfloor$ and $k=\lfloor a\log_2 n\rfloor$. Using Lemma 3.3 and Stirling's formula, we obtain, for sufficiently large n,

$$\begin{aligned}\mu(n,k)&\geqslant\binom{l}{k}>\frac{C_1\sqrt{l}\,l^l\mathrm{e}^{-l}}{\sqrt{k}\,k^k\mathrm{e}^{-k}\,\sqrt{l-k}\,(l-k)^{l-k}\mathrm{e}^{k-l}}\\&=C_1(k(1-k/l))^{-1/2}((k/l)^{k/l}(1-k/l)^{1-k/l})^{-l}\\&>c(a)(\log_2 n)^{-1/2}(a^a(1-a)^{1-a})^{-\log_2 n}\\&=c(a)(\log_2 n)^{-1/2}\,n^{-a\log_2 a-(1-a)\log_2(1-a)},\end{aligned}$$

where C_1 and $c(a)$ are absolute constants. $\square$

4. Large generators in normalized union-closed families

For $n\geqslant 1$, let $u(n)$ be the maximal number such that for each union-closed family $\mathscr{F}$ with n sets there exists an element $x\in\bigcup\mathscr{F}$ such that $|\mathscr{F}_{\ni x}|\geqslant u(n)$. Thus, the union-closed sets conjecture says that $u(n)\geqslant n/2$, while Knill's observation mentioned in the Introduction implies that $u(n)\geqslant(n-1)/\log_2 n$. We improve this estimate by a constant factor.

Theorem 4.1. *For large* n,

$$u(n)>\frac{1+\mathrm{o}(1)}{\log_2(\frac{4}{3})}\frac{n}{\log_2 n}>\frac{2.40942n}{\log_2 n}.$$

Proof. We shall use the equivalence of assertions (i) and (iv) of Theorem 2.1. Let $\mathscr{F}$ be a normalized union-closed family with $|\mathscr{F}|=n$. Let M be the size of the largest generator of $\mathscr{F}$ and $m=\max\{M,\lceil n/\log_2 n\rceil\}$. We shall show that if n is sufficiently large, then $m>(1+\mathrm{o}(1))n/(\log_2(\frac{4}{3})\log_2 n)$. By the definition of normalized family, $|\bigcup\mathscr{F}|=n-1$. Since $\bigcup\mathscr{F}$ is a union of generators, there exists a set $X\in\mathscr{F}$ such that

$$n-2m\leqslant|X|<n-m.$$

Consider the family $\mathscr{F}_{\supseteq X}$. By Lemma 2.4,

$$|\mathscr{F}_{\supseteq X}|=|\mathscr{F}|-|X|\geqslant m. \tag{4.2}$$

Clearly, $\mathscr{F}_{\supseteq X}$ is union-closed. Let $\mathscr{I} \subseteq 2^{J(\mathscr{F}_{\supseteq X})\setminus\{\emptyset\}}$ be the ideal satisfying Lemma 2.5 for the family $\mathscr{F}_{\supseteq X}$. Let $k = \lfloor (\frac{1}{3}) \log_2 m \rfloor$ and

$$\mathscr{F}_1 = \{\bigcup \mathscr{G} \mid \mathscr{G} \in \mathscr{I}_{\leqslant k}\}.$$

From Lemma 2.5 it follows that $|\mathscr{I}| = |\mathscr{F}_{\supseteq X}| \geqslant m$ (by (4.2)) and $|\mathscr{F}_1| = |\mathscr{I}_{\leqslant k}|$. Using Theorem 3.1, we obtain $|\mathscr{F}_1| \geqslant \mu(|\mathscr{I}|, k) \geqslant \mu(m,k)$. By the definition of m, m can be made arbitrary large by choosing n large enough. Thus, Lemma 3.4 for $a = \frac{1}{3}$ implies that if n is sufficiently large, then for some absolute constant c,

$$|\mathscr{F}_1| > c\,(\log_2 m)^{-1/2}\, m^{\log_2 3 - 2/3}, \tag{4.3}$$

because $(\frac{1}{3}) \log_2(\frac{1}{3}) + (\frac{2}{3}) \log_2(\frac{2}{3}) = -(\frac{1}{3}) \log_2 3 + (\frac{2}{3})(1 - \log_2 3) = \frac{2}{3} - \log_2 3$.

