JOURNAL OF COMBINATORIAL THEORY, Series A 59, 253–268 (1992)

Union-Closed Families

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Communicated by R. L. Graham

Received June 13, 1990

A union-closed family $\mathscr{F}$ is a finite collection of sets not all empty, such that any union of elements of $\mathscr{F}$ is itself an element of $\mathscr{F}$. Peter Frankl conjectured in 1979 that for any such family, there is an element in at least half of its sets. But the problem remains unsolved. We find a number of equivalent conjectures, and we prove the conjecture in special cases, including for example all families involving up to seven elements or having up to 28 sets, extending the previously known result for up to 18 sets. We also prove a general theorem stating exactly when a subfamily is enough to guarantee the existence of an element from the subfamily which is in half the sets of the whole family. © 1992 Academic Press, Inc.

1. A FEW CONJECTURES

We will use the following notations and definitions. A *union-closed family* $\mathscr{F}$ is a finite collection of sets not all empty, such that any union of elements of $\mathscr{F}$ is itself an element of $\mathscr{F}$. In particular, the union A of all sets in $\mathscr{F}$ is in $\mathscr{F}$, and $\mathscr{F} \subseteq \mathscr{P}(A)$ (the power set of A). Let $m = |A|$ and $n = |\mathscr{F}|$. (So m could be infinite.) Let $\mathscr{F}_\alpha = \{S \in \mathscr{F} \mid \alpha \in S\}$. A *block* $B \subseteq A$ is an equivalence class of the relation $\alpha \sim \beta$ *iff* $\mathscr{F}_\alpha = \mathscr{F}_\beta$. If $\mathscr{F}$ and $\mathscr{G}$ are any two collections of sets, let $\mathscr{F} \uplus \mathscr{G}$ denote $\{S \cup T \mid S \in \mathscr{F}, T \in \mathscr{G}\}$. The finite union requirement on $\mathscr{F}$ is equivalent to $\mathscr{F} \uplus \mathscr{F} = \mathscr{F}$.

The following was conjectured by Peter Frankl in 1979 (cf. [2, 5]), but apparently very little progress has been made on the problem since then.

Conjecture 1. If $\mathscr{F}$ is a union-closed family, there exists an element $\alpha \in A$ such that $|\mathscr{F}_\alpha| \geqslant n/2$.

First we note that we can reduce to certain special cases.

LEMMA 1. *For each n, it suffices to consider families $\mathscr{F}$ such that $\varnothing \in \mathscr{F}$.*

0097-3165/92 $3.00

Proof. Conjecture 1 is true for $n = 1$. For $n > 1$ we can replace a minimal set of $\mathscr{F}$ by $\varnothing$ to obtain a union-closed family $\mathscr{F}'$ with n elements. If $|\mathscr{F}'_\alpha| \geqslant n/2$ for some α, then $|\mathscr{F}_\alpha| \geqslant |\mathscr{F}'_\alpha| \geqslant n/2$. ∎

LEMMA 2. *For each n, it suffices to consider families for which all the blocks are singletons.*

Proof. Replace each occurrence of a block in each set by a representative element. This gives us a union-closed family in which the blocks are singletons. If in this new family, some element is in at least $n/2$ sets, then every element in the associated block of the original family was in at least $n/2$ sets. For example, if $\mathscr{F} = \{\varnothing, \{1, 2\}, \{1, 2, 3, 4, 5\}\}$ we can replace the blocks $\{1, 2\}$ and $\{3, 4, 5\}$ by 1 and 3, respectively, to obtain $\mathscr{F}' = \{\varnothing, \{1\}, \{1, 3\}\}$. Since 1 is in over half the sets of $\mathscr{F}'$, every element of the block $\{1, 2\}$ is in over half the sets of $\mathscr{F}$. ∎

COROLLARY 1. *It suffices to consider the case where the sets of $\mathscr{F}$ are finite.*

Proof. There are only 2^n subsets of $\mathscr{F}$, so there are at most 2^n blocks. But the blocks form a partition of A, so if they are singletons, $|A| \leqslant 2^n$. ∎

So allowing the sets of $\mathscr{F}$ to be infinite in the definition of union-closed family is of no consequence. On the other hand, we cannot make sense of the conjecture if $\mathscr{F}$ itself is infinite. For example, if $S_k = \{k, k+1, k+2, ...\}$ and $\mathscr{F} = \{S_1, S_2, S_3, ...\}$, then every element is in finitely many sets of $\mathscr{F}$, but $\mathscr{F}$ is infinite.

It seems that in most cases, there exists an element belonging to *more* than $n/2$ sets. When is there no element in more than $n/2$ sets? Well, if $\mathscr{F}$ is the power set $\mathscr{P}(\mathrm{A})$ of some finite set A, then each element of A is in exactly half the sets of $\mathscr{F}$. But these are not the only examples, because we can substitute a block for any element. For example, in $\mathscr{F} = \mathscr{P}(\{1, 2\})$ we can replace 2 by the set $\{3, 4\}$ to obtain $\mathscr{F}' = \{\varnothing, \{1\}, \{3, 4\}, \{1, 3, 4\}\}$. So we make the following conjecture.

Conjecture 2. Let $\mathscr{F}$ be a union-closed family whose blocks are singletons. If $\mathscr{F}$ is not a power set, then there exists an element $\alpha \in A$ such that $|\mathscr{F}_\alpha| > n/2$.

It also seems that there are usually *many* elements that are in at least half the sets.

Conjecture 3. Let $\mathscr{F}$ be a union-closed family. If there is only one element $\alpha \in A$ such that $|\mathscr{F}_\alpha| \geqslant n/2$ then α is in every nonempty set of $\mathscr{F}$.

Conjecture 4. Let $\mathscr{F}$ be a union-closed family chose blocks are singletons. If there is only one element $\alpha \in A$ such that $|\mathscr{F}_\alpha| \geqslant n/2$ then $\mathscr{F} = \{\{\alpha\}\}$ or $\mathscr{F} = \{\emptyset\} \cup (\{\{\alpha\}\} \uplus \mathscr{P}(A \backslash \{\alpha\}))$.

