

GDA HW 3

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1 Metric Tree Curvature

Want to show that a metric tree space has negative curvature, where a metric tree space (M, d) is a metric space such that:

$$\forall x, y \in M, \exists \text{ a unique path } x \rightsquigarrow y \text{ that is homeomorphic to } [0, 1]$$

Proof. We will show that a metric tree space (M, d) has negative curvature by showing that for all $x, y, z \in M$:

$$d(x, y) + d(y, z) - d(x, z) \geq 0$$

We will prove this by cases. First, consider the case where x, y, z are all on the same path. Then, $d(x, y) + d(y, z) = d(x, z)$, so $d(x, y) + d(y, z) - d(x, z) = 0$. Thus, the inequality holds.

Now, consider the case where x, y, z are not all on the same path. Then, there are two cases: either x and y are on the same path, or y and z are on the same path. Without loss of generality, assume that x and y are on the same path. Then, $d(x, y) = d(x, z) - d(y, z)$. Thus, $d(x, y) + d(y, z) - d(x, z) = d(x, z) - d(y, z) + d(y, z) - d(x, z) = 0$. Thus, the inequality holds. \square

2 Hausdorff and Gromov-Hausdorff Metrics

Want to show that Hausdorff and Gromov-Hausdorff metrics are indeed metrics. Recall that a metric space is defined as (M, d) for set M and metric (distance) function d such that for all $x, y, z \in M$:

$$\begin{array}{ll} d_M(x, y) = 0 \iff x = y & \text{(equality)} \\ d_M(x, y) > 0 \text{ for } x \neq y & \text{(positivity)} \\ d_M(x, y) = d_M(y, x) & \text{(symmetry)} \\ d_M(x, z) \leq d_M(x, y) + d_M(y, z) & \text{(triangle inequality)} \end{array}$$

2.1 Hausdorff Metric

The Hausdorff distance is defined on two non-empty subsets X, Y of a metric space (M, d_M) as:

$$d_H(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} d_M(x, y), \sup_{y \in Y} \inf_{x \in X} d_M(x, y) \right\}$$

Want to show that the Hausdorff distance d_H is a metric that satisfies the four properties above.

Proof. We will prove all four properties of a metric in a metric space for the Hausdorff distance.

1. **Equality:** To show the property of equality we prove both directions. If $X = Y$ then:

$$\begin{aligned} d_H(X, Y) &= d_H(X, X) \\ &= \max \left\{ \sup_{x \in X} \inf_{x' \in X} d_M(x, x'), \sup_{x \in X} \inf_{x' \in X} d_M(x, x') \right\} \\ &= \max \{0, 0\} \quad \text{by } d_M(x, x') = 0 \text{ for } x = x' \\ &= 0 \end{aligned}$$

If $d_H(X, Y) = 0$ then:

$$\begin{aligned} d_H(X, Y) &= 0 \\ &= \max \left\{ \sup_{x \in X} \inf_{y \in Y} d_M(x, y), \sup_{y \in Y} \inf_{x \in X} d_M(x, y) \right\} \end{aligned}$$

By the max operation, one or both arguments must be equal to zero. However, since metric-distances are non-negative, both arguments must be zero:

$$\sup_{x \in X} \inf_{y \in Y} d_M(x, y) = \sup_{y \in Y} \inf_{x \in X} d_M(x, y) = 0$$

By the definition of the sup and inf operators, this implies that for all $x \in X$ and $y \in Y$, $d_M(x, y) = 0$. Since d_M is a metric, this implies that $x = y$ for all $x \in X$ and $y \in Y$. Thus, $X = Y$.

Therefore d_H satisfies the property of equality.

2. **Positivity:** To show the property of positivity we prove the following: if $X \neq Y$ then $d_H(X, Y) > 0$. We will prove this by contradiction. Suppose $d_H(X, Y) = 0$ for $X \neq Y$. Then:

$$\begin{aligned} d_H(X, Y) &= 0 \\ &= \max \left\{ \sup_{x \in X} \inf_{y \in Y} d_M(x, y), \sup_{y \in Y} \inf_{x \in X} d_M(x, y) \right\} \end{aligned}$$

By the max operation, one or both arguments must be equal to zero. However, since metric-distances are non-negative, both arguments must be zero:

$$\sup_{x \in X} \inf_{y \in Y} d_M(x, y) = \sup_{y \in Y} \inf_{x \in X} d_M(x, y) = 0$$

By the definition of the sup and inf operators, this implies that for all $x \in X$ and $y \in Y$, $d_M(x, y) = 0$. Since d_M is a metric, this implies that $x = y$ for all $x \in X$ and $y \in Y$. Thus, $X = Y$.

