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Strong semimodular lattices and Frankl's conjecture

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Abstract. In this paper, we show that Frankl's conjecture holds for strong semimodular lattices. The result is the first step to deal with the case of *upper* semimodular lattices.

1. Introduction

Recently, there are some results for Frankl's conjecture: every nontrivial finite lattice L has a join-irreducible element such that the principal filter generated by the element has at most $|L|/2$ elements. If a lattice is either lower semimodular or section-complemented, then the conjecture holds [2, 3, 4, 5]. However, it is unknown whether the conjecture is true for *upper* semimodular lattices. In this paper, we deal with the case of strong semimodular lattices as the first step. We use the terminology on lattices in [6]. Throughout the paper, all lattices are assumed to be finite.

Let L be a lattice. We denote by j' the unique cocover of j for each join-irreducible element j. Then define for $x > 0$

$$x' = \bigvee\{j' : x \geq j \text{ and } j \text{ is join-irreducible}\}.$$

It is called the *derivation* of x. On the other hand, if $x > 0$ then we can define the meet of all cocovers of x. We write x_+ for it.

THEOREM 1.1. *Let L be a nontrivial lattice. Suppose that $x_+ \leq x' < x$ for all $x > 0$. Then*

$$\eta(L) := \min\{|V_j| : j \text{ is a join-irreducible element of } L\} \leq |L|/2,$$

where V_j is the principal filter generated by j. Moreover, equality holds if and only if L is Boolean.

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Note that x' and x_+ are not always comparable in general. However, when L is semimodular, $x_+ \leq x'$ for each $x > 0$. (See Lemma 6.5.2 in [6], which is called *Butterfly Lemma*). In particular, a semimodular lattice is strong if and only if $x_+ = x'$ holds for each $x > 0$ (See Theorem 6.5.5 in [6]). Hence we have the following corollary.

COROLLARY 1.2. *If a nontrivial lattice is strong and semimodular, then Frankl's conjecture holds for it.*

Next suppose that L is section-complemented. Let x be an element with $x > 0$. Since L is atomistic, $x' = 0$. On the other hand, since the interval $[0, x]$ is complemented, $x_+ = 0$. Hence L satisfies the assumption in Theorem 1.1.

COROLLARY 1.3. *Frankl's conjecture is true for nontrivial, section-complemented lattices.*

Therefore Theorem 1.1 is stronger than results in [4] and [2].

2. Proof of Theorem 1.1

Since $1' < 1$, there is a join-irreducible element j such that $j \not\leq 1'$. Take an element x with $x \geq j$. If $j \leq x_+$, then $j \leq x_+ \leq x' \leq 1'$. This is a contradiction. Since $j \not\leq x_+$, there is a cocover x_1 of x such that $x_1 \vee j = x$. Hence $\eta(L) \leq |V_j| \leq (|L \setminus V_j| + |V_j|)/2 \leq |L|/2$.

Next suppose that $\eta(L) = |L|/2$. The above argument implies that for every join-irreducible element j with $j \not\leq 1'$, j is an atom since $|L \setminus V_j| = |V_j|$ and $0 \vee j = j$. We will show that $1' = 0$. Suppose on the contrary that $1' > 0$. If a join-irreducible element j satisfies that $j' > 0$, then $j \leq 1'$. Hence $(1')' = 1'$. This contradicts the assumption that $x' < x$ for all $x > 0$. Hence $1' = 0$. In particular, L is atomistic.

Let a be an atom. Since $1_+ \leq 1' = 0$ and $\eta(L) = |L|/2$, there is a unique coatom b such that $a \vee b = 1$. We claim that L is coatomistic. Suppose on the contrary that there is a meet-irreducible element m which is not a coatom. Let m^* be a unique cover of m. Take a cover x of m^* and an atom a such that $a \leq x$ and $a \not\leq m^*$. Then $a \vee m = a \vee m^* = x$. This contradicts that $|V_a| = |L \setminus V_a|$. Hence L is coatomistic. Note that for each atom a there is a unique coatom b such that $I_b = L \setminus V_a$, where I_b is the principal ideal generated by b.

Finally, we will show that L is Boolean. Suppose that L contains a pentagon $\{x \wedge z, x, y, z, x \vee y\}$ with $x \wedge z < y < z < x \vee y$. Take an atom a such that $a \leq z$ and $a \not\leq y$. Then there is a coatom b such that $V_a \cup I_b = L$. However, this implies that $x \leq b$ and $y \leq b$, and so $a \leq z \leq x \vee y \leq b$. Hence L is modular. By similar argument, we see that L contains no diamond. Therefore L is distributive and atomistic, and so it is Boolean.

3. Remarks

In [1] we introduced the *excess* of a lattice in the following way. For a join-irreducible element j and a meet-irreducible element m, if $j \not\leq m$, then we call the pair (j, m) *mismatching*. The excess of L is defined by

$$\mathrm{ex}(L) := |L| - \min\{|V_j| + |I_m| : (j, m) \text{ is mismatching}\},$$

where I_m is the principal ideal generated by m. Theorem 2.1 of [1] proves that a lattice has excess 0 if and only if it is Boolean. This result can likewise be used to show the final step of the above proof.

An element x is called *singular* if $x = x'$ and $x > 0$. The lattices L_1, L_2 of Figure 1 are semimodular but not strong. It is clear that L_1 has no singular element. On the other hand, L_2 has a singular element since $1 = 1'$. In particular, L_2 is also upper locally distributive. Theorem 1.1 implies that if a nontrivial, semimodular lattice L is nonsingular, that is, L has no singular element, then Frankl's conjecture holds for L. Therefore, our future research is to analyze singular semimodular lattices.

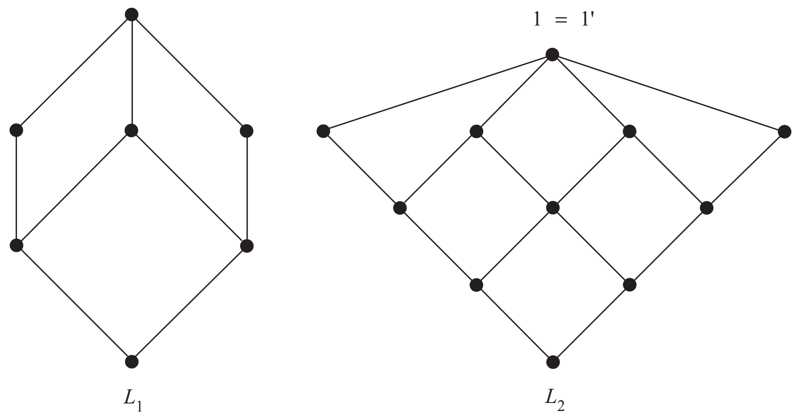

Figure 1

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