The Method of Moments for Antenna Simulation

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December 23, 2024

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2.1 Maxwell's Equations

In a homogeneous region with permittivity ε and permeability μ, Maxwell's equations describe the electric fields $\mathbf{E}, \mathbf{D}$ and magnetic fields $\mathbf{H}, \mathbf{B}$.

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}, \tag{1}$$
$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J}_0, \tag{2}$$
$$\nabla \cdot \mathbf{D} = \rho, \tag{3}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

where $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \varepsilon\mathbf{E}$. We assume a time-harmonic formulation with time dependence $e^{j\omega t}$ throughout the text.

2.2 Green's Functions

A Green's function is the impulse response of a linear differential equations on a specified domain. Using the superposition principle, the solution of a differential equation can be expressed as a sum of Green's functions. This property will allow us to write an integral equation for the electric field in Section 2.3.

2.2.1 Electrostatic Green's Function

Consider the electrostatic Poisson equation

$$\nabla^2\phi = -\frac{\rho}{\varepsilon}, \tag{5}$$

where ρ is the charge density. To find the electric potential due to an arbitrary charge configuration, we can apply the method using the electrostatic Green's function as outlined above. First, we find the impulse response $G(r)$ such that

$$\nabla^2 G(\mathbf{r}) = -\frac{\delta(\mathbf{r})}{\varepsilon}. \tag{6}$$

Expanding the Laplace operator in spherical coordinates, we find

$$\nabla^2 G = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial G}{\partial r}\right] = \frac{\partial^2 G}{\partial r^2} + \frac{2}{r}\frac{\partial G}{\partial r} = \frac{1}{r}\frac{\partial^2 (rG)}{\partial r^2}$$

Outside of the point $r = 0$, the Dirac delta function is zero, such that

$$\frac{\partial^2(rG)}{\partial r^2} = 0 \qquad r > 0 \tag{7}$$

Integrating this differential equation twice gives us the general solution $G(r) = a + \frac{b}{r}$. We must have $G \to 0$ as $r \to \infty$, so the coefficient $a = 0$. To find coefficient b, we integrate the differential equation over a sphere of radius R:

$$\begin{aligned} b \iiint_V \nabla^2 \left(\frac{1}{r}\right) dV &= -\iiint_V \frac{\delta(r)}{\varepsilon} dV = -\frac{1}{\varepsilon} \\ \oiint_{\partial V} \nabla \left(\frac{1}{r}\right) \cdot \hat{\mathbf{r}} ds &= -\frac{1}{\varepsilon b} \\ \oiint_{\partial V} \frac{1}{r^2} ds &= \frac{1}{\varepsilon b} \\ 4\pi &= \frac{1}{\varepsilon b} \end{aligned}$$

Therefore, the electrostatic Green's function becomes

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\varepsilon r}, \qquad r = |\mathbf{r} - \mathbf{r}'|. \tag{8}$$

Then, the electric potential due to an arbitrary charge distribution $\rho(\mathbf{r})$ is

$$\phi(\mathbf{r}) = \iiint G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' = \frac{1}{4\pi\varepsilon} \iiint \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \tag{9}$$

2.2.2 Electrodynamic Green's Function

Similarly, we can derive a Green's function which satisfies the scalar Helmholtz equation. It will be shown later that the electrodynamic equations resemble the (vector) Helmholtz equation. Consider

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') + k^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r}, \mathbf{r}') \tag{10}$$

Using the same derivation as before, we can equate the left-hand side to zero for $r > 0$.

$$\frac{\partial^2(rG)}{\partial r^2} + k^2(rG) = 0 \qquad r > 0 \tag{11}$$

The solution to this differential equation is

$$G(r) = \frac{ae^{-jkr}}{r} + \frac{be^{+jkr}}{r}.$$

By requiring that $G \to 0$ as $r \to \infty$, we find that $b = 0$. To find coefficient a, we again integrate over a sphere of radius R.

$$a \iiint_V \nabla^2 \left(\frac{e^{-jkr}}{r} \right) + k^2 \left(\frac{e^{-jkr}}{r} \right) dV = -1$$