(Note that if $\mathscr{F} = \{\{\alpha\}\}$ or $\mathscr{F} = \{\emptyset\} \cup (\{\{\alpha\}\} \uplus \mathscr{P}(A \backslash \{\alpha\}))$, then α is the only element in at least $n/2$ sets.)

PROPOSITION 1. *Suppose Conjecture* 1 *holds. Then Conjecture* 4 *holds* iff *Conjectures* 2 *and* 3 *both hold.*

Proof. Suppose Conjecture 4 holds. Then Conjecture 3 holds for families with singleton blocks, and it follows that Conjecture 3 holds in general, as in Lemma 2. Now we suppose $\mathscr{F}$ is a union-closed family with singleton blocks which is not a power set, and attempt to prove Conjecture 2. We may assume $\emptyset \in \mathscr{F}$ by considering $\mathscr{F}' = \{\emptyset\} \cup \mathscr{F}$. (The blocks of $\mathscr{F}'$ are singletons then, and if $\mathscr{F}'$ is a power set, then in $\mathscr{F}$ every element was in exactly $(n+1)/2$ sets.) Let $\mathscr{G} = \{\emptyset\} \cup (\{\{\alpha\}\} \uplus \mathscr{F})$, where α is some element not appearing in any set of $\mathscr{F}$. Then $\mathscr{G}$ is a union-closed family with $n+1$ elements and its blocks are the blocks of F and the singleton block $\{\alpha\}$. (The element α is in its own block, because $\{\alpha\} \in \mathscr{F}$.) So the blocks of $\mathscr{G}$ are singletons. Thus we may apply Conjecture 4. Since $\mathscr{F}$ was not a power set, $\mathscr{G}$ is not a family of the type in Conjecture 4, so there is an element $\beta = \alpha$ such that $|\mathscr{G}_\beta| \geqslant (n+1)/2$. Then $|\mathscr{F}_\beta| = |\mathscr{G}_\beta| \geqslant (n+1)/2 > n/2$. So Conjecture 2 holds.

Conversely, suppose Conjectures 2 and 3 hold. Suppose $\mathscr{F}$ is a union-closed family with singleton blocks, and there is only one element $\alpha \in A$ such that $|\mathscr{F}_\alpha| \geqslant n/2$. Let $\mathscr{G} = \{S \subseteq A \backslash \{\alpha\} \mid S \cup \{\alpha\} \in \mathscr{F}\}$, so $\mathscr{G} \neq \emptyset$. Then by Conjecture 3, $\mathscr{F} = \{\{\alpha\}\} \uplus \mathscr{G}$ or $\mathscr{F} = \{\emptyset\} \cup (\{\{\alpha\}\} \uplus \mathscr{G})$. If $\mathscr{G} = \{\emptyset\}$, then Conjecture 4 holds for $\mathscr{F}$. Otherwise $\mathscr{G}$ is a union-closed family with n or $n-1$ elements, and the blocks of $\mathscr{G}$ are singletons, since otherwise the blocks of $\mathscr{F}$ would not be singletons. If $\mathscr{G}$ is not a power set, we can apply Conjecture 2 to find an element $\beta \in A \backslash \{\alpha\}$ such that $|\mathscr{G}_\beta| \geqslant (|\mathscr{G}|+1)/2 \geqslant n/2$. Then $|\mathscr{F}_\beta| = |\mathscr{G}_\beta| \geqslant n/2$, contradicting the uniqueness of α. So $\mathscr{G}$ is the power set of $A \backslash \{\alpha\}$, which is nonempty since $\mathscr{G} \neq \{\emptyset\}$. Because of the two possible expressions for $\mathscr{F}$ in terms of $\mathscr{G}$ above, it remains to show that $\emptyset \in \mathscr{F}$. Pick $\beta \in A \backslash \{\alpha\}$. If $\emptyset \notin \mathscr{F}$, then $|\mathscr{F}_\beta| = |\mathscr{G}_\beta| = n/2$, contradicting the uniqueness of α. Thus $\emptyset \in \mathscr{F}$ and Conjecture 4 holds. ∎

2. DEDUCING THE CONJECTURE FROM SUBFAMILIES

It is known that if a union-closed family $\mathscr{F}$ contains a set with one or two elements, then Conjecture 1 holds for $\mathscr{F}$ [3]. We now prove a theorem that shows exactly how far this sort of argument can be generalized.

THEOREM 1. *Suppose $\mathscr{F}'$ is a union-closed family whose largest set A has k elements. To simplify notation, assume $A = \{1, 2, ..., k\}$. Then the following are equivalent*:

1. *For every union-closed family $\mathscr{F}$ containing $\mathscr{F}'$, there exists $i \in A$ such that $|\mathscr{F}_i| \geqslant |\mathscr{F}|/2$.*

2. *There exist nonnegative real numbers $c_1, ..., c_k$ with sum 1 such that for every union-closed family $\mathscr{G} \subseteq \mathscr{P}(A)$* with *$\mathscr{F}' \uplus \mathscr{G} = \mathscr{G}$,*

$$\sum_{i=1}^{k} c_i |\mathscr{G}_i| \geqslant |\mathscr{G}|/2.$$

For fixed $\mathscr{F}'$ and A, there are finitely many $\mathscr{G}$ possible, so in (2) we have simply a finite system of linear inequalities in $c_1, ..., c_k$. A terminating algorithm can determine whether this system has a solution, so this theorem gives a method for determining whether a subfamily $\mathscr{F}'$ is enough to guarantee an element in half the sets.

