
NOTAS DE ESTUDO EM ANÁLISE I (ANÁLISE REAL)

UM GUIA DE TEOREMAS, RESULTADOS IMPORTANTES E EXERCÍCIOS

Gil S. M. Neto

Graduando em Matemática Aplicada - UFRJ
gilsmneto@gmail.com, gil.neto@ufrj.br
<http://mirandagil.github.io>

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1 Teoria Ingênua dos Conjuntos

Definição 1.1 (Informal de conjuntos). *Um conjunto é uma coleção não ordenada de objetos. Se x é um objeto do conjunto A , dizemos $x \in A$, caso contrário dizemos $x \notin A$.*

Exemplo: $3 \in \{1, 2, 3, 4, 5\}$; $7 \notin \{1, 2, 3, 4, 5\}$

Axioma 1.1 (Conjuntos são objetos). *Se A é um conjunto, então A também é um objeto, ou seja, se existe outro conjunto B , então faz sentido inferir $A \in B$ ou $A \notin B$*

Exemplo. Seja $B = \{1, 3, \{4, 5\}, 8\}$; $A = \{4, 5\}$, então $A \in B$

Seja $C = \{1, 3, 4, 5, 8\}$; $D = \{4, 5\}$, então $C \subset D$

é importante notar que apesar de $4 \in A, 5 \in A$, é verdade que $4 \notin B, 5 \notin B$ (verificar)

Definição 1.2 (Subconjuntos). $A \subset B \iff x \in A \implies x \in B, \forall x \in A$

Definição 1.3 (Igualdade de Conjuntos). *Definimos dois conjuntos $A = B \iff A \subset B \wedge B \subset A$*

Ou seja, $x \in A \implies x \in B, \forall x \in A \wedge y \in B \implies y \in A, \forall y \in B$

Axioma 1.2 (Conjunto Vazio). *Existe um conjunto ao qual nenhum objeto pertence. A este grupo denominamos \emptyset . Para qualquer objeto x , temos $x \notin \emptyset$.*

Lema 1.1 (O Conjunto vazio é subconjunto de todo conjunto). *Seja A um conjunto qualquer, então $\emptyset \subset A$*

Proof. Suponha que $\emptyset \not\subset A$, para qualquer conjunto A . Para negar a Definição 1.2 teremos: $A \not\subset B \iff \exists x \in A; x \notin B$

Logo, para termos $\emptyset \not\subset A$, deve existir um objeto em \emptyset que não está contido em A , mas não há nenhum objeto em \emptyset , logo uma contradição, e temos $\emptyset \subset A, \forall A$ ■

Lema 1.2 (O conjunto vazio é único). *Proof.* Seja \emptyset, \emptyset' conjuntos vazios, então do Lema 1.1 temos $\emptyset \subset \emptyset'$ e $\emptyset' \subset \emptyset$, e pela Definição 1.3 $\emptyset = \emptyset'$. ■

Lema 1.3 (Escolha única). *Seja A um conjunto não vazio, então existe ao menos um x tal que $x \in A$*

Proof. Suponha que não exista nenhum objeto x pertencente a A , então: $x \notin A, \forall x$, mas isso implicaria que A é um conjunto vazio, o que contraria a hipótese. ■

Este lema nos permite escolher algum elemento de A . Ainda mais, dado uma família finita de Conjuntos A_1, A_2, \dots, A_n , podemos escolher um elemento de cada conjunto x_1, x_2, \dots, x_n . Para o caso infinito cairá no Axioma da Escolha, assunto a ser desenvolvido em outro momento.

Axioma 1.3 (Singleton). *Dado um objeto a , então existe um conjunto de apenas um elemento $\{a\}$. Ou seja, para todo objeto $x, x \in \{a\} \iff x = a$. Ainda mais, para todo objeto a, b existe um conjunto $\{a, b\}$ onde $\forall y, y \in \{a, b\} \iff y = a \vee y = b$*

2 A construção dos números

2.1 Números Naturais e Inteiros \mathbb{N}, \mathbb{Z}

In this section, we list some analytic statements regarding the convergence of Dirichlet series. We omit the proof of most theorems in this section; they generally reduce to extensive computation. Still, they make good exercises for the reader.

