$\boldsymbol{R}$: {⟨x,y⟩ for ⟨x,y⟩ ∈ A^2 if x$\boldsymbol{R}$y}

$\boldsymbol{T}\cdot\boldsymbol{R}$: {⟨a,c⟩ | ∃b ∈ B (⟨a,b⟩ ∈ $\boldsymbol{T}$ ∧ ⟨b,c⟩ ∈ $\boldsymbol{R}$)}

$\boldsymbol{R}^2$: a$\boldsymbol{R}^2$c ⇔ {⟨a,c⟩ | ∃b ∈ A (⟨a,b⟩ ∈ $\boldsymbol{R}$ ∧ ⟨b,c⟩ ∈ $\boldsymbol{R}$)}

an ordered pair ⟨a,c⟩∈$\boldsymbol{R}^2$ means there's a "middle" b∈B that satisfie
⟨b,c⟩∈$\boldsymbol{R}$

Examples

- $(a=-b)^2 = \boldsymbol{I}_{\mathbb{R}}$
- ⟨a,b⟩ ∈ $\boldsymbol{R}^2$ ⇔ ⟨a,c⟩,⟨c,b⟩ ∈ $\boldsymbol{R}$

Empty $\varnothing_A$

$\boldsymbol{R}$:=rel(A×B) = ∅

No pair ∈ A×B satisfies ⟨a,b⟩ ∈ $\boldsymbol{R}$

Properties

- $\boldsymbol{S}\cdot\varnothing_A = \varnothing$
- anti-symmetric
- symmetric ?

Examples

- {⟨x,y⟩ ∈ $\mathbb{N}^2$ | x+y < x}

Identity $\boldsymbol{I}_A$

Properties

- $\boldsymbol{R}\cdot\boldsymbol{I}_A = \boldsymbol{R}$

Reflexivity

$\boldsymbol{R}$:=rel(A) is reflexive if ∀a ∈ A(⟨a,a⟩ ∈ $\boldsymbol{R}$)

$\boldsymbol{R}$ is reflexive if every a ∈ A satisfies ⟨a,a⟩∈$\boldsymbol{R}$. In other words:
$\boldsymbol{I}_A \subseteq \boldsymbol{R}$

A = { -1, 0, 1 }. Is ⋱ contained ∈ $\boldsymbol{R}$?

$\boldsymbol{R}$ = lambda a,b: a⊙b; all(R(x,x) for x ∈ A)?

Properties

- ⇔ $\boldsymbol{R}^{-1}$ is reflexive
- → $\boldsymbol{R} \subseteq \boldsymbol{R}^2$ (and $\boldsymbol{R}^2$ is reflexive)

- → $\boldsymbol{R}\subseteq\boldsymbol{R}^2$
- if $\boldsymbol{S}\subseteq\boldsymbol{R}$ then $\boldsymbol{S}$ is reflexive
- if $\boldsymbol{S}$ is reflexive then both $\boldsymbol{R}\cup\boldsymbol{S}$ ∧ $\boldsymbol{R}\cap\boldsymbol{S}$ are reflexive

Examples

- $\boldsymbol{U}_A$: ∀a ∈ A(⟨a,a⟩ ∈ A×A = $\boldsymbol{U}$A)
- $\boldsymbol{I}_A$: ∀a ∈ A(⟨a,a⟩ ∈ {⟨−1, −1⟩, ⟨0, 0⟩, ⟨1, 1⟩})
- ≤, ≥ both contain ⋱

Counter Examples

- ≠ (which is $\boldsymbol{U}_A$ − $\boldsymbol{I}$A)
- <, >, $\varnothing_A$
- a=−b ⋰

Anti-Reflexivity

$\boldsymbol{R}$:=rel(A) is anti-reflexive iff ¬∃a ∈ A(⟨a,a⟩ ∈ $\boldsymbol{R}$)

$\boldsymbol{R}$ is reflexive if every a ∈ A satisfies ⟨a,a⟩ ∉ $\boldsymbol{R}$. In other words:

$\boldsymbol{I}_A$ ∩ $\boldsymbol{R}$ = ∅ just $\boldsymbol{I}_A$ ⊈ $\boldsymbol{R}$ isn't enough; $\boldsymbol{I}_A$ = {⟨1,1⟩, ⟨2,2⟩} ⊈ $\boldsymbol{R}$ = {⟨

but ⟨1,1⟩ ∈ $\boldsymbol{R}$ so isn't anti-reflexive

Examples

- ≠, <, >, $\varnothing_A$

Counter Examples

- $\boldsymbol{U}_A$, $\boldsymbol{I}_A$, a=−b ⋰, ≤, ≥

Symmetry

$\boldsymbol{R}$:=rel(A) is symmetric iff $\boldsymbol{R}$ = $\boldsymbol{R}^{-1}$

$\boldsymbol{R}$ is symmetric if every ⟨x,y⟩ ∈ $\boldsymbol{R}$ satisfies ⟨y,x⟩ ∈ $\boldsymbol{R}$
  assuming both x ∧ y exist ∈ A

∀x∀y(⟨x,y⟩∈$\boldsymbol{R}$ → ⟨y,x⟩∈$\boldsymbol{R}$)

$\boldsymbol{R}$ = lambda a,b: a⊙b; all(rel(y,x) for x,y ∈ R)?

