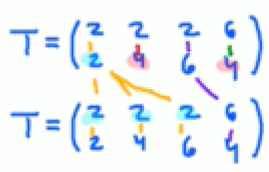
$\boldsymbol{R}$: {⟨x,y⟩ for ⟨x,y⟩ ∈ A^2 if x$\boldsymbol{R}$y}

$\boldsymbol{T}\cdot\boldsymbol{R}$: {⟨a,c⟩ | ∃b ∈ B (⟨a,b⟩ ∈ $\boldsymbol{T}$ ∧ ⟨b,c⟩ ∈ $\boldsymbol{R}$)}

$\boldsymbol{R}^2$: a$\boldsymbol{R}^2$c ⇔ {⟨a,c⟩ | ∃b ∈ A (⟨a,b⟩ ∈ $\boldsymbol{R}$ ∧ ⟨b,c⟩ ∈ $\boldsymbol{R}$)}

> An ordered pair ⟨a,c⟩∈$\boldsymbol{R}^2$ means there's a "middle" b∈B that satisfies ⟨a,b⟩∈$\boldsymbol{R}$ and ⟨b,c⟩∈$\boldsymbol{R}$



Examples

- $(a=-b)^2 = \boldsymbol{I}_{\mathbb{R}}$
- ⟨a,b⟩ ∈ $\boldsymbol{R}^2$ ⇔ ⟨a,c⟩,⟨c,b⟩ ∈ $\boldsymbol{R}$

Empty $\emptyset_A$

> $\boldsymbol{R}$:=rel(A×B) = ∅
> No pair in A×B satisfies ⟨a,b⟩ ∈ $\boldsymbol{R}$

Properties

- $\boldsymbol{S}\cdot\emptyset_A = \emptyset$
- anti-symmetric
- symmetric ?

Examples

- {⟨x,y⟩ ∈ $\mathbb{N}^2$ | x+y < x}

Identity $\boldsymbol{I}_A$

Properties

- $\boldsymbol{R}\cdot\boldsymbol{I}_A = \boldsymbol{R}$

Reflexivity

> $\boldsymbol{R}$:=rel(A) is reflexive if ∀a ∈ A(⟨a,a⟩ ∈ $\boldsymbol{R}$)
> $\boldsymbol{R}$ is reflexive if every a in A satisfies ⟨a,a⟩∈$\boldsymbol{R}$. In other words:
> $\boldsymbol{I}_A \subseteq \boldsymbol{R}$

A = { -1, 0, 1 }. Is ⋱ contained in $\boldsymbol{R}$?

$\boldsymbol{R}$ = lambda a,b: a⨀b; all(R(x,x) for x in A)?

Properties

- ⟺ $\boldsymbol{R}^{-1}$ is reflexive
- → $\boldsymbol{R} \subseteq \boldsymbol{R}^2$ (and $\boldsymbol{R}^2$ is reflexive)
- → $\boldsymbol{R} \subseteq \boldsymbol{R}^2$
- if $\boldsymbol{S} \subseteq \boldsymbol{R}$ then $\boldsymbol{S}$ is reflexive
- if $\boldsymbol{S}$ is reflexive then both $\boldsymbol{R} \cup \boldsymbol{S}$ and $\boldsymbol{R} \cap \boldsymbol{S}$ are reflexive

Examples

- $\boldsymbol{U}_A$: ∀a ∈ A(⟨a,a⟩ ∈ A×A = $\boldsymbol{U}$A)
- $\boldsymbol{I}_A$: ∀a ∈ A(⟨a,a⟩ ∈ {⟨-1, -1⟩, ⟨0, 0⟩, ⟨1, 1⟩})
- ≤, ≥ `both contain ⋱`

Counter Examples

- ≠ (which is $\boldsymbol{U}_A$ - $\boldsymbol{I}$A)
- <, >, $\emptyset_A$
- a=$-b$ ⋰