The first part of the integral can be tackled by applying the divergence theorem:

$$\iiint_V \nabla^2 \left(\frac{e^{-jkr}}{r} \right) dV = \oiint_{\partial V} \nabla \left(\frac{e^{-jkr}}{r} \right) \cdot \hat{\mathbf{r}} ds = 4\pi a^2 \left[\frac{\partial}{\partial r} \left(\frac{e^{-jkr}}{r} \right) \right]_{r=R}$$

$$\lim_{R \to 0} 4\pi a^2 \left[\frac{\partial}{\partial r} \left(\frac{e^{-jkr}}{r} \right) \right]_{r=R} = -4\pi$$

The second part of the integral is calculated by inspection:

$$\iiint_V k^2 \left(\frac{e^{-jkr}}{r} \right) dV = 4\pi k^2 R^2 \int_0^R \frac{e^{-jkr}}{r} dr$$

$$\lim_{R \to 0} 4\pi k^2 R^2 \int_0^R \frac{e^{-jkr}}{r} dr = 0$$

Therefore, we find that the coefficient $a = \frac{1}{4\pi}$ and the electrodynamic Green's function is

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jkr}}{4\pi r}, \qquad r = |\mathbf{r} - \mathbf{r}'|. \tag{12}$$

2.3 Electric Field Integral Equation (EFIE)

Taking the curl of (1) we obtain

$$\nabla \times \nabla \times \mathbf{E} = -j\omega\mu(j\omega\varepsilon\mathbf{E} + \mathbf{J}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$$

Gathering the unknown $\mathbf{E}$ on the left-hand side, and the source terms involving $\mathbf{J}$ on the right:

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = j\omega\mu\mathbf{J} - \frac{\nabla(\nabla \cdot \mathbf{J})}{j\omega\varepsilon} \tag{13}$$

This equation has the vector Helmholtz form with wavenumber $k = \omega/c = \omega\sqrt{\mu\varepsilon}$. Using the electrodynamic Green's function derived above, we get an integral equation for the electric field.

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \iiint G(\mathbf{r}, \mathbf{r}') \left[1 + \frac{\nabla'\nabla'}{k^2} \right] \mathbf{J}(\mathbf{r}') d\mathbf{r}' \tag{14}$$

Equation (14) is called the *electric field integral equation* (EFIE) and can also be written more compactly in terms of the operator $\mathcal{L}$.

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu(\mathcal{L}\mathbf{J})(\mathbf{r}), \tag{15}$$

where

$$(\mathcal{L}\mathbf{X})(\mathbf{r}) = \iiint G(\mathbf{r},\mathbf{r}')\left[1+\frac{\nabla'\nabla'}{k^2}\right]\mathbf{X}(\mathbf{r}')d\mathbf{r}' \tag{16}$$

3 Method of Moments

3.1 Discretization

Starting from the EFIE, we expand the unknown current distribution $\mathbf{J}(\mathbf{r})$ into a series of basis functions:

$$\mathbf{J}(\mathbf{r}) = \sum_{n=1}^{N} I_n \mathbf{f}_n(\mathbf{r}), \tag{17}$$

where I_n are the unknown coefficients representing the current, and $\mathbf{f}_n(\mathbf{r})$ are the vector basis functions. Many choices of basis are possible, but we will use the triangular basis functions shown in Figure 1. The orientation of the basis vectors $\hat{\mathbf{a}}_m$ is further discussed in Section 3.3.

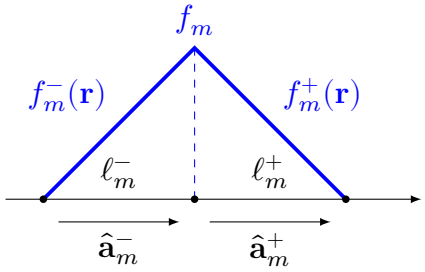

Figure 1: Triangular basis function

Substituting (17) into the EFIE, the $\mathcal{L}$ operator now operates on the (known) basis functions.

$$\mathbf{E}(\mathbf{r}) = j\omega\mu \sum_{n=1}^{N} I_n(\mathcal{L}\mathbf{f}_n)(\mathbf{r}) \tag{18}$$

Next, we can test (18) with the same basis functions $\mathbf{f}_m(\mathbf{r})$ with $m = 1, \ldots, N$.

$$\langle \mathbf{f}_m, \mathbf{E}\rangle = j\omega\mu \sum_{n=1}^{N} I_n \langle \mathbf{f}_m, \mathcal{L}\mathbf{f}_n\rangle \qquad m = 1, \ldots, N \tag{19}$$

