CHAPTER

2 Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Much of discrete mathematics is devoted to the study of discrete structures, used to represent discrete objects. Many important discrete structures are built using sets, which are collections of objects. Among the discrete structures built from sets are combinations, unordered collections of objects used extensively in counting; relations, sets of ordered pairs that represent relationships between objects; graphs, sets of vertices and edges that connect vertices; and finite state machines, used to model computing machines. These are some of the topics we will study in later chapters.

The concept of a function is extremely important in discrete mathematics. A function assigns to each element of a first set exactly one element of a second set, where the two sets are not necessarily distinct. Functions play important roles throughout discrete mathematics. They are used to represent the computational complexity of algorithms, to study the size of sets, to count objects, and in a myriad of other ways. Useful structures such as sequences and strings are special types of functions. In this chapter, we will introduce the notion of sequences, which represent ordered lists of elements. Furthermore, we will introduce some important types of sequences and we will show how to define the terms of a sequence using earlier terms. We will also address the problem of identifying a sequence from its first few terms.

In our study of discrete mathematics, we will often add consecutive terms of a sequence of numbers. Because adding terms from a sequence, as well as other indexed sets of numbers, is such a common occurrence, a special notation has been developed for adding such terms. In this chapter, we will introduce the notation used to express summations. We will develop formulae for certain types of summations that appear throughout the study of discrete mathematics. For instance, we will encounter such summations in the analysis of the number of steps used by an algorithm to sort a list of numbers so that its terms are in increasing order.

The relative sizes of infinite sets can be studied by introducing the notion of the size, or cardinality, of a set. We say that a set is countable when it is finite or has the same size as the set of positive integers. In this chapter we will establish the surprising result that the set of rational numbers is countable, while the set of real numbers is not. We will also show how the concepts we discuss can be used to show that there are functions that cannot be computed using a computer program in any programming language.

Matrices are used in discrete mathematics to represent a variety of discrete structures. We will review the basic material about matrices and matrix arithmetic needed to represent relations and graphs. The matrix arithmetic we study will be used to solve a variety of problems involving these structures.

2.1 Sets

2.1.1 Introduction

In this section, we study the fundamental discrete structure on which all other discrete structures are built, namely, the set. Sets are used to group objects together. Often, but not always, the objects in a set have similar properties. For instance, all the students who are currently enrolled in your school make up a set. Likewise, all the students currently taking a course in discrete mathematics at any school make up a set. In addition, those students enrolled in your school who are taking a course in discrete mathematics form a set that can be obtained by taking the elements common to the first two collections. The language of sets is a means to study such collections in an organized fashion. We now provide a definition of a set. This definition is an intuitive definition, which is not part of a formal theory of sets.

Definition 1 A *set* is an unordered collection of distinct objects, called *elements* or *members* of the set. A set is said to *contain* its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

It is common for sets to be denoted using uppercase letters. Lowercase letters are usually used to denote elements of sets.

There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces. For example, the notation $\{a, b, c, d\}$ represents the set with the four elements a, b, c, and d. This way of describing a set is known as the **roster method**.

EXAMPLE 1 The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$. ◀

EXAMPLE 2 The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$. ◀

EXAMPLE 3 Although sets are usually used to group together elements with common properties, there is nothing that prevents a set from having seemingly unrelated elements. For instance, $\{a$, 2, Fred, New Jersey$\}$ is the set containing the four elements a, 2, Fred, and New Jersey. ◀

Sometimes the roster method is used to describe a set without listing all its members. Some members of the set are listed, and then *ellipses* (…) are used when the general pattern of the elements is obvious.

EXAMPLE 4 The set of positive integers less than 100 can be denoted by $\{1, 2, 3, \ldots, 99\}$. ◀

Extra Examples ❯

Another way to describe a set is to use **set builder** notation. We characterize all those elements in the set by stating the property or properties they must have to be members. The general form of this notation is $\{x \mid x$ has property $P\}$ and is read "the set of all x such that x has property P." For instance, the set O of all odd positive integers less than 10 can be written as

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\},$$

or, specifying the universe as the set of positive integers, as

$$O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}.$$

We often use this type of notation to describe sets when it is impossible to list all the elements of the set. For instance, the set $\mathbf{Q}^+$ of all positive rational numbers can be written as

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = \tfrac{p}{q}, \text{ for some positive integers } p \text{ and } q\}.$$

Beware that mathematicians disagree whether 0 is a natural number. We consider it quite natural.

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of all **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of all **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of all **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of all **rational numbers**
$\mathbf{R}$, the set of all **real numbers**

$\mathbf{R}^+$, the set of all **positive real numbers**
C, the set of all **complex numbers**.

(Note that some people do not consider 0 a natural number, so be careful to check how the term *natural numbers* is used when you read other books.)

Among the sets studied in calculus and other subjects are **intervals**, sets of all the real numbers between two numbers a and b, with or without a and b. If a and b are real numbers with $a \leq b$, we denote these intervals by

$$[a, b] = \{x \mid a \leq x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$(a, b) = \{x \mid a < x < b\}.$$

Note that $[a, b]$ is called the **closed interval** from a to b and (a, b) is called the **open interval** from a to b. Each of the intervals $[a, b]$, $[a, b)$, $(a, b]$, and (a, b) contains all the real numbers strictly between a and b. The first two of these contain a and the first and third contain b.

Remark: Some books use the notations $[a, b[$, $]a, b]$, and $]a, b[$ for $[a, b)$, $(a, b]$, and (a, b), respectively.

Sets can have other sets as members, as Example 5 illustrates.

EXAMPLE 5 The set $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$ is a set containing four elements, each of which is a set. The four elements of this set are **N**, the set of natural numbers; **Z**, the set of integers; **Q**, the set of rational numbers; and **R**, the set of real numbers. ◀

Remark: Note that the concept of a datatype, or type, in computer science is built upon the concept of a set. In particular, a **datatype** or **type** is the name of a set, together with a set of operations that can be performed on objects from that set. For example, *boolean* is the name of the set $\{0, 1\}$, together with operators on one or more elements of this set, such as AND, OR, and NOT.

Because many mathematical statements assert that two differently specified collections of objects are really the same set, we need to understand what it means for two sets to be equal.

Definition 2 Two sets are *equal* if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

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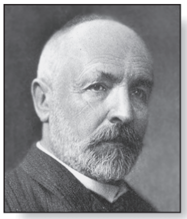

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GEORG CANTOR (1845–1918) Georg Cantor was born in St. Petersburg, Russia, where his father was a successful merchant. Cantor developed his interest in mathematics in his teens. He began his university studies in Zurich in 1862, but when his father died he left Zurich. He continued his university studies at the University of Berlin in 1863, where he studied under the eminent mathematicians Weierstrass, Kummer, and Kronecker. He received his doctor's degree in 1867, after having written a dissertation on number theory. Cantor assumed a position at the University of Halle in 1869, where he continued working until his death.

Cantor is considered the founder of set theory. His contributions in this area include the discovery that the set of real numbers is uncountable. He is also noted for his many important contributions to analysis. Cantor also was interested in philosophy and wrote papers relating his theory of sets with metaphysics.

Cantor married in 1874 and had six children. His melancholy temperament was balanced by his wife's happy disposition. Although he received a large inheritance from his father, he was poorly paid as a professor. To mitigate this, he tried to obtain a better-paying position at the University of Berlin. His appointment there was blocked by Kronecker, who did not agree with Cantor's views on set theory. Cantor suffered from mental illness throughout the later years of his life. He died in 1918 from a heart attack.

EXAMPLE 6 The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements. ◀

THE EMPTY SET There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is denoted by $\emptyset$. The empty set can also be denoted by $\{\ \}$ (that is, we represent the empty set with a pair of braces that encloses all the elements in this set). Often, a set of elements with certain properties turns out to be the null set. For instance, the set of all positive integers that are greater than their squares is the null set.

$\{\emptyset\}$ has one more element than $\emptyset$.

A set with one element is called a **singleton set**. A common error is to confuse the empty set $\emptyset$ with the set $\{\emptyset\}$, which is a singleton set. The single element of the set $\{\emptyset\}$ is the empty set itself! A useful analogy for remembering this difference is to think of folders in a computer file system. The empty set can be thought of as an empty folder and the set consisting of just the empty set can be thought of as a folder with exactly one folder inside, namely, the empty folder.

Links

NAIVE SET THEORY Note that the term *object* has been used in the definition of a set, Definition 1, without specifying what an object is. This description of a set as a collection of objects, based on the intuitive notion of an object, was first stated in 1895 by the German mathematician Georg Cantor. The theory that results from this intuitive definition of a set, and the use of the intuitive notion that for any property whatever, there is a set consisting of exactly the objects with this property, leads to **paradoxes**, or logical inconsistencies. This was shown by the English philosopher Bertrand Russell in 1902 (see Exercise 50 for a description of one of these paradoxes). These logical inconsistencies can be avoided by building set theory beginning with axioms. However, we will use Cantor's original version of set theory, known as **naive set theory**, in this book because all sets considered in this book can be treated consistently using Cantor's original theory. Students will find familiarity with naive set theory helpful if they go on to learn about axiomatic set theory. They will also find the development of axiomatic set theory much more abstract than the material in this text. We refer the interested reader to [Su72] to learn more about axiomatic set theory.

2.1.2 Venn Diagrams

Sets can be represented graphically using Venn diagrams, named after the English mathematician John Venn, who introduced their use in 1881. In Venn diagrams the **universal set** U, which contains all the objects under consideration, is represented by a rectangle. (Note that the universal set varies depending on which objects are of interest.) Inside this rectangle, circles or other geometrical figures are used to represent sets. Sometimes points are used to represent the particular elements of the set. Venn diagrams are often used to indicate the relationships between sets. We show how a Venn diagram can be used in Example 7.

Assessment

EXAMPLE 7 Draw a Venn diagram that represents V, the set of vowels in the English alphabet.

Solution: We draw a rectangle to indicate the universal set U, which is the set of the 26 letters of the English alphabet. Inside this rectangle we draw a circle to represent V. Inside this circle we indicate the elements of V with points (see Figure 1). ◀

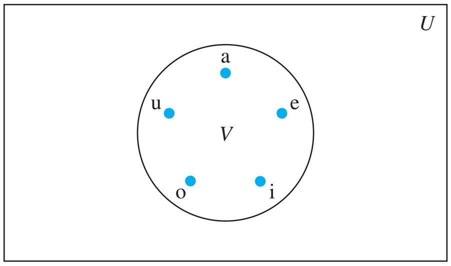

FIGURE 1 **Venn diagram for the set of vowels.**

2.1.3 Subsets

It is common to encounter situations where the elements of one set are also the elements of a second set. We now introduce some terminology and notation to express such relationships between sets.

Definition 3

The set A is a *subset* of B, and B is a *superset* of A, if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B. If, instead, we want to stress that B is a superset of A, we use the equivalent notation $B \supseteq A$. (So, $A \subseteq B$ and $B \supseteq A$ are equivalent statements.)

We see that $A \subseteq B$ if and only if the quantification

$$\forall x(x \in A \rightarrow x \in B)$$

is true. Note that to show that A is not a subset of B we need only find one element $x \in A$ with $x \notin B$. Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.

We have these useful rules for determining whether one set is a subset of another:

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $A \nsubseteq B$, find a single $x \in A$ such that $x \notin B$.

EXAMPLE 8 The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10, the set of rational numbers is a subset of the set of real numbers, the set of all computer

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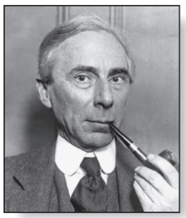

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BERTRAND RUSSELL (1872–1970) Bertrand Russell was born into a prominent English family active in the progressive movement and having a strong commitment to liberty. He became an orphan at an early age and was placed in the care of his father's parents, who had him educated at home. He entered Trinity College, Cambridge, in 1890, where he excelled in mathematics and in moral science. He won a fellowship on the basis of his work on the foundations of geometry. In 1910 Trinity College appointed him to a lectureship in logic and the philosophy of mathematics.

Russell fought for progressive causes throughout his life. He held strong pacifist views, and his protests against World War I led to dismissal from his position at Trinity College. He was imprisoned for 6 months in 1918 because of an article he wrote that was branded as seditious. Russell fought for women's suffrage in Great Britain. In 1961, at the age of 89, he was imprisoned for the second time for his protests advocating nuclear disarmament.

Russell's greatest work was in his development of principles that could be used as a foundation for all of mathematics. His most famous work is *Principia Mathematica,* written with Alfred North Whitehead, which attempts to deduce all of mathematics using a set of primitive axioms. He wrote many books on philosophy, physics, and his political ideas. Russell won the Nobel Prize for Literature in 1950.

science majors at your school is a subset of the set of all students at your school, and the set of all people in China is a subset of the set of all people in China (that is, it is a subset of itself). Each of these facts follows immediately by noting that an element that belongs to the first set in each pair of sets also belongs to the second set in that pair. ◀

EXAMPLE 9 The set of integers with squares less than 100 is not a subset of the set of nonnegative integers because -1 is in the former set [as $(-1)^2 < 100$], but not the latter set. The set of people who have taken discrete mathematics at your school is not a subset of the set of all computer science majors at your school if there is at least one student who has taken discrete mathematics who is not a computer science major. ◀

Theorem 1 shows that every nonempty set S is guaranteed to have at least two subsets, the empty set and the set S itself, that is, $\emptyset \subseteq S$ and $S \subseteq S$.

THEOREM 1 For every set S, $(i)\ \emptyset \subseteq S$ and $(ii)\ S \subseteq S$.

Proof: We will prove (i) and leave the proof of (ii) as an exercise.

Let S be a set. To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. Because the empty set contains no elements, it follows that $x \in \emptyset$ is always false. It follows that the conditional statement $x \in \emptyset \rightarrow x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true. Therefore, $\forall x(x \in \emptyset \rightarrow x \in S)$ is true. This completes the proof of (i). Note that this is an example of a vacuous proof. ◁

When we wish to emphasize that a set A is a subset of a set B but that $A \neq B$, we write $A \subset B$ and say that A is a **proper subset** of B. For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A. That is, A is a proper subset of B if and only if

$$\forall x(x \in A \rightarrow x \in B) \land \exists x(x \in B \land x \notin A)$$

is true. Venn diagrams can be used to illustrate that a set A is a subset of a set B. We draw the universal set U as a rectangle. Within this rectangle we draw a circle for B. Because A is a subset of B, we draw the circle for A within the circle for B. This relationship is shown in Figure 2.

Recall from Definition 2 that sets are equal if they have the same elements. A useful way to show that two sets have the same elements is to show that each set is a subset of the other. In other words, we can show that if A and B are sets with $A \subseteq B$ and $B \subseteq A$, then $A = B$. That is, $A = B$ if and only if $\forall x(x \in A \rightarrow x \in B)$ and $\forall x(x \in B \rightarrow x \in A)$ or equivalently if and only if $\forall x(x \in A \leftrightarrow x \in B)$, which is what it means for the A and B to be equal. Because this method of showing two sets are equal is so useful, we highlight it here.

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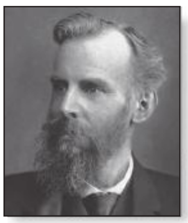

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JOHN VENN (1834–1923) John Venn was born into a London suburban family noted for its philanthropy. He attended London schools and got his mathematics degree from Caius College, Cambridge, in 1857. He was elected a fellow of this college and held his fellowship there until his death. He took holy orders in 1859 and, after a brief stint of religious work, returned to Cambridge, where he developed programs in the moral sciences. Besides his mathematical work, Venn had an interest in history and wrote extensively about his college and family.

Venn's book *Symbolic Logic* clarifies ideas originally presented by Boole. In this book, Venn presents a systematic development of a method that uses geometric figures, known now as *Venn diagrams*. Today these diagrams are primarily used to analyze logical arguments and to illustrate relationships between sets. In addition to his work on symbolic logic, Venn made contributions to probability theory described in his widely used textbook on that subject.

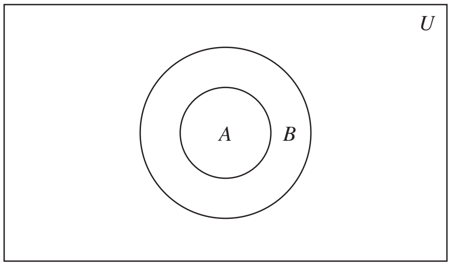

FIGURE 2 **Venn diagram showing that A is a subset of B.**

Showing Two Sets are Equal To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Sets may have other sets as members. For instance, we have the sets

$$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \qquad \text{and} \qquad B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}.$$

Note that these two sets are equal, that is, $A = B$. Also note that $\{a\} \in A$, but $a \notin A$.

2.1.4 The Size of a Set

Sets are used extensively in counting problems, and for such applications we need to discuss the sizes of sets.

Definition 4 Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

Remark: The term *cardinality* comes from the common usage of the term *cardinal number* as the size of a finite set.

EXAMPLE 10 Let A be the set of odd positive integers less than 10. Then $|A| = 5$. ◀

EXAMPLE 11 Let S be the set of letters in the English alphabet. Then $|S| = 26$. ◀

EXAMPLE 12 Because the null set has no elements, it follows that $|\emptyset| = 0$. ◀

We will also be interested in sets that are not finite.

Definition 5 A set is said to be *infinite* if it is not finite.

EXAMPLE 13 The set of positive integers is infinite. ◀

Extra Examples

We will extend the notion of cardinality to infinite sets in Section 2.5, a challenging topic full of surprising results.

2.1.5 Power Sets

Many problems involve testing all combinations of elements of a set to see if they satisfy some property. To consider all such combinations of elements of a set S, we build a new set that has as its members all the subsets of S.

Definition 6 Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

EXAMPLE 14 What is the power set of the set $\{0, 1, 2\}$?

Extra Examples

Solution: The power set $\mathcal{P}(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Note that the empty set and the set itself are members of this set of subsets. ◀

EXAMPLE 15 What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Solution: The empty set has exactly one subset, namely, itself. Consequently,

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

The set $\{\emptyset\}$ has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore,

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

◀

If a set has n elements, then its power set has 2^n elements. We will demonstrate this fact in several ways in subsequent sections of the text.

2.1.6 Cartesian Products

The order of elements in a collection is often important. Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by **ordered *n*-tuples**.

Definition 7 The *ordered n-tuple* $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, …, and a_n as its nth element.

We say that two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal. In other words, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$, for $i = 1, 2, \ldots, n$. In particular, ordered 2-tuples are called **ordered pairs**. The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$. Note that (a, b) and (b, a) are not equal unless $a = b$.

Many of the discrete structures we will study in later chapters are based on the notion of the *Cartesian product* of sets (named after René Descartes). We first define the Cartesian product of two sets.

Definition 8

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

EXAMPLE 16

Extra Examples

Let A represent the set of all students at a university, and let B represent the set of all courses offered at the university. What is the Cartesian product $A \times B$ and how can it be used?

Solution: The Cartesian product $A \times B$ consists of all the ordered pairs of the form (a, b), where a is a student at the university and b is a course offered at the university. One way to use the set $A \times B$ is to represent all possible enrollments of students in courses at the university. Furthermore, observe that each subset of $A \times B$ represents one possible total enrollment configuration, and $\mathcal{P}(A \times B)$ represents all possible enrollment configurations. ◀

EXAMPLE 17

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution: The Cartesian product $A \times B$ is

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$ ◀

Note that the Cartesian products $A \times B$ and $B \times A$ are not equal unless $A = \emptyset$ or $B = \emptyset$ (so that $A \times B = \emptyset$) or $A = B$ (see Exercises 33 and 40). This is illustrated in Example 18.

EXAMPLE 18

Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$, where A and B are as in Example 17.

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RENÉ DESCARTES (1596–1650) René Descartes was born into a noble family near Tours, France, about 130 miles southwest of Paris. He was the third child of his father's first wife; she died several days after his birth. Because of René's poor health, his father, a provincial judge, let his son's formal lessons slide until, at the age of 8, René entered the Jesuit college at La Flèche. The rector of the school took a liking to him and permitted him to stay in bed until late in the morning because of his frail health. From then on, Descartes spent his mornings in bed; he considered these times his most productive hours for thinking.

Descartes left school in 1612, moving to Paris, where he spent 2 years studying mathematics. He earned a law degree in 1616 from the University of Poitiers. At 18 Descartes became disgusted with studying and decided to see the world. He moved to Paris and became a successful gambler. However, he grew tired of bawdy living and moved to the suburb of Saint-Germain, where he devoted himself to mathematical study. When his gambling friends found him, he decided to leave France and undertake a military career. However, he never did any fighting. One day, while escaping the cold in an overheated room at a military encampment, he had several feverish dreams, which revealed his future career as a mathematician and philosopher.

After ending his military career, he traveled throughout Europe. He then spent several years in Paris, where he studied mathematics and philosophy and constructed optical instruments. Descartes decided to move to Holland, where he spent 20 years wandering around the country, accomplishing his most important work. During this time he wrote several books, including the *Discours,* which contains his contributions to analytic geometry, for which he is best known. He also made fundamental contributions to philosophy.

In 1649 Descartes was invited by Queen Christina to visit her court in Sweden to tutor her in philosophy. Although he was reluctant to live in what he called "the land of bears amongst rocks and ice," he finally accepted the invitation and moved to Sweden. Unfortunately, the winter of 1649–1650 was extremely bitter. Descartes caught pneumonia and died in mid-February.

Solution: The Cartesian product $B \times A$ is

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

This is not equal to $A \times B$, which was found in Example 17. ◀

The Cartesian product of more than two sets can also be defined.

Definition 9 The *Cartesian product* of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$. In other words,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}.$$

EXAMPLE 19 What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

Solution: The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Hence,

$$\begin{aligned} A \times B \times C = \{&(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), \\ &(1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}. \end{aligned}$$ ◀

Remark: Note that when A, B, and C are sets, $(A \times B) \times C$ is not the same as $A \times B \times C$ (see Exercise 41).

We use the notation A^2 to denote $A \times A$, the Cartesian product of the set A with itself. Similarly, $A^3 = A \times A \times A$, $A^4 = A \times A \times A \times A$, and so on. More generally,

$$A^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for } i = 1, 2, \ldots, n\}.$$

EXAMPLE 20 Suppose that $A = \{1, 2\}$. It follows that $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ and $A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$. ◀

A subset R of the Cartesian product $A \times B$ is called a **relation** from the set A to the set B. The elements of R are ordered pairs, where the first element belongs to A and the second to B. For example, $R = \{(a, 0), (a, 1), (a, 3), (b, 1), (b, 2), (c, 0), (c, 3)\}$ is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$, and it is also a relation from the set $\{a, b, c, d, e\}$ to the set $\{0, 1, 3, 4)$. (This illustrates that a relation need not contain a pair (x, y) for every element x of A.) A relation from a set A to itself is called a relation on A.

EXAMPLE 21 What are the ordered pairs in the less than or equal to relation, which contains (a, b) if $a \le b$, on the set $\{0, 1, 2, 3\}$?

Solution: The ordered pair (a, b) belongs to R if and only if both a and b belong to $\{0, 1, 2, 3\}$ and $a \le b$. Consequently, $R = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$. ◀

We will study relations and their properties at length in Chapter 9.

2.1.7 Using Set Notation with Quantifiers

Sometimes we restrict the domain of a quantified statement explicitly by making use of a particular notation. For example, $\forall x \in S(P(x))$ denotes the universal quantification of $P(x)$ over all elements in the set S. In other words, $\forall x \in S(P(x))$ is shorthand for $\forall x(x \in S \rightarrow P(x))$. Similarly, $\exists x \in S(P(x))$ denotes the existential quantification of $P(x)$ over all elements in S. That is, $\exists x \in S(P(x))$ is shorthand for $\exists x(x \in S \wedge P(x))$.

EXAMPLE 22 What do the statements $\forall x \in \mathbf{R}\ (x^2 \geq 0)$ and $\exists x \in \mathbf{Z}\ (x^2 = 1)$ mean?

Solution: The statement $\forall x \in \mathbf{R}(x^2 \geq 0)$ states that for every real number x, $x^2 \geq 0$. This statement can be expressed as "The square of every real number is nonnegative." This is a true statement.

The statement $\exists x \in \mathbf{Z}(x^2 = 1)$ states that there exists an integer x such that $x^2 = 1$. This statement can be expressed as "There is an integer whose square is 1." This is also a true statement because $x = 1$ is such an integer (as is -1). ◀

2.1.8 Truth Sets and Quantifiers

We will now tie together concepts from set theory and from predicate logic. Given a predicate P, and a domain D, we define the **truth set** of P to be the set of elements x in D for which $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$.

EXAMPLE 23 What are the truth sets of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers and $P(x)$ is "$|x| = 1$," $Q(x)$ is "$x^2 = 2$," and $R(x)$ is "$|x| = x$."

Solution: The truth set of P, $\{x \in \mathbf{Z} \mid |x| = 1\}$, is the set of integers for which $|x| = 1$. Because $|x| = 1$ when $x = 1$ or $x = -1$, and for no other integers x, we see that the truth set of P is the set $\{-1, 1\}$.

The truth set of Q, $\{x \in \mathbf{Z} \mid x^2 = 2\}$, is the set of integers for which $x^2 = 2$. This is the empty set because there are no integers x for which $x^2 = 2$.

The truth set of R, $\{x \in \mathbf{Z} \mid |x| = x\}$, is the set of integers for which $|x| = x$. Because $|x| = x$ if and only if $x \geq 0$, it follows that the truth set of R is $\mathbf{N}$, the set of nonnegative integers. ◀

Note that $\forall x P(x)$ is true over the domain U if and only if the truth set of P is the set U. Likewise, $\exists x P(x)$ is true over the domain U if and only if the truth set of P is nonempty.

Exercises

1. List the members of these sets.
- **a)** $\{x \mid x \text{ is a real number such that } x^2 = 1\}$
- **b)** $\{x \mid x \text{ is a positive integer less than } 12\}$
- **c)** $\{x \mid x \text{ is the square of an integer and } x < 100\}$
- **d)** $\{x \mid x \text{ is an integer such that } x^2 = 2\}$

2. Use set builder notation to give a description of each of these sets.
- **a)** $\{0, 3, 6, 9, 12\}$
- **b)** $\{-3, -2, -1, 0, 1, 2, 3\}$
- **c)** $\{m, n, o, p\}$

3. Which of the intervals $(0, 5)$, $(0, 5]$, $[0, 5)$, $[0, 5]$, $(1, 4]$, $[2, 3]$, $(2, 3)$ contains
- **a)** 0?
- **b)** 1?
- **c)** 2?
- **d)** 3?
- **e)** 4?
- **f)** 5?

4. For each of these intervals, list all its elements or explain why it is empty.
- **a)** $[a, a]$
- **b)** $[a, a)$
- **c)** $(a, a]$
- **d)** (a, a)
- **e)** (a, b), where $a > b$
- **f)** $[a, b]$, where $a > b$

5. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.

a) the set of airline flights from New York to New Delhi, the set of nonstop airline flights from New York to New Delhi
b) the set of people who speak English, the set of people who speak Chinese
c) the set of flying squirrels, the set of living creatures that can fly

6. For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.

a) the set of people who speak English, the set of people who speak English with an Australian accent
b) the set of fruits, the set of citrus fruits
c) the set of students studying discrete mathematics, the set of students studying data structures

7. Determine whether each of these pairs of sets are equal.

a) $\{1, 3, 3, 3, 5, 5, 5, 5, 5\}$, $\{5, 3, 1\}$
b) $\{\{1\}\}$, $\{1, \{1\}\}$ **c)** $\emptyset$, $\{\emptyset\}$

8. Suppose that $A = \{2, 4, 6\}$, $B = \{2, 6\}$, $C = \{4, 6\}$, and $D = \{4, 6, 8\}$. Determine which of these sets are subsets of which other of these sets.

9. For each of the following sets, determine whether 2 is an element of that set.

a) $\{x \in \mathbf{R} \mid x \text{ is an integer greater than } 1\}$
b) $\{x \in \mathbf{R} \mid x \text{ is the square of an integer}\}$
c) $\{2, \{2\}\}$ **d)** $\{\{2\}, \{\{2\}\}\}$
e) $\{\{2\}, \{2, \{2\}\}\}$ **f)** $\{\{\{2\}\}\}$

10. For each of the sets in Exercise 9, determine whether $\{2\}$ is an element of that set.

11. Determine whether each of these statements is true or false.

a) $0 \in \emptyset$ **b)** $\emptyset \in \{0\}$
c) $\{0\} \subset \emptyset$ **d)** $\emptyset \subset \{0\}$
e) $\{0\} \in \{0\}$ **f)** $\{0\} \subset \{0\}$
g) $\{\emptyset\} \subseteq \{\emptyset\}$

12. Determine whether these statements are true or false.

a) $\emptyset \in \{\emptyset\}$ **b)** $\emptyset \in \{\emptyset, \{\emptyset\}\}$
c) $\{\emptyset\} \in \{\emptyset\}$ **d)** $\{\emptyset\} \in \{\{\emptyset\}\}$
e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$ **f)** $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

13. Determine whether each of these statements is true or false.

a) $x \in \{x\}$ **b)** $\{x\} \subseteq \{x\}$ **c)** $\{x\} \in \{x\}$
d) $\{x\} \in \{\{x\}\}$ **e)** $\emptyset \subseteq \{x\}$ **f)** $\emptyset \in \{x\}$

14. Use a Venn diagram to illustrate the subset of odd integers in the set of all positive integers not exceeding 10.

15. Use a Venn diagram to illustrate the set of all months of the year whose names do not contain the letter R in the set of all months of the year.

16. Use a Venn diagram to illustrate the relationship $A \subseteq B$ and $B \subseteq C$.

17. Use a Venn diagram to illustrate the relationships $A \subset B$ and $B \subset C$.

18. Use a Venn diagram to illustrate the relationships $A \subset B$ and $A \subset C$.

19. Suppose that A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Show that $A \subseteq C$.

20. Find two sets A and B such that $A \in B$ and $A \subseteq B$.

21. What is the cardinality of each of these sets?

a) $\{a\}$ **b)** $\{\{a\}\}$
c) $\{a, \{a\}\}$ **d)** $\{a, \{a\}, \{a, \{a\}\}\}$

22. What is the cardinality of each of these sets?

a) $\emptyset$ **b)** $\{\emptyset\}$
c) $\{\emptyset, \{\emptyset\}\}$ **d)** $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

23. Find the power set of each of these sets, where a and b are distinct elements.

a) $\{a\}$ **b)** $\{a, b\}$ **c)** $\{\emptyset, \{\emptyset\}\}$

24. Can you conclude that $A = B$ if A and B are two sets with the same power set?

25. How many elements does each of these sets have where a and b are distinct elements?

a) $\mathcal{P}(\{a, b, \{a, b\}\})$
b) $\mathcal{P}(\{\emptyset, a, \{a\}, \{\{a\}\}\})$
c) $\mathcal{P}(\mathcal{P}(\emptyset))$

26. Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$ **b)** $\{\emptyset, \{a\}\}$
c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$ **d)** $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

27. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

28. Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$

29. Let $A = \{a, b, c, d\}$ and $B = \{y, z\}$. Find

a) $A \times B$. **b)** $B \times A$.

30. What is the Cartesian product $A \times B$, where A is the set of courses offered by the mathematics department at a university and B is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used.

31. What is the Cartesian product $A \times B \times C$, where A is the set of all airlines and B and C are both the set of all cities in the United States? Give an example of how this Cartesian product can be used.

32. Suppose that $A \times B = \emptyset$, where A and B are sets. What can you conclude?

33. Let A be a set. Show that $\emptyset \times A = A \times \emptyset = \emptyset$.

34. Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

a) $A \times B \times C$. **b)** $C \times B \times A$.
c) $C \times A \times B$. **d)** $B \times B \times B$.

35. Find A^2 if

a) $A = \{0, 1, 3\}$. **b)** $A = \{1, 2, a, b\}$.

36. Find A^3 if

a) $A = \{a\}$. **b)** $A = \{0, a\}$.

37. How many different elements does $A \times B$ have if A has m elements and B has n elements?

38. How many different elements does $A \times B \times C$ have if A has m elements, B has n elements, and C has p elements?

39. How many different elements does A^n have when A has m elements and n is a positive integer?

40. Show that $A \times B \neq B \times A$, when A and B are nonempty, unless $A = B$.

41. Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

42. Explain why $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ are not the same.

43. Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

44. Prove or disprove that if A, B, and C are nonempty sets and $A \times B = A \times C$, then $B = C$.

45. Translate each of these quantifications into English and determine its truth value.

a) $\forall x \in \mathbf{R}\ (x^2 \neq -1)$ **b)** $\exists x \in \mathbf{Z}\ (x^2 = 2)$
c) $\forall x \in \mathbf{Z}\ (x^2 > 0)$ **d)** $\exists x \in \mathbf{R}\ (x^2 = x)$

46. Translate each of these quantifications into English and determine its truth value.

a) $\exists x \in \mathbf{R}\ (x^3 = -1)$ **b)** $\exists x \in \mathbf{Z}\ (x + 1 > x)$
c) $\forall x \in \mathbf{Z}\ (x - 1 \in \mathbf{Z})$ **d)** $\forall x \in \mathbf{Z}\ (x^2 \in \mathbf{Z})$

47. Find the truth set of each of these predicates where the domain is the set of integers.

a) $P(x)$: $x^2 < 3$ **b)** $Q(x)$: $x^2 > x$
c) $R(x)$: $2x + 1 = 0$

48. Find the truth set of each of these predicates where the domain is the set of integers.

a) $P(x)$: $x^3 \geq 1$ **b)** $Q(x)$: $x^2 = 2$
c) $R(x)$: $x < x^2$

***49.** The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair (a, b) to be $\{\{a\}, \{a, b\}\}$, then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. [*Hint:* First show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$.]

***50.** This exercise presents **Russell's paradox**. Let S be the set that contains a set x if the set x does not belong to itself, so that $S = \{x \mid x \notin x\}$.

Links

a) Show the assumption that S is a member of S leads to a contradiction.

b) Show the assumption that S is not a member of S leads to a contradiction.

By parts (a) and (b) it follows that the set S cannot be defined as it was. This paradox can be avoided by restricting the types of elements that sets can have.

***51.** Describe a procedure for listing all the subsets of a finite set.

2.2 Set Operations

2.2.1 Introduction

Links

Two, or more, sets can be combined in many different ways. For instance, starting with the set of mathematics majors at your school and the set of computer science majors at your school, we can form the set of students who are mathematics majors or computer science majors, the set of students who are joint majors in mathematics and computer science, the set of all students not majoring in mathematics, and so on.

Definition 1

Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

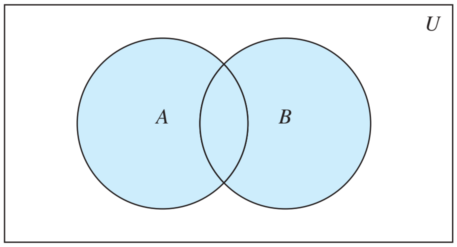
An element x belongs to the union of the sets A and B if and only if x belongs to A or x belongs to B. This tells us that

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

The Venn diagram shown in Figure 1 represents the union of two sets A and B. The area that represents $A \cup B$ is the shaded area within either the circle representing A or the circle representing B.

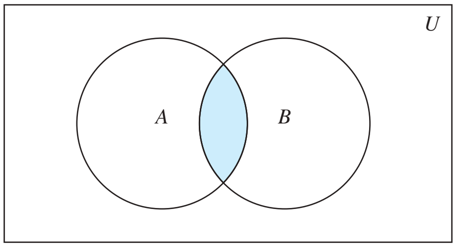
We will give some examples of the union of sets.

EXAMPLE 1 The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$. ◀



$A \cup B$ is shaded.

FIGURE 1 **Venn diagram of the union of A and B.**



$A \cap B$ is shaded.

FIGURE 2 **Venn diagram of the intersection of A and B.**

EXAMPLE 2 The union of the set of all computer science majors at your school and the set of all mathematics majors at your school is the set of students at your school who are majoring either in mathematics or in computer science (or in both). ◀

Definition 2 Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B. This tells us that

$$A \cap B = \{x \mid x \in A \land x \in B\}.$$

The Venn diagram shown in Figure 2 represents the intersection of two sets A and B. The shaded area that is within both the circles representing the sets A and B is the area that represents the intersection of A and B.

We give some examples of the intersection of sets.

EXAMPLE 3 The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$; that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$. ◀

EXAMPLE 4 The intersection of the set of all computer science majors at your school and the set of all mathematics majors is the set of all students who are joint majors in mathematics and computer science. ◀

Definition 3 Two sets are called *disjoint* if their intersection is the empty set.

EXAMPLE 5 Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint. ◀

Be careful not to overcount!

We are often interested in finding the cardinality of a union of two finite sets A and B. Note that $|A| + |B|$ counts each element that is in A but not in B or in B but not in A exactly once, and each element that is in both A and B exactly twice. Thus, if the number of elements that are in both A and B is subtracted from $|A| + |B|$, elements in $A \cap B$ will be counted only once. Hence,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

The generalization of this result to unions of an arbitrary number of sets is called the **principle of inclusion–exclusion**. The principle of inclusion–exclusion is an important technique used in enumeration. We will discuss this principle and other counting techniques in detail in Chapters 6 and 8.

There are other important ways to combine sets.

Definition 4 Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.

Remark: The difference of sets A and B is sometimes denoted by $A \backslash B$.

An element x belongs to the difference of A and B if and only if $x \in A$ and $x \notin B$. This tells us that

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$

The Venn diagram shown in Figure 3 represents the difference of the sets A and B. The shaded area inside the circle that represents A and outside the circle that represents B is the area that represents $A - B$.

We give some examples of differences of sets.

EXAMPLE 6 The difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{5\}$; that is, $\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$. This is different from the difference of $\{1, 2, 3\}$ and $\{1, 3, 5\}$, which is the set $\{2\}$. ◀

EXAMPLE 7 The difference of the set of computer science majors at your school and the set of mathematics majors at your school is the set of all computer science majors at your school who are not also mathematics majors. ◀

Once the universal set U has been specified, the **complement** of a set can be defined.

Definition 5 Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

Remark: The definition of the complement of A depends on a particular universal set U. This definition makes sense for any superset U of A. If we want to identify the universal set U, we would write "the complement of A with respect to the set U."

An element belongs to $\overline{A}$ if and only if $x \notin A$. This tells us that

$$\overline{A} = \{x \in U \mid x \notin A\}.$$

In Figure 4 the shaded area outside the circle representing A is the area representing $\overline{A}$.

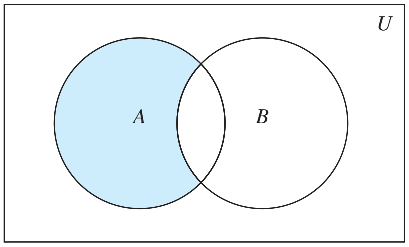
We give some examples of the complement of a set.

EXAMPLE 8 Let $A = \{a, e, i, o, u\}$ (where the universal set is the set of letters of the English alphabet). Then $\overline{A} = \{b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z\}$. ◀

EXAMPLE 9 Let A be the set of positive integers greater than 10 (with universal set the set of all positive integers). Then $\overline{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. ◀

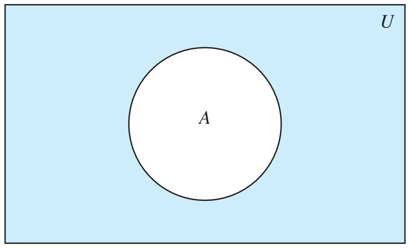
It is left to the reader (Exercise 21) to show that we can express the difference of A and B as the intersection of A and the complement of B. That is,

$$A - B = A \cap \overline{B}.$$



$A - B$ is shaded.

FIGURE 3 Venn diagram for the difference of A and B.



$\overline{A}$ is shaded.

FIGURE 4 Venn diagram for the complement of the set A.

2.2.2 Set Identities

Set identities and propositional equivalences are just special cases of identities for Boolean algebra.

Table 1 lists the most important identities of unions, intersections, and complements of sets. We will prove several of these identities here, using three different methods. These methods are presented to illustrate that there are often many different approaches to the solution of a problem. The proofs of the remaining identities will be left as exercises. The reader should note the similarity between these set identities and the logical equivalences discussed in Section 1.3. (Compare Table 6 of Section 1.6 and Table 1.) In fact, the set identities given can be proved directly from the corresponding logical equivalences. Furthermore, both are special cases of identities that hold for Boolean algebra (discussed in Chapter 12).

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Before we discuss different approaches for proving set identities, we briefly discuss the role of Venn diagrams. Although these diagrams can help us understand sets constructed using two or three **atomic sets** (the sets used to construct more complicated combinations of these sets), they provide far less insight when four or more atomic sets are involved. Venn diagrams for four or more sets are quite complex because it is necessary to use ellipses rather than circles to represent the sets. This is necessary to ensure that every possible combination of the sets is represented by a nonempty region. Although Venn diagrams can provide an informal proof for some identities, such proofs should be formalized using one of the three methods we will now describe.

This identity says that the complement of the intersection of two sets is the union of their complements.

One way to show that two sets are equal is to show that each is a subset of the other. Recall that to show that one set is a subset of a second set, we can show that if an element belongs to the first set, then it must also belong to the second set. We generally use a direct proof to do this. We illustrate this type of proof by establishing the first of De Morgan's laws.

EXAMPLE 10 Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Extra Examples

Solution: We will prove that the two sets $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$ are equal by showing that each set is a subset of the other.

First, we will show that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. We do this by showing that if x is in $\overline{A \cap B}$, then it must also be in $\overline{A} \cup \overline{B}$. Now suppose that $x \in \overline{A \cap B}$. By the definition of complement, $x \notin A \cap B$. Using the definition of intersection, we see that the proposition $\neg((x \in A) \wedge (x \in B))$ is true.

By applying De Morgan's law for propositions, we see that $\neg(x \in A)$ or $\neg(x \in B)$. Using the definition of negation of propositions, we have $x \notin A$ or $x \notin B$. Using the definition of the complement of a set, we see that this implies that $x \in \overline{A}$ or $x \in \overline{B}$. Consequently, by the definition of union, we see that $x \in \overline{A} \cup \overline{B}$. We have now shown that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Next, we will show that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. We do this by showing that if x is in $\overline{A} \cup \overline{B}$, then it must also be in $\overline{A \cap B}$. Now suppose that $x \in \overline{A} \cup \overline{B}$. By the definition of union, we know that $x \in \overline{A}$ or $x \in \overline{B}$. Using the definition of complement, we see that $x \notin A$ or $x \notin B$. Consequently, the proposition $\neg(x \in A) \vee \neg(x \in B)$ is true.

By De Morgan's law for propositions, we conclude that $\neg((x \in A) \wedge (x \in B))$ is true. By the definition of intersection, it follows that $\neg(x \in A \cap B)$. We now use the definition of complement to conclude that $x \in \overline{A \cap B}$. This shows that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

Because we have shown that each set is a subset of the other, the two sets are equal, and the identity is proved. ◀

We can more succinctly express the reasoning used in Example 10 using set builder notation, as Example 11 illustrates.

EXAMPLE 11 Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: We can prove this identity with the following steps.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement}\\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol}\\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection}\\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences}\\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol}\\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement}\\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union}\\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Note that besides the definitions of complement, union, set membership, and set builder notation, this proof uses the second De Morgan law for logical equivalences. ◀

Proving a set identity involving more than two sets by showing each side of the identity is a subset of the other often requires that we keep track of different cases, as illustrated by the proof in Example 12 of one of the distributive laws for sets.

EXAMPLE 12 Prove the second distributive law from Table 1, which states that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C.

Solution: We will prove this identity by showing that each side is a subset of the other side.

Suppose that $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. By the definition of union, it follows that $x \in A$, and $x \in B$ or $x \in C$ (or both). In other words, we know that the compound proposition $(x \in A) \wedge ((x \in B) \vee (x \in C))$ is true. By the distributive law for conjunction over disjunction, it follows that $((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C))$. We conclude that either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. By the definition of intersection, it follows that $x \in A \cap B$ or $x \in A \cap C$. Using the definition of union, we conclude that $x \in (A \cap B) \cup (A \cap C)$. We conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Now suppose that $x \in (A \cap B) \cup (A \cap C)$. Then, by the definition of union, $x \in A \cap B$ or $x \in A \cap C$. By the definition of intersection, it follows that $x \in A$ and $x \in B$ or that $x \in A$ and $x \in C$. From this we see that $x \in A$, and $x \in B$ or $x \in C$. Consequently, by the definition of union we see that $x \in A$ and $x \in B \cup C$. Furthermore, by the definition of intersection, it follows that $x \in A \cap (B \cup C)$. We conclude that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. This completes the proof of the identity. ◀

Set identities can also be proved using **membership tables**. We consider each combination of the atomic sets (that is, the original sets used to produce the sets on each side) that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity. To indicate that an element is in a set, a 1 is used; to indicate that an element is not in a set, a 0 is used. (The reader should note the similarity between membership tables and truth tables.)

EXAMPLE 13 Use a membership table to show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: The membership table for these combinations of sets is shown in Table 2. This table has eight rows. Because the columns for $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are the same, the identity is valid. ◀

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Once we have proved set identities, we can use them to prove new identities. In particular, we can apply a string of identities, one in each step, to take us from one side of a desired identity to the other. It is helpful to explicitly state the identity that is used in each step, as we do in Example 14.

EXAMPLE 14 Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

Solution: We have

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

◀

We summarize the three different ways for proving set identities in Table 3.

TABLE 3 Methods of Proving Set Identities.

Description	***Method***
Subset method	Show that each side of the identity is a subset of the other side.
Membership table	For each possible combination of the atomic sets, show that an element in exactly these atomic sets must either belong to both sides or belong to neither side
Apply existing identities	Start with one side, transform it into the other side using a sequence of steps by applying an established identity.

2.2.3 Generalized Unions and Intersections

Because unions and intersections of sets satisfy associative laws, the sets $A \cup B \cup C$ and $A \cap B \cap C$ are well defined; that is, the meaning of this notation is unambiguous when A, B, and C are sets. That is, we do not have to use parentheses to indicate which operation comes first because $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$. Note that $A \cup B \cup C$ contains those elements that are in at least one of the sets A, B, and C, and that $A \cap B \cap C$ contains those elements that are in all of A, B, and C. These combinations of the three sets, A, B, and C, are shown in Figure 5.

EXAMPLE 15 Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

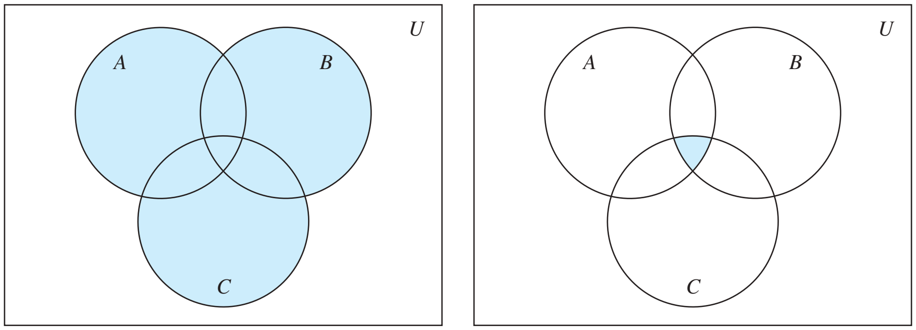
Solution: The set $A \cup B \cup C$ contains those elements in at least one of A, B, and C. Hence,

$$A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}.$$

The set $A \cap B \cap C$ contains those elements in all three of A, B, and C. Thus,

$$A \cap B \cap C = \{0\}.$$

◀



(a) $A \cup B \cup C$ is shaded. (b) $A \cap B \cap C$ is shaded.

FIGURE 5 The union and intersection of A, B, and C.

We can also consider unions and intersections of an arbitrary number of sets. We introduce these definitions.

Definition 6 The *union* of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

We use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

to denote the union of the sets $A_1, A_2, \ldots, A_n$.

Definition 7 The *intersection* of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

We use the notation

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

to denote the intersection of the sets $A_1, A_2, \ldots, A_n$. We illustrate generalized unions and intersections with Example 16.

EXAMPLE 16 For $i = 1, 2, \ldots,$ let $A_i = \{i, i+1, i+2, \ldots\}$. Then,

Extra Examples

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\},$$

and

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n.$$

◀

We can extend the notation we have introduced for unions and intersections to other families of sets. In particular, to denote the union of the infinite family of sets $A_1, A_2, \ldots, A_n, \ldots$, we use the notation

$$A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots = \bigcup_{i=1}^{\infty} A_i.$$

Similarly, the intersection of these sets is denoted by

$$A_1 \cap A_2 \cap \cdots \cap A_n \cap \cdots = \bigcap_{i=1}^{\infty} A_i.$$

More generally, when I is a set, the notations $\bigcap_{i \in I} A_i$ and $\bigcup_{i \in I} A_i$ are used to denote the intersection and union of the sets A_i for $i \in I$, respectively. Note that we have $\bigcap_{i \in I} A_i = \{x \mid \forall i \in I\,(x \in A_i)\}$ and $\bigcup_{i \in I} A_i = \{x \mid \exists i \in I\,(x \in A_i)\}$.

EXAMPLE 17 Suppose that $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Then,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1, 2, 3, \ldots\} = \mathbf{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \{1\}.$$

To see that the union of these sets is the set of positive integers, note that every positive integer n is in at least one of the sets, because it belongs to $A_n = \{1, 2, \ldots, n\}$, and every element of the sets in the union is a positive integer. To see that the intersection of these sets is the set $\{1\}$, note that the only element that belongs to all the sets $A_1, A_2, \ldots$ is 1. To see this note that $A_1 = \{1\}$ and $1 \in A_i$ for $i = 1, 2, \ldots$. ◀

2.2.4 Computer Representation of Sets

There are various ways to represent sets using a computer. One method is to store the elements of the set in an unordered fashion. However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time consuming, because each of these operations would require a large amount of searching for elements. We will present a method for storing elements using an arbitrary ordering of the elements of the universal set. This method of representing sets makes computing combinations of sets easy.

Assume that the universal set U is finite (and of reasonable size so that the number of elements of U is not larger than the memory size of the computer being used). First, specify an arbitrary ordering of the elements of U, for instance $a_1, a_2, \ldots, a_n$. Represent a subset A of U with the bit string of length n, where the ith bit in this string is 1 if a_i belongs to A and is 0 if a_i does not belong to A. Example 18 illustrates this technique.

