Algorithms - Ex10 Solutions

1 Question 1:

$$\sum_{k=0}^{n-1}|\hat{f}(k)|^2=\sum_{k=0}^{n-1}\hat{f}(k)\overline{\hat{f}(k)}=\sum_{k=0}^{n-1}\hat{f}(k)\overline{\sum_{j=0}^{n-1}f(j)\omega_n^{-kj}}=\sum_{k=0}^{n-1}\hat{f}(k)\sum_{j=0}^{n-1}\overline{f(j)}\omega_n^{kj}=\sum_{j=0}^{n-1}\overline{f(j)}\sum_{k=0}^{n-1}\hat{f}(k)\omega_n^{kj}=$$

$$=n\sum_{j=0}^{n-1}\overline{f(j)}f(j)=n\sum_{j=0}^{n-1}|f(j)|^2$$

Notice that if V is as defined in the question, $f=\frac{1}{\sqrt{n}}V\hat{f}$. Therefore, given a vector $x\in\mathbb{C}^n$, let y be a vector such that $\hat{y}=x$. Then

$$\frac{1}{n}\|Vx\|^2=\left\|\frac{1}{\sqrt{n}}V\hat{y}\right\|^2=\|y\|^2=\frac{1}{n}||\hat{y}||^2=\frac{1}{n}||x||^2$$

Therefore,

$$\|Vx\|=||x||$$

2 Question 2:

We are given an $m\times n$ matrix A, $m<n$, and we wish to find a vector x that is a solution to $Ax=b$, with minimal l_2 norm. Let U,D,V be the SV decomposition of A, i.e. $A=UDV^T$. Then solving $Ax=b$ is equivalent to solving $UDVx=b$. Denote $y=Vx$. Since V is an orthogonal matrix, for any vector x $||Vx||_2=||x||_2$. Therefore if we find a vector y which is a solution to $UDy=b$ with minimal l_2 norm, then $x=V^{-1}y$ is a solution to $UDVx=b$, $||x||_2=||y||_2$, and therefore x is also a solution with minimal l_2 norm.

U is orthogonal, therefore U^{-1} exists. Denote $b'=U^{-1}b$. We want to solve $Dy=b'$. D is an $m\times n$ matrix, therefore we want to solve the m following equations:

$$d_{11}y_1=b'_1$$

$$d_{22}y_2=b'_2$$

$$\vdots$$

$$d_{mm}y_m=b'_m$$

Let $y_i=b'_i/d_{ii}$ for $i=1,\ldots,m$, and choose any value for the variables $y_{m+1},\ldots,y_n$. This is a solution to $Dy=b'$. The solution with minimal l_2 norm is obviously the vector with $y_{m+1}=\cdots=y_n=0$. Now we can find the solution: $x=V^{-1}y$.