203627310 גלעד ברנע ת.ז.

ממ"ן 13

1

Find the power of each of the following sets

א

> The set of all real numbers $A \subseteq (0,1)$ that when expanded as an infinite fraction, each digit appears adjacent to an equal digit.

First, $A \subseteq (0,1) \implies |A| \leq |(0,1)|$.

On the other hand, function $f:(0,1) \to A$, e.g: $f(0.a_1a_2...) = 0.a_1a_1a_2a_2...$ is one-to-one, therefore:

$|(0,1)| \leq |A|$

Considering Cantor-Bernstein: $|(0,1)| \leq |A| \implies$ **$|(0,1)| = |A| = \aleph$**

ב

> $((\mathbb{N}) \times (0,1)) \cap (\mathbb{R} \times \mathbb{Q})$

The intersection means that:
$\langle x,y \rangle \in ((\mathbb{N}) \times (0,1)) \cap (\mathbb{R} \times \mathbb{Q}) \iff \langle x,y \rangle \in ((\mathbb{N}) \times (0,1)) \wedge \langle x,y \rangle \in (\mathbb{R} \times \mathbb{Q})$

Expanding the pairs:
$y \in (0,1) \wedge x \in \mathbb{N} \wedge y \in \mathbb{Q} \wedge x \in \mathbb{R} \implies x \in \mathbb{N} \cap \mathbb{R} = \mathbb{N} \wedge y \in (0,1) \cap \mathbb{Q}$

Therefore:
$\mathbb{N} \times ((0,1) \cap \mathbb{Q}) = (\mathbb{N} \times (0,1)) \cap (\mathbb{R} \times \mathbb{Q})$

Since $(0,1) \cap \mathbb{Q} \subseteq Q \implies$ **$|(0,1) \cap \mathbb{Q}| \leq |\mathbb{Q}| = \aleph_0$**

On the other hand,

$\{^1/_1, {}^1/_2, {}^1/_3, ...\} \subseteq (0,1) \cap \mathbb{Q} \implies \aleph_0 = |\{^1/_1, {}^1/_2, {}^1/_3, ...\}| \leq |(0,1) \cap \mathbb{Q}|$

From Cantor-Bernstein: $|(0,1) \cap \mathbb{Q}| = \aleph_0$

Since:
- $A \times A$ is countable if A is countable
- $|A| = \aleph_0$ if A is countable and infinite
- $A \times A$ is equivalent to A if A is countable and infinite

$\Longrightarrow$

$|\mathbb{N}\times((0,1)\cap\mathbb{Q})|=\aleph_0$

Finally:

$\mathbf{|(\mathbb{N}\times(0,1)\cap(\mathbb{R}\times\mathbb{Q}))|=\aleph_0}$

ג

> $\mathcal{P}((0,1)\setminus\boldsymbol{I})$ where $\boldsymbol{I}$ is the set of all real, irrational numbers (I think it's marked $\mathbb{J}$?)

$(0,1)\setminus\boldsymbol{I}$ is the set of all rational numbers contained in the interval (0,1).

So $(0,1)\setminus\boldsymbol{I} = (0,1)\cap\mathbb{Q} \Longrightarrow |(0,1)\setminus\boldsymbol{I}|=\aleph_0 \Longrightarrow |(0,1)\setminus\boldsymbol{I}|=\aleph_0$.

Since:
- given $|A|=\aleph_0$, then $\mathcal{P}(A)$ is not countable and $\mathcal{P}(A) \sim \{0,1\}^A$
- if n is a positive natural number, then $\{0,1,...,n\}^{\mathbb{N}} \sim \mathbb{R}$, so $|\{0,1,...,n\}^{\mathbb{N}}| = \aleph$

$\Longrightarrow$

$\mathcal{P}((0,1)\setminus\boldsymbol{I}) \sim \{0,1\}^{(0,1)\setminus\boldsymbol{I}} \Longrightarrow$ $\mathbf{|\mathcal{P}((0,1)\setminus\boldsymbol{I})|=\aleph}$

ד

> $\mathcal{P}((0,10^{-10})\setminus\mathbb{Q})$

The power of $(0,10^{-10})$:

Since every non-degenerate interval is equivalent to $\mathbb{R}$, $\Longrightarrow$ $(0,10^{-10}) \sim \mathbb{R} \Longrightarrow |(0,10^{-10})|=\aleph$.

