GDA HW 3

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1 Metric Tree Curvature

2 Hausdorff and Gromov-Hausdorff Metrics

Want to show that Hausdorff and Gromov-Hausdorff metrics are indeed metrics. Recall that a metric space is defined as (M, d) for set M and metric (distance) function d such that for all $x, y, z \in M$:

$$\begin{aligned}
&d_M(x, y) = 0 \iff x = y && \text{(equality)}\\
&d_M(x, y) > 0 \text{ for } x \neq y && \text{(positivity)}\\
&d_M(x, y) = d_M(y, x) && \text{(symmetry)}\\
&d_M(x, z) \leq d_M(x, y) + d_M(y, z) && \text{(triangle inequality)}
\end{aligned}$$

2.1 Hausdorff Metric

The Hausdorff distance is defined on two non-empty subsets X, Y of a metric space (M, d_M) as:

$$d_H(X, Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y} d_M(x, y), \sup_{y\in Y}\inf_{x\in X} d_M(x, y)\right\}$$

Want to show that the Hausdorff distance d_H is a metric that satisifes the four properties above.

Proof. We will prove all four properties of a metric in a metric space for the Hausdorff distance.

1. **Equality:** To show the property of equality we prove both directions. If $X = Y$ then:

$$
\begin{aligned}
d_H(X,Y) &= d_H(X,X) \\
&= \max\left\{\sup_{x\in X}\inf_{x'\in X} d_M(x,x'), \sup_{x\in X}\inf_{x'\in X} d_M(x,x')\right\} \\
&= \max\{0,0\} \qquad\qquad \text{by } d_M(x,x')=0 \text{ for } x=x' \\
&= 0
\end{aligned}
$$

If $d_H(X,Y)=0$ then:

$$
\begin{aligned}
d_H(X,Y) &= 0 \\
&= \max\left\{\sup_{x\in X}\inf_{y\in Y} d_M(x,y), \sup_{y\in Y}\inf_{x\in X} d_M(x,y)\right\}
\end{aligned}
$$

By the max operation, one or both arguments must be equal to zero. However, since metric-distances are non-negative, both arguments must be zero:

$$
\sup_{x\in X}\inf_{y\in Y} d_M(x,y) = \sup_{y\in Y}\inf_{x\in X} d_M(x,y) = 0
$$

By the definition of the sup and inf operators, this implies that for all $x \in X$ and $y \in Y$, $d_M(x,y)=0$. Since d_M is a metric, this implies that $x=y$ for all $x \in X$ and $y \in Y$. Thus, $X=Y$.

Therefore d_H satisfies the property of equality.

2. **Positivity:** To show the property of positivity we prove the following: if $X \neq Y$ then $d_H(X,Y)>0$. We will prove this by contradiction. Suppose $d_H(X,Y)=0$ for $X \neq Y$. Then:

$$
\begin{aligned}
d_H(X,Y) &= 0 \\
&= \max\left\{\sup_{x\in X}\inf_{y\in Y} d_M(x,y), \sup_{y\in Y}\inf_{x\in X} d_M(x,y)\right\}
\end{aligned}
$$

By the max operation, one or both arguments must be equal to zero. However, since metric-distances are non-negative, both arguments must be zero:

$$\sup_{x\in X}\inf_{y\in Y} d_M(x,y) = \sup_{y\in Y}\inf_{x\in X} d_M(x,y) = 0$$

By the definition of the sup and inf operators, this implies that for all $x \in X$ and $y \in Y$, $d_M(x,y) = 0$. Since d_M is a metric, this implies that $x = y$ for all $x \in X$ and $y \in Y$. Thus, $X = Y$.

This contradicts the assumption that $X \neq Y$. Therefore d_H satisfies the property of positivity.

3. **Symmetry:** To show the property of symmetry, prove that $d_H(X,Y) = d_H(Y,X)$. If $X = Y$, this proof is trivial because $d_H(X,Y) = 0$ and $d_H(Y,X) = 0$. If $X \neq Y$, then:

$$\begin{aligned}
d_H(X,Y) &= \max\left\{\sup_{x\in X}\inf_{y\in Y} d_M(x,y), \sup_{y\in Y}\inf_{x\in X} d_M(x,y)\right\} \\
&= \max\left\{\sup_{y\in Y}\inf_{x\in X} d_M(x,y), \sup_{x\in X}\inf_{y\in Y} d_M(x,y)\right\} \\
&= d_H(Y,X)
\end{aligned}$$

