$\boldsymbol{R}$: {⟨x,y⟩ for ⟨x,y⟩ ∈ A^2 if x$\boldsymbol{R}$y}

$\boldsymbol{T}\cdot\boldsymbol{R}$: {⟨a,c⟩ | ∃b ∈ B (⟨a,b⟩ ∈ $\boldsymbol{T}$ ∧ ⟨b,c⟩ ∈ $\boldsymbol{R}$)}

$\boldsymbol{R}^2$: a$\boldsymbol{R}^2$c ⇔ {⟨a,c⟩ | ∃b ∈ A (⟨a,b⟩ ∈ $\boldsymbol{R}$ ∧ ⟨b,c⟩ ∈ $\boldsymbol{R}$)}

an ordered pair ⟨a,c⟩∈$\boldsymbol{R}^2$ means there's a "middle" b∈B that satisfies ⟨a,b⟩∈$\boldsymbol{R}$ ∧ ⟨b,c⟩∈$\boldsymbol{R}$

Examples

- $(a=-b)^2 = \boldsymbol{I}_{\mathbb{R}}$
- ⟨a,b⟩ ∈ $\boldsymbol{R}^2$ ⇔ ⟨a,c⟩,⟨c,b⟩ ∈ $\boldsymbol{R}$

Empty $\varnothing_A$

$\boldsymbol{R}$:=rel(A×B) = ∅

No pair ∈ A×B satisfies ⟨a,b⟩ ∈ $\boldsymbol{R}$

Properties

- $\boldsymbol{S}\cdot\varnothing_A = \varnothing$
- anti-symmetric
- symmetric ?

Examples

- {⟨x,y⟩ ∈ $\mathbb{N}^2$ | x+y < x}

Identity $\boldsymbol{I}_A$

Properties

- $\boldsymbol{R}\cdot\boldsymbol{I}_A = \boldsymbol{R}$

Reflexivity

$\boldsymbol{R}$:=rel(A) is reflexive if ∀a ∈ A(⟨a,a⟩ ∈ $\boldsymbol{R}$)

$\boldsymbol{R}$ is reflexive if every a ∈ A satisfies ⟨a,a⟩∈$\boldsymbol{R}$. In other words:
$\boldsymbol{I}_A \subseteq \boldsymbol{R}$

A = { −1, 0, 1 }. Is ⋱ contained ∈ $\boldsymbol{R}$?

$\boldsymbol{R}$ = lambda a,b: a⊙b; all(R(x,x) for x ∈ A)?

Properties

- ⇔ $\boldsymbol{R}^{-1}$ is reflexive
- → $\boldsymbol{R} \subseteq \boldsymbol{R}^2$ (and $\boldsymbol{R}^2$ is reflexive)

- → $\boldsymbol{R} \subseteq \boldsymbol{R}^2$
- if $\boldsymbol{S} \subseteq \boldsymbol{R}$ then $\boldsymbol{S}$ is reflexive
- if $\boldsymbol{S}$ is reflexive then both $\boldsymbol{R} \cup \boldsymbol{S}$ ∧ $\boldsymbol{R} \cap \boldsymbol{S}$ are reflexive

Examples

- $\boldsymbol{U}_A$: ∀a ∈ A(⟨a,a⟩ ∈ A×A = $\boldsymbol{U}$A)
- $\boldsymbol{I}_A$: ∀a ∈ A(⟨a,a⟩ ∈ {⟨−1, −1⟩, ⟨0, 0⟩, ⟨1, 1⟩})
- ≤, ≥ `both contain ⋱`

Counter Examples

- ≠ (which is $\boldsymbol{U}_A$ − $\boldsymbol{I}$A)
- <, >, $\varnothing_A$
- a=−b ⋰

Anti-Reflexivity

$\boldsymbol{R}$:=rel(A) is anti-reflexive iff ¬∃a ∈ A(⟨a,a⟩ ∈ $\boldsymbol{R}$)

$\boldsymbol{R}$ is reflexive if every a ∈ A satisfies ⟨a,a⟩ ∉ $\boldsymbol{R}$. In other words:

$\boldsymbol{I}_A \cap \boldsymbol{R} = \varnothing$

`just` $\boldsymbol{I}_A \not\subseteq \boldsymbol{R}$ `isn't enough;` $\boldsymbol{I}_A$ = {⟨1,1⟩, ⟨2,2⟩} ⊈ $\boldsymbol{R}$ = {⟨1,1⟩, ⟨1,2⟩} `but ⟨1,1⟩ ∈` $\boldsymbol{R}$ `so isn't anti-reflexive`

Examples

- ≠, <, >, $\varnothing_A$

Counter Examples

- $\boldsymbol{U}_A$, $\boldsymbol{I}_A$, a=−b ⋰, ≤, ≥

Symmetry

$\boldsymbol{R}$:=rel(A) is symmetric iff $\boldsymbol{R} = \boldsymbol{R}^{-1}$

$\boldsymbol{R}$ is symmetric if every ⟨x,y⟩ ∈ $\boldsymbol{R}$ satisfies ⟨y,x⟩ ∈ $\boldsymbol{R}$
`assuming both x ∧ y exist ∈ A`

∀x∀y(⟨x,y⟩∈$\boldsymbol{R}$ → ⟨y,x⟩∈$\boldsymbol{R}$)

$\boldsymbol{R}$ = lambda a,b: a⊙b; all(rel(y,x) for x,y ∈ R)?

Properties

- if $\boldsymbol{S}$ is symmetric then both $\boldsymbol{R} \cup \boldsymbol{S}$ ∧ $\boldsymbol{R} \cap \boldsymbol{S}$ are reflexive
- if $\boldsymbol{S}$ is symmetric then $\boldsymbol{R} \setminus \boldsymbol{S}$ is symmetric

Examples

- $\varnothing_A$ `can't point at ⟨x,y⟩ ∧ say ⟨y,x⟩ is ¬ ∈` $\varnothing^{-1}$

- $\boldsymbol{U}_A$, $\boldsymbol{I}_A$, a=−b ∴, ≠

Counter Examples

- ≤, ≥, <, >

Anti-Symmetry

$\boldsymbol{R}$:=rel(A) is anti-symmetric iff $\boldsymbol{R} \cap \boldsymbol{R}^{-1} = \varnothing$

$\boldsymbol{R}$ is anti-symmetric if every ⟨x,y⟩ ∈ $\boldsymbol{R}$ satisfies ⟨y,x⟩ ∉ $\boldsymbol{R}$

$\forall x \forall y((x,y) \in \boldsymbol{R} \rightarrow (y,x) \notin \boldsymbol{R})$

$\boldsymbol{R} \cap \boldsymbol{R}^{-1} = \varnothing$ means there can't be a ⟨x,x⟩

Properties

- → $\boldsymbol{R}$ is anti-reflexive
- → $\boldsymbol{R}^{-1}$ is anti-symmetric
- if $\boldsymbol{S} \subseteq \boldsymbol{R}$ then $\boldsymbol{S}$ is anti-symmetric
- if $\boldsymbol{S} \cup \boldsymbol{T}$ is anti-symmetric then both $\boldsymbol{S}$ ∧ $\boldsymbol{T}$ are anti-symmetric
- → $\boldsymbol{R} \cap \boldsymbol{S}$ is anti-symmetric
- if $\boldsymbol{R}$ is antireflexive ∧ transitive then it's asymmetric ∧ anti-symmetric

Examples

- <, >, $\varnothing_A$
- $b > a^2$

Counter Examples

- ≠, ≤, ≥, $\boldsymbol{U}_A$, $\boldsymbol{I}_A$, a=−b ∴, ≠
- $b < a^2$ ⟨3,4⟩ ∧ ⟨4,3⟩ are symmetric