Proposição 1. *Let*

$$f(n) = \sum_{n \geq 1} \frac{a(n)}{n^s}$$

be a Dirichlet series and let $S(x) = \sum_{n \leq x} a(n)$, and suppose there exist constants a and b such that $|S(x)| \leq ax^b$ for all large x . Then, $f(s)$ converges uniformly for s in

$$D(b, \delta, \epsilon) = \{ \Re(s) \geq b + \delta, \arg(s - b) \leq \pi/2 - \epsilon \}$$

for all $\delta, \epsilon \geq 0$, and it converges to an analytic function on the half plane $\Re(s) > b$. (Note that $\Re(s)$ denotes the real part of s .)

Lema 2.1. The Riemann zeta function $\zeta(s)$ has a meromorphic continuation to the half plane $\Re(s) > 0$ with a simple pole at $s = 1$.

Lema 2.2. For s real and $s > 1$,

$$\frac{1}{s-1} \leq \zeta(s) \leq 1 + \frac{1}{s-1}$$

Hence, $\zeta(s)$ has a simple pole at $s = 1$ and

$$\zeta(s) = \frac{1}{s-1} + \text{function holomorphic near } 1$$

Proof. This is left as an exercise to the reader. (Hint: Look at the graph of $y = x^{-s}$ and relate $\zeta(s)$ to the area under the curve.) ■

Armed with this fact, we can look at other interesting Dirichlet series.

Proposição 2. Let $f(n)$ be a Dirichlet series for which there exists constants C , a , and $b < 1$ such that $|S(n) - an| \leq Cx^b$. Then, f extends to a meromorphic function on $\Re(s) > b$ with a simple pole at $s = 1$ with residue a .

Proof. For the Dirichlet series $f(s) - a\zeta(s)$, $|S(n)| \leq Cx^b$, so by Proposition 1, this series converges for $\Re(s) > b$. The result readily follows. ■

Before we move on, we encounter one last lemma that will prove to be useful soon.

Lema 2.3. Let u_1, u_2, \dots be a sequence of real numbers ≥ 2 for which

$$f(s) = \prod_{j=1}^{\infty} \frac{1}{1 - u_j^{-s}}$$

is uniformly convergent on each region $D(1, \delta, \epsilon)$ (with $\delta, \epsilon > 0$). Then,

$$\log f(s) \sim \sum \frac{1}{u_j^s}$$

as $s \rightarrow 1^+$ (i.e., from the right side of the plane).

Proof. This is a simple exercise in manipulating sums. (Hint: use the Maclaurin series for $\log(1 - x)$ and then break the double sum apart.) ■

2.2 Números Racionais \mathbb{Q}

Now, we introduce some basic character theory. In particular, knowing certain statements about characters - namely, the orthogonality relations - will aid us in our study of L-functions.

Definição 2.1. A one-dimensional representation of a group G , i.e. $\chi : G \rightarrow \mathbb{C}^\times$ is a character of G . Note that this map is a homomorphism.

Proposição 3. For a character χ of G , we have that $\sum_{a \in G} \chi(a) = \begin{cases} |G| & \text{if } \chi = \chi_0 \text{ (the trivial character)} \\ 0 & \text{otherwise} \end{cases}$

Proof. The first part is obvious. If we have a nontrivial character χ , then for some $g \in G$, $\chi(g) \neq 1$. Then,

$$\chi(g) \sum_{a \in G} \chi(a) = \sum_{a \in G} \chi(ga) = \sum_{a \in G} \chi(a),$$

meaning $\sum_{a \in G} \chi(a) = 0$, as desired. ■

Proposição 4. Suppose the group G is abelian. Fix some $a \in G$. Then,

$$\sum_{\chi \in \hat{G}} \chi(a) = \begin{cases} |G| & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

Here, $\hat{G} = \text{Hom}(G, \mathbb{C}^\times)$ is the character group of G .

Proof. Using the fact that G is noncanonically isomorphic to \hat{G} , this proof becomes identical to that of the previous proposition. ■

Before we introduce some new tools, let us provide some motivation to our treatment of L-functions. Let K be a number field and m be some modulus. Begin with the Dedekind zeta function, $\zeta_K(s)$. For some class $\mathfrak{t} \in C_m$ (i.e., the class group), define the partial zeta function to be

$$\zeta(s, \mathfrak{t}) = \sum_{\mathfrak{a} \geq 0, \mathfrak{a} \in \mathfrak{t}} \frac{1}{N\mathfrak{a}^s}$$

Note that for every character χ of the class group,

$$\zeta_K(s) = \sum_{\mathfrak{t} \in C_m} \zeta(s, \mathfrak{t}) \text{ and}$$

$$L(s, \chi) = \sum_{\mathfrak{t} \in C_m} \chi(\mathfrak{t}) \zeta(s, \mathfrak{t})$$

In other words, knowing about $\zeta(s, \mathfrak{t})$ can tell us about the Dedekind zeta function as well as the corresponding L-function.