This results in a dense, square $N \times N$ system of equations.

$$\{V\} = [Z]\{I\}, \tag{20}$$

with

$$\begin{aligned} V_m &= \int_{f_m} \mathbf{f}_m(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) d\mathbf{r} \\ Z_{mn} &= j\omega\mu \int_{f_m} \int_{f_n} \mathbf{f}_m(\mathbf{r}) \cdot \mathbf{f}_n(\mathbf{r}') G_k(\mathbf{r}, \mathbf{r}') d\mathbf{r}' d\mathbf{r} \\ &\quad - \frac{1}{j\omega} \int_{f_m} \int_{f_n} \nabla \cdot \mathbf{f}_m(\mathbf{r}) \nabla' \cdot \mathbf{f}_n(\mathbf{r}') G_k(\mathbf{r}, \mathbf{r}') d\mathbf{r}' d\mathbf{r} \end{aligned}$$

For ease of notation and to clarify the physical role each integral term plays, we redefine the impedance contribution in terms of the vector potential $\mathbf{A}$ and the scalar potential Φ.

$$Z_{mn} = j\omega\mu A_{mn} - \frac{j}{\omega\varepsilon}\Phi_{mn} = jk\eta_0 \left(A_{mn} - \frac{1}{k^2}\Phi_{mn} \right), \tag{21}$$

where

$$A_{mn} = \int_{f_m} \int_{f_n} \mathbf{f}_m(\mathbf{r}) \cdot \mathbf{f}_n(\mathbf{r}') G_k(\mathbf{r}, \mathbf{r}') d\mathbf{r}' d\mathbf{r} \tag{22}$$

$$\Phi_{mn} = \int_{f_m} \int_{f_n} \nabla \cdot \mathbf{f}_m(\mathbf{r}) \nabla' \cdot \mathbf{f}_n(\mathbf{r}') G_k(\mathbf{r}, \mathbf{r}') d\mathbf{r}' d\mathbf{r} \tag{23}$$

3.1.1 Quadrature

$$\int_{-1}^{+1} I(x) dx \approx \sum_{p=1}^{M} w_p I(x_p), \tag{24}$$

with quadrature weights w_p and quadrature points x_p. In the Julia code, Gauss-Legendre quadrature is used.

3.1.2 Non-Self Terms

The basis and testing functions are defined on (two-dimensional) segments of thin wire in three-dimensional space. Considering a pair of non-overlapping basis functions n, m, we can calculate the integrals numerically by applying a quadrature rule to both integrals. Because each shape function consists of two parts (for triangular basis functions), we can split the contributions into four parts.

$$A_{mn} = A_{mn}^{++} + A_{mn}^{+-} + A_{mn}^{-+} + A_{mn}^{--} \tag{25}$$

$$\Phi_{mn} = \Phi_{mn}^{++} + \Phi_{mn}^{+-} + \Phi_{mn}^{-+} + \Phi_{mn}^{--} \tag{26}$$

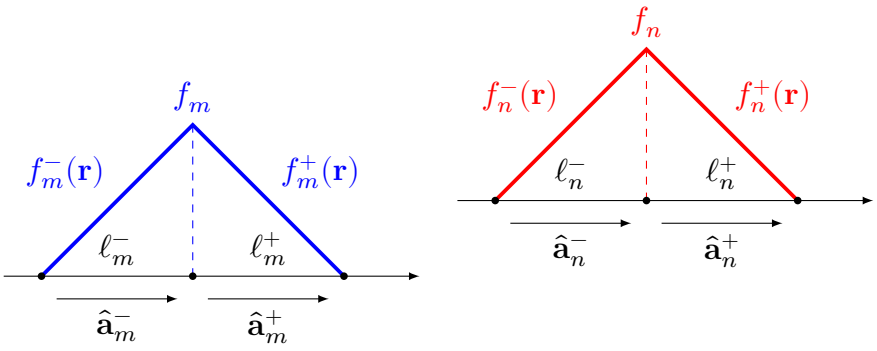

Figure 2: Definition of (non-overlapping) basis functions.

