Notas de estudo em Análise I (Análise Real)
Um guia de teoremas, resultados importantes e exercícios

Gil S. M. Neto
Graduando em Matemática Aplicada - UFRJ
gilsmneto@gmail.com, gil.neto@ufrj.br
http://mirandagil.github.io

Criado em 22 de Julho de 2019

Atualizado em August 23, 2019

Contents

1 Teoria Ingênua dos Conjuntos

Definição 1.1 (Informal de conjuntos)**.** *Um conjunto é uma coleção não ordenada de objetos. Se x é um objeto do conjunto A, dizemos $x \in A$, caso contrário dizemos $x \notin A$.*
Exemplo: $3 \in \{1, 2, 3, 4, 5\};\ 7 \notin \{1, 2, 3, 4, 5\}$

Axioma 1.1 (Conjuntos são objetos)**.** *Se A é um conjunto, então A também é um objeto, ou seja, se existe outro conjunto B, então faz sentido inferir $A \in B$ ou $A \notin B$*

Exemplo. Seja $B = \{1, 3, \{4, 5\}, 8\};\ A = \{4, 5\}$, então $A \in B$
Seja $C = \{1, 3, 4, 5, 8\};\ D = \{4, 5\}$, então $C \subset D$
é importante notar que apesar de $4 \in A, 5 \in A$, é verdade que $4 \notin B, 5 \notin B$ (verificar)

Definição 1.2 (Subconjuntos)**.** $A \subset B \iff x \in A \implies x \in B, \forall x \in A$

Definição 1.3 (Igualdade de Conjuntos)**.** *Definimos dois conjuntos $A = B \iff A \subset B \wedge B \subset A$*
Ou seja, $x \in A \implies x \in B, \forall x \in A \wedge y \in B \implies y \in A, \forall y \in B$

Axioma 1.2 (Conjunto Vazio)**.** *Existe um conjunto ao qual nenhum objeto pertence. A este grupo denominamos $\emptyset$. Para qualquer objeto x, temos $x \notin \emptyset$.*

Lema 1.1 (O Conjunto vazio é subconjunto de todo conjunto)**.** *Seja A um conjunto qualquer, então $\emptyset \subset A$*

Proof. Suponha que $\emptyset \not\subset A$, para qualquer conjunto A. Para negar a Definição 1.2 teremos: $A \not\subset B \iff \exists x \in A; x \notin B$
Logo, para termos $\emptyset \not\subset A$, deve existir um objeto em $\emptyset$ que não está contido em A, mas não há nenhum objeto em $\emptyset$, logo uma contradição, e temos $\emptyset \subset A, \forall A$ ∎

Lema 1.2 (O conjunto vazio é único)**.** *Proof.* Seja $\emptyset, \emptyset'$ conjuntos vazios, então do Lema 1.1 temos $\emptyset \subset \emptyset'$ e $\emptyset' \subset \emptyset$, e pela Definição 1.3 $\emptyset = \emptyset'$. ∎

Lema 1.3 (Escolha única)**.** *Seja A um conjunto não vazio, então existe ao menos um x tal que $x \in A$*

Proof. Suponha que não exista nenhum objeto x pertencente a A, então: $x \notin A, \forall x$, mas isso implicaria que A é um conjunto vazio, o que contraria a hipótese. ∎

Este lema nos permite escolher algum elemento de A. Ainda mais, dado uma família finita de Conjuntos $A_1, A_2, \dots, A_n$, podemos escolher um elemento de cada conjunto $x_1, x_2, \dots, x_n$. Para o caso infinito cairá no Axioma da Escolha, assunto a ser desenvolvido em outro momento.

Axioma 1.3 (Singleton)**.** *Dado um objeto a, então existe um conjunto de apenas um elemento $\{a\}$. Ou seja, para todo objeto $x, x \in \{a\} \iff y = a$. Ainda mais, para todo objeto a, b existe um conjunto $\{a, b\}$ onde $\forall y, y \in \{a, b\} \iff y = a \vee y = b$*

2 A construção dos números

2.1 Números Naturais e Inteiros $\mathbb{N}, \mathbb{Z}$

In this section, we list some analytic statements regarding the convergence of Dirichlet series. We omit the proof of most theorems in this section; they generally reduce to extensive computation. Still, they make good exercises for the reader.