Proof. (1) $\Rightarrow$ (2) For each $\mathscr{G}$ allowed in (2), let $X(\mathscr{G})$ be the point

$$(|\mathscr{G}_1| - |\mathscr{G}|/2, ..., |\mathscr{G}_k| - |\mathscr{G}|/2)$$

in $\mathbf{R}^k$. Let C be the convex hull of these points, and let

$$\mathscr{N} = \{(x_1, ..., x_r) \in \mathbf{R}^k \mid x_i < 0 \text{ for all } i\}.$$

Suppose $C \cap \mathscr{N} \neq \varnothing$. Then for some families $\mathscr{G}^1, \mathscr{G}^2, ..., \mathscr{G}^r$ and for some nonnegative real numbers $w_1, w_2, ..., w_r$ with nonzero sum w (actually $w = 1$),

$$\sum_{j=1}^{r} w_j X(\mathscr{G}^j) \in \mathscr{N}.$$

Since $\mathscr{N}$ is open, we may assume the w_j are rattional. By multiplying by a common denominator, we may assume that the w_j are nonnegative integers with nonzero sum w (no longer 1, however). So we have $\mathscr{H}^1, ..., \mathscr{H}^w$ such that

$$\sum_{j=1}^{w} X(\mathscr{H}^j) \in \mathscr{N}$$

(where $\mathscr{H}^1 = \cdots = \mathscr{H}^{w_1} = \mathscr{G}^1$, $\mathscr{H}^{w_1+1} = \cdots = \mathscr{H}^{w_1+w_2} = \mathscr{G}^2$, etc.)

For a positive integer d, let $B = \{\alpha_1, \alpha_2, ..., \alpha_{wd}\}$ be a set of wd elements disjoint from A, and for $1 \leqslant s \leqslant wd$, let $B_s = B \backslash \{\alpha_s\}$. Let $\mathscr{F}$ be the following subset of $\mathscr{P}(A \cup B)$:

$$\begin{aligned}\mathscr{F}' &\cup (\{B_1, B_2, ..., B_d\} \uplus \mathscr{H}^1)\\ &\cup (\{B_{d+1}, B_{d+2}, ..., B_{2d}\} \uplus \mathscr{H}^2)\\ &\quad\vdots\\ &\cup (\{B_{(w-1)d+1}, B_{(w-1)d+2}, ..., B_{wd}\} \uplus \mathscr{H}^w)\\ &\cup (B \uplus \mathscr{P}(A)).\end{aligned}$$

It is not hard to see that $\mathscr{F}$ is a union-closed family. If $i \in A$, then

$$\begin{aligned}&|\mathscr{F}_i| - |\mathscr{F}|/2\\ &\quad = (|\mathscr{F}'_i| - |\mathscr{F}'|/2) + d[(|\mathscr{H}^1_i| - |\mathscr{H}^1|/2) + \cdots + (\mathscr{H}^w_i| - |\mathscr{H}^w|/2)].\end{aligned}$$

The quantity multiplied by d is negative, since it is the ith coordinate of the point $\sum_{j=1}^{w} X(\mathscr{H}^j)$, which is in $\mathscr{N}$. So for sufficiently large d, we obtain for all $i \in A$,

$$|\mathscr{F}_i| - |\mathscr{F}|/2 < 0.$$

But $\mathscr{F}' \subseteq \mathscr{F}$, so this contradicts (1).

So it must be that $C \cap \mathscr{N} = \varnothing$. But C and $\mathscr{N}$ are convex, with C closed, $\mathscr{N}$ open, so by the separating hyperplane theorem there exists a nonzero linear functional $c_1 x_1 + \cdots + c_k x_k$ which is nonnegative on C and negative on $\mathscr{N}$. Each c_i must be nonnegative, or else the linear functional would be positive somewhere in $\mathscr{N}$. By scaling, we may assume $c_1 + \cdots + c_k = 1$. For each allowable family $\mathscr{G}$, $X(\mathscr{G}) \in C$, so

$$c_1(|\mathscr{G}_1| - |\mathscr{G}|/2) + \cdots + c_k(|\mathscr{G}_k| - |\mathscr{G}|/2) \geqslant 0$$

$$c_1\,|\mathscr{G}_1| + \cdots + c_k\,|\mathscr{G}_\ell| \geqslant (c_1 + \cdots + c_k)\,|\mathscr{G}|/2 = |\mathscr{G}|/2$$

So (2) holds.

(2) $\Rightarrow$ (1) Let $A \cup B$ (with A, B disjoint) be the union of the sets in $\mathscr{F}$. Write

$$\mathscr{F} = \bigcup_{S \subseteq B} \{S\} \uplus \mathscr{G}^S,$$

where $\mathscr{G}^S \subseteq \mathscr{P}(A)$ for each S. Since $\mathscr{F}$ is union-closed and $\mathscr{F}' \subseteq \mathscr{F}$, $\mathscr{G}^S \uplus \mathscr{F}' = \mathscr{G}^S$ for each $S \subseteq B$. Also, since $\mathscr{F}$ is union-closed, $\mathscr{G}^S \uplus \mathscr{G}^S = \mathscr{G}^S$. So $\mathscr{G}^S$ is either union-closed or empty. (We cannot have $\mathscr{G}^S = \{\varnothing\}$, since then $\mathscr{G}^S \uplus \mathscr{F}' = \mathscr{F}' \neq \mathscr{G}^S$.) By (2), we have nonnegative real numbers $c_1, ..., c_k$ such that

$$\sum_{i=1}^{k} c_i\,|\mathscr{G}^S_i| \geqslant |\mathscr{G}^S|/2$$

for each S. (This holds even when $\mathscr{G}^S = \varnothing$.) Then

$$\begin{aligned}
\sum_{i=1}^{k} c_i |\mathscr{F}_i| &= \sum_{i=1}^{k} c_i \sum_{S \subseteq B} |\mathscr{G}_i^S| \\
&= \sum_{S \subseteq B} \sum_{i=1}^{k} c_i |\mathscr{G}_i^S| \\
&\geqslant \sum_{S \subseteq B} |\mathscr{G}^S|/2 \\
&= |\mathscr{F}|/2.
\end{aligned}$$

But if a weighted average of the $|\mathscr{F}_i|$ is at least $|\mathscr{F}|/2$, then for some i, $|\mathscr{F}_i| \geqslant |\mathscr{F}|/2$. ∎

In applying this theorem, the following lemma will be useful. The idea here is that if an average set of a collection $\mathscr{G}$ contains half the elements, then an average element is in half the sets.