This contradicts the assumption that $X \neq Y$. Therefore d_H satisfies the property of positivity since d_M is always non-negative.

3. **Symmetry:** To show the property of symmetry, prove that $d_H(X, Y) = d_H(Y, X)$. If $X = Y$, this proof is trivial because $d_H(X, Y) = 0$ and $d_H(Y, X) = 0$. If $X \neq Y$, then:

$$\begin{aligned} d_H(X, Y) &= \max \left\{ \sup_{x \in X} \inf_{y \in Y} d_M(x, y), \sup_{y \in Y} \inf_{x \in X} d_M(x, y) \right\} \\ &= \max \left\{ \sup_{y \in Y} \inf_{x \in X} d_M(x, y), \sup_{x \in X} \inf_{y \in Y} d_M(x, y) \right\} \\ &= d_H(Y, X) \end{aligned}$$

4. **Triangle Inequality:** To show the property of triangle inequality, prove that $d_H(X, Z) \leq d_H(X, Y) + d_H(Y, Z)$. If $X = Y$ or $Y = Z$, this proof is trivial because $d_H(X, Y) = 0$ or $d_H(Y, Z) = 0$ and $d_H(X, Z) = 0$. If $X \neq Y$ and $Y \neq Z$, then:

$$d_H(X, Z) = \max \left\{ \sup_{x \in X} \inf_{z \in Z} d_M(x, z), \sup_{z \in Z} \inf_{x \in X} d_M(x, z) \right\}$$

Because d_M is a metric, it satisfies the triangle inequality for x, y, z for all choices of x, y, z . In particular, any choice of y holds, letting us pick $y' := \inf_{y \in Y} d_M(x, y)$ and $y'' := \inf_{y \in Y} d_M(y, z)$:

$$\begin{aligned}
d_H(X, Z) &= \max \left\{ \sup_{x \in X} \inf_{z \in Z} d_M(x, z), \sup_{z \in Z} \inf_{x \in X} d_M(x, z) \right\} \\
&\leq \max \left\{ \sup_{x \in X} \inf_{z \in Z} (d_M(x, y') + d_M(y', z)), \sup_{z \in Z} \inf_{x \in X} (d_M(x, y'') + d_M(y'', z)) \right\} \\
&= \max \left\{ \sup_{x \in X} d_M(x, y') + \inf_{z \in Z} d_M(y', z), \sup_{z \in Z} d_M(y'', z) + \inf_{x \in X} d_M(x, y'') \right\} \\
&\leq \max \left\{ \sup_{x \in X} d_M(x, y'), \inf_{x \in X} d_M(x, y'') \right\} + \max \left\{ \inf_{z \in Z} d_M(y', z), \sup_{z \in Z} d_M(y'', z) \right\} \\
&\leq \max \left\{ \sup_{x \in X} \inf_{y \in Y} d_M(x, y), \inf_{x \in X} \sup_{y \in Y} d_M(x, y) \right\} + \max \left\{ \inf_{z \in Z} \sup_{y \in Y} d_M(y, z), \sup_{z \in Z} \inf_{y \in Y} d_M(y, z) \right\} \\
&\leq \max \left\{ \sup_{x \in X} \inf_{y \in Y} d_M(x, y), \inf_{x \in X} \sup_{y \in Y} d_M(x, y) \right\} + \max \left\{ \inf_{z \in Z} \sup_{y \in Y} d_M(y, z), \sup_{z \in Z} \inf_{y \in Y} d_M(y, z) \right\} \\
&= d_H(X, Y) + d_H(Y, Z)
\end{aligned}$$

Therefore, because the Hausdorff distance has all four properties of a metric, it is indeed a metric. \square

2.2 Gromov-Hausdorff Metric

The Gromov-Hausdorff distance is defined for two metric spaces (X, d_X) and (Y, d_Y) with two isometric functions $\phi : X \rightarrow A$ and $\psi : Y \rightarrow A$ as:

$$d_{GH}(X, Y) := \inf_{\phi, \psi} d_H(\phi(X), \psi(Y))$$

Want to show that the Gromov-Hausdorff distance d_{GH} is a metric that satisfies the four metric properties above.

Proof. We will prove all four properties of a metric in a metric space for the Gromov-Hausdorff distance.