Note that for every generator U of $\mathscr{F}_{\supseteq X}$ there exists a generator $G_{\mathscr{F}}(U)$ of $\mathscr{F}$ such that $X \cup G_{\mathscr{F}}(U) = U$. Let $Y \in \mathscr{F}_1$. Then $Y = \bigcup \mathscr{G}(Y)$, where $\mathscr{G}(Y)$ is a subfamily of generators of $\mathscr{F}_{\supseteq X}$ such that $|\mathscr{G}(Y)| \leqslant k$. Let $E(Y) = \bigcup_{U \in \mathscr{G}(Y)} G_{\mathscr{F}}(U)$. Then $E(Y) \in \mathscr{F}$,

$$X \cup E(Y) = Y$$

and $|E(Y)| \leqslant km$. Hence,

$$|X \setminus E(Y)| \geqslant |X| - |E(Y)| \geqslant n - 2m - km. \tag{4.4}$$

Since $Y \in \mathscr{F}$, there exists $\mathscr{B}(Y) \subseteq J(\mathscr{F})$ such that

$$\bigcup \mathscr{B}(Y) \cup E(Y) = Y$$

and for every $B \in \mathscr{B}(Y)$,

$$\bigcup (\mathscr{B}(Y) \setminus \{B\}) \cup E(Y) \neq Y.$$

Thus, for any $\mathscr{A}_1, \mathscr{A}_2 \subseteq \mathscr{B}(Y)$, if $\mathscr{A}_1 \neq \mathscr{A}_2$, then

$$\bigcup \mathscr{A}_1 \cup E(Y) \neq \bigcup \mathscr{A}_2 \cup E(Y),$$

which implies that there exist $2^{|\mathscr{B}(Y)|}$ distinct sets in $\mathscr{F}$ of the form $\bigcup \mathscr{A} \cup E(Y)$, where $\mathscr{A} \subseteq \mathscr{B}(Y)$. Moreover, if $Y_1, Y_2 \in \mathscr{F}_1$, $Y_1 \neq Y_2$, $\mathscr{A}_1 \subseteq \mathscr{B}(Y_1)$ and $\mathscr{A}_2 \subseteq \mathscr{B}(Y_2)$, then, since $Y_1 \setminus X \neq Y_2 \setminus X$, we have $\bigcup \mathscr{A}_1 \cup E(Y_1) \neq \bigcup \mathscr{A}_2 \cup E(Y_2)$. Therefore, in $\mathscr{F}$ one can find at least $\sum_{Y \in \mathscr{F}_1} 2^{|\mathscr{B}(Y)|}$ distinct sets. For every $Y \in \mathscr{F}_1$, since each generator from $\mathscr{B}(Y)$ has size at most m and $\bigcup \mathscr{B}(Y) \supseteq X \setminus E(Y)$, we have $|\mathscr{B}(Y)| m \geqslant |\bigcup \mathscr{B}(Y)| \geqslant |X \setminus E(Y)|$, which together with (4.3) gives $|\mathscr{B}(Y)| \geqslant n/m - k - 2$. It follows that

$$|\mathscr{F}_1| 2^{n/m - k - 2} \leqslant \sum_{Y \in \mathscr{F}_1} 2^{|\mathscr{B}(Y)|} \leqslant |\mathscr{F}| = n.$$

Combining the last inequality with (4.2), we get, for sufficiently large n,

$$\log_2 c - (\tfrac{1}{2}) \log_2 \log_2 m + (\log_2 3 - \tfrac{2}{3}) \log_2 m + n/m - \lfloor (\tfrac{1}{3}) \log_2 m \rfloor - 2 < \log_2 n.$$

By definition, $m \geqslant n/\log_2 n$. Thus, for large n,

$$\begin{aligned} n/m &< \log_2 n + (1-\log_2 3)\log_2 m + \mathrm{o}(\log_2 n) \\ &\leqslant \log_2 n + (1-\log_2 3)\log_2(n/\log_2 n) + \mathrm{o}(\log_2 n) \\ &= (2-\log_2 3)\log_2 n + \mathrm{o}(\log_2 n) \\ &= (1+\mathrm{o}(1))\log_2(4/3)\log_2 n, \end{aligned}$$

which completes the proof of Theorem 4.1. □

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