Teorema 2.1. *The partial zeta function $\zeta(s, \mathfrak{t})$ is analytic for $\Re(s) > 1 - \frac{1}{[K:\mathbb{Q}]}$ except for a simple pole at $s = 1$. If we let g_m denote the residue at $s = 1$, then g_m is independent of \mathfrak{t} .*

Proof. We omit the proof of this theorem, mainly because it relies on the famous class number formula. It allows us to determine exactly what g_m is. ■

Corolário 1. *If χ is not the trivial character, the L-function $L(s, \chi)$ is analytic for $\Re(s) > 1 - \frac{1}{[K:\mathbb{Q}]}$.*

Proof. Near $s = 1$,

$$L(s, \chi) = \sum_{\mathfrak{t} \in C_m} \chi(\mathfrak{t}) \zeta(s, \mathfrak{t}) = \frac{\sum_{\mathfrak{t} \in C_m} \chi(\mathfrak{t}) g_m}{s - 1} + \text{holomorphic function}$$

and Proposition 3 shows us that the numerator of the first term is 0. ■

2.3 Números Reais \mathbb{R}

At last, we come across one type of density. For a set T of prime ideals of K , we define $\zeta_{K,T}(s) = \prod_{\mathfrak{p} \in T} \frac{1}{1 - N\mathfrak{p}^{-s}}$.

Definição 2.2. *If some positive integral power $\zeta_{K,T}(s)^n$ of $\zeta_{K,T}(s)$ extends to a meromorphic function on a neighborhood of 1 having a pole of order m at 1, we say that T has polar density $\delta(T) = \frac{m}{n}$.*

Proposição 5 (Properties of Polar Density). *We have the following assertions:*

1. *The set of all prime ideals of K has polar density 1.*
2. *The polar density of every set is nonnegative.*
3. *If T is the disjoint union of T_1 and T_2 , and two of the three polar densities exist, then so does the third, and we have $\delta(T) = \delta(T_1) + \delta(T_2)$.*
4. *If $T \subset T'$, then $\delta(T) \leq \delta(T')$.*
5. *A finite set has density zero.*

Proof.

1. We know that $\zeta_{K,T}(s)$ extends to a neighborhood of 1, where it has a simple pole. Thus $\frac{m}{n} = 1$, as desired.
2. Having a negative polar density means $m < 0$, i.e., $\zeta_{K,T}(s)$ is holomorphic in a neighborhood of $s = 1$ and zero there. However, $\zeta_{K,T}(1) = \prod_{\mathfrak{p} \in T} \frac{1}{1 - N\mathfrak{p}^{-1}} > 0$, meaning polar density is nonnegative.

3. Observe that $\zeta_{K,T}(s) = \zeta_{K,T_1}(s) \cdot \zeta_{K,T_2}(s)$. Suppose $\zeta_{K,T}(s)^n$ and $\zeta_{K,T_1}(s)^{n_1}$ extend to meromorphic functions with poles of order m and m_1 , respectively; the other two cases are identical. Then

$$\zeta_{K,T_2}(s)^{nn_1} = \frac{\zeta_{K,T}(s)^{nn_1}}{\zeta_{K,T_1}(s)^{nn_1}}$$

extends to a meromorphic function in a neighborhood of $s = 1$ and has a pole there of order $mn_1 - m_1n$. Thus, $\delta(T_2) = \frac{mn_1 - m_1n}{nn_1} = \frac{m}{n} - \frac{m_1}{n_1} = \delta(T) - \delta(T_1)$, as desired.