The contributions to the vector potential are derived from (22). The superscripts refer to the positive and negative elements belonging to the basis functions m and n as defined in Figure 2.

$$A_{mn}^{++} = \hat{\mathbf{a}}_m^+ \cdot \hat{\mathbf{a}}_n^+ \frac{\ell_m^+ \ell_n^+}{4} \sum_{p=1}^{M} \sum_{q=1}^{M} w_p w_q f_m(\mathbf{r}_p^+) f_n(\mathbf{r}_q^+) G(\mathbf{r}_p^+, \mathbf{r}_q^+)$$

$$A_{mn}^{+-} = \hat{\mathbf{a}}_m^+ \cdot \hat{\mathbf{a}}_n^- \frac{\ell_m^+ \ell_n^-}{4} \sum_{p=1}^{M} \sum_{q=1}^{M} w_p w_q f_m(\mathbf{r}_p^+) f_n(\mathbf{r}_q^-) G(\mathbf{r}_p^+, \mathbf{r}_q^-)$$

$$A_{mn}^{-+} = \hat{\mathbf{a}}_m^- \cdot \hat{\mathbf{a}}_n^+ \frac{\ell_m^- \ell_n^+}{4} \sum_{p=1}^{M} \sum_{q=1}^{M} w_p w_q f_m(\mathbf{r}_p^-) f_n(\mathbf{r}_q^+) G(\mathbf{r}_p^-, \mathbf{r}_q^+)$$

$$A_{mn}^{--} = \hat{\mathbf{a}}_m^- \cdot \hat{\mathbf{a}}_n^- \frac{\ell_m^- \ell_n^-}{4} \sum_{p=1}^{M} \sum_{q=1}^{M} w_p w_q f_m(\mathbf{r}_p^-) f_n(\mathbf{r}_q^-) G(\mathbf{r}_p^-, \mathbf{r}_q^-)$$

Similarly, for the scalar potential contributions. Here, we can pre-calculate the divergence of the basis functions, which is constant over the element (for triangular basis functions).

$$\nabla \cdot \mathbf{f}_m^+(\mathbf{r}) = \frac{-1}{\ell_m^+} \qquad \mathbf{r} \in \mathrm{supp}(f_m^+) \tag{27}$$

$$\nabla \cdot \mathbf{f}_m^-(\mathbf{r}) = \frac{+1}{\ell_m^-} \qquad \mathbf{r} \in \mathrm{supp}(f_m^-) \tag{28}$$

Therefore, the scalar potential terms become

$$\begin{aligned}
\Phi_{mn}^{++} &= \frac{-1}{\ell_m^+}\frac{-1}{\ell_m^+}\frac{\ell_m^+\ell_n^+}{4}\sum_{p=1}^{M}\sum_{q=1}^{M} w_p w_q G(\mathbf{r}_p^+, \mathbf{r}_q^+) \\
&= \frac{1}{4}\sum_{p=1}^{M}\sum_{q=1}^{M} w_p w_q G(\mathbf{r}_p^+, \mathbf{r}_q^+) \\
\Phi_{mn}^{+-} &= -\frac{1}{4}\sum_{p=1}^{M}\sum_{q=1}^{M} w_p w_q G(\mathbf{r}_p^+, \mathbf{r}_q^-) \\
\Phi_{mn}^{-+} &= -\frac{1}{4}\sum_{p=1}^{M}\sum_{q=1}^{M} w_p w_q G(\mathbf{r}_p^-, \mathbf{r}_q^+) \\
\Phi_{mn}^{--} &= \frac{1}{4}\sum_{p=1}^{M}\sum_{q=1}^{M} w_p w_q G(\mathbf{r}_p^-, \mathbf{r}_q^-)
\end{aligned}$$

3.1.3 Self Terms

If the basis functions n, m overlap (see Figure 3), then we must treat the singularity that arises in the integral of the Green's function. We can do this by evaluating the inner integral analytically, and the outer integral numerically using an M-point quadrature rule. Assume that the singularity is in the $++$ direction (only possible when $m = n$). For the vector potential term,

$$\begin{aligned}
A_{mn}^{++} &= \hat{\mathbf{a}}_m^+ \cdot \hat{\mathbf{a}}_n^+ \frac{\ell_m^+}{2}\sum_{p=1}^{M} w_p f_m(\mathbf{r}_p^+)\int_{f_n} f_n(\mathbf{r}')G(\mathbf{r}_p^+, \mathbf{r}')d\mathbf{r}' \\
&= \hat{\mathbf{a}}_m^+ \cdot \hat{\mathbf{a}}_n^+ \frac{\ell_m^+}{2}\sum_{p=1}^{M} w_p f_m(\mathbf{r}_p^+) S_1(\mathbf{r}_p^+)
\end{aligned}$$