Proposição 1. *Let*

$$f(n) = \sum_{n \geq 1} \frac{a(n)}{n^s}$$

be a Dirichlet series and let $S(x) = \sum_{n \leq x} a(n)$, and suppose there exist constants a and b such that $|S(x)| \leq ax^b$ for all large x. Then, $f(s)$ converges uniformly for s in

$$D(b, \delta, \epsilon) = \{\Re(s) \geq b + \delta, \arg(s - b) \leq \pi/2 - \epsilon\}$$

for all $\delta, \epsilon \geq 0$, and it converges to an analytic function on the half plane $\Re(s) > b$. (Note that $\Re(s)$ denotes the real part of s.)

Lema 2.1. *The Riemann zeta function $\zeta(s)$ has a meromorphic continuation to the half plane $\Re(s) > 0$ with a simple pole at $s = 1$.*

Lema 2.2. *For s real and $s > 1$,*

$$\frac{1}{s-1} \leq \zeta(s) \leq 1 + \frac{1}{s-1}$$

Hence, $\zeta(s)$ has a simple pole at $s = 1$ and

$$\zeta(s) = \frac{1}{s-1} + \text{function holomorphic near } 1$$

Proof. This is left as an exercise to the reader. (Hint: Look at the graph of $y = x^{-s}$ and relate $\zeta(s)$ to the area under the curve.) ■

Armed with this fact, we can look at other interesting Dirichlet series.

Proposição 2. *Let $f(n)$ be a Dirichlet series for which there exists constants C, a, and $b < 1$ such that $|S(n) - an| \leq Cx^b$. Then, f extends to a meromorphic function on $\Re(s) > b$ with a simple pole at $s = 1$ with residue a.*

Proof. For the Dirichlet series $f(s) - a\zeta(s)$, $|S(n)| \leq Cx^b$, so by Proposition 1, this series converges for $\Re(s) > b$. The result readily follows. ■

Before we move on, we encounter one last lemma that will prove to be useful soon.

Lema 2.3. *Let $u_1, u_2, \cdots$ be a sequence of real numbers ≥ 2 for which*

$$f(s) = \prod_{j=1}^{\infty} \frac{1}{1 - u_j^{-s}}$$

is uniformly convergent on each region $D(1, \delta, \epsilon)$ (with $\delta, \epsilon > 0$). Then,

$$\log f(s) \sim \sum \frac{1}{u_j^s}$$

as $s \to 1^+$ (i.e., from the right side of the plane).

Proof. This is a simple exercise in manipulating sums. (Hint: use the Maclaurin series for $\log(1-x)$ and then break the double sum apart.) ■

2.2 Números Racionais $\mathbb{Q}$

Now, we introduce some basic character theory. In particular, knowing certain statements about characters - namely, the orthogonality relations - will aid us in our study of L-functions.

Definição 2.1. *A one-dimensional representation of a group G, i.e. $\chi : G \longrightarrow \mathbb{C}^\times$ is a character of G. Note that this map is a homomorphism.*

Proposição 3. *For a character χ of G, we have that $\sum_{a \in G} \chi(a) = \begin{cases} |G| & \text{if } \chi = \chi_0 \text{ (the trivial character)} \\ 0 & \text{otherwise} \end{cases}$*

Proof. The first part is obvious. If we have a nontrivial character χ, then for some $g \in G$, $\chi(g) \neq 1$. Then,

$$\chi(g) \sum_{a \in G} \chi(a) = \sum_{a \in G} \chi(ga) = \sum_{a \in G} \chi(a),$$

meaning $\sum_{a \in G} \chi(a) = 0$, as desired. ■

Proposição 4. *Suppose the group G is abelian. Fix some $a \in G$. Then,*

$$\sum_{\chi \in \hat{G}} \chi(a) = \begin{cases} |G| & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, $\hat{G} = \mathrm{Hom}(G, C^\times)$ is the character group of G.

Proof. Using the fact that G is noncanonically isomorphic to $\hat{G}$, this proof becomes identical to that of the previous proposition. ■

Before we introduce some new tools, let us provide some motivation to our treatment of L-functions. Let K be a number field and $\mathfrak{m}$ be some modulus. Begin with the Dedekind zeta function, $\zeta_K(s)$. For some class $\mathfrak{t} \in C_{\mathfrak{m}}$ (i.e., the class group), define the partial zeta function to be

$$\zeta(s, \mathfrak{t}) = \sum_{\mathfrak{a} \geq 0, \mathfrak{a} \in \mathfrak{t}} \frac{1}{N\mathfrak{a}^s}$$

Note that for every character χ of the class group,

$$\zeta_K(s) = \sum_{\mathfrak{t} \in C_{\mathfrak{m}}} \zeta(s, \mathfrak{t}) \text{ and}$$

$$L(s, \chi) = \sum_{\mathfrak{t} \in C_{\mathfrak{m}}} \chi(\mathfrak{t}) \zeta(s, \mathfrak{t})$$

In other words, knowing about $\zeta(s, \mathfrak{t})$ can tell us about the Dedekind zeta function as well as the corresponding L-function.