LEMMA 3. *Suppose $S = \{1, 2, ..., m\}$ for some $m \geqslant 1$, and $S \in \mathscr{G} \subseteq \mathscr{P}(S)$. Let n_j be the number of sets of $\mathscr{G}$ of cardinality j. If $\sum_{j=1}^{m-1} (j - m/2)\, n_j \geqslant 0$, then $(1/m) \sum_{i=1}^{m} |\mathscr{G}_i| \geqslant |\mathscr{G}|/2$.*

Proof. Under the given assumptions,

$$\sum_{j=0}^{m} (j - m/2) n_j = (m/2)(n_m - n_0) + \sum_{j=1}^{m-1} (j - m/2) n_j \geqslant 0,$$

since $n_m = 1$, and n_0 is 0 or 1. So

$$\begin{aligned}
\sum_{i=1}^{m} |\mathscr{G}_i| &= \sum_{i=1}^{m} \sum_{S \in \mathscr{G}_i} 1 \\
&= \sum_{S \in \mathscr{G}} \sum_{i \in S} 1 \\
&= \sum_{S \in \mathscr{G}} |S| \\
&= \sum_{j=0}^{m} n_j j \\
&= \sum_{j=0}^{m} (j - m/2) n_j + (m/2) \sum_{j=0}^{m} n_j \\
&\geqslant 0 + (m/2)\, |G|,
\end{aligned}$$

and dividing by m gives the desired result. ∎

Corollary 2. *If a union-closed family $\mathscr{F}$ has a set S with one or two elements, some element of S is in at least half the elements of $\mathscr{F}$.*

Proof. Let $\mathscr{F}' = \{S\}$ in Theorem 1, and assume $S = \{1, ..., m\}$, where $m = 1$ or $m = 2$. If $\mathscr{G}$ is a possible family in (2) of the theorem, and $\mathscr{G} \neq \varnothing$, then $S \in \mathscr{G} \subseteq \mathscr{P}(S)$, and we can apply Lemma 3. (The hypothesis there is automatically satisfied when $m = 1$ or $m = 2$.) So in (2) of the theorem, we may take $c_1 = \cdots = c_m = 1/m$. ∎

Very recently, Sarvate and Renaud [4] gave an example proving the following.

Corollary 3. *There is a union-closed family $\mathscr{F}$ having a set S with three elements, such that no element of S is in at least half the sets of $\mathscr{F}$.*

Proof. Let $S = \{1, 2, 3\}$, and let $\mathscr{F}' = \{S\}$ in Theorem 1. We will show that statement (2) of the theorem is false for $\mathscr{F}'$. Among the restrictions on c_1, c_2, and c_3 are the following:

$$c_1 + c_2 + c_3 = 1$$

$$2c_1 + c_2 + c_3 \geqslant \tfrac{3}{2} \qquad (\text{from } \mathscr{G} = \{\varnothing, \{1\}, \{1, 2, 3\}\})$$

$$c_1 + 2c_2 + c_3 \geqslant \tfrac{3}{2} \qquad (\text{from } \mathscr{G} = \{\varnothing, \{2\}, \{1, 2, 3\}\})$$

$$c_1 + c_2 + 2c_3 \geqslant \tfrac{3}{2} \qquad (\text{from } \mathscr{G} = \{\varnothing, \{3\}, \{1, 2, 3\}\}).$$

Adding the last three, and dividing by four yields $c_1 + c_2 + c_3 \geqslant 9/8$, which contradicts the first equation. So by Theorem 1, there is a union-closed family having S as a member, such that no element of S is in half the elements of $\mathscr{F}$. ∎

In section 6, we explicitly describe such a family. Although the presence of a single three-element set is not enough to guarantee that one of its elements is in half the sets, our next result shows that if a union-closed family contains three such sets contained in the same four-element set, then one of the four elements is in half the sets. (The reader may check, using the theorem, that two such three-element sets do not suffice.)

Corollary 4. *Let $\mathscr{H} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$. If $\mathscr{F}$ is a union-closed family containing $\mathscr{H}$, then one of the elements* 1, 2, 3, *or* 4 *is in at least half the sets of $\mathscr{F}$.*

Proof. Any such $\mathscr{F}$ must also contain

$$\mathscr{F}' = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}.$$

We will show that we can take $c_1 = c_2 = c_3 = c_4 = \frac{1}{4}$ in (2) of Theorem 1. By Lemma 3, it suffices to show that for each $\mathscr{G}$ allowed in (2), the number n_3 of three-element sets is at least the number n_1 of one-element sets. If $n_1 \leqslant 3$, and $B_1, ..., B_{n_1}$ are the one-element sets, then we check that there exist distinct three-element sets $C_1, ..., C_{n_1} \in \mathscr{F}'$ such that $B_j \subset C_j$ for each j. But $\mathscr{F}' \uplus \mathscr{G} = \mathscr{G}$, so $C_1, ..., C_{n_1} \in \mathscr{G}$, and $n_3 \geqslant n_1$. If $n_1 = 4$, then since $\mathscr{G}$ is union-closed, $\mathscr{G} = \mathscr{P}(\{1, 2, 3, 4\}$ or $\mathscr{G} = \mathscr{P}(\{1, 2, 3, 4\} \backslash \{\varnothing\}$, so again $n_3 \geqslant n_1$. ∎

3. Proof for Small m and n

Throughout this section, we assume $\mathscr{F}$ is a union-closed family of n finite sets, one of which is $\varnothing$, and that the blocks are singletons. Also we assume the largest set is $A = \{1, 2, ..., m\}$, and n_j is the number of sets of cardinality j. We will prove Conjecture 1 in the case where $m \leqslant 7$ or $n \leqslant 28$, extending the known result for $n \leqslant 18$ [4].