EXAMPLE 18 Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Solution: The bit string that represents the set of odd integers in U, namely, $\{1, 3, 5, 7, 9\}$, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is

10 1010 1010.

(We have split this bit string of length ten into blocks of length four for easy reading.) Similarly, we represent the subset of all even integers in U, namely, $\{2, 4, 6, 8, 10\}$, by the string

01 0101 0101.

The set of all integers in U that do not exceed 5, namely, $\{1, 2, 3, 4, 5\}$, is represented by the string

11 1110 0000. ◀

Using bit strings to represent sets, it is easy to find complements of sets and unions, intersections, and differences of sets. To find the bit string for the complement of a set from the bit string for that set, we simply change each 1 to a 0 and each 0 to 1, because $x \in A$ if and only if $x \notin \overline{A}$. Note that this operation corresponds to taking the negation of each bit when we associate a bit with a truth value—with 1 representing true and 0 representing false.

EXAMPLE 19 We have seen that the bit string for the set $\{1, 3, 5, 7, 9\}$ (with universal set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$) is

10 1010 1010.

What is the bit string for the complement of this set?

Solution: The bit string for the complement of this set is obtained by replacing 0s with 1s and vice versa. This yields the string

01 0101 0101,

which corresponds to the set $\{2, 4, 6, 8, 10\}$. ◀

To obtain the bit string for the union and intersection of two sets we perform bitwise Boolean operations on the bit strings representing the two sets. The bit in the ith position of the bit string of the union is 1 if either of the bits in the ith position in the two strings is 1 (or both are 1), and is 0 when both bits are 0. Hence, the bit string for the union is the bitwise *OR* of the bit strings for the two sets. The bit in the ith position of the bit string of the intersection is 1 when the bits in the corresponding position in the two strings are both 1, and is 0 when either of the two bits is 0 (or both are). Hence, the bit string for the intersection is the bitwise *AND* of the bit strings for the two sets.

EXAMPLE 20 The bit strings for the sets $\{1, 2, 3, 4, 5\}$ and $\{1, 3, 5, 7, 9\}$ are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

Solution: The bit string for the union of these sets is

$$11\ 1110\ 0000 \vee 10\ 1010\ 1010 = 11\ 1110\ 1010,$$

which corresponds to the set $\{1, 2, 3, 4, 5, 7, 9\}$. The bit string for the intersection of these sets is

$$11\ 1110\ 0000 \wedge 10\ 1010\ 1010 = 10\ 1010\ 0000,$$

which corresponds to the set $\{1, 3, 5\}$. ◀

2.2.5 Multisets

Sometimes the number of times that an element occurs in an unordered collection matters. A **multiset** (short for multiple-membership set) is an unordered collection of elements where an element can occur as a member more than once. We can use the same notation for a multiset as we do for a set, but each element is listed the number of times it occurs. (Recall that in a set, an element either belongs to a set or it does not. Listing it more than once does not affect the membership of this element in the set.) So, the multiset denoted by $\{a, a, a, b, b\}$ is the multiset that contains the element a thrice and the element b twice. When we use this notation, it must be clear that we are working with multisets and not ordinary sets. We can avoid this ambiguity by using an alternate notation for multisets. The notation $\{m_1 \cdot a_1, m_2 \cdot a_2, \ldots, m_r \cdot a_r\}$ denotes the multiset with element a_1 occurring m_1 times, element a_2 occurring m_2 times, and so on. The numbers m_i, $i = 1, 2, \ldots, r$, are called the **multiplicities** of the elements a_i, $i = 1, 2, \ldots, r$. (Elements not in a multiset are assigned 0 as their multiplicity in this set.) The cardinality of a multiset is defined to be the sum of the multiplicities of its elements. The word *multiset* was introduced by Nicolaas Govert de Bruijn in the 1970s, but the concept dates back to the 12th century work of the Indian mathematician Bhaskaracharya.

Let P and Q be multisets. The **union** of the multisets P and Q is the multiset in which the multiplicity of an element is the maximum of its multiplicities in P and Q. The **intersection** of P and Q is the multiset in which the multiplicity of an element is the minimum of its multiplicities in P and Q. The **difference** of P and Q is the multiset in which the multiplicity of an element is the multiplicity of the element in P less its multiplicity in Q unless this difference is negative, in which case the multiplicity is 0. The **sum** of P and Q is the multiset in which the multiplicity of an element is the sum of multiplicities in P and Q. The union, intersection, and difference of P and Q are denoted by $P \cup Q$, $P \cap Q$, and $P - Q$, respectively (where these operations should not be confused with the analogous operations for sets). The sum of P and Q is denoted by $P + Q$.

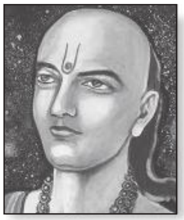
EXAMPLE 21 Suppose that P and Q are the multisets $\{4 \cdot a, 1 \cdot b, 3 \cdot c\}$ and $\{3 \cdot a, 4 \cdot b, 2 \cdot d\}$, respectively. Find $P \cup Q$, $P \cap Q$, $P - Q$, and $P + Q$.

Solution: We have

$$\begin{aligned} P \cup Q &= \{\max(4, 3) \cdot a, \max(1, 4) \cdot b, \max(3, 0) \cdot c, \max(0, 2) \cdot d\} \\ &= \{4 \cdot a, 4 \cdot b, 3 \cdot c, 2 \cdot d\}, \end{aligned}$$

$$\begin{aligned} P \cap Q &= \{\min(4, 3) \cdot a, \min(1, 4) \cdot b, \min(3, 0) \cdot c, \min(0, 2) \cdot d\} \\ &= \{3 \cdot a, 1 \cdot b, 0 \cdot c, 0 \cdot d\} = \{3 \cdot a, 1 \cdot b\}, \end{aligned}$$

Links



BHASKARACHARYA (1114–1185) Bhaskaracharya was born in Bijapur in the Indian state of Karnataka. (Bhaskaracharya's name was actually Bhaskara, but the title Acharya, which means teacher, was added honorifically.) His father was a well-known scholar and a famous astrologer. Bhaskaracharya was head of the astronomical observatory at Ujjain, the leading Indian mathematical center of the day. He is considered to be the greatest mathematician of medieval India. Bhaskaracharya made discoveries in many parts of mathematics, including geometry, plane and spherical trigonometry, algebra, number theory, and combinatorics. Bhaskaracharya described the principles of differential calculus, which he applied to astronomical problems, predating the works of Newton and Leibniz by more than 500 years. In number theory he made many discoveries about Diophantine equations, the study of the solution in integers of equations, which were rediscovered more than 600 years later. His greatest work is the *Crown of Treatises* (*Siddhanta Shiromani*), which includes four main parts, covering arithmetic, algebra, mathematics of the planets, and spheres.

$$P - Q = \{\max(4-3,0)\cdot a, \max(1-4,0)\cdot b, \max(3-0,0)\cdot c, \max(0-2,0)\cdot d\}$$
$$= \{1\cdot a, 0\cdot b, 3\cdot c, 0\cdot d\} = \{1\cdot a, 3\cdot c\}, \text{ and}$$

$$P + Q = \{(4+3)\cdot a, (1+4)\cdot b, (3+0)\cdot c, (0+2)\cdot d\}$$
$$= \{7\cdot a, 5\cdot b, 3\cdot c, 2\cdot d\}.$$

Exercises

1. Let A be the set of students who live within one mile of school and let B be the set of students who walk to classes. Describe the students in each of these sets.
   a) $A \cap B$ b) $A \cup B$
   c) $A - B$ d) $B - A$
2. Suppose that A is the set of sophomores at your school and B is the set of students in discrete mathematics at your school. Express each of these sets in terms of A and B.
   a) the set of sophomores taking discrete mathematics in your school
   b) the set of sophomores at your school who are not taking discrete mathematics
   c) the set of students at your school who either are sophomores or are taking discrete mathematics
   d) the set of students at your school who either are not sophomores or are not taking discrete mathematics
3. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find
   a) $A \cup B$. b) $A \cap B$.
   c) $A - B$. d) $B - A$.
4. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find
   a) $A \cup B$. b) $A \cap B$.
   c) $A - B$. d) $B - A$.

In Exercises 5–10 assume that A is a subset of some underlying universal set U.

5. Prove the complementation law in Table 1 by showing that $\overline{\overline{A}} = A$.
6. Prove the identity laws in Table 1 by showing that
   a) $A \cup \emptyset = A$. b) $A \cap U = A$.
7. Prove the domination laws in Table 1 by showing that
   a) $A \cup U = U$. b) $A \cap \emptyset = \emptyset$.
8. Prove the idempotent laws in Table 1 by showing that
   a) $A \cup A = A$. b) $A \cap A = A$.
9. Prove the complement laws in Table 1 by showing that
   a) $A \cup \overline{A} = U$. b) $A \cap \overline{A} = \emptyset$.
10. Show that
    a) $A - \emptyset = A$. b) $\emptyset - A = \emptyset$.
11. Let A and B be sets. Prove the commutative laws from Table 1 by showing that
    a) $A \cup B = B \cup A$.
    b) $A \cap B = B \cap A$.
12. Prove the first absorption law from Table 1 by showing that if A and B are sets, then $A \cup (A \cap B) = A$.
13. Prove the second absorption law from Table 1 by showing that if A and B are sets, then $A \cap (A \cup B) = A$.
14. Find the sets A and B if $A - B = \{1, 5, 7, 8\}$, $B - A = \{2, 10\}$, and $A \cap B = \{3, 6, 9\}$.
15. Prove the second De Morgan law in Table 1 by showing that if A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$
    a) by showing each side is a subset of the other side.
    b) using a membership table.
16. Let A and B be sets. Show that
    a) $(A \cap B) \subseteq A$. b) $A \subseteq (A \cup B)$.
    c) $A - B \subseteq A$. d) $A \cap (B - A) = \emptyset$.
    e) $A \cup (B - A) = A \cup B$.
17. Show that if A and B are sets in a universe U then $A \subseteq B$ if and only if $\overline{A} \cup B = U$.
18. Given sets A and B in a universe U, draw the Venn diagrams of each of these sets.
    a) $A \to B = \{x \in U \mid x \in A \to x \in B\}$
    b) $A \leftrightarrow B = \{x \in U \mid x \in A \leftrightarrow x \in B\}$
19. Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$
    a) by showing each side is a subset of the other side.
    b) using a membership table.
20. Let A, B, and C be sets. Show that
    a) $(A \cup B) \subseteq (A \cup B \cup C)$.
    b) $(A \cap B \cap C) \subseteq (A \cap B)$.
    c) $(A - B) - C \subseteq A - C$.
    d) $(A - C) \cap (C - B) = \emptyset$.
    e) $(B - A) \cup (C - A) = (B \cup C) - A$.
21. Show that if A and B are sets, then
    a) $A - B = A \cap \overline{B}$.
    b) $(A \cap B) \cup (A \cap \overline{B}) = A$.
22. Show that if A and B are sets with $A \subseteq B$, then
    a) $A \cup B = B$.
    b) $A \cap B = A$.
23. Prove the first associative law from Table 1 by showing that if A, B, and C are sets, then $A \cup (B \cup C) = (A \cup B) \cup C$.
24. Prove the second associative law from Table 1 by showing that if A, B, and C are sets, then $A \cap (B \cap C) = (A \cap B) \cap C$.
25. Prove the first distributive law from Table 1 by showing that if A, B, and C are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

26. Let A, B, and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$.

27. Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{0, 1, 2, 3, 4, 5, 6\}$, and $C = \{4, 5, 6, 7, 8, 9, 10\}$. Find

a) $A \cap B \cap C$.
b) $A \cup B \cup C$.
c) $(A \cup B) \cap C$.
d) $(A \cap B) \cup C$.

28. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B \cup C)$
b) $\overline{A} \cap \overline{B} \cap \overline{C}$
c) $(A - B) \cup (A - C) \cup (B - C)$

29. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

a) $A \cap (B - C)$
b) $(A \cap B) \cup (A \cap C)$
c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$

30. Draw the Venn diagrams for each of these combinations of the sets A, B, C, and D.

a) $(A \cap B) \cup (C \cap D)$
b) $\overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D}$
c) $A - (B \cap C \cap D)$

31. What can you say about the sets A and B if we know that

a) $A \cup B = A$?
b) $A \cap B = A$?
c) $A - B = A$?
d) $A \cap B = B \cap A$?
e) $A - B = B - A$?

32. Can you conclude that $A = B$ if A, B, and C are sets such that

a) $A \cup C = B \cup C$?
b) $A \cap C = B \cap C$?
c) $A \cup C = B \cup C$ and $A \cap C = B \cap C$?

33. Let A and B be subsets of a universal set U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

34. Let A, B, and C be sets. Use the the identity $A - B = A \cap \overline{B}$, which holds for any sets A and B, and the identities from Table 1 to show that $(A - B) \cap (B - C) \cap (A - C) = \emptyset$.

35. Let A, B, and C be sets. Use the identities in Table 1 to show that $\overline{(A \cup B)} \cap \overline{(B \cup C)} \cap \overline{(A \cup C)} = \overline{A} \cap \overline{B} \cap \overline{C}$.

36. Prove or disprove that for all sets A, B, and C, we have

a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

37. Prove or disprove that for all sets A, B, and C, we have

a) $A \times (B - C) = (A \times B) - (A \times C)$.
b) $\overline{A} \times \overline{(B \cup C)} = \overline{A \times (B \cup C)}$.

The **symmetric difference** of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

38. Find the symmetric difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$.

39. Find the symmetric difference of the set of computer science majors at a school and the set of mathematics majors at this school.

40. Draw a Venn diagram for the symmetric difference of the sets A and B.

41. Show that $A \oplus B = (A \cup B) - (A \cap B)$.

42. Show that $A \oplus B = (A - B) \cup (B - A)$.

43. Show that if A is a subset of a universal set U, then

a) $A \oplus A = \emptyset$.
b) $A \oplus \emptyset = A$.
c) $A \oplus U = \overline{A}$.
d) $A \oplus \overline{A} = U$.

44. Show that if A and B are sets, then

a) $A \oplus B = B \oplus A$.
b) $(A \oplus B) \oplus B = A$.

45. What can you say about the sets A and B if $A \oplus B = A$?

***46.** Determine whether the symmetric difference is associative; that is, if A, B, and C are sets, does it follow that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$?

***47.** Suppose that A, B, and C are sets such that $A \oplus C = B \oplus C$. Must it be the case that $A = B$?

48. If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D)$?

49. If A, B, C, and D are sets, does it follow that $(A \oplus B) \oplus (C \oplus D) = (A \oplus D) \oplus (B \oplus C)$?

50. Show that if A and B are finite sets, then $A \cup B$ is a finite set.

51. Show that if A is an infinite set, then whenever B is a set, $A \cup B$ is also an infinite set.

***52.** Show that if A, B, and C are sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

(This is a special case of the inclusion–exclusion principle, which will be studied in Chapter 8.)

53. Let $A_i = \{1, 2, 3, \ldots, i\}$ for $i = 1, 2, 3, \ldots$. Find

a) $\bigcup_{i=1}^{n} A_i$.
b) $\bigcap_{i=1}^{n} A_i$.

54. Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find

a) $\bigcup_{i=1}^{n} A_i$.
b) $\bigcap_{i=1}^{n} A_i$.

55. Let A_i be the set of all nonempty bit strings (that is, bit strings of length at least one) of length not exceeding i. Find

a) $\bigcup_{i=1}^{n} A_i$.
b) $\bigcap_{i=1}^{n} A_i$.

56. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{i, i+1, i+2, \ldots\}$.
b) $A_i = \{0, i\}$.
c) $A_i = (0, i)$, that is, the set of real numbers x with $0 < x < i$.
d) $A_i = (i, \infty)$, that is, the set of real numbers x with $x > i$.

57. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{-i, -i+1, \ldots, -1, 0, 1, \ldots, i-1, i\}$.
b) $A_i = \{-i, i\}$.
c) $A_i = [-i, i]$, that is, the set of real numbers x with $-i \le x \le i$.
d) $A_i = [i, \infty)$, that is, the set of real numbers x with $x \ge i$.

58. Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the ith bit in the string is 1 if i is in the set and 0 otherwise.

a) $\{3, 4, 5\}$
b) $\{1, 3, 6, 10\}$
c) $\{2, 3, 4, 7, 8, 9\}$

59. Using the same universal set as in the last exercise, find the set specified by each of these bit strings.

a) 11 1100 1111
b) 01 0111 1000
c) 10 0000 0001

60. What subsets of a finite universal set do these bit strings represent?

a) the string with all zeros
b) the string with all ones

61. What is the bit string corresponding to the difference of two sets?

62. What is the bit string corresponding to the symmetric difference of two sets?

63. Show how bitwise operations on bit strings can be used to find these combinations of $A = \{a, b, c, d, e\}$, $B = \{b, c, d, g, p, t, v\}$, $C = \{c, e, i, o, u, x, y, z\}$, and $D = \{d, e, h, i, n, o, t, u, x, y\}$.

a) $A \cup B$
b) $A \cap B$
c) $(A \cup D) \cap (B \cup C)$
d) $A \cup B \cup C \cup D$

64. How can the union and intersection of n sets that all are subsets of the universal set U be found using bit strings?

The **successor** of the set A is the set $A \cup \{A\}$.

65. Find the successors of the following sets.

a) $\{1, 2, 3\}$
b) $\emptyset$
c) $\{\emptyset\}$
d) $\{\emptyset, \{\emptyset\}\}$

66. How many elements does the successor of a set with n elements have?

67. Let A and B be the multisets $\{3 \cdot a, 2 \cdot b, 1 \cdot c\}$ and $\{2 \cdot a, 3 \cdot b, 4 \cdot d\}$, respectively. Find

a) $A \cup B$.
b) $A \cap B$.
c) $A - B$.
d) $B - A$.
e) $A + B$.

68. Assume that $a \in A$, where A is a set. Which of these statements are true and which are false, where all sets shown are ordinary sets, and not multisets. Explain each answer.

a) $\{a, a\} \cup \{a, a, a\} = \{a, a, a, a, a\}$
b) $\{a, a\} \cup \{a, a, a\} = \{a\}$
c) $\{a, a\} \cap \{a, a, a\} = \{a, a\}$
d) $\{a, a\} \cap \{a, a, a\} = \{a\}$
e) $\{a, a, a\} - \{a, a\} = \{a\}$

69. Answer the same questions as posed in Exercise 68 where all sets are multisets, and not ordinary sets.

70. Suppose that A is the multiset that has as its elements the types of computer equipment needed by one department of a university and the multiplicities are the number of pieces of each type needed, and B is the analogous multiset for a second department of the university. For instance, A could be the multiset $\{107 \cdot$ personal computers, $44 \cdot$ routers, $6 \cdot$ servers$\}$ and B could be the multiset $\{14 \cdot$ personal computers, $6 \cdot$ routers, $2 \cdot$ mainframes$\}$.

a) What combination of A and B represents the equipment the university should buy assuming both departments use the same equipment?
b) What combination of A and B represents the equipment that will be used by both departments if both departments use the same equipment?
c) What combination of A and B represents the equipment that the second department uses, but the first department does not, if both departments use the same equipment?
d) What combination of A and B represents the equipment that the university should purchase if the departments do not share equipment?

The **Jaccard similarity** $J(A, B)$ of the finite sets A and B is $J(A, B) = |A \cap B|/|A \cup B|$, with $J(\emptyset, \emptyset) = 1$. The **Jaccard distance** $d_J(A, B)$ between A and B equals $d_J(A, B) = 1 - J(A, B)$.

71. Find $J(A, B)$ and $d_J(A, B)$ for these pairs of sets.

a) $A = \{1, 3, 5\}, B = \{2, 4, 6\}$
b) $A = \{1, 2, 3, 4\}, B = \{3, 4, 5, 6\}$
c) $A = \{1, 2, 3, 4, 5, 6\}, B = \{1, 2, 3, 4, 5, 6\}$
d) $A = \{1\}, B = \{1, 2, 3, 4, 5, 6\}$

72. Prove that each of the properties in parts (a)–(d) holds whenever A and B are finite sets.

a) $J(A, A) = 1$ and $d_J(A, A) = 0$
b) $J(A, B) = J(B, A)$ and $d_J(A, B) = d_J(B, A)$
c) $J(A, B) = 1$ and $d_J(A, B) = 0$ if and only if $A = B$
d) $0 \le J(A, B) \le 1$ and $0 \le d_J(A, B) \le 1$
e) ** Show that if A, B, and C are sets, then $d_J(A, C) \le d_J(A, B) + d_J(B, C)$. (This inequality is known as the **triangle inequality** and together with parts (a), (b), and (c) implies that d_J is a **metric**.)

Fuzzy sets are used in artificial intelligence. Each element in the universal set U has a **degree of membership**, which is a real number between 0 and 1 (including 0 and 1), in a fuzzy set S. The fuzzy set S is denoted by listing the elements Links with their degrees of membership (elements with 0 degree of membership are not listed). For instance, we write {0.6 Alice, 0.9 Brian, 0.4 Fred, 0.1 Oscar, 0.5 Rita} for the set F (of famous people) to indicate that Alice has a 0.6 degree of membership in F, Brian has a 0.9 degree of membership in F, Fred has a 0.4 degree of membership in F, Oscar has a 0.1 degree of membership in F, and Rita has a 0.5 degree of membership in F (so that Brian is the most famous and Oscar is the least famous of these people). Also suppose that R is the set of rich people with $R = \{0.4$ Alice, 0.8 Brian, 0.2 Fred, 0.9 Oscar, 0.7 Rita$\}$.

73. The **complement** of a fuzzy set S is the set $\overline{S}$, with the degree of the membership of an element in $\overline{S}$ equal to 1 minus the degree of membership of this element in S. Find $\overline{F}$ (the fuzzy set of people who are not famous) and $\overline{R}$ (the fuzzy set of people who are not rich).

74. The **union** of two fuzzy sets S and T is the fuzzy set $S \cup T$, where the degree of membership of an element in $S \cup T$ is the maximum of the degrees of membership of this element in S and in T. Find the fuzzy set $F \cup R$ of rich or famous people.

75. The **intersection** of two fuzzy sets S and T is the fuzzy set $S \cap T$, where the degree of membership of an element in $S \cap T$ is the minimum of the degrees of membership of this element in S and in T. Find the fuzzy set $F \cap R$ of rich and famous people.

2.3 Functions

2.3.1 Introduction

In many instances we assign to each element of a set a particular element of a second set (which may be the same as the first). For example, suppose that each student in a discrete mathematics class is assigned a letter grade from the set $\{A, B, C, D, F\}$. And suppose that the grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens. This assignment of grades is illustrated in Figure 1.

This assignment is an example of a function. The concept of a function is extremely important in mathematics and computer science. For example, in discrete mathematics functions are used in the definition of such discrete structures as sequences and strings. Functions are also used to represent how long it takes a computer to solve problems of a given size. Many computer programs and subroutines are designed to calculate values of functions. Recursive functions, which are functions defined in terms of themselves, are used throughout computer science; they will be studied in Chapter 5. This section reviews the basic concepts involving functions needed in discrete mathematics.

Definition 1

Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

Remark: Functions are sometimes also called **mappings** or **transformations**.

Assessment

Functions are specified in many different ways. Sometimes we explicitly state the assignments, as in Figure 1. Often we give a formula, such as $f(x) = x + 1$, to define a function. Other times we use a computer program to specify a function.

A function $f : A \to B$ can also be defined in terms of a relation from A to B. Recall from Section 2.1 that a relation from A to B is just a subset of $A \times B$. A relation from A to B that contains one, and only one, ordered pair (a, b) for every element $a \in A$, defines a function f from A to B. This function is defined by the assignment $f(a) = b$, where (a, b) is the unique ordered pair in the relation that has a as its first element.

Definition 2

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range*, or *image*, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f *maps* A to B.

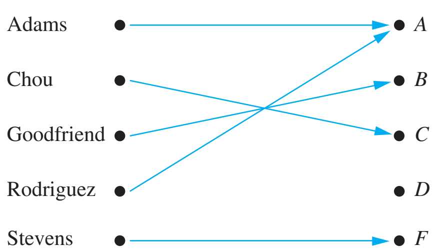


FIGURE 1 Assignment of grades in a discrete mathematics class.

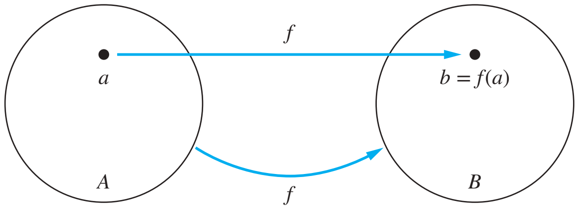


FIGURE 2 The function f maps A to B.

Figure 2 represents a function f from A to B.

Remark: Note that the codomain of a function from A to B is the set of all possible values of such a function (that is, all elements of B), and the range is the set of all values of $f(a)$ for $a \in A$, and is always a subset of the codomain. That is, the codomain is the set of possible values of the function and the range is the set of all elements of the codomain that are achieved as the value of f for at least one element of the domain.

When we define a function we specify its domain, its codomain, and the mapping of elements of the domain to elements in the codomain. Two functions are **equal** when they have the same domain, have the same codomain, and map each element of their common domain to the same element in their common codomain. Note that if we change either the domain or the codomain of a function, then we obtain a different function. If we change the mapping of elements, then we also obtain a different function.

Examples 1–5 provide examples of functions. In each case, we describe the domain, the codomain, the range, and the assignment of values to elements of the domain.

EXAMPLE 1 What are the domain, codomain, and range of the function that assigns grades to students described in the first paragraph of the introduction of this section?

Solution: Let G be the function that assigns a grade to a student in our discrete mathematics class. Note that $G(\text{Adams}) = A$, for instance. The domain of G is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}, and the codomain is the set $\{A, B, C, D, F\}$. The range of G is the set $\{A, B, C, F\}$, because each grade except D is assigned to some student. ◀

EXAMPLE 2 Let R be the relation with ordered pairs (Abdul, 22), (Brenda, 24), (Carla, 21), (Desire, 22), (Eddie, 24), and (Felicia, 22). Here each pair consists of a graduate student and this student's age. Specify a function determined by this relation.

Solution: If f is a function specified by R, then $f(\text{Abdul}) = 22$, $f(\text{Brenda}) = 24$, $f(\text{Carla}) = 21$, $f(\text{Desire}) = 22$, $f(\text{Eddie}) = 24$, and $f(\text{Felicia}) = 22$. [Here, $f(x)$ is the age of x, where x is a student.] For the domain, we take the set {Abdul, Brenda, Carla, Desire, Eddie, Felicia}. We also need to specify a codomain, which needs to contain all possible ages of students. Because it is highly likely that all students are less than 100 years old, we can take the set of positive integers less than 100 as the codomain. (Note that we could choose a different codomain, such as the set of all positive integers or the set of positive integers between 10 and 90, but that would change the function. Using this codomain will also allow us to extend the function by adding the names and ages of more students later.) The range of the function we have specified is the set of different ages of these students, which is the set $\{21, 22, 24\}$. ◀

EXAMPLE 3 *Extra Examples* Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string. For example, $f(11010) = 10$. Then, the domain of f is the set of all bit strings of length 2 or greater, and both the codomain and range are the set $\{00, 01, 10, 11\}$. ◀

EXAMPLE 4 Let $f: \mathbf{Z} \to \mathbf{Z}$ assign the square of an integer to this integer. Then, $f(x) = x^2$, where the domain of f is the set of all integers, the codomain of f is the set of all integers, and the range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, \dots\}$. ◀

EXAMPLE 5 The domain and codomain of functions are often specified in programming languages. For instance, the Java statement

int **floor**(float real){…}

and the C++ function statement

int **function** (float x){…}

both tell us that the domain of the floor function is the set of real numbers (represented by floating point numbers) and its codomain is the set of integers. ◀

A function is called **real-valued** if its codomain is the set of real numbers, and it is called **integer-valued** if its codomain is the set of integers. Two real-valued functions or two integer-valued functions with the same domain can be added, as well as multiplied.

Definition 3 Let f_1 and f_2 be functions from A to $\mathbf{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbf{R}$ defined for all $x \in A$ by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x),$$
$$(f_1 f_2)(x) = f_1(x) f_2(x).$$

Note that the functions $f_1 + f_2$ and $f_1 f_2$ have been defined by specifying their values at x in terms of the values of f_1 and f_2 at x.

EXAMPLE 6 Let f_1 and f_2 be functions from $\mathbf{R}$ to $\mathbf{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

Solution: From the definition of the sum and product of functions, it follows that

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

and

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4.$$ ◀

When f is a function from A to B, the image of a subset of A can also be defined.

Definition 4 Let f be a function from A to B and let S be a subset of A. The *image* of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S \, (t = f(s))\}.$$

We also use the shorthand $\{f(s) \mid s \in S\}$ to denote this set.

Remark: The notation $f(S)$ for the image of the set S under the function f is potentially ambiguous. Here, $f(S)$ denotes a set, and not the value of the function f for the set S.

EXAMPLE 7 Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$. ◀

2.3.2 One-to-One and Onto Functions

Some functions never assign the same value to two different domain elements. These functions are said to be **one-to-one**.

Definition 5 A function f is said to be *one-to-one,* or an *injection,* if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one.

Note that a function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$. This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition.

Remark: We can express that f is one-to-one using quantifiers as $\forall a \forall b (f(a) = f(b) \rightarrow a = b)$ or equivalently $\forall a \forall b (a \neq b \rightarrow f(a) \neq f(b))$, where the universe of discourse is the domain of the function.

Assessment ❯

We illustrate this concept by giving examples of functions that are one-to-one and other functions that are not one-to-one.

EXAMPLE 8 Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Extra Examples ❯

Solution: The function f is one-to-one because f takes on different values at the four elements of its domain. This is illustrated in Figure 3. ◀

EXAMPLE 9 Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution: The function $f(x) = x^2$ is not one-to-one because, for instance, $f(1) = f(-1) = 1$, but $1 \neq -1$. ◀

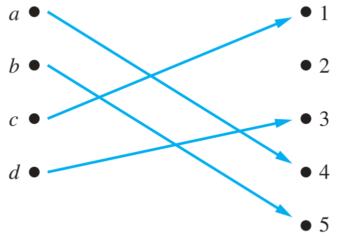


FIGURE 3 **A one-to-one function.**

Remark: The function $f(x) = x^2$ with domain $\mathbf{Z}^+$ is one-to-one. (See the explanation in Example 12 to see why.) This is a different function from the function in Example 9 because of the difference in their domains.

EXAMPLE 10 Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.

Solution: Suppose that x and y are real numbers with $f(x) = f(y)$, so that $x + 1 = y + 1$. This means that $x = y$. Hence, $f(x) = x + 1$ is a one-to-one function from $\mathbf{R}$ to $\mathbf{R}$. ◀

EXAMPLE 11 Suppose that each worker in a group of employees is assigned a job from a set of possible jobs, each to be done by a single worker. In this situation, the function f that assigns a job to each worker is one-to-one. To see this, note that if x and y are two different workers, then $f(x) \neq f(y)$ because the two workers x and y must be assigned different jobs. ◀

We now give some conditions that guarantee that a function is one-to-one.

Definition 6 A function f whose domain and codomain are subsets of the set of real numbers is called *increasing* if $f(x) \leq f(y)$, and *strictly increasing* if $f(x) < f(y)$, whenever $x < y$ and x and y are in the domain of f. Similarly, f is called *decreasing* if $f(x) \geq f(y)$, and *strictly decreasing* if $f(x) > f(y)$, whenever $x < y$ and x and y are in the domain of f. (The word *strictly* in this definition indicates a strict inequality.)

Remark: A function f is increasing if $\forall x \forall y(x < y \rightarrow f(x) \leq f(y))$, strictly increasing if $\forall x \forall y(x < y \rightarrow f(x) < f(y))$, decreasing if $\forall x \forall y(x < y \rightarrow f(x) \geq f(y))$, and strictly decreasing if $\forall x \forall y(x < y \rightarrow f(x) > f(y))$, where the universe of discourse is the domain of f.

EXAMPLE 12 The function $f(x) = x^2$ from $\mathbf{R}^+$ to $\mathbf{R}^+$ is strictly increasing. To see this, suppose that x and y are positive real numbers with $x < y$. Multiplying both sides of this inequality by x gives $x^2 < xy$. Similarly, multiplying both sides by y gives $xy < y^2$. Hence, $f(x) = x^2 < xy < y^2 = f(y)$. However, the function $f(x) = x^2$ from $\mathbf{R}$ to the set of nonnegative real numbers is not strictly increasing because $-1 < 0$, but $f(-1) = (-1)^2 = 1$ is not less than $f(0) = 0^2 = 0$. ◀

From these definitions, it can be shown (see Exercises 26 and 27) that a function that is either strictly increasing or strictly decreasing must be one-to-one. However, a function that is increasing, but not strictly increasing, or decreasing, but not strictly decreasing, is not one-to-one.

For some functions the range and the codomain are equal. That is, every member of the codomain is the image of some element of the domain. Functions with this property are called **onto** functions.

Definition 7 A function f from A to B is called *onto,* or a *surjection,* if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

Remark: A function f is onto if $\forall y \exists x(f(x) = y)$, where the domain for x is the domain of the function and the domain for y is the codomain of the function.

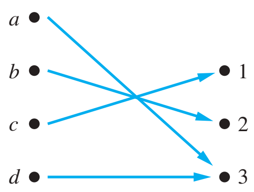


FIGURE 4 **An onto function.**

We now give examples of onto functions and functions that are not onto.

EXAMPLE 13 Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Extra Examples

Solution: Because all three elements of the codomain are images of elements in the domain, we see that f is onto. This is illustrated in Figure 4. Note that if the codomain were $\{1, 2, 3, 4\}$, then f would not be onto. ◀

EXAMPLE 14 Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution: The function f is not onto because there is no integer x with $x^2 = -1$, for instance. ◀

EXAMPLE 15 Is the function $f(x) = x + 1$ from the set of integers to the set of integers onto?

Solution: This function is onto, because for every integer y there is an integer x such that $f(x) = y$. To see this, note that $f(x) = y$ if and only if $x + 1 = y$, which holds if and only if $x = y - 1$. (Note that $y - 1$ is also an integer, and so, is in the domain of f.) ◀

EXAMPLE 16 Consider the function f in Example 11 that assigns jobs to workers. The function f is onto if for every job there is a worker assigned this job. The function f is not onto when there is at least one job that has no worker assigned it. ◀

Definition 8 The function f is a *one-to-one correspondence,* or a *bijection,* if it is both one-to-one and onto. We also say that such a function is *bijective*.

Examples 16 and 17 illustrate the concept of a bijection.

EXAMPLE 17 Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a bijection?

Solution: The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection. ◀

Figure 5 displays four functions where the first is one-to-one but not onto, the second is onto but not one-to-one, the third is both one-to-one and onto, and the fourth is neither one-to-one nor onto. The fifth correspondence in Figure 5 is not a function, because it sends an element to two different elements.

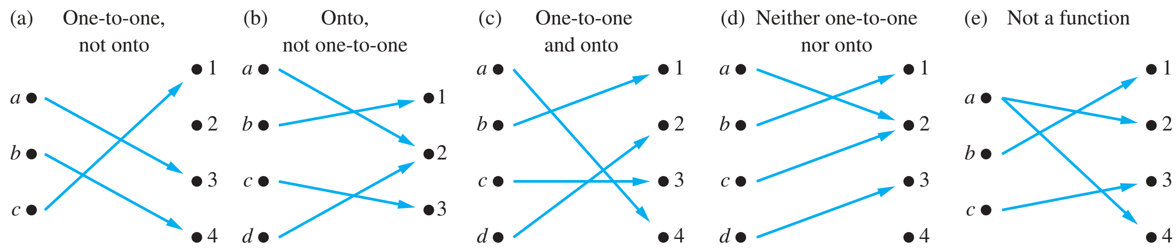


FIGURE 5 Examples of different types of correspondences.

Suppose that f is a function from a set A to itself. If A is finite, then f is one-to-one if and only if it is onto. (This follows from the result in Exercise 74.) This is not necessarily the case if A is infinite (as will be shown in Section 2.5).

EXAMPLE 18 Let A be a set. The *identity function* on A is the function $\iota_A : A \to A$, where

$$\iota_A(x) = x$$

for all $x \in A$. In other words, the identity function ι_A is the function that assigns each element to itself. The function ι_A is one-to-one and onto, so it is a bijection. (Note that ι is the Greek letter iota.) ◀

For future reference, we summarize what needs be to shown to establish whether a function is one-to-one and whether it is onto. It is instructive to review Examples 8–17 in light of this summary.

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

2.3.3 Inverse Functions and Compositions of Functions

Now consider a one-to-one correspondence f from the set A to the set B. Because f is an onto function, every element of B is the image of some element in A. Furthermore, because f is also a one-to-one function, every element of B is the image of a *unique* element of A. Consequently, we can define a new function from B to A that reverses the correspondence given by f. This leads to Definition 9.

Definition 9 Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

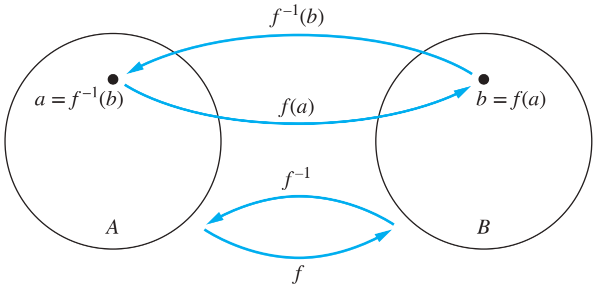


FIGURE 6 The function f^{-1} is the inverse of function f.

Remark: Be sure not to confuse the function f^{-1} with the function $1/f$, which is the function that assigns to each x in the domain the value $1/f(x)$. Notice that the latter makes sense only when $f(x)$ is a nonzero real number.

Figure 6 illustrates the concept of an inverse function.

If a function f is not a one-to-one correspondence, we cannot define an inverse function of f. When f is not a one-to-one correspondence, either it is not one-to-one or it is not onto. If f is not one-to-one, some element b in the codomain is the image of more than one element in the domain. If f is not onto, for some element b in the codomain, no element a in the domain exists for which $f(a) = b$. Consequently, if f is not a one-to-one correspondence, we cannot assign to each element b in the codomain a unique element a in the domain such that $f(a) = b$ (because for some b there is either more than one such a or no such a).

A one-to-one correspondence is called **invertible** because we can define an inverse of this function. A function is **not invertible** if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

EXAMPLE 19 Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$. ◀

EXAMPLE 20 Let $f : \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f has an inverse because it is a one-to-one correspondence, as follows from Examples 10 and 15. To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbf{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$. ◀

EXAMPLE 21 Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ with $f(x) = x^2$. Is f invertible?

Solution: Because $f(-2) = f(2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible. (Note we can also show that f is not invertible because it is not onto.) ◀

Sometimes we can restrict the domain or the codomain of a function, or both, to obtain an invertible function, as Example 22 illustrates.

EXAMPLE 22 Show that if we restrict the function $f(x) = x^2$ in Example 21 to a function from the set of all nonnegative real numbers to the set of all nonnegative real numbers, then f is invertible.

Solution: The function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one. To see this, note that if $f(x) = f(y)$, then $x^2 = y^2$, so $x^2 - y^2 = (x + y)(x - y) = 0$. This means that $x + y = 0$ or $x - y = 0$, so $x = -y$ or $x = y$. Because both x and y are nonnegative, we must have $x = y$. So, this function is one-to-one. Furthermore, $f(x) = x^2$ is onto when the codomain is the set of all nonnegative real numbers, because each nonnegative real number has a square root. That is, if y is a nonnegative real number, there exists a nonnegative real number x such that $x = \sqrt{y}$, which means that $x^2 = y$. Because the function $f(x) = x^2$ from the set of nonnegative real numbers to the set of nonnegative real numbers is one-to-one and onto, it is invertible. Its inverse is given by the rule $f^{-1}(y) = \sqrt{y}$. ◀

Definition 10 Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is the function from A to C defined by

$$(f \circ g)(a) = f(g(a)).$$

In other words, $f \circ g$ is the function that assigns to the element a of A the element assigned by f to $g(a)$. The domain of $f \circ g$ is the domain of g. The range of $f \circ g$ is the image of the range of g with respect to the function f. That is, to find $(f \circ g)(a)$ we first apply the function g to a to obtain $g(a)$ and then we apply the function f to the result $g(a)$ to obtain $(f \circ g)(a) = f(g(a))$. Note that the composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f. In Figure 7 the composition of functions is shown.

EXAMPLE 23 Let g be the function from the set $\{a, b, c\}$ to itself such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let f be the function from the set $\{a, b, c\}$ to the set $\{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f and g, and what is the composition of g and f?

Solution: The composition $f \circ g$ is defined by $(f \circ g)(a) = f(g(a)) = f(b) = 2$, $(f \circ g)(b) = f(g(b)) = f(c) = 1$, and $(f \circ g)(c) = f(g(c)) = f(a) = 3$.

Note that $g \circ f$ is not defined, because the range of f is not a subset of the domain of g. ◀

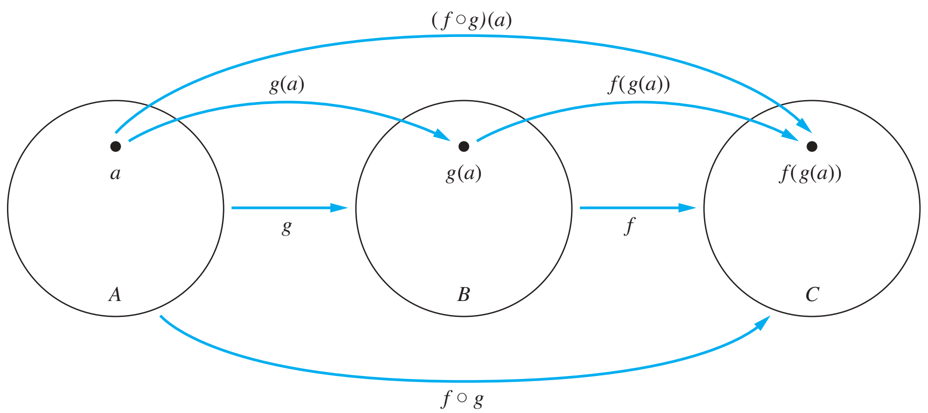


FIGURE 7 **The composition of the functions f and g.**

EXAMPLE 24 Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$ ◀

Remark: Note that even though $f \circ g$ and $g \circ f$ are defined for the functions f and g in Example 24, $f \circ g$ and $g \circ f$ are not equal. In other words, the commutative law does not hold for the composition of functions.

EXAMPLE 25 Let f and g be the functions defined by $f : \mathbf{R} \to \mathbf{R}^+ \cup \{0\}$ with $f(x) = x^2$ and $g : \mathbf{R}^+ \cup \{0\} \to \mathbf{R}$ with $g(x) = \sqrt{x}$ (where $\sqrt{x}$ is the nonnegative square root of x). What is the function $(f \circ g)(x)$?

Solution: The domain of $(f \circ g)(x) = f(g(x))$ is the domain of g, which is $\mathbf{R}^+ \cup \{0\}$, the set of nonnegative real numbers. If x is a nonnegative real number, we have $(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \left(\sqrt{x}\right)^2 = x$. The range of $f \circ g$ is the image of the range of g with respect to the function f. This is the set $\mathbf{R}^+ \cup \{0\}$, the set of nonnegative real numbers. Summarizing, $f : \mathbf{R}^+ \cup \{0\} \to \mathbf{R}^+ \cup \{0\}$ and $f(g(x)) = x$ for all x. ◀

When the composition of a function and its inverse is formed, in either order, an identity function is obtained. To see this, suppose that f is a one-to-one correspondence from the set A to the set B. Then the inverse function f^{-1} exists and is a one-to-one correspondence from B to A. The inverse function reverses the correspondence of the original function, so $f^{-1}(b) = a$ when $f(a) = b$, and $f(a) = b$ when $f^{-1}(b) = a$. Hence,

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(b) = a,$$

and

$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b.$$

Consequently $f^{-1} \circ f = \iota_A$ and $f \circ f^{-1} = \iota_B$, where ι_A and ι_B are the identity functions on the sets A and B, respectively. That is, $(f^{-1})^{-1} = f$.

2.3.4 The Graphs of Functions

We can associate a set of pairs in $A \times B$ to each function from A to B. This set of pairs is called the **graph** of the function and is often displayed pictorially to aid in understanding the behavior of the function.

Definition 11 Let f be a function from the set A to the set B. The *graph* of the function f is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.

From the definition, the graph of a function f from A to B is the subset of $A \times B$ containing the ordered pairs with the second entry equal to the element of B assigned by f to the first entry. Also, note that the graph of a function f from A to B is the same as the relation from A to B determined by the function f, as described on Section 2.3.1.

EXAMPLE 26 Display the graph of the function $f(n) = 2n + 1$ from the set of integers to the set of integers.

Solution: The graph of f is the set of ordered pairs of the form $(n, 2n + 1)$, where n is an integer. This graph is displayed in Figure 8. ◀

EXAMPLE 27 Display the graph of the function $f(x) = x^2$ from the set of integers to the set of integers.

Solution: The graph of f is the set of ordered pairs of the form $(x, f(x)) = (x, x^2)$, where x is an integer. This graph is displayed in Figure 9. ◀

2.3.5 Some Important Functions

Next, we introduce two important functions in discrete mathematics, namely, the floor and ceiling functions. Let x be a real number. The floor function rounds x down to the closest integer less than or equal to x, and the ceiling function rounds x up to the closest integer greater than or equal to x. These functions are often used when objects are counted. They play an important role in the analysis of the number of steps used by procedures to solve problems of a particular size.

Definition 12 The *floor function* assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The *ceiling function* assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

Remark: The floor function is often also called the *greatest integer function*. It is often denoted by $[x]$.

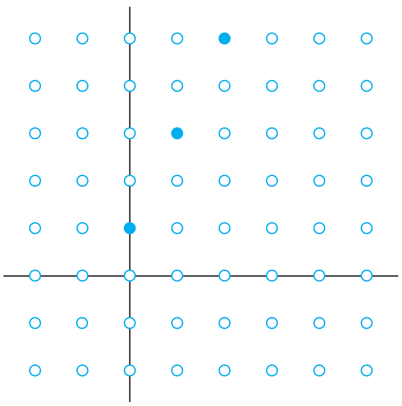


FIGURE 8 The graph of $f(n) = 2n + 1$ from Z to Z.

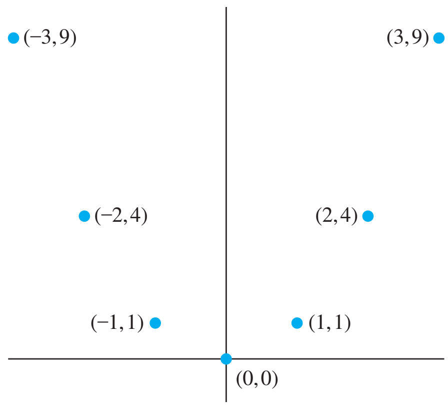


FIGURE 9 The graph of $f(x) = x^2$ from Z to Z.

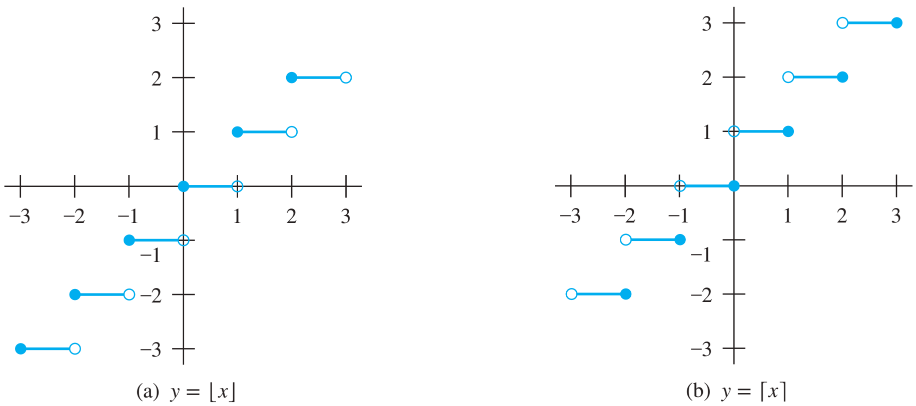


FIGURE 10 Graphs of the (a) floor and (b) ceiling functions.

EXAMPLE 28 These are some values of the floor and ceiling functions:

$$\lfloor \tfrac{1}{2} \rfloor = 0,\ \lceil \tfrac{1}{2} \rceil = 1,\ \lfloor -\tfrac{1}{2} \rfloor = -1,\ \lceil -\tfrac{1}{2} \rceil = 0,\ \lfloor 3.1 \rfloor = 3,\ \lceil 3.1 \rceil = 4,\ \lfloor 7 \rfloor = 7,\ \lceil 7 \rceil = 7.$$ ◀

Links

We display the graphs of the floor and ceiling functions in Figure 10. In Figure 10(a) we display the graph of the floor function $\lfloor x \rfloor$. Note that this function has the same value throughout the interval $[n, n+1)$, namely n, and then it jumps up to $n+1$ when $x = n+1$. In Figure 10(b) we display the graph of the ceiling function $\lceil x \rceil$. Note that this function has the same value throughout the interval $(n, n+1]$, namely $n+1$, and then jumps to $n+2$ when x is a little larger than $n+1$.

The floor and ceiling functions are useful in a wide variety of applications, including those involving data storage and data transmission. Consider Examples 29 and 30, typical of basic calculations done when database and data communications problems are studied.

EXAMPLE 29 Data stored on a computer disk or transmitted over a data network are usually represented as a string of bytes. Each byte is made up of 8 bits. How many bytes are required to encode 100 bits of data?

Solution: To determine the number of bytes needed, we determine the smallest integer that is at least as large as the quotient when 100 is divided by 8, the number of bits in a byte. Consequently, $\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13$ bytes are required. ◀

EXAMPLE 30 In asynchronous transfer mode (ATM) (a communications protocol used on backbone networks), data are organized into cells of 53 bytes. How many ATM cells can be transmitted in 1 minute over a connection that transmits data at the rate of 500 kilobits per second?