The power of $(0,10^{-10})\setminus\mathbb{Q}$:

We know that for any two sets A and B:
given $B\subseteq A$ and $|B|=\aleph_0$, the two sets satisfy $|A\setminus B|=|A|$ if $A\setminus B$ is infinite or A is non-countable (either is sufficient).

Let $A=(0,10^{-10})$; $B=(0,10^{-10})\cap\mathbb{Q}$. It follows that $B\subseteq A$ and $|B|=\aleph_0$, therefore $|A\setminus B|=|A|$.

Since $|A|=\aleph$ and $A\setminus B=(0,10^{-10})\setminus\mathbb{Q} \Longrightarrow |(0,10^{-10})\setminus\mathbb{Q}|=\aleph$.

Because $|\mathcal{P}(A)|=2^{|A|} \Longrightarrow |\mathcal{P}((0,10^{-10})\setminus\mathbb{Q})| = 2^{|(0,10^{-10})\setminus Q|}=\mathbf{2^{\aleph}}$

2

> Let:
> $M = \{A \in \mathcal{P}(\mathbb{N}) \mid |A| = \aleph_0 \land |\overline{A}| = \aleph_0\}$
> $K = \{A \in \mathcal{P}(\mathbb{N}) \mid |\overline{A}| = \aleph_0\}$
>
> Prove or disprove the following:

א

> $|K| = \aleph_0$

The statement is false.

Let X be the set of all odd, natural numbers. It follows that for every set Y that satisfies $Y \subseteq X$, the following holds: $\overline{X} \subseteq \overline{Y}$.

$\overline{Y} \subseteq \mathbb{N}$ and is infinite $\Longrightarrow |\overline{Y}| = \aleph_0 \Longrightarrow$ every subset of X belongs to K.

Therefore $\mathcal{P}(X) \subseteq K \Rightarrow |\mathcal{P}(X)| \leq |K|$.

Since $|X| = \aleph_0 \Longrightarrow |\mathcal{P}(X)| = 2^{|X|} = 2^{\aleph_0}$, therefore $2^{\aleph_0} \leq |K| \Longrightarrow$ $\mathbf{|K| \neq \aleph_0}$

ב

> $|M| = \aleph_0$

The statement is false.

Let:
X be the set of all odd, natural numbers
$Y = \{4n+2 \mid n \in \mathbb{N}\}$
Z is a set that satisfies $Y \leq Z \leq X$ (and so is infinite and contained in $\mathbb{N}$)

It follows that $|\overline{Z}| = \aleph_0$ and that $|Z| = \aleph_0 \Longrightarrow Z \in M \Longrightarrow \{Z \in \mathcal{P}(\mathbb{N}) \mid Y \subseteq Z \subseteq X\} \subseteq M$.

For convenience, let $Q = \{Z \in \mathcal{P}(\mathbb{N}) \mid Y \subseteq Z \subseteq X\}$.

Let's define function f from Q to $\mathcal{P}(X \setminus Y)$, defined by $f(B) = B \setminus F$. f is onto and one-to-one $\Longrightarrow Q \sim \mathcal{P}(X \setminus Y)$.

Since $X \setminus Y = \{4n \mid n \in \mathbb{N}\} \Longrightarrow |X \setminus Y| = \aleph_0$, therefore $|\mathcal{P}(X \setminus Y)| = 2^{\aleph_0} = \aleph \Longrightarrow |Q| = \aleph$.

Q is contained in M therefore $|M| \geq \aleph$ therefore $\mathbf{|M| \neq \aleph_0}$

ג

> $|\mathcal{P}(\mathbb{N})\setminus K|=\aleph_0$

The statement is true.

K is the set of all sets partial to $\mathbb{N}$ whose complement is infinite, therefore $\mathcal{P}(\mathbb{N})\setminus K$ is the set of all set partial to $\mathbb{N}$ whose complement is **finite**.

Let $X(\mathbb{N})$ be the set of all finite sets that are partial to $\mathbb{N}$. In other words, the union of all finite sets of natural numbers.

A finite set S of natural numbers is countable, because S is contained in a set of natural numbers ranging from $\varnothing$ to n, where n if the greatest element in S.

Therefore $X(\mathbb{N})$ consist of a countable number of countable sets $\Longrightarrow$ $X(\mathbb{N})$ itself is countable.