Antireflexivity

> $\boldsymbol{R}$:=rel(A) is antireflexive iff ¬∃a ∈ A(⟨a,a⟩ ∈ $\boldsymbol{R}$)
> $\boldsymbol{R}$ is antireflexive if every a in A satisfies ⟨a,a⟩ ∉ $\boldsymbol{R}$. In other words:
> $\boldsymbol{I}_A \cap \boldsymbol{R} = \emptyset$ `just` $\boldsymbol{I}_A \not\subseteq \boldsymbol{R}$ `isn't enough;` $\boldsymbol{I}_A$ `= {⟨1,1⟩, ⟨2,2⟩}⊈`$\boldsymbol{R}$ `= {⟨1,1⟩, ⟨1,2⟩} but ⟨1,1⟩∈`$\boldsymbol{R}$ `so isn't antireflexive`

Examples

- < `never:` $n < n$
- ≠, >, $\emptyset_A$

Counter Examples

- $\boldsymbol{U}_A$, $\boldsymbol{I}_A$, a=-b ⋰, ≤, ≥

Symmetry

> $\boldsymbol{R}$:=rel(A) is symmetric iff $\boldsymbol{R} = \boldsymbol{R}^{-1}$
>
> $\boldsymbol{R}$ is symmetric if every ⟨x,y⟩ in $\boldsymbol{R}$ satisfies ⟨y,x⟩ ∈ $\boldsymbol{R}$
>
> ∀x∀y(⟨x,y⟩∈$\boldsymbol{R}$ → ⟨y,x⟩∈$\boldsymbol{R}$)
>
> $\boldsymbol{R}$ = lambda a,b: a⊙b; all(rel(y,x) for x,y in R)?

Properties

- if $\boldsymbol{S}$ is symmetric then both $\boldsymbol{R}$∪$\boldsymbol{S}$ and $\boldsymbol{R}$∩$\boldsymbol{S}$ are reflexive
- if $\boldsymbol{S}$ is symmetric then $\boldsymbol{RS}$ is symmetric

Examples

- $\emptyset_A$ `can't point at ⟨x,y⟩ and say ⟨y,x⟩ is not in ∅⁻¹`
- $\boldsymbol{U}_A$, $\boldsymbol{I}_A$, a=-b ∴, ≠

Counter Examples

- ≤, ≥, <, >

Antisymmetry

> $\boldsymbol{R}$:=rel(A) is antisymmetric iff $\boldsymbol{R} \cap \boldsymbol{R}^{-1} = \emptyset$ `R ∩ R⁻¹ = ∅ means there can't be a ⟨x,x⟩`
>
> $\boldsymbol{R}$ is antisymmetric if every ⟨x,y⟩ in $\boldsymbol{R}$ satisfies ⟨y,x⟩ ∉ $\boldsymbol{R}$
>
> ∀x∀y((x,y) ∈ $\boldsymbol{R}$ → (y,x) ∉ $\boldsymbol{R}$)

Properties

- → $\boldsymbol{R}$ is antireflexive
- → $\boldsymbol{R}^{-1}$ is antisymmetric
- if $\boldsymbol{S}$⊆$\boldsymbol{R}$ then $\boldsymbol{S}$ is antisymmetric
- if $\boldsymbol{S}$∪$\boldsymbol{T}$ is antisymmetric then both $\boldsymbol{S}$ and $\boldsymbol{T}$ are antisymmetric
- → $\boldsymbol{R}$∩$\boldsymbol{S}$ is antisymmetric
- [if $\boldsymbol{R}$ is antireflexive and transitive then it's asymmetric and antisymmetric]()
- No set is a ⊂ of itself, so ⊂ is antisymmetric `lesson 7 00:27:40`

Examples

- <, >, $\emptyset_A$
- $b > a^2$

Counter Examples

- ≠, ≤, ≥, $\boldsymbol{U}_A$, $\boldsymbol{I}_A$, a=-b ∴, ≠

- $b < a^2$ `⟨3,4⟩ and ⟨4,3⟩ are symmetric`

Weak Antisymmetry

> $\boldsymbol{R} \cap \boldsymbol{R}^{-1} \subseteq \boldsymbol{I}_{\mathrm{A}}$
> $\forall x \forall y(\langle x,y\rangle \in \boldsymbol{R} \wedge \langle y,x\rangle \in \boldsymbol{R} \rightarrow x=y)$
> if both ⟨x,y⟩ ∈ $\boldsymbol{R}$ and ⟨y,x⟩ ∈ $\boldsymbol{R}$ it's only because they're equal
> for x,y ∈ A: if x≠y and ⟨x,y⟩ ∈ $\boldsymbol{R}$ then must ⟨y,x⟩ ∉ $\boldsymbol{R}$