Solution: In 1 minute, this connection can transmit $500{,}000 \cdot 60 = 30{,}000{,}000$ bits. Each ATM cell is 53 bytes long, which means that it is $53 \cdot 8 = 424$ bits long. To determine the number of cells that can be transmitted in 1 minute, we determine the largest integer not exceeding the quotient when 30,000,000 is divided by 424. Consequently, $\lfloor 30{,}000{,}000/424 \rfloor = 70{,}754$ ATM cells can be transmitted in 1 minute over a 500 kilobit per second connection. ◀

Table 1, with x denoting a real number, displays some simple but important properties of the floor and ceiling functions. Because these functions appear so frequently in discrete

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x+1$
(2)	$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

mathematics, it is useful to look over these identities. Each property in this table can be established using the definitions of the floor and ceiling functions. Properties (1a), (1b), (1c), and (1d) follow directly from these definitions. For example, (1a) states that $\lfloor x \rfloor = n$ if and only if the integer n is less than or equal to x and $n+1$ is larger than x. This is precisely what it means for n to be the greatest integer not exceeding x, which is the definition of $\lfloor x \rfloor = n$. Properties (1b), (1c), and (1d) can be established similarly. We will prove property (4a) using a direct proof.

Proof: Suppose that $\lfloor x \rfloor = m$, where m is a positive integer. By property (1a), it follows that $m \leq x < m+1$. Adding n to all three quantities in this chain of two inequalities shows that $m+n \leq x+n < m+n+1$. Using property (1a) again, we see that $\lfloor x+n \rfloor = m+n = \lfloor x \rfloor + n$. This completes the proof. Proofs of the other properties are left as exercises. ◁

The floor and ceiling functions enjoy many other useful properties besides those displayed in Table 1. There are also many statements about these functions that may appear to be correct, but actually are not. We will consider statements about the floor and ceiling functions in Examples 31 and 32.

A useful approach for considering statements about the floor function is to let $x = n + \epsilon$, where $n = \lfloor x \rfloor$ is an integer, and ϵ, the fractional part of x, satisfies the inequality $0 \leq \epsilon < 1$. Similarly, when considering statements about the ceiling function, it is useful to write $x = n - \epsilon$, where $n = \lceil x \rceil$ is an integer and $0 \leq \epsilon < 1$.

EXAMPLE 31 Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

Extra Examples

Solution: To prove this statement we let $x = n + \epsilon$, where n is an integer and $0 \leq \epsilon < 1$. There are two cases to consider, depending on whether ϵ is less than, or greater than or equal to $\frac{1}{2}$. (The reason we choose these two cases will be made clear in the proof.)

Links

We first consider the case when $0 \leq \epsilon < \frac{1}{2}$. In this case, $2x = 2n + 2\epsilon$ and $\lfloor 2x \rfloor = 2n$ because $0 \leq 2\epsilon < 1$. Similarly, $x + \frac{1}{2} = n + (\frac{1}{2} + \epsilon)$, so $\lfloor x + \frac{1}{2} \rfloor = n$, because $0 < \frac{1}{2} + \epsilon < 1$. Consequently, $\lfloor 2x \rfloor = 2n$ and $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = n + n = 2n$.

Next, we consider the case when $\frac{1}{2} \leq \epsilon < 1$. In this case, $2x = 2n + 2\epsilon = (2n+1) + (2\epsilon - 1)$. Because $0 \leq 2\epsilon - 1 < 1$, it follows that $\lfloor 2x \rfloor = 2n+1$. Because

$\lfloor x + \frac{1}{2} \rfloor = \lfloor n + (\frac{1}{2} + \epsilon) \rfloor = \lfloor n + 1 + (\epsilon - \frac{1}{2}) \rfloor$ and $0 \le \epsilon - \frac{1}{2} < 1$, it follows that $\lfloor x + \frac{1}{2} \rfloor = n + 1$. Consequently, $\lfloor 2x \rfloor = 2n + 1$ and $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = n + (n + 1) = 2n + 1$. This concludes the proof. ◀

EXAMPLE 32 Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.

Solution: Although this statement may appear reasonable, it is false. A counterexample is supplied by $x = \frac{1}{2}$ and $y = \frac{1}{2}$. With these values we find that $\lceil x + y \rceil = \lceil \frac{1}{2} + \frac{1}{2} \rceil = \lceil 1 \rceil = 1$, but $\lceil x \rceil + \lceil y \rceil = \lceil \frac{1}{2} \rceil + \lceil \frac{1}{2} \rceil = 1 + 1 = 2$. ◀

There are certain types of functions that will be used throughout the text. These include polynomial, logarithmic, and exponential functions. A brief review of the properties of these functions needed in this text is given in Appendix 2. In this book the notation $\log x$ will be used to denote the logarithm to the base 2 of x, because 2 is the base that we will usually use for logarithms. We will denote logarithms to the base b, where b is any real number greater than 1, by $\log_b x$, and the natural logarithm by $\ln x$.

Another function we will use throughout this text is the **factorial function** $f : \mathbf{N} \to \mathbf{Z}^+$, denoted by $f(n) = n!$. The value of $f(n) = n!$ is the product of the first n positive integers, so $f(n) = 1 \cdot 2 \cdots (n-1) \cdot n$ [and $f(0) = 0! = 1$].

EXAMPLE 33 We have $f(1) = 1! = 1$, $f(2) = 2! = 1 \cdot 2 = 2$, $f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$, and $f(20) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 \cdot 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20 =$ 2,432,902,008,176,640,000. ◀

Example 33 illustrates that the factorial function grows extremely rapidly as n grows. The rapid growth of the factorial function is made clearer by Stirling's formula, a result from higher mathematics that tells us that $n! \sim \sqrt{2\pi n}(n/e)^n$. Here, we have used the notation $f(n) \sim g(n)$, which means that the ratio $f(n)/g(n)$ approaches 1 as n grows without bound (that is, $\lim_{n\to\infty} f(n)/g(n) = 1$). The symbol $\sim$ is read "is asymptotic to." Stirling's formula is named after James Stirling, a Scottish mathematician of the eighteenth century.

JAMES STIRLING (1692–1770) James Stirling was born near the town of Stirling, Scotland. His family strongly supported the Jacobite cause of the Stuarts as an alternative to the British crown. The first information known about James is that he entered Balliol College, Oxford, on a scholarship in 1711. However, he later lost his scholarship when he refused to pledge his allegiance to the British crown. The first Jacobean rebellion took place in 1715, and Stirling was accused of communicating with rebels. He was charged with cursing King George, but he was acquitted of these charges. Even though he could not graduate from Oxford because of his politics, he remained there for several years. Stirling published his first work, which extended Newton's work on plane curves, in 1717. He traveled to Venice, where a chair of mathematics had been promised to him, an appointment that unfortunately fell through. Nevertheless, Stirling stayed in Venice, continuing his mathematical work. He attended the University of Padua in 1721, and in 1722 he returned to Glasgow. Stirling apparently fled Italy after learning the secrets of the Italian glass industry, avoiding the efforts of Italian glass makers to assassinate him to protect their secrets.

In late 1724 Stirling moved to London, staying there 10 years teaching mathematics and actively engaging in research. In 1730 he published *Methodus Differentialis*, his most important work, presenting results on infinite series, summations, interpolation, and quadrature. It is in this book that his asymptotic formula for $n!$ appears. Stirling also worked on gravitation and the shape of the earth; he stated, but did not prove, that the earth is an oblate spheroid. Stirling returned to Scotland in 1735, when he was appointed manager of a Scottish mining company. He was very successful in this role and even published a paper on the ventilation of mine shafts. He continued his mathematical research, but at a reduced pace, during his years in the mining industry. Stirling is also noted for surveying the River Clyde with the goal of creating a series of locks to make it navigable. In 1752 the citizens of Glasgow presented him with a silver teakettle as a reward for this work.

2.3.6 Partial Functions

A program designed to evaluate a function may not produce the correct value of the function for all elements in the domain of this function. For example, a program may not produce a correct value because evaluating the function may lead to an infinite loop or an overflow. Similarly, in abstract mathematics, we often want to discuss functions that are defined only for a subset of the real numbers, such as $1/x$, $\sqrt{x}$, and $\arcsin(x)$. Also, we may want to use such notions as the "youngest child" function, which is undefined for a couple having no children, or the "time of sunrise," which is undefined for some days above the Arctic Circle. To study such situations, we use the concept of a partial function.

Definition 13 A *partial function* f from a set A to a set B is an assignment to each element a in a subset of A, called the *domain of definition* of f, of a unique element b in B. The sets A and B are called the *domain* and *codomain* of f, respectively. We say that f is *undefined* for elements in A that are not in the domain of definition of f. When the domain of definition of f equals A, we say that f is a *total function.*

Remark: We write $f : A \to B$ to denote that f is a partial function from A to B. Note that this is the same notation as is used for functions. The context in which the notation is used determines whether f is a partial function or a total function.

EXAMPLE 34 The function $f : \mathbf{Z} \to \mathbf{R}$ where $f(n) = \sqrt{n}$ is a partial function from $\mathbf{Z}$ to $\mathbf{R}$ where the domain of definition is the set of nonnegative integers. Note that f is undefined for negative integers. ◀

Exercises

1. Why is f not a function from $\mathbf{R}$ to $\mathbf{R}$ if

a) $f(x) = 1/x$?
b) $f(x) = \sqrt{x}$?
c) $f(x) = \pm\sqrt{(x^2 + 1)}$?

2. Determine whether f is a function from $\mathbf{Z}$ to $\mathbf{R}$ if

a) $f(n) = \pm n$.
b) $f(n) = \sqrt{n^2 + 1}$.
c) $f(n) = 1/(n^2 - 4)$.

3. Determine whether f is a function from the set of all bit strings to the set of integers if

a) $f(S)$ is the position of a 0 bit in S.
b) $f(S)$ is the number of 1 bits in S.
c) $f(S)$ is the smallest integer i such that the ith bit of S is 1 and $f(S) = 0$ when S is the empty string, the string with no bits.

4. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) the function that assigns to each nonnegative integer its last digit
b) the function that assigns the next largest integer to a positive integer
c) the function that assigns to a bit string the number of one bits in the string
d) the function that assigns to a bit string the number of bits in the string

5. Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) the function that assigns to each bit string the number of ones in the string minus the number of zeros in the string
b) the function that assigns to each bit string twice the number of zeros in that string
c) the function that assigns the number of bits left over when a bit string is split into bytes (which are blocks of 8 bits)
d) the function that assigns to each positive integer the largest perfect square not exceeding this integer

6. Find the domain and range of these functions.

a) the function that assigns to each pair of positive integers the first integer of the pair
b) the function that assigns to each positive integer its largest decimal digit
c) the function that assigns to a bit string the number of ones minus the number of zeros in the string
d) the function that assigns to each positive integer the largest integer not exceeding the square root of the integer
e) the function that assigns to a bit string the longest string of ones in the string

7. Find the domain and range of these functions.

a) the function that assigns to each pair of positive integers the maximum of these two integers

b) the function that assigns to each positive integer the number of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 that do not appear as decimal digits of the integer

c) the function that assigns to a bit string the number of times the block 11 appears

d) the function that assigns to a bit string the numerical position of the first 1 in the string and that assigns the value 0 to a bit string consisting of all 0s

8. Find these values.

a) $\lfloor 1.1 \rfloor$ **b)** $\lceil 1.1 \rceil$

c) $\lfloor -0.1 \rfloor$ **d)** $\lceil -0.1 \rceil$

e) $\lceil 2.99 \rceil$ **f)** $\lceil -2.99 \rceil$

g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$ **h)** $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$

9. Find these values.

a) $\lceil \frac{3}{4} \rceil$ **b)** $\lfloor \frac{7}{8} \rfloor$

c) $\lceil -\frac{3}{4} \rceil$ **d)** $\lfloor -\frac{7}{8} \rfloor$

e) $\lceil 3 \rceil$ **f)** $\lfloor -1 \rfloor$

g) $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor$ **h)** $\lfloor \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \rfloor$

10. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

a) $f(a) = b, f(b) = a, f(c) = c, f(d) = d$

b) $f(a) = b, f(b) = b, f(c) = d, f(d) = c$

c) $f(a) = d, f(b) = b, f(c) = c, f(d) = d$

11. Which functions in Exercise 10 are onto?

12. Determine whether each of these functions from **Z** to **Z** is one-to-one.

a) $f(n) = n - 1$ **b)** $f(n) = n^2 + 1$

c) $f(n) = n^3$ **d)** $f(n) = \lceil n/2 \rceil$

13. Which functions in Exercise 12 are onto?

14. Determine whether $f\colon \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = 2m - n$.

b) $f(m, n) = m^2 - n^2$.

c) $f(m, n) = m + n + 1$.

d) $f(m, n) = |m| - |n|$.

e) $f(m, n) = m^2 - 4$.

15. Determine whether the function $f\colon \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = m + n$.

b) $f(m, n) = m^2 + n^2$.

c) $f(m, n) = m$.

d) $f(m, n) = |n|$.

e) $f(m, n) = m - n$.

16. Consider these functions from the set of students in a discrete mathematics class. Under what conditions is the function one-to-one if it assigns to a student his or her

a) mobile phone number.

b) student identification number.

c) final grade in the class.

d) home town.

17. Consider these functions from the set of teachers in a school. Under what conditions is the function one-to-one if it assigns to a teacher his or her

a) office.

b) assigned bus to chaperone in a group of buses taking students on a field trip.

c) salary.

d) social security number.

18. Specify a codomain for each of the functions in Exercise 16. Under what conditions is each of these functions with the codomain you specified onto?

19. Specify a codomain for each of the functions in Exercise 17. Under what conditions is each of the functions with the codomain you specified onto?

20. Give an example of a function from **N** to **N** that is

a) one-to-one but not onto.

b) onto but not one-to-one.

c) both onto and one-to-one (but different from the identity function).

d) neither one-to-one nor onto.

21. Give an explicit formula for a function from the set of integers to the set of positive integers that is

a) one-to-one, but not onto.

b) onto, but not one-to-one.

c) one-to-one and onto.

d) neither one-to-one nor onto.

22. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = -3x + 4$

b) $f(x) = -3x^2 + 7$

c) $f(x) = (x + 1)/(x + 2)$

d) $f(x) = x^5 + 1$

23. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = 2x + 1$

b) $f(x) = x^2 + 1$

c) $f(x) = x^3$

d) $f(x) = (x^2 + 1)/(x^2 + 2)$

24. Let $f\colon \mathbf{R} \to \mathbf{R}$ and let $f(x) > 0$ for all $x \in \mathbf{R}$. Show that $f(x)$ is strictly increasing if and only if the function $g(x) = 1/f(x)$ is strictly decreasing.

25. Let $f\colon \mathbf{R} \to \mathbf{R}$ and let $f(x) > 0$ for all $x \in \mathbf{R}$. Show that $f(x)$ is strictly decreasing if and only if the function $g(x) = 1/f(x)$ is strictly increasing.

26. **a)** Prove that a strictly increasing function from **R** to itself is one-to-one.

b) Give an example of an increasing function from **R** to itself that is not one-to-one.

27. **a)** Prove that a strictly decreasing function from **R** to itself is one-to-one.

b) Give an example of a decreasing function from **R** to itself that is not one-to-one.

28. Show that the function $f(x) = e^x$ from the set of real numbers to the set of real numbers is not invertible, but if the codomain is restricted to the set of positive real numbers, the resulting function is invertible.

29. Show that the function $f(x) = |x|$ from the set of real numbers to the set of nonnegative real numbers is not invertible, but if the domain is restricted to the set of nonnegative real numbers, the resulting function is invertible.

30. Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

a) $f(x) = 1$. **b)** $f(x) = 2x + 1$.
c) $f(x) = \lceil x/5 \rceil$. **d)** $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.

31. Let $f(x) = \lfloor x^2/3 \rfloor$. Find $f(S)$ if

a) $S = \{-2, -1, 0, 1, 2, 3\}$.
b) $S = \{0, 1, 2, 3, 4, 5\}$.
c) $S = \{1, 5, 7, 11\}$.
d) $S = \{2, 6, 10, 14\}$.

32. Let $f(x) = 2x$ where the domain is the set of real numbers. What is

a) $f(\mathbf{Z})$? **b)** $f(\mathbf{N})$? **c)** $f(\mathbf{R})$?

33. Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.
b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

34. Suppose that g is a function from A to B and f is a function from B to C. Prove each of these statements.

a) If $f \circ g$ is onto, then f must also be onto.
b) If $f \circ g$ is one-to-one, then g must also be one-to-one.
c) If $f \circ g$ is a bijection, then g is onto if and only if f is one-to-one.

35. Find an example of functions f and g such that $f \circ g$ is a bijection, but g is not onto and f is not one-to-one.

***36.** If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.

***37.** If f and $f \circ g$ are onto, does it follow that g is onto? Justify your answer.

38. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from $\mathbf{R}$ to $\mathbf{R}$.

39. Find $f + g$ and fg for the functions f and g given in Exercise 36.

40. Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine necessary and sufficient conditions on the constants a, b, c, and d so that $f \circ g = g \circ f$.

41. Show that the function $f(x) = ax + b$ from $\mathbf{R}$ to $\mathbf{R}$, where a and b are constants with $a \neq 0$ is invertible, and find the inverse of f.

42. Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that

a) $f(S \cup T) = f(S) \cup f(T)$.
b) $f(S \cap T) \subseteq f(S) \cap f(T)$.

43. a) Give an example to show that the inclusion in part (b) in Exercise 42 may be proper.
b) Show that if f is one-to-one, the inclusion in part (b) in Exercise 42 is an equality.

Let f be a function from the set A to the set B. Let S be a subset of B. We define the **inverse image** of S to be the subset of A whose elements are precisely all preimages of all elements of S. We denote the inverse image of S by $f^{-1}(S)$, so $f^{-1}(S) = \{a \in A \mid f(a) \in S\}$. [*Beware:* The notation f^{-1} is used in two different ways. Do not confuse the notation introduced here with the notation $f^{-1}(y)$ for the value at y of the inverse of the invertible function f. Notice also that $f^{-1}(S)$, the inverse image of the set S, makes sense for all functions f, not just invertible functions.]

44. Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ defined by $f(x) = x^2$. Find

a) $f^{-1}(\{1\})$. **b)** $f^{-1}(\{x \mid 0 < x < 1\})$.
c) $f^{-1}(\{x \mid x > 4\})$.

45. Let $g(x) = \lfloor x \rfloor$. Find

a) $g^{-1}(\{0\})$. **b)** $g^{-1}(\{-1, 0, 1\})$.
c) $g^{-1}(\{x \mid 0 < x < 1\})$.

46. Let f be a function from A to B. Let S and T be subsets of B. Show that

a) $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.
b) $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$.

47. Let f be a function from A to B. Let S be a subset of B. Show that $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$.

48. Show that $\lfloor x + \frac{1}{2} \rfloor$ is the closest integer to the number x, except when x is midway between two integers, when it is the larger of these two integers.

49. Show that $\lceil x - \frac{1}{2} \rceil$ is the closest integer to the number x, except when x is midway between two integers, when it is the smaller of these two integers.

50. Show that if x is a real number, then $\lceil x \rceil - \lfloor x \rfloor = 1$ if x is not an integer and $\lceil x \rceil - \lfloor x \rfloor = 0$ if x is an integer.

51. Show that if x is a real number, then $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$.

52. Show that if x is a real number and m is an integer, then $\lceil x + m \rceil = \lceil x \rceil + m$.

53. Show that if x is a real number and n is an integer, then

a) $x < n$ if and only if $\lfloor x \rfloor < n$.
b) $n < x$ if and only if $n < \lceil x \rceil$.

54. Show that if x is a real number and n is an integer, then

a) $x \leq n$ if and only if $\lceil x \rceil \leq n$.
b) $n \leq x$ if and only if $n \leq \lfloor x \rfloor$.

55. Prove that if n is an integer, then $\lfloor n/2 \rfloor = n/2$ if n is even and $(n - 1)/2$ if n is odd.

56. Prove that if x is a real number, then $\lfloor -x \rfloor = -\lceil x \rceil$ and $\lceil -x \rceil = -\lfloor x \rfloor$.

57. The function INT is found on some calculators, where $\text{INT}(x) = \lfloor x \rfloor$ when x is a nonnegative real number and $\text{INT}(x) = \lceil x \rceil$ when x is a negative real number. Show that this INT function satisfies the identity $\text{INT}(-x) = -\text{INT}(x)$.

58. Let a and b be real numbers with $a < b$. Use the floor and/or ceiling functions to express the number of integers n that satisfy the inequality $a \le n \le b$.

59. Let a and b be real numbers with $a < b$. Use the floor and/or ceiling functions to express the number of integers n that satisfy the inequality $a < n < b$.

60. How many bytes are required to encode n bits of data where n equals

a) 4? **b)** 10? **c)** 500? **d)** 3000?

61. How many bytes are required to encode n bits of data where n equals

a) 7? **b)** 17? **c)** 1001? **d)** 28,800?

62. How many ATM cells (described in Example 30) can be transmitted in 10 seconds over a link operating at the following rates?

a) 128 kilobits per second (1 kilobit = 1000 bits)
b) 300 kilobits per second
c) 1 megabit per second (1 megabit = 1,000,000 bits)

63. Data are transmitted over a particular Ethernet network in blocks of 1500 octets (blocks of 8 bits). How many blocks are required to transmit the following amounts of data over this Ethernet network? (Note that a byte is a synonym for an octet, a kilobyte is 1000 bytes, and a megabyte is 1,000,000 bytes.)

a) 150 kilobytes of data
b) 384 kilobytes of data
c) 1.544 megabytes of data
d) 45.3 megabytes of data

64. Draw the graph of the function $f(n) = 1 - n^2$ from **Z** to **Z**.

65. Draw the graph of the function $f(x) = \lfloor 2x \rfloor$ from **R** to **R**.

66. Draw the graph of the function $f(x) = \lfloor x/2 \rfloor$ from **R** to **R**.

67. Draw the graph of the function $f(x) = \lfloor x \rfloor + \lfloor x/2 \rfloor$ from **R** to **R**.

68. Draw the graph of the function $f(x) = \lceil x \rceil + \lfloor x/2 \rfloor$ from **R** to **R**.

69. Draw graphs of each of these functions.

a) $f(x) = \lfloor x + \frac{1}{2} \rfloor$ **b)** $f(x) = \lfloor 2x + 1 \rfloor$
c) $f(x) = \lceil x/3 \rceil$ **d)** $f(x) = \lceil 1/x \rceil$
e) $f(x) = \lceil x - 2 \rceil + \lfloor x + 2 \rfloor$
f) $f(x) = \lfloor 2x \rfloor \lceil x/2 \rceil$ **g)** $f(x) = \lceil \lfloor x - \frac{1}{2} \rfloor + \frac{1}{2} \rceil$

70. Draw graphs of each of these functions.

a) $f(x) = \lceil 3x - 2 \rceil$ **b)** $f(x) = \lceil 0.2x \rceil$
c) $f(x) = \lfloor -1/x \rfloor$ **d)** $f(x) = \lfloor x^2 \rfloor$
e) $f(x) = \lceil x/2 \rceil \lfloor x/2 \rfloor$ **f)** $f(x) = \lfloor x/2 \rfloor + \lceil x/2 \rfloor$
g) $f(x) = \lfloor 2 \lceil x/2 \rceil + \frac{1}{2} \rfloor$

71. Find the inverse function of $f(x) = x^3 + 1$.

72. Suppose that f is an invertible function from Y to Z and g is an invertible function from X to Y. Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

73. Let S be a subset of a universal set U. The **characteristic function** f_S of S is the function from U to the set $\{0, 1\}$ such that $f_S(x) = 1$ if x belongs to S and $f_S(x) = 0$ if x does not belong to S. Let A and B be sets. Show that for all $x \in U$,

a) $f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$
b) $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x)$
c) $f_{\overline{A}}(x) = 1 - f_A(x)$
d) $f_{A \oplus B}(x) = f_A(x) + f_B(x) - 2f_A(x)f_B(x)$

☞ **74.** Suppose that f is a function from A to B, where A and B are finite sets with $|A| = |B|$. Show that f is one-to-one if and only if it is onto.

75. Prove or disprove each of these statements about the floor and ceiling functions.

a) $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$ for all real numbers x.
b) $\lfloor 2x \rfloor = 2\lfloor x \rfloor$ whenever x is a real number.
c) $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0$ or 1 whenever x and y are real numbers.
d) $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$ for all real numbers x and y.
e) $\left\lceil \frac{x}{2} \right\rceil = \left\lfloor \frac{x+1}{2} \right\rfloor$ for all real numbers x.

76. Prove or disprove each of these statements about the floor and ceiling functions.

a) $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$ for all real numbers x.
b) $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ for all real numbers x and y.
c) $\lceil \lceil x/2 \rceil /2 \rceil = \lceil x/4 \rceil$ for all real numbers x.
d) $\lfloor \sqrt{\lceil x \rceil} \rfloor = \lfloor \sqrt{x} \rfloor$ for all positive real numbers x.
e) $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \le \lfloor 2x \rfloor + \lfloor 2y \rfloor$ for all real numbers x and y.

77. Prove that if x is a positive real number, then

a) $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$.
b) $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$.

78. Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

79. For each of these partial functions, determine its domain, codomain, domain of definition, and the set of values for which it is undefined. Also, determine whether it is a total function.

a) $f: \mathbf{Z} \to \mathbf{R}, f(n) = 1/n$
b) $f: \mathbf{Z} \to \mathbf{Z}, f(n) = \lceil n/2 \rceil$
c) $f: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Q}, f(m, n) = m/n$
d) $f: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}, f(m, n) = mn$
e) $f: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}, f(m, n) = m - n$ if $m > n$

80. **a)** Show that a partial function from A to B can be viewed as a function f^* from A to $B \cup \{u\}$, where u is not an element of B and

$$f^*(a) = \begin{cases} f(a) & \text{if } a \text{ belongs to the domain} \\ & \text{of definition of } f \\ u & \text{if } f \text{ is undefined at } a. \end{cases}$$

b) Using the construction in (a), find the function f^* corresponding to each partial function in Exercise 79.

☞ **81.** **a)** Show that if a set S has cardinality m, where m is a positive integer, then there is a one-to-one correspondence between S and the set $\{1, 2, \ldots, m\}$.

b) Show that if S and T are two sets each with m elements, where m is a positive integer, then there is a one-to-one correspondence between S and T.

***82.** Show that a set S is infinite if and only if there is a proper subset A of S such that there is a one-to-one correspondence between A and S.

2.4 Sequences and Summations

2.4.1 Introduction

Sequences are ordered lists of elements, used in discrete mathematics in many ways. For example, they can be used to represent solutions to certain counting problems, as we will see in Chapter 8. They are also an important data structure in computer science. We will often need to work with sums of terms of sequences in our study of discrete mathematics. This section reviews the use of summation notation, basic properties of summations, and formulas for the sums of terms of some particular types of sequences.

The terms of a sequence can be specified by providing a formula for each term of the sequence. In this section we describe another way to specify the terms of a sequence using a recurrence relation, which expresses each term as a combination of the previous terms. We will introduce one method, known as iteration, for finding a closed formula for the terms of a sequence specified via a recurrence relation. Identifying a sequence when the first few terms are provided is a useful skill when solving problems in discrete mathematics. We will provide some tips, including a useful tool on the Web, for doing so.

2.4.2 Sequences

A sequence is a discrete structure used to represent an ordered list. For example, 1, 2, 3, 5, 8 is a sequence with five terms and $1, 3, 9, 27, 81, \ldots, 3^n, \ldots$ is an infinite sequence.

Definition 1 A *sequence* is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \ldots\}$ or the set $\{1, 2, 3, \ldots\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a *term* of the sequence.

We use the notation $\{a_n\}$ to describe the sequence. (Note that a_n represents an individual term of the sequence $\{a_n\}$. Be aware that the notation $\{a_n\}$ for a sequence conflicts with the notation for a set. However, the context in which we use this notation will always make it clear when we are dealing with sets and when we are dealing with sequences. Moreover, although we have used the letter a in the notation for a sequence, other letters or expressions may be used depending on the sequence under consideration. That is, the choice of the letter a is arbitrary.)

We describe sequences by listing the terms of the sequence in order of increasing subscripts.

EXAMPLE 1 Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}.$$

The list of the terms of this sequence, beginning with a_1, namely,

$$a_1, a_2, a_3, a_4, \ldots,$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

◀

Definition 2

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term a* and the *common ratio r* are real numbers.

Remark: A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

EXAMPLE 2 The sequences $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2 \cdot 5^n$, and $\{d_n\}$ with $d_n = 6 \cdot (1/3)^n$ are geometric progressions with initial term and common ratio equal to 1 and -1, 2 and 5, and 6 and $1/3$, respectively, if we start at $n = 0$. The list of terms $b_0, b_1, b_2, b_3, b_4, \ldots$ begins with

$$1, -1, 1, -1, 1, \ldots;$$

the list of terms $c_0, c_1, c_2, c_3, c_4, \ldots$ begins with

$$2, 10, 50, 250, 1250, \ldots;$$

and the list of terms $d_0, d_1, d_2, d_3, d_4, \ldots$ begins with

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots.$$

◄

Definition 3

An *arithmetic progression* is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the *initial term a* and the *common difference d* are real numbers.

Remark: An arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$.

EXAMPLE 3 The sequences $\{s_n\}$ with $s_n = -1 + 4n$ and $\{t_n\}$ with $t_n = 7 - 3n$ are both arithmetic progressions with initial terms and common differences equal to -1 and 4, and 7 and -3, respectively, if we start at $n = 0$. The list of terms $s_0, s_1, s_2, s_3, \ldots$ begins with

$$-1, 3, 7, 11, \ldots,$$

and the list of terms $t_0, t_1, t_2, t_3, \ldots$ begins with

$$7, 4, 1, -2, \ldots.$$

◄

Sequences of the form $a_1, a_2, \ldots, a_n$ are often used in computer science. These finite sequences are also called **strings**. This string is also denoted by $a_1a_2 \ldots a_n$. (Recall that bit strings, which are finite sequences of bits, were introduced in Section 1.1.) The **length** of a string is the number of terms in this string. The **empty string**, denoted by λ, is the string that has no terms. The empty string has length zero.

EXAMPLE 4 The string *abcd* is a string of length four. ◄

2.4.3 Recurrence Relations

In Examples 1–3 we specified sequences by providing explicit formulas for their terms. There are many other ways to specify a sequence. For example, another way to specify a sequence is to provide one or more initial terms together with a rule for determining subsequent terms from those that precede them.

Definition 4

A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation. (A recurrence relation is said to *recursively define* a sequence. We will explain this alternative terminology in Chapter 5.)

EXAMPLE 5 Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$, and suppose that $a_0 = 2$. What are a_1, a_2, and a_3?

Solution: We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$. It then follows that $a_2 = 5 + 3 = 8$ and $a_3 = 8 + 3 = 11$. ◀

EXAMPLE 6 Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Solution: We see from the recurrence relation that $a_2 = a_1 - a_0 = 5 - 3 = 2$ and $a_3 = a_2 - a_1 = 2 - 5 = -3$. We can find a_4, a_5, and each successive term in a similar way. ◀

The **initial conditions** for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect. For instance, the initial condition in Example 5 is $a_0 = 2$, and the initial conditions in Example 6 are $a_0 = 3$ and $a_1 = 5$. Using mathematical induction, a proof technique introduced in Chapter 5, it can be shown that a recurrence relation together with its initial conditions determines a unique solution.

Hop along to Chapter 8 to learn how to find a formula for the Fibonacci numbers.

Links

Next, we define a particularly useful sequence defined by a recurrence relation, known as the **Fibonacci sequence**, after the Italian mathematician Fibonacci who was born in the 12th century (see Chapter 5 for his biography). We will study this sequence in depth in Chapters 5 and 8, where we will see why it is important for many applications, including modeling the population growth of rabbits. Fibonacci numbers occur naturally in the structures of plants and animals, such as in the arrangement of sunflower seeds in a seed head and in the shell of the chambered nautilus.

Definition 5

The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \ldots$.

EXAMPLE 7 Find the Fibonacci numbers f_2, f_3, f_4, f_5, and f_6.

Solution: The recurrence relation for the Fibonacci sequence tells us that we find successive terms by adding the previous two terms. Because the initial conditions tell us that $f_0 = 0$ and $f_1 = 1$, using the recurrence relation in the definition we find that

$$
\begin{aligned}
f_2 &= f_1 + f_0 = 1 + 0 = 1,\\
f_3 &= f_2 + f_1 = 1 + 1 = 2,\\
f_4 &= f_3 + f_2 = 2 + 1 = 3,\\
f_5 &= f_4 + f_3 = 3 + 2 = 5,\\
f_6 &= f_5 + f_4 = 5 + 3 = 8.
\end{aligned}
$$

◀

EXAMPLE 8 Suppose that $\{a_n\}$ is the sequence of integers defined by $a_n = n!$, the value of the factorial function at the integer n, where $n = 1, 2, 3, \ldots$. Because $n! = n((n-1)(n-2)\ldots 2 \cdot 1) = n(n-1)! = na_{n-1}$, we see that the sequence of factorials satisfies the recurrence relation $a_n = na_{n-1}$, together with the initial condition $a_1 = 1$. ◀

We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a **closed formula**, for the terms of the sequence.

EXAMPLE 9 Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$. Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

Solution: Suppose that $a_n = 3n$ for every nonnegative integer n. Then, for $n \geq 2$, we see that $2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n = a_n$. Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the recurrence relation.

Suppose that $a_n = 2^n$ for every nonnegative integer n. Note that $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$. Because $2a_1 - a_0 = 2 \cdot 2 - 1 = 3 \neq a_2$, we see that $\{a_n\}$, where $a_n = 2^n$, is not a solution of the recurrence relation.

Suppose that $a_n = 5$ for every nonnegative integer n. Then for $n \geq 2$, we see that $a_n = 2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$. Therefore, $\{a_n\}$, where $a_n = 5$, is a solution of the recurrence relation. ◀

Many methods have been developed for solving recurrence relations. Here, we will introduce a straightforward method known as iteration via several examples. In Chapter 8 we will study recurrence relations in depth. In that chapter we will show how recurrence relations can be used to solve counting problems and we will introduce several powerful methods that can be used to solve many different recurrence relations.

EXAMPLE 10 Solve the recurrence relation and initial condition in Example 5.

Solution: We can successively apply the recurrence relation in Example 5, starting with the initial condition $a_1 = 2$, and working upward until we reach a_n to deduce a closed formula for the sequence. We see that

$$
\begin{aligned}
a_2 &= 2 + 3\\
a_3 &= (2 + 3) + 3 = 2 + 3 \cdot 2\\
a_4 &= (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3\\
&\vdots\\
a_n &= a_{n-1} + 3 = (2 + 3 \cdot (n-2)) + 3 = 2 + 3(n-1).
\end{aligned}
$$

We can also successively apply the recurrence relation in Example 5, starting with the term a_n and working downward until we reach the initial condition $a_1 = 2$ to deduce this same formula. The steps are

$$\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\
&= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\
&\ \vdots \\
&= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1).
\end{aligned}$$

At each iteration of the recurrence relation, we obtain the next term in the sequence by adding 3 to the previous term. We obtain the nth term after $n - 1$ iterations of the recurrence relation. Hence, we have added $3(n - 1)$ to the initial term $a_0 = 2$ to obtain a_n. This gives us the closed formula $a_n = 2 + 3(n - 1)$. Note that this sequence is an arithmetic progression. ◀

The technique used in Example 10 is called **iteration**. We have iterated, or repeatedly used, the recurrence relation. The first approach is called **forward substitution**—we found successive terms beginning with the initial condition and ending with a_n. The second approach is called **backward substitution**, because we began with a_n and iterated to express it in terms of falling terms of the sequence until we found it in terms of a_1. Note that when we use iteration, we essentialy guess a formula for the terms of the sequence. To prove that our guess is correct, we need to use mathematical induction, a technique we discuss in Chapter 5.

In Chapter 8 we will show that recurrence relations can be used to model a wide variety of problems. We provide one such example here, showing how to use a recurrence relation to find compound interest.

EXAMPLE 11

Extra Examples

Compound Interest Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Solution: To solve this problem, let P_n denote the amount in the account after n years. Because the amount in the account after n years equals the amount in the account after $n - 1$ years plus interest for the nth year, we see that the sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}.$$

The initial condition is $P_0 = 10{,}000$.

We can use an iterative approach to find a formula for P_n. Note that

$$\begin{aligned}
P_1 &= (1.11)P_0 \\
P_2 &= (1.11)P_1 = (1.11)^2 P_0 \\
P_3 &= (1.11)P_2 = (1.11)^3 P_0 \\
&\ \vdots \\
P_n &= (1.11)P_{n-1} = (1.11)^n P_0.
\end{aligned}$$

When we insert the initial condition $P_0 = 10{,}000$, the formula $P_n = (1.11)^n 10{,}000$ is obtained.

Inserting $n = 30$ into the formula $P_n = (1.11)^n 10{,}000$ shows that after 30 years the account contains

$$P_{30} = (1.11)^{30} 10{,}000 = \$228{,}922.97.$$

◀

2.4.4 Special Integer Sequences

A common problem in discrete mathematics is finding a closed formula, a recurrence relation, or some other type of general rule for constructing the terms of a sequence. Sometimes only a few terms of a sequence solving a problem are known; the goal is to identify the sequence. Even though the initial terms of a sequence do not determine the entire sequence (after all, there are infinitely many different sequences that start with any finite set of initial terms), knowing the first few terms may help you make an educated conjecture about the identity of your sequence. Once you have made this conjecture, you can try to verify that you have the correct sequence.

When trying to deduce a possible formula, recurrence relation, or some other type of rule for the terms of a sequence when given the initial terms, try to find a pattern in these terms. You might also see whether you can determine how a term might have been produced from those preceding it. There are many questions you could ask, but some of the more useful are:

- Are there runs of the same value? That is, does the same value occur many times in a row?
- Are terms obtained from previous terms by adding the same amount or an amount that depends on the position in the sequence?
- Are terms obtained from previous terms by multiplying by a particular amount?
- Are terms obtained by combining previous terms in a certain way?
- Are there cycles among the terms?

EXAMPLE 12 Find formulae for the sequences with the following first five terms: (a) 1, 1/2, 1/4, 1/8, 1/16 (b) 1, 3, 5, 7, 9 (c) 1, −1, 1, −1, 1.

Extra Examples

Solution: (a) We recognize that the denominators are powers of 2. The sequence with $a_n = 1/2^n$, $n = 0, 1, 2, \ldots$ is a possible match. This proposed sequence is a geometric progression with $a = 1$ and $r = 1/2$.

(b) We note that each term is obtained by adding 2 to the previous term. The sequence with $a_n = 2n + 1$, $n = 0, 1, 2, \ldots$ is a possible match. This proposed sequence is an arithmetic progression with $a = 1$ and $d = 2$.

(c) The terms alternate between 1 and −1. The sequence with $a_n = (-1)^n$, $n = 0, 1, 2 \ldots$ is a possible match. This proposed sequence is a geometric progression with $a = 1$ and $r = -1$. ◀

Examples 13–15 illustrate how we can analyze sequences to find how the terms are constructed.

EXAMPLE 13 How can we produce the terms of a sequence if the first 10 terms are 1, 2, 2, 3, 3, 3, 4, 4, 4, 4?

Solution: In this sequence, the integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, and the integer 4 appears four times. A reasonable rule for generating this sequence is that the integer n appears exactly n times, so the next five terms of the sequence would all be 5, the following six terms would all be 6, and so on. The sequence generated this way is a possible match. ◀

EXAMPLE 14 How can we produce the terms of a sequence if the first 10 terms are 5, 11, 17, 23, 29, 35, 41, 47, 53, 59?

Solution: Note that each of the first 10 terms of this sequence after the first is obtained by adding 6 to the previous term. (We could see this by noticing that the difference between consecutive terms is 6.) Consequently, the nth term could be produced by starting with 5 and adding 6 a total

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, …
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, …
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, …
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, …
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, …

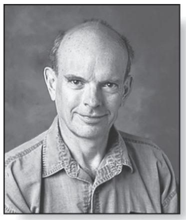
of $n - 1$ times; that is, a reasonable guess is that the nth term is $5 + 6(n - 1) = 6n - 1$. (This is an arithmetic progression with $a = 5$ and $d = 6$.) ◀

EXAMPLE 15 How can we produce the terms of a sequence if the first 10 terms are 1, 3, 4, 7, 11, 18, 29, 47, 76, 123?

Solution: Observe that each successive term of this sequence, starting with the third term, is the sum of the two previous terms. That is, $4 = 3 + 1$, $7 = 4 + 3$, $11 = 7 + 4$, and so on. Consequently, if L_n is the nth term of this sequence, we guess that the sequence is determined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with initial conditions $L_1 = 1$ and $L_2 = 3$ (the same recurrence relation as the Fibonacci sequence, but with different initial conditions). This sequence is known as the **Lucas sequence**, after the French mathematician François Édouard Lucas. Lucas studied this sequence and the Fibonacci sequence in the nineteenth century. ◀

Another useful technique for finding a rule for generating the terms of a sequence is to compare the terms of a sequence of interest with the terms of a well-known integer sequence, such as terms of an arithmetic progression, terms of a geometric progression, perfect squares, perfect cubes, and so on. The first 10 terms of some sequences you may want to keep in mind are displayed in Table 1. Note that we have listed these sequences so that the terms of each sequence grow faster than those in the preceding sequence in the list. The rates of growth of these terms will be studied in Section 3.2.

Links



Courtesy of Neil Sloane

NEIL SLOANE (BORN 1939) Neil Sloane studied mathematics and electrical engineering at the University of Melbourne on a scholarship from the Australian state telephone company. He mastered many telephone-related jobs, such as erecting telephone poles, in his summer work. After graduating, he designed minimal-cost telephone networks in Australia. In 1962 he came to the United States and studied electrical engineering at Cornell University. His Ph.D. thesis was on what are now called neural networks. He took a job at Bell Labs in 1969, working in many areas, including network design, coding theory, and sphere packing. He moved to AT&T Labs in 1996 when it was split off from Bell Labs, working there until his retirement in 2012. One of his favorite problems is the **kissing problem** (a name he coined), which asks how many spheres can be arranged in n dimensions so that they all touch a central sphere of the same size. (In two dimensions the answer is 6, because 6 pennies can be placed so that they touch a central penny. In three dimensions, 12 billiard balls can be placed so that they touch a central billiard ball. Two billiard balls that just touch are said to "kiss," giving rise to the terminology "kissing problem" and "kissing number.") Sloane, together with Andrew Odlyzko, showed that in 8 and 24 dimensions, the optimal kissing numbers are, respectively, 240 and 196,560. The kissing number is known in dimensions 1, 2, 3, 4, 8, and 24, but not in any other dimensions. Sloane's books include *Sphere Packings, Lattices and Groups,* 3d ed., with John Conway; *The Theory of Error-Correcting Codes* with Jessie MacWilliams; *The Encyclopedia of Integer Sequences* with Simon Plouffe (which has grown into the popular OEIS website); and *The Rock-Climbing Guide to New Jersey Crags* with Paul Nick. The last book demonstrates his interest in rock climbing; it includes more than 50 climbing sites in New Jersey.

EXAMPLE 16 Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

Solution: To attack this problem, we begin by looking at the difference of consecutive terms, but we do not see a pattern. When we form the ratio of consecutive terms to see whether each term is a multiple of the previous term, we find that this ratio, although not a constant, is close to 3. So it is reasonable to suspect that the terms of this sequence are generated by a formula involving 3^n. Comparing these terms with the corresponding terms of the sequence $\{3^n\}$, we notice that the nth term is 2 less than the corresponding power of 3. We see that $a_n = 3^n - 2$ for $1 \leq n \leq 10$ and conjecture that this formula holds for all n. ◀

We will see throughout this text that integer sequences appear in a wide range of contexts in discrete mathematics. Sequences we have encountered or will encounter include the sequence of prime numbers (Chapter 4), the number of ways to order n discrete objects (Chapter 6), the number of moves required to solve the Tower of Hanoi puzzle with n disks (Chapter 8), and the number of rabbits on an island after n months (Chapter 8).

Check out the puzzles at the OEIS site.

Links

Integer sequences appear in an amazingly wide range of subject areas besides discrete mathematics, including biology, engineering, chemistry, and physics, as well as in puzzles. An amazing database of over 250,000 different integer sequences (as of 2017) can be found in the *On-Line Encyclopedia of Integer Sequences (OEIS)*. This database was originated by Neil Sloane in 1964 and is now maintained by the OEIS Foundation. The last printed version of this database was published in 1995 ([SIPI95]); the current encyclopedia would occupy more than 900 volumes of the size of the 1995 book with more than 10,000 new submissions a year. You can use a program on the OEIS website to find sequences from the encyclopedia that match initial terms you provide, if there is a match. For instance, when you enter 1, 1, 2, 3, 5, 8, OEIS displays a page that identifies these numbers as successive terms of the Fibonacci sequence, provides the recurrence relation that generates this sequence, lists an extensive set of comments about the many ways the Fibonacci sequence arises including references, and displays information about quite a few other sequences that begin with these same terms.

2.4.5 Summations

Next, we consider the addition of the terms of a sequence. For this we introduce **summation notation**. We begin by describing the notation used to express the sum of the terms

$$a_m, a_{m+1}, \ldots, a_n$$

from the sequence $\{a_n\}$. We use the notation

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or} \qquad \textstyle\sum_{m \leq j \leq n} a_j$$

(read as the sum from $j = m$ to $j = n$ of a_j) to represent

$$a_m + a_{m+1} + \cdots + a_n.$$

Here, the variable j is called the **index of summation**, and the choice of the letter j as the variable is arbitrary; that is, we could have used any other letter, such as i or k. Or, in notation,

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k.$$

Here, the index of summation runs through all integers starting with its **lower limit** m and ending with its **upper limit** n. A large uppercase Greek letter sigma, $\sum$, is used to denote summation.

The usual laws for arithmetic apply to summations. For example, when a and b are real numbers, we have $\sum_{j=1}^{n}(ax_j + by_j) = a\sum_{y=1}^{n} x_j + b\sum_{j=1}^{n} y_j$, where $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are real numbers. (We do not present a formal proof of this identity here. Such a proof can be constructed using mathematical induction, a proof method we introduce in Chapter 5. The proof also uses the commutative and associative laws for addition and the distributive law of multiplication over addition.)

We give some examples of summation notation.

EXAMPLE 17 Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where $a_j = 1/j$ for $j = 1, 2, 3, \ldots$.

Extra Examples

Solution: The lower limit for the index of summation is 1, and the upper limit is 100. We write this sum as

$$\sum_{j=1}^{100} \frac{1}{j}.$$

◀

EXAMPLE 18 What is the value of $\sum_{j=1}^{5} j^2$?

Solution: We have

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$
$$= 1 + 4 + 9 + 16 + 25$$
$$= 55.$$

◀

EXAMPLE 19 What is the value of $\sum_{k=4}^{8}(-1)^k$?

Solution: We have

$$\sum_{k=4}^{8}(-1)^k = (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8$$
$$= 1 + (-1) + 1 + (-1) + 1$$
$$= 1.$$

◀

Sometimes it is useful to shift the index of summation in a sum. This is often done when two sums need to be added but their indices of summation do not match. When shifting an index of summation, it is important to make the appropriate changes in the corresponding summand. This is illustrated by Example 20.

EXAMPLE 20 Suppose we have the sum

$$\sum_{j=1}^{5} j^2$$

Extra Examples

but want the index of summation to run between 0 and 4 rather than from 1 to 5. To do this, we let $k = j - 1$. Then the new summation index runs from 0 (because $k = 1 - 0 = 0$ when $j = 1$) to 4 (because $k = 5 - 1 = 4$ when $j = 5$), and the term j^2 becomes $(k + 1)^2$. Hence,

$$\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2.$$

It is easily checked that both sums are $1 + 4 + 9 + 16 + 25 = 55$. ◀

Sums of terms of geometric progressions commonly arise (such sums are called **geometric series**). Theorem 1 gives us a formula for the sum of terms of a geometric progression.

THEOREM 1 If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Proof: Let

$$S_n = \sum_{j=0}^{n} ar^j.$$

To compute S, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$\begin{aligned} rS_n &= r \sum_{j=0}^{n} ar^j && \text{substituting summation formula for } S \\ &= \sum_{j=0}^{n} ar^{j+1} && \text{by the distributive property} \\ &= \sum_{k=1}^{n+1} ar^k && \text{shifting the index of summation, with } k = j + 1 \\ &= \left(\sum_{k=0}^{n} ar^k \right) + (ar^{n+1} - a) && \text{removing } k = n + 1 \text{ term and adding } k = 0 \text{ term} \\ &= S_n + (ar^{n+1} - a) && \text{substituting } S \text{ for summation formula} \end{aligned}$$

From these equalities, we see that

$$rS_n = S_n + (ar^{n+1} - a).$$

Solving for S_n shows that if $r \neq 1$, then

$$S_n = \frac{ar^{n+1} - a}{r - 1}.$$

If $r = 1$, then the $S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n + 1)a$. ◁

EXAMPLE 21 Double summations arise in many contexts (as in the analysis of nested loops in computer programs). An example of a double summation is

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij.$$

To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:

$$\begin{aligned}\sum_{i=1}^{4}\sum_{j=1}^{3} ij &= \sum_{i=1}^{4}(i + 2i + 3i)\\ &= \sum_{i=1}^{4} 6i\\ &= 6 + 12 + 18 + 24 = 60.\end{aligned}$$

◀

We can also use summation notation to add all values of a function, or terms of an indexed set, where the index of summation runs over all values in a set. That is, we write

$$\sum_{s \in S} f(s)$$

to represent the sum of the values $f(s)$, for all members s of S.

EXAMPLE 22 What is the value of $\sum_{s \in \{0,2,4\}} s$?

Solution: Because $\sum_{s \in \{0,2,4\}} s$ represents the sum of the values of s for all the members of the set $\{0, 2, 4\}$, it follows that

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6.$$

◀

Certain sums arise repeatedly throughout discrete mathematics. Having a collection of formulae for such sums can be useful; Table 2 provides a small table of formulae for commonly occurring sums.

We derived the first formula in this table in Theorem 1. The next three formulae give us the sum of the first n positive integers, the sum of their squares, and the sum of their cubes. These three formulae can be derived in many different ways (for example, see Exercises 37 and 38). Also note that each of these formulae, once known, can be proved using mathematical induction, the subject of Section 5.1. The last two formulae in the table involve infinite series and will be discussed shortly.

