

ממ"ץ 11

Note : Sometimes I'll be using e.g. $\neg A$ to represent the complement of A
(My editor doesn't fully support superscript or overline)

2

8

Prove:

$$(A \setminus B) \cup (B \setminus C) = (A \cup B) \setminus (B \cap C)$$

First: expand left-hand side $(A \setminus B) \cup (B \setminus C)$

$$\begin{aligned} (A \cap \neg B) \cup (B \cap \neg C) & \quad // \text{diff} \\ (A \cup B) \cap (A \cup \neg C) \cap (\neg B \cup B) \cap (\neg B \cup \neg C) & \quad // \text{distributivity} \\ (A \cup B) \cap (A \cup \neg C) \cap (\neg B \cup \neg C) & \quad // (\neg B \cup B) \equiv T \\ (A \cup B) \cap [(A \cap \neg B) \cup \neg C] & \quad // \text{dist.} \end{aligned}$$

Second: expand right-hand side $(A \cup B) \setminus (B \cap C)$

$$\begin{aligned} (A \cup B) \cap \overline{(B \cap C)} \\ (A \cup B) \cap (\neg B \cup \neg C) \\ (A \cap \neg B) \cup (A \cap \neg C) \cup (B \cap \neg B) \cup (B \cap \neg C) & \quad // \text{dist} \\ (A \setminus B) \cup (A \cap \neg C) \cup (B \cap \neg C) & \quad // (B \cap \neg B) \equiv \emptyset \\ (A \setminus B) \cup [(A \cup B) \cap \neg C] & \quad // \text{dist} \\ [(A \setminus B) \cup (A \cup B)] \cap [(A \cap \neg B) \cup \neg C] & \quad // \text{dist} \\ // \text{I'll now prove that } [(A \setminus B) \cup (A \cup B)] \equiv (A \cup B), \\ // \text{then get back to expanding the full statement} \end{aligned}$$

Since $(A \setminus B) \subseteq A$ and $A \subseteq (A \cup B) \Rightarrow$
 $(A \setminus B) \subseteq (A \cup B)$
 Therefore
 $(A \setminus B) \cup (A \cup B) = (A \cup B)$

$$(A \cup B) \cap [(A \cap \neg B) \cup \neg C]$$

We see that left-hand side \equiv right-hand side, therefore

$$(A \setminus B) \cup (B \setminus C) = (A \cup B) \setminus (B \cap C)$$

2

Prove:

if $P(A) \vee P(B) = P(C)$, then $(C=A) \vee (C=B)$

I'll be proving:

$(C \subseteq A \wedge A \subseteq C) \vee (C \subseteq B \wedge B \subseteq C)$

Since it's equivalent to

$(C=A) \vee (C=B)$

First: proof that $C \subseteq A \vee C \subseteq B$

$C \in P(C)$ // power set definition

$P(C) = P(A) \vee P(B) \Rightarrow C \in (P(A) \vee P(B))$

$C \in P(A) \vee C \in P(B)$

$C \subseteq A \vee C \subseteq B$

Second: proof that $A \subseteq C \vee B \subseteq C$

$A \in P(A)$

$P(A) \subseteq P(A) \cup P(B)$ // union definition

$A \in P(A) \cup P(B)$

Given $P(C) = (P(A) \cup P(B)) \Rightarrow A \in P(C)$

$A \subseteq C$

$B \in P(B)$

$P(B) \subseteq P(A) \cup P(B)$ // union definition

$B \in P(A) \cup P(B)$

Given $P(C) = (P(A) \cup P(B)) \Rightarrow B \in P(C)$

$B \subseteq C$

Since $C \subseteq A \vee C \subseteq B$ and $A \subseteq C$ and $B \subseteq C$,

we conclude that:

$C \subseteq A \vee C \subseteq B \wedge A \subseteq C \wedge B \subseteq C$

Therefore

$(C=A) \vee (C=B)$

3

Prove:

if A, B are finite and $|P(A)| = 2 \cdot |P(A \setminus B)|$, then $|A \cap B| = 1$

(1)

$$A \setminus B \equiv A \setminus (A \cap B) \quad // \text{ by definition}$$

(2)

We know that for any two sets X, Y , if $Y \subseteq X$ then $|X \setminus Y| = |X| - |X \cap Y|$

Certainly $(A \cap B) \subseteq A$, so

$$|A \setminus (A \cap B)| = |A| - |A \cap B|.$$

(3)

Assuming $|A \cap B| = 1$, it follows that:

$|A| - |A \cap B| = |A| - 1$, therefore using (1) and (2):

$$|A \setminus B| = |A \setminus (A \cap B)| = |A| - |A \cap B| = |A| - 1, \text{ so}$$

$$|P(A \setminus B)| = 2^{|A \setminus B|} = 2^{|A| - 1}$$

(4): Expanding $2 \cdot |P(A \setminus B)|$

$$2 \cdot |P(A \setminus B)| = 2 \cdot 2^{|A| - 1} = 2^{|A|}$$

(5)

$$|P(A)| = 2^{|A|} \quad // \text{ by definition}$$

(6)

$$|P(A)| = 2 \cdot |P(A \setminus B)|$$

3

⌘

Prove: if $(A \subset B)$, then $(A \cup \neg B) \neq U$

Since A is a **proper** subset of B , then $(B \setminus A) \neq \emptyset$.

Expanding $(B \setminus A)$:

$$(B \setminus A) =$$

$$(B \cap \neg A) =$$

$$(\neg A \cap B) = \quad // \text{ comm.}$$

$$\overline{(A \cup \neg B)} \quad // \text{ DeMorgan}$$

$$\text{Therefore } \overline{(A \cup \neg B)} \neq \emptyset$$

Since the complement of a given set X is the universal set (U) if and only if $X = \emptyset$, it follows that the complement of a given set Y is **not** U if and only if $Y \neq \emptyset$.

In other words:

$$(\neg X = U) \Leftrightarrow (X = \emptyset) \equiv$$

$$[(\neg X = U) \wedge (X = \emptyset)] \vee [\overline{(\neg X = U)} \wedge \neg (X = \emptyset)]$$

$$\neg(\neg X = U) \equiv (\neg X \neq U) \vee (X = \emptyset)$$

Solving for $X = (A \cup \neg B)$:

$$\neg(\neg(A \cup \neg B) = U) \equiv (\neg(A \cup \neg B) \neq U) \vee ((A \cup \neg B) = \emptyset)$$

$$\neg(\overline{(A \cup \neg B)} = U) \equiv (\overline{(A \cup \neg B)} \neq U) \vee ((A \cup \neg B) = \emptyset)$$