Geometric Data Analysis HW 2

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1 LLE

2 Non-Manifold Spaces

An example of a non-manifold space is two unit circles, one centered at $(-1, 0)$ and the other at $(1, 0)$, that touch at a single point $(0, 0)$:

$$(x-1)^2 + y^2 = 1$$
$$(x+1)^2 + y^2 = 1$$

This space is not a manifold because at the overlapping point $(0, 0)$, a propert chart cannot be constructed. It's neighborhood looks like a rotated '+' sign which is not homeomorphic to a line segment in $\mathbb{R}^1$.

3 Chart Mapping to Zero

Show that given any point $m \in \mathcal{M}$, where $\mathcal{M}$ is a topoligical manifold, we can choose a chart $\theta : U \to \mathbb{R}^n$ such that $\theta(m) = 0$.

Proof. Consider all neighborhoods that contain m in its domain i.e. $m \in U_1, U_2, \ldots$. By definition of a manifold, for each neighborhood U_i, there exists some corresponding chart $\theta_i : U_i \to \mathbb{R}^n$ where m is mapped to some arbitrary location $p \in \mathbb{R}^n$.

To instead make $\theta_i(m) = 0$, we can simply translate the chart by $-p$. Thus we can construct entirely new charts:

$$\theta_i'(x) = \theta_i(x) - p = \theta_i(x) - \theta_i(m)$$

which ensures $\theta_i(m) = 0$ always.

□

4 Coordinate Charts and Transition Functions for Circles

We will explicitly construct coordinate charts and transition functions for a circle defined by the angle and x, y projections respectively. To do so, consider S, the unit circle $x^2 + y^2 = 1$ defined for $[0, 2\pi]$.

4.1 Angle Construction

Let α be some angle less than π and let ϕ be the angle between a point $(x, y) \in S$ and the x axis. Now divide the circle into two coordinate patches:

$$\begin{aligned} U_1 &= \{(x, y) \in S | -\alpha \leq \phi \leq \pi + \alpha\} \\ U_2 &= \{(x, y) \in S | \pi \leq \phi \leq 2\pi\} \end{aligned}$$

Now define two corresponding charts $\theta : U \to \mathbb{R}$. I do so based on the arctan2 function, which uniqley computes the angle between a point (x, y) and the x axis.

$$\theta_1(x, y) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0, y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0, y < 0 \\ \frac{\pi}{2} & \text{if } x = 0, y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0, y < 0 \\ \text{undefined} & \text{if } x = 0, y = 0 \end{cases}$$

$$\theta_2(x, y) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0, y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0, y < 0 \\ \frac{\pi}{2} & \text{if } x = 0, y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0, y < 0 \\ \text{undefined} & \text{if } x = 0, y = 0 \end{cases}$$

These charts sufficiently cover the circle, thus forming at atlas. We now construct two transition functions from $\mathbb{R} \to S \to \mathbb{R}$ for some point $p \in \mathbb{R}$. These overlaps occur below the x-axis on the left and right hand sides of the circle, between $[\pi, \pi + \alpha]$ and $[-\alpha, 2\pi]$ respectively.

$$\begin{aligned} T_1 &: [\pi, \pi + \alpha] \to (x, y) \to [\pi, \pi + \alpha] \\ T_1(p) &= \theta_1\left(\theta_2^{-1}(p)\right) = \theta_2\left(\theta_1^{-1}(p)\right) = p \end{aligned}$$

$$T_1 : [-\alpha, 2\pi] \to (x,y) \to [-\alpha, 2\pi]$$
$$T_1(p) = \theta_1\left(\theta_2^{-1}(p)\right) = \theta_2\left(\theta_1^{-1}(p)\right) = p$$

4.2 Projection Construction

Divide the circle into four coordinate patches, each of which is a half-plane:

$$\begin{aligned} U_{top} &= \{(x,y) \in S | y \geq 0\} \\ U_{bottom} &= \{(x,y) \in S | y \leq 0\} \\ U_{right} &= \{(x,y) \in S | x \geq 0\} \\ U_{left} &= \{(x,y) \in S | x \leq 0\} \end{aligned}$$

Now define four corresponding charts $\theta : U \to \mathbb{R}$ and their inverses $X^{-1} : \mathbb{R} \to U$:

$$\begin{aligned} X_{top}(x,y) &= x & X_{top}^{-1}(x) &= (x, \sqrt{1-x^2}) \\ X_{bottom}(x,y) &= x & X_{bottom}^{-1}(x) &= (x, \sqrt{1-x^2}) \\ X_{left}(x,y) &= y & X_{left}^{-1}(y) &= (\sqrt{1-y^2}, y) \\ X_{right}(x,y) &= y & X_{right}^{-1}(y) &= (\sqrt{1-y^2}, y) \end{aligned}$$

This sufficently covers the entire circle, forming an atlas. Where the charts overlap, we construct four transition functions $T : R \to R$ for some point $p \in \mathbb{R}$:

$$\begin{aligned} T_1(p) &= X_{right}(X_{top}^{-1}(p)) = X_{right}(p, \sqrt{1-p^2}) = \sqrt{1-p^2} \\ T_2(p) &= X_{right}(X_{bottom}^{-1}(p)) = X_{right}(p, \sqrt{1-p^2}) = \sqrt{1-p^2} \\ T_3(p) &= X_{left}(X_{bottom}^{-1}(p)) = X_{left}(p, \sqrt{1-p^2}) = \sqrt{1-p^2} \\ T_4(p) &= X_{left}(X_{top}^{-1}(p)) = X_{left}(p, \sqrt{1-p^2}) = \sqrt{1-p^2} \end{aligned}$$

Note that the order of function composition does not matter here.
sub-matrix along the diagonal is the same dissimilarity matrix via paper.