Example 23 illustrates how the formulae in Table 2 can be useful.

EXAMPLE 23 Find $\sum_{k=50}^{100} k^2$.

Solution: First note that because $\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$, we have

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2.$$

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Using the formula $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$ from Table 2 (and proved in Exercise 38), we see that

$$\sum_{k=50}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338{,}350 - 40{,}425 = 297{,}925.$$

◀

SOME INFINITE SERIES Although most of the summations in this book are finite sums, infinite series are important in some parts of discrete mathematics. Infinite series are usually studied in a course in calculus and even the definition of these series requires the use of calculus, but sometimes they arise in discrete mathematics, because discrete mathematics deals with infinite collections of discrete elements. In particular, in our future studies in discrete mathematics, we will find the closed forms for the infinite series in Examples 24 and 25 to be quite useful.

EXAMPLE 24 (*Requires calculus*) Let x be a real number with $|x| < 1$. Find $\sum_{n=0}^{\infty} x^n$.

Extra Examples

Solution: By Theorem 1 with $a = 1$ and $r = x$ we see that $\sum_{n=0}^{k} x^n = \frac{x^{k+1} - 1}{x - 1}$. Because $|x| < 1$, x^{k+1} approaches 0 as k approaches infinity. It follows that

$$\sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} \frac{x^{k+1} - 1}{x - 1} = \frac{0 - 1}{x - 1} = \frac{1}{1 - x}.$$

◀

We can produce new summation formulae by differentiating or integrating existing formulae.

EXAMPLE 25 (*Requires calculus*) Differentiating both sides of the equation

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x},$$

from Example 24 we find that

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}.$$

(This differentiation is valid for $|x| < 1$ by a theorem about infinite series.) ◀

Exercises

1. Find these terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)^n + 5^n$.

a) a_0 **b)** a_1 **c)** a_4 **d)** a_5

2. What is the term a_8 of the sequence $\{a_n\}$ if a_n equals

a) 2^{n-1}? **b)** 7?
c) $1 + (-1)^n$? **d)** $-(-2)^n$?

3. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals

a) $2^n + 1$? **b)** $(n+1)^{n+1}$?
c) $\lfloor n/2 \rfloor$? **d)** $\lfloor n/2 \rfloor + \lceil n/2 \rceil$?

4. What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$, where a_n equals

a) $(-2)^n$? **b)** 3?
c) $7 + 4^n$? **d)** $2^n + (-2)^n$?

5. List the first 10 terms of each of these sequences.

a) the sequence that begins with 2 and in which each successive term is 3 more than the preceding term
b) the sequence that lists each positive integer three times, in increasing order
c) the sequence that lists the odd positive integers in increasing order, listing each odd integer twice
d) the sequence whose nth term is $n! - 2^n$
e) the sequence that begins with 3, where each succeeding term is twice the preceding term
f) the sequence whose first term is 2, second term is 4, and each succeeding term is the sum of the two preceding terms
g) the sequence whose nth term is the number of bits in the binary expansion of the number n (defined in Section 4.2)
h) the sequence where the nth term is the number of letters in the English word for the index n

6. List the first 10 terms of each of these sequences.

a) the sequence obtained by starting with 10 and obtaining each term by subtracting 3 from the previous term
b) the sequence whose nth term is the sum of the first n positive integers
c) the sequence whose nth term is $3^n - 2^n$
d) the sequence whose nth term is $\lfloor \sqrt{n} \rfloor$
e) the sequence whose first two terms are 1 and 5 and each succeeding term is the sum of the two previous terms
f) the sequence whose nth term is the largest integer whose binary expansion (defined in Section 4.2) has n bits (Write your answer in decimal notation.)
g) the sequence whose terms are constructed sequentially as follows: start with 1, then add 1, then multiply by 1, then add 2, then multiply by 2, and so on
h) the sequence whose nth term is the largest integer k such that $k! \le n$

7. Find at least three different sequences beginning with the terms 1, 2, 4 whose terms are generated by a simple formula or rule.

8. Find at least three different sequences beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule.

9. Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = 6a_{n-1}, a_0 = 2$
b) $a_n = a_{n-1}^2, a_1 = 2$
c) $a_n = a_{n-1} + 3a_{n-2}, a_0 = 1, a_1 = 2$
d) $a_n = na_{n-1} + n^2 a_{n-2}, a_0 = 1, a_1 = 1$
e) $a_n = a_{n-1} + a_{n-3}, a_0 = 1, a_1 = 2, a_2 = 0$

10. Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = -2a_{n-1}, a_0 = -1$
b) $a_n = a_{n-1} - a_{n-2}, a_0 = 2, a_1 = -1$
c) $a_n = 3a_{n-1}^2, a_0 = 1$
d) $a_n = na_{n-1} + a_{n-2}^2, a_0 = -1, a_1 = 0$
e) $a_n = a_{n-1} - a_{n-2} + a_{n-3}, a_0 = 1, a_1 = 1, a_2 = 2$

11. Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \ldots$.

a) Find a_0, a_1, a_2, a_3, and a_4.
b) Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$.
c) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \ge 2$.

12. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if

a) $a_n = 0$. **b)** $a_n = 1$.
c) $a_n = (-4)^n$. **d)** $a_n = 2(-4)^n + 3$.

13. Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if

a) $a_n = 0$? **b)** $a_n = 1$?
c) $a_n = 2^n$? **d)** $a_n = 4^n$?
e) $a_n = n4^n$? **f)** $a_n = 2 \cdot 4^n + 3n4^n$?
g) $a_n = (-4)^n$? **h)** $a_n = n^2 4^n$?

14. For each of these sequences find a recurrence relation satisfied by this sequence. (The answers are not unique because there are infinitely many different recurrence relations satisfied by any sequence.)

a) $a_n = 3$
b) $a_n = 2n$
c) $a_n = 2n + 3$
d) $a_n = 5^n$
e) $a_n = n^2$
f) $a_n = n^2 + n$
g) $a_n = n + (-1)^n$
h) $a_n = n!$

15. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if

a) $a_n = -n + 2$.
b) $a_n = 5(-1)^n - n + 2$.
c) $a_n = 3(-1)^n + 2^n - n + 2$.
d) $a_n = 7 \cdot 2^n - n + 2$.

16. Find the solution to each of these recurrence relations with the given initial conditions. Use an iterative approach such as that used in Example 10.

a) $a_n = -a_{n-1}$, $a_0 = 5$
b) $a_n = a_{n-1} + 3$, $a_0 = 1$
c) $a_n = a_{n-1} - n$, $a_0 = 4$
d) $a_n = 2a_{n-1} - 3$, $a_0 = -1$
e) $a_n = (n + 1)a_{n-1}$, $a_0 = 2$
f) $a_n = 2na_{n-1}$, $a_0 = 3$
g) $a_n = -a_{n-1} + n - 1$, $a_0 = 7$

17. Find the solution to each of these recurrence relations and initial conditions. Use an iterative approach such as that used in Example 10.

a) $a_n = 3a_{n-1}$, $a_0 = 2$
b) $a_n = a_{n-1} + 2$, $a_0 = 3$
c) $a_n = a_{n-1} + n$, $a_0 = 1$
d) $a_n = a_{n-1} + 2n + 3$, $a_0 = 4$
e) $a_n = 2a_{n-1} - 1$, $a_0 = 1$
f) $a_n = 3a_{n-1} + 1$, $a_0 = 1$
g) $a_n = na_{n-1}$, $a_0 = 5$
h) $a_n = 2na_{n-1}$, $a_0 = 1$

18. A person deposits \$1000 in an account that yields 9% interest compounded annually.

a) Set up a recurrence relation for the amount in the account at the end of n years.
b) Find an explicit formula for the amount in the account at the end of n years.
c) How much money will the account contain after 100 years?

19. Suppose that the number of bacteria in a colony triples every hour.

a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.
b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

20. Assume that the population of the world in 2017 was 7.6 billion and is growing at the rate of 1.12% a year.

Links
a) Set up a recurrence relation for the population of the world n years after 2017.
b) Find an explicit formula for the population of the world n years after 2017.
c) What will the population of the world be in 2050?

21. A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with n cars made in the nth month.

a) Set up a recurrence relation for the number of cars produced in the first n months by this factory.
b) How many cars are produced in the first year?
c) Find an explicit formula for the number of cars produced in the first n months by this factory.

22. An employee joined a company in 2017 with a starting salary of \$50,000. Every year this employee receives a raise of \$1000 plus 5% of the salary of the previous year.

a) Set up a recurrence relation for the salary of this employee n years after 2017.
b) What will the salary of this employee be in 2025?
c) Find an explicit formula for the salary of this employee n years after 2017.

23. Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan of \$5000 at a rate of 7% if a payment of \$100 is made each month. [*Hint:* Express $B(k)$ in terms of $B(k - 1)$; the monthly interest is $(0.07/12)B(k - 1)$.]

24. **a)** Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan at a rate of r if a payment P is made on the loan each month. [*Hint:* Express $B(k)$ in terms of $B(k - 1)$ and note that the monthly interest rate is $r/12$.]

Links
b) Determine what the monthly payment P should be so that the loan is paid off after T months.

25. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …
b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …
c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …
d) 3, 6, 12, 24, 48, 96, 192, …
e) 15, 8, 1, −6, −13, −20, −27, …
f) 3, 5, 8, 12, 17, 23, 30, 38, 47, …
g) 2, 16, 54, 128, 250, 432, 686, …
h) 2, 3, 7, 25, 121, 721, 5041, 40321, …

26. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

a) 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, …
b) 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, …
c) 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, …
d) 1, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, 5, 5, …
e) 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, …
f) 1, 3, 15, 105, 945, 10395, 135135, 2027025, 34459425, …
g) 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, …
h) 2, 4, 16, 256, 65536, 4294967296, …

∗∗27. Show that if a_n denotes the nth positive integer that is not a perfect square, then $a_n = n + \{\sqrt{n}\}$, where $\{x\}$ denotes the integer closest to the real number x.

***28.** Let a_n be the nth term of the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, …, constructed by including the integer k exactly k times. Show that $a_n = \lfloor\sqrt{2n} + \frac{1}{2}\rfloor$.

29. What are the values of these sums?

a) $\sum_{k=1}^{5}(k+1)$ **b)** $\sum_{j=0}^{4}(-2)^j$

c) $\sum_{i=1}^{10} 3$ **d)** $\sum_{j=0}^{8}(2^{j+1}-2^j)$

30. What are the values of these sums, where $S = \{1, 3, 5, 7\}$?

a) $\sum_{j\in S} j$ **b)** $\sum_{j\in S} j^2$

c) $\sum_{j\in S}(1/j)$ **d)** $\sum_{j\in S} 1$

31. What is the value of each of these sums of terms of a geometric progression?

a) $\sum_{j=0}^{8} 3\cdot 2^j$ **b)** $\sum_{j=1}^{8} 2^j$

c) $\sum_{j=2}^{8}(-3)^j$ **d)** $\sum_{j=0}^{8} 2\cdot(-3)^j$

32. Find the value of each of these sums.

a) $\sum_{j=0}^{8}(1+(-1)^j)$ **b)** $\sum_{j=0}^{8}(3^j-2^j)$

c) $\sum_{j=0}^{8}(2\cdot 3^j+3\cdot 2^j)$ **d)** $\sum_{j=0}^{8}(2^{j+1}-2^j)$

33. Compute each of these double sums.

a) $\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j)$ **b)** $\sum_{i=0}^{2}\sum_{j=0}^{3}(2i+3j)$

c) $\sum_{i=1}^{3}\sum_{j=0}^{2} i$ **d)** $\sum_{i=0}^{2}\sum_{j=1}^{3} ij$

34. Compute each of these double sums.

a) $\sum_{i=1}^{3}\sum_{j=1}^{2}(i-j)$ **b)** $\sum_{i=0}^{3}\sum_{j=0}^{2}(3i+2j)$

c) $\sum_{i=1}^{3}\sum_{j=0}^{2} j$ **d)** $\sum_{i=0}^{2}\sum_{j=0}^{3} i^2j^3$

35. Show that $\sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - a_0$, where $a_0, a_1, \ldots, a_n$ is a sequence of real numbers. This type of sum is called **telescoping**.

36. Use the identity $1/(k(k+1)) = 1/k - 1/(k+1)$ and Exercise 35 to compute $\sum_{k=1}^{n} 1/(k(k+1))$.

37. Sum both sides of the identity $k^2 - (k-1)^2 = 2k - 1$ from $k = 1$ to $k = n$ and use Exercise 35 to find

a) a formula for $\sum_{k=1}^{n}(2k-1)$ (the sum of the first n odd natural numbers).

b) a formula for $\sum_{k=1}^{n} k$.

***38.** Use the technique given in Exercise 35, together with the result of Exercise 37b, to derive the formula for $\sum_{k=1}^{n} k^2$ given in Table 2. [*Hint*: Take $a_k = k^3$ in the telescoping sum in Exercise 35.]

39. Find $\sum_{k=100}^{200} k$. (Use Table 2.)

40. Find $\sum_{k=99}^{200} k^3$. (Use Table 2.)

41. Find $\sum_{k=10}^{20} k^2(k-3)$. (Use Table 2.)

42. Find $\sum_{k=10}^{20}(k-1)(2k^2+1)$. (Use Table 2.)

***43.** Find a formula for $\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor$, when m is a positive integer.

***44.** Find a formula for $\sum_{k=0}^{m}\lfloor\sqrt[3]{k}\rfloor$, when m is a positive integer.

There is also a special notation for products. The product of $a_m, a_{m+1}, \ldots, a_n$ is represented by $\prod_{j=m}^{n} a_j$, read as the product from $j = m$ to $j = n$ of a_j.

45. What are the values of the following products?

a) $\prod_{i=0}^{10} i$ **b)** $\prod_{i=5}^{8} i$

c) $\prod_{i=1}^{100}(-1)^i$ **d)** $\prod_{i=1}^{10} 2$

Recall that the value of the factorial function at a positive integer n, denoted by $n!$, is the product of the positive integers from 1 to n, inclusive. Also, we specify that $0! = 1$.

46. Express $n!$ using product notation.

47. Find $\sum_{j=0}^{4} j!$.

48. Find $\prod_{j=0}^{4} j!$.

2.5 Cardinality of Sets

2.5.1 Introduction

In Definition 4 of Section 2.1 we defined the cardinality of a finite set as the number of elements in the set. We use the cardinalities of finite sets to tell us when they have the same size, or when one is bigger than the other. In this section we extend this notion to infinite sets. That is, we will define what it means for two infinite sets to have the same cardinality, providing us with a way to measure the relative sizes of infinite sets.

We will be particularly interested in countably infinite sets, which are sets with the same cardinality as the set of positive integers. We will establish the surprising result that the set of rational numbers is countably infinite. We will also provide an example of an uncountable set when we show that the set of real numbers is not countable.

The concepts developed in this section have important applications to computer science. A function is called uncomputable if no computer program can be written to find all its values, even with unlimited time and memory. We will use the concepts in this section to explain why uncomputable functions exist.

We now define what it means for two sets to have the same size, or cardinality. In Section 2.1, we discussed the cardinality of finite sets and we defined the size, or cardinality, of such sets. In Exercise 81 of Section 2.3 we showed that there is a one-to-one correspondence between any two finite sets with the same number of elements. We use this observation to extend the concept of cardinality to all sets, both finite and infinite.

Definition 1 The sets A and B have the same *cardinality* if and only if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write $|A| = |B|$.

For infinite sets the definition of cardinality provides a relative measure of the sizes of two sets, rather than a measure of the size of one particular set. We can also define what it means for one set to have a smaller cardinality than another set.

Definition 2 If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.

Remark: In Definitions 1 and 2 we introduced the notations $|A| = |B|$ and $|A| < |B|$ to denote that A and B have the same cardinality and that the cardinality of A is less than the cardinality of B. However, these definitions do not give any separate meaning to $|A|$ and $|B|$ when A and B are arbitrary infinite sets.

2.5.2 Countable Sets

We will now split infinite sets into two groups, those with the same cardinality as the set of natural numbers and those with a different cardinality.

Definition 3 A set that is either finite or has the same cardinality as the set of positive integers is called *countable*. A set that is not countable is called *uncountable*. When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$ (where $\aleph$ is aleph, the first letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

We illustrate how to show a set is countable in the next example.

EXAMPLE 1 Show that the set of odd positive integers is a countable set.

Solution: To show that the set of odd positive integers is countable, we will exhibit a one-to-one correspondence between this set and the set of positive integers. Consider the function

$$f(n) = 2n - 1$$

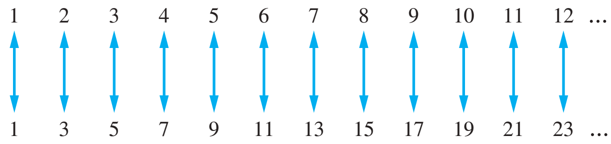


FIGURE 1 A One-to-One Correspondence Between $\mathbf{Z}^+$ and the Set of Odd Positive Integers.

from $\mathbf{Z}^+$ to the set of odd positive integers. We show that f is a one-to-one correspondence by showing that it is both one-to-one and onto. To see that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n - 1 = 2m - 1$, so $n = m$. To see that it is onto, suppose that t is an odd positive integer. Then t is 1 less than an even integer $2k$, where k is a natural number. Hence $t = 2k - 1 = f(k)$. We display this one-to-one correspondence in Figure 1. ◀

An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers). The reason for this is that a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$, where $a_1 = f(1), a_2 = f(2), \ldots, a_n = f(n), \ldots$.

You can always get a room at Hilbert's Grand Hotel!

Links

HILBERT'S GRAND HOTEL We now describe a paradox that shows that something impossible with finite sets may be possible with infinite sets. The famous mathematician David Hilbert invented the notion of the **Grand Hotel**, which has a countably infinite number of rooms, each occupied by a guest. When a new guest arrives at a hotel with a finite number of rooms, and all rooms are occupied, this guest cannot be accommodated without evicting a current guest. However, we can always accommodate a new guest at the Grand Hotel, even when all rooms are already occupied, as we show in Example 2. Exercises 5 and 8 ask you to show that we can accommodate a finite number of new guests and a countable number of new guests, respectively, at the fully occupied Grand Hotel.

EXAMPLE 2 How can we accommodate a new guest arriving at the fully occupied Grand Hotel without removing any of the current guests?

Solution: Because the rooms of the Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general, the guest in Room n to Room $n + 1$, for all positive integers n. This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms. We illustrate this situation in Figure 2. ◀

When there are finitely many rooms in a hotel, the notion that all rooms are occupied is equivalent to the notion that no new guests can be accommodated. However, Hilbert's paradox

Links

DAVID HILBERT (1862–1943) Hilbert, born in Königsberg, the city famous in mathematics for its seven bridges, was the son of a judge. During his tenure at Göttingen University, from 1892 to 1930, he made many fundamental contributions to a wide range of mathematical subjects. He almost always worked on one area of mathematics at a time, making important contributions, then moving to a new mathematical subject. Some areas in which Hilbert worked are the calculus of variations, geometry, algebra, number theory, logic, and mathematical physics. Besides his many outstanding original contributions, Hilbert is remembered for his important and influential list of 23 unsolved problems. He described these problems at the 1900 International Congress of Mathematicians, as a challenge to mathematicians at the birth of the twentieth century. Since that time, they have spurred a tremendous amount and variety of research. Although many of these problems have now been solved, several remain open, including the Riemann hypothesis, which is part of Problem 8 on Hilbert's list. Hilbert was also the author of several important textbooks in number theory and geometry.

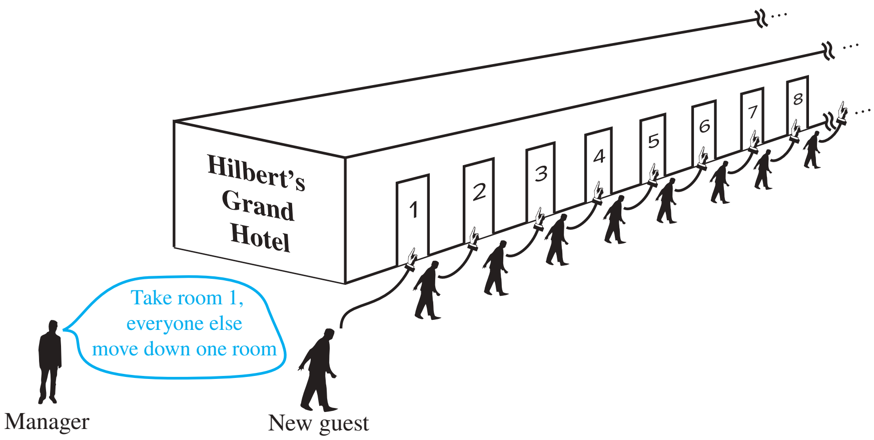


FIGURE 2 **A new guest arrives at Hilbert's Grand Hotel.**

of the Grand Hotel can be explained by noting that this equivalence no longer holds when there are infinitely many rooms.

EXAMPLES OF COUNTABLE AND UNCOUNTABLE SETS We will now show that certain sets of numbers are countable. We begin with the set of all integers. Note that we can show that the set of all integers is countable by listing its members.

EXAMPLE 3 Show that the set of all integers is countable.

Solution: We can list all integers in a sequence by starting with 0 and alternating between positive and negative integers: $0, 1, -1, 2, -2, \ldots$. Alternatively, we could find a one-to-one correspondence between the set of positive integers and the set of all integers. We leave it to the reader to show that the function $f(n) = n/2$ when n is even and $f(n) = -(n-1)/2$ when n is odd is such a function. Consequently, the set of all integers is countable. ◀

It is not surprising that the set of odd integers and the set of all integers are both countable sets (as shown in Examples 1 and 3). Many people are amazed to learn that the set of rational numbers is countable, as Example 4 demonstrates.

EXAMPLE 4 Show that the set of positive rational numbers is countable.

Solution: It may seem surprising that the set of positive rational numbers is countable, but we will show how we can list the positive rational numbers as a sequence $r_1, r_2, \ldots, r_n, \ldots$. First, note that every positive rational number is the quotient p/q of two positive integers.

We can arrange the positive rational numbers by listing those with denominator $q = 1$ in the first row, those with denominator $q = 2$ in the second row, and so on, as displayed in Figure 3.

The key to listing the rational numbers in a sequence is to first list the positive rational numbers p/q with $p + q = 2$, followed by those with $p + q = 3$, followed by those with $p + q = 4$, and so on, following the path shown in Figure 3. Whenever we encounter a number p/q that is already listed, we do not list it again. For example, when we come to $2/2 = 1$ we do not list it because we have already listed $1/1 = 1$. The initial terms in the list of positive rational numbers we have constructed are $1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4, 5$, and so on. These numbers are shown circled; the uncircled numbers in the list are those we leave out because they are

Terms not circled are not listed because they repeat previously listed terms

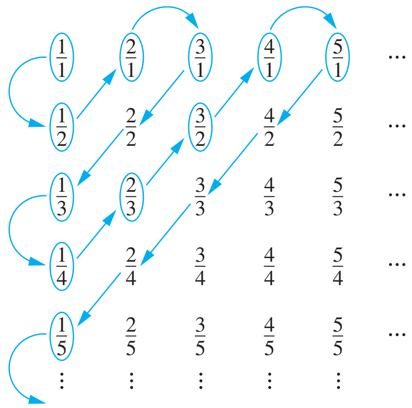


FIGURE 3 **The positive rational numbers are countable.**

already listed. Because all positive rational numbers are listed once, as the reader can verify, we have shown that the set of positive rational numbers is countable. ◀

2.5.3 An Uncountable Set

Not all infinite sets have the same size!

Links

We have seen that the set of positive rational numbers is a countable set. Do we have a promising candidate for an uncountable set? The first place we might look is the set of real numbers. In Example 5 we use an important proof method, introduced in 1879 by Georg Cantor and known as the **Cantor diagonalization argument**, to prove that the set of real numbers is not countable. This proof method is used extensively in mathematical logic and in the theory of computation.

EXAMPLE 5 Show that the set of real numbers is an uncountable set.

Extra Examples

Solution: To show that the set of real numbers is uncountable, we suppose that the set of real numbers is countable and arrive at a contradiction. Then, the subset of all real numbers that fall between 0 and 1 would also be countable (because any subset of a countable set is also countable; see Exercise 16). Under this assumption, the real numbers between 0 and 1 can be listed in some order, say, $r_1, r_2, r_3, \ldots$. Let the decimal representation of these real numbers be

$$\begin{aligned}
r_1 &= 0.d_{11}d_{12}d_{13}d_{14} \ldots \\
r_2 &= 0.d_{21}d_{22}d_{23}d_{24} \ldots \\
r_3 &= 0.d_{31}d_{32}d_{33}d_{34} \ldots \\
r_4 &= 0.d_{41}d_{42}d_{43}d_{44} \ldots \\
&\ \ \vdots
\end{aligned}$$

where $d_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. (For example, if $r_1 = 0.23794102 \ldots$, we have $d_{11} = 2$, $d_{12} = 3$, $d_{13} = 7$, and so on.) Then, form a new real number with decimal expansion $r = 0.d_1d_2d_3d_4 \ldots$, where the decimal digits are determined by the following rule:

$$d_i = \begin{cases} 4 \text{ if } d_{ii} \neq 4 \\ 5 \text{ if } d_{ii} = 4. \end{cases}$$

(As an example, suppose that $r_1 = 0.23794102 \ldots$, $r_2 = 0.44590138 \ldots$, $r_3 = 0.09118764 \ldots$, $r_4 = 0.80553900 \ldots$, and so on. Then we have $r = 0.d_1d_2d_3d_4 \ldots = 0.4544 \ldots$, where $d_1 = 4$ because $d_{11} \neq 4$, $d_2 = 5$ because $d_{22} = 4$, $d_3 = 4$ because $d_{33} \neq 4$, $d_4 = 4$ because $d_{44} \neq 4$, and so on.)

A number with a decimal expansion that terminates has a second decimal expansion ending with an infinite sequence of 9s because $1 = 0.999\ldots$.

Every real number has a unique decimal expansion (when the possibility that the expansion has a tail end that consists entirely of the digit 9 is excluded). Therefore, the real number r is not equal to any of $r_1, r_2, \ldots$ because the decimal expansion of r differs from the decimal expansion of r_i in the ith place to the right of the decimal point, for each i.

Because there is a real number r between 0 and 1 that is not in the list, the assumption that all the real numbers between 0 and 1 could be listed must be false. Therefore, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable. Any set with an uncountable subset is uncountable (see Exercise 15). Hence, the set of real numbers is uncountable. ◀

RESULTS ABOUT CARDINALITY We will now discuss some results about the cardinality of sets. First, we will prove that the union of two countable sets is also countable.

THEOREM 1 If A and B are countable sets, then $A \cup B$ is also countable.

This proof uses WLOG and cases.

Proof: Suppose that A and B are both countable sets. Without loss of generality, we can assume that A and B are disjoint. (If they are not, we can replace B by $B - A$, because $A \cap (B - A) = \emptyset$ and $A \cup (B - A) = A \cup B$.) Furthermore, without loss of generality, if one of the two sets is countably infinite and other finite, we can assume that B is the one that is finite.

There are three cases to consider: (i) A and B are both finite, (ii) A is infinite and B is finite, and (iii) A and B are both countably infinite.

Case (i): Note that when A and B are finite, $A \cup B$ is also finite, and therefore, countable.

Case (ii): Because A is countably infinite, its elements can be listed in an infinite sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ and because B is finite, its terms can be listed as $b_1, b_2, \ldots, b_m$ for some positive integer m. We can list the elements of $A \cup B$ as $b_1, b_2, \ldots, b_m, a_1, a_2, a_3, \ldots, a_n, \ldots$. This means that $A \cup B$ is countably infinite.

Case (iii): Because both A and B are countably infinite, we can list their elements as $a_1, a_2, a_3, \ldots, a_n, \ldots$ and $b_1, b_2, b_3, \ldots, b_n, \ldots$, respectively. By alternating terms of these two sequences we can list the elements of $A \cup B$ in the infinite sequence $a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_n, b_n, \ldots$. This means $A \cup B$ must be countably infinite.

We have completed the proof, as we have shown that $A \cup B$ is countable in all three cases. ◁

Because of its importance, we now state a key theorem in the study of cardinality.

THEOREM 2 **SCHRÖDER-BERNSTEIN THEOREM** If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

Because Theorem 2 seems to be quite straightforward, we might expect that it has an easy proof. However, this is not the case because when you have an injection from a set A to a set B that may not be onto, and another injection from B to A that also may not be onto, there is no obvious way to construct a bijection from A to B. Moreover, even though it has been proved in a variety of ways without using advanced mathematics, all known proofs are rather subtle and have twists and turns that are tricky to explain. One of these proofs is developed in Exercise 41 and students are invited to complete the details. Motivated readers can also find proofs in [AiZiHo09] and [Ve06]. This result is called the Schröder-Bernstein theorem after Ernst Schröder who published a flawed proof of it in 1898 and Felix Bernstein, a student of Georg Cantor, who presented a proof in 1897. However, a proof of this theorem was found in notes of Richard Dedekind dated 1887. Dedekind was a German mathematician who made important contributions to the foundations of mathematics, abstract algebra, and number theory.

We illustrate the use of Theorem 2 with an example.

EXAMPLE 6 Show that the $|(0, 1)| = |(0, 1]|$.

Solution: It is not at all obvious how to find a one-to-one correspondence between $(0, 1)$ and $(0, 1]$ to show that $|(0, 1)| = |(0, 1]|$. Fortunately, we can use the Schröder-Bernstein theorem instead. Finding a one-to-one function from $(0, 1)$ to $(0, 1]$ is simple. Because $(0, 1) \subset (0, 1]$, $f(x) = x$ is a one-to-one function from $(0, 1)$ to $(0, 1]$. Finding a one-to-one function from $(0, 1]$ to $(0, 1)$ is also not difficult. The function $g(x) = x/2$ is clearly one-to-one and maps $(0, 1]$ to $(0, 1/2] \subset (0, 1)$. As we have found one-to-one functions from $(0, 1)$ to $(0, 1]$ and from $(0, 1]$ to $(0, 1)$, the Schröder-Bernstein theorem tells us that $|(0, 1)| = |(0, 1]|$. ◀

UNCOMPUTABLE FUNCTIONS We will now describe an important application of the concepts of this section to computer science. In particular, we will show that there are functions whose values cannot be computed by any computer program.

Definition 4 We say that a function is **computable** if there is a computer program in some programming language that finds the values of this function. If a function is not computable we say it is **uncomputable**.

To show that there are uncomputable functions, we need to establish two results. First, we need to show that the set of all computer programs in any particular programming language is countable. This can be proved by noting that a computer program in a particular language can be thought of as a string of characters from a finite alphabet (see Exercise 37). Next, we show that there are uncountably many different functions from a particular countably infinite set to itself. In particular, Exercise 38 shows that the set of functions from the set of positive integers to itself is uncountable. This is a consequence of the uncountability of the real numbers between 0 and 1 (see Example 5). Putting these two results together (Exercise 39) shows that there are uncomputable functions.

THE CONTINUUM HYPOTHESIS We conclude this section with a brief discussion of a famous open question about cardinality. It can be shown that the power set of $\mathbf{Z}^+$ and the set of real numbers $\mathbf{R}$ have the same cardinality (see Exercise 38). In other words, we know that $|\mathcal{P}(\mathbf{Z}^+)| = |\mathbf{R}| = \mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of the set of real numbers.

An important theorem of Cantor (Exercise 40) states that the cardinality of a set is always less than the cardinality of its power set. Hence, $|\mathbf{Z}^+| < |\mathcal{P}(\mathbf{Z}^+)|$. We can rewrite this as $\aleph_0 < 2^{\aleph_0}$, using the notation $2^{|S|}$ to denote the cardinality of the power set of the set S. Also, note that the relationship $|\mathcal{P}(\mathbf{Z}^+)| = |\mathbf{R}|$ can be expressed as $2^{\aleph_0} = \mathfrak{c}$.

$\mathfrak{c}$ is the lowercase Fraktur c.

This leads us to the **continuum hypothesis**, which asserts that there is no cardinal number X between $\aleph_0$ and $\mathfrak{c}$. In other words, the continuum hypothesis states that there is no set A such that $\aleph_0$, the cardinality of the set of positive integers, is less than $|A|$ and $|A|$ is less than $\mathfrak{c}$, the cardinality of the set of real numbers. It can be shown that the smallest infinite cardinal numbers form an infinite sequence $\aleph_0 < \aleph_1 < \aleph_2 < \ldots$. If we assume that the continuum hypothesis is true, it would follow that $\mathfrak{c} = \aleph_1$, so that $2^{\aleph_0} = \aleph_1$.

The continuum hypothesis was stated by Cantor in 1877. He labored unsuccessfully to prove it, becoming extremely dismayed that he could not. By 1900, settling the continuum hypothesis was considered to be among the most important unsolved problems in mathematics. It was the first problem posed by David Hilbert in his 1900 list of open problems in mathematics.

The continuum hypothesis is still an open question and remains an area for active research. However, it has been shown that it can be neither proved nor disproved under the standard set theory axioms in modern mathematics, the Zermelo-Fraenkel axioms. The Zermelo-Fraenkel axioms were formulated to avoid the paradoxes of naive set theory, such as Russell's paradox, but there is much controversy whether they should be replaced by some other set of axioms for set theory.

Exercises

1. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) the negative integers
b) the even integers
c) the integers less than 100
d) the real numbers between 0 and $\frac{1}{2}$
e) the positive integers less than 1,000,000,000
f) the integers that are multiples of 7

2. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) the integers greater than 10
b) the odd negative integers
c) the integers with absolute value less than 1,000,000
d) the real numbers between 0 and 2
e) the set $A \times \mathbf{Z}^+$ where $A = \{2, 3\}$
f) the integers that are multiples of 10

3. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) all bit strings not containing the bit 0
b) all positive rational numbers that cannot be written with denominators less than 4
c) the real numbers not containing 0 in their decimal representation
d) the real numbers containing only a finite number of 1s in their decimal representation

4. Determine whether each of these sets is countable or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) integers not divisible by 3
b) integers divisible by 5 but not by 7
c) the real numbers with decimal representations consisting of all 1s
d) the real numbers with decimal representations of all 1s or 9s

5. Show that a finite group of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without evicting any current guest.

6. Suppose that Hilbert's Grand Hotel is fully occupied, but the hotel closes all the even numbered rooms for maintenance. Show that all guests can remain in the hotel.

7. Suppose that Hilbert's Grand Hotel is fully occupied on the day the hotel expands to a second building which also contains a countably infinite number of rooms. Show that the current guests can be spread out to fill every room of the two buildings of the hotel.

8. Show that a countably infinite number of guests arriving at Hilbert's fully occupied Grand Hotel can be given rooms without evicting any current guest.

∗9. Suppose that a countably infinite number of buses, each containing a countably infinite number of guests, arrive at Hilbert's fully occupied Grand Hotel. Show that all the arriving guests can be accommodated without evicting any current guest.

10. Give an example of two uncountable sets A and B such that $A - B$ is

a) finite.
b) countably infinite.
c) uncountable.

11. Give an example of two uncountable sets A and B such that $A \cap B$ is

a) finite.
b) countably infinite.
c) uncountable.

12. Show that if A and B are sets and $A \subset B$ then $|A| \leq |B|$.

13. Explain why the set A is countable if and only if $|A| \leq |\mathbf{Z}^+|$.

14. Show that if A and B are sets with the same cardinality, then $|A| \leq |B|$ and $|B| \leq |A|$.

☞ **15.** Show that if A and B are sets, A is uncountable, and $A \subseteq B$, then B is uncountable.

☞ **16.** Show that a subset of a countable set is also countable.

17. If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?

18. Show that if A and B are sets $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

19. Show that if A, B, C, and D are sets with $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$.

20. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

21. Show that if A, B, and C are sets such that $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

22. Suppose that A is a countable set. Show that the set B is also countable if there is an onto function f from A to B.

23. Show that if A is an infinite set, then it contains a countably infinite subset.

24. Show that there is no infinite set A such that $|A| < |\mathbf{Z}^+| = \aleph_0$.

25. Prove that if it is possible to label each element of an infinite set S with a finite string of keyboard characters, from a finite list characters, where no two elements of S have the same label, then S is a countably infinite set.

26. Use Exercise 25 to provide a proof different from that in the text that the set of rational numbers is countable. [*Hint:* Show that you can express a rational number as a string of digits with a slash and possibly a minus sign.]

***27.** Show that the union of a countable number of countable sets is countable.

28. Show that the set $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable.

***29.** Show that the set of all finite bit strings is countable.

***30.** Show that the set of real numbers that are solutions of quadratic equations $ax^2 + bx + c = 0$, where a, b, and c are integers, is countable.

***31.** Show that $\mathbf{Z}^+ \times \mathbf{Z}^+$ is countable by showing that the polynomial function $f : \mathbf{Z}^+ \times \mathbf{Z}^+ \to \mathbf{Z}^+$ with $f(m, n) = (m + n - 2)(m + n - 1)/2 + m$ is one-to-one and onto.

***32.** Show that when you substitute $(3n + 1)^2$ for each occurrence of n and $(3m + 1)^2$ for each occurrence of m in the right-hand side of the formula for the function $f(m, n)$ in Exercise 31, you obtain a one-to-one polynomial function $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$. It is an open question whether there is a one-to-one polynomial function $\mathbf{Q} \times \mathbf{Q} \to \mathbf{Q}$.

33. Use the Schröder-Bernstein theorem to show that $(0, 1)$ and $[0, 1]$ have the same cardinality.

34. Show that $(0, 1)$ and $\mathbf{R}$ have the same cardinality by

a) showing that $f(x) = \frac{2x-1}{2x(1-x)}$ is a bijection from $(0, 1)$ to $\mathbf{R}$.

b) using the Schröder-Bernstein theorem.

35. Show that there is no one-to-one correspondence from the set of positive integers to the power set of the set of positive integers. [*Hint:* Assume that there is such a one-to-one correspondence. Represent a subset of the set of positive integers as an infinite bit string with ith bit 1 if i belongs to the subset and 0 otherwise. Suppose that you can list these infinite strings in a sequence indexed by the positive integers. Construct a new bit string with its ith bit equal to the complement of the ith bit of the ith string in the list. Show that this new bit string cannot appear in the list.]

***36.** Show that there is a one-to-one correspondence from the set of subsets of the positive integers to the set real numbers between 0 and 1. Use this result and Exercises 34 and 35 to conclude that $\aleph_0 < |\mathcal{P}(\mathbf{Z}^+)| = |\mathbf{R}|$. [*Hint:* Look at the first part of the hint for Exercise 35.]

***37.** Show that the set of all computer programs in a particular programming language is countable. [*Hint:* A computer program written in a programming language can be thought of as a string of symbols from a finite alphabet.]

***38.** Show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable. [*Hint:* First set up a one-to-one correspondence between the set of real numbers between 0 and 1 and a subset of these functions. Do this by associating to the real number $0.d_1d_2 \ldots d_n \ldots$ the function f with $f(n) = d_n$.]

***39.** We say that a function is **computable** if there is a computer program that finds the values of this function. Use Exercises 37 and 38 to show that there are functions that are not computable.

***40.** Show that if S is a set, then there does not exist an onto function f from S to $\mathcal{P}(S)$, the power set of S. Conclude that $|S| < |\mathcal{P}(S)|$. This result is known as **Cantor's theorem**. [*Hint:* Suppose such a function f existed. Let $T = \{s \in S \mid s \notin f(s)\}$ and show that no element s can exist for which $f(s) = T$.]

***41.** In this exercise, we prove the Schröder-Bernstein theorem. Suppose that A and B are sets where $|A| \leq |B|$ and $|B| \leq |A|$. This means that there are injections $f : A \to B$ and $g : B \to A$. To prove the theorem, we must show that there is a bijection $h : A \to B$, implying that $|A| = |B|$.

To build h, we construct the *chain* of an element $a \in A$. This chain contains the elements $a, f(a), g(f(a)), f(g(f(a))), g(f(g(f(a)))), \ldots$. It also may contain more elements that precede a, extending the chain backwards. So, if there is a $b \in B$ with $g(b) = a$, then b will be the term of the chain just before a. Because g may not be a surjection, there may not be any such b, so that a is the first element of the chain. If such a b exists, because g is an injection, it is the unique element of B mapped by g to a; we denote it by $g^{-1}(a)$. (Note that this defines g^{-1} as a partial function from B to A.) We extend the chain backwards as long as possible in the same way, adding $f^{-1}(g^{-1}(a))$, $g^{-1}(f^{-1}(g^{-1}(a)))$, ... To construct the proof, complete these five parts.

a) Show that every element in A or in B belongs to exactly one chain.

b) Show that there are four types of chains: chains that form a loop, that is, carrying them forward from every element in the chain will eventually return to this element (type 1), chains that go backwards without stopping (type 2), chains that go backwards and end in the set A (type 3), and chains that go backwards and end in the set B (type 4).

c) We now define a function $h : A \to B$. We set $h(a) = f(a)$ when a belongs to a chain of type 1, 2, or 3. Show that we can define $h(a)$ when a is in a chain of type 4, by taking $h(a) = g^{-1}(a)$. In parts (d) and (e), we show that this function is a bijection from A to B, proving the theorem.

d) Show that h is one-to-one. (You can consider chains of types 1, 2, and 3 together, but chains of type 4 separately.)

e) Show that h is onto. (You need to consider chains of types 1, 2, and 3 together, but chains of type 4 separately.)

2.6 Matrices

2.6.1 Introduction

Matrices are used throughout discrete mathematics to express relationships between elements in sets. In subsequent chapters we will use matrices in a wide variety of models. For instance, matrices will be used in models of communications networks and transportation systems. Many algorithms will be developed that use these matrix models. This section reviews matrix arithmetic that will be used in these algorithms.

Definition 1 A *matrix* is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix. The plural of matrix is *matrices*. A matrix with the same number of rows as columns is called *square*. Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

EXAMPLE 1 The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$ is a 3×2 matrix. ◀

We now introduce some terminology about matrices. Boldface uppercase letters will be used to represent matrices.

Definition 2 Let m and n be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The ith *row* of $\mathbf{A}$ is the $1 \times n$ matrix $[a_{i1}, a_{i2}, \dots, a_{in}]$. The jth *column* of $\mathbf{A}$ is the $m \times 1$ matrix

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{mj} \end{bmatrix}.$$

The (i, j)th *element* or *entry* of $\mathbf{A}$ is the element a_{ij}, that is, the number in the ith row and jth column of $\mathbf{A}$. A convenient shorthand notation for expressing the matrix $\mathbf{A}$ is to write $\mathbf{A} = [a_{ij}]$, which indicates that $\mathbf{A}$ is the matrix with its (i, j)th element equal to a_{ij}.

2.6.2 Matrix Arithmetic

The basic operations of matrix arithmetic will now be discussed, beginning with a definition of matrix addition.

Definition 3 Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ matrices. The *sum* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} + \mathbf{B}$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i, j)th element. In other words, $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$.

The sum of two matrices of the same size is obtained by adding elements in the corresponding positions. Matrices of different sizes cannot be added, because such matrices will not both have entries in some of their positions.

EXAMPLE 2 We have $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$. ◀

We now discuss matrix products. A product of two matrices is defined only when the number of columns in the first matrix equals the number of rows of the second matrix.

Definition 4 Let $\mathbf{A}$ be an $m \times k$ matrix and $\mathbf{B}$ be a $k \times n$ matrix. The *product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{AB}$, is the $m \times n$ matrix with its (i, j)th entry equal to the sum of the products of the corresponding elements from the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$. In other words, if $\mathbf{AB} = [c_{ij}]$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}.$$

In Figure 1 the colored row of $\mathbf{A}$ and the colored column of $\mathbf{B}$ are used to compute the element c_{ij} of $\mathbf{AB}$. The product of two matrices is not defined when the number of columns in the first matrix and the number of rows in the second matrix are not the same.

We now give some examples of matrix products.

EXAMPLE 3 Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

Find $\mathbf{AB}$ if it is defined.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & c_{ij} & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

FIGURE 1 The product of $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$.

Extra Examples

Solution: Because $\mathbf{A}$ is a 4×3 matrix and $\mathbf{B}$ is a 3×2 matrix, the product $\mathbf{AB}$ is defined and is a 4×2 matrix. To find the elements of $\mathbf{AB}$, the corresponding elements of the rows of $\mathbf{A}$ and the columns of $\mathbf{B}$ are first multiplied and then these products are added. For instance, the element in the (3, 1)th position of $\mathbf{AB}$ is the sum of the products of the corresponding elements of the third row of $\mathbf{A}$ and the first column of $\mathbf{B}$; namely, $3 \cdot 2 + 1 \cdot 1 + 0 \cdot 3 = 7$. When all the elements of $\mathbf{AB}$ are computed, we see that

$$\mathbf{AB} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}.$$

◀

Although matrix multiplication is associative, as can easily be proved using the associativity of addition and multiplication of real numbers, matrix multiplication is *not* commutative. That is, if $\mathbf{A}$ and $\mathbf{B}$ are two matrices, it is not necessarily true that $\mathbf{AB}$ and $\mathbf{BA}$ are the same. In fact, it may be that only one of these two products is defined. For instance, if $\mathbf{A}$ is 2×3 and $\mathbf{B}$ is 3×4, then $\mathbf{AB}$ is defined and is 2×4; however, $\mathbf{BA}$ is not defined, because it is impossible to multiply a 3×4 matrix and a 2×3 matrix.

In general, suppose that $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $r \times s$ matrix. Then $\mathbf{AB}$ is defined only when $n = r$ and $\mathbf{BA}$ is defined only when $s = m$. Moreover, even when $\mathbf{AB}$ and $\mathbf{BA}$ are both defined, they will not be the same size unless $m = n = r = s$. Hence, if both $\mathbf{AB}$ and $\mathbf{BA}$ are defined and are the same size, then both $\mathbf{A}$ and $\mathbf{B}$ must be square and of the same size. Furthermore, even with $\mathbf{A}$ and $\mathbf{B}$ both $n \times n$ matrices, $\mathbf{AB}$ and $\mathbf{BA}$ are not necessarily equal, as Example 4 demonstrates.

EXAMPLE 4 Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Does $\mathbf{AB} = \mathbf{BA}$?

Solution: We find that

$$\mathbf{AB} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}.$$

Hence, $\mathbf{AB} \neq \mathbf{BA}$. ◀

2.6.3 Transposes and Powers of Matrices

We now introduce an important matrix with entries that are zeros and ones.

Definition 5 The *identity matrix of order n* is the $n \times n$ matrix $\mathbf{I}_n = [\delta_{ij}]$, (the *Kronecker delta*) where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Hence,

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Multiplying a matrix by an appropriately sized identity matrix does not change this matrix. In other words, when $\mathbf{A}$ is an $m \times n$ matrix, we have

$$\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}.$$

Powers of square matrices can be defined because matrix multiplication is associative. When $\mathbf{A}$ is an $n \times n$ matrix, we have

$$\mathbf{A}^0 = \mathbf{I}_n, \qquad \mathbf{A}^r = \underbrace{\mathbf{AAA}\cdots\mathbf{A}}_{r\ \text{times}}.$$

The operation of interchanging the rows and columns of a square matrix arises in many contexts.

Definition 6 Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix. The *transpose* of $\mathbf{A}$, denoted by $\mathbf{A}^t$, is the $n \times m$ matrix obtained by interchanging the rows and columns of $\mathbf{A}$. In other words, if $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

EXAMPLE 5 The transpose of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$. ◀

Matrices that do not change when their rows and columns are interchanged are often important.

Definition 7 A square matrix $\mathbf{A}$ is called *symmetric* if $\mathbf{A} = \mathbf{A}^t$. Thus, $\mathbf{A} = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for all i and j with $1 \le i \le n$ and $1 \le j \le n$.

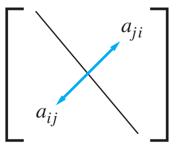


FIGURE 2 A symmetric matrix.

Note that a matrix is symmetric if and only if it is square and it is symmetric with respect to its main diagonal (which consists of entries that are in the ith row and ith column for some i). This symmetry is displayed in Figure 2.

EXAMPLE 6 The matrix $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is symmetric. ◀

2.6.4 Zero–One Matrices

A matrix all of whose entries are either 0 or 1 is called a **zero–one matrix**. Zero–one matrices are often used to represent discrete structures, as we will see in Chapters 9 and 10. Algorithms using these structures are based on Boolean arithmetic with zero–one matrices. This arithmetic is based on the Boolean operations $\wedge$ and $\vee$, which operate on pairs of bits, defined by

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 8 Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ be $m \times n$ zero–one matrices. Then the *join* of $\mathbf{A}$ and $\mathbf{B}$ is the zero–one matrix with (i, j)th entry $a_{ij} \vee b_{ij}$. The join of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \vee \mathbf{B}$. The *meet* of $\mathbf{A}$ and $\mathbf{B}$ is the zero–one matrix with (i, j)th entry $a_{ij} \wedge b_{ij}$. The meet of $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \wedge \mathbf{B}$.

EXAMPLE 7 Find the join and meet of the zero–one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: We find that the join of $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$ ◀

We now define the **Boolean product** of two matrices.

Definition 9 Let $\mathbf{A} = [a_{ij}]$ be an $m \times k$ zero–one matrix and $\mathbf{B} = [b_{ij}]$ be a $k \times n$ zero–one matrix. Then the *Boolean product* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \odot \mathbf{B}$, is the $m \times n$ matrix with (i, j)th entry c_{ij} where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Note that the Boolean product of $\mathbf{A}$ and $\mathbf{B}$ is obtained in an analogous way to the ordinary product of these matrices, but with addition replaced with the operation $\vee$ and with multiplication replaced with the operation $\wedge$. We give an example of the Boolean products of matrices.

EXAMPLE 8 Find the Boolean product of $\mathbf{A}$ and $\mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution: The Boolean product $\mathbf{A} \odot \mathbf{B}$ is given by

$$\begin{aligned} \mathbf{A} \odot \mathbf{B} &= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix} \\ &= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \end{aligned}$$ ◀

We can also define the Boolean powers of a square zero–one matrix. These powers will be used in our subsequent studies of paths in graphs, which are used to model such things as communications paths in computer networks.