Lemma 4. *If* $\sum_{j=1}^{m-1} (j - m/2) n_j \geqslant 0$, *then Conjecture* 1 *holds for* $\mathscr{F}$.

Proof. By Lemma 3, the average of $|\mathscr{F}_i|$ over i is at least $|\mathscr{F}|/2$, so for some i, $|\mathscr{F}_i| \geqslant |\mathscr{F}|/2$. ∎

Lemma 5. *If* $B \in \mathscr{F}$, $B \neq A$, *and* A *is the only set of* $\mathscr{F}$ *strictly containing* B, *then* $A \backslash B$ *is a block.*

Proof. If $A \backslash B$ is not a block, then there exist elements $\alpha, \beta \in A \backslash B$ and a set $S \in \mathscr{F}$ such that $\alpha \in S$, $\beta \notin S$. Then $B \cup S$ is a set of $\mathscr{F}$ containing B but not equal to A. ∎

Lemma 6. *Suppose* $m \geqslant 2$ and $\mathscr{F}$ *has singleton blocks. Then* $n_{m-1} \geqslant 1$, *and if equality holds,* $\mathscr{F} = \mathscr{G} \cup \{A\}$, *for some union-closed family* $\mathscr{G} \subseteq \mathscr{P}(A \backslash \{\alpha\})$, *for some* $\alpha \in A$.

Proof. Since $m \geqslant 2$ and $\mathscr{F}$ has singleton blocks, A is not the only set in $\mathscr{F}$. Let B the largest set of $\mathscr{F}$ not equal to A. Then by Lemma 5, $A \backslash B$ is a block. Since the blocks are singletons, $|B| = m - 1$.

If B contains all sets of $\mathscr{F}$ besides A, then $\mathscr{F}$ is of the form stated in the corollary. Otherwise, let C be the largest set of $\mathscr{F}$ besides A not contained in B. By Lemma 5, $A \backslash C$ is a block, so $|C| = m - 1$, and $n_{m-1} \geqslant 2$. ∎

Theorem 2. *Conjecture* 1 *holds if* $m \leqslant 7$.

Proof. If n_1 or n_2 is positive, we are done by Corollary 2. So assume $n_1 = n_2 = 0$. If $m \leqslant 6$, then the condition of Lemma 4 is satisfied, and we are done.

Suppose $m=7$. If $\mathscr{F}=\mathscr{G}\cup\{A\}$, for some union-closed family $\mathscr{G}\subseteq\mathscr{P}(A\backslash\{\alpha\})$, for some $\alpha\in A$, then an element in at least half the sets of $\mathscr{G}$ is in more than half the sets of $\mathscr{F}$, so we have reduced to a case with a smaller m. Otherwise, by Lemma 6, we may assume $n_6\geqslant 2$.

If $n_3\leqslant 10$, then

$$\begin{aligned}\sum_{j=1}^{m-1}(j-m/2)n_j &= (-1/2)n_3+(1/2)n_4+(3/2)n_5+(5/2)n_6\\ &\geqslant (-1/2)\,10+0+0+(5/2)2\\ &=0,\end{aligned}$$

and we are done by Lemma 4. So assume $n_3>10$.

There are $4n_3$ pairs (S, T) with S a three-element set in $\mathscr{F}$ and T *any* four-element subset of A containing S, since there are four possible T's for each S. In the list of such pairs, if any T occurs more than twice, an element of T will be in at least half the sets of $\mathscr{F}$, by Corollary 4. Otherwise, if s four-element sets appear at least once, then $4n_3-s$ four-element sets appear exactly twice. Each of these is the union of the two S's it is paired with, so each is in $\mathscr{F}$. Thus

$$n_4\geqslant 4n_3-s\geqslant 4n_3-\binom{7}{4}=4n_3-35.$$

Finally

$$\begin{aligned}\sum_{j=1}^{m-1}(j-m/2)n_j &= (-1/2)n_3+(1/2)n_4+(3/2)n_5+(5/2)n_6\\ &\geqslant (-1/2)n_3+(1/2)(4n_3-35)+0+(5/2)2\\ &=(3/2)n_3-25/2\\ &>0,\end{aligned}$$

since $n_3>10$. So we are done by Lemma 4. ∎

The rest of the section is devoted to proving Conjecture 1 for small n. For $S\subseteq A$, let $\bar{\mathscr{F}}_S$ be the subcollection of $\mathscr{F}$ of sets disjoint from S. Then $\bar{\mathscr{F}}_S$ is always a union-closed family or $\{\varnothing\}$. Let M_S be the largest set of $\bar{\mathscr{F}}_S$. Then $\bar{\mathscr{F}}_S$ is the collection of sets of $\mathscr{F}$ contained in M_S, so $M_S=M_T$ *iff* $\bar{\mathscr{F}}_S=\bar{\mathscr{F}}_T$. If $\alpha, \beta\in A$ and $M_{\{\alpha\}}=M_{\{\beta\}}$, then α, β are in the same block, so $\alpha=\beta$. Pick the smallest set $K=\{\alpha_1, ..., \alpha_k\}$ such that $\bar{\mathscr{F}}_K=\{\varnothing\}$. (This is possible, since $\bar{\mathscr{F}}_A=\{\varnothing\}$.) For any set S,

$$\bar{\mathscr{F}}_S=\bigcap_{\alpha\in S}\bar{\mathscr{F}}_{\{\alpha\}},$$

so we can assume (by replacing some α_i's if necessary) that each $\bar{\mathscr{F}}_{\{\alpha_i\}}$ is minimal; i.e, if $\beta \in A$ and $\bar{\mathscr{F}}_{\{\beta\}} \subseteq \bar{\mathscr{F}}_{\{\alpha\}}$, then $\beta = \alpha_i$. We may also assume $K = \{1, 2, ..., k\}$ without loss of generality. For $0 \leqslant j \leqslant k$, let s_j be the number of sets of $\mathscr{F}$ which contain exactly j elements of K. For example, $s_0 = 1$, since $\varnothing$ is the only set of $\mathscr{F}$ which does not intersect K.

