1 Question 1:

First we notice that the product $p(x)q(x)$ has degree 2, meaning the length of its coefficient vector is 3. Since we know how to do FFT on n's that are a power of two, we will choose $n = 4$. Therefore we will need to know the roots of unity of order 4 and 2 (for the recursion). These are $1, i, -1, -i$ (order 4) and $1, -1$ (order 2).

Let $p = (9, 7, 0, 0), q = (5, 4, 0, 0)$, and we are looking for $s = (s_0, s_1, s_2, s_3)$ which are the coefficients of $s(x) = p(x)q(x)$ (we hope s_3 will be zero).

First we will find the Fourier transform of the coefficients of $p(x)$:

$FFT(9, 7, 0, 0)$:

- $f_{even} = (9, 0)$
- $f_{odd} = (7, 0)$
- $y_{even} = FFT(9, 0)$:
  - $f_{even} = (9)$
  - $f_{odd} = (0)$
  - $y_{even} = FFT(9) = (9)$
  - $y_{odd} = FFT(0) = (0)$
  - $k = 0, \omega = 1$:
    - $y_0 = y_{even}(0) + \omega y_{odd}(0) = 9 + 0 = 9$
    - $y_1 = y_{even}(0) - \omega y_{odd}(0) = 9 - 0 = 9$
  - return $(9, 9)$
- $y_{odd} = FFT(7, 0)$:
  - $f_{even} = (7)$
  - $f_{odd} = (0)$
  - $y_{even} = FFT(7) = (7)$
  - $y_{odd} = FFT(0) = (0)$
  - $k = 0, \omega = 1$:
    - $y_0 = y_{even}(0) + \omega y_{odd}(0) = 7 + 0 = 7$
    - $y_1 = y_{even}(0) - \omega y_{odd}(0) = 7 - 0 = 7$
  - return $(7, 7)$
- $k = 0, \omega = 1$:
  - $y_0 = y_{even}(0) + \omega y_{odd}(0) = 9 + 7 = 16$

- $y_1 = y_{even}(0) - \omega y_{odd}(0) = 9 - 7 = 2$

- $k = 1, \omega = \omega_4^{-1} = -i$:
  - $y_1 = y_{even}(1) + \omega y_{odd}(1) = 9 - 7i$
  - $y_3 = y_{even}(1) - \omega y_{odd}(1) = 9 + 7i$
- return $(16, 9 - 7i, 2, 9 + 7i)$

And the Fourier transform of the coefficients of $q(x)$: $FFT(5, 4, 0, 0)$:

- $f_{even} = (5, 0)$
- $f_{odd} = (4, 0)$
- $y_{even} = FFT(5, 0)$:
  - $f_{even} = (5)$
  - $f_{odd} = (0)$
  - $y_{even} = FFT(5) = (5)$
  - $y_{odd} = FFT(0) = (0)$
  - $k = 0, \omega = 1$:
    * $y_0 = y_{even}(0) + \omega y_{odd}(0) = 5 + 0 = 5$
    * $y_1 = y_{even}(0) - \omega y_{odd}(0) = 5 - 0 = 5$
  - return $(5, 5)$
- $y_{odd} = FFT(4, 0)$:
  - $f_{even} = (4)$
  - $f_{odd} = (0)$
  - $y_{even} = FFT(4) = (4)$
  - $y_{odd} = FFT(0) = (0)$
  - $k = 0, \omega = 1$:
    * $y_0 = y_{even}(0) + \omega y_{odd}(0) = 4 + 0 = 4$
    * $y_1 = y_{even}(0) - \omega y_{odd}(0) = 4 - 0 = 4$
  - return $(4, 4)$
- $k = 0, \omega = 1$:
  - $y_0 = y_{even}(0) + \omega y_{odd}(0) = 5 + 4 = 9$
  - $y_1 = y_{even}(0) - \omega y_{odd}(0) = 5 - 4 = 1$
- $k = 1, \omega = \omega_4^{-1} = -i$:
  - $y_1 = y_{even}(1) + \omega y_{odd}(1) = 5 - 4i$
  - $y_3 = y_{even}(1) - \omega y_{odd}(1) = 5 + 4i$
- return $(9, 5 - 4i, 1, 5 + 4i)$

Now we can do point wise multiplication and get the vector $(144, 17-71i, 2, 17+71i)$. (Notice that since p, q are real then so is their product $s(x) = p(x)q(x)$. Indeed, as we mentioned in class, $\hat{p}_0, \hat{p}_{n/2}$ are real, and $\hat{p}_k = \overline{\hat{p}_{n-k}}$ for $k = 1, \ldots, n-1$ which in this specific case is only $k = 1$. This is also the case for $\hat{q}$ and $\hat{s}$.)