Definition 10 Let $\mathbf{A}$ be a square zero–one matrix and let r be a positive integer. The rth *Boolean power* of $\mathbf{A}$ is the Boolean product of r factors of $\mathbf{A}$. The rth Boolean product of $\mathbf{A}$ is denoted by $\mathbf{A}^{[r]}$. Hence,

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \mathbf{A} \odot \cdots \odot \mathbf{A}}_{r \text{ times}}.$$

(This is well defined because the Boolean product of matrices is associative.) We also define $\mathbf{A}^{[0]}$ to be $\mathbf{I}_n$.

EXAMPLE 9 Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Find $\mathbf{A}^{[n]}$ for all positive integers n.

Solution: We find that

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

We also find that

$$\mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Additional computation shows that

$$\mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The reader can now see that $\mathbf{A}^{[n]} = \mathbf{A}^{[5]}$ for all positive integers n with $n \geq 5$. ◀

Exercises

1. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 4 & 6 \\ 1 & 1 & 3 & 7 \end{bmatrix}$.

a) What size is $\mathbf{A}$?

b) What is the third column of $\mathbf{A}$?

c) What is the second row of $\mathbf{A}$?

d) What is the element of $\mathbf{A}$ in the (3, 2)th position?

e) What is $\mathbf{A}^t$?

2. Find $\mathbf{A} + \mathbf{B}$, where

a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & 2 \\ 0 & -2 & -3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 2 & -3 \\ 2 & -3 & 0 \end{bmatrix}$.

b) $\mathbf{A} = \begin{bmatrix} -1 & 0 & 5 & 6 \\ -4 & -3 & 5 & -2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -3 & 9 & -3 & 4 \\ 0 & -2 & -1 & 2 \end{bmatrix}$.

3. Find $\mathbf{AB}$ if

a) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$.

b) $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$.

c) $\mathbf{A} = \begin{bmatrix} 4 & -3 \\ 3 & -1 \\ 0 & -2 \\ -1 & 5 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & 3 & 2 & -2 \\ 0 & -1 & 4 & -3 \end{bmatrix}$.

4. Find the product **AB**, where

a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

b) $\mathbf{A} = \begin{bmatrix} 1 & -3 & 0 \\ 1 & 2 & 2 \\ 2 & 1 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -1 & 0 & 3 & -1 \\ -3 & -2 & 0 & 2 \end{bmatrix}$.

c) $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 7 & 2 \\ -4 & -3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 & 2 & 3 & 0 \\ -2 & 0 & 3 & 4 & 1 \end{bmatrix}$.

5. Find a matrix **A** such that

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

[*Hint:* Finding **A** requires that you solve systems of linear equations.]

6. Find a matrix **A** such that

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 4 & 0 & 3 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 7 & 1 & 3 \\ 1 & 0 & 3 \\ -1 & -3 & 7 \end{bmatrix}.$$

7. Let **A** be an $m \times n$ matrix and let **0** be the $m \times n$ matrix that has all entries equal to zero. Show that $\mathbf{A} = \mathbf{0} + \mathbf{A} = \mathbf{A} + \mathbf{0}$.

8. Show that matrix addition is commutative; that is, show that if **A** and **B** are both $m \times n$ matrices, then $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

9. Show that matrix addition is associative; that is, show that if **A**, **B**, and **C** are all $m \times n$ matrices, then $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.

10. Let **A** be a 3×4 matrix, **B** be a 4×5 matrix, and **C** be a 4×4 matrix. Determine which of the following products are defined and find the size of those that are defined.

a) **AB** **b)** **BA** **c)** **AC**
d) **CA** **e)** **BC** **f)** **CB**

11. What do we know about the sizes of the matrices **A** and **B** if both of the products **AB** and **BA** are defined?

12. In this exercise we show that matrix multiplication is distributive over matrix addition.

a) Suppose that **A** and **B** are $m \times k$ matrices and that **C** is a $k \times n$ matrix. Show that $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.

b) Suppose that **C** is an $m \times k$ matrix and that **A** and **B** are $k \times n$ matrices. Show that $\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$.

13. In this exercise we show that matrix multiplication is associative. Suppose that **A** is an $m \times p$ matrix, **B** is a $p \times k$ matrix, and **C** is a $k \times n$ matrix. Show that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

14. The $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is called a **diagonal matrix** if $a_{ij} = 0$ when $i \neq j$. Show that the product of two $n \times n$ diagonal matrices is again a diagonal matrix. Give a simple rule for determining this product.

15. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find a formula for $\mathbf{A}^n$, whenever n is a positive integer.

16. Show that $(\mathbf{A}^t)^t = \mathbf{A}$.

17. Let **A** and **B** be two $n \times n$ matrices. Show that

a) $(\mathbf{A} + \mathbf{B})^t = \mathbf{A}^t + \mathbf{B}^t$.

b) $(\mathbf{AB})^t = \mathbf{B}^t\mathbf{A}^t$.

If **A** and **B** are $n \times n$ matrices with $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$, then **B** is called the **inverse** of **A** (this terminology is appropriate because such a matrix **B** is unique) and **A** is said to be **invertible**. The notation $\mathbf{B} = \mathbf{A}^{-1}$ denotes that **B** is the inverse of **A**.

18. Show that

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$$

is the inverse of

$$\begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}.$$

19. Let **A** be the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that if $ad - bc \neq 0$, then

$$\mathbf{A}^{-1} = \begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}.$$

20. Let

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}.$$

a) Find $\mathbf{A}^{-1}$. [*Hint:* Use Exercise 19.]

b) Find $\mathbf{A}^3$.

c) Find $(\mathbf{A}^{-1})^3$.

d) Use your answers to (b) and (c) to show that $(\mathbf{A}^{-1})^3$ is the inverse of $\mathbf{A}^3$.

21. Let **A** be an invertible matrix. Show that $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$ whenever n is a positive integer.

22. Let **A** be a matrix. Show that the matrix $\mathbf{AA}^t$ is symmetric. [*Hint:* Show that this matrix equals its transpose with the help of Exercise 17b.]

23. Suppose that $\mathbf{A}$ is an $n \times n$ matrix where n is a positive integer. Show that $\mathbf{A} + \mathbf{A}^t$ is symmetric.

24. a) Show that the system of simultaneous linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n. \end{aligned}$$

in the variables $x_1, x_2, \ldots, x_n$ can be expressed as $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A} = [a_{ij}]$, $\mathbf{X}$ is an $n \times 1$ matrix with x_i the entry in its ith row, and $\mathbf{B}$ is an $n \times 1$ matrix with b_i the entry in its ith row.

b) Show that if the matrix $\mathbf{A} = [a_{ij}]$ is invertible (as defined in the preamble to Exercise 18), then the solution of the system in part (a) can be found using the equation $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

25. Use Exercises 18 and 24 to solve the system

$$\begin{aligned} 7x_1 - 8x_2 + 5x_3 &= 5 \\ -4x_1 + 5x_2 - 3x_3 &= -3 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}$$

26. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Find

a) $\mathbf{A} \vee \mathbf{B}$. **b)** $\mathbf{A} \wedge \mathbf{B}$. **c)** $\mathbf{A} \odot \mathbf{B}$.

27. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find

a) $\mathbf{A} \vee \mathbf{B}$. **b)** $\mathbf{A} \wedge \mathbf{B}$. **c)** $\mathbf{A} \odot \mathbf{B}$.

28. Find the Boolean product of $\mathbf{A}$ and $\mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

29. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find

a) $\mathbf{A}^{[2]}$. **b)** $\mathbf{A}^{[3]}$.

c) $\mathbf{A} \vee \mathbf{A}^{[2]} \vee \mathbf{A}^{[3]}$.

30. Let $\mathbf{A}$ be a zero–one matrix. Show that

a) $\mathbf{A} \vee \mathbf{A} = \mathbf{A}$. **b)** $\mathbf{A} \wedge \mathbf{A} = \mathbf{A}$.

31. In this exercise we show that the meet and join operations are commutative. Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ zero–one matrices. Show that

a) $\mathbf{A} \vee \mathbf{B} = \mathbf{B} \vee \mathbf{A}$. **b)** $\mathbf{B} \wedge \mathbf{A} = \mathbf{A} \wedge \mathbf{B}$.

32. In this exercise we show that the meet and join operations are associative. Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be $m \times n$ zero–one matrices. Show that

a) $(\mathbf{A} \vee \mathbf{B}) \vee \mathbf{C} = \mathbf{A} \vee (\mathbf{B} \vee \mathbf{C})$.

b) $(\mathbf{A} \wedge \mathbf{B}) \wedge \mathbf{C} = \mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C})$.

33. We will establish distributive laws of the meet over the join operation in this exercise. Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be $m \times n$ zero–one matrices. Show that

a) $\mathbf{A} \vee (\mathbf{B} \wedge \mathbf{C}) = (\mathbf{A} \vee \mathbf{B}) \wedge (\mathbf{A} \vee \mathbf{C})$.

b) $\mathbf{A} \wedge (\mathbf{B} \vee \mathbf{C}) = (\mathbf{A} \wedge \mathbf{B}) \vee (\mathbf{A} \wedge \mathbf{C})$.

34. Let $\mathbf{A}$ be an $n \times n$ zero–one matrix. Let $\mathbf{I}$ be the $n \times n$ identity matrix. Show that $\mathbf{A} \odot \mathbf{I} = \mathbf{I} \odot \mathbf{A} = \mathbf{A}$.

35. In this exercise we will show that the Boolean product of zero–one matrices is associative. Assume that $\mathbf{A}$ is an $m \times p$ zero–one matrix, $\mathbf{B}$ is a $p \times k$ zero–one matrix, and $\mathbf{C}$ is a $k \times n$ zero–one matrix. Show that $\mathbf{A} \odot (\mathbf{B} \odot \mathbf{C}) = (\mathbf{A} \odot \mathbf{B}) \odot \mathbf{C}$.

Key Terms and Results

TERMS

set: an unordered collection of distinct objects

axiom: a basic assumption of a theory

paradox: a logical inconsistency

element, member of a set: an object in a set

roster method: a method that describes a set by listing its elements

set builder notation: the notation that describes a set by stating a property an element must have to be a member

multiset: an unordered collection of objects where objects can occur multiple times

∅ (empty set, null set): the set with no members

universal set: the set containing all objects under consideration

Venn diagram: a graphical representation of a set or sets

$S = T$ (set equality): S and T have the same elements

$S \subseteq T$ (S is a subset of T): every element of S is also an element of T

$S \subset T$ (S is a proper subset of T): S is a subset of T and $S \neq T$

finite set: a set with n elements, where n is a nonnegative integer

infinite set: a set that is not finite

$|S|$ (the cardinality of S): the number of elements in S

$P(S)$ (the power set of S): the set of all subsets of S

$A \cup B$ (the union of A and B): the set containing those elements that are in at least one of A and B

$A \cap B$ (the intersection of A and B): the set containing those elements that are in both A and B.

$A - B$ (the difference of A and B): the set containing those elements that are in A but not in B

$\overline{A}$ (the complement of A): the set of elements in the universal set that are not in A

$A \oplus B$ (the symmetric difference of A and B): the set containing those elements in exactly one of A and B

membership table: a table displaying the membership of elements in sets

function from A to B: an assignment of exactly one element of B to each element of A

domain of f: the set A, where f is a function from A to B

codomain of f: the set B, where f is a function from A to B

b is the image of a under f: $b = f(a)$

a is a preimage of b under f: $f(a) = b$

range of f: the set of images of f

onto function, surjection: a function from A to B such that every element of B is the image of some element in A

one-to-one function, injection: a function such that the images of elements in its domain are distinct

one-to-one correspondence, bijection: a function that is both one-to-one and onto

inverse of f: the function that reverses the correspondence given by f (when f is a bijection)

$f \circ g$ (composition of f and g): the function that assigns $f(g(x))$ to x

$\lfloor x \rfloor$ (floor function): the largest integer not exceeding x

$\lceil x \rceil$ (ceiling function): the smallest integer greater than or equal to x

partial function: an assignment to each element in a subset of the domain a unique element in the codomain

sequence: a function with domain that is a subset of the set of integers

geometric progression: a sequence of the form $a, ar, ar^2, \ldots,$ where a and r are real numbers

arithmetic progression: a sequence of the form a, $a + d$, $a + 2d, \ldots,$ where a and d are real numbers

string: a finite sequence

empty string: a string of length zero

recurrence relation: a equation that expresses the nth term a_n of a sequence in terms of one or more of the previous terms of the sequence for all integers n greater than a particular integer

$\sum_{i=1}^{n} a_i$: the sum $a_1 + a_2 + \cdots + a_n$

$\prod_{i=1}^{n} a_i$: the product $a_1 a_2 \cdots a_n$

cardinality: two sets A and B have the same cardinality if there is a one-to-one correspondence from A to B

countable set: a set that either is finite or can be placed in one-to-one correspondence with the set of positive integers

uncountable set: a set that is not countable

$\aleph_0$ (aleph null): the cardinality of a countable set

$\mathfrak{c}$: the cardinality of the set of real numbers

Cantor diagonalization argument: a proof technique used to show that the set of real numbers is uncountable

computable function: a function for which there is a computer program in some programming language that finds its values

uncomputable function: a function for which no computer program in a programming language exists that finds its values

continuum hypothesis: the statement that no set A exists such that $\aleph_0 < |A| < \mathfrak{c}$

matrix: a rectangular array of numbers

matrix addition: see page 188

matrix multiplication: see page 189

$\mathbf{I}_n$ (identity matrix of order n): the $n \times n$ matrix that has entries equal to 1 on its diagonal and 0s elsewhere

$\mathbf{A}^t$ (transpose of $\mathbf{A}$): the matrix obtained from $\mathbf{A}$ by interchanging the rows and columns

symmetric matrix: a matrix is symmetric if it equals its transpose

zero–one matrix: a matrix with each entry equal to either 0 or 1

$\mathbf{A} \vee \mathbf{B}$ (the join of $\mathbf{A}$ and $\mathbf{B}$): see page 191

$\mathbf{A} \wedge \mathbf{B}$ (the meet of $\mathbf{A}$ and $\mathbf{B}$): see page 191

$\mathbf{A} \odot \mathbf{B}$ (the Boolean product of $\mathbf{A}$ and $\mathbf{B}$): see page 192

RESULTS

The set identities given in Table 1 in Section 2.2

The summation formulae in Table 2 in Section 2.4

The set of rational numbers is countable.

The set of real numbers is uncountable.

Review Questions

1. Explain what it means for one set to be a subset of another set. How do you prove that one set is a subset of another set?
2. What is the empty set? Show that the empty set is a subset of every set.
3. a) Define $|S|$, the cardinality of the set S.
   b) Give a formula for $|A \cup B|$, where A and B are sets.
4. a) Define the power set of a set S.
   b) When is the empty set in the power set of a set S?
   c) How many elements does the power set of a set S with n elements have?

5. a) Define the union, intersection, difference, and symmetric difference of two sets.
   b) What are the union, intersection, difference, and symmetric difference of the set of positive integers and the set of odd integers?
6. a) Explain what it means for two sets to be equal.
   b) Describe as many of the ways as you can to show that two sets are equal.
   c) Show in at least two different ways that the sets $A - (B \cap C)$ and $(A - B) \cup (A - C)$ are equal.
7. Explain the relationship between logical equivalences and set identities.
8. a) Define the domain, codomain, and range of a function.
   b) Let $f(n)$ be the function from the set of integers to the set of integers such that $f(n) = n^2 + 1$. What are the domain, codomain, and range of this function?
9. a) Define what it means for a function from the set of positive integers to the set of positive integers to be one-to-one.
   b) Define what it means for a function from the set of positive integers to the set of positive integers to be onto.
   c) Give an example of a function from the set of positive integers to the set of positive integers that is both one-to-one and onto.
   d) Give an example of a function from the set of positive integers to the set of positive integers that is one-to-one but not onto.
   e) Give an example of a function from the set of positive integers to the set of positive integers that is not one-to-one but is onto.
   f) Give an example of a function from the set of positive integers to the set of positive integers that is neither one-to-one nor onto.
10. a) Define the inverse of a function.
    b) When does a function have an inverse?
    c) Does the function $f(n) = 10 - n$ from the set of integers to the set of integers have an inverse? If so, what is it?
11. a) Define the floor and ceiling functions from the set of real numbers to the set of integers.
    b) For which real numbers x is it true that $\lfloor x \rfloor = \lceil x \rceil$?
12. Conjecture a formula for the terms of the sequence that begins 8, 14, 32, 86, 248 and find the next three terms of your sequence.
13. Suppose that $a_n = a_{n-1} - 5$ for $n = 1, 2, \ldots$. Find a formula for a_n.
14. What is the sum of the terms of the geometric progression $a + ar + \cdots + ar^n$ when $r \neq 1$?
15. Show that the set of odd integers is countable.
16. Give an example of an uncountable set.
17. Define the product of two matrices **A** and **B**. When is this product defined?
18. Show that matrix multiplication is not commutative.

Supplementary Exercises

1. Let A be the set of English words that contain the letter x, and let B be the set of English words that contain the letter q. Express each of these sets as a combination of A and B.
   a) The set of English words that do not contain the letter x.
   b) The set of English words that contain both an x and a q.
   c) The set of English words that contain an x but not a q.
   d) The set of English words that do not contain either an x or a q.
   e) The set of English words that contain an x or a q, but not both.
2. Show that if A is a subset of B, then the power set of A is a subset of the power set of B.
3. Suppose that A and B are sets such that the power set of A is a subset of the power set of B. Does it follow that A is a subset of B?
4. Let **E** denote the set of even integers and **O** denote the set of odd integers. As usual, let **Z** denote the set of all integers. Determine each of these sets.

   a) $\mathbf{E} \cup \mathbf{O}$ b) $\mathbf{E} \cap \mathbf{O}$ c) $\mathbf{Z} - \mathbf{E}$ d) $\mathbf{Z} - \mathbf{O}$
5. Show that if A and B are sets, then $A - (A - B) = A \cap B$.
6. Let A and B be sets. Show that $A \subseteq B$ if and only if $A \cap B = A$.
7. Let A, B, and C be sets. Show that $(A - B) - C$ is not necessarily equal to $A - (B - C)$.
8. Suppose that A, B, and C are sets. Prove or disprove that $(A - B) - C = (A - C) - B$.
9. Suppose that A, B, C, and D are sets. Prove or disprove that $(A - B) - (C - D) = (A - C) - (B - D)$.
10. Show that if A and B are finite sets, then $|A \cap B| \leq |A \cup B|$. Determine when this relationship is an equality.
11. Let A and B be sets in a finite universal set U. List the following in order of increasing size.
    a) $|A|, |A \cup B|, |A \cap B|, |U|, |\emptyset|$
    b) $|A - B|, |A \oplus B|, |A| + |B|, |A \cup B|, |\emptyset|$
12. Let A and B be subsets of the finite universal set U. Show that $|\overline{A} \cap \overline{B}| = |U| - |A| - |B| + |A \cap B|$.
13. Let f and g be functions from $\{1, 2, 3, 4\}$ to $\{a, b, c, d\}$ and from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$, respectively, with $f(1) = d$, $f(2) = c$, $f(3) = a$, and $f(4) = b$, and $g(a) = 2$, $g(b) = 1$, $g(c) = 3$, and $g(d) = 2$.
    a) Is f one-to-one? Is g one-to-one?
    b) Is f onto? Is g onto?
    c) Does either f or g have an inverse? If so, find this inverse.

14. Suppose that f is a function from A to B where A and B are finite sets. Explain why $|f(S)| \leq |S|$ for all subsets S of A.

15. Suppose that f is a function from A to B where A and B are finite sets. Explain why $|f(S)| = |S|$ for all subsets S of A if and only if f is one-to-one.

Suppose that f is a function from A to B. We define the function S_f from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ by the rule $S_f(X) = f(X)$ for each subset X of A. Similarly, we define the function $S_{f^{-1}}$ from $\mathcal{P}(B)$ to $\mathcal{P}(A)$ by the rule $S_{f^{-1}}(Y) = f^{-1}(Y)$ for each subset Y of B. Here, we are using Definition 4, and the definition of the inverse image of a set found in the preamble to Exercise 44, both in Section 2.3.

***16.** Suppose that f is a function from the set A to the set B. Prove that

- **a)** if f is one-to-one, then S_f is a one-to-one function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.
- **b)** if f is onto function, then S_f is an onto function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.
- **c)** if f is onto function, then $S_{f^{-1}}$ is a one-to-one function from $\mathcal{P}(B)$ to $\mathcal{P}(A)$.
- **d)** if f is one-to-one, then $S_{f^{-1}}$ is an onto function from $\mathcal{P}(B)$ to $\mathcal{P}(A)$.
- **e)** if f is a one-to-one correspondence, then S_f is a one-to-one correspondence from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ and $S_{f^{-1}}$ is a one-to-one correspondence from $\mathcal{P}(B)$ to $\mathcal{P}(A)$. [*Hint:* Use parts (a)–(d).]

17. Prove that if f and g are functions from A to B and $S_f = S_g$ (using the definition in the preamble to Exercise 16), then $f(x) = g(x)$ for all $x \in A$.

18. Show that if n is an integer, then $n = \lceil n/2 \rceil + \lfloor n/2 \rfloor$.

19. For which real numbers x and y is it true that $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$?

20. For which real numbers x and y is it true that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$?

21. For which real numbers x and y is it true that $\lceil x + y \rceil = \lceil x \rceil + \lfloor y \rfloor$?

22. Prove that $\lfloor n/2 \rfloor \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$ for all integers n.

23. Prove that if m is an integer, then $\lfloor x \rfloor + \lfloor m - x \rfloor = m - 1$, unless x is an integer, in which case, it equals m.

24. Prove that if x is a real number, then $\lfloor \lfloor x/2 \rfloor /2 \rfloor = \lfloor x/4 \rfloor$.

25. Prove that if n is an odd integer, then $\lceil n^2/4 \rceil = (n^2 + 3)/4$.

26. Prove that if m and n are positive integers and x is a real number, then

$$\left\lfloor \frac{\lfloor x \rfloor + n}{m} \right\rfloor = \left\lfloor \frac{x + n}{m} \right\rfloor .$$

***27.** Prove that if m is a positive integer and x is a real number, then

$$\lfloor mx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{m} \right\rfloor + \left\lfloor x + \frac{2}{m} \right\rfloor + \cdots + \left\lfloor x + \frac{m-1}{m} \right\rfloor .$$

***28.** We define the **Ulam numbers** by setting $u_1 = 1$ and $u_2 = 2$. Furthermore, after determining whether the integers less than n are Ulam numbers, we set n equal to the next Ulam number if it can be written uniquely as the sum of two different Ulam numbers. Note that $u_3 = 3$, $u_4 = 4$, $u_5 = 6$, and $u_6 = 8$.

- **a)** Find the first 20 Ulam numbers.
- **b)** Prove that there are infinitely many Ulam numbers.

29. Determine the value of $\prod_{k=1}^{100} \frac{k+1}{k}$. (The notation used here for products is defined in the preamble to Exercise 45 in Section 2.4.)

***30.** Determine a rule for generating the terms of the sequence that begins 1, 3, 4, 8, 15, 27, 50, 92, …, and find the next four terms of the sequence.

***31.** Determine a rule for generating the terms of the sequence that begins 2, 3, 3, 5, 10, 13, 39, 43, 172, 177, 885, 891, …, and find the next four terms of the sequence.

32. Show that the set of irrational numbers is an uncountable set.

33. Show that the set S is a countable set if there is a function f from S to the positive integers such that $f^{-1}(j)$ is countable whenever j is a positive integer.

34. Show that the set of all finite subsets of the set of positive integers is a countable set.

****35.** Show that $|\mathbf{R} \times \mathbf{R}| = |\mathbf{R}|$. [*Hint:* Use the Schröder-Bernstein theorem to show that $|(0, 1) \times (0, 1)| = |(0, 1)|$. To construct an injection from $(0, 1) \times (0, 1)$ to $(0, 1)$, suppose that $(x, y) \in (0, 1) \times (0, 1)$. Map (x, y) to the number with decimal expansion formed by alternating between the digits in the decimal expansions of x and y, which do not end with an infinite string of 9s.]

****36.** Show that $\mathbf{C}$, the set of complex numbers, has the same cardinality as $\mathbf{R}$, the set of real numbers.

37. Find $\mathbf{A}^n$ if $\mathbf{A}$ is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

38. Show that if $\mathbf{A} = c\mathbf{I}$, where c is a real number and $\mathbf{I}$ is the $n \times n$ identity matrix, then $\mathbf{AB} = \mathbf{BA}$ whenever $\mathbf{B}$ is an $n \times n$ matrix.

39. Show that if $\mathbf{A}$ is a 2×2 matrix such that $\mathbf{AB} = \mathbf{BA}$ whenever $\mathbf{B}$ is a 2×2 matrix, then $\mathbf{A} = c\mathbf{I}$, where c is a real number and $\mathbf{I}$ is the 2×2 identity matrix.

40. Show that if $\mathbf{A}$ and $\mathbf{B}$ are invertible matrices and $\mathbf{AB}$ exists, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

41. Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathbf{0}$ be the $n \times n$ matrix all of whose entries are zero. Show that the following are true.

- **a)** $\mathbf{A} \odot \mathbf{0} = \mathbf{0} \odot \mathbf{A} = \mathbf{0}$
- **b)** $\mathbf{A} \vee \mathbf{0} = \mathbf{0} \vee \mathbf{A} = \mathbf{A}$
- **c)** $\mathbf{A} \wedge \mathbf{0} = \mathbf{0} \wedge \mathbf{A} = \mathbf{0}$

Computer Projects

Write programs with the specified input and output.

1. Given subsets A and B of a set with n elements, use bit strings to find $\overline{A}$, $A \cup B$, $A \cap B$, $A - B$, and $A \oplus B$.
2. Given multisets A and B from the same universal set, find $A \cup B$, $A \cap B$, $A - B$, and $A + B$ (see Section 2.2.5).
3. Given fuzzy sets A and B, find $\overline{A}$, $A \cup B$, and $A \cap B$ (see preamble to Exercise 73 of Section 2.2).
4. Given a function f from $\{1, 2, \ldots, n\}$ to the set of integers, determine whether f is one-to-one.
5. Given a function f from $\{1, 2, \ldots, n\}$ to itself, determine whether f is onto.
6. Given a bijection f from the set $\{1, 2, \ldots, n\}$ to itself, find f^{-1}.
7. Given an $m \times k$ matrix **A** and a $k \times n$ matrix **B**, find **AB**.
8. Given a square matrix **A** and a positive integer n, find $\mathbf{A}^n$.
9. Given a square matrix, determine whether it is symmetric.
10. Given two $m \times n$ Boolean matrices, find their meet and join.
11. Given an $m \times k$ Boolean matrix **A** and a $k \times n$ Boolean matrix **B**, find the Boolean product of **A** and **B**.
12. Given a square Boolean matrix **A** and a positive integer n, find $\mathbf{A}^{[n]}$.

Computations and Explorations

Use a computational program or programs you have written to do these exercises.

1. Given two finite sets, list all elements in the Cartesian product of these two sets.
2. Given a finite set, list all elements of its power set.
3. Calculate the number of one-to-one functions from a set S to a set T, where S and T are finite sets of various sizes. Can you determine a formula for the number of such functions? (We will find such a formula in Chapter 6.)
4. Calculate the number of onto functions from a set S to a set T, where S and T are finite sets of various sizes. Can you determine a formula for the number of such functions? (We will find such a formula in Chapter 8.)
5. Given an positive integer n, generate the first n Fibonacci numbers. Looking at the terms, formulate conjectures about them. (For instance look at their size, divisibility by different factors, or possible identities relating different terms.)

*6. Develop a collection of different rules for generating the terms of a sequence and a program for randomly selecting one of these rules and the particular sequence generated using these rules. Make this part of an interactive program that prompts for the next term of the sequence and determines whether the response is the intended next term.

Writing Projects

Respond to these with essays using outside sources.

1. Discuss how an axiomatic set theory can be developed to avoid Russell's paradox. (See Exercise 50 of Section 2.1.)
2. Research where the concept of a function first arose, and describe how this concept was first used.
3. Explain the different ways in which the *Encyclopedia of Integer Sequences* has been found useful. Also, describe a few of the more unusual sequences in this encyclopedia and how they arise.
4. Define the recently invented EKG sequence and describe some of its properties and open questions about it.
5. Look up the definition of a transcendental number. Explain how to show that such numbers exist and how such numbers can be constructed. Which famous numbers can be shown to be transcendental and for which famous numbers is it still unknown whether they are transcendental?
6. Expand the discussion of the continuum hypothesis in the text.

CHAPTER

6 Counting

Combinatorics, the study of arrangements of objects, is an important part of discrete mathematics. This subject was studied as long ago as the seventeenth century, when combinatorial questions arose in the study of gambling games. Enumeration, the counting of objects with certain properties, is an important part of combinatorics. We must count objects to solve many different types of problems. For instance, counting is used to determine the complexity of algorithms. Counting is also required to determine whether there are enough telephone numbers or Internet protocol addresses to meet demand. Recently, it has played a key role in mathematical biology, especially in sequencing DNA. Furthermore, counting techniques are used extensively when probabilities of events are computed.

The basic rules of counting, which we will study in Section 6.1, can solve a tremendous variety of problems. For instance, we can use these rules to enumerate the different telephone numbers possible in the United States, the allowable passwords on a computer system, and the different orders in which the runners in a race can finish. Another important combinatorial tool is the pigeonhole principle, which we will study in Section 6.2. This states that when objects are placed in boxes and there are more objects than boxes, then there is a box containing at least two objects. For instance, we can use this principle to show that among a set of 15 or more students, at least 3 were born on the same day of the week.

We can phrase many counting problems in terms of ordered or unordered arrangements of the objects of a set with or without repetitions. These arrangements, called permutations and combinations, are used in many counting problems. For instance, suppose the 100 top finishers on a competitive exam taken by 2000 students are invited to a banquet. We can count the possible sets of 100 students that will be invited, as well as the ways in which the top 10 prizes can be awarded.

Another problem in combinatorics involves generating all the arrangements of a specified kind. This is often important in computer simulations. We will devise algorithms to generate arrangements of various types.

6.1 The Basics of Counting

6.1.1 Introduction

Suppose that a password on a computer system consists of six, seven, or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain at least one digit. How many such passwords are there? The techniques needed to answer this question and a wide variety of other counting problems will be introduced in this section.

Counting problems arise throughout mathematics and computer science. For example, we must count the successful outcomes of experiments and all the possible outcomes of these experiments to determine probabilities of discrete events. We need to count the number of operations used by an algorithm to study its time complexity.

We will introduce the basic techniques of counting in this section. These methods serve as the foundation for almost all counting techniques.

6.1.2 Basic Counting Principles

Assessment

We first present two basic counting principles, the **product rule** and the **sum rule**. Then we will show how they can be used to solve many different counting problems.

The product rule applies when a procedure is made up of separate tasks.

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Extra Examples

Examples 1–10 show how the product rule is used.

EXAMPLE 1 A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution: The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees. ◀

EXAMPLE 2 The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Solution: The procedure of labeling a chair consists of two tasks, namely, assigning to the seat one of the 26 uppercase English letters, and then assigning to it one of the 100 possible integers. The product rule shows that there are $26 \cdot 100 = 2600$ different ways that a chair can be labeled. Therefore, the largest number of chairs that can be labeled differently is 2600. ◀

EXAMPLE 3 There are 32 computers in a data center in the cloud. Each of these computers has 24 ports. How many different computer ports are there in this data center?

Solution: The procedure of choosing a port consists of two tasks, first picking a computer and then picking a port on this computer. Because there are 32 ways to choose the computer and 24 ways to choose the port no matter which computer has been selected, the product rule shows that there are $32 \cdot 24 = 768$ ports. ◀

An extended version of the product rule is often useful. Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots, T_m$ in sequence. If each task T_i, $i = 1, 2, \ldots, n$, can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 \cdot n_2 \cdot \cdots \cdot n_m$ ways to carry out the procedure. This version of the product rule can be proved by mathematical induction from the product rule for two tasks (see Exercise 76).

EXAMPLE 4 How many different bit strings of length seven are there?

Solution: Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven. ◀

EXAMPLE 5 How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?

$\underbrace{\_\ \_\ \_}_{\text{26 choices for each letter}}\ \underbrace{\_\ \_\ \_}_{\text{10 choices for each digit}}$

Solution: There are 26 choices for each of the three uppercase English letters and 10 choices for each of the three digits. Hence, by the product rule there are a total of $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17{,}576{,}000$ possible license plates. ◀

EXAMPLE 6 **Counting Functions** How many functions are there from a set with m elements to a set with n elements?

Solution: A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain. Hence, by the product rule there are $n \cdot n \cdot \cdots \cdot n = n^m$ functions from a set with m elements to one with n elements. For example, there are $5^3 = 125$ different functions from a set with three elements to a set with five elements. ◀

EXAMPLE 7 **Counting One-to-One Functions** How many one-to-one functions are there from a set with m elements to one with n elements?

Solution: First note that when $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements.

Counting the number of onto functions is harder. We'll do this in Chapter 8.

Now let $m \le n$. Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of the function at a_1. Because the function is one-to-one, the value of the function at a_2 can be picked in $n - 1$ ways (because the value used for a_1 cannot be used again). In general, the value of the function at a_k can be chosen in $n - k + 1$ ways. By the product rule, there are $n(n - 1)(n - 2) \cdots (n - m + 1)$ one-to-one functions from a set with m elements to one with n elements.

For example, there are $5 \cdot 4 \cdot 3 = 60$ one-to-one functions from a set with three elements to a set with five elements. ◀

EXAMPLE 8 **The Telephone Numbering Plan** The *North American numbering plan (NANP)* specifies the format of telephone numbers in the U.S., Canada, and many other parts of North America. A telephone number in this plan consists of 10 digits, which are split into a three-digit area code, a three-digit office code, and a four-digit station code. Because of signaling considerations, there are certain restrictions on some of these digits. To specify the allowable format, let X denote a digit that can take any of the values 0 through 9, let N denote a digit that can take any of the values 2 through 9, and let Y denote a digit that must be a 0 or a 1. Two numbering plans, which will be called the old plan, and the new plan, will be discussed. (The old plan, in use in the 1960s, has been replaced by the new plan, but the recent rapid growth in demand for new numbers for mobile phones and devices will eventually make even this new plan obsolete. In this example, the letters used to represent digits follow the conventions of the *North American Numbering Plan*.) As will be shown, the new plan allows the use of more numbers.

Links

Current projections are that by 2038, it will be necessary to add one or more digits to North American telephone numbers.

In the old plan, the formats of the area code, office code, and station code are *NYX*, *NNX*, and *XXXX*, respectively, so that telephone numbers had the form *NYX-NNX-XXXX*. In the new plan, the formats of these codes are *NXX*, *NXX*, and *XXXX*, respectively, so that telephone numbers have the form *NXX-NXX-XXXX*. How many different North American telephone numbers are possible under the old plan and under the new plan?

Solution: By the product rule, there are $8 \cdot 2 \cdot 10 = 160$ area codes with format *NYX* and $8 \cdot 10 \cdot 10 = 800$ area codes with format *NXX*. Similarly, by the product rule, there are

$8 \cdot 8 \cdot 10 = 640$ office codes with format *NNX*. The product rule also shows that there are $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes with format *XXXX*.

Note that we have ignored restrictions that rule out N11 station codes for most area codes.

Consequently, applying the product rule again, it follows that under the old plan there are

$$160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000$$

different numbers available in North America. Under the new plan, there are

$$800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000$$

different numbers available. ◀

EXAMPLE 9 What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

```
k := 0
for i_1 := 1 to n_1
    for i_2 := 1 to n_2
            .
            .
            .
        for i_m := 1 to n_m
            k := k + 1
```

Solution: The initial value of k is zero. Each time the nested loop is traversed, 1 is added to k. Let T_i be the task of traversing the ith loop. Then the number of times the loop is traversed is the number of ways to do the tasks $T_1, T_2, \ldots, T_m$. The number of ways to carry out the task T_j, $j = 1, 2, \ldots, m$, is n_j, because the jth loop is traversed once for each integer i_j with $1 \le i_j \le n_j$. By the product rule, it follows that the nested loop is traversed $n_1 n_2 \cdots n_m$ times. Hence, the final value of k is $n_1 n_2 \cdots n_m$. ◀

EXAMPLE 10 **Counting Subsets of a Finite Set** Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.

Solution: Let S be a finite set. List the elements of S in arbitrary order. Recall from Section 2.2 that there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$. Namely, a subset of S is associated with the bit string with a 1 in the ith position if the ith element in the list is in the subset, and a 0 in this position otherwise. By the product rule, there are $2^{|S|}$ bit strings of length $|S|$. Hence, $|P(S)| = 2^{|S|}$. (Recall that we used mathematical induction to prove this fact in Example 10 of Section 5.1.) ◀

The product rule is often phrased in terms of sets in this way: If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set. To relate this to the product rule, note that the task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2, …, and an element in A_m. By the product rule it follows that

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|.$$

EXAMPLE 11 **DNA and Genomes** The hereditary information of a living organism is encoded using deoxyribonucleic acid (DNA), or in certain viruses, ribonucleic acid (RNA). DNA and RNA are extremely complex molecules, with different molecules interacting in a vast variety of ways to enable living process. For our purposes, we give only the briefest description of how DNA and RNA encode genetic information.

DNA molecules consist of two strands consisting of blocks known as nucleotides. Each nucleotide contains subcomponents called **bases**, each of which is adenine (A), cytosine (C), guanine (G), or thymine (T). The two strands of DNA are held together by hydrogen bonds connecting different bases, with A bonding only with T, and C bonding only with G. Unlike DNA, RNA is single stranded, with uracil (U) replacing thymine as a base. So, in DNA the possible base pairs are A-T and C-G, while in RNA they are A-U, and C-G. The DNA of a living creature consists of multiple pieces of DNA forming separate chromosomes. A **gene** is a segment of a DNA molecule that encodes a particular protein. The entirety of genetic information of an organism is called its **genome**.

Sequences of bases in DNA and RNA encode long chains of proteins called amino acids. There are 22 essential amino acids for human beings. We can quickly see that a sequence of at least three bases are needed to encode these 22 different amino acid. First note, that because there are four possibilities for each base in DNA, A, C, G, and T, by the product rule there are $4^2 = 16 < 22$ different sequences of two bases. However, there are $4^3 = 64$ different sequences of three bases, which provide enough different sequences to encode the 22 different amino acids (even after taking into account that several different sequences of three bases encode the same amino acid).

The DNA of simple living creatures such as algae and bacteria have between 10^5 and 10^7 links, where each link is one of the four possible bases. More complex organisms, such as insects, birds, and mammals, have between 10^8 and 10^{10} links in their DNA. So, by the product rule, there are at least 4^{10^5} different sequences of bases in the DNA of simple organisms and at least 4^{10^8} different sequences of bases in the DNA of more complex organisms. These are both incredibly huge numbers, which helps explain why there is such tremendous variability among living organisms. In the past several decades techniques have been developed for determining the genome of different organisms. The first step is to locate each gene in the DNA of an organism. The next task, called **gene sequencing**, is the determination of the sequence of links on each gene. (The specific sequence of links on these genes depends on the particular individual representative of a species whose DNA is analyzed.) For example, the human genome includes approximately 23,000 genes, each with 1000 or more links. Gene sequencing techniques take advantage of many recently developed algorithms and are based on numerous new ideas in combinatorics. Many mathematicians and computer scientists work on problems involving genomes, taking part in the fast moving fields of bioinformatics and computational biology. ◀

Soon it won't be that costly to have your own genetic code found.

We now introduce the sum rule.

> THE SUM RULE If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example 12 illustrates how the sum rule is used.

EXAMPLE 12 Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Solution: There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one is both a faculty member and a student. By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick this representative. ◀

We can extend the sum rule to more than two tasks. Suppose that a task can be done in one of n_1 ways, in one of n_2 ways, … , or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq m$. Then the number of ways to do the task is $n_1 + n_2 + \cdots + n_m$. This extended version of the sum rule is often useful in counting problems, as Examples 13 and 14 show. This version of the sum rule can be proved using mathematical induction from the sum rule for two sets. (See Exercise 75.)

EXAMPLE 13 A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution: The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project. ◀

EXAMPLE 14 What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

```
k := 0
for i_1 := 1 to n_1
    k := k + 1
for i_2 := 1 to n_2
    k := k + 1
        .
        .
        .
for i_m := 1 to n_m
    k := k + 1
```

Solution: The initial value of k is zero. This block of code is made up of m different loops. Each time a loop is traversed, 1 is added to k. To determine the value of k after this code has been executed, we need to determine how many times we traverse a loop. Note that there are n_i ways to traverse the ith loop. Because we only traverse one loop at a time, the sum rule shows that the final value of k, which is the number of ways to traverse one of the m loops is $n_1 + n_2 + \cdots + n_m$. ◀

The sum rule can be phrased in terms of sets as: If $A_1, A_2, \ldots, A_m$ are pairwise disjoint finite sets, then the number of elements in the union of these sets is the sum of the numbers of elements in the sets. To relate this to our statement of the sum rule, note there are $|A_i|$ ways to choose an element from A_i for $i = 1, 2, \ldots, m$. Because the sets are pairwise disjoint, when we select an element from one of the sets A_i, we do not also select an element from a different set A_j. Consequently, by the sum rule, because we cannot select an element from two of these sets at the same time, the number of ways to choose an element from one of the sets, which is the number of elements in the union, is

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m| \text{ when } A_i \cap A_j = \ \text{ for all } i, j.$$

This equality applies only when the sets in question are pairwise disjoint. The situation is much more complicated when these sets have elements in common. That situation will be briefly discussed later in this section and discussed in more depth in Chapter 8.

6.1.3 More Complex Counting Problems

Many counting problems cannot be solved using just the sum rule or just the product rule. However, many complicated counting problems can be solved using both of these rules in combination. We begin by counting the number of variable names in the programming language BASIC. (In the exercises, we consider the number of variable names in JAVA.) Then we will count the number of valid passwords subject to a particular set of restrictions.

EXAMPLE 15 In a version of the computer language BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. (An *alphanumeric* character is either one of the 26 English letters or one of the 10 digits.) Moreover, a variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC?

Extra Examples

Solution: Let V equal the number of different variable names in this version of BASIC. Let V_1 be the number of these that are one character long and V_2 be the number of these that are two characters long. Then by the sum rule, $V = V_1 + V_2$. Note that $V_1 = 26$, because a one-character variable name must be a letter. Furthermore, by the product rule there are $26 \cdot 36$ strings of length two that begin with a letter and end with an alphanumeric character. However, five of these are excluded, so $V_2 = 26 \cdot 36 - 5 = 931$. Hence, there are $V = V_1 + V_2 = 26 + 931 = 957$ different names for variables in this version of BASIC. ◀

EXAMPLE 16 Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively. By the sum rule, $P = P_6 + P_7 + P_8$. We will now find P_6, P_7, and P_8. Finding P_6 directly is difficult. To find P_6 it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits. By the product rule, the number of strings of six characters is 36^6, and the number of strings with no digits is 26^6. Hence,

$$P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$$

Similarly, we have

$$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920$$

and

$$\begin{aligned} P_8 &= 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 \\ &= 2{,}612{,}282{,}842{,}880. \end{aligned}$$

Consequently,

$$P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$$ ◀

Bit Number	0	1	2	3	4 · 8 · 16 · 24 · 31
Class A	0	netid			hostid
Class B	1	0	netid		hostid
Class C	1	1	0	netid	hostid
Class D	1	1	1	0	Multicast Address
Class E	1	1	1	1	0 · Address

FIGURE 1 **Internet addresses (IPv4).**

EXAMPLE 17

Links

Counting Internet Addresses In the Internet, which is made up of interconnected physical networks of computers, each computer (or more precisely, each network connection of a computer) is assigned an *Internet address*. In Version 4 of the Internet Protocol (IPv4), still in use today, an address is a string of 32 bits. It begins with a *network number* (*netid*). The netid is followed by a *host number* (*hostid*), which identifies a computer as a member of a particular network.

Three forms of addresses are used, with different numbers of bits used for netids and hostids. **Class A addresses**, used for the largest networks, consist of 0, followed by a 7-bit netid and a 24-bit hostid. **Class B addresses**, used for medium-sized networks, consist of 10, followed by a 14-bit netid and a 16-bit hostid. **Class C addresses**, used for the smallest networks, consist of 110, followed by a 21-bit netid and an 8-bit hostid. There are several restrictions on addresses because of special uses: 1111111 is not available as the netid of a Class A network, and the hostids consisting of all 0s and all 1s are not available for use in any network. A computer on the Internet has either a Class A, a Class B, or a Class C address. (Besides Class A, B, and C addresses, there are also Class D addresses, reserved for use in multicasting when multiple computers are addressed at a single time, consisting of 1110 followed by 28 bits, and Class E addresses, reserved for future use, consisting of 11110 followed by 27 bits. Neither Class D nor Class E addresses are assigned as the IPv4 address of a computer on the Internet.) Figure 1 illustrates IPv4 addressing. (Limitations on the number of Class A and Class B netids have made IPv4 addressing inadequate; IPv6, a new version of IP, uses 128-bit addresses to solve this problem.)

The lack of available IPv4 address has become a crisis!

How many different IPv4 addresses are available for computers on the Internet?

Solution: Let x be the number of available addresses for computers on the Internet, and let x_A, x_B, and x_C denote the number of Class A, Class B, and Class C addresses available, respectively. By the sum rule, $x = x_A + x_B + x_C$.

To find x_A, note that there are $2^7 - 1 = 127$ Class A netids, recalling that the netid 1111111 is unavailable. For each netid, there are $2^{24} - 2 = 16{,}777{,}214$ hostids, recalling that the hostids consisting of all 0s and all 1s are unavailable. Consequently, $x_A = 127 \cdot 16{,}777{,}214 = 2{,}130{,}706{,}178$.

To find x_B and x_C, note that there are $2^{14} = 16{,}384$ Class B netids and $2^{21} = 2{,}097{,}152$ Class C netids. For each Class B netid, there are $2^{16} - 2 = 65{,}534$ hostids, and for each Class C netid, there are $2^8 - 2 = 254$ hostids, recalling that in each network the hostids consisting of all 0s and all 1s are unavailable. Consequently, $x_B = 1{,}073{,}709{,}056$ and $x_C = 532{,}676{,}608$.

We conclude that the total number of IPv4 addresses available is $x = x_A + x_B + x_C = 2{,}130{,}706{,}178 + 1{,}073{,}709{,}056 + 532{,}676{,}608 = 3{,}737{,}091{,}842$. ◀

6.1.4 The Subtraction Rule (Inclusion–Exclusion for Two Sets)

Suppose that a task can be done in one of two ways, but some of the ways to do it are common to both ways. In this situation, we cannot use the sum rule to count the number of ways to do the task. If we add the number of ways to do the tasks in these two ways, we get an overcount of the total number of ways to do it, because the ways to do the task that are common to the two ways are counted twice.

Overcounting is perhaps the most common enumeration error.

To correctly count the number of ways to do the two tasks, we must subtract the number of ways that are counted twice. This leads us to an important counting rule.

THE SUBTRACTION RULE If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

The subtraction rule is also known as the **principle of inclusion–exclusion**, especially when it is used to count the number of elements in the union of two sets. Suppose that A_1 and A_2 are sets. Then, there are $|A_1|$ ways to select an element from A_1 and $|A_2|$ ways to select an element from A_2. The number of ways to select an element from A_1 or from A_2, that is, the number of ways to select an element from their union, is the sum of the number of ways to select an element from A_1 and the number of ways to select an element from A_2, minus the number of ways to select an element that is in both A_1 and A_2. Because there are $|A_1 \cup A_2|$ ways to select an element in either A_1 or in A_2, and $|A_1 \cap A_2|$ ways to select an element common to both sets, we have

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

This is the formula given in Section 2.2 for the number of elements in the union of two sets.

Example 18 illustrates how we can solve counting problems using the subtraction principle.

EXAMPLE 18 How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Extra Examples

Solution: Figure 2 illustrates the three counting problems we need to solve before we can apply the principle of inclusion–exclusion. We can construct a bit string of length eight that either starts with a 1 bit or ends with the two bits 00, by constructing a bit string of length eight beginning with a 1 bit or by constructing a bit string of length eight that ends with the two bits 00. We can construct a bit string of length eight that begins with a 1 in $2^7 = 128$ ways. This follows by the product rule, because the first bit can be chosen in only one way and each of the other seven bits can be chosen in two ways. Similarly, we can construct a bit string of length eight ending with the two bits 00, in $2^6 = 64$ ways. This follows by the product rule, because each of the first six bits can be chosen in two ways and the last two bits can be chosen in only one way.

Some of the ways to construct a bit string of length eight starting with a 1 are the same as the ways to construct a bit string of length eight that ends with the two bits 00. There are $2^5 = 32$ ways to construct such a string. This follows by the product rule, because the first bit can be chosen in only one way, each of the second through the sixth bits can be chosen in two ways, and the last two bits can be chosen in one way. Consequently, the number of bit strings of length eight that begin with a 1 or end with a 00, which equals the number of ways to construct a bit string of length eight that begins with a 1 or that ends with 00, equals $128 + 64 - 32 = 160$. ◀

$$\underline{1}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{2^7 = 128 \text{ ways}}$$

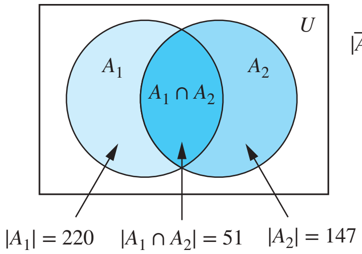
$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{2^6 = 64 \text{ ways}}\ \underline{0}\ \underline{0}$$

$$\underline{1}\ \underbrace{\_\ \_\ \_\ \_\ \_}_{2^5 = 32 \text{ ways}}\ \underline{0}\ \underline{0}$$

FIGURE 2 8-Bit strings starting with 1 or ending with 00.

We present an example that illustrates how the formulation of the principle of inclusion–exclusion can be used to solve counting problems.

