1. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

True or false? In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

הטענה לא נכונה.

W2	W1	M2	M1
M1	M2	W2	W1
M2	M1	W1	W2

2. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

True or false? *Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Then in every stable matching S for this instance, the pair (m, w) belongs to S.*

הטענה נכונה.

ברגע שיגיע תורו שלה הגבר להציע לאישה כלשהי – הוא יציע לזאת שהיא ראשונה ברשימת ההעדפות שלו. ברגע שהוא יציע לה – לא משנה אם היא מאורסת כבר אם לאו, היא תתארס איתו ותישאר כך עד לסיום הריצה.

4. Gale and Shapley published their paper on the Stable Matching Problem in 1962; but a version of their algorithm had already been in use for ten years by the National Resident Matching Program, for the problem of assigning medical residents to hospitals.

Basically, the situation was the following. There were m hospitals, each with a certain number of available positions for hiring residents. There were n medical students graduating in a given year, each interested in joining one of the hospitals. Each hospital had a ranking of the students in order of preference, and each student had a ranking of the hospitals in order of preference. We will assume that there were more students graduating than there were slots available in the m hospitals.

The interest, naturally, was in finding a way of assigning each student to at most one hospital, in such a way that all available positions in all hospitals were filled. (Since we are assuming a surplus of students, there would be some students who do not get assigned to any hospital.)

We say that an assignment of students to hospitals is *stable* if neither of the following situations arises.

- First type of instability: There are students s and s', and a hospital h, so that
  - s is assigned to h, and
  - s' is assigned to no hospital, and
  - h prefers s' to s.
- Second type of instability: There are students s and s', and hospitals h and h', so that
  - s is assigned to h, and
  - s' is assigned to h', and
  - h prefers s' to s, and
  - s' prefers h to h'.

So we basically have the Stable Matching Problem, except that (i) hospitals generally want more than one resident, and (ii) there is a surplus of medical students.

Show that there is always a stable assignment of students to hospitals, and give an algorithm to find one.

While some hospital h_i has available positions
  h_i offers a position to the next student s_j on its preference list
  if s_j is free then
    s_j accepts the offer
  else (s_j is already committed to a hospital h_k)
    if s_j prefers h_k to h_i then
      s_j remains committed to h_k
    else s_j becomes committed to h_i
        the number of available positions at h_k increases by one.
        the number of available positions at h_i decreases by one.

8. For this problem, we will explore the issue of *truthfulness* in the Stable Matching Problem and specifically in the Gale-Shapley algorithm. The basic question is: Can a man or a woman end up better off by lying about his or her preferences? More concretely, we suppose each participant has a true preference order. Now consider a woman w. Suppose w prefers man m to m', but both m and m' are low on her list of preferences. Can it be the case that by switching the order of m and m' on her list of preferences (i.e., by falsely claiming that she prefers m' to m) and running the algorithm with this false preference list, w will end up with a man m'' that she truly prefers to both m and m'? (We can ask the same question for men, but will focus on the case of women for purposes of this question.)

Resolve this question by doing one of the following two things:

(a) Give a proof that, for any set of preference lists, switching the order of a pair on the list cannot improve a woman's partner in the Gale-Shapley algorithm; or

(b) Give an example of a set of preference lists for which there is a switch that would improve the partner of a woman who switched preferences.

M1	M2	M3	W1	W2	W3	W3’
W3	W1	W3	M1	M1	M2	M2
W1	W3	W1	M2	M2	M1	M3
W2	W2	W2	M3	M3	M3	M1

בעיית השידוך היציב (שאלה 1.8 בספר הקורס). בשאלה זו נברר האם אחת הנשים יכולה, באמצעות דיווח שקרי על העדפותיה, לשפר את השידוך (=הזיווג), המתקבל עבורה באלגוריתם של **Gale-Shapley** (GS). הגדרה: קלט לבעיית השידוך היציב ייקרא "מעודד שקרים", אם ישנה אשה w עבורה מתקיימים שלושת התנאים הבאים:

(א) w משודכת ע"י האלגוריתם GS לגבר m.

(ב) w מעדיפה גבר אחר m' על פניו של m.

(ג) אם w תשקר, באמצעות החלפת מקומותיהם של m ושל m' ברשימת ההעדפות שהיא מוסרת לאלגוריתם (ואף שקר אחר לא יופיע ברשימת ההעדפות שלה, או של הנשים והגברים האחרים), אזי באלגוריתם GS, w תשודך לשמחתה עם m' במקום עם m.

הוכיחו או הפריכו את הטענה הבאה: אין אף קלט מעודד-שקרים לבעיית השידוך היציב.

הטענה נכונה.

נריץ במקביל את האלגוריתם עם הקלט המקורי ועם הקלט השקרי.

ברור שבהרצה האמיתית אף גבר שמופיע ברשימת ההעדפות של w מעל m לא הציע לה, כיון שאם אחד כזה היה מציע לה – היא היתה נשארת איתו ולא מסיימת את האלגוריתם עם m.

בהרצה השקרית הדירוג של m אפילו יותר גבוה מאשר בהרצה האמיתית. לכן גם בהרצה הזאת ההצעה שלו ל-w תתקבל, והיא תישאר איתו עד סוף האלגוריתם.