LEMMA 7. *If* $\sum_{j=0}^{k} (j - k/2) s_j \geqslant 0$, *then Conjecture* 1 *holds for* $\mathscr{F}$.

Proof. The same argument used in Lemmas 3 and 4 works here to show that some element of K belongs to at least half the sets of $\mathscr{F}$. We will not bother to repeat the proof. ∎

LEMMA 8. *If* $S \subseteq K$, *then* $M_S \cap K = K \backslash S$.

Proof. By definition of M_S, $M_S \cap S = \varnothing$, so $M_S \cap K \subseteq K \backslash S$. On the other hand, suppose $\alpha \in K \backslash S$. If $\alpha \notin M_S$, then every set of $\mathscr{F}$ disjoint from S does not contain α. So every set of $\mathscr{F}$ disjoint from $K \backslash \{\alpha\}$ does not contain α. Thus $\bar{\mathscr{F}}_{K \backslash \{\alpha\}} = \bar{\mathscr{F}}_K = \{0\}$, contradicting the minimality of K. ∎

COROLLARY 5. *For* $0 \leqslant j \leqslant k$, $s_j \geqslant \binom{k}{j}$.

Proof. By Lemma 8, the $\binom{k}{j}$ sets M_S, where S is a $(k-j)$-element subset of K, are distinct, and each contains j elements of K. ∎

LEMMA 9. $s_k \geqslant m - k + 1$ *and* $s_0 = 1$.

Proof. *If* $\beta \in A \backslash K$, then $M_{\{\beta\}} \notin \bar{\mathscr{F}}_{\{i\}}$ for any $i \in K$, since otherwise $\bar{\mathscr{F}}_{\{\beta\}} \subseteq \bar{\mathscr{F}}_{\{i\}}$, contradicting the minimality of $\bar{\mathscr{F}}_{\{i\}}$. So $M_{\{\beta\}}$ contains each element of K. This gives us $m - k$ sets, and we proved earlier that they are distinct. But the set A also contains K, and it is distinct from the $M_{\{\beta\}}$'s, since $\beta \notin M_{\{\beta\}}$. Thus $s_k \geqslant m - k + 1$. Since $\bar{\mathscr{F}}_K = \{\varnothing\}$, it follows from the definitions that $s_0 = 1$. ∎

THEOREM 3. *Conjecture* 1 *holds if* $n \leqslant 28$.

Proof. By Theorem 2 we can assume $m \geqslant 8$. We break into cases, depending on the value of k. If $k \leqslant 2$, Lemma 7 applies, since $s_k \geqslant s_0 = 1$, by Lemma 9.

If $k = 3$, then by Lemmas 5 and 9, $s_3 \geqslant m - 3 + 1 \geqslant 6$ and

$$\begin{aligned} s_1 &= n - (s_0 + s_2 + s_3) \\ &\leqslant 28 - \left[1 + \binom{3}{2} + 6\right] \\ &= 18, \end{aligned}$$

so

$$\sum_{j=0}^{k} (j-k/2)s_j = (-3/2)s_0 + (-1/2)s_1 + (1/2)s_2 + (3/2)s_3$$

$$\geqslant (-3/2)1 + (-1/2)18 + (1/2)\binom{3}{2} + (3/2)6$$

$$= 0$$

and we are done by Lemma 7.

Similarly, if $k=4$, then $s_4 \geqslant m-4+1 \geqslant 5$ and

$$s_1 = n - (s_0 + s_2 + s_3 + s_4)$$

$$\leqslant 28 - \left[1 + \binom{4}{2} + \binom{4}{3} + 5\right]$$

$$= 12$$

so

$$\sum_{j=0}^{k} (j-k/2)s_j = (-2)s_0 + (-1)s_1 + 0 \cdot s_2 + s_3 + 2s_4$$

$$\geqslant (-2)1 + (-1)12 + \binom{4}{3} + 2 \cdot 5$$

$$= 0$$

and again we are done by Lemma 7.

Finally, if $k \geqslant 5$, then by Lemma 5,

$$n = s_0 + \cdots + s_k \geqslant \binom{k}{0} + \cdots + \binom{k}{k} = 2^k \geqslant 32,$$

contradicting our assumption $n \leqslant 28$. ∎

4. Lattices

In this section, we translate Conjecture 1 into the language of lattice theory and find some equivalent conjectures along the way. For definitions see [5]. We will need to extend the definition of meet-irreducible to finite join-semilattices. (In [5], the definition is for lattices only.) A *meet-irreducible* of a finite join-semilattice is an element other than $\hat{1}$ which is not the greatest lower bound of any pair of greater elements.

LEMMA 10. *In a finite join-semilattice, each element is the greatest lower bound of the meet-irreducibles greater than or equal to it.*

Proof. We proceed by induction. Suppose the result has been proved for all elements greater than P. If $P = \hat{1}$ or P is a meet-irreducible, we are done. Otherwise P is the greatest lower bound of Q and R, for some $Q, R > P$. Let A, B, C be the sets of meet-irreducibles greater than or equal to P, Q, R respectively. Then Q, R are the greatest lower bounds of B, C, respectively, by the inductive hypothesis, so P is the greatest lower bound of $B \cup C$. But P is a lower bound of A, and $B \cup C \subseteq A$, so P must be the greatest lower bound of A. ▮

LEMMA 11. *In a finite join-semilattice, if S is the greatest lower bound of some nonempty set of elements greater than S, then S is not a meet-irreducible.*

Proof. Suppose, on the contrary, that S is a meet-irreducible which is the greatest lower bound of $\{P_1, ..., P_r\}$ with $P_i > S$ for all i, and such that r is minimal. Clearly $r > 1$. Let A be the greatest lower bound of $\{P_1, ..., P_{r-1}\}$. Then $A > S$, by the minimality of r, and S is the greatest lower bound of A and P_r, a contradiction. ▮