EXAMPLE 19 A computer company receives 350 applications from college graduates for a job planning a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147



$$\begin{aligned}\overline{|A_1 \cup A_2|} &= |U| - |A_1 \cup A_2| \\ &= |U| - (|A_1| + |A_2| - |A_1 \cap A_2|) \\ &= 350 - (220 + 147 - 51) \\ &= 350 - 316 \\ &= 34\end{aligned}$$

FIGURE 3 **Applicants who majored in neither computer science nor business.**

majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Solution: To find the number of these applicants who majored neither in computer science nor in business, we can subtract the number of students who majored either in computer science or in business (or both) from the total number of applicants. Let A_1 be the set of students who majored in computer science and A_2 the set of students who majored in business. Then $A_1 \cup A_2$ is the set of students who majored in computer science or business (or both), and $A_1 \cap A_2$ is the set of students who majored both in computer science and in business. By the subtraction rule the number of students who majored either in computer science or in business (or both) equals

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 220 + 147 - 51 = 316.$$

We conclude that $350 - 316 = 34$ of the applicants majored neither in computer science nor in business. A Venn diagram for this example is shown in Figure 3. ◀

The subtraction rule, or the principle of inclusion–exclusion, can be generalized to find the number of ways to do one of n different tasks or, equivalently, to find the number of elements in the union of n sets, whenever n is a positive integer. We will study the inclusion–exclusion principle and some of its many applications in Chapter 8.

6.1.5 The Division Rule

We have introduced the product, sum, and subtraction rules for counting. You may wonder whether there is also a division rule for counting. In fact, there is such a rule, which can be useful when solving certain types of enumeration problems.

THE DIVISION RULE There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

We can restate the division rule in terms of sets: "If the finite set A is the union of n pairwise disjoint subsets each with d elements, then $n = |A|/d$."

We can also formulate the division rule in terms of functions: "If f is a function from A to B where A and B are finite sets, and that for every value $y \in B$ there are exactly d values $x \in A$ such that $f(x) = y$ (in which case, we say that f is d-to-one), then $|B| = |A|/d$."

Remark: The division rule comes in handy when it appears that a task can be done in n different ways, but it turns out that for each way of doing the task, there are d equivalent ways of doing it. Under these circumstances, we can conclude that there are n/d inequivalent ways of doing the task.

We illustrate the use of the division rule for counting with two examples.

EXAMPLE 20 Suppose that an automated system has been developed that counts the legs of cows in a pasture. Suppose that this system has determined that in a farmer's pasture there are exactly 572 legs. How many cows are there is this pasture, assuming that each cow has four legs and that there are no other animals present?

Solution: Let n be the number of cow legs counted in a pasture. Because each cow has four legs, by the division rule we know that the pasture contains $n/4$ cows. Consequently, the pasture with 572 cow legs has $572/4 = 143$ cows in it. ◀

EXAMPLE 21 How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

Solution: We arbitrarily select a seat at the table and label it seat 1. We number the rest of the seats in numerical order, proceeding clockwise around the table. Note that are four ways to select the person for seat 1, three ways to select the person for seat 2, two ways to select the person for seat 3, and one way to select the person for seat 4. Thus, there are $4! = 24$ ways to order the given four people for these seats. However, each of the four choices for seat 1 leads to the same arrangement, as we distinguish two arrangements only when one of the people has a different immediate left or immediate right neighbor. Because there are four ways to choose the person for seat 1, by the division rule there are $24/4 = 6$ different seating arrangements of four people around the circular table. ◀

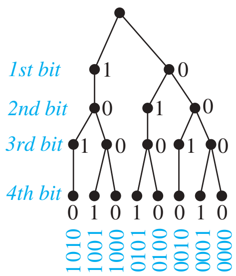


FIGURE 4 **Bit strings of length four without consecutive 1s.**

6.1.6 Tree Diagrams

Counting problems can be solved using **tree diagrams**. A tree consists of a root, a number of branches leaving the root, and possible additional branches leaving the endpoints of other branches. (We will study trees in detail in Chapter 11.) To use trees in counting, we use a branch to represent each possible choice. We represent the possible outcomes by the leaves, which are the endpoints of branches not having other branches starting at them.

Note that when a tree diagram is used to solve a counting problem, the number of choices of which branch to follow to reach a leaf can vary as in Example 22.

EXAMPLE 22 How many bit strings of length four do not have two consecutive 1s?

Solution: The tree diagram in Figure 4 displays all bit strings of length four without two consecutive 1s. We see that there are eight bit strings of length four without two consecutive 1s. ◀

EXAMPLE 23 A playoff between two teams consists of at most five games. The first team that wins three games wins the playoff. In how many different ways can the playoff occur?

Solution: The tree diagram in Figure 5 displays all the ways the playoff can proceed, with the winner of each game shown. We see that there are 20 different ways for the playoff to occur. ◀

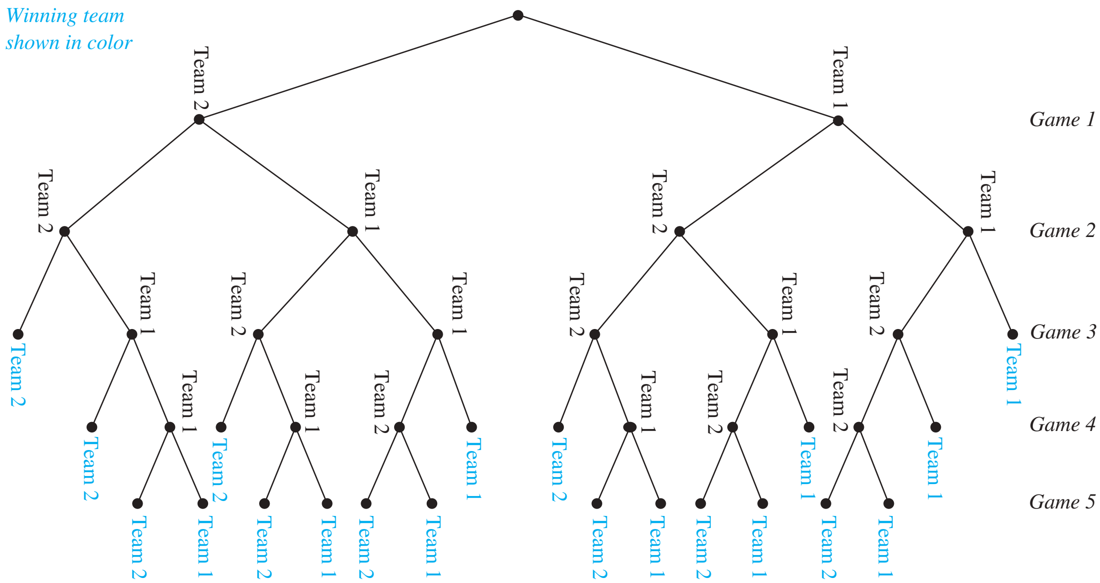


FIGURE 5 **Best three games out of five playoffs.**

EXAMPLE 24 Suppose that "I Love New Jersey" T-shirts come in five different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in four colors, white, red, green, and black, except for XL, which comes only in red, green, and black, and XXL, which comes only in green and black. How many different shirts does a souvenir shop have to stock to have at least one of each available size and color of the T-shirt?

Solution: The tree diagram in Figure 6 displays all possible size and color pairs. It follows that the souvenir shop owner needs to stock 17 different T-shirts. ◀

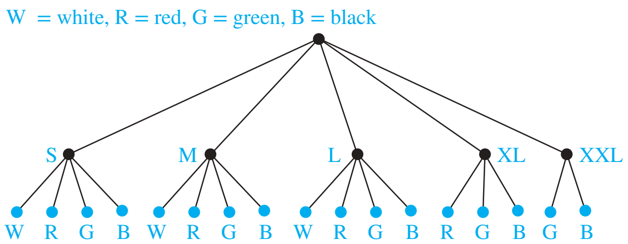


FIGURE 6 **Counting varieties of T-shirts.**

Exercises

1. There are 18 mathematics majors and 325 computer science majors at a college.

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

3. A multiple-choice test contains 10 questions. There are four possible answers for each question.

a) In how many ways can a student answer the questions on the test if the student answers every question?

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

5. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. How many different pairs of airlines can you choose on which to book a trip from New York to San Francisco via Denver, when you pick an airline for the flight to Denver and an airline for the continuation flight to San Francisco?

6. There are four major auto routes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?

7. How many different three-letter initials can people have?

8. How many different three-letter initials with none of the letters repeated can people have?

9. How many different three-letter initials are there that begin with an A?

10. How many bit strings are there of length eight?

11. How many bit strings of length ten both begin and end with a 1?

12. How many bit strings are there of length six or less, not counting the empty string?

13. How many bit strings with length not exceeding n, where n is a positive integer, consist entirely of 1s, not counting the empty string?

14. How many bit strings of length n, where n is a positive integer, start and end with 1s?

15. How many strings are there of lowercase letters of length four or less, not counting the empty string?

16. How many strings are there of four lowercase letters that have the letter x in them?

17. How many strings of five ASCII characters contain the character @ ("at" sign) at least once? [*Note:* There are 128 different ASCII characters.]

18. How many 5-element DNA sequences

a) end with A?
b) start with T and end with G?
c) contain only A and T?
d) do not contain C?

19. How many 6-element RNA sequences

a) do not contain U?
b) end with GU?
c) start with C?
d) contain only A or U?

20. How many positive integers between 5 and 31

a) are divisible by 3? Which integers are these?
b) are divisible by 4? Which integers are these?
c) are divisible by 3 and by 4? Which integers are these?

21. How many positive integers between 50 and 100

a) are divisible by 7? Which integers are these?
b) are divisible by 11? Which integers are these?
c) are divisible by both 7 and 11? Which integers are these?

22. How many positive integers less than 1000

a) are divisible by 7?
b) are divisible by 7 but not by 11?
c) are divisible by both 7 and 11?
d) are divisible by either 7 or 11?
e) are divisible by exactly one of 7 and 11?
f) are divisible by neither 7 nor 11?
g) have distinct digits?
h) have distinct digits and are even?

23. How many positive integers between 100 and 999 inclusive

a) are divisible by 7?
b) are odd?
c) have the same three decimal digits?
d) are not divisible by 4?
e) are divisible by 3 or 4?
f) are not divisible by either 3 or 4?
g) are divisible by 3 but not by 4?
h) are divisible by 3 and 4?

24. How many positive integers between 1000 and 9999 inclusive

a) are divisible by 9?
b) are even?
c) have distinct digits?
d) are not divisible by 3?
e) are divisible by 5 or 7?
f) are not divisible by either 5 or 7?
g) are divisible by 5 but not by 7?
h) are divisible by 5 and 7?

25. How many strings of three decimal digits

a) do not contain the same digit three times?
b) begin with an odd digit?
c) have exactly two digits that are 4s?

26. How many strings of four decimal digits

a) do not contain the same digit twice?
b) end with an even digit?
c) have exactly three digits that are 9s?

27. A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

28. How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

29. How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

30. How many license plates can be made using either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?

31. How many license plates can be made using either two or three uppercase English letters followed by either two or three digits?

32. How many strings of eight uppercase English letters are there

a) if letters can be repeated?
b) if no letter can be repeated?
c) that start with X, if letters can be repeated?
d) that start with X, if no letter can be repeated?
e) that start and end with X, if letters can be repeated?
f) that start with the letters BO (in that order), if letters can be repeated?
g) that start and end with the letters BO (in that order), if letters can be repeated?
h) that start or end with the letters BO (in that order), if letters can be repeated?

33. How many strings of eight English letters are there

a) that contain no vowels, if letters can be repeated?
b) that contain no vowels, if letters cannot be repeated?
c) that start with a vowel, if letters can be repeated?
d) that start with a vowel, if letters cannot be repeated?
e) that contain at least one vowel, if letters can be repeated?
f) that contain exactly one vowel, if letters can be repeated?
g) that start with X and contain at least one vowel, if letters can be repeated?
h) that start and end with X and contain at least one vowel, if letters can be repeated?

34. How many different functions are there from a set with 10 elements to sets with the following numbers of elements?

a) 2 **b)** 3 **c)** 4 **d)** 5

35. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a) 4 **b)** 5 **c)** 6 **d)** 7

36. How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$?

37. How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$

a) that are one-to-one?
b) that assign 0 to both 1 and n?
c) that assign 1 to exactly one of the positive integers less than n?

38. How many partial functions (see Section 2.3) are there from a set with five elements to sets with each of these number of elements?

a) 1 **b)** 2 **c)** 5 **d)** 9

39. How many partial functions (see Definition 13 of Section 2.3) are there from a set with m elements to a set with n elements, where m and n are positive integers?

40. How many subsets of a set with 100 elements have more than one element?

41. A **palindrome** is a string whose reversal is identical to the string. How many bit strings of length n are palindromes?

42. How many 4-element DNA sequences

a) do not contain the base T?
b) contain the sequence ACG?
c) contain all four bases A, T, C, and G?
d) contain exactly three of the four bases A, T, C, and G?

43. How many 4-element RNA sequences

a) contain the base U?
b) do not contain the sequence CUG?
c) do not contain all four bases A, U, C, and G?
d) contain exactly two of the four bases A, U, C, and G?

44. On each of the 22 work days in a particular month, every employee of a start-up venture was sent a company communication. If a total of 4642 total company communications were sent, how many employees does the company have, assuming that no staffing changes were made that month?

45. At a large university, 434 freshmen, 883 sophomores, and 43 juniors are enrolled in an introductory algorithms course. How many sections of this course need to be scheduled to accommodate all these students if each section contains 34 students?

46. How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?

47. How many ways are there to seat six people around a circular table where two seatings are considered the same when everyone has the same two neighbors without regard to whether they are right or left neighbors?

48. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if

a) the bride must be in the picture?
b) both the bride and groom must be in the picture?
c) exactly one of the bride and the groom is in the picture?

49. In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if

a) the bride must be next to the groom?
b) the bride is not next to the groom?
c) the bride is positioned somewhere to the left of the groom?

50. How many bit strings of length seven either begin with two 0s or end with three 1s?

51. How many bit strings of length 10 either begin with three 0s or end with two 0s?

***52.** How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

****53.** How many bit strings of length eight contain either three consecutive 0s or four consecutive 1s?

54. Every student in a discrete mathematics class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class if there are 38 computer science majors (including joint majors), 23 mathematics majors (including joint majors), and 7 joint majors?

55. How many positive integers not exceeding 100 are divisible either by 4 or by 6?

56. How many different initials can someone have if a person has at least two, but no more than five, different initials? Assume that each initial is one of the 26 uppercase letters of the English language.

57. Suppose that a password for a computer system must have at least 8, but no more than 12, characters, where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the six special characters $*$, $>$, $<$, !, $+$, and $=$.

a) How many different passwords are available for this computer system?

b) How many of these passwords contain at least one occurrence of at least one of the six special characters?

c) Using your answer to part (a), determine how long it takes a hacker to try every possible password, assuming that it takes one nanosecond for a hacker to check each possible password.

58. The name of a variable in the C programming language is a string that can contain uppercase letters, lowercase letters, digits, or underscores. Further, the first character in the string must be a letter, either uppercase or lowercase, or an underscore. If the name of a variable is determined by its first eight characters, how many different variables can be named in C? (Note that the name of a variable may contain fewer than eight characters.)

59. The name of a variable in the JAVA programming language is a string of between 1 and 65,535 characters, inclusive, where each character can be an uppercase or a lowercase letter, a dollar sign, an underscore, or a digit, except that the first character must not be a digit. Determine the number of different variable names in JAVA.

60. The International Telecommunications Union (ITU) specifies that a telephone number must consist of a country code with between 1 and 3 digits, except that the code 0 is not available for use as a country code, followed by a number with at most 15 digits. How many available possible telephone numbers are there that satisfy these restrictions?

61. Suppose that at some future time every telephone in the world is assigned a number that contains a country code 1 to 3 digits long, that is, of the form *X*, *XX*, or *XXX*, followed by a 10-digit telephone number of the form *NXX-NXX-XXXX* (as described in Example 8). How many different telephone numbers would be available worldwide under this numbering plan?

62. A key in the Vigenère cryptosystem is a string of English letters, where the case of the letters does not matter. How many different keys for this cryptosystem are there with three, four, five, or six letters?

63. A wired equivalent privacy (WEP) key for a wireless fidelity (WiFi) network is a string of either 10, 26, or 58 hexadecimal digits. How many different WEP keys are there?

64. Suppose that p and q are prime numbers and that $n = pq$. Use the principle of inclusion–exclusion to find the number of positive integers not exceeding n that are relatively prime to n.

65. Use the principle of inclusion–exclusion to find the number of positive integers less than 1,000,000 that are not divisible by either 4 or by 6.

66. Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s.

67. How many ways are there to arrange the letters a, b, c, and d such that a is not followed immediately by b?

68. Use a tree diagram to find the number of ways that the World Series can occur, where the first team that wins four games out of seven wins the series.

69. Use a tree diagram to determine the number of subsets of $\{3, 7, 9, 11, 24\}$ with the property that the sum of the elements in the subset is less than 28.

70. a) Suppose that a store sells six varieties of soft drinks: cola, ginger ale, orange, root beer, lemonade, and cream soda. Use a tree diagram to determine the number of different types of bottles the store must stock to have all varieties available in all size bottles if all varieties are available in 12-ounce bottles, all but lemonade are available in 20-ounce bottles, only cola and ginger ale are available in 32-ounce bottles, and all but lemonade and cream soda are available in 64-ounce bottles?

b) Answer the question in part (a) using counting rules.

71. a) Suppose that a popular style of running shoe is available for both men and women. The woman's shoe comes in sizes 6, 7, 8, and 9, and the man's shoe comes in sizes 8, 9, 10, 11, and 12. The man's shoe comes in white and black, while the woman's shoe comes in white, red, and black. Use a tree diagram to determine the number of different shoes that a store has to stock to have at least one pair of this type of running shoe for all available sizes and colors for both men and women.

b) Answer the question in part (a) using counting rules.

72. Determine the number of matches played in a single-elimination tournament with n players, where for each game between two players the winner goes on, but the loser is eliminated.

73. Determine the minimum and the maximum number of matches that can be played in a double-elimination tournament with n players, where after each game between two players, the winner goes on and the loser goes on if and only if this is not a second loss.

***74.** Use the product rule to show that there are 2^{2^n} different truth tables for propositions in n variables.

75. Use mathematical induction to prove the sum rule for m tasks from the sum rule for two tasks.

76. Use mathematical induction to prove the product rule for m tasks from the product rule for two tasks.

77. How many diagonals does a convex polygon with n sides have? (Recall that a polygon is convex if every line segment connecting two points in the interior or boundary of the polygon lies entirely within this set and that a diagonal of a polygon is a line segment connecting two vertices that are not adjacent.)

78. Data are transmitted over the Internet in **datagrams**, which are structured blocks of bits. Each datagram contains header information organized into a maximum of 14 different fields (specifying many things, including the source and destination addresses) and a data area that contains the actual data that are transmitted. One of the 14 header fields is the **header length field** (denoted by HLEN), which is specified by the protocol to be 4 bits long and that specifies the header length in terms of 32-bit blocks of bits. For example, if HLEN $= 0110$, the header is made up of six 32-bit blocks. Another of the 14 header fields is the 16-bit-long **total length field** (denoted by TOTAL LENGTH), which specifies the length in bits of the entire datagram, including both the header fields and the data area. The length of the data area is the total length of the datagram minus the length of the header.

a) The largest possible value of TOTAL LENGTH (which is 16 bits long) determines the maximum total length in octets (blocks of 8 bits) of an Internet datagram. What is this value?

b) The largest possible value of HLEN (which is 4 bits long) determines the maximum total header length in 32-bit blocks. What is this value? What is the maximum total header length in octets?

c) The minimum (and most common) header length is 20 octets. What is the maximum total length in octets of the data area of an Internet datagram?

d) How many different strings of octets in the data area can be transmitted if the header length is 20 octets and the total length is as long as possible?

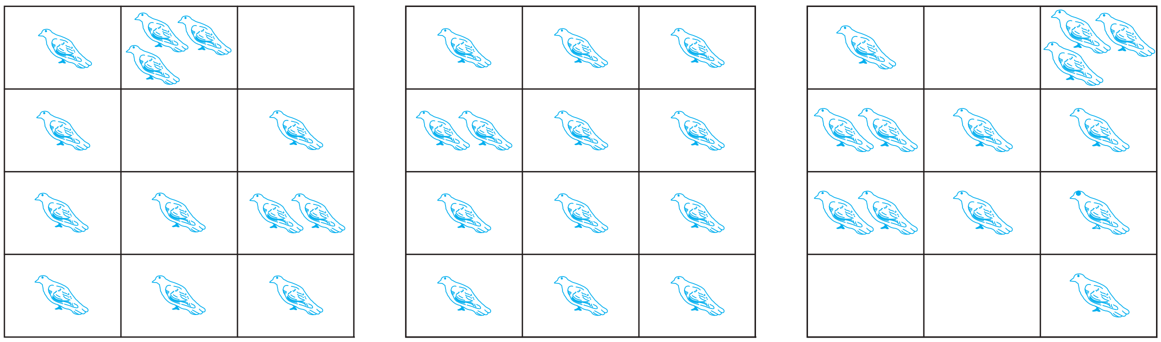
6.2 The Pigeonhole Principle

6.2.1 Introduction

Links

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it. To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most 19 pigeons, one per hole, could be accommodated. This illustrates a general principle called the **pigeonhole principle**, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it (see Figure 1). This principle is extremely useful; it applies to much more than pigeons and pigeonholes.

THEOREM 1 **THE PIGEONHOLE PRINCIPLE** If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.



(a) (b) (c)

FIGURE 1 **There are more pigeons than pigeonholes.**

Proof: We prove the pigeonhole principle using a proof by contraposition. Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k. This is a contradiction, because there are at least $k + 1$ objects. ◁

The pigeonhole principle is also called the **Dirichlet drawer principle**, after the nineteenth-century German mathematician G. Lejeune Dirichlet, who often used this principle in his work. (Dirichlet was not the first person to use this principle; a demonstration that there were at least two Parisians with the same number of hairs on their heads dates back to the 17th century—see Exercise 35.) It is an important additional proof technique supplementing those we have developed in earlier chapters. We introduce it in this chapter because of its many important applications to combinatorics.

We will illustrate the usefulness of the pigeonhole principle. We first show that it can be used to prove a useful corollary about functions.

COROLLARY 1 A function f from a set with $k + 1$ or more elements to a set with k elements is not one-to-one.

Proof: Suppose that for each element y in the codomain of f we have a box that contains all elements x of the domain of f such that $f(x) = y$. Because the domain contains $k + 1$ or more elements and the codomain contains only k elements, the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain. This means that f cannot be one-to-one. ◁

Examples 1–3 show how the pigeonhole principle is used.

EXAMPLE 1 Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays. ◀

EXAMPLE 2 In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet. ◀

EXAMPLE 3 How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score. ◀

Links

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G. LEJEUNE DIRICHLET (1805–1859) G. Lejeune Dirichlet was born into a Belgian family living near Cologne, Germany. His father was a postmaster. He became passionate about mathematics at a young age. He was spending all his spare money on mathematics books by the time he entered secondary school in Bonn at the age of 12. At 14 he entered the Jesuit College in Cologne, and at 16 he began his studies at the University of Paris. In 1825 he returned to Germany and was appointed to a position at the University of Breslau. In 1828 he moved to the University of Berlin. In 1855 he was chosen to succeed Gauss at the University of Göttingen. Dirichlet is said to be the first person to master Gauss's *Disquisitiones Arithmeticae*, which appeared 20 years earlier. He is said to have kept a copy at his side even when he traveled. Dirichlet made many important discoveries in number theory, including the theorem that there are infinitely many primes in arithmetical progressions $an + b$ when a and b are relatively prime. He proved the $n = 5$ case of Fermat's last theorem, that there are no nontrivial solutions in integers to $x^5 + y^5 = z^5$. Dirichlet also made many contributions to analysis. Dirichlet was considered to be an excellent teacher who could explain ideas with great clarity. He was married to Rebecka Mendelssohn, one of the sisters of the composer Felix Mendelssohn.

The pigeonhole principle is a useful tool in many proofs, including proofs of surprising results, such as that given in Example 4.

EXAMPLE 4 Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Extra Examples

Solution: Let n be a positive integer. Consider the $n+1$ integers $1, 11, 111, \dots, 11\dots1$ (where the last integer in this list is the integer with $n+1$ 1s in its decimal expansion). Note that there are n possible remainders when an integer is divided by n. Because there are $n+1$ integers in this list, by the pigeonhole principle there must be two with the same remainder when divided by n. The larger of these integers less the smaller one is a multiple of n, which has a decimal expansion consisting entirely of 0s and 1s. ◀

6.2.2 The Generalized Pigeonhole Principle

The pigeonhole principle states that there must be at least two objects in the same box when there are more objects than boxes. However, even more can be said when the number of objects exceeds a multiple of the number of boxes. For instance, among any set of 21 decimal digits there must be 3 that are the same. This follows because when 21 objects are distributed into 10 boxes, one box must have more than 2 objects.

THEOREM 2 **THE GENERALIZED PIGEONHOLE PRINCIPLE** If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: We will use a proof by contraposition. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then, the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < (N/k) + 1$ has been used. Thus, the total number of objects is less than N. This completes the proof by contraposition. ◁

A common type of problem asks for the minimum number of objects such that at least r of these objects must be in one of k boxes when these objects are distributed among the boxes. When we have N objects, the generalized pigeonhole principle tells us there must be at least r objects in one of the boxes as long as $\lceil N/k \rceil \geq r$. The smallest integer N with $N/k > r - 1$, namely, $N = k(r-1) + 1$, is the smallest integer satisfying the inequality $\lceil N/k \rceil \geq r$. Could a smaller value of N suffice? The answer is no, because if we had $k(r-1)$ objects, we could put $r-1$ of them in each of the k boxes and no box would have at least r objects.

When thinking about problems of this type, it is useful to consider how you can avoid having at least r objects in one of the boxes as you add successive objects. To avoid adding a rth object to any box, you eventually end up with $r-1$ objects in each box. There is no way to add the next object without putting an rth object in that box.

Examples 5–8 illustrate how the generalized pigeonhole principle is applied.

EXAMPLE 5 Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month. ◀

EXAMPLE 6 What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Extra Examples

Solution: The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$. The smallest such integer is $N = 5 \cdot 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade. ◀

EXAMPLE 7 **a)** How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are selected?
b) How many must be selected from a standard deck of 52 cards to guarantee that at least three hearts are selected?

A standard deck of 52 cards has 13 kinds of cards, with four cards of each of kind, one in each of the four suits, hearts, diamonds, spades, and clubs.

Solution: **a)** Suppose there are four boxes, one for each suit, and as cards are selected they are placed in the box reserved for cards of that suit. Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least one box containing at least $\lceil N/4 \rceil$ cards. Consequently, we know that at least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$, so nine cards suffice. Note that if eight cards are selected, it is possible to have two cards of each suit, so more than eight cards are needed. Consequently, nine cards must be selected to guarantee that at least three cards of one suit are chosen. One good way to think about this is to note that after the eighth card is chosen, there is no way to avoid having a third card of some suit.

b) We do not use the generalized pigeonhole principle to answer this question, because we want to make sure that there are three hearts, not just three cards of one suit. Note that in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next three cards will be all hearts, so we may need to select 42 cards to get three hearts. ◀

EXAMPLE 8 What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers? (Assume that telephone numbers are of the form *NXX-NXX-XXXX*, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

Solution: There are eight million different phone numbers of the form *NXX-XXXX* (as shown in Example 8 of Section 6.1). Hence, by the generalized pigeonhole principle, among 25 million telephones, at least $\lceil 25{,}000{,}000/8{,}000{,}000 \rceil = 4$ of them must have identical phone numbers. Hence, at least four area codes are required to ensure that all 10-digit numbers are different. ◀

Example 9, although not an application of the generalized pigeonhole principle, makes use of similar principles.

EXAMPLE 9 Suppose that a computer science laboratory has 15 workstations and 10 servers. A cable can be used to directly connect a workstation to a server. For each server, only one direct connection to that server can be active at any time. We want to guarantee that at any time any set of 10 or fewer workstations can simultaneously access different servers via direct connections. Although we could do this by connecting every workstation directly to every server (using 150 connections), what is the minimum number of direct connections needed to achieve this goal?

Solution: Suppose that we label the workstations $W_1, W_2, \ldots, W_{15}$ and the servers $S_1, S_2, \ldots, S_{10}$. First, we would like to find a way for there to be far fewer than 150 direct connections between workstations and servers to achieve our goal. One promising approach is to directly connect W_k to S_k for $k = 1, 2, \ldots, 10$ and then to connect each of $W_{11}, W_{12}, W_{13}, W_{14}$, and W_{15} to all

10 servers. This gives us a total of $10 + 5 \cdot 10 = 60$ direct connections. We need to determine whether with this configuration any set of 10 or fewer workstations can simultaneously access different servers. We note that if workstation W_j is included with $1 \le j \le 10$, it can access server S_j, and for each workstation W_k with $k \ge 11$ included, there must be a corresponding workstation W_j with $1 \le j \le 10$ not included, so W_k can access server S_j. (This follows because there are at least as many available servers S_j as there are workstations W_j with $1 \le j \le 10$ not included.) So, any set of 10 or fewer workstations are able to simultaneously access different servers.

But can we use fewer than 60 direct connections? Suppose there are fewer than 60 direct connections between workstations and servers. Then some server would be connected to at most $\lfloor 59/10 \rfloor = 5$ workstations. (If all servers were connected to at least six workstations, there would be at least $6 \cdot 10 = 60$ direct connections.) This means that the remaining nine servers are not enough for the other 10 or more workstations to simultaneously access different servers. Consequently, at least 60 direct connections are needed. It follows that 60 is the answer. ◀

6.2.3 Some Elegant Applications of the Pigeonhole Principle

In many interesting applications of the pigeonhole principle, the objects to be placed in boxes must be chosen in a clever way. A few such applications will be described here.

EXAMPLE 10 During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution: Let a_j be the number of games played on or before the jth day of the month. Then $a_1, a_2, \ldots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \le a_j \le 45$. Moreover, $a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$ is also an increasing sequence of distinct positive integers, with $15 \le a_j + 14 \le 59$.

The 60 positive integers $a_1, a_2, \ldots, a_{30}, a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$ are all less than or equal to 59. Hence, by the pigeonhole principle two of these integers are equal. Because the integers $a_j, j = 1, 2, \ldots, 30$ are all distinct and the integers $a_j + 14, j = 1, 2, \ldots, 30$ are all distinct, there must be indices i and j with $a_i = a_j + 14$. This means that exactly 14 games were played from day $j + 1$ to day i. ◀

EXAMPLE 11 Show that among any $n + 1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Solution: Write each of the $n + 1$ integers $a_1, a_2, \ldots, a_{n+1}$ as a power of 2 times an odd integer. In other words, let $a_j = 2^{k_j} q_j$ for $j = 1, 2, \ldots, n + 1$, where k_j is a nonnegative integer and q_j is odd. The integers $q_1, q_2, \ldots, q_{n+1}$ are all odd positive integers less than $2n$. Because there are only n odd positive integers less than $2n$, it follows from the pigeonhole principle that two of the integers $q_1, q_2, \ldots, q_{n+1}$ must be equal. Therefore, there are distinct integers i and j such that $q_i = q_j$. Let q be the common value of q_i and q_j. Then, $a_i = 2^{k_i} q$ and $a_j = 2^{k_j} q$. It follows that if $k_i < k_j$, then a_i divides a_j; while if $k_i > k_j$, then a_j divides a_i. ◀

A clever application of the pigeonhole principle shows the existence of an increasing or a decreasing subsequence of a certain length in a sequence of distinct integers. We review some definitions before this application is presented. Suppose that $a_1, a_2, \ldots, a_N$ is a sequence of real numbers. A **subsequence** of this sequence is a sequence of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$, where $1 \le i_1 < i_2 < \cdots < i_m \le N$. Hence, a subsequence is a sequence obtained from the original sequence by including some of the terms of the original sequence in their original order, and perhaps not including other terms. A sequence is called **strictly increasing** if each term is larger than the one that precedes it, and it is called **strictly decreasing** if each term is smaller than the one that precedes it.

THEOREM 3 Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

We give an example before presenting the proof of Theorem 3.

EXAMPLE 12 The sequence 8, 11, 9, 1, 4, 6, 12, 10, 5, 7 contains 10 terms. Note that $10 = 3^2 + 1$. There are four strictly increasing subsequences of length four, namely, 1, 4, 6, 12; 1, 4, 6, 7; 1, 4, 6, 10; and 1, 4, 5, 7. There is also a strictly decreasing subsequence of length four, namely, 11, 9, 6, 5. ◀

The proof of the theorem will now be given.

Proof: Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of $n^2 + 1$ distinct real numbers. Associate an ordered pair with each term of the sequence, namely, associate (i_k, d_k) to the term a_k, where i_k is the length of the longest increasing subsequence starting at a_k, and d_k is the length of the longest decreasing subsequence starting at a_k.

Suppose that there are no increasing or decreasing subsequences of length $n + 1$. Then i_k and d_k are both positive integers less than or equal to n, for $k = 1, 2, \ldots, n^2 + 1$. Hence, by the product rule there are n^2 possible ordered pairs for (i_k, d_k). By the pigeonhole principle, two of these $n^2 + 1$ ordered pairs are equal. In other words, there exist terms a_s and a_t, with $s < t$ such that $i_s = i_t$ and $d_s = d_t$. We will show that this is impossible. Because the terms of the sequence are distinct, either $a_s < a_t$ or $a_s > a_t$. If $a_s < a_t$, then, because $i_s = i_t$, an increasing subsequence of length $i_t + 1$ can be built starting at a_s, by taking a_s followed by an increasing subsequence of length i_t beginning at a_t. This is a contradiction. Similarly, if $a_s > a_t$, the same reasoning shows that d_s must be greater than d_t, which is a contradiction. ◁

Links

The final example shows how the generalized pigeonhole principle can be applied to an important part of combinatorics called **Ramsey theory**, after the English mathematician F. P. Ramsey. In general, Ramsey theory deals with the distribution of subsets of elements of sets.

EXAMPLE 13 Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that there are either three mutual friends or three mutual enemies in the group.

Solution: Let A be one of the six people. Of the five other people in the group, there are either three or more who are friends of A, or three or more who are enemies of A. This follows from

Links

Courtesy of Stephen Frank Burch

FRANK PLUMPTON RAMSEY (1903–1930) Frank Plumpton Ramsey, son of the president of Magdalene College, Cambridge, was educated at Winchester and Trinity Colleges. After graduating in 1923, he was elected a fellow of King's College, Cambridge, where he spent the remainder of his life. Ramsey made important contributions to mathematical logic. What we now call Ramsey theory began with his clever combinatorial arguments, published in the paper "On a Problem of Formal Logic." Ramsey also made contributions to the mathematical theory of economics. He was noted as an excellent lecturer on the foundations of mathematics. According to one of his brothers, he was interested in almost everything, including English literature and politics. Ramsey was married and had two daughters. His death at the age of 26 resulting from chronic liver problems deprived the mathematical community and Cambridge University of a brilliant young scholar.

the generalized pigeonhole principle, because when five objects are divided into two sets, one of the sets has at least $\lceil 5/2 \rceil = 3$ elements. In the former case, suppose that B, C, and D are friends of A. If any two of these three individuals are friends, then these two and A form a group of three mutual friends. Otherwise, B, C, and D form a set of three mutual enemies. The proof in the latter case, when there are three or more enemies of A, proceeds in a similar manner. ◀

The **Ramsey number** $R(m, n)$, where m and n are positive integers greater than or equal to 2, denotes the minimum number of people at a party such that there are either m mutual friends or n mutual enemies, assuming that every pair of people at the party are friends or enemies. Example 13 shows that $R(3, 3) \leq 6$. We conclude that $R(3, 3) = 6$ because in a group of five people where every two people are friends or enemies, there may not be three mutual friends or three mutual enemies (see Exercise 28).

It is possible to prove some useful properties about Ramsey numbers, but for the most part it is difficult to find their exact values. Note that by symmetry it can be shown that $R(m, n) = R(n, m)$ (see Exercise 32). We also have $R(2, n) = n$ for every positive integer $n \geq 2$ (see Exercise 31). The exact values of only nine Ramsey numbers $R(m, n)$ with $3 \leq m \leq n$ are known, including $R(4, 4) = 18$. Only bounds are known for many other Ramsey numbers, including $R(5, 5)$, which is known to satisfy $43 \leq R(5, 5) \leq 49$. The reader interested in learning more about Ramsey numbers should consult [MiRo91] or [GrRoSp90].

Exercises

1. Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on weekends.

2. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

3. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

a) How many socks must he take out to be sure that he has at least two socks of the same color?

b) How many socks must he take out to be sure that he has at least two black socks?

4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

a) How many balls must she select to be sure of having at least three balls of the same color?

b) How many balls must she select to be sure of having at least three blue balls?

5. Undergraduate students at a college belong to one of four groups depending on the year in which they are expected to graduate. Each student must choose one of 21 different majors. How many students are needed to assure that there are two students expected to graduate in the same year who have the same major?

6. There are six professors teaching the introductory discrete mathematics class at a university. The same final exam is given by all six professors. If the lowest possible score on the final is 0 and the highest possible score is 100, how many students must there be to guarantee that there are two students with the same professor who earned the same final examination score?

7. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

8. Let d be a positive integer. Show that among any group of $d + 1$ (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.

9. Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

10. Show that if f is a function from S to T, where S and T are finite sets with $|S| > |T|$, then there are elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$, or in other words, f is not one-to-one.

11. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

***12.** Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

***13.** Let (x_i, y_i, z_i), $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$, be a set of nine distinct points with integer coordinates in xyz space. Show that the midpoint of at least one pair of these points has integer coordinates.

14. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \textbf{ mod } 5 = a_2 \textbf{ mod } 5$ and $b_1 \textbf{ mod } 5 = b_2 \textbf{ mod } 5$?

15. a) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.

b) Is the conclusion in part (a) true if four integers are selected rather than five?

16. a) Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11.

b) Is the conclusion in part (a) true if six integers are selected rather than seven?

17. How many numbers must be selected from the set $\{1, 2, 3, 4, 5, 6\}$ to guarantee that at least one pair of these numbers add up to 7?

18. How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

19. A company stores products in a warehouse. Storage bins in this warehouse are specified by their aisle, location in the aisle, and shelf. There are 50 aisles, 85 horizontal locations in each aisle, and 5 shelves throughout the warehouse. What is the least number of products the company can have so that at least two products must be stored in the same bin?

20. Suppose that there are nine students in a discrete mathematics class at a small college.

a) Show that the class must have at least five male students or at least five female students.

b) Show that the class must have at least three male students or at least seven female students.

21. Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior.

a) Show that there are at least nine freshmen, at least nine sophomores, or at least nine juniors in the class.

b) Show that there are either at least three freshmen, at least 19 sophomores, or at least five juniors in the class.

22. Find an increasing subsequence of maximal length and a decreasing subsequence of maximal length in the sequence 22, 5, 7, 2, 23, 10, 15, 21, 3, 17.

23. Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of five terms.

24. Show that if there are 101 people of different heights standing in a line, it is possible to find 11 people in the order they are standing in the line with heights that are either increasing or decreasing.

***25.** Show that whenever 25 girls and 25 boys are seated around a circular table there is always a person both of whose neighbors are boys.

****26.** Suppose that 21 girls and 21 boys enter a mathematics competition. Furthermore, suppose that each entrant solves at most six questions, and for every boy-girl pair, there is at least one question that they both solved. Show that there is a question that was solved by at least three girls and at least three boys.

***27.** Describe an algorithm in pseudocode for producing the largest increasing or decreasing subsequence of a sequence of distinct integers.

28. Show that in a group of five people (where any two people are either friends or enemies), there are not necessarily three mutual friends or three mutual enemies.

29. Show that in a group of 10 people (where any two people are either friends or enemies), there are either three mutual friends or four mutual enemies, and there are either three mutual enemies or four mutual friends.

30. Use Exercise 29 to show that among any group of 20 people (where any two people are either friends or enemies), there are either four mutual friends or four mutual enemies.

31. Show that if n is an integer with $n \geq 2$, then the Ramsey number $R(2, n)$ equals n. (Recall that Ramsey numbers were discussed after Example 13 in Section 6.2.)

32. Show that if m and n are integers with $m \geq 2$ and $n \geq 2$, then the Ramsey numbers $R(m, n)$ and $R(n, m)$ are equal. (Recall that Ramsey numbers were discussed after Example 13 in Section 6.2.)

33. Show that there are at least six people in California (population: 39 million) with the same three initials who were born on the same day of the year (but not necessarily in the same year). Assume that everyone has three initials.

34. Show that if there are 100,000,000 wage earners in the United States who earn less than 1,000,000 dollars (but at least a penny), then there are two who earned exactly the same amount of money, to the penny, last year.

35. In the 17th century, there were more than 800,000 inhabitants of Paris. At the time, it was believed that no one had more than 200,000 hairs on their head. Assuming these numbers are correct and that everyone has at least one hair on their head (that is, no one is completely bald), use the pigeonhole principle to show, as the French writer Pierre Nicole did, that there had to be two Parisians with the same number of hairs on their heads. Then use the generalized pigeonhole principle to show that there had to be at least five Parisians at that time with the same number of hairs on their heads.

36. Assuming that no one has more than 1,000,000 hairs on their head and that the population of New York City was 8,537,673 in 2016, show there had to be at least nine people in New York City in 2016 with the same number of hairs on their heads.

37. There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

38. A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

39. A computer network consists of six computers. Each computer is directly connected to zero or more of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers. [*Hint:* It is impossible to have a computer linked to none of the others and a computer linked to all the others.]

40. Find the least number of cables required to connect eight computers to four printers to guarantee that for every choice of four of the eight computers, these four computers can directly access four different printers. Justify your answer.

41. Find the least number of cables required to connect 100 computers to 20 printers to guarantee that every subset of 20 computers can directly access 20 different printers. (Here, the assumptions about cables and computers are the same as in Example 9.) Justify your answer.

***42.** Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

43. An arm wrestler is the champion for a period of 75 hours. (Here, by an hour, we mean a period starting from an exact hour, such as 1 P.M., until the next hour.) The arm wrestler had at least one match an hour, but no more than 125 total matches. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches.

***44.** Is the statement in Exercise 43 true if 24 is replaced by

a) 2? **b)** 23? **c)** 25? **d)** 30?

45. Show that if f is a function from S to T, where S and T are nonempty finite sets and $m = \lceil |S| / |T| \rceil$, then there are at least m elements of S mapped to the same value of T. That is, show that there are distinct elements $s_1, s_2, \ldots, s_m$ of S such that $f(s_1) = f(s_2) = \cdots = f(s_m)$.

46. There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.

***47.** Let x be an irrational number. Show that for some positive integer j not exceeding the positive integer n, the absolute value of the difference between jx and the nearest integer to jx is less than $1/n$.

48. Let $n_1, n_2, \ldots, n_t$ be positive integers. Show that if $n_1 + n_2 + \cdots + n_t - t + 1$ objects are placed into t boxes, then for some i, $i = 1, 2, \ldots, t$, the ith box contains at least n_i objects.

***49.** An alternative proof of Theorem 3 based on the generalized pigeonhole principle is outlined in this exercise. The notation used is the same as that used in the proof in the text.

a) Assume that $i_k \le n$ for $k = 1, 2, \ldots, n^2 + 1$. Use the generalized pigeonhole principle to show that there are $n + 1$ terms $a_{k_1}, a_{k_2}, \ldots, a_{k_{n+1}}$ with $i_{k_1} = i_{k_2} = \cdots = i_{k_{n+1}}$, where $1 \le k_1 < k_2 < \cdots < k_{n+1}$.

b) Show that $a_{k_j} > a_{k_{j+1}}$ for $j = 1, 2, \ldots, n$. [*Hint:* Assume that $a_{k_j} < a_{k_{j+1}}$, and show that this implies that $i_{k_j} > i_{k_{j+1}}$, which is a contradiction.]

c) Use parts (a) and (b) to show that if there is no increasing subsequence of length $n + 1$, then there must be a decreasing subsequence of this length.

6.3 Permutations and Combinations

6.3.1 Introduction

Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters. Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter. For example, in how many ways can we select three students from a group of five students to stand in line for a picture? How many different committees of three students can be formed from a group of four students? In this section we will develop methods to answer questions such as these.

6.3.2 Permutations

We begin by solving the first question posed in the introduction to this section, as well as related questions.

EXAMPLE 1 In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Extra Examples

Solution: First, note that the order in which we select the students matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $5 \cdot 4 \cdot 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.

To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to arrange all five students in a line for a picture. ◀

Example 1 illustrates how ordered arrangements of distinct objects can be counted. This leads to some terminology.

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A **permutation** of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an ***r*-permutation**.

EXAMPLE 2 Let $S = \{1, 2, 3\}$. The ordered arrangement 3, 1, 2 is a permutation of S. The ordered arrangement 3, 2 is a 2-permutation of S. ◀

The number of r-permutations of a set with n elements is denoted by $P(n, r)$. We can find $P(n, r)$ using the product rule.

EXAMPLE 3 Let $S = \{a, b, c\}$. The 2-permutations of S are the ordered arrangements a, b; a, c; b, a; b, c; c, a; and c, b. Consequently, there are six 2-permutations of this set with three elements. There are always six 2-permutations of a set with three elements. There are three ways to choose the first element of the arrangement. There are two ways to choose the second element of the arrangement, because it must be different from the first element. Hence, by the product rule, we see that $P(3, 2) = 3 \cdot 2 = 6$. the first element. By the product rule, it follows that $P(3, 2) = 3 \cdot 2 = 6$. ◀

We now use the product rule to find a formula for $P(n, r)$ whenever n and r are positive integers with $1 \leq r \leq n$.

THEOREM 1 If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Proof: We will use the product rule to prove that this formula is correct. The first element of the permutation can be chosen in n ways because there are n elements in the set. There are $n-1$ ways to choose the second element of the permutation, because there are $n-1$ elements left in the set after using the element picked for the first position. Similarly, there are $n-2$ ways to choose the third element, and so on, until there are exactly $n-(r-1) = n-r+1$ ways to choose the rth element. Consequently, by the product rule, there are

$$n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of the set. ◁

Note that $P(n, 0) = 1$ whenever n is a nonnegative integer because there is exactly one way to order zero elements. That is, there is exactly one list with no elements in it, namely the empty list.

We now state a useful corollary of Theorem 1.

COROLLARY 1 If n and r are integers with $0 \le r \le n$, then $P(n, r) = \dfrac{n!}{(n-r)!}$.

Proof: When n and r are integers with $1 \le r \le n$, by Theorem 1 we have

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Because $\dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = 1$ whenever n is a nonnegative integer, we see that the formula $P(n, r) = \dfrac{n!}{(n-r)!}$ also holds when $r = 0$. ◁

By Theorem 1 we know that if n is a positive integer, then $P(n, n) = n!$. We will illustrate this result with some examples.

EXAMPLE 4 How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution: Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is

$$P(100, 3) = 100 \cdot 99 \cdot 98 = 970{,}200.$$ ◀

EXAMPLE 5 Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

Solution: The number of different ways to award the medals is the number of 3-permutations of a set with eight elements. Hence, there are $P(8, 3) = 8 \cdot 7 \cdot 6 = 336$ possible ways to award the medals. ◀

EXAMPLE 6 Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The number of possible paths between the cities is the number of permutations of seven elements, because the first city is determined, but the remaining seven can be ordered arbitrarily. Consequently, there are $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ ways for the saleswoman to choose her tour. If, for instance, the saleswoman wishes to find the path between the cities with minimum distance, and she computes the total distance for each possible path, she must consider a total of 5040 paths! ◀

EXAMPLE 7 How many permutations of the letters *ABCDEFGH* contain the string *ABC*?

Solution: Because the letters *ABC* must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block *ABC* and the individual letters *D*, *E*, *F*, *G*, and *H*. Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters *ABCDEFGH* in which *ABC* occurs as a block. ◀

6.3.3 Combinations

We now turn our attention to counting unordered selections of objects. We begin by solving a question posed in the introduction to this section of the chapter.

EXAMPLE 8 How many different committees of three students can be formed from a group of four students?

Solution: To answer this question, we need only find the number of subsets with three elements from the set containing the four students. We see that there are four such subsets, one for each of the four students, because choosing three students is the same as choosing one of the four students to leave out of the group. This means that there are four ways to choose the three students for the committee, where the order in which these students are chosen does not matter. ◀

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Example 8 illustrates that many counting problems can be solved by finding the number of subsets of a particular size of a set with n elements, where n is a positive integer.

An ***r*-combination** of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

EXAMPLE 9 Let S be the set $\{1, 2, 3, 4\}$. Then $\{1, 3, 4\}$ is a 3-combination from S. (Note that $\{4, 1, 3\}$ is the same 3-combination as $\{1, 3, 4\}$, because the order in which the elements of a set are listed does not matter.) ◀

The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$. Note that $C(n, r)$ is also denoted by $\binom{n}{r}$ and is called a **binomial coefficient**. We will learn where this terminology comes from in Section 6.4.

EXAMPLE 10 We see that $C(4, 2) = 6$, because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$. ◀

We can determine the number of r-combinations of a set with n elements using the formula for the number of r-permutations of a set. To do this, note that the r-permutations of a set can be obtained by first forming r-combinations and then ordering the elements in these combinations. The proof of Theorem 2, which gives the value of $C(n, r)$, is based on this observation.

THEOREM 2 The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals

$$C(n, r) = \frac{n!}{r!\,(n-r)!}.$$

Proof: The $P(n, r)$ r-permutations of the set can be obtained by forming the $C(n, r)$ r-combinations of the set, and then ordering the elements in each r-combination, which can be done in $P(r, r)$ ways. Consequently, by the product rule,

$$P(n, r) = C(n, r) \cdot P(r, r).$$

This implies that

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!\,(n-r)!}.$$

We can also use the division rule for counting to construct a proof of this theorem. Because the order of elements in a combination does not matter and there are $P(r, r)$ ways to order r elements in an r-combination of n elements, each of the $C(n, r)$ r-combinations of a set with n elements corresponds to exactly $P(r, r)$ r-permutations. Hence, by the division rule, $C(n, r) = \frac{P(n,r)}{P(r,r)}$, which implies as before that $C(n, r) = \frac{n!}{r!\,(n-r)!}$. ◁

The formula in Theorem 2, although explicit, is not helpful when $C(n, r)$ is computed for large values of n and r. The reasons are that it is practical to compute exact values of factorials exactly only for small integer values, and when floating point arithmetic is used, the formula in Theorem 2 may produce a value that is not an integer. When computing $C(n, r)$, first note that when we cancel out $(n - r)!$ from the numerator and denominator of the expression for $C(n, r)$ in Theorem 2, we obtain

$$C(n, r) = \frac{n!}{r!\,(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

Consequently, to compute $C(n, r)$ you can cancel out all the terms in the larger factorial in the denominator from the numerator and denominator, then multiply all the terms that do not cancel in the numerator and finally divide by the smaller factorial in the denominator. [When doing this calculation by hand, instead of by machine, it is also worthwhile to factor out common factors in the numerator $n(n - 1)\cdots(n - r + 1)$ and in the denominator $r!$.] Note that many computational programs can be used to find $C(n, r)$. [Such functions may be called *choose*(n, k) or *binom*(n, k).]