Properties

- if $\boldsymbol{S}$ is symmetric then both $\boldsymbol{R}\cup\boldsymbol{S}$ ∧ $\boldsymbol{R}\cap\boldsymbol{S}$ are reflexive
- if $\boldsymbol{S}$ is symmetric then $\boldsymbol{R}\setminus\boldsymbol{S}$ is symmetric

Examples

- $\varnothing_A$ can't point at ⟨x,y⟩ ∧ say ⟨y,x⟩ is ¬ ∈ $\varnothing^{-1}$
- $\boldsymbol{U}_A$, $\boldsymbol{I}_A$, a=−b ⋰, ≠

Counter Examples

- ≤, ≥, <, >

Anti-Symmetry

R:=rel(A) is anti-symmetric iff $R \cap R^{-1} = \varnothing$

R is anti-symmetric if every ⟨x,y⟩ ∈ R satisfies ⟨y,x⟩ ∉ R

$\forall x \forall y((x,y) \in R \rightarrow (y,x) \notin R)$

$R \cap R^{-1} = \varnothing$ means there can't be a ⟨x,x⟩

Properties

- → R is anti-reflexive
- → R^{-1} is anti-symmetric
- if $S \subseteq R$ then S is anti-symmetric
- if $S \cup T$ is anti-symmetric then both S ∧ T are anti-symmetric
- → $R \cap S$ is anti-symmetric
- [if R is antireflexive ∧ transitive then it's asymmetric ∧ anti-s]

Examples

- <, >, $\varnothing_A$
- $b > a^2$

Counter Examples

- ≠, ≤, ≥, U_A, I_A, a=-b ∴, ≠
- $b < a^2$ ⟨3,4⟩ ∧ ⟨4,3⟩ are symmetric

Weak Anti-Symmetry

$R \cap R^{-1} \subseteq I_A$

$\forall x \forall y(\langle x,y \rangle \in R \land \langle y,x \rangle \in R \rightarrow x=y)$

if both ⟨x,y⟩ ∈ R ∧ ⟨y,x⟩ ∈ R it's only because they're equal

for x,y ∈ A: if x≠y ∧ ⟨x,y⟩ ∈ R then must ⟨y,x⟩ ∉ R

A_S vs WA_S: A_S requires every pair's opposite to ¬ be ∈ R, whereas W
the same only for pairs that x=y

Examples

- I_A

Transitivity

$R^2 \subseteq R$

$\forall x \forall y \forall z((R(x,y) \land R(y,z)) \rightarrow R(x,z))$

Every (x,y,z)∈A that satisfy ⟨x,y⟩∈$\boldsymbol{R}$ ∧ ⟨y,z⟩∈$\boldsymbol{R}$ also satisfy ⟨x,z⟩∈$\boldsymbol{R}$
If you see an x that leads to y that leads to z, then expect x to le

this is why $\boldsymbol{R}^2 \subseteq \boldsymbol{R}$

Properties

- if $\boldsymbol{T}$ is symmetric ∧ anti-symmetric then it's also transitive

Examples

- A={1,2,3}; $\boldsymbol{R}$ = {⟨**1**,2⟩, ⟨2,**3**⟩, **⟨1,3⟩**} ⇒ $\boldsymbol{R}^2$ = {⟨1,3⟩} ⊆ $\boldsymbol{R}$
- A={1,2,3}; $\boldsymbol{T}$ = {⟨1,2⟩} ⇒ $\boldsymbol{T}^2$ = ∅ ⊆ $\boldsymbol{T}$
- $\boldsymbol{W}$ = {⟨1,1⟩} ⇒ $\boldsymbol{W}^2$ = {⟨1,1⟩} ⊆ $\boldsymbol{W}$
- $\boldsymbol{I}_A$
- $\varnothing_A$
- $\boldsymbol{U}_A$ if ⟨a,b⟩ ∈ A^2 ∧ ⟨b,a⟩ ∈ A^2 then ⟨a,c⟩ ∈ A^2
- if |A| > 1 then ≠ is trans
- <, ≤

Counter Examples

- $\boldsymbol{P}$={⟨1,2⟩, ⟨2,1⟩} ⇒ $\boldsymbol{P}^2$ = {⟨1,1⟩, ⟨2,2⟩} ⊈ $\boldsymbol{P}$
  iow: 1 leads to 2 leads to 1, but ⟨1,1⟩ ⊈ $\boldsymbol{P}$
- ∃x∃y∃z($\boldsymbol{R}$(x,y) ∧ $\boldsymbol{R}$(y,z) ∧ ¬$\boldsymbol{R}$(x,z))