Weak Anti-Symmetry

$\boldsymbol{R} \cap \boldsymbol{R}^{-1} \subseteq \boldsymbol{I}_A$

$\forall x \forall y(\langle x,y \rangle \in \boldsymbol{R} \wedge \langle y,x \rangle \in \boldsymbol{R} \rightarrow x=y)$

if both ⟨x,y⟩ ∈ $\boldsymbol{R}$ ∧ ⟨y,x⟩ ∈ $\boldsymbol{R}$ it's only because they're equal

for x,y ∈ A: if x≠y ∧ ⟨x,y⟩ ∈ $\boldsymbol{R}$ then must ⟨y,x⟩ ∉ $\boldsymbol{R}$

A_S vs WA_S: A_S requires every pair's opposite to ¬ be ∈ $\boldsymbol{R}$, whereas WA_S requires the same only for pairs that x=y

Examples

- $\boldsymbol{I}_A$

Transitivity

$\boldsymbol{R}^2 \subseteq \boldsymbol{R}$

$\forall x \forall y \forall z((\boldsymbol{R}(x,y) \land \boldsymbol{R}(y,z)) \to \boldsymbol{R}(x,z))$

Every $(x,y,z) \in A$ that satisfy $\langle x,y\rangle \in \boldsymbol{R} \land \langle y,z\rangle \in \boldsymbol{R}$ also satisfy $\langle x,z\rangle \in \boldsymbol{R}$

If you see an x that leads to y that leads to z, then expect x to lead to z   this is why $\boldsymbol{R}^2 \subseteq \boldsymbol{R}$

Properties

- if $\boldsymbol{T}$ is symmetric ∧ anti-symmetric then it's also transitive

Examples

- $A=\{1,2,3\}$; $\boldsymbol{R} = \{\langle \mathbf{1},2\rangle, \langle 2,\mathbf{3}\rangle, \boldsymbol{\langle 1,3\rangle}\} \Rightarrow \boldsymbol{R}^2 = \{\langle 1,3\rangle\} \subseteq \boldsymbol{R}$
- $A=\{1,2,3\}$; $\boldsymbol{T} = \{\langle 1,2\rangle\} \Rightarrow \boldsymbol{T}^2 = \varnothing \subseteq \boldsymbol{T}$
- $\boldsymbol{W} = \{\langle 1,1\rangle\} \Rightarrow \boldsymbol{W}^2 = \{\langle 1,1\rangle\} \subseteq \boldsymbol{W}$
- $\boldsymbol{I}_A$
- $\varnothing_A$
- $\boldsymbol{U}_A$ if $\langle a,b\rangle \in A^2 \land \langle b,a\rangle \in A^2$ then $\langle a,c\rangle \in A^2$
- if $|A| > 1$ then $\neq$ is trans
- $<$, $\leq$

Counter Examples

- $\boldsymbol{P}=\{\langle 1,2\rangle, \langle 2,1\rangle\} \Rightarrow \boldsymbol{P}^2 = \{\langle 1,1\rangle, \langle 2,2\rangle\} \nsubseteq \boldsymbol{P}$
  iow: 1 leads to 2 leads to 1, but $\langle 1,1\rangle \notin \boldsymbol{P}$
- $\exists x \exists y \exists z(\boldsymbol{R}(x,y) \land \boldsymbol{R}(y,z) \land \neg\boldsymbol{R}(x,z))$

Equivalance

$\boldsymbol{R}$ over A is equivalence iff $\boldsymbol{R}$ is reflexive, symmetric ∧ transitive

Examples

- $\boldsymbol{U}_A$, $\boldsymbol{I}_A$, equality
- "Has the same absolute value" on the set of real numbers
- if $A=\varnothing$ then $\varnothing_A$ is symmetric, transitive ∧ reflexive

Counter Examples

- $\geq$ reflexive ∧ transitive but ¬ symmetric
- if $A\neq\varnothing$ then $\varnothing_A$ is symmetric ∧ transitive, but ¬ reflexive

Connexivity

`lesson 7 00:06:00`

$\boldsymbol{R}$ over A is connexive iff ∀(x,y)∈A (x≠y → ⟨x,y⟩∈$\boldsymbol{R}$ ∨ ⟨y,x⟩∈$\boldsymbol{R}$)

Order

`lesson 7 00:00:00`

Partial Order

$\boldsymbol{R}$ over A (≤) is a partial order iff it's antireflexive ∧ transitive

Properties

- Antisymmetric `because antireflexive ∧ transitive`

Examples

- ⊂ over $\mathcal{P}$(A)

???
for all a, b, ∧ c:

- a ≤ a `reflex`
- if a ≤ b ∧ b ≤ a, then a = b `antisymm`
- if a ≤ b ∧ b ≤ c, then a ≤ c `trans`

Examples

- equality
  ???