Example 11 illustrates how $C(n, k)$ is computed when k is relatively small compared to n and when k is close to n. It also illustrates a key identity enjoyed by the numbers $C(n, k)$.

EXAMPLE 11 How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Solution: Because the order in which the five cards are dealt from a deck of 52 cards does not matter, there are

$$C(52, 5) = \frac{52!}{5!47!}$$

different hands of five cards that can be dealt. To compute the value of $C(52, 5)$, first divide the numerator and denominator by 47! to obtain

$$C(52, 5) = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}.$$

This expression can be simplified by first dividing the factor 5 in the denominator into the factor 50 in the numerator to obtain a factor 10 in the numerator, then dividing the factor 4 in the denominator into the factor 48 in the numerator to obtain a factor of 12 in the numerator, then dividing the factor 3 in the denominator into the factor 51 in the numerator to obtain a factor of 17 in the numerator, and finally, dividing the factor 2 in the denominator into the factor 52 in the numerator to obtain a factor of 26 in the numerator. We find that

$$C(52, 5) = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2{,}598{,}960.$$

Consequently, there are 2,598,960 different poker hands of five cards that can be dealt from a standard deck of 52 cards.

Note that there are

$$C(52, 47) = \frac{52!}{47!5!}$$

different ways to select 47 cards from a standard deck of 52 cards. We do not need to compute this value because $C(52, 47) = C(52, 5)$. (Only the order of the factors 5! and 47! is different in the denominators in the formulae for these quantities.) It follows that there are also 2,598,960 different ways to select 47 cards from a standard deck of 52 cards. ◀

In Example 11 we observed that $C(52, 5) = C(52, 47)$. This is not surprising because selecting five cards out of 52 is the same as selecting the 47 that we leave out. The identity $C(52, 5) = C(52, 47)$ is a special case of the useful identity for the number of r-combinations of a set given in Corollary 2.

COROLLARY 2 Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2 it follows that

$$C(n, r) = \frac{n!}{r!\,(n - r)!}$$

and

$$C(n, n - r) = \frac{n!}{(n - r)!\,[n - (n - r)]!} = \frac{n!}{(n - r)!\,r!}.$$

Hence, $C(n, r) = C(n, n - r)$. ◁

We can also prove Corollary 2 without relying on algebraic manipulation. Instead, we can use a combinatorial proof. We describe this important type of proof in Definition 1.

Definition 1 A *combinatorial proof* of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity. These two types of proofs are called *double counting proofs* and *bijective proofs*, respectively.

Combinatorial proofs are almost always much shorter and provide more insights than proofs based on algebraic manipulation.

Many identities involving binomial coefficients can be proved using combinatorial proofs. We now show how to prove Corollary 2 using a combinatorial proof. We will provide both a double counting proof and a bijective proof, both based on the same basic idea.

Proof: We will use a bijective proof to show that $C(n, r) = C(n, n - r)$ for all integers n and r with $0 \leq r \leq n$. Suppose that S is a set with n elements. The function that maps a subset A of S to $\overline{A}$ is a bijection between subsets of S with r elements and subsets with $n - r$ elements (as the reader should verify). The identity $C(n, r) = C(n, n - r)$ follows because when there is a bijection between two finite sets, the two sets must have the same number of elements.

Alternatively, we can reformulate this argument as a double counting proof. By definition, the number of subsets of S with r elements equals $C(n, r)$. But each subset A of S is also determined by specifying which elements are not in A, and so are in $\overline{A}$. Because the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements. It follows that $C(n, r) = C(n, n - r)$. ◁

EXAMPLE 12 How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

Extra Examples

Solution: The answer is given by the number of 5-combinations of a set with 10 elements. By Theorem 2, the number of such combinations is

$$C(10, 5) = \frac{10!}{5!\,5!} = 252.$$

◀

EXAMPLE 13 A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

Solution: The number of ways to select a crew of six from the pool of 30 people is the number of 6-combinations of a set with 30 elements, because the order in which these people are chosen does not matter. By Theorem 2, the number of such combinations is

$$C(30, 6) = \frac{30!}{6!\,24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593{,}775.$$

◀

EXAMPLE 14 How many bit strings of length n contain exactly r 1s?

Solution: The positions of r 1s in a bit string of length n form an r-combination of the set $\{1, 2, 3, \ldots, n\}$. Hence, there are $C(n, r)$ bit strings of length n that contain exactly r 1s. ◀

EXAMPLE 15 Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution: By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By Theorem 2, the number of ways to select the committee is

$$C(9, 3) \cdot C(11, 4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 84 \cdot 330 = 27{,}720.$$

◀

Exercises

1. List all the permutations of $\{a, b, c\}$.
2. How many different permutations are there of the set $\{a, b, c, d, e, f, g\}$?
3. How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?
4. Let $S = \{1, 2, 3, 4, 5\}$.
   a) List all the 3-permutations of S.
   b) List all the 3-combinations of S.
5. Find the value of each of these quantities.
   a) $P(6, 3)$
   b) $P(6, 5)$
   c) $P(8, 1)$
   d) $P(8, 5)$
   e) $P(8, 8)$
   f) $P(10, 9)$
6. Find the value of each of these quantities.
   a) $C(5, 1)$
   b) $C(5, 3)$
   c) $C(8, 4)$
   d) $C(8, 8)$
   e) $C(8, 0)$
   f) $C(12, 6)$
7. Find the number of 5-permutations of a set with nine elements.
8. In how many different orders can five runners finish a race if no ties are allowed?
9. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?
10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?
11. How many bit strings of length 10 contain
    a) exactly four 1s?
    b) at most four 1s?
    c) at least four 1s?
    d) an equal number of 0s and 1s?
12. How many bit strings of length 12 contain
    a) exactly three 1s?
    b) at most three 1s?
    c) at least three 1s?
    d) an equal number of 0s and 1s?
13. A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?
14. In how many ways can a set of two positive integers less than 100 be chosen?
15. In how many ways can a set of five letters be selected from the English alphabet?
16. How many subsets with an odd number of elements does a set with 10 elements have?
17. How many subsets with more than two elements does a set with 100 elements have?
18. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes
    a) are there in total?
    b) contain exactly three heads?
    c) contain at least three heads?
    d) contain the same number of heads and tails?
19. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes
    a) are there in total?
    b) contain exactly two heads?
    c) contain at most three tails?
    d) contain the same number of heads and tails?
20. How many bit strings of length 10 have
    a) exactly three 0s?
    b) more 0s than 1s?
    c) at least seven 1s?
    d) at least three 1s?
21. How many permutations of the letters *ABCDEFG* contain
    a) the string *BCD*?
    b) the string *CFGA*?
    c) the strings *BA* and *GF*?
    d) the strings *ABC* and *DE*?
    e) the strings *ABC* and *CDE*?
    f) the strings *CBA* and *BED*?
22. How many permutations of the letters *ABCDEFGH* contain
    a) the string *ED*?
    b) the string *CDE*?
    c) the strings *BA* and *FGH*?
    d) the strings *AB*, *DE*, and *GH*?
    e) the strings *CAB* and *BED*?
    f) the strings *BCA* and *ABF*?
23. How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [*Hint:* First position the men and then consider possible positions for the women.]
24. How many ways are there for 10 women and six men to stand in a line so that no two men stand next to each other? [*Hint:* First position the women and then consider possible positions for the men.]
25. How many ways are there for four men and five women to stand in a line so that
    a) all men stand together?
    b) all women stand together?
26. How many ways are there for three penguins and six puffins to stand in a line so that
    a) all puffins stand together?
    b) all penguins stand together?
27. One hundred tickets, numbered $1, 2, 3, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if
    a) there are no restrictions?
    b) the person holding ticket 47 wins the grand prize?
    c) the person holding ticket 47 wins one of the prizes?
    d) the person holding ticket 47 does not win a prize?
    e) the people holding tickets 19 and 47 both win prizes?
    f) the people holding tickets 19, 47, and 73 all win prizes?

g) the people holding tickets 19, 47, 73, and 97 all win prizes?
h) none of the people holding tickets 19, 47, 73, and 97 wins a prize?
i) the grand prize winner is a person holding ticket 19, 47, 73, or 97?
j) the people holding tickets 19 and 47 win prizes, but the people holding tickets 73 and 97 do not win prizes?

28. Thirteen people on a softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?
b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?
c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

29. A club has 25 members.

a) How many ways are there to choose four members of the club to serve on an executive committee?
b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

30. A professor writes 40 discrete mathematics true/false questions. Of the statements in these questions, 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

***31.** How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers k, $k+1$, $k+2$, in the correct order

a) where these consecutive integers can perhaps be separated by other integers in the permutation?
b) where they are in consecutive positions in the permutation?

32. Seven women and nine men are on the faculty in the mathematics department at a school.

a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?
b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

33. The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain

a) exactly one vowel?
b) exactly two vowels?
c) at least one vowel?
d) at least two vowels?

34. How many strings of six lowercase letters from the English alphabet contain

a) the letter a?
b) the letters a and b?
c) the letters a and b in consecutive positions with a preceding b, with all the letters distinct?
d) the letters a and b, where a is somewhere to the left of b in the string, with all the letters distinct?

35. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

36. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

37. How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

38. How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

39. How many bit strings of length 10 contain at least three 1s and at least three 0s?

40. How many ways are there to select 12 countries in the United Nations to serve on a council if 3 are selected from a block of 45, 4 are selected from a block of 57, and the others are selected from the remaining 69 countries?

41. How many license plates consisting of three letters followed by three digits contain no letter or digit twice?

A **circular *r*-permutation of *n*** people is a seating of r of these n people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table.

42. Find the number of circular 3-permutations of 5 people.

43. Find a formula for the number of circular r-permutations of n people.

44. Find a formula for the number of ways to seat r of n people around a circular table, where seatings are considered the same if every person has the same two neighbors without regard to which side these neighbors are sitting on.

45. How many ways are there for a horse race with three horses to finish if ties are possible? [*Note:* Two or three horses may tie.]

***46.** How many ways are there for a horse race with four horses to finish if ties are possible? [*Note:* Any number of the four horses may tie.]

***47.** There are six runners in the 100-yard dash. How many ways are there for three medals to be awarded if ties are possible? (The runner or runners who finish with the fastest time receive gold medals, the runner or runners who finish with exactly one runner ahead receive silver medals, and the runner or runners who finish with exactly two runners ahead receive bronze medals.)

***48.** This procedure is used to break ties in games in the championship round of the World Cup soccer tournament. Each team selects five players in a prescribed order. Each of these players takes a penalty kick, with a player from the first team followed by a player from the second team and so on, following the order of players specified. If the score is still tied at the end of the 10 penalty kicks, this procedure is repeated. If the score is still tied after 20 penalty kicks, a sudden-death shootout occurs, with the first team scoring an unanswered goal victorious.

a) How many different scoring scenarios are possible if the game is settled in the first round of 10 penalty kicks, where the round ends once it is impossible for a team to equal the number of goals scored by the other team?

b) How many different scoring scenarios for the first and second groups of penalty kicks are possible if the game is settled in the second round of 10 penalty kicks?

c) How many scoring scenarios are possible for the full set of penalty kicks if the game is settled with no more than 10 total additional kicks after the two rounds of five kicks for each team?

6.4 Binomial Coefficients and Identities

As we remarked in Section 6.3, the number of r-combinations from a set with n elements is often denoted by $\binom{n}{r}$. This number is also called a **binomial coefficient** because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a+b)^n$. We will discuss the **binomial theorem**, which gives a power of a binomial expression as a sum of terms involving binomial coefficients. We will prove this theorem using a combinatorial proof. We will also show how combinatorial proofs can be used to establish some of the many different identities that express relationships among binomial coefficients.

6.4.1 The Binomial Theorem

Links

The binomial theorem gives the coefficients of the expansion of powers of binomial expressions. A **binomial** expression is simply the sum of two terms, such as $x + y$. (The terms can be products of constants and variables, but that does not concern us here.)

Example 1 illustrates how the coefficients in a typical expansion can be found and prepares us for the statement of the binomial theorem.

EXAMPLE 1 The expansion of $(x+y)^3$ can be found using combinatorial reasoning instead of multiplying the three terms out. When $(x+y)^3 = (x+y)(x+y)(x+y)$ is expanded, all products of a term in the first sum, a term in the second sum, and a term in the third sum are added. Terms of the form x^3, x^2y, xy^2, and y^3 arise. To obtain a term of the form x^3, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^3 term in the product has a coefficient of 1. To obtain a term of the form x^2y, an x must be chosen in two of the three sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 2-combinations of three objects, namely, $\binom{3}{2}$. Similarly, the number of terms of the form xy^2 is the number of ways to pick one of the three sums to obtain an x (and consequently take a y from each of the other two sums). This can be done in $\binom{3}{1}$ ways. Finally, the only way to obtain a y^3 term is to choose the y for each of the three sums in the product, and this can be done in exactly one way. Consequently, it follows that

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) = (xx+xy+yx+yy)(x+y)\\ &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy\\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

◀

We now state the binomial theorem.

THEOREM 1 3.1 p.66

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}y^j = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

Proof: We use a combinatorial proof. The terms in the product when it is expanded are of the form $x^{n-j}y^j$ for $j = 0, 1, 2, \ldots, n$. To count the number of terms of the form $x^{n-j}y^j$, note that to obtain such a term it is necessary to choose $n - j$ xs from the n binomial factors (so that the other j terms in the product are ys). Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$, which is equal to $\binom{n}{j}$. This proves the theorem. ◁

Some computational uses of the binomial theorem are illustrated in Examples 2–4.

EXAMPLE 2 What is the expansion of $(x + y)^4$?

Extra Examples

Solution: From the binomial theorem it follows that

$$\begin{aligned}(x+y)^4 &= \sum_{j=0}^{4}\binom{4}{j}x^{4-j}y^j\\ &= \overset{1}{\binom{4}{0}}x^4 + \overset{4}{\binom{4}{1}}x^3y + \overset{6}{\binom{4}{2}}x^2y^2 + \overset{4}{\binom{4}{3}}xy^3 + \overset{1}{\binom{4}{4}}y^4\\ &= 1(x^4y^0) + 4(x^3y^1) + 6(x^2y^2) + 4(x^1y^3) + 1(x^0y^4)\\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

◀

EXAMPLE 3 What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

Solution: From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.$$

◀

EXAMPLE 4 What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: First, note that this expression equals $(2x + (-3y))^{25}$. By the binomial theorem, we have

$$(2x + (-3y))^{25} = \sum_{j=0}^{25}\binom{25}{j}(2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$, namely,

$$\binom{25}{13}2^{12}(-3)^{13} = -\frac{25!}{13!\,12!}2^{12}3^{13}.$$

Note that another way to find the solution is to first use the binomial theorem to see that

$$(u + v)^{25} = \sum_{j=0}^{25}\binom{25}{j}u^{25-j}v^j.$$

Setting $u = 2x$ and $v = -3y$ in this equation yields the same result. ◀

We can prove some useful identities using the binomial theorem, as Corollaries 1, 2, and 3 demonstrate.

COROLLARY 1

3.2 p.70

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof: Using the binomial theorem with $x = 1$ and $y = 1$, we see that

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$$

This is the desired result. ◁

There is also a nice combinatorial proof of Corollary 1, which we now present.

Proof: A set with n elements has a total of 2^n different subsets. Each subset has zero elements, one element, two elements, …, or n elements in it. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ subsets with one element, $\binom{n}{2}$ subsets with two elements, …, and $\binom{n}{n}$ subsets with n elements. Therefore,

$$\sum_{k=0}^{n} \binom{n}{k}$$

counts the total number of subsets of a set with n elements. By equating the two formulas we have for the number of subsets of a set with n elements, we see that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

◁

COROLLARY 2

3.8 p.71

Let n be a positive integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Proof: When we use the binomial theorem with $x = -1$ and $y = 1$, we see that

$$0 = 0^n = ((-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

This proves the corollary. ◁

Remark: Corollary 2 implies that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots.$$

COROLLARY 3 Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

Proof: We recognize that the left-hand side of this formula is the expansion of $(1+2)^n$ provided by the binomial theorem. Therefore, by the binomial theorem, we see that

$$(1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Hence

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

◁

6.4.2 Pascal's Identity and Triangle

The binomial coefficients satisfy many different identities. We introduce one of the most important of these now.

THEOREM 2 **PASCAL'S IDENTITY** Let n and k be positive integers with $n \geq k$. Then

3.1 p.68

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof: We will use a combinatorial proof. Suppose that T is a set containing $n+1$ elements. Let a be an element in T, and let $S = T - \{a\}$. Note that there are $\binom{n+1}{k}$ subsets of T containing k elements. However, a subset of T with k elements either contains a together with $k-1$ elements of S, or contains k elements of S and does not contain a. Because there are $\binom{n}{k-1}$ subsets of $k-1$ elements of S, there are $\binom{n}{k-1}$ subsets of k elements of T that contain a. And there are $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S. Consequently,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

◁

Remark: It is also possible to prove this identity by algebraic manipulation from the formula for $\binom{n}{r}$ (see Exercise 23).

Remark: Pascal's identity, together with the initial conditions $\binom{n}{0} = \binom{n}{n} = 1$ for all integers n, can be used to recursively define binomial coefficients. This recursive definition is useful in the

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\dots
\end{array}
$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\dots
\end{array}
$$

(b)

FIGURE 1 **Pascal's triangle.**

computation of binomial coefficients because only addition, and not multiplication, of integers is needed to use this recursive definition.

Pascal's identity is the basis for a geometric arrangement of the binomial coefficients in a triangle, as shown in Figure 1.

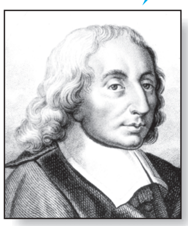
The nth row in the triangle consists of the binomial coefficients

$$\binom{n}{k}, \ k = 0, 1, \dots, n.$$

This triangle is known as **Pascal's triangle**, named after the French mathematician Blaise Pascal. Pascal's identity shows that when two adjacent binomial coefficients in this triangle are added, the binomial coefficient in the next row between these two coefficients is produced.

Pascal's triangle has a long and ancient history, predating Pascal by many centuries. In the East, binomial coefficients and Pascal's identity were known in the second century B.C.E. by the Indian mathematician Pingala. Later, Indian mathematicians included commentaries relating to Pascal's triangle in their books written in the first half of the last millennium. The Persian

Links



Source: National Library of Medicine

BLAISE PASCAL (1623–1662) Blaise Pascal was taught by his father, a tax collector in Rouen, France. He exhibited his talents at an early age, although his father, who had made discoveries in analytic geometry, kept mathematics books away from him to encourage other interests. At 16 Pascal discovered an important result concerning conic sections. At 18 he designed a calculating machine, which he built and sold. Pascal, along with Fermat, laid the foundations for the modern theory of probability. In this work, he made new discoveries concerning what is now called Pascal's triangle. In 1654, Pascal abandoned his mathematical pursuits to devote himself to theology. After this, he returned to mathematics only once. One night, distracted by a severe toothache, he sought comfort by studying the mathematical properties of the cycloid. Miraculously, his pain subsided, which he took as a sign of divine approval of the study of mathematics.

mathematician Al-Karaji and the multitalented Omar Khayyám wrote about Pascal's triangle in the eleventh and twelfth centuries, respectively; in Iran, Pascal's triangle is known as Khayyám's triangle. The triangle was known by the Chinese mathematician Jia Xian in the eleventh century and was written about in the 13th century by Yang Hui; in Chinese Pascal's triangle is often known as Yang Hui's triangle.

In the West, Pascal's triangle appears on the frontispiece of a 1527 book on business calculation written by the German scholar Petrus Apianus. In Italy, Pascal's triangle is called Tartaglia's triangle, after the Italian mathematician Niccolò Fontana Tartaglia who published the first few rows of the triangle in 1556. In his book *Traitè du triangle arithmétique*, published posthumously 1665, Pascal presented results about Pascal's triangle and used them to solve probability theory problems. Later French mathematicians named this triangle after Pascal; in 1730 Abraham de Moivre coined the name "Pascal's Arithmetic Triangle," which later became "Pascal's Triangle."

6.4.3 Other Identities Involving Binomial Coefficients

We conclude this section with combinatorial proofs of two of the many identities enjoyed by the binomial coefficients.

THEOREM 3 3.12 p.72

VANDERMONDE'S IDENTITY Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Links

Remark: This identity was discovered by mathematician Alexandre-Théophile Vandermonde in the eighteenth century.

Proof: Suppose that there are m items in one set and n items in a second set. Then the total number of ways to pick r elements from the union of these sets is $\binom{m+n}{r}$.

Another way to pick r elements from the union is to pick k elements from the second set and then $r-k$ elements from the first set, where k is an integer with $0 \le k \le r$. Because there are $\binom{n}{k}$ ways to choose k elements from the second set and $\binom{m}{r-k}$ ways to choose $r-k$ elements from the first set, the product rule tells us that this can be done in $\binom{m}{r-k}\binom{n}{k}$ ways. Hence, the total number of ways to pick r elements from the union also equals $\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$.

We have found two expressions for the number of ways to pick r elements from the union of a set with m items and a set with n items. Equating them gives us Vandermonde's identity. ◁

Corollary 4 follows from Vandermonde's identity.

ALEXANDRE-THÉOPHILE VANDERMONDE (1735–1796) Because Alexandre-Théophile Vandermonde was a sickly child, his physician father directed him to a career in music. However, he later developed an interest in mathematics. His complete mathematical work consists of four papers published in 1771–1772. These papers include fundamental contributions on the roots of equations, on the theory of determinants, and on the knight's tour problem (introduced in the exercises in Section 10.5). Vandermonde's interest in mathematics lasted for only 2 years. Afterward, he published papers on harmony, experiments with cold, and the manufacture of steel. He also became interested in politics, joining the cause of the French revolution and holding several different positions in government.

COROLLARY 4

3.16 א p.74

If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Proof: We use Vandermonde's identity with $m = r = n$ to obtain

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

The last equality was obtained using the identity $\binom{n}{k} = \binom{n}{n-k}$. ◁

We can prove combinatorial identities by counting bit strings with different properties, as the proof of Theorem 4 will demonstrate.

THEOREM 4

3.15 p.73

Let n and r be nonnegative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Proof: We use a combinatorial proof. By Example 14 in Section 6.3, the left-hand side, $\binom{n+1}{r+1}$, counts the bit strings of length $n+1$ containing $r+1$ ones.

We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ ones. This final one must occur at position $r+1, r+2, \ldots,$ or $n+1$. Furthermore, if the last one is the kth bit there must be r ones among the first $k-1$ positions. Consequently, by Example 14 in Section 6.3, there are $\binom{k-1}{r}$ such bit strings. Summing over k with $r+1 \leq k \leq n+1$, we find that there are

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

bit strings of length n containing exactly $r+1$ ones. (Note that the last step follows from the change of variables $j = k-1$.) Because the left-hand side and the right-hand side count the same objects, they are equal. This completes the proof. ◁

Exercises

1. Find the expansion of $(x+y)^4$
  a) using combinatorial reasoning, as in Example 1.
  b) using the binomial theorem.

2. Find the expansion of $(x+y)^5$
  a) using combinatorial reasoning, as in Example 1.
  b) using the binomial theorem.

3. Find the expansion of $(x+y)^6$.

4. Find the coefficient of x^5y^8 in $(x+y)^{13}$.

5. How many terms are there in the expansion of $(x+y)^{100}$ after like terms are collected?

6. What is the coefficient of x^7 in $(1+x)^{11}$?

7. What is the coefficient of x^9 in $(2-x)^{19}$?

8. What is the coefficient of x^8y^9 in the expansion of $(3x+2y)^{17}$?

9. What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$?

10. Use the binomial theorem to expand $(3x - y^2)^4$ into a sum of terms of the form $cx^a y^b$, where c is a real number and a and b are nonnegative integers.

11. Use the binomial theorem to expand $(3x^4 - 2y^3)^5$ into a sum of terms of the form $cx^a y^b$, where c is a real number and a and b are nonnegative integers.

12. Use the binomial theorem to find the coefficient of $x^a y^b$ in the expansion of $(5x^2 + 2y^3)^6$, where

a) $a = 6, b = 9$.
b) $a = 2, b = 15$.
c) $a = 3, b = 12$.
d) $a = 12, b = 0$.
e) $a = 8, b = 9$.

13. Use the binomial theorem to find the coefficient of $x^a y^b$ in the expansion of $(2x^3 - 4y^2)^7$, where

a) $a = 9, b = 8$.
b) $a = 8, b = 9$.
c) $a = 0, b = 14$.
d) $a = 12, b = 6$.
e) $a = 18, b = 2$.

* **14.** Give a formula for the coefficient of x^k in the expansion of $(x + 1/x)^{100}$, where k is an integer.

* **15.** Give a formula for the coefficient of x^k in the expansion of $(x^2 - 1/x)^{100}$, where k is an integer.

16. The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \le k \le 10$, is:

1 10 45 120 210 252 210 120 45 10 1

Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.

17. What is the row of Pascal's triangle containing the binomial coefficients $\binom{9}{k}$, $0 \le k \le 9$?

18. Show that if n is a positive integer, then $1 = \binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n} = 1$.

19. Show that $\binom{n}{k} \le 2^n$ for all positive integers n and all integers k with $0 \le k \le n$.

20. **a)** Use Exercise 18 and Corollary 1 to show that if n is an integer greater than 1, then $\binom{n}{\lfloor n/2 \rfloor} \ge 2^n/n$.
b) Conclude from part (a) that if n is a positive integer, then $\binom{2n}{n} \ge 4^n/2n$.

21. Show that if n and k are integers with $1 \le k \le n$, then $\binom{n}{k} \le n^k/2^{k-1}$.

22. Suppose that b is an integer with $b \ge 7$. Use the binomial theorem and the appropriate row of Pascal's triangle to find the base-b expansion of $(11)_b^4$ [that is, the fourth power of the number $(11)_b$ in base-b notation].

23. Prove Pascal's identity, using the formula for $\binom{n}{r}$.

24. Suppose that k and n are integers with $1 \le k < n$. Prove the **hexagon identity**

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1},$$

which relates terms in Pascal's triangle that form a hexagon.

25. Prove that if n and k are integers with $1 \le k \le n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$,

a) using a combinatorial proof. [*Hint:* Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.]
b) using an algebraic proof based on the formula for $\binom{n}{r}$ given in Theorem 2 in Section 6.3.

26. Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and
3.10 k are nonnegative integers with $r \le n$ and $k \le r$,
p.72 **a)** using a combinatorial argument.
b) using an argument based on the formula for the number of r-combinations of a set with n elements.

27. Show that if n and k are positive integers, then

$$\binom{n+1}{k} = (n+1)\binom{n}{k-1} \Big/ k.$$

Use this identity to construct an inductive definition of the binomial coefficients.

28. Show that if p is a prime and k is an integer such that $1 \le k \le p - 1$, then p divides $\binom{p}{k}$.

29. Let n be a positive integer. Show that

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+2}{n+1}\Big/2.$$

* **30.** Let n and k be integers with $1 \le k \le n$. Show that

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \binom{2n+2}{n+1}\Big/2 - \binom{2n}{n}.$$

* **31.** Prove the **hockeystick identity**

3.17 p.74
$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

a) using a combinatorial argument.
b) using Pascal's identity.

32. Show that if n is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$
3.13 **a)** using a combinatorial argument.
p.73 **b)** by algebraic manipulation.

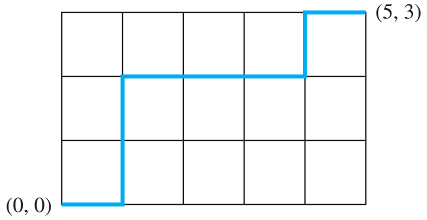
* **33.** Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.
3.9 p.71 [*Hint:* Count in two ways the number of ways to select a committee and to then select a leader of the committee.] answers in p. 1033

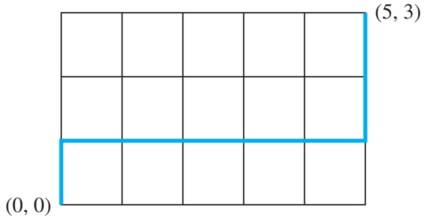
* **34.** Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$. [*Hint:* Count in two ways the number of ways to select a committee, with n members from a group of n mathematics professors and n computer science professors, such that the chairperson of the committee is a mathematics professor.]

35. Show that a nonempty set has the same number of subsets with an odd number of elements as it does subsets with an even number of elements.

* **36.** Prove the binomial theorem using mathematical induction.

37. In this exercise we will count the number of paths in the xy plane between the origin $(0, 0)$ and point (m, n), where m and n are nonnegative integers, such that each path is made up of a series of steps, where each step is a move one unit to the right or a move one unit upward. (No moves to the left or downward are allowed.) Two such paths from $(0, 0)$ to $(5, 3)$ are illustrated here.





a) Show that each path of the type described can be represented by a bit string consisting of m 0s and n 1s, where a 0 represents a move one unit to the right and a 1 represents a move one unit upward.

b) Conclude from part (a) that there are $\binom{m+n}{n}$ paths of the desired type.

38. Use Exercise 37 to give an alternative proof of Corollary 2 in Section 6.3, which states that $\binom{n}{k} = \binom{n}{n-k}$ whenever k is an integer with $0 \le k \le n$. [*Hint:* Consider the number of paths of the type described in Exercise 37 from $(0, 0)$ to $(n - k, k)$ and from $(0, 0)$ to $(k, n - k)$.]

39. Use Exercise 37 to prove Theorem 4. [*Hint:* Count the number of paths with n steps of the type described in Exercise 37. Every such path must end at one of the points $(n - k, k)$ for $k = 0, 1, 2, \ldots, n$.]

40. Use Exercise 37 to prove Pascal's identity. [*Hint:* Show that a path of the type described in Exercise 37 from $(0, 0)$ to $(n + 1 - k, k)$ passes through either $(n + 1 - k, k - 1)$ or $(n - k, k)$, but not through both.]

41. Use Exercise 37 to prove the hockeystick identity from Exercise 31. [*Hint:* First, note that the number of paths from $(0, 0)$ to $(n + 1, r)$ equals $\binom{n+1+r}{r}$. Second, count the number of paths by summing the number of these paths that start by going k units upward for $k = 0, 1, 2, \ldots, r$.]

42. Give a combinatorial proof that if n is a positive integer then $\sum_{k=0}^{n} k^2 \binom{n}{k} = n(n + 1)2^{n-2}$. [*Hint:* Show that both sides count the ways to select a subset of a set of n elements together with two not necessarily distinct elements from this subset. Furthermore, express the right-hand side as $n(n - 1)2^{n-2} + n2^{n-1}$.]

***43.** Determine a formula involving binomial coefficients for the nth term of a sequence if its initial terms are those listed. [*Hint:* Looking at Pascal's triangle will be helpful. Although infinitely many sequences start with a specified set of terms, each of the following lists is the start of a sequence of the type desired.]

a) 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, …

b) 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, …

c) 1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, …

d) 1, 1, 2, 3, 6, 10, 20, 35, 70, 126, …

e) 1, 1, 1, 3, 1, 5, 15, 35, 1, 9, …

f) 1, 3, 15, 84, 495, 3003, 18564, 116280, 735471, 4686825, …

6.5 Generalized Permutations and Combinations

6.5.1 Introduction

Links

In many counting problems, elements may be used repeatedly. For instance, a letter or digit may be used more than once on a license plate. When a dozen donuts are selected, each variety can be chosen repeatedly. This contrasts with the counting problems discussed earlier in the chapter where we considered only permutations and combinations in which each item could be used at most once. In this section we will show how to solve counting problems where elements may be used more than once.

Also, some counting problems involve indistinguishable elements. For instance, to count the number of ways the letters of the word *SUCCESS* can be rearranged, the placement of identical letters must be considered. This contrasts with the counting problems discussed earlier where all elements were considered distinguishable. In this section we will describe how to solve counting problems in which some elements are indistinguishable.

Moreover, in this section we will explain how to solve another important class of counting problems, problems involving counting the ways distinguishable elements can be placed in boxes. An example of this type of problem is the number of different ways poker hands can be dealt to four players.

Taken together, the methods described earlier in this chapter and the methods introduced in this section form a useful toolbox for solving a wide range of counting problems. When the additional methods discussed in Chapter 8 are added to this arsenal, you will be able to solve a large percentage of the counting problems that arise in a wide range of areas of study.

6.5.2 Permutations with Repetition

Counting permutations when repetition of elements is allowed can easily be done using the product rule, as Example 1 shows.

EXAMPLE 1 How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: By the product rule, because there are 26 uppercase English letters, and because each letter can be used repeatedly, we see that there are 26^r strings of uppercase English letters of length r. ◀

The number of r-permutations of a set with n elements when repetition is allowed is given in Theorem 1.

THEOREM 1 The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed, because for each choice all n objects are available. Hence, by the product rule there are n^r r-permutations when repetition is allowed. ◁

6.5.3 Combinations with Repetition

Consider these examples of combinations with repetition of elements allowed.

EXAMPLE 2 How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

Solution: To solve this problem we list all the ways possible to select the fruit. There are 15 ways:

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

The solution is the number of 4-combinations with repetition allowed from a three-element set, $\{apple, orange, pear\}$. ◀

To solve more complex counting problems of this type, we need a general method for counting the r-combinations of an n-element set. In Example 3 we will illustrate such a method.

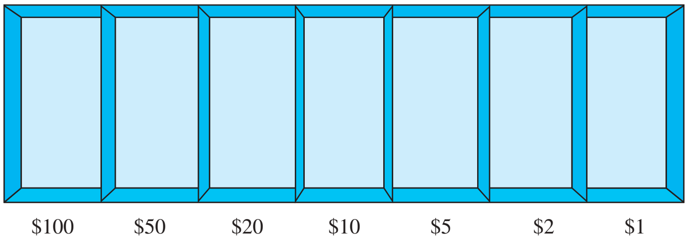


FIGURE 1 **Cash box with seven types of bills.**

EXAMPLE 3 How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills? Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least five bills of each type.

Solution: Because the order in which the bills are selected does not matter and seven different types of bills can be selected as many as five times, this problem involves counting 5-combinations with repetition allowed from a set with seven elements. Listing all possibilities would be tedious, because there are a large number of solutions. Instead, we will illustrate the use of a technique for counting combinations with repetition allowed.

Suppose that a cash box has seven compartments, one to hold each type of bill, as illustrated in Figure 1. These compartments are separated by six dividers, as shown in the picture. The choice of five bills corresponds to placing five markers in the compartments holding different types of bills. Figure 2 illustrates this correspondence for three different ways to select five bills, where the six dividers are represented by bars and the five bills by stars.

The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row with a total of 11 positions. Consequently, the number of ways to select the five bills is the number of ways to select the positions of the five stars from the 11

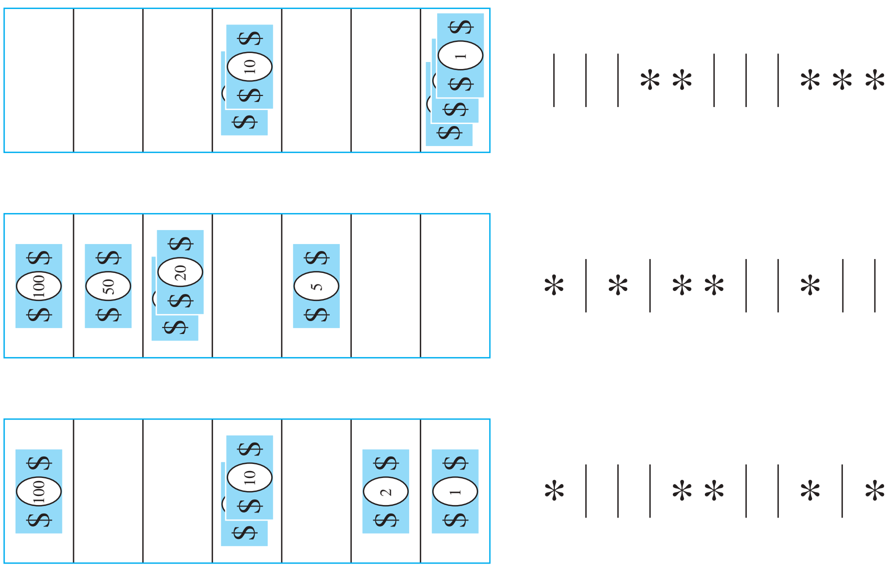


FIGURE 2 **Examples of ways to select five bills.**

positions. This corresponds to the number of unordered selections of 5 objects from a set of 11 objects, which can be done in $C(11, 5)$ ways. Consequently, there are

$$C(11, 5) = \frac{11!}{5!\,6!} = 462$$

ways to choose five bills from the cash box with seven types of bills. ◀

Theorem 2 generalizes this discussion.

THEOREM 2

D(n,k)

There are $C(n + r - 1, r) = C(n + r - 1, n - 1)$ r-combinations from a set with n elements when repetition of elements is allowed.

Proof: Each r-combination of a set with n elements when repetition is allowed can be represented by a list of $n - 1$ bars and r stars. The $n - 1$ bars are used to mark off n different cells, with the ith cell containing a star for each time the ith element of the set occurs in the combination. For instance, a 6-combination of a set with four elements is represented with three bars and six stars. Here

$$**|*||***$$

represents the combination containing exactly two of the first element, one of the second element, none of the third element, and three of the fourth element of the set.

As we have seen, each different list containing $n - 1$ bars and r stars corresponds to an r-combination of the set with n elements, when repetition is allowed. The number of such lists is $C(n - 1 + r, r)$, because each list corresponds to a choice of the r positions to place the r stars from the $n - 1 + r$ positions that contain r stars and $n - 1$ bars. The number of such lists is also equal to $C(n - 1 + r, n - 1)$, because each list corresponds to a choice of the $n - 1$ positions to place the $n - 1$ bars. ◁

Examples 4–6 show how Theorem 2 is applied.

EXAMPLE 4 Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

Extra Examples

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. From Theorem 2 this equals $C(4 + 6 - 1, 6) = C(9, 6)$. Because

$$C(9, 6) = C(9, 3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84,$$

there are 84 different ways to choose the six cookies. ◀

Theorem 2 can also be used to find the number of solutions of certain linear equations where the variables are integers subject to constraints. This is illustrated by Example 5.

EXAMPLE 5 How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers?

Solution: To count the number of solutions, we note that a solution corresponds to a way of selecting 11 items from a set with three elements so that x_1 items of type one, x_2 items of type two, and x_3 items of type three are chosen. Hence, the number of solutions is equal to the number of 11-combinations with repetition allowed from a set with three elements. From Theorem 2 it follows that there are

$$C(3 + 11 - 1, 11) = C(13, 11) = C(13, 2) = \frac{13 \cdot 12}{1 \cdot 2} = 78$$

solutions.

The number of solutions of this equation can also be found when the variables are subject to constraints. For instance, we can find the number of solutions where the variables are integers with $x_1 \geq 1$, $x_2 \geq 2$, and $x_3 \geq 3$. A solution to the equation subject to these constraints corresponds to a selection of 11 items with x_1 items of type one, x_2 items of type two, and x_3 items of type three, where, in addition, there is at least one item of type one, two items of type two, and three items of type three. So, a solution corresponds to a choice of one item of type one, two of type two, and three of type three, together with a choice of five additional items of any type. By Theorem 2 this can be done in

$$C(3 + 5 - 1, 5) = C(7, 5) = C(7, 2) = \frac{7 \cdot 6}{1 \cdot 2} = 21$$

ways. Thus, there are 21 solutions of the equation subject to the given constraints. ◀

Example 6 shows how counting the number of combinations with repetition allowed arises in determining the value of a variable that is incremented each time a certain type of nested loop is traversed.

EXAMPLE 6 What is the value of k after the following pseudocode has been executed?

```
k := 0
for i_1 := 1 to n
   for i_2 := 1 to i_1
      .
      .
      .
         for i_m := 1 to i_{m-1}
            k := k + 1
```

Solution: Note that the initial value of k is 0 and that 1 is added to k each time the nested loop is traversed with a sequence of integers $i_1, i_2, \ldots, i_m$ such that

$$1 \leq i_m \leq i_{m-1} \leq \cdots \leq i_1 \leq n.$$

The number of such sequences of integers is the number of ways to choose m integers from $\{1, 2, \ldots, n\}$, with repetition allowed. (To see this, note that once such a sequence has been selected, if we order the integers in the sequence in nondecreasing order, this uniquely defines an assignment of $i_m, i_{m-1}, \ldots, i_1$. Conversely, every such assignment corresponds to a unique unordered set.) Hence, from Theorem 2, it follows that $k = C(n + m - 1, m)$ after this code has been executed. ◀

The formulae for the numbers of ordered and unordered selections of r elements, chosen with and without repetition allowed from a set with n elements, are shown in Table 1.

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

6.5.4 Permutations with Indistinguishable Objects

Some elements may be indistinguishable in counting problems. When this is the case, care must be taken to avoid counting things more than once. Consider Example 7.

EXAMPLE 7 How many different strings can be made by reordering the letters of the word *SUCCESS*?

Extra Examples

Solution: Because some of the letters of *SUCCESS* are the same, the answer is *not* given by the number of permutations of seven letters. This word contains three *S*s, two *C*s, one *U*, and one *E*. To determine the number of different strings that can be made by reordering the letters, first note that the three *S*s can be placed among the seven positions in $C(7, 3)$ different ways, leaving four positions free. Then the two *C*s can be placed in $C(4, 2)$ ways, leaving two free positions. The *U* can be placed in $C(2, 1)$ ways, leaving just one position free. Hence *E* can be placed in $C(1, 1)$ way. Consequently, from the product rule, the number of different strings that can be made is

$$\begin{aligned} C(7,3)C(4,2)C(2,1)C(1,1) &= \frac{7!}{3!\,4!}\cdot\frac{4!}{2!\,2!}\cdot\frac{2!}{1!\,1!}\cdot\frac{1!}{1!\,0!} \\ &= \frac{7!}{3!\,2!\,1!\,1!} \\ &= 420. \end{aligned}$$

◀

We can prove Theorem 3 using the same sort of reasoning as in Example 7.

THEOREM 3 The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …, and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}.$$

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Proof: To determine the number of permutations, first note that the n_1 objects of type one can be placed among the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions free. Then the objects of type two can be placed in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions free. Continue placing the objects of type three, …, type $k - 1$, until at the last stage, n_k objects of type k can be placed in $C(n - n_1 - n_2 - \cdots - n_{k-1}, n_k)$ ways. Hence, by the product rule, the total number of different permutations is

$$C(n, n_1)C(n - n_1, n_2) \cdots C(n - n_1 - \cdots - n_{k-1}, n_k)$$
$$= \frac{n!}{n_1!\,(n - n_1)!} \frac{(n - n_1)!}{n_2!\,(n - n_1 - n_2)!} \cdots \frac{(n - n_1 - \cdots - n_{k-1})!}{n_k!\,0!}$$
$$= \frac{n!}{n_1!\,n_2! \cdots n_k!}.$$

◁

6.5.5 Distributing Objects into Boxes

Links

Many counting problems can be solved by enumerating the ways objects can be placed into boxes (where the order these objects are placed into the boxes does not matter). The objects can be either *distinguishable*, that is, different from each other, or *indistinguishable*, that is, considered identical. Distinguishable objects are sometimes said to be *labeled*, whereas indistinguishable objects are said to be *unlabeled*. Similarly, boxes can be *distinguishable*, that is, different, or *indistinguishable*, that is, identical. Distinguishable boxes are often said to be *labeled*, while indistinguishable boxes are said to be *unlabeled*. When you solve a counting problem using the model of distributing objects into boxes, you need to determine whether the objects are distinguishable and whether the boxes are distinguishable. Although the context of the counting problem makes these two decisions clear, counting problems are sometimes ambiguous and it may be unclear which model applies. In such a case it is best to state whatever assumptions you are making and explain why the particular model you choose conforms to your assumptions.

Extra Examples

We will see that there are closed formulae for counting the ways to distribute objects, distinguishable or indistinguishable, into distinguishable boxes. We are not so lucky when we count the ways to distribute objects, distinguishable or indistinguishable, into indistinguishable boxes; there are no closed formulae to use in these cases.

Remark: A **closed formula** is an expression that can be evaluated using a finite number of operations and that includes numbers, variables, and values of functions, where the operations and functions belong to a generally accepted set that can depend on the context. In this book, we include the usual arithmetic operations, rational powers, exponential and logarithmic functions, trigonometric functions, and the factorial function. We do not allow infinite series to be included in closed formulae.

DISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES We first consider the case when distinguishable objects are placed into distinguishable boxes. Consider Example 8 in which the objects are cards and the boxes are hands of players.

EXAMPLE 8 How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Solution: We will use the product rule to solve this problem. To begin, note that the first player can be dealt 5 cards in $C(52, 5)$ ways. The second player can be dealt 5 cards in $C(47, 5)$ ways, because only 47 cards are left. The third player can be dealt 5 cards in $C(42, 5)$ ways. Finally, the fourth player can be dealt 5 cards in $C(37, 5)$ ways. Hence, the total number of ways to deal four players 5 cards each is

$$C(52, 5)C(47, 5)C(42, 5)C(37, 5) = \frac{52!}{47!\,5!} \cdot \frac{47!}{42!\,5!} \cdot \frac{42!}{37!\,5!} \cdot \frac{37!}{32!\,5!}$$
$$= \frac{52!}{5!\,5!\,5!\,5!\,32!}.$$

◀

Remark: The solution to Example 8 equals the number of permutations of 52 objects, with 5 indistinguishable objects of each of four different types, and 32 objects of a fifth type. This equality can be seen by defining a one-to-one correspondence between permutations of this type and distributions of cards to the players. To define this correspondence, first order the cards from 1 to 52. Then cards dealt to the first player correspond to the cards in the positions assigned to objects of the first type in the permutation. Similarly, cards dealt to the second, third, and fourth players, respectively, correspond to cards in the positions assigned to objects of the second, third, and fourth type, respectively. The cards not dealt to any player correspond to cards in the positions assigned to objects of the fifth type. The reader should verify that this is a one-to-one correspondence.

Example 8 is a typical problem that involves distributing distinguishable objects into distinguishable boxes. The distinguishable objects are the 52 cards, and the five distinguishable boxes are the hands of the four players and the rest of the deck. Counting problems that involve distributing distinguishable objects into boxes can be solved using Theorem 4.

THEOREM 4 The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1, 2, \ldots, k$, equals

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}.$$

Theorem 4 can be proved using the product rule. We leave the details as Exercise 49. It can also be proved (see Exercise 50) by setting up a one-to-one correspondence between the permutations counted by Theorem 3 and the ways to distribute objects counted by Theorem 4.

INDISTINGUISHABLE OBJECTS AND DISTINGUISHABLE BOXES Counting the number of ways of placing n indistinguishable objects into k distinguishable boxes turns out to be the same as counting the number of n-combinations for a set with k elements when repetitions are allowed. The reason behind this is that there is a one-to-one correspondence between n-combinations from a set with k elements when repetition is allowed and the ways to place n indistinguishable balls into k distinguishable boxes. To set up this correspondence, we put a ball in the ith bin each time the ith element of the set is included in the n-combination.

EXAMPLE 9 How many ways are there to place 10 indistinguishable balls into eight distinguishable bins?

Solution: The number of ways to place 10 indistinguishable balls into eight bins equals the number of 10-combinations from a set with eight elements when repetition is allowed. Consequently, there are

$$C(8 + 10 - 1, 10) = C(17, 10) = \frac{17!}{10!7!} = 19{,}448.$$

◀

This means that there are $C(n + r - 1, n - 1)$ ways to place r indistinguishable objects into n distinguishable boxes.

DISTINGUISHABLE OBJECTS AND INDISTINGUISHABLE BOXES Counting the ways to place n distinguishable objects into k indistinguishable boxes is more difficult than counting the ways to place objects, distinguishable or indistinguishable objects, into distinguishable boxes. We illustrate this with an example.

EXAMPLE 10 How many ways are there to put four different employees into three indistinguishable offices, when each office can contain any number of employees?

Solution: We will solve this problem by enumerating all the ways these employees can be placed into the offices. We represent the four employees by A, B, C, and D. First, we note that we can distribute employees so that all four are put into one office, three are put into one office and a fourth is put into a second office, two employees are put into one office and two put into a second office, and finally, two are put into one office, and one each put into the other two offices. Each way to distribute these employees to these offices can be represented by a way to partition the elements A, B, C, and D into disjoint subsets.

We can put all four employees into one office in exactly one way, represented by $\{\{A, B, C, D\}\}$. We can put three employees into one office and the fourth employee into a different office in exactly four ways, represented by $\{\{A, B, C\}, \{D\}\}$, $\{\{A, B, D\}, \{C\}\}$, $\{\{A, C, D\}, \{B\}\}$, and $\{\{B, C, D\}, \{A\}\}$. We can put two employees into one office and two into a second office in exactly three ways, represented by $\{\{A, B\}, \{C, D\}\}$, $\{\{A, C\}, \{B, D\}\}$, and $\{\{A, D\}, \{B, C\}\}$. Finally, we can put two employees into one office, and one each into each of the remaining two offices in six ways, represented by $\{\{A, B\}, \{C\}, \{D\}\}$, $\{\{A, C\}, \{B\}, \{D\}\}$, $\{\{A, D\}, \{B\}, \{C\}\}$, $\{\{B, C\}, \{A\}, \{D\}\}$, $\{\{B, D\}, \{A\}, \{C\}\}$, and $\{\{C, D\}, \{A\}, \{B\}\}$.