Equality: We will prove both directions of $d_{GH}(X, Y) = 0 \iff X = Y$. First, let $d_{GH}(X, Y) = 0$. Because $d_H \geq 0$, we know by the def of infimum that $d_H(\phi(X), \psi(Y)) = 0$. Because d_H is a metric, we know that $\phi(X) = \psi(Y)$ only when $X = Y$. Therefore, $d_{GH}(X, Y) = 0 \implies X = Y$.

Now let $X = Y$. By the metric property of the Hausdorff distance, we know $d_H(X, Y) = 0$ for $X = Y$. Therefore, no matter the isometric embeddings ϕ and ψ , there is always an isometric embedding $\phi = \psi$ such that $d_H(\phi(X), \psi(Y)) = 0$. Therefore, $d_{GH}(X, Y) = 0$.

2. **Positivity:** We will prove that $d_{GH}(X, Y) > 0$ for all $X \neq Y$.

Because d_H is a metric, we know that $d_H(\phi(X), \psi(Y)) > 0$ is always true when $X \neq Y$. Therefore, no matter the isometric embedding, the infimum of $d_H(\phi(X), \psi(Y))$ is always positive. Thus $d_{GH}(X, Y) > 0$.

3. **Symmetry:** We will prove that $d_{GH}(X, Y) = d_{GH}(Y, X)$. This is trivially true by relabeling X and Y since the isometric embedding functions ϕ, ψ can be anything.

4. **Triangle Inequality:** We will prove that $d_{GH}(X, Z) \leq d_{GH}(X, Y) + d_{GH}(Y, Z)$ for all X, Y, Z .

By the metric property of the Hausdorff distance we know for spaces $\phi(X), \theta(Y), \psi(Z)$ that:

$$d_H(\phi(X), \psi(Z)) \leq d_H(\phi(X), \theta(Y)) + d_H(\theta(Y), \psi(Z))$$

Taking the infimum of both sides, we get:

$$d_{GH}(X, Z) \leq d_{GH}(X, Y) + d_{GH}(Y, Z)$$

Thus we have shown that the Gromov-Hausdorff distance is a metric. \square

3 Bounding Gromov-Hausdorff Distance With Diameter

Want to bound the Gromov-Hausdorff distance d_{GH} in terms of the diameter of metric spaces X and Y :

$$\text{diam}(X) := \max_{x, x' \in X} d_X(x, x') \quad \text{diam}(Y) := \max_{y, y' \in Y} d_Y(y, y')$$

Proof. Let \tilde{R} be an arbitrary relation.

$$\begin{aligned}
d_{GH}(X, Y) &= \inf_{\phi, \psi} d_H(\phi(X), \psi(Y)) \\
&= \frac{1}{2} \inf_{\tilde{R}} \sup_{\substack{(x, y) \in R \\ (x', y') \in R}} |d_X(x, x') - d_Y(y, y')| \\
&\leq \frac{1}{2} \sup_{\substack{(x, y) \in \tilde{R} \\ (x', y') \in \tilde{R}}} |d_X(x, x') - d_Y(y, y')| \\
&\leq \frac{1}{2} \sup_{\substack{(x, y) \in \tilde{R} \\ (x', y') \in \tilde{R}}} \{d_X(x, x'), d_Y(y, y')\} \\
&= \frac{1}{2} \max \{\text{diam}(X), \text{diam}(Y)\}
\end{aligned}$$

□

4 Bounding Gromov-Hausdorff Distance With ϵ -Nets

Want to bound the Gromov Hausdorff distance $d_{GH}(X, Y)$ for metric spaces X and Y . Here, $Y \subset X$ is an ϵ -net such that:

$$\forall x \in X, \exists y \in Y \text{ such that } d(x, y) < \epsilon$$

Proof.

$$\begin{aligned}
d_{GH}(X, Y) &= \inf_{\phi, \psi} d_H(\phi(X), \psi(Y)) \\
&= \inf_{\phi, \psi} \max \left\{ \sup_{x \in \phi(X)} \inf_{y \in \psi(Y)} d(x, y), \sup_{y \in \psi(Y)} \inf_{x \in \phi(X)} d(x, y) \right\} \\
&\leq \inf_{\phi, \psi} \max \left\{ \sup_{x \in \phi(X)} \inf_{y \in \psi(Y)} \epsilon, \sup_{y \in \psi(Y)} \inf_{x \in \phi(X)} \epsilon \right\} \\
&= \inf_{\phi, \psi} \max \{\epsilon, \epsilon\} \\
&= \epsilon
\end{aligned}$$

□