Total Order

Partial order ∧ connexive (aka "linearly ordered")
∀(x,y)∈A (x≠y → ⟨x,y⟩∈$\boldsymbol{R}$ ⊻ ⟨y,x⟩∈$\boldsymbol{R}$) `note the xor. verify`

Examples

- ⊂ over ℕ `also over ℝ?`

Counter Examples

- if A≠∅ then $\boldsymbol{I}_A$ isn't total order because for all a∈A: a=a

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Partitions

Partition of A is a set of non-empty, non-overlapping subsets of A whose ∪ = A

Properties

- every a∈A is ∈ exactly one block
- no block contains ∅
- ∪ of blocks = A
- ∩ of any two blocks = ∅
- → A is finite ⇒ rank of P is |X| − |P| ?

Examples

- {A} is partition of A `trivial`
- ∅'s only partition is ∅
- {1,2,3} has five partitions: {{1},{2},{3}}, {{1, 2},{3}}, {{1, 3},{2}}, {{1},{2, 3}}, {{1, 2, 3}}

Counter Examples

- ¬ partitions of {1,2,3}:
  - {{}, {1,3}, {2}} `contains ∅`
  - {{1, 2}, {2, 3}} `2 exists ∈ more than one block`
  - {{1}, {2}} `no block contains 3`

Equivalence Class: {x∈$\mathcal{S}$ | x≡a} where a∈$\mathcal{S}$

Given $\mathcal{R}$ is an equivalence relation on $\mathcal{S}$, the equivalence class of an element a ∈ $\mathcal{S}$ is the set {x∈$\mathcal{S}$ | ⟨x,a⟩∈$\mathcal{R}$}

⟦a⟧ = {b|a$\mathcal{R}$b} = {b|⟨a,b⟩ ∈ $\mathcal{R}$} `all elements ∈ S that when paired with a, exist ∈ R`

In other words: going over $\mathcal{R}$, the elements ∈ ⟦a⟧ are all the elements that a is paired with

Properties

- ⋃ of all equivalence classes = $\mathcal{S}$?
- a ∈ ⟦a⟧ `every element exists ∈ its equivalence class`
- the items ∈ each equivalence class of $\mathcal{S}$ exist only ∈ their equivalence class ?
- every possible pair of eq. classes is zar ?

Examples

- $\mathcal{X}$ = all cars; relation $\equiv_{\mathcal{X}}$ = "has the same color as"; one particular equivlance class consists of all green cars
- Relation $\equiv_{\mathbb{Z}}$ is ⟨a,b⟩∈$\equiv_{\mathbb{Z}}$ ⇔ (a − b)%2==∅ ⇒ two equivalence classes: even numbers ∧ odd numbers

- S = {1,2,3,4,5}
  - $\equiv_S$ = {⟨1,**1**⟩, ⟨1,**2**⟩, ⟨1,**3**⟩, ⟨2,2⟩, ⟨3,3⟩, ⟨4,4⟩, ⟨5,5⟩, ⟨2,1⟩, ⟨2,3⟩, ⟨3,2⟩, ⟨3,1⟩}
  - ⟦1⟧ = **{1, 2, 3}** `everything that 1 is related to`
  - ⟦2⟧ = {2, 1, 3}
  - ⟦3⟧ = {3, 2, 1} `note that ⟦1⟧ ≡ ⟦2⟧ ≡ ⟦3⟧`
  - ⟦4⟧ = {4}
  - ⟦5⟧ = {5}