Counting all the possibilities, we find that there are 14 ways to put four different employees into three indistinguishable offices. Another way to look at this problem is to look at the number of offices into which we put employees. Note that there are six ways to put four different employees into three indistinguishable offices so that no office is empty, seven ways to put four different employees into two indistinguishable offices so that no office is empty, and one way to put four employees into one office so that it is not empty. ◀

There is no simple closed formula for the number of ways to distribute n distinguishable objects into j indistinguishable boxes. However, there is a formula involving a summation, which we will now describe. Let $S(n, j)$ denote the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no box is empty. The numbers $S(n, j)$ are called **Stirling numbers of the second kind**. For instance, Example 10 shows that $S(4, 3) = 6$, $S(4, 2) = 7$, and $S(4, 1) = 1$. We see that the number of ways to distribute n distinguishable objects into k indistinguishable boxes (where the number of boxes that are nonempty equals k, $k-1, \ldots, 2$, or 1) equals $\sum_{j=1}^{k} S(n, j)$. For instance, following the reasoning in Example 10, the number of ways to distribute four distinguishable objects into three indistinguishable boxes equals $S(4, 1) + S(4, 2) + S(4, 3) = 1 + 7 + 6 = 14$. Using the inclusion–exclusion principle (see Section 8.6) it can be shown that

$$S(n, j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n.$$

Consequently, the number of ways to distribute n distinguishable objects into k indistinguishable boxes equals

$$\sum_{j=1}^{k} S(n, j) = \sum_{j=1}^{k}\frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n.$$

Remark: The reader may be curious about the Stirling numbers of the first kind. A combinatorial definition of the **signless Stirling numbers of the first kind**, the absolute values of the Stirling numbers of the first kind, can be found in the preamble to Exercise 47 in the Supplementary Exercises. For the definition of Stirling numbers of the first kind, for more information about Stirling numbers of the second kind, and to learn more about Stirling numbers of the first kind and the relationship between Stirling numbers of the first and second kind, see combinatorics textbooks such as [Bó07], [Br99], and [RoTe05], and Chapter 6 in [MiRo91].

INDISTINGUISHABLE OBJECTS AND INDISTINGUISHABLE BOXES Some counting problems can be solved by determining the number of ways to distribute indistinguishable objects into indistinguishable boxes. We illustrate this principle with an example.

EXAMPLE 11 How many ways are there to pack six copies of the same book into four identical boxes, where a box can contain as many as six books?

Solution: We will enumerate all ways to pack the books. For each way to pack the books, we will list the number of books in the box with the largest number of books, followed by the numbers of books in each box containing at least one book, in order of decreasing number of books in a box. The ways we can pack the books are

6	3, 2, 1
5, 1	3, 1, 1, 1
4, 2	2, 2, 2
4, 1, 1	2, 2, 1, 1.
3, 3	

For example, 4, 1, 1 indicates that one box contains four books, a second box contains a single book, and a third box contains a single book (and the fourth box is empty). We conclude that there are nine allowable ways to pack the books, because we have listed them all. ◀

Observe that distributing n indistinguishable objects into k indistinguishable boxes is the same as writing n as the sum of at most k positive integers in nonincreasing order. If $a_1 + a_2 + \cdots + a_j = n$, where $a_1, a_2, \ldots, a_j$ are positive integers with $a_1 \geq a_2 \geq \cdots \geq a_j$, we say that $a_1, a_2, \ldots, a_j$ is a **partition** of the positive integer n into j positive integers. We see that if $p_k(n)$ is the number of partitions of n into at most k positive integers, then there are $p_k(n)$ ways to distribute n indistinguishable objects into k indistinguishable boxes. No simple closed formula exists for this number. For more information about partitions of positive integers, see [Ro11].

Exercises

1. In how many different ways can five elements be selected in order from a set with three elements when repetition is allowed?

2. In how many different ways can five elements be selected in order from a set with five elements when repetition is allowed?

3. How many strings of six letters are there?

4. Every day a student randomly chooses a sandwich for lunch from a pile of wrapped sandwiches. If there are six kinds of sandwiches, how many different ways are there for the student to choose sandwiches for the seven days of a week if the order in which the sandwiches are chosen matters?

5. How many ways are there to assign three jobs to five employees if each employee can be given more than one job?

6. How many ways are there to select five unordered elements from a set with three elements when repetition is allowed?

7. How many ways are there to select three unordered elements from a set with five elements when repetition is allowed?

8. How many different ways are there to choose a dozen donuts from the 21 varieties at a donut shop?

9. A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose

a) six bagels?
b) a dozen bagels?
c) two dozen bagels?
d) a dozen bagels with at least one of each kind?
e) a dozen bagels with at least three egg bagels and no more than two salty bagels?

10. A croissant shop has plain croissants, cherry croissants, chocolate croissants, almond croissants, apple croissants, and broccoli croissants. How many ways are there to choose

a) a dozen croissants?
b) three dozen croissants?
c) two dozen croissants with at least two of each kind?
d) two dozen croissants with no more than two broccoli croissants?
e) two dozen croissants with at least five chocolate croissants and at least three almond croissants?
f) two dozen croissants with at least one plain croissant, at least two cherry croissants, at least three chocolate croissants, at least one almond croissant, at least two apple croissants, and no more than three broccoli croissants?

11. How many ways are there to choose eight coins from a piggy bank containing 100 identical pennies and 80 identical nickels?

12. How many different combinations of pennies, nickels, dimes, quarters, and half dollars can a piggy bank contain if it has 20 coins in it?

13. A book publisher has 3000 copies of a discrete mathematics book. How many ways are there to store these books in their three warehouses if the copies of the book are indistinguishable?

14. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 17,$$

where x_1, x_2, x_3, and x_4 are nonnegative integers?

15. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i, $i = 1, 2, 3, 4, 5$, is a nonnegative integer such that

a) $x_1 \geq 1$?
b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?
c) $0 \leq x_1 \leq 10$?
d) $0 \leq x_1 \leq 3$, $1 \leq x_2 < 4$, and $x_3 \geq 15$?

16. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29,$$

where x_i, $i = 1, 2, 3, 4, 5, 6$, is a nonnegative integer such that

a) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$?
b) $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, $x_5 > 5$, and $x_6 \geq 6$?
c) $x_1 \leq 5$?
d) $x_1 < 8$ and $x_2 > 8$?

17. How many strings of 10 ternary digits (0, 1, or 2) are there that contain exactly two 0s, three 1s, and five 2s?

18. How many strings of 20-decimal digits are there that contain two 0s, four 1s, three 2s, one 3, two 4s, three 5s, two 7s, and three 9s?

19. Suppose that a large family has 14 children, including two sets of identical triplets, three sets of identical twins, and two individual children. How many ways are there to seat these children in a row of chairs if the identical triplets or twins cannot be distinguished from one another?

20. How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where x_1, x_2, and x_3 are nonnegative integers? [*Hint:* Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$.]

21. A Swedish tour guide has devised a clever way for his clients to recognize him. He owns 13 pairs of shoes of the same style, customized so that each pair has a unique color. How many ways are there for him to choose a left shoe and a right shoe from these 13 pairs

a) without restrictions and which color is on which foot matters?
b) so that the colors of the left and right shoe are different and which color is on which foot matters?
c) so that the colors of the left and right shoe are different but which color is on which foot does not matter?
d) without restrictions, but which color is on which foot does not matter?

***22.** In how many ways can an airplane pilot be scheduled for five days of flying in October if he cannot work on consecutive days?

23. How many ways are there to distribute six indistinguishable balls into nine distinguishable bins?

24. How many ways are there to distribute 12 indistinguishable balls into six distinguishable bins?

25. How many ways are there to distribute 12 distinguishable objects into six distinguishable boxes so that two objects are placed in each box?

26. How many ways are there to distribute 15 distinguishable objects into five distinguishable boxes so that the boxes have one, two, three, four, and five objects in them, respectively.

27. How many positive integers less than 1,000,000 have the sum of their digits equal to 19?

28. How many positive integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 13?

29. There are 10 questions on a discrete mathematics final exam. How many ways are there to assign scores to the problems if the sum of the scores is 100 and each question is worth at least 5 points?

30. Show that there are $C(n + r - q_1 - q_2 - \cdots - q_r - 1, n - q_1 - q_2 - \cdots - q_r)$ different unordered selections of n objects of r different types that include at least q_1 objects of type one, q_2 objects of type two, $\ldots$, and q_r objects of type r.

31. How many different bit strings can be transmitted if the string must begin with a 1 bit, must include three additional 1 bits (so that a total of four 1 bits is sent), must include a total of 12 0 bits, and must have at least two 0 bits following each 1 bit?

32. How many different strings can be made from the letters in *MISSISSIPPI*, using all the letters?

33. How many different strings can be made from the letters in *ABRACADABRA*, using all the letters?

34. How many different strings can be made from the letters in *AARDVARK*, using all the letters, if all three *A*s must be consecutive?

35. How many different strings can be made from the letters in *ORONO*, using some or all of the letters?

36. How many strings with five or more characters can be formed from the letters in *SEERESS*?

37. How many strings with seven or more characters can be formed from the letters in *EVERGREEN*?

38. How many different bit strings can be formed using six 1s and eight 0s?

39. A student has three mangos, two papayas, and two kiwi fruits. If the student eats one piece of fruit each day, and only the type of fruit matters, in how many different ways can these fruits be consumed?

40. A professor packs her collection of 40 issues of a mathematics journal in four boxes with 10 issues per box. How many ways can she distribute the journals if

a) each box is numbered, so that they are distinguishable?

b) the boxes are identical, so that they cannot be distinguished?

41. How many ways are there to travel in xyz space from the origin $(0, 0, 0)$ to the point $(4, 3, 5)$ by taking steps one unit in the positive x direction, one unit in the positive y direction, or one unit in the positive z direction? (Moving in the negative x, y, or z direction is prohibited, so that no backtracking is allowed.)

42. How many ways are there to travel in $xyzw$ space from the origin $(0, 0, 0, 0)$ to the point $(4, 3, 5, 4)$ by taking steps one unit in the positive x, positive y, positive z, or positive w direction?

43. How many ways are there to deal hands of seven cards to each of five players from a standard deck of 52 cards?

44. In bridge, the 52 cards of a standard deck are dealt to four players. How many different ways are there to deal bridge hands to four players?

45. How many ways are there to deal hands of five cards to each of six players from a deck containing 48 different cards?

46. In how many ways can a dozen books be placed on four distinguishable shelves

a) if the books are indistinguishable copies of the same title?

b) if no two books are the same, and the positions of the books on the shelves matter? [*Hint:* Break this into 12 tasks, placing each book separately. Start with the sequence 1, 2, 3, 4 to represent the shelves. Represent the books by b_i, $i = 1, 2, \ldots, 12$. Place b_1 to the right of one of the terms in 1, 2, 3, 4. Then successively place $b_2, b_3, \ldots,$ and b_{12}.]

47. How many ways can n books be placed on k distinguishable shelves

a) if the books are indistinguishable copies of the same title?

b) if no two books are the same, and the positions of the books on the shelves matter?

48. A shelf holds 12 books in a row. How many ways are there to choose five books so that no two adjacent books are chosen? [*Hint:* Represent the books that are chosen by bars and the books not chosen by stars. Count the number of sequences of five bars and seven stars so that no two bars are adjacent.]

***49.** Use the product rule to prove Theorem 4, by first placing objects in the first box, then placing objects in the second box, and so on.

***50.** Prove Theorem 4 by first setting up a one-to-one correspondence between permutations of n objects with n_i indistinguishable objects of type i, $i = 1, 2, 3, \ldots, k$, and the distributions of n objects in k boxes such that n_i objects are placed in box i, $i = 1, 2, 3, \ldots, k$ and then applying Theorem 3.

***51.** In this exercise we will prove Theorem 2 by setting up a one-to-one correspondence between the set of r-combinations with repetition allowed of $S = \{1, 2, 3, \ldots, n\}$ and the set of r-combinations of the set $T = \{1, 2, 3, \ldots, n + r - 1\}$.

a) Arrange the elements in an r-combination, with repetition allowed, of S into an increasing sequence $x_1 \leq x_2 \leq \cdots \leq x_r$. Show that the sequence formed by adding $k - 1$ to the kth term is strictly increasing. Conclude that this sequence is made up of r distinct elements from T.

b) Show that the procedure described in (a) defines a one-to-one correspondence between the set of r-combinations, with repetition allowed, of S and the r-combinations of T. [*Hint:* Show the correspondence can be reversed by associating to the r-combination $\{x_1, x_2, \ldots, x_r\}$ of T, with $1 \leq x_1 < x_2 < \cdots < x_r \leq n + r - 1$, the r-combination with repetition allowed from S, formed by subtracting $k - 1$ from the kth element.]

c) Conclude that the number of r-combinations with repetition allowed from a set with n elements is $C(n + r - 1, r)$.

52. How many ways are there to distribute five distinguishable objects into three indistinguishable boxes?

53. How many ways are there to distribute six distinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

54. How many ways are there to put five temporary employees into four identical offices?

55. How many ways are there to put six temporary employees into four identical offices so that there is at least one temporary employee in each of these four offices?

56. How many ways are there to distribute five indistinguishable objects into three indistinguishable boxes?

57. How many ways are there to distribute six indistinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

58. How many ways are there to pack eight identical DVDs into five indistinguishable boxes so that each box contains at least one DVD?

59. How many ways are there to pack nine identical DVDs into three indistinguishable boxes so that each box contains at least two DVDs?

60. How many ways are there to distribute five balls into seven boxes if each box must have at most one ball in it if

a) both the balls and boxes are labeled?
b) the balls are labeled, but the boxes are unlabeled?
c) the balls are unlabeled, but the boxes are labeled?
d) both the balls and boxes are unlabeled?

61. How many ways are there to distribute five balls into three boxes if each box must have at least one ball in it if

a) both the balls and boxes are labeled?
b) the balls are labeled, but the boxes are unlabeled?
c) the balls are unlabeled, but the boxes are labeled?
d) both the balls and boxes are unlabeled?

62. Suppose that a basketball league has 32 teams, split into two conferences of 16 teams each. Each conference is split into three divisions. Suppose that the North Central Division has five teams. Each of the teams in the North Central Division plays four games against each of the other teams in this division, three games against each of the 11 remaining teams in the conference, and two games against each of the 16 teams in the other conference. In how many different orders can the games of one of the teams in the North Central Division be scheduled?

***63.** Suppose that a weapons inspector must inspect each of five different sites twice, visiting one site per day. The inspector is free to select the order in which to visit these sites, but cannot visit site X, the most suspicious site, on two consecutive days. In how many different orders can the inspector visit these sites?

64. How many different terms are there in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$ after all terms with identical sets of exponents are added?

***65.** Prove the **Multinomial Theorem:** If n is a positive integer, then

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1 + n_2 + \cdots + n_m = n} C(n; n_1, n_2, \ldots, n_m) x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m},$$

where

$$C(n; n_1, n_2, \ldots, n_m) = \frac{n!}{n_1!\, n_2! \cdots n_m!}$$

is a **multinomial coefficient**.

66. Find the expansion of $(x + y + z)^4$.

67. Find the coefficient of $x^3 y^2 z^5$ in $(x + y + z)^{10}$.

68. How many terms are there in the expansion of

$$(x + y + z)^{100}?$$

6.6 Generating Permutations and Combinations

6.6.1 Introduction

Methods for counting various types of permutations and combinations were described in the previous sections of this chapter, but sometimes permutations or combinations need to be generated, not just counted. Consider the following three problems. First, suppose that a salesperson must visit six different cities. In which order should these cities be visited to minimize total travel time? One way to determine the best order is to determine the travel time for each of the $6! = 720$ different orders in which the cities can be visited and choose the one with the smallest travel time. Second, suppose we are given a set of six positive integers and wish to find a subset of them that has 100 as their sum, if such a subset exists. One way to find these numbers is to generate all $2^6 = 64$ subsets and check the sum of their elements. Third, suppose a laboratory has 95 employees. A group of 12 of these employees with a particular set of 25 skills is needed for a project. (Each employee can have one or more of these skills.) One way to find such a set of employees is to generate all sets of 12 of these employees and check whether they have the desired skills. These examples show that it is often necessary to generate permutations and combinations to solve problems.

6.6.2 Generating Permutations

Links

Any set with n elements can be placed in one-to-one correspondence with the set $\{1, 2, 3, \ldots, n\}$. We can list the permutations of any set of n elements by generating the permutations of the n smallest positive integers and then replacing these integers with the corresponding elements. Many different algorithms have been developed to generate the $n!$ permutations of this set. We will describe one of these that is based on the **lexicographic** (or **dictionary**) **ordering** of the set of permutations of $\{1, 2, 3, \ldots, n\}$. In this ordering, the permutation $a_1a_2 \cdots a_n$ precedes the permutation of $b_1b_2 \cdots b_n$, if for some k, with $1 \leq k \leq n$, $a_1 = b_1$, $a_2 = b_2$, …, $a_{k-1} = b_{k-1}$, and $a_k < b_k$. In other words, a permutation of the set of the n smallest positive integers precedes (in lexicographic order) a second permutation if the number in this permutation in the first position where the two permutations disagree is smaller than the number in that position in the second permutation.

EXAMPLE 1 The permutation 23415 of the set $\{1, 2, 3, 4, 5\}$ precedes the permutation 23514, because these permutations agree in the first two positions, but the number in the third position in the first permutation, 4, is smaller than the number in the third position in the second permutation, 5. Similarly, the permutation 41532 precedes 52143. ◀

An algorithm for generating the permutations of $\{1, 2, \ldots, n\}$ can be based on a procedure that constructs the next permutation in lexicographic order following a given permutation $a_1a_2 \cdots a_n$. We will show how this can be done. First, suppose that $a_{n-1} < a_n$. Interchange a_{n-1} and a_n to obtain a larger permutation. No other permutation is both larger than the original permutation and smaller than the permutation obtained by interchanging a_{n-1} and a_n. For instance, the next larger permutation after 234156 is 234165. On the other hand, if $a_{n-1} > a_n$, then a larger permutation cannot be obtained by interchanging these last two terms in the permutation. Look at the last three integers in the permutation. If $a_{n-2} < a_{n-1}$, then the last three integers in the permutation can be rearranged to obtain the next largest permutation. Put the smaller of the two integers a_{n-1} and a_n that is greater than a_{n-2} in position $n - 2$. Then, place the remaining integer and a_{n-2} into the last two positions in increasing order. For instance, the next larger permutation after 234165 is 234516.

On the other hand, if $a_{n-2} > a_{n-1}$ (and $a_{n-1} > a_n$), then a larger permutation cannot be obtained by permuting the last three terms in the permutation. Based on these observations, a general method can be described for producing the next larger permutation in increasing order following a given permutation $a_1a_2 \cdots a_n$. First, find the integers a_j and a_{j+1} with $a_j < a_{j+1}$ and

$$a_{j+1} > a_{j+2} > \cdots > a_n,$$

that is, the last pair of adjacent integers in the permutation where the first integer in the pair is smaller than the second. Then, the next larger permutation in lexicographic order is obtained by putting in the jth position the least integer among $a_{j+1}, a_{j+2}, \ldots,$ and a_n that is greater than a_j and listing in increasing order the rest of the integers $a_j, a_{j+1}, \ldots, a_n$ in positions $j + 1$ to n. It is easy to see that there is no other permutation larger than the permutation $a_1a_2 \cdots a_n$ but smaller than the new permutation produced. (The verification of this fact is left as an exercise for the reader.)

EXAMPLE 2 What is the next permutation in lexicographic order after 362541?

Extra Examples

Solution: The last pair of integers a_j and a_{j+1} where $a_j < a_{j+1}$ is $a_3 = 2$ and $a_4 = 5$. The least integer to the right of 2 that is greater than 2 in the permutation is $a_5 = 4$. Hence, 4 is placed in the third position. Then the integers 2, 5, and 1 are placed in order in the last three positions, giving 125 as the last three positions of the permutation. Hence, the next permutation is 364125. ◀

To produce the $n!$ permutations of the integers $1, 2, 3, \ldots, n$, begin with the smallest permutation in lexicographic order, namely, $123 \cdots n$, and successively apply the procedure described for producing the next larger permutation of $n! - 1$ times. This yields all the permutations of the n smallest integers in lexicographic order.

EXAMPLE 3 Generate the permutations of the integers 1, 2, 3 in lexicographic order.

Solution: Begin with 123. The next permutation is obtained by interchanging 3 and 2 to obtain 132. Next, because $3 > 2$ and $1 < 3$, permute the three integers in 132. Put the smaller of 3 and 2 in the first position, and then put 1 and 3 in increasing order in positions 2 and 3 to obtain 213. This is followed by 231, obtained by interchanging 1 and 3, because $1 < 3$. The next larger permutation has 3 in the first position, followed by 1 and 2 in increasing order, namely, 312. Finally, interchange 1 and 2 to obtain the last permutation, 321. We have generated the permutations of 1, 2, 3 in lexicographic order. They are 123, 132, 213, 231, 312, and 321. ◀

Algorithm 1 displays the procedure for finding the next permutation in lexicographic order after a permutation that is not $n\ \ n-1\ \ n-2\ \ \ldots\ \ 2\ \ 1$, which is the largest permutation.

ALGORITHM 1 Generating the Next Permutation in Lexicographic Order.

procedure *next permutation*($a_1a_2 \ldots a_n$: permutation of
    $\{1, 2, \ldots, n\}$ not equal to $n\ \ n-1\ \ \ldots\ \ 2\ \ 1$)
$j := n - 1$
while $a_j > a_{j+1}$
  $j := j - 1$
$\{j$ is the largest subscript with $a_j < a_{j+1}\}$
$k := n$
while $a_j > a_k$
  $k := k - 1$
$\{a_k$ is the smallest integer greater than a_j to the right of $a_j\}$
interchange a_j and a_k
$r := n$
$s := j + 1$
while $r > s$
  interchange a_r and a_s
  $r := r - 1$
  $s := s + 1$
{this puts the tail end of the permutation after the jth position in increasing order}
$\{a_1a_2 \ldots a_n$ is now the next permutation$\}$

6.6.3 Generating Combinations

Links

How can we generate all the combinations of the elements of a finite set? Because a combination is just a subset, we can use the correspondence between subsets of $\{a_1, a_2, \ldots, a_n\}$ and bit strings of length n.

Recall that the bit string corresponding to a subset has a 1 in position k if a_k is in the subset, and has a 0 in this position if a_k is not in the subset. If all the bit strings of length n can be listed, then by the correspondence between subsets and bit strings, a list of all the subsets is obtained.

Recall that a bit string of length n is also the binary expansion of an integer between 0 and $2^n - 1$. The 2^n bit strings can be listed in order of their increasing size as integers in their binary expansions. To produce all binary expansions of length n, start with the bit string $000 \ldots 00$, with n zeros. Then, successively find the next expansion until the bit string $111 \ldots 11$ is obtained. At each stage the next binary expansion is found by locating the first position from the right that is not a 1, then changing all the 1s to the right of this position to 0s and making this first 0 (from the right) a 1.

EXAMPLE 4 Find the next bit string after 10 0010 0111.

Solution: The first bit from the right that is not a 1 is the fourth bit from the right. Change this bit to a 1 and change all the following bits to 0s. This produces the next larger bit string, 10 0010 1000. ◀

The procedure for producing the next larger bit string after $b_{n-1}b_{n-2}\dots b_1b_0$ is given as Algorithm 2.

ALGORITHM 2 Generating the Next Larger Bit String.

procedure *next bit string*($b_{n-1}\,b_{n-2}\dots b_1b_0$: bit string not equal to $11\dots11$)
$i := 0$
while $b_i = 1$
  $b_i := 0$
  $i := i + 1$
$b_i := 1$
$\{b_{n-1}\,b_{n-2}\dots b_1b_0$ is now the next bit string$\}$

Next, an algorithm for generating the r-combinations of the set $\{1, 2, 3, \dots, n\}$ will be given. An r-combination can be represented by a sequence containing the elements in the subset in increasing order. The r-combinations can be listed using lexicographic order on these sequences. In this lexicographic ordering, the first r-combination is $\{1, 2, \dots, r-1, r\}$ and the last r-combination is $\{n-r+1, n-r+2, \dots, n-1, n\}$. The next r-combination after $a_1a_2 \cdots a_r$ can be obtained in the following way: First, locate the last element a_i in the sequence such that $a_i \neq n - r + i$. Then, replace a_i with $a_i + 1$ and a_j with $a_i + j - i + 1$, for $j = i + 1, i + 2, \dots, r$. It is left for the reader to show that this produces the next larger r-combination in lexicographic order. This procedure is illustrated with Example 5.

EXAMPLE 5 Find the next larger 4-combination of the set $\{1, 2, 3, 4, 5, 6\}$ after $\{1, 2, 5, 6\}$.

Solution: The last term among the terms a_i with $a_1 = 1$, $a_2 = 2$, $a_3 = 5$, and $a_4 = 6$ such that $a_i \neq 6 - 4 + i$ is $a_2 = 2$. To obtain the next larger 4-combination, increment a_2 by 1 to obtain $a_2 = 3$. Then set $a_3 = 3 + 1 = 4$ and $a_4 = 3 + 2 = 5$. Hence the next larger 4-combination is $\{1, 3, 4, 5\}$. ◀

Algorithm 3 displays pseudocode for this procedure.

ALGORITHM 3 Generating the Next *r*-Combination in Lexicographic Order.

procedure *next r-combination*($\{a_1, a_2, \dots, a_r\}$: proper subset of
    $\{1, 2, \dots, n\}$ not equal to $\{n-r+1, \dots, n\}$ with
    $a_1 < a_2 < \cdots < a_r$)
$i := r$
while $a_i = n - r + i$
  $i := i - 1$
$a_i := a_i + 1$
for $j := i + 1$ **to** r
  $a_j := a_i + j - i$
$\{\{a_1, a_2, \dots, a_r\}$ is now the next combination$\}$

Exercises

1. Place these permutations of $\{1, 2, 3, 4, 5\}$ in lexicographic order: 43521, 15432, 45321, 23451, 23514, 14532, 21345, 45213, 31452, 31542.
2. Place these permutations of $\{1,2,3,4,5,6\}$ in lexicographic order: 234561, 231456, 165432, 156423, 543216, 541236, 231465, 314562, 432561, 654321, 654312, 435612.
3. The name of a file in a computer directory consists of three uppercase letters followed by a digit, where each letter is either A, B, or C, and each digit is either 1 or 2. List the name of these files in lexicographic order, where we order letters using the usual alphabetic order of letters.
4. Suppose that the name of a file in a computer directory consists of three digits followed by two lowercase letters and each digit is 0, 1, or 2, and each letter is either a or b. List the name of these files in lexicographic order, where we order letters using the usual alphabetic order of letters.
5. Find the next larger permutation in lexicographic order after each of these permutations.
   a) 1432 **b)** 54123 **c)** 12453
   d) 45231 **e)** 6714235 **f)** 31528764
6. Find the next larger permutation in lexicographic order after each of these permutations.
   a) 1342 **b)** 45321 **c)** 13245
   d) 612345 **e)** 1623547 **f)** 23587416
7. Use Algorithm 1 to generate the 24 permutations of the first four positive integers in lexicographic order.
8. Use Algorithm 2 to list all the subsets of the set $\{1, 2, 3, 4\}$.
9. Use Algorithm 3 to list all the 3-combinations of $\{1, 2, 3, 4, 5\}$.
10. Show that Algorithm 1 produces the next larger permutation in lexicographic order.
11. Show that Algorithm 3 produces the next larger r-combination in lexicographic order after a given r-combination.
12. Develop an algorithm for generating the r-permutations of a set of n elements.
13. List all 3-permutations of $\{1, 2, 3, 4, 5\}$.

The remaining exercises in this section develop another algorithm for generating the permutations of $\{1, 2, 3, \ldots, n\}$. This algorithm is based on Cantor expansions of integers. Every nonnegative integer less than $n!$ has a unique Cantor expansion

$$a_1 1! + a_2 2! + \cdots + a_{n-1}(n-1)!$$

where a_i is a nonnegative integer not exceeding i, for $i = 1, 2, \ldots, n-1$. The integers $a_1, a_2, \ldots, a_{n-1}$ are called the **Cantor digits** of this integer.

Given a permutation of $\{1, 2, \ldots, n\}$, let a_{k-1}, $k = 2, 3, \ldots, n$, be the number of integers less than k that follow k in the permutation. For instance, in the permutation 43215, a_1 is the number of integers less than 2 that follow 2, so $a_1 = 1$. Similarly, for this example $a_2 = 2$, $a_3 = 3$, and $a_4 = 0$. Consider the function from the set of permutations of $\{1, 2, 3, \ldots, n\}$ to the set of nonnegative integers less than $n!$ that sends a permutation to the integer that has $a_1, a_2, \ldots, a_{n-1}$, defined in this way, as its Cantor digits.

14. Find the Cantor digits $a_1, a_2, \ldots, a_{n-1}$ that correspond to these permutations.
    a) 246531 **b)** 12345 **c)** 654321
*15. Show that the correspondence described in the preamble is a bijection between the set of permutations of $\{1, 2, 3, \ldots, n\}$ and the nonnegative integers less than $n!$.
16. Find the permutations of $\{1, 2, 3, 4, 5\}$ that correspond to these integers with respect to the correspondence between Cantor expansions and permutations as described in the preamble to Exercise 14.
    a) 3 **b)** 89 **c)** 111
17. Develop an algorithm for producing all permutations of a set of n elements based on the correspondence described in the preamble to Exercise 14.

Key Terms and Results

TERMS

combinatorics: the study of arrangements of objects

enumeration: the counting of arrangements of objects

tree diagram: a diagram made up of a root, branches leaving the root, and other branches leaving some of the endpoints of branches

permutation: an ordered arrangement of the elements of a set

***r*-permutation:** an ordered arrangement of r elements of a set

$\boldsymbol{P(n,r)}$**:** the number of r-permutations of a set with n elements

***r*-combination:** an unordered selection of r elements of a set

$\boldsymbol{C(n,r)}$**:** the number of r-combinations of a set with n elements

binomial coefficient $\boldsymbol{\binom{n}{r}}$**:** also the number of r-combinations of a set with n elements

combinatorial proof: a proof that uses counting arguments rather than algebraic manipulation to prove a result

Pascal's triangle: a representation of the binomial coefficients where the ith row of the triangle contains $\binom{i}{j}$ for $j = 0, 1, 2, \ldots, i$

$\boldsymbol{S(n,j)}$**:** the Stirling number of the second kind denoting the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no box is empty

RESULTS

product rule for counting: The number of ways to do a procedure that consists of two tasks is the product of the number of ways to do the first task and the number of ways to do the second task after the first task has been done.

product rule for sets: The number of elements in the Cartesian product of finite sets is the product of the number of elements in each set.

sum rule for counting: The number of ways to do a task in one of two ways is the sum of the number of ways to do these tasks if they cannot be done simultaneously.

sum rule for sets: The number of elements in the union of pairwise disjoint finite sets is the sum of the numbers of elements in these sets.

subtraction rule for counting or **inclusion–exclusion for sets:** If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

subtraction rule or **inclusion–exclusion for sets:** The number of elements in the union of two sets is the sum of the number of elements in these sets minus the number of elements in their intersection.

division rule for counting: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

division rule for sets: Suppose that a finite set A is the union of n disjoint subsets each with d elements. Then $n = |A|/d$.

the pigeonhole principle: When more than k objects are placed in k boxes, there must be a box containing more than one object.

the generalized pigeonhole principle: When N objects are placed in k boxes, there must be a box containing at least $\lceil N/k \rceil$ objects.

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Pascal's identity: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

the binomial theorem: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

There are n^r r-permutations of a set with n elements when repetition is allowed.

There are $C(n + r - 1, r)$ r-combinations of a set with n elements when repetition is allowed.

There are $n!/(n_1! n_2! \cdots n_k!)$ permutations of n objects of k types where there are n_i indistinguishable objects of type i for $i = 1, 2, 3, \ldots, k$.

the algorithm for generating the permutations of the set $\{1, 2, \ldots, n\}$

Review Questions

1. Explain how the sum and product rules can be used to find the number of bit strings with a length not exceeding 10.

2. Explain how to find the number of bit strings of length not exceeding 10 that have at least one 0 bit.

3. **a)** How can the product rule be used to find the number of functions from a set with m elements to a set with n elements?
   b) How many functions are there from a set with five elements to a set with 10 elements?
   c) How can the product rule be used to find the number of one-to-one functions from a set with m elements to a set with n elements?
   d) How many one-to-one functions are there from a set with five elements to a set with 10 elements?
   e) How many onto functions are there from a set with five elements to a set with 10 elements?

4. How can you find the number of possible outcomes of a playoff between two teams where the first team that wins four games wins the playoff?

5. How can you find the number of bit strings of length ten that either begin with 101 or end with 010?

6. **a)** State the pigeonhole principle.
   b) Explain how the pigeonhole principle can be used to show that among any 11 integers, at least two must have the same last digit.

7. **a)** State the generalized pigeonhole principle.
   b) Explain how the generalized pigeonhole principle can be used to show that among any 91 integers, there are at least ten that end with the same digit.

8. **a)** What is the difference between an r-combination and an r-permutation of a set with n elements?
   b) Derive an equation that relates the number of r-combinations and the number of r-permutations of a set with n elements.
   c) How many ways are there to select six students from a class of 25 to serve on a committee?
   d) How many ways are there to select six students from a class of 25 to hold six different executive positions on a committee?

9. **a)** What is Pascal's triangle?
   b) How can a row of Pascal's triangle be produced from the one above it?

10. What is meant by a combinatorial proof of an identity? How is such a proof different from an algebraic one?

11. Explain how to prove Pascal's identity using a combinatorial argument.

12. **a)** State the binomial theorem.

b) Explain how to prove the binomial theorem using a combinatorial argument.

c) Find the coefficient of $x^{100}y^{101}$ in the expansion of $(2x + 5y)^{201}$.

13. **a)** Explain how to find a formula for the number of ways to select r objects from n objects when repetition is allowed and order does not matter.

b) How many ways are there to select a dozen objects from among objects of five different types if objects of the same type are indistinguishable?

c) How many ways are there to select a dozen objects from these five different types if there must be at least three objects of the first type?

d) How many ways are there to select a dozen objects from these five different types if there cannot be more than four objects of the first type?

e) How many ways are there to select a dozen objects from these five different types if there must be at least two objects of the first type, but no more than three objects of the second type?

14. **a)** Let n and r be positive integers. Explain why the number of solutions of the equation $x_1 + x_2 + \cdots + x_n = r$, where x_i is a nonnegative integer for $i = 1, 2, 3, \ldots, n$, equals the number of r-combinations of a set with n elements.

b) How many solutions in nonnegative integers are there to the equation $x_1 + x_2 + x_3 + x_4 = 17$?

c) How many solutions in positive integers are there to the equation in part (b)?

15. **a)** Derive a formula for the number of permutations of n objects of k different types, where there are n_1 indistinguishable objects of type one, n_2 indistinguishable objects of type two, …, and n_k indistinguishable objects of type k.

b) How many ways are there to order the letters of the word *INDISCREETNESS*?

16. Describe an algorithm for generating all the permutations of the set of the n smallest positive integers.

17. **a)** How many ways are there to deal hands of five cards to six players from a standard 52-card deck?

b) How many ways are there to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed in box i?

18. Describe an algorithm for generating all the combinations of the set of the n smallest positive integers.

Supplementary Exercises

1. How many ways are there to choose 6 items from 10 distinct items when

a) the items in the choices are ordered and repetition is not allowed?

b) the items in the choices are ordered and repetition is allowed?

c) the items in the choices are unordered and repetition is not allowed?

d) the items in the choices are unordered and repetition is allowed?

2. How many ways are there to choose 10 items from 6 distinct items when

a) the items in the choices are ordered and repetition is not allowed?

b) the items in the choices are ordered and repetition is allowed?

c) the items in the choices are unordered and repetition is not allowed?

d) the items in the choices are unordered and repetition is allowed?

3. A test contains 100 true/false questions. How many different ways can a student answer the questions on the test, if answers may be left blank?

4. How many strings of length 10 either start with 000 or end with 1111?

5. How many bit strings of length 10 over the alphabet $\{a, b, c\}$ have either exactly three as or exactly four bs?

6. The internal telephone numbers in the phone system on a campus consist of five digits, with the first digit not equal to zero. How many different numbers can be assigned in this system?

7. An ice cream parlor has 28 different flavors, 8 different kinds of sauce, and 12 toppings.

a) In how many different ways can a dish of three scoops of ice cream be made where each flavor can be used more than once and the order of the scoops does not matter?

b) How many different kinds of small sundaes are there if a small sundae contains one scoop of ice cream, a sauce, and a topping?

c) How many different kinds of large sundaes are there if a large sundae contains three scoops of ice cream, where each flavor can be used more than once and the order of the scoops does not matter; two kinds of sauce, where each sauce can be used only once and the order of the sauces does not matter; and three toppings, where each topping can be used only once and the order of the toppings does not matter?

8. How many positive integers less than 1000

a) have exactly three decimal digits?

b) have an odd number of decimal digits?

c) have at least one decimal digit equal to 9?

d) have no odd decimal digits?

e) have two consecutive decimal digits equal to 5?

f) are palindromes (that is, read the same forward and backward)?

9. When the numbers from 1 to 1000 are written out in decimal notation, how many of each of these digits are used?

a) 0 **b)** 1 **c)** 2 **d)** 9

10. There are 12 signs of the zodiac. How many people are needed to guarantee that at least six of these people have the same sign?

11. A fortune cookie company makes 213 different fortunes. A student eats at a restaurant that uses fortunes from this company and gives each customer one fortune cookie at the end of a meal. What is the largest possible number of times that the student can eat at the restaurant without getting the same fortune four times?

12. How many people are needed to guarantee that at least two were born on the same day of the week and in the same month (perhaps in different years)?

13. Show that given any set of 10 positive integers not exceeding 50 there exist at least two different five-element subsets of this set that have the same sum.

14. A package of baseball cards contains 20 cards. How many packages must be purchased to ensure that two cards in these packages are identical if there are a total of 550 different cards?

15. **a)** How many cards must be chosen from a standard deck of 52 cards to guarantee that at least two of the four aces are chosen?

b) How many cards must be chosen from a standard deck of 52 cards to guarantee that at least two of the four aces and at least two of the 13 kinds are chosen?

c) How many cards must be chosen from a standard deck of 52 cards to guarantee that there are at least two cards of the same kind?

d) How many cards must be chosen from a standard deck of 52 cards to guarantee that there are at least two cards of each of two different kinds?

***16.** Show that in any set of $n + 1$ positive integers not exceeding $2n$ there must be two that are relatively prime.

***17.** Show that in a sequence of m integers there exists one or more consecutive terms with a sum divisible by m.

18. Show that if five points are picked in the interior of a square with a side length of 2, then at least two of these points are no farther than $\sqrt{2}$ apart.

19. Show that the decimal expansion of a rational number must repeat itself from some point onward.

20. Once a computer worm infects a personal computer via an infected e-mail message, it sends a copy of itself to 100 e-mail addresses it finds in the electronic message mailbox on this personal computer. What is the maximum number of different computers this one computer can infect in the time it takes for the infected message to be forwarded five times?

21. How many ways are there to choose a dozen donuts from 20 varieties

a) if there are no two donuts of the same variety?

b) if all donuts are of the same variety?

c) if there are no restrictions?

d) if there are at least two varieties among the dozen donuts chosen?

e) if there must be at least six blueberry-filled donuts?

f) if there can be no more than six blueberry-filled donuts?

22. Find n if

a) $P(n, 2) = 110$. **b)** $P(n, n) = 5040$.

c) $P(n, 4) = 12P(n, 2)$.

23. Find n if

a) $C(n, 2) = 45$. **b)** $C(n, 3) = P(n, 2)$.

c) $C(n, 5) = C(n, 2)$.

24. Show that if n and r are nonnegative integers and $n \geq r$, then

$$P(n + 1, r) = P(n, r)(n + 1)/(n + 1 - r).$$

***25.** Suppose that S is a set with n elements. How many ordered pairs (A, B) are there such that A and B are subsets of S with $A \subseteq B$? [*Hint:* Show that each element of S belongs to A, $B - A$, or $S - B$.]

26. Give a combinatorial proof of Corollary 2 of Section 6.4 by setting up a correspondence between the subsets of a set with an even number of elements and the subsets of this set with an odd number of elements. [*Hint:* Take an element a in the set. Set up the correspondence by putting a in the subset if it is not already in it and taking it out if it is in the subset.]

27. Let n and r be integers with $1 \leq r < n$. Show that

$$C(n, r - 1) = C(n + 2, r + 1) \\ - 2C(n + 1, r + 1) + C(n, r + 1).$$

28. Prove using mathematical induction that $\sum_{j=2}^{n} C(j, 2) = C(n + 1, 3)$ whenever n is an integer greater than 1.

29. Show that if n is an integer then

$$\sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n.$$

30. Show that $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = \binom{n}{2}$ if n is an integer with $n \geq 2$.

31. Show that $\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} 1 = \binom{n}{3}$ if n is an integer with $n \geq 3$.

32. In this exercise we will derive a formula for the sum of the squares of the n smallest positive integers. We will count the number of triples (i, j, k) where i, j, and k are integers such that $0 \leq i < k$, $0 \leq j < k$, and $1 \leq k \leq n$ in two ways.

a) Show that there are k^2 such triples with a fixed k. Deduce that there are $\sum_{k=1}^{n} k^2$ such triples.

b) Show that the number of such triples with $0 \leq i < j < k$ and the number of such triples with $0 \leq j < i < k$ both equal $C(n + 1, 3)$.

c) Show that the number of such triples with $0 \leq i = j < k$ equals $C(n + 1, 2)$.

d) Combining part (a) with parts (b) and (c), conclude that

$$\sum_{k=1}^{n} k^2 = 2C(n+1,3) + C(n+1,2)$$
$$= n(n+1)(2n+1)/6.$$

***33.** How many bit strings of length n, where $n \geq 4$, contain exactly two occurrences of 01?

34. Let S be a set. We say that a collection of subsets $A_1, A_2, \ldots, A_n$ each containing d elements, where $d \geq 2$, is 2-*colorable* if it is possible to assign to each element of S one of two different colors so that in every subset A_i there are elements that have been assigned each color. Let $m(d)$ be the largest integer such that every collection of fewer than $m(d)$ sets each containing d elements is 2-colorable.

a) Show that the collection of all subsets with d elements of a set S with $2d-1$ elements is not 2-colorable.

b) Show that $m(2) = 3$.

****c)** Show that $m(3) = 7$. [*Hint:* Show that the collection $\{1,3,5\}$, $\{1,2,6\}$, $\{1,4,7\}$, $\{2,3,4\}$, $\{2,5,7\}$, $\{3,6,7\}$, $\{4,5,6\}$ is not 2-colorable. Then show that all collections of six sets with three elements each are 2-colorable.]

35. A professor writes 20 multiple-choice questions, each with the possible answer a, b, c, or d, for a discrete mathematics test. If the number of questions with a, b, c, and d as their answer is 8, 3, 4, and 5, respectively, how many different answer keys are possible, if the questions can be placed in any order?

36. How many different arrangements are there of eight people seated at a round table, where two arrangements are considered the same if one can be obtained from the other by a rotation?

37. How many ways are there to assign 24 students to five faculty advisors?

38. How many ways are there to choose a dozen apples from a bushel containing 20 indistinguishable Delicious apples, 20 indistinguishable Macintosh apples, and 20 indistinguishable Granny Smith apples, if at least three of each kind must be chosen?

39. How many solutions are there to the equation $x_1 + x_2 + x_3 = 17$, where x_1, x_2, and x_3 are nonnegative integers with

a) $x_1 > 1$, $x_2 > 2$, and $x_3 > 3$?

b) $x_1 < 6$ and $x_3 > 5$?

c) $x_1 < 4$, $x_2 < 3$, and $x_3 > 5$?

40. a) How many different strings can be made from the word *PEPPERCORN* when all the letters are used?

b) How many of these strings start and end with the letter *P*?

c) In how many of these strings are the three letter *P*s consecutive?

41. How many subsets of a set with ten elements

a) have fewer than five elements?

b) have more than seven elements?

c) have an odd number of elements?

42. A witness to a hit-and-run accident tells the police that the license plate of the car in the accident, which contains three letters followed by three digits, starts with the letters *AS* and contains both the digits 1 and 2. How many different license plates can fit this description?

43. How many ways are there to put n identical objects into m distinct containers so that no container is empty?

44. How many ways are there to seat six boys and eight girls in a row of chairs so that no two boys are seated next to each other?

45. How many ways are there to distribute six objects to five boxes if

a) both the objects and boxes are labeled?

b) the objects are labeled, but the boxes are unlabeled?

c) the objects are unlabeled, but the boxes are labeled?

d) both the objects and the boxes are unlabeled?

46. How many ways are there to distribute five objects into six boxes if

a) both the objects and boxes are labeled?

b) the objects are labeled, but the boxes are unlabeled?

c) the objects are unlabeled, but the boxes are labeled?

d) both the objects and the boxes are unlabeled?

The **signless Stirling number of the first kind** $c(n,k)$, where k and n are integers with $1 \leq k \leq n$, equals the number of ways to arrange n people around k circular tables with at least one person seated at each table, where two seatings of m people around a circular table are considered the same if everyone has the same left neighbor and the same right neighbor.

47. Find these signless Stirling numbers of the first kind.

a) $c(3,2)$ **b)** $c(4,2)$

c) $c(4,3)$ **d)** $c(5,4)$

48. Show that if n is a positive integer, then $\sum_{j=1}^{n} c(n,j) = n!$.

49. Show that if n is a positive integer with $n \geq 3$, then $c(n, n-2) = (3n-1)C(n,3)/4$.

***50.** Show that if n and k are integers with $1 \leq k < n$, then $c(n+1,k) = c(n,k-1) + nc(n,k)$.

51. Give a combinatorial proof that 2^n divides $n!$ whenever n is an even positive integer. [*Hint:* Use Theorem 3 in Section 6.5 to count the number of permutations of $2n$ objects where there are two indistinguishable objects of n different types.]

52. How many 11-element RNA sequences consist of 4 As, 3 Cs, 2 Us, and 2 Gs, and end with CAA?

Exercises 53 and 54 are based on a discussion in [RoTe09]. A method used in the 1960s for sequencing RNA chains used enzymes to break chains after certain links. Some enzymes break RNA chains after each G link, while others break them after each C or U link. Using these enzymes it is sometimes possible to correctly sequence all the bases in an RNA chain.

***53.** Suppose that when an enzyme that breaks RNA chains after each G link is applied to a 12-link chain, the fragments obtained are G, CCG, AAAG, and UCCG, and when an enzyme that breaks RNA chains after each C or U link is applied, the fragments obtained are C, C, C, C, GGU, and GAAAG. Can you determine the entire 12-link RNA chain from these two sets of fragments? If so, what is this RNA chain?

***54.** Suppose that when an enzyme that breaks RNA chains after each G link is applied to a 12-link chain, the fragments obtained are AC, UG, and ACG and when an enzyme that breaks RNA chains after each C or U link is applied, the fragments obtained are U, GAC, and GAC. Can you determine the entire RNA chain from these two sets of fragments? If so, what is this RNA chain?

55. Devise an algorithm for generating all the r-permutations of a finite set when repetition is allowed.

56. Devise an algorithm for generating all the r-combinations of a finite set when repetition is allowed.

***57.** Show that if m and n are integers with $m \geq 3$ and $n \geq 3$, then $R(m, n) \leq R(m, n-1) + R(m-1, n)$.

***58.** Show that $R(3, 4) \geq 7$ by showing that in a group of six people, where any two people are friends or enemies, there are not necessarily three mutual friends or four mutual enemies.

Computer Projects

Write programs with these input and output.

1. Given a positive integer n and a nonnegative integer not exceeding n, find the number of r-permutations and r-combinations of a set with n elements.

2. Given positive integers n and r, find the number of r-permutations when repetition is allowed and r-combinations when repetition is allowed of a set with n elements.

3. Given a sequence of positive integers, find the longest increasing and the longest decreasing subsequence of the sequence.

***4.** Given an equation $x_1 + x_2 + \cdots + x_n = C$, where C is a constant, and $x_1, x_2, \ldots, x_n$ are nonnegative integers, list all the solutions.

5. Given a positive integer n, list all the permutations of the set $\{1, 2, 3, \ldots, n\}$ in lexicographic order.

6. Given a positive integer n and a nonnegative integer r not exceeding n, list all the r-combinations of the set $\{1, 2, 3, \ldots, n\}$ in lexicographic order.

7. Given a positive integer n and a nonnegative integer r not exceeding n, list all the r-permutations of the set $\{1, 2, 3, \ldots, n\}$ in lexicographic order.

8. Given a positive integer n, list all the combinations of the set $\{1, 2, 3, \ldots, n\}$.

9. Given positive integers n and r, list all the r-permutations, with repetition allowed, of the set $\{1, 2, 3, \ldots, n\}$.

10. Given positive integers n and r, list all the r-combinations, with repetition allowed, of the set $\{1, 2, 3, \ldots, n\}$.

Computations and Explorations

Use a computational program or programs you have written to do these exercises.

1. Find the number of possible outcomes in a two-team playoff when the winner is the first team to win 5 out of 9, 6 out of 11, 7 out of 13, and 8 out of 15.

2. Which binomial coefficients are odd? Can you formulate a conjecture based on numerical evidence?

3. Verify that $C(2n, n)$ is divisible by the square of a prime, when $n \neq 1, 2,$ or 4, for as many positive integers n as you can. [That $C(2n, n)$ is divisible by the square of a prime with $n \neq 1, 2,$ or 4 was proved in 1996 by Andrew Granville and Olivier Ramaré, settling a conjecture made in 1980 by Paul Erdős and Ron Graham.]

4. Find as many odd integers n less than 200 as you can for which $C(n, \lfloor n/2 \rfloor)$ is not divisible by the square of a prime. Formulate a conjecture based on your evidence.

***5.** For each integer less than 100 determine whether $C(2n, n)$ is divisible by 3. Can you formulate a conjecture that tells us for which integers n the binomial coefficient $C(2n, n)$ is divisible by 3 based on the digits in the base three