A_S vs WA_S: A_S requires every pair's opposite to not be in $\boldsymbol{R}$, whereas WA_S requires the same only for pairs that x=y

Examples

- $\boldsymbol{I}_{\mathrm{A}}$

Transitivity

> $\boldsymbol{R}^2 \subseteq \boldsymbol{R}$
> $\forall x \forall y \forall z((\boldsymbol{R}(x,y) \wedge \boldsymbol{R}(y,z)) \rightarrow \boldsymbol{R}(x,z))$
> Every (x,y,z)∈A that satisfy ⟨x,y⟩∈$\boldsymbol{R}$ and ⟨y,z⟩∈$\boldsymbol{R}$ also satisfy ⟨x,z⟩∈$\boldsymbol{R}$
> If you see an x that leads to y that leads to z, then expect x to lead to z `this is why` $\boldsymbol{R}^2 \subseteq \boldsymbol{R}$

Properties

- if $\boldsymbol{T}$ is symmetric and antisymmetric then it's also transitive

Examples

- A={1,2,3}; $\boldsymbol{R} = \{\langle \mathbf{1},2\rangle, \langle 2,\mathbf{3}\rangle, \langle \mathbf{1},\mathbf{3}\rangle\} \implies \boldsymbol{R}^2 = \{\langle 1,3\rangle\} \subseteq \boldsymbol{R}$
- A={1,2,3}; $\boldsymbol{T} = \{\langle 1,2\rangle\} \implies \boldsymbol{T}^2 = \emptyset \subseteq \boldsymbol{T}$
- $\boldsymbol{W} = \{\langle 1,1\rangle\} \implies \boldsymbol{W}^2 = \{\langle 1,1\rangle\} \subseteq \boldsymbol{W}$
- $\boldsymbol{I}_{\mathrm{A}}$
- $\emptyset_{\mathrm{A}}$
- $\boldsymbol{U}_{\mathrm{A}}$ `if` $\langle a,b\rangle \in A^2$ `and` $\langle b,a\rangle \in A^2$ `then` $\langle a,c\rangle \in A^2$
- if $|A| > 1$ then ≠ is trans
- < over $\mathbb{N}$ $l < m \wedge m < n \implies l < n$
- ≤
- $\boldsymbol{T} = (\langle 2,1\rangle, \langle 2,3\rangle) \implies \boldsymbol{T}^2 = \emptyset \subseteq \boldsymbol{T}$

Counter Examples

- $\boldsymbol{P}=\{\langle 1,2\rangle, \langle 2,1\rangle\} \Rightarrow \boldsymbol{P}^2 = \{\langle 1,1\rangle, \langle 2,2\rangle\} \nsubseteq \boldsymbol{P}$ `iow: 1 leads to 2 leads to 1, but` $\langle 1,1\rangle \notin \boldsymbol{P}$
- $\exists x \exists y \exists z(\boldsymbol{R}(x,y) \land \boldsymbol{R}(y,z) \land \neg\boldsymbol{R}(x,z))$

Equivalance

> $\boldsymbol{R}$ over A is equivalence iff $\boldsymbol{R}$ is reflexive, symmetric and transitive

Examples

- $\boldsymbol{U}_A$, $\boldsymbol{I}_A$, equality
- "Has the same absolute value" on the set of real numbers
- if $A=\emptyset$ then $\emptyset_A$ is symmetric, transitive and reflexive

Counter Examples

- $\geq$ `reflexive and transitive but not symmetric`
- if $A\neq\emptyset$ then $\emptyset_A$ is symmetric and transitive, but not reflexive

Connexivity

`lesson 7 00:06:00`

> $\boldsymbol{R}$ over A is connexive iff $\forall(x,y)\in A\ (x\neq y \rightarrow \langle x,y\rangle\in\boldsymbol{R} \lor \langle y,x\rangle\in\boldsymbol{R})$