Equivalance

$\boldsymbol{R}$ over A is equivalence iff $\boldsymbol{R}$ is reflexive, symmetric ∧ transitive

Examples

- $\boldsymbol{U}_A$, $\boldsymbol{I}_A$, equality
- "Has the same absolute value" on the set of real numbers
- if A=∅ then $\varnothing_A$ is symmetric, transitive ∧ reflexive

Counter Examples

- ≥ reflexive ∧ transitive but ¬ symmetric
- if A≠∅ then $\varnothing_A$ is symmetric ∧ transitive, but ¬ reflexive

Connexivity

lesson 7 00:06:00

$\boldsymbol{R}$ over A is connexive iff ∀(x,y)∈A (x≠y → ⟨x,y⟩∈$\boldsymbol{R}$ ∨ ⟨y,x⟩∈$\boldsymbol{R}$)

Order

lesson 7 00:00:00

Partial Order

$\boldsymbol{R}$ over A (≤) is a partial order iff it's antireflexive ∧ transitive

Properties

- Antisymmetric   because antireflexive ∧ transitive

Examples

- ⊂ over $\mathcal{P}$(A)

???
for all a, b, ∧ c:

- a ≤ a   reflex
- if a ≤ b ∧ b ≤ a, then a = b   antisymm
- if a ≤ b ∧ b ≤ c, then a ≤ c   trans

Examples

- equality
  ???

Total Order

Partial order ∧ connexive (aka "linearly ordered")
∀(x,y)∈A (x≠y → ⟨x,y⟩∈$\boldsymbol{R}$ ⊻ ⟨y,x⟩∈$\boldsymbol{R}$)   note the xor. verify

Examples

- ⊂ over ℕ   also over ℝ?

Counter Examples

- if A≠∅ then $\boldsymbol{I}_A$ isn't total order because for all a∈A: a=a

---

Partitions

Partition of A is a set of non-empty, non-overlapping subsets of A w

Properties

- every a∈A is ∈ exactly one block
- no block contains ∅
- ∪ of blocks = A
- ∩ of any two blocks = ∅
- → A is finite ⇒ rank of P is |X| − |P| ?

Examples

- {A} is partition of A   trivial
- ∅'s only partition is ∅

- {1,2,3} has five partitions: {{1},{2},{3}}, {{1, 2}, {3}}, {{1,
  {{1},{2, 3}}, {{1, 2, 3}}

Counter Examples

- ¬ partitions of {1,2,3}:
  - {{}, {1,3}, {2}}   contains ∅
  - {{1, 2}, {2, 3}}   2 exists ∈ more than one block
  - {{1}, {2}}   no block contains 3

Equivalence Class: {x∈$\boldsymbol{S}$ | x≡a} where a∈$\boldsymbol{S}$

Given $\boldsymbol{R}$ is an equivalence relation on $\boldsymbol{S}$, the equivalence class of ar
∈ $\boldsymbol{S}$ is the set {x∈$\boldsymbol{S}$ | ⟨x,a⟩∈$\boldsymbol{R}$}

⟦a⟧ = {b|a$\boldsymbol{R}$b} = {b|⟨a,b⟩ ∈ $\boldsymbol{R}$}
  all elements ∈ $\boldsymbol{S}$ that when paired with a, exist ∈ $\boldsymbol{R}$

In other words: going over $\boldsymbol{R}$, the elements ∈ ⟦a⟧ are all the elemen
paired with

Properties

- ⋃ of all equivalence classes = $\boldsymbol{S}$?
- a ∈ ⟦a⟧   every element exists ∈ its equivalence class
- the items ∈ each equivalence class of $\boldsymbol{S}$ exist only ∈ their equiv
  ?
- every possible pair of eq. classes is zar ?

Examples

- $\boldsymbol{X}$ = all cars; relation $\equiv_{\boldsymbol{X}}$ = "has the same color as"; one particu
  equivlance class consists of all green cars
- Relation $\equiv_{\mathbb{Z}}$ is ⟨a,b⟩∈$\equiv_{\mathbb{Z}}$ ⇔ (a − b)%2==∅ ⇒ two equivalence classes
  numbers ∧ odd numbers
- $\boldsymbol{S}$ = {1,2,3,4,5}
  - $\equiv_{\boldsymbol{S}}$ = {⟨1,**1**⟩, ⟨1,**2**⟩, ⟨1,**3**⟩, ⟨2,2⟩, ⟨3,3⟩, ⟨4,4⟩, ⟨5,5⟩, ⟨2,1⟩
    ⟨3,2⟩, ⟨3,1⟩}
  - ⟦1⟧ = **{1, 2, 3}**   everything that 1 is related to
  - ⟦2⟧ = {2, 1, 3}
  - ⟦3⟧ = {3, 2, 1}   note that ⟦1⟧ ≡ ⟦2⟧ ≡ ⟦3⟧
  - ⟦4⟧ = {4}
  - ⟦5⟧ = {5}