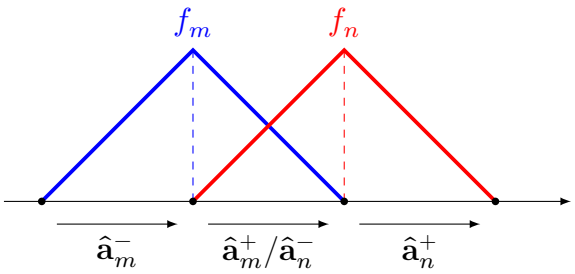

Figure 3: Definition of (overlapping) basis functions.

And for the scalar potential term,

$$\Phi_{mn}^{++} = \frac{\ell_m^+}{2} \sum_{p=1}^{M} w_p \frac{1}{\ell_m^+ \ell_n^+} \int_{f_n} G(\mathbf{r}_p^+, \mathbf{r}') d\mathbf{r}' = \frac{\ell_m^+}{2} \sum_{p=1}^{M} w_p S_2(\mathbf{r}_p^+)$$

where we define S_1 and S_2 as follows

$$S_1(\mathbf{r}) = \int_{f_n} f_n(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') d\mathbf{r}' \tag{29}$$

$$S_2(\mathbf{r}) = \frac{1}{s^2} \int_{f_n} G(\mathbf{r}, \mathbf{r}') d\mathbf{r}' \tag{30}$$

where s is the length of the segment containing the singularity.

3.1.4 Thin-Wire Approximation

To find an analytical expression for the integrals S_1 and S_2, we apply the thin-wire approximation. That is, we assume that the one-dimensional elements are in fact wires with a radius a which is small compared to the length of the element ℓ_n and the wavelength λ. We redefine the distance between $\mathbf{r}$ and $\mathbf{r}'$ as

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{(r - r')^2 + a^2}$$

Then, the two integrals can be evaluated analytically. The detailed derivation is presented in Gibson, 2022.

$$\begin{aligned} S_1(x, s) = \frac{1}{4\pi} \Bigg[& \frac{1}{s} \sqrt{a^2 + (x - s)^2} - \frac{1}{s} \sqrt{a^2 + x^2} \\ & + \frac{x}{s} \frac{\log\left(x + \sqrt{x^2 + a^2}\right)}{x - s + \sqrt{a^2 + (x - s)^2}} - \frac{jks}{2} \Bigg] \end{aligned} \tag{31}$$

$$S_2(x, s) = \frac{1}{4\pi s^2} \left[\frac{\log\left(x + \sqrt{x^2 + a^2}\right)}{x - s + \sqrt{a^2 + (x - s)^2}} - jks \right] \tag{32}$$

3.2 Practical Element-Wise Matrix Assembly

To assemble the system matrix equation (20), we loop over all pairs of nodal basis functions and calculate the contributions of each part using (25), (26). A more practical method of matrix assembly is to loop over all pairs of elements, each with pre-defined basis functions (see Figure 4).

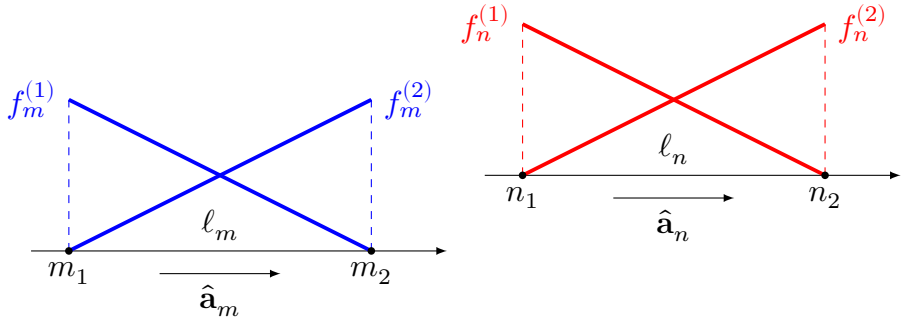

Figure 4: Definition of element-local basis functions.