4. This follows readily from 3.

5. This is obvious; $m = 0$ because $\zeta_{K,T}(s)$ is finite and positive. Moreover, there is no pole at $s = 1$.

■

Proposição 6. *If T contains no primes \mathfrak{p} for which $N\mathfrak{p}$ is prime (in \mathbb{Z}), then $\delta(T) = 0$.*

Proof. Let \mathfrak{p} be a prime in T . Since $N\mathfrak{p} = p^f$ (where p lies under \mathfrak{p} in \mathbb{Z} and f denotes the inertial degree of \mathfrak{p}), we must have $f \geq 2$; if $f = 1$, $N\mathfrak{p}$ would be prime. Moreover, for any given prime $p \in \mathbb{Z}$, there are at most $[K : \mathbb{Q}]$ primes of K lying over p . Thus, $\zeta_{K,T}(s)$ can be decomposed into a product $\prod_{1 \leq i \leq [K:\mathbb{Q}]} g_i(s)$ of d infinite products over the prime numbers, with each factor of g_i being either a 1 or a $\frac{1}{1-p^{-fs}}$ (for every prime p). Thus, for any i , $g_i(1) \leq \prod_p \frac{1}{1-p^{-fs}} \leq \prod_p \frac{1}{1-p^{-2}} = \zeta(2) = \frac{\pi^2}{6}$. Thus, $g_i(s)$ is holomorphic at $s = 1$, meaning that the order of the pole there must be 0 (recall that polar density cannot be negative). We conclude that $\delta(T) = 0$. ■

Corolário 2. *Let T_1 and T_2 be sets of prime ideals in K . If the sets differ only by primes \mathfrak{p} for which $N\mathfrak{p}$ is not prime and one of the two sets has polar density, then so does the other, and the densities are equal.*

At last, the time has come to exploit the power of polar density. It turns out we can derive some important analytic results.

Teorema 2.2. *Let $L \supset K$ be a field extension of finite degree and let M be its Galois closure. Then the set of prime ideals of K that split completely in L has density $\frac{1}{[M:K]}$.*

Proof. The first thing to notice is that a prime ideal \mathfrak{p} of K splits completely in L if and only if it splits completely in M . One direction is easy: if it splits completely in M , it must split completely in the subfield L . If it splits completely in L , then it also splits completely in every conjugate field L' . All of these conjugate fields must lie under the decomposition field (the fixed field of the decomposition group of $\text{Gal}(M/K)$), and so their compositum is a field lying under the decomposition field as well. This field is just M ! \mathfrak{p} splits completely only up to and including the decomposition field, so we conclude that it splits completely in M as well.

Thus, it suffices to prove this theorem with the assumption that L is Galois over K . Let S be the set of prime ideals of K that split completely in L and let T be the primes of L lying over a prime ideal in S . For each $\mathfrak{p} \in S$, there are exactly $[L : K]$ prime ideals $\mathfrak{P} \in T$, and for each of them, $N_K^L(\mathfrak{P}) = \mathfrak{p}$ (where N_K^L denotes norm). Thus, $N\mathfrak{P} = N\mathfrak{p}$ (where N denotes norm over \mathbb{Q}). This tells us that $\zeta_{L,T}(s) = \zeta_{K,S}(s)^{[L:K]}$. Also, T contains every prime ideal of L that is unramified over K and for which $N\mathfrak{P}$ is prime (in \mathbb{Z}). Thus, T differs from the set of all prime ideals in L by a set of polar density 0 (using Corollary 2), and so T has density 1. Moreover, this shows that $\zeta_{K,S}$ has the property signifying that S is a set of polar density $\frac{1}{[L:K]}$, as desired. ■

Corolário 3. *If $f(x) \in K[x]$ splits into linear factors modulo \mathfrak{p} for all but finitely many prime ideals \mathfrak{p} of K , then f splits into linear factors in K .*

Proof. If L is the splitting field of f , then L is Galois over K . Now, use Theorem 2.2 on L/K . For more interesting details, see Bhandarkar [?], Section 4. ■

Corolário 4. *For every abelian extension L/K and every finite set S of primes of K including those that ramify in L , let I_K^S denote the fractional ideals that are prime to all ideals in S . Then, the Artin map*

$$\left(\frac{L/K}{\cdot} \right) : I_K^S \longrightarrow \text{Gal}(L/K)$$

is surjective.

Proof. Let H be the image of the Artin map; it is some subgroup of $\text{Gal}(L/K)$. If its fixed field is L^H , then we see that $H = \text{Gal}(L/L^H)$ is the image. For all $\mathfrak{p} \notin S$, $\left(\frac{L^H/K}{\mathfrak{p}}\right) = \left(\frac{L/K}{\mathfrak{p}}\right) |_{L^H} = 1$, which implies that \mathfrak{p} splits completely in L^H . Thus, all but finitely many prime ideals of \mathcal{O}_K split completely in L^H , so Theorem 2.2 tells us that $[L^H : K] = 1$; in other words, the Artin map is surjective. ■

Bibliografia

References

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