Examples

- Any two numbers $\mathbb{N}$

Order (יחסי סדר)

`lesson 7 00:00:00`

Partial Order (יחס סדר חלקי)

> $\boldsymbol{R}$ over A ($\leq$) is a partial order iff it's antireflexive and transitive

Properties

- Antisymmetric `because antireflexive and transitive`
- set A with partial order is a קבוצה סדורה חלקית

Examples

- ⊂ over $\mathcal{P}(\mathbb{N})$ `A∈𝓟(ℕ) is antisym because A⊄A, and trans because A⊂B⊂C ⇒ A⊂C`

```
???
for all a, b, and c:
- a ≤ a    reflex
- if a ≤ b and b ≤ a, then a = b    antisymm
- if a ≤ b and b ≤ c, then a ≤ c    trans
```

Examples

- equality
  ???

Total Order (יחס סדר מלא)

> Partial order and connexive (aka "linearly ordered")
>
> $\forall (x,y)\in A\ (x\neq y \rightarrow \langle x,y\rangle\in \boldsymbol{R}\ \veebar\ \langle y,x\rangle\in \boldsymbol{R})$ `note the xor. verify`

Properties

- set A with total order is a קבוצה סדורה לינארית

Examples

- < over $\mathbb{N}$ `also over ℝ?`
- < over every subgroup of $\mathbb{R}$

Counter Examples

- if $A\neq\emptyset$ then $\boldsymbol{I}_A$ isn't total order because for all $a\in A$: $a=a$

Yahas Mashve יחס משווה, או תכונת ההשוואה

`lesson 8 00:00:50 / p97`

> Each $\langle a,b\rangle\in A$ safisfies exactly one of:
>
> - $a\boldsymbol{R}b$
> - $b\boldsymbol{R}a$
> - $a=b$

Properties

- If we can't find $a\boldsymbol{R}b$ nor $b\boldsymbol{R}a$, then $a=b$

Examples

- < over $\mathbb{N}$

Element: Minimal/Maximal, Least/Greatest

lesson 7 00:36:00

Minimal Element (איבר מינימלי)

> Element a in partially ordered set ⟨A,≺⟩ is a minimal element if there's no other element x∈A that x≺a

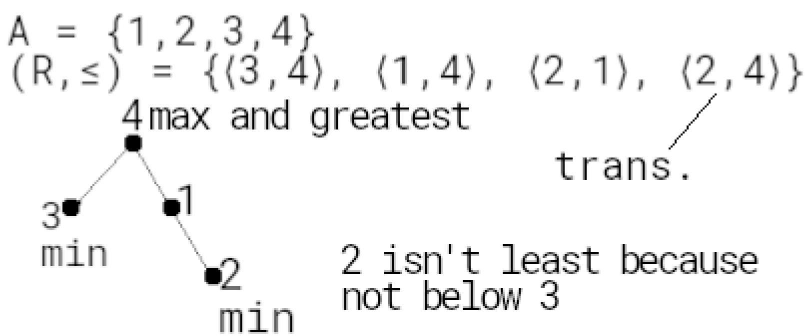
Maximal Element (איבר מקסימלי)

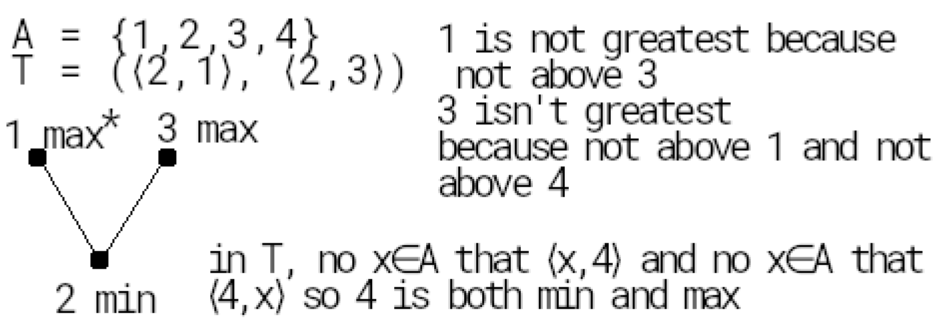
> Element a in partially ordered set ⟨A,≺⟩ is a maximal element if there's no other element x∈A that a≺x