Each element contributes to four locations in the system matrix ($[m_1, n_1]$, $[m_1, n_2]$, $[m_2, n_1]$, $[m_2, n_2]$), allowing the local contribution to be written conveniently as a 2×2 matrix. This way of constructing the local contribution is very similar to the way this is done in the finite element method.

$$A_{mn} = \hat{\mathbf{a}}_m \cdot \hat{\mathbf{a}}_n \frac{\ell_m \ell_n}{4} \sum_{p=1}^{M} \sum_{q=1}^{M} w_p w_q \begin{bmatrix} f_{m,p}^{(1)} f_{n,q}^{(1)} & f_{m,p}^{(1)} f_{n,q}^{(2)} \\ f_{m,p}^{(2)} f_{n,q}^{(1)} & f_{m,p}^{(2)} f_{n,q}^{(2)} \end{bmatrix} G(\mathbf{r}_p, \mathbf{r}_q) \tag{33}$$

$$\Phi_{mn} = \frac{1}{4} \sum_{p=1}^{M} \sum_{q=1}^{M} w_p w_q \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} G(\mathbf{r}_p, \mathbf{r}_q) \tag{34}$$

where $f_{m,p}^{(1)}$ is basis function $f_m^{(1)}$, evaluated at $\mathbf{r}_p$, etc.

It is now very easy to determine whether the self or non-self terms must be used, since there is only an overlap of basis functions when elements m and n are the same, i.e., $m = n$. The self-terms become

$$A_{mm} = \frac{\ell_m}{2} \sum_{p=1}^{M} w_p \begin{bmatrix} f_{m,p}^{(1)} & f_{m,p}^{(1)} \\ f_{m,p}^{(2)} & f_{m,p}^{(2)} \end{bmatrix} S_1(\mathbf{r}_p) \tag{35}$$

$$\Phi_{mm} = \frac{\ell_m}{2} \sum_{p=1}^{M} w_p \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} S_2(\mathbf{r}_p) \tag{36}$$

3.3 Boundary Conditions

Two boundary conditions arise from the equation of charge conservation:

1. The basis vectors $\mathbf{a}_n$ should be oriented such that (37) holds at every node.
2. A condition $I = 0$ must be imposed on the end-points of every segment, as this is the only way to enforce (37).

$$\nabla \cdot \mathbf{J} = 0 \tag{37}$$

3.4 Post-processing

3.4.1 Antenna impedance

To calculate the impedance seen by a source embedded in segment e, with nodes n_1, n_2, we make use of the current coefficient vector $\{I\}$. The impedance is defined as

$$Z = \frac{V_{src}}{I_{src}}, \tag{38}$$

where V_{src} is the known source voltage and I_{src} is the solved current averaged over segment e:

$$I_{src} = \frac{1}{2}\left(I[n_1] + I[n_2]\right)$$

3.4.2 Far-field radiation

The far-field radiation pattern can also be derived from the solved current vector. The EFIE can be simplified by assuming that the $1/r$ are dominant, and that higher order terms have negligible amplitude in the far field.

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \int \mathbf{J}(\mathbf{r}')G_k(\mathbf{r}, \mathbf{r}')d\mathbf{r}' \tag{39}$$

This expression can be further simplified by considering that $\mathbf{r} \gg \mathbf{r}'$, such that

$$|\mathbf{r} - \mathbf{r}'| = \begin{cases} r & \text{for amplitude variations} \\ r - \mathbf{r}' \cdot \hat{\mathbf{r}} & \text{for phase variations} \end{cases} \tag{40}$$

where $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. Applying (40) to (39) yields

$$\mathbf{E}(\mathbf{r}) = j\omega\mu\frac{e^{-jkr}}{4\pi r} \int ((\hat{\mathbf{r}} \cdot \mathbf{J}(\mathbf{r}'))\hat{\mathbf{r}} - \mathbf{J}(\mathbf{r}'))e^{jk\mathbf{r}'\cdot\hat{\mathbf{r}}}d\mathbf{r}' \tag{41}$$

Applying the same discretization procedure,

$$\mathbf{E}(\mathbf{r}) = j\omega\mu\frac{e^{-jkr}}{4\pi r} \sum_{n=1}^{N} I_n \int_{f_n} ((\hat{\mathbf{r}} \cdot \hat{\mathbf{a}}_n)\hat{\mathbf{r}} - \hat{\mathbf{a}}_n)f_n(\mathbf{r}')e^{jk\mathbf{r}'\cdot\hat{\mathbf{r}}}d\mathbf{r}' \tag{42}$$

The integral can be tackled using quadrature and all the terms are known.