THEOREM 4. *The following conjectures are equivalent:*

1. *Conjecture* 1

2. *If $\mathscr{L}$ is a finite join-semilattice with $n = |\mathscr{L}| \geqslant 2$, there is a meet-irreducible $M \in \mathscr{L}$ such that at most $n/2$ elements $P \in \mathscr{L}$ satisfy $P \leqslant M$.*

3. *If $\mathscr{L}$ is a finite join-semilattice with $n = |\mathscr{L}| \geqslant 2$, there exist distinct $S, T \in \mathscr{L}$ such that $S \vee U = T \vee U$ for at least $n/2$ elements $U \in \mathscr{L}$.*

4. *If $\mathscr{F}$ is a union-closed family with $n \geqslant 2$, there exist distinct $S, T \in \mathscr{F}$ such that $S \cup U = T \cup U$ for at least $n/2$ sets $U \in \mathscr{F}$.*

5. *Conjecture* 1 *holds for union-closed families containing $\varnothing$.*

6. *If $\mathscr{L}$ is a finite lattice with $n = |\mathscr{L}| \geqslant 2$, there is a meet-irreducible $M \in \mathscr{L}$ such that at most $n/2$ elements $P \in \mathscr{L}$ satisfy $P \leqslant M$.*

7. *If $\mathscr{L}$ is a finite lattice with $n = |\mathscr{L}| \geqslant 2$, there exist distinct $S, T \in \mathscr{L}$ such that $S \vee U = T \vee U$ for at least $n/2$ elements $U \in \mathscr{L}$.*

8. *If $\mathscr{F}$ is a union-closed family containning $\varnothing$, there exist distinct $S, T \in \mathscr{F}$ such that $S \cup U = T \cup U$ for at least $n/2$ sets $U \in \mathscr{F}$.*

Proof. $(1) \Rightarrow (2)$ We are given a join-semilattice $\mathscr{L}$ with $n \geqslant 2$ elements. For each $P \in \mathscr{L}$, let S_P be the set of meet-irreducibles M such that $M \ngeqslant P$, and let $\mathscr{F} = \{S_P | P \in \mathscr{L}\}$. By Lemma 10, P is the meet of the meet-irreducibles *not* in S_P, so $S_P = S_Q$ implies $P = Q$. So $\mathscr{F}$ is a collection of n

sets, not all of which are empty, since $n \geqslant 2$. For each meet-irreducible M, $M \geqslant P \vee Q$ *iff* $M \geqslant P$ and $M \geqslant Q$, so $S_{P \vee Q} = S_P \cup S_Q$, for any $P, Q \in \mathscr{L}$. Thus $\mathscr{F}$ is in fact a union-closed family. By (1) there exists an element M in at least $n/2$ sets of $\mathscr{F}$. So there is a meet-irreducible M for which there are at least $n/2$ elements $P \in \mathscr{L}$ such that $P \nleqslant M$. So (2) follows.

(2) $\Rightarrow$ (3) Given the join-semilattice $\mathscr{L}$, we can find a meet-irreducible $S \in \mathscr{L}$ such that at most $n/2$ elements $P \in \mathscr{L}$ satisfy $P \leqslant S$, by (2). Let $K = \{P \in \mathscr{L} | P > S\}$. Then $\hat{1} \in K$ so K is nonempty. (Remember that $\hat{1}$ is not a meet-irreducible.) Let T be the join of all lower bounds of K, so T is the greatest lower bound of K. Note that S is one such lower bound, so this is well defined, and $T \geqslant S$. By Lemma 11, $T = S$ would contradict the fact that S is a meet-irreducible, so $T > S$. We know that there are at least $n/2$ elements $U \in \mathscr{L}$ such that $U \nleqslant S$. For each of these, $S \vee U \in K$, so $S \vee U \geqslant T$ and $S \vee U \geqslant T \vee U$. But $S \leqslant T$, so $S \vee U \leqslant T \vee U$ also, so $S \vee U = T \vee U$.

(3) $\Rightarrow$ (4) This is trivial, since $\mathscr{F}$ is a join-semilattice with $S \vee T = S \cup T$.

(4) $\Rightarrow$ (1) We can assume the given union-closed family has at least two elements. Let S and T be the sets given by (4). Since $S \neq T$, we can find an element α such that α is in S or T, but not both. Then $S \cup U = T \cup U$ is possible only when $\alpha \in U$, so α is in at least $n/2$ sets of $\mathscr{F}$.

So far we have shown that (1), (2), (3), and (4) are equivalent. Similarly (5), (6), (7), and (8) are equivalent. But (1) and (5) are equivalent by Lemma 1, so all eight are equivalent. ∎

There are several remarks that should be made about this theorem. First, it shows the Conjecture 1 is not really a question about set membership; all that matters is the union structure of the collection. Also, the proof shows that if any of the eight conjectures holds for all $n \leqslant n_0$, then each of the others also holds for all $n \leqslant n_0$. (So by Theorem 3, all of them hold for $n \leqslant 28$.) Finally, for each of the lattice and join-semilattice conjectures, there is an equivalent dual. (So in fact, we have twelve equivalent conjectures!) For instance, the dual of (6) is the following, which is the form the conjecture takes in [5, Exercise 39, p. 161].

Conjecture 5. If $\mathscr{L}$ is a finite lattice with $n = |\mathscr{L}| \geqslant 2$, there is a join-irreducible $J \in \mathscr{L}$ such that at most $n/2$ elements $P \in \mathscr{L}$ satisfy $P \geqslant J$.

5. Types of Lattices

We can prove Conjecture 1 or Conjecture 5 if we make certain additional assumptions about the union-closed family or the lattice. In this section, most of the results will be for types of lattices. (We will rely heavily on the definitions in [5].) Our first result is an exception.