Properties

- a partially ordered, **finite** set must have a min element and a max element (or more) `p. 110`
- a partially ordered, **infinite** set **may** have min / max elements

Examples





Least Element (איבר ראשון)

> Element a in partially ordered set ⟨A,≺⟩ is the least element if for all x∈A: a≺x ∨ a=x
>
> ($\boldsymbol{P}$, ≤) is partially ordered set ⟹ { y∈$\boldsymbol{P}$ | ∀x∈$\boldsymbol{P}$, y≤x} → y is least element `x is all the elements in P`

`lesson 7 01:05:00`

Properties

- The least element is necessarily a minimal element `the only min el? 01:10:50`

Greatest Element (איבר אחרון)

> Element a in partially ordered set ⟨A,≺⟩ is the greatest element if for all x∈A: x≺a ∨ x=a
>
> ($\boldsymbol{P}$, ≤) is partially ordered set ⟹ {y∈$\boldsymbol{P}$ | ∀x∈$\boldsymbol{P}$, x≤y} → y is greatest element

Properties

- The greatest element is necessarily a maximal element `the only max el? 01:10:50`

Partitions

Partition of A is a set of non-empty, non-overlapping subsets of A whose union = A

Properties

- every a∈A is in exactly one block
- no block contains ∅
- union of blocks = A
- \intersection of any two blocks = ∅
- → A is finite ⇒ rank of P is |X| - |P| ?

Examples

- {A} is partition of A `trivial`
- ∅'s only partition is ∅
- {1,2,3} has five partitions: {{1},{2},{3}}, {{1, 2}, {3}}, {{1, 3}, {2}}, {{1},{2, 3}}, {{1, 2, 3}}

Counter Examples

- not partitions of {1,2,3}:
- {{}, {1,3}, {2}} `contains ∅`
- {{1, 2}, {2, 3}} `2 exists in more than one block`
- {{1}, {2}} `no block contains 3`

Equivalence Class: $\{x \in \boldsymbol{S} \mid x \equiv a\}$ where $a \in \boldsymbol{S}$

> Given $\boldsymbol{R}$ is an equivalence relation on $\boldsymbol{S}$, the equivalence class of an element a in $\boldsymbol{S}$ is the set $\{x \in \boldsymbol{S} \mid \langle x,a\rangle \in \boldsymbol{R}\}$
> $[\![a]\!] = \{b \mid a\boldsymbol{R}b\} = \{b \mid \langle a,b\rangle \in \boldsymbol{R}\}$ `all elements in S that when paired with a, exist in R`
> In other words: going over $\boldsymbol{R}$, the elements in $[\![a]\!]$ are all the elements that a is paired with

Properties

- $\cup$ of all equivalence classes = $\boldsymbol{S}$?
- $a \in [\![a]\!]$ `every element exists in its equivalence class`
- the items in each equivalence class of $\boldsymbol{S}$ exist only in their equivalence class ?
- every possible pair of eq. classes is zar ?

Examples

- $\boldsymbol{X}$ = all cars; relation $\equiv_{\boldsymbol{X}}$ = "has the same color as"; one particular equivlance class consists of all green cars
- Relation $\equiv_{\mathbb{Z}}$ is $\langle a,b\rangle \in \equiv_{\mathbb{Z}} \Leftrightarrow (a - b)\%2==0 \Longrightarrow$ two equivalence classes: even numbers and odd numbers
- $\boldsymbol{S} = \{1,2,3,4,5\}$
- $\equiv_{\boldsymbol{S}} = \{\langle 1,\mathbf{1}\rangle, \langle 1,\mathbf{2}\rangle, \langle 1,\mathbf{3}\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 5,5\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,1\rangle\}$
- $[\![1]\!]$ = **{1, 2, 3}** `everything that 1 is related to`
- $[\![2]\!]$ = {2, 1, 3}
- $[\![3]\!]$ = {3, 2, 1} `note that [[1]] ≡ [[2]] ≡ [[3]]`
- $[\![4]\!]$ = {4}
- $[\![5]\!]$ = {5}