Moreover, $X(\mathbb{N})$ is infinite $\Longrightarrow$ $|X(\mathbb{N})|=\aleph_0$.   (0)

Let's define function f from $\mathcal{P}(\mathbb{N})\setminus K$ to $X(\mathbb{N})$, defined by $f(A)=\overline{A}$. f is onto and one-to-one $\Longrightarrow$ $|\mathcal{P}(\mathbb{N})\setminus K| = |X(\mathbb{N})|$.   (1)

Combining (0) and (1), we get $\boldsymbol{|\mathcal{P}(\mathbb{N})\setminus K|=\aleph_0}$

ד

> $|\mathcal{P}(\mathbb{N})\setminus M|=\aleph_0$

The statement is true.

Based on the definition of M:
$\mathcal{P}(\mathbb{N})\setminus M$ is the set of all sets partial to $\mathbb{N}$ which are finite or that their complement is finite.

Let $X = \{A\in\mathcal{P}(\mathbb{N}) \mid |A|=\aleph_0\}$ $\Longrightarrow$
$\mathcal{P}(\mathbb{N})\setminus X$ is the set of all finite sets partial to $\mathbb{N}$ $\Longrightarrow$
$\mathcal{P}(\mathbb{N})\setminus M = (\mathcal{P}(\mathbb{N})\setminus K)\cup(\mathcal{P}(\mathbb{N})\setminus X)$.

Because $\mathcal{P}(\mathbb{N})\setminus X$ is the same as $X(\mathbb{N})$ from gimel, $|\mathcal{P}(\mathbb{N})\setminus X|=\aleph_0$.
Considering the conclusion of gimel ($|\mathcal{P}(\mathbb{N})\setminus K|=\aleph_0$), and given that the union of sets with cardinality of $\aleph_0$ has itself a cardinality of $\aleph_0$ $\Longrightarrow$ $\boldsymbol{|\mathcal{P}(\mathbb{N})\setminus M|=\aleph_0}$.

3

> Let:
> $A = \{A_i \mid i\in\mathbb{N}\}$ where $A_i\subseteq\mathbb{N}$, $A_i\neq A_j$, $A_i\cap A_j=\varnothing$, for all $i,j\in\mathbb{N}$, $i\neq j$

> B is a set of non-empty, open intervals in $\mathbb{R}$ such that no two intervals overlap
> C is an uncountable, infinite set of open intervals in $\mathbb{R}$.
>
> Prove:

א

> |B|≤|A|

First let's prove that $|A|=\aleph_0$.

Function f:ℕ→A defined by $f(i)=A_i$ for all $i\in\mathbb{N}$ is onto.
Also, because $i,j\in\mathbb{N}$ and $i\neq j$, the following holds: $A_i\neq A_j \implies f(i)\neq f(j)$ $\implies A\sim\mathbb{N} \implies |A|=\aleph_0$.

Let $B=\{B_\alpha \mid \alpha\in I\}$ where I is a matching set of indices.
Let f be a function from I→B defined by $f(\alpha)=B_\alpha$ for all $\alpha\in I$.
f is one-to-one and onto $\implies |B|=|I|$.

Because every non-empty open interval contains all rational numbers, and because for all $\alpha\in I$, set B_α is a non-empty open interval $\implies$ for all $\alpha\in I$ there exists a number $q_\alpha\in\mathbb{Q}$ such that $q_\alpha\in B_\alpha$.

Let g be a function from I→ℚ defined by $f(\alpha)=q_\alpha$ for all $\alpha\in I$.
To prove (by negation) that g is one-to-one, let's assume that $f(\alpha)=f(\beta)$ for α,b∈I where $\alpha\neq\beta \implies q_\alpha = q_\beta$.
$q_\alpha\in B_\alpha$ and $q_\beta\in B_\beta \implies q_\alpha\in(B_\alpha\cap B_\beta)$ which negates the definition of B (no two intervals in B overlap).

Having concluded that g is one-to-one $\implies |I|\leq|\mathbb{Q}|=\aleph_0 \implies |B|\leq\aleph_0 \implies$ **|B|≤|A|**.

ב

> Prove that two intervals I,J∈C exist such that |I∩J|=|ℝ|

We have concluded in alef that if two intervals are don't overlap, then their set is countable.

Combining the definition of C together with the aforementioned conclusion, it follows that there exist I,J∈C such that I∩J≠∅.

Let I=(a,b); J=(c,d). Assuming a≤c, since I∩J≠∅ $\implies$ c < b $\implies$ I∩J=(c,b).

We know that every non-degenerate is equivalent to $\mathbb{R}$, therefore $\mathbf{|I \cap J| = |(b,c)| = |\mathbb{R}|}$.