All that is left to do is find the inverse Fourier transform of the result vector:

$FFT^{-1}(144, 17-71i, 2, 17+71i)$:

- $f_{even} = (144, 2)$
- $f_{odd} = (17-71i, 17+71i)$
- $y_{even} = FFT^{-1}(144, 2)$:
  - $f_{even} = (144)$
  - $f_{odd} = (2)$
  - $y_{even} = FFT^{-1}(144) = (144)$
  - $y_{odd} = FFT^{-1}(2) = (2)$
  - $k = 0, \omega = 1$:
    - $y_0 = y_{even}(0) + \omega y_{odd}(0) = 144 + 2 = 146$
    - $y_1 = y_{even}(0) - \omega y_{odd}(0) = 144 - 2 = 142$
  - return $(146, 142)$
- $y_{odd} = FFT^{-1}(17-71i, 17+71i)$:
  - $f_{even} = (17-71i)$
  - $f_{odd} = (17+71i)$
  - $y_{even} = FFT^{-1}(17-71i) = (17-71i)$
  - $y_{odd} = FFT^{-1}(17+71i) = (17+71i)$
  - $k = 0, \omega = 1$:
    - $y_0 = y_{even}(0) + \omega y_{odd}(0) = 17 - 71i + 17 + 71i = 34$
    - $y_1 = y_{even}(0) - \omega y_{odd}(0) = 17 - 71i - 17 - 71i = -142i$
  - return $(34, -142i)$
- $k = 0, \omega = 1$:
  - $y_0 = y_{even}(0) + \omega y_{odd}(0) = 146 + 34 = 180$
  - $y_1 = y_{even}(0) - \omega y_{odd}(0) = 146 - 34 = 112$
- $k = 1, \omega = \omega_4 = i$:
  - $y_1 = y_{even}(1) + \omega y_{odd}(1) = 142 + 142 = 284$
  - $y_3 = y_{even}(1) - \omega y_{odd}(1) = 142 - 142 = 0$
- return $(180, 284, 112, 0)$

This vector is supposed to be the coefficients of $s(x)$ multiplied by $n = 4$, and indeed: $p(x)q(x) = (9+7x)(5+4x) = 45 + 71x + 28x^2$, and $180/4 = 45, 284/4 = 71$ and $112/4 = 28$.

2 Question 2:

Let $p(x) = \sum_{i=0}^{n-1} a_i x^i$. The k'th derivative of $p(x)$ at $x = x_0$ is

$$p^{(k)}(x_0) = \sum_{j=k}^{n-1} \frac{j!}{(j-k)!} a_j x_0^{j-k}$$

We will define two vectors whose convolution contains all the derivatives we are looking for.

$$f_i = \begin{cases} a_i i! & 0 \le i \le n-1 \\ 0 & n \le i \le 2n-1 \end{cases}$$

$$g_i = \begin{cases} 1 & i = 0 \\ 0 & 1 \le i \le n \\ \frac{1}{(2n-i)!} x_0^{2n-i} & n+1 \le i \le 2n-1 \end{cases}$$

Let $h = f * g$, then for $0 \le k \le n-1$

$$h_k = \sum_{j=0}^{2n-1} f_j g_{(k-j) \mod 2n} = \sum_{j=0}^{k} f_j g_{k-j} + \sum_{j=k+1}^{2n-1} f_j g_{k-j+2n} =$$

$$= f_k g_0 + \sum_{j=k+1}^{n-1} a_j j! \frac{x_0^{j-k}}{(j-k)!} = \sum_{j=k}^{n-1} \frac{j!}{(j-k)!} a_j x_0^{j-k} = p^{(k)}(x_0)$$

3 Question 3:

Let s be the number of question marks in P.

Replace all the zeros in T, P by -1 and all the question marks in P by zero. Now the algorithm is identical to the one we saw in class: Define the vectors $f_i = (t_{(i-1)m}, \ldots, t_{(i+1)m-1})$ for $i = 1, \ldots, \lfloor \frac{n}{m} \rfloor$ (the last one may be padded with zeros if T is too short). Define also the vector $g = (p_m, 0, \ldots, 0, p_{m-1}, \ldots, p_0)$ (the number of zeros in the middle is $m-1$). Compute $h_i = f_i * g$ for every i, then for every $0 \le k \le m-1$ $h_k = m+1-s$ if and only if $t_{(i-1)m+k} t_{(i-1)m+k+1} \cdots t_{im+k}$ is equal to P for all the indices where $p_i \ne '?'$. The proof of this is very similar to the proof we saw in class, so we leave it out.

4 Question 4:

According to the definition of the Fourier transform, we know the value of $h(\omega_n^{-k})$ for all $k = 0, \ldots, n-1$:

$$h(\omega_n^{-k}) = \sum_{j=0}^{n-1} h_j \omega_n^{-kj} = \hat{h}_k$$

The value of $r(\omega_n^{-k})$ is:

$$r(\omega_n^{-k}) = f(\omega_n^{-k}) g(\omega_n^{-k}) - q(\omega_n^{-k})(\omega_n^{-kn} - 1) = f(\omega_n^{-k}) g(\omega_n^{-k})$$

and since we also know that

$$f(\omega_n^{-k}) = \sum_{j=0}^{n-1} f_j \omega_n^{-kj} = \hat{f}_k$$

$$g(\omega_n^{-k}) = \sum_{j=0}^{n-1} g_j \omega_n^{-kj} = \hat{g}_k$$

we get that $r(\omega_n^{-k}) = \hat{f}_k \hat{g}_k = \hat{h}_k$. Therefore we have that $r(\omega_n^{-k}) = h(\omega_n^{-k})$ for $k = 0, \ldots, n-1$, r, h are polynomials of degree $n-1$, so we conclude that they are equal.