Teorema 2.1. *The partial zeta function $\zeta(s, \mathfrak{t})$ is analytic for $\Re(s) > 1 - \frac{1}{[K:\mathbb{Q}]}$ except for a simple pole at $s = 1$. If we let $g_{\mathfrak{m}}$ denote the residue at $s = 1$, then $g_{\mathfrak{m}}$ is independent of $\mathfrak{t}$.*

Proof. We omit the proof of this theorem, mainly because it relies on the famous class number formula. It allows us to determine exactly what $g_{\mathfrak{m}}$ is. ■

Corolário 1. *If χ is not the trivial character, the L-function $L(s, \chi)$ is analytic for $\Re(s) > 1 - \frac{1}{[K:\mathbb{Q}]}$.*

Proof. Near $s = 1$,

$$L(s, \chi) = \sum_{\mathfrak{t} \in C_{\mathfrak{m}}} \chi(\mathfrak{t}) \zeta(s, \mathfrak{t}) = \frac{\sum_{\mathfrak{t} \in C_{\mathfrak{m}}} \chi(\mathfrak{t}) g_{\mathfrak{m}}}{s - 1} + \text{ holomorphic function}$$

and Proposition 3 shows us that the numerator of the first term is 0. ■

2.3 Números Reais $\mathbb{R}$

At last, we come across one type of density. For a set T of prime ideals of K, we define $\zeta_{K,T}(s) = \prod_{\mathfrak{p} \in T} \frac{1}{1 - N\mathfrak{p}^{-s}}$.

Definição 2.2. *If some positive integral power $\zeta_{K,T}(s)^n$ of $\zeta_{K,T}(s)$ extends to a meromorphic function on a neighborhood of 1 having a pole of order m at 1, we say that T has polar density $\delta(T) = \frac{m}{n}$.*

Proposição 5 (Properties of Polar Density)**.** *We have the following assertions:*

1. *The set of all prime ideals of K has polar density 1.*
2. *The polar density of every set is nonnegative.*
3. *If T is the disjoint union of T_1 and T_2, and two of the three polar densities exist, then so does the third, and we have $\delta(T) = \delta(T_1) + \delta(T_2)$.*
4. *If $T \subset T'$, then $\delta(T) \leq \delta(T')$.*
5. *A finite set has density zero.*

Proof.

1. We know that $\zeta_{K,T}(s)$ extends to a neighborhood of 1, where it has a simple pole. Thus $\frac{m}{n} = 1$, as desired.
2. Having a negative polar density means $m < 0$, i.e., $\zeta_{K,T}(s)$ is holomorphic in a neighborhood of $s = 1$ and zero there. However, $\zeta_{K,T}(1) = \prod_{\mathfrak{p} \in T} \frac{1}{1 - N\mathfrak{p}^{-1}} > 0$, meaning polar density is nonnegative.

3. Observe that $\zeta_{K,T}(s) = \zeta_{K,T_1}(s) \cdot \zeta_{K,T_2}(s)$. Suppose $\zeta_{K,T}(s)^n$ and $\zeta_{K,T_1}(s)^{n_1}$ extend to meromorphic functions with poles of order m and m_1, respectively; the other two cases are identical. Then

$$\zeta_{K,T_2}(s)^{nn_1} = \frac{\zeta_{K,T}(s)^{nn_1}}{\zeta_{K,T_1}(s)^{nn_1}}$$

extends to a meromorphic function in a neighborhood of $s = 1$ and has a pole there of order $mn_1 - m_1 n$. Thus, $\delta(T_2) = \frac{mn_1 - m_1 n}{nn_1} = \frac{m}{n} - \frac{m_1}{n_1} = \delta(T) - \delta(T_1)$, as desired.

