Exercise 5 - Solutions

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Question 1

a) Denote by G' the flow network after changing the capacity of e, and assume f is a maximal flow in G. f is also a flow in G' (the capacity of e is higher, and other edges did not change). The value of a minimal cut in G' differs in at most 1 from that of G and thus also the value of a maximal flow. So, it is anough to search once for an augmenting path, i.e. for a path from s to t in the residual network. This can be done in time $O(V + E)$ (for example by using BFS)

b) The solution is similar to the one in section (a). Only in this case, f may not be a valid flow in G' (it might break the capacity constraint in the edge e). To fix this we must find a path form s to t through e and reduce the flow by 1 in this path. If there is no such path, $f(e) = 0$ and f is a valid flow in G'.

Question 2

We are looking for a subset $A \subset \{1, \ldots, n\}$ that minimizes the expression

$$\sum_{i \in A} \alpha_i + \sum_{j \in \overline{A}} \beta_j + \sum_{i \in A, j \in \overline{A}} c_{ij}$$

Define the following flow network. The vertices of the network are $s, t, v_1, \ldots, v_n$. The edges are: (s, v_i) with capacity α_i, (v_i, t) with capacity β_i, $i = 1, \ldots, n$ and (v_i, v_j) with capacity c_{ij}, $1 \leq i, j \leq n$.

A cut (S, T) in this network is defined uniquely by the subset of the v_i's that belongs to T. Denote by $A = T \setminus \{t\}$. The capacity of this cut is

$$c(S, T) = \sum_{u \in S, v \in T} c(u, v) = \sum_{i \in A} \alpha_i + \sum_{j \in \overline{A}} \beta_j + \sum_{i \in A, j \in \overline{A}} c_{ij},$$

where $\sum_{i \in A} \alpha_i$ is the capacity of the edges from s to T, $\sum_{j \in \overline{A}} \beta_j$ is the capacity of the edges from S to t. And the third summand is the capacity $c(A, \overline{A})$. Thus we have reduced our problem to the problem of finding a minimal cut in a flow network which we know how to solve (see exercise 6, if this does not hold for you).

Question 3

Theorem 1 *Let $G = (V, E)$ be an undirected graph, and let $u, v \in V$. There are d edge disjoint paths from u to v if and only if we cannot disconnect u from v by taking out of G a set of edges with $d-1$ edges or less.*

outline of the proof: we define a flow network in which the value of the maximal flow equals the maximal number of edge disjoint paths for u to v, and the capacity of a minimal cut equals the number of edges in a minimal cut in G having u in one set and v in its complement. We then use the MIN-CUT MAX-FLOW theorem which can be stated as follows: there is a flow of value d in the flow network if and only if there is no cut with capacity $d-1$ or less.

Proof Define the following flow network $G_{u,v}$. The set of vertices of the network is V. There are edges (v_i, v_j) and (v_j, v_i) with capacity 1, for every edge $(v_i, v_j) \in E$. $s = u$ is the source, and $t = v$ is the target.

Claim 2 *There is a flow with value d in $G_{u,v}$ if and only if there are d disjoint paths from u to v in G.*

Proof $\Leftarrow$ we define d flows along each of the edge disjoint paths, and denote them by f_i, $1 \le i \le n$. A flow along a path $v_1 \to v_2 \to v_3 \ldots \to v_k$, assignes 1 to each of the edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$ in the path , -1 to each of the edges $(v_2, v_1), (v_3, v_2), \ldots, (v_k, v_{k-1})$ in the oposite direction, and 0 to all other edges. It is easy to check that this is a valid flow. Consider $f = \sum_{i=1}^{d} f_i$, as a sum of edge disjoint flows f is also a flow, and $|f| = \sum_{i=1}^{d} |f_i| = d$.

$\Rightarrow$ we use induction: It is true when $d = 0$. For $d > 0$, assume f is a flow with value d. Consider all edges in $G_{u,v}$, with nonzero flow. Since $d > 0$, there is a path from u to v along these edges, call it p. Reduce the flow along p to zero. We are left with a flow of value $d-1$, that does not use the edges of p, and we can take these edges out. By the induction hypotheses there are $d-1$ edge disjoint paths in the graph without the edges of p, adding p we get d edge-disjoint paths. ∎

Claim 3 *There is a cut in $G_{u,v}$ with capacity $d-1$ or less if and only if there is a set with less than d edges in G that taking them out disconnects u form v*

Proof $\Rightarrow$ A cut with capacity k in $G_{u,v}$ implies a cut with k edges in G (for any k). $\Leftarrow$ Assume there are $k < d$ edges in G that when they are taken out, G breaks into 2 or more connected componenets with u in one componenet and v in another. Define the following cut (S, T), S contains the vertices of the connected component u is in, and T contains all other vertices. This is a valid cut since v is not in the connected component u is in. The capacity of this cut is the number of edges between the componnent u is in and other connected components. But the edges we took out are the only edges between connected components, thus the capacity of this cut is at most $k < d$. ■

Using the two claims Theorem (1) is equivalent to the MIN-CUT MAX-FLOW theorem on $G_{u,v}$.

■

Theorem 4 *Let $G = (V, E)$ be an undirected graph, and let $u, v \in V$. There are d vertex disjoint paths from u to v if and only if we cannot disconnect u from v by taking out of G a set of $d-1$ vertices or less.*

The proof of this theorem is similar to the proof of Theorem 1, but we need to define a different flow network. The flow network we build for this case is as follows: The vertices are $s = u$, $t = v$ and w^{in}, w^{out} for all $w \in V \backslash \{u, v\}$. The edges are (w^{in}, w^{out}) with capacity 1 for all $w \in V \backslash \{u, v\}$, (w_i^{out}, w_j^{in}) and (w_j^{out}, w_i^{in}) with capacity ∞ for all $(w_i, w_j) \in E$, (s, w^{in}) and (w^{out}, t) with infinite capacity for all $(u, w) \in E$ and $(w, v) \in E$. In this network there are d edge disjoint paths form s to t if and only if there are d vertex disjoint paths for u to v in G. Also, there is a cut with capacity k in G' if and only if there is a set of vertices of size k that disconnects u from v in G.

Note, we can assume that (u, v) is not an edge in G, since otherwise u and v cannot be disconnected by taking out any set of vertices.