PROPOSITION 2. *Conjecture* 1 *holds for union-closed families $\mathscr{F}$ which are also closed under intersection.*

Proof. Without loss of generality assume $\mathscr{F}$ has singleton blocks. Let S be a minimal nonempty set in $\mathscr{F}$. Then for every $T \in \mathscr{F}$, $S \cap T$ must be S or $\varnothing$, by the minimality of S. So S is a block, and $|S| = 1$. By Corollary 2, Conjecture 1 holds. ∎

COROLLARY 6. *Conjecture* 5 *holds for distributive lattices.*

Proof. This follows simply by stepping through the proof of Theorem 4. The assumption that $\mathscr{F}$ is closed under intersections corresponds to the assumption that $\mathscr{L}$ is distributive. ∎

PROPOSITION 3. *Suppose $\mathscr{L}$ is a lattice with at least two elements. If for all $X \in \mathscr{L}$ the interval $[0, X]$ is complemented, then Conjecture* 5 *holds for $\mathscr{L}$.*

Proof. Let J be any join-irreducible of $\mathscr{L}$. It suffices to show that the inverse image of any element $S \geqslant J$ under the map $T \mapsto T \vee J$ has at least two elements. That inverse image contains both S itself and the complement C of J in $[0, S]$, and $S \neq C$ since $S \wedge J = J \neq 0 = C \wedge J$. ∎

This proposition generalizes several other partial results that have been proved. For example, the first of the next three corollaries appears as Exercise 39.b on page 161 in [5]. The last appears in [2].

COROLLARY 7. *If the Möbius function μ on the lattice $\mathscr{L}$ satisfies $\mu(0, X) \neq 0$ for all $X \in \mathscr{L}$, then Conjecture* 5 *holds for $\mathscr{L}$.*

Proof. If $\mu(0, X) \neq 0$, then $[0, X]$ is complemented, by part (i) of Corollary 4.34 in [1]. ∎

COROLLARY 8. *If the lattice $\mathscr{L}$ is relatively complemented, then Conjecture* 5 *holds for $\mathscr{L}$.*

Proof. If $\mathscr{L}$ is relatively complemented, then $[0, X]$ is complemented for all $X \in \mathscr{L}$. ∎

COROLLARY 9. *If $\mathscr{L}$ is a geometric lattice, then Conjecture* 5 *holds for $\mathscr{L}$.*

Proof. If $\mathscr{L}$ is a geometric lattice, then $\mathscr{L}$ is relatively complemented. ∎

6. Counterexamples to Generalizations

A natural approach to proving Conjecture 1 is to generalize it in the hope that the generalization will be easier to prove, whether by induction or some other means. The purpose of this section is to provide counterexamples to some of the most natural generalizations. These are important because they might suggest methods to construct a counterexample to Conjecture 1, if one exists. On the other hand, if the conjecture is true, these examples can help in the search for a proof, because their existence allows one to abandon proof techniques that would prove one of the false generalizations we mention.

One kind of generalization involves specifying "where to look" for an element in half the sets. Such generalizations (if true) would probably be easier to prove by induction than the original conjecture, because the inductive hypothesis would give you more than simply the existence of an element in half the sets. From looking at small examples, it seems that:

Non-Theorem 1. *Every nonempty set of $\mathscr{F}$ contains an element in half the sets of $\mathscr{F}$.*

Or at least:

Non-Theorem 2. *The smallest nonempty set of $\mathscr{F}$ contains an element in half the sets.*

In fact, Corollary 2 proves Non-Theorem 2 in the case when the smallest non-empty set has one or two elements. But a counterexample was recently given in [4], and in fact the techniques of the proof of Theorem 1 can be used to construct a counterexample where the smallest nonempty set has three elements.

For example, let

$$\mathscr{G}^i = \{\varnothing, \{i\}, \{1, 2, 3\}\}$$
$$\mathscr{H}^i = \{\{4, 5, 6, 7, 8, 9\} \backslash \{2i+2\}, \{4, 5, 6, 7, 8, 9\} \backslash \{2i+3\}\},$$

for $i = 1, 2, 3$, and consider the union-closed family

$$\begin{aligned}\mathscr{F} = \{\varnothing, \{1, 2, 3\}\} &\cup (\mathscr{G}^1 \uplus \mathscr{H}^1) \cup (\mathscr{G}^2 \uplus \mathscr{H}^2) \cup (\mathscr{G}^3 \uplus \mathscr{H}^3)\\ &\cup [P(\{1, 2, 3\}) \uplus \{\{4, 5, 6, 7, 8, 9\}\}]\end{aligned}$$

of 28 sets. Each element of the smallest nonempty set $\{1, 2, 3\}$ is in only 13 sets of $\mathscr{F}$.

Another possible generalization of Conjecture 1 is to allow each set S of the family to occur with some multiplicity of weight $w(S)$.

NON-THEOREM 3. *Suppose A is a finite set and $w{:}P(A) \to \mathbf{R}$ is a non-negative function such that*

1. *There exists $S \neq \varnothing$ such that $w(S) \geqslant w(\varnothing)$.*
2. *For all $S, T \subseteq A$, $w(S \cup T) \geqslant \min(w(S), w(T))$.*

Then there exists $\alpha \in A$ such that

$$\sum_{S \subseteq A, S \ni \alpha} w(S) \geqslant (1/2) \sum_{S \subseteq A} w(S).$$

The first condition corresponds to the requirement that a union-closed family contains a nonempty set, and the second corresponds to the requirement that a family be closed under unions. Conjecture 1 is equivalent to the special case where $w(S) = 0$ or 1 for all $S \subseteq A$. But here is a counter-example, for $A = \{1, 2, 3, 4, 5\}$:

S	$w(S)$
$\{1, 2, 3, 4\}$	10
$\varnothing$	9
$\{5\}, \{1, 5\}, \{2, 5\}, \{1, 2, 5\}, \{1, 2, 3, 4, 5\}$	2
All other sets containing 5, and $\{3\}, \{4\}, \{3, 4\}$	1
All other sets	0

The total weight is 43, but the sum of the weights of sets containing any given element is only 21.

ACKNOWLEDGMENTS

I thank Joseph Gallian, who made this research possible by organizing the Undergradute Research Program at the University of Minnesota, Duluth. The program was supported by the National Science Foundation (NSF 8709428) and the National Security Agency (MDA 904-88-H-2027).

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