4. This follows readily from 3.

5. This is obvious; $m = 0$ because $\zeta_{K,T}(s)$ is finite and positive. Moreover, there is no pole at $s = 1$.

■

Proposição 6. *If T contains no primes $\mathfrak{p}$ for which $N\mathfrak{p}$ is prime (in $\mathbb{Z}$), then $\delta(T) = 0$.*

Proof. Let $\mathfrak{p}$ be a prime in T. Since $N\mathfrak{p} = p^f$ (where p lies under $\mathfrak{p}$ in $\mathbb{Z}$ and f denotes the inertial degree of $\mathfrak{p}$), we must have $f \geq 2$; if $f = 1$, $N\mathfrak{p}$ would be prime. Moreover, for any given prime $p \in \mathbb{Z}$, there are at most $[K : \mathbb{Q}]$ primes of K lying over p. Thus, $\zeta_{K,T}(s)$ can be decomposed into a product $\prod_{1 \leq i \leq [K:\mathbb{Q}]} g_i(s)$ of d infinite products over the prime numbers, with each factor of g_i being either a 1 or a $\frac{1}{1-p^{-fs}}$ (for every prime p). Thus, for any i, $g_i(1) \leq \prod_p \frac{1}{1-p^{-f_p}} \leq \prod_p \frac{1}{1-p^{-2}} = \zeta(2) = \frac{\pi^2}{6}$. Thus, $g_i(s)$ is holomorphic at $s = 1$, meaning that the order of the pole there must be 0 (recall that polar density cannot be negative). We conclude that $\delta(T) = 0$. ■

Corolário 2. *Let T_1 and T_2 be sets of prime ideals in K. If the sets differ only by primes $\mathfrak{p}$ for which $N\mathfrak{p}$ is not prime and one of the two sets has polar density, then so does the other, and the densities are equal.*

At last, the time has come to exploit the power of polar density. It turns out we can derive some important analytic results.

Teorema 2.2. *Let $L \supset K$ be a field extension of finite degree and let M be its Galois closure. Then the set of prime ideals of K that split completely in L has density $\frac{1}{[M:K]}$.*

Proof. The first thing to notice is that a prime ideal $\mathfrak{p}$ of K splits completely in L if and only if it splits completely in M. One direction is easy: if it splits completely in M, it must split completely in the subfield L. If it splits completely in L, then it also splits completely in every conjugate field L'. All of these conjugate fields must lie under the decomposition field (the fixed field of the decomposition group of $\mathrm{Gal}(M/K)$), and so their compositum is a field lying under the decomposition field as well. This field is just M! $\mathfrak{p}$ splits completely only up to and including the decomposition field, so we conclude that it splits completely in M as well.

Thus, it suffices to prove this theorem with the assumption that L is Galois over K. Let S be the set of prime ideals of K that split completely in L and let T be the primes of L lying over a prime ideal in S. For each $\mathfrak{p} \in S$, there are exactly $[L : K]$ prime ideals $\mathfrak{P} \in T$, and for each of them, $N_K^L(\mathfrak{P}) = \mathfrak{p}$ (where N_K^L denotes norm). Thus, $N\mathfrak{P} = N\mathfrak{p}$ (where N denotes norm over $\mathbb{Q}$). This tells us that $\zeta_{L,T}(s) = \zeta_{K,S}(s)^{[L:K]}$. Also, T contains every prime ideal of L that is unramified over K and for which $N\mathfrak{P}$ is prime (in $\mathbb{Z}$). Thus, T differs from the set of all prime ideals in L by a set of polar density 0 (using Corollary 2), and so T has density 1. Moreover, this shows that $\zeta_{K,S}$ has the property signifying that S is a set of polar density $\frac{1}{[L:K]}$, as desired. ■

Corolário 3. *If $f(x) \in K[x]$ splits into linear factors modulo $\mathfrak{p}$ for all but finitely many prime ideals $\mathfrak{p}$ of K, then f splits into linear factors in K.*

Proof. If L is the splitting field of f, then L is Galois over K. Now, use Theorem 2.2 on L/K. For more interesting details, see Bhandarkar [**?**], Section 4. ■

Corolário 4. *For every abelian extension L/K and every finite set S of primes of K including those that ramify in L, let I_K^S denote the fractional ideals that are prime to all ideals in S. Then, the Artin map*

$$\left(\frac{L/K}{\cdot}\right) : I_K^S \longrightarrow \mathrm{Gal}(L/K)$$

is surjective.

Proof. Let H be the image of the Artin map; it is some subgroup of $\mathrm{Gal}(L/K)$. If its fixed field is L^H, then we see that $H = \mathrm{Gal}(L/L^H)$ is the image. For all $\mathfrak{p} \notin S$, $\left(\frac{L^H/K}{\mathfrak{p}}\right) = \left(\frac{L/K}{\mathfrak{p}}\right)|_{L^H} = 1$, which implies that $\mathfrak{p}$ splits completely in L^H. Thus, all but finitely many prime ideals of $\mathcal{O}_K$ split completely in L^H, so Theorem 2.2 tells us that $[L^H : K] = 1$; in other words, the Artin map is surjective. ■

Bibliografia

References

[1] Tao, T.: *Analysis I*. 1st ed. Hindustan Book Agency (2006)