4. **Triangle Inequality:** To show the property of triangle inequality, prove that $d_H(X,Z) \leq d_H(X,Y) + d_H(Y,Z)$. If $X = Y$ or $Y = Z$, this proof is trivial because $d_H(X,Y) = 0$ or $d_H(Y,Z) = 0$ and $d_H(X,Z) = 0$. If $X \neq Y$ and $Y \neq Z$, then:

$$d_H(X,Z) = \max\left\{\sup_{x\in X}\inf_{z\in Z} d_M(x,z), \sup_{z\in Z}\inf_{x\in X} d_M(x,z)\right\}$$

Because d_M is a metric, it satisfies the triangle inequality for x, y, z for all choices of x, y, z. In particular, any choice of y holds, letting us pick $y' := \inf_{y\in Y} d_M(x,y)$ and $y'' := \inf_{y\in Y} d_M(y,z)$:

$$
\begin{aligned}
d_H(X,Z) &= \max\left\{\sup_{x\in X}\inf_{z\in Z} d_M(x,z), \sup_{z\in Z}\inf_{x\in X} d_M(x,z)\right\} \\
&\leq \max\left\{\sup_{x\in X}\inf_{z\in Z}\Big(d_M(x,y')+d_M(y',z)\Big), \sup_{z\in Z}\inf_{x\in X}\Big(d_M(x,y'')+d_M(y'',z)\Big)\right\} \\
&= \max\left\{\sup_{x\in X} d_M(x,y') + \inf_{z\in Z} d_M(y',z), \sup_{z\in Z} d_M(y'',z) + \inf_{x\in X} d_M(x,y'')\right\} \\
&\leq \max\left\{\sup_{x\in X} d_M(x,y'), \inf_{x\in X} d_M(x,y'')\right\} + \max\left\{\inf_{z\in Z} d_M(y',z), \sup_{z\in Z} d_M(y'',z)\right\} \\
&\leq \max\left\{\sup_{x\in X}\inf_{y\in Y} d_M(x,y), \inf_{x\in X} d_M(x,y'')\right\} + \max\left\{\inf_{z\in Z} d_M(y',z), \sup_{z\in Z}\inf_{y\in Y} d_M(y,z)\right\} \\
&\leq \max\left\{\sup_{x\in X}\inf_{y\in Y} d_M(x,y), \inf_{x\in X}\sup_{y\in Y} d_M(x,y)\right\} + \max\left\{\inf_{z\in Z}\sup_{y\in Y} d_M(y,z), \sup_{z\in Z}\inf_{y\in Y} d_M(y,z)\right\} \\
&= d_H(X,Y) + d_H(Y,Z)
\end{aligned}
$$

max(a+b,c+d) = max(a,c) + max(b,d)

$$
\begin{aligned}
d_H(X,Y) + d_H(Y,Z) &= \max\left\{\sup_{x\in X}\inf_{y\in Y} d_M(x,y), \sup_{y\in Y}\inf_{x\in X} d_M(x,y)\right\} \\
&\quad + \max\left\{\sup_{y\in Y}\inf_{z\in Z} d_M(y,z), \sup_{z\in Z}\inf_{y\in Y} d_M(y,z)\right\} \\
&= \max\Big\{ \sup_{x\in X}\inf_{y\in Y} d_M(x,y) + \sup_{y\in Y}\inf_{z\in Z} d_M(y,z), \\
&\qquad \sup_{x\in X}\inf_{y\in Y} d_M(x,y) + \sup_{z\in Z}\inf_{y\in Y} d_M(y,z), \\
&\qquad \sup_{y\in Y}\inf_{x\in X} d_M(x,y) + \sup_{y\in Y}\inf_{z\in Z} d_M(y,z), \\
&\qquad \sup_{y\in Y}\inf_{x\in X} d_M(x,y) + \sup_{z\in Z}\inf_{y\in Y} d_M(y,z)\Big\}
\end{aligned}
$$

Let $y_{inf} := \inf_{y\in Y}$ and $y_{sup} := \sup_{y\in Y}$.

$$
\begin{aligned}
d_H(X,Y) + d_H(Y,Z) &= \max\Big\{ \sup_{x\in X}\inf_{y\in Y} d_M(x,y) + \sup_{y\in Y}\inf_{z\in Z} d_M(y,z), \\
&\qquad \sup_{x\in X} d_M(x,y_{inf}) + \sup_{z\in Z} d_M(y_{inf},z), \\
&\qquad \inf_{x\in X} d_M(x,y_{sup}) + \inf_{z\in Z} d_M(y_{sup},z), \\
&\qquad \sup_{y\in Y}\inf_{x\in X} d_M(x,y) + \sup_{z\in Z}\inf_{y\in Y} d_M(y,z)\Big\}
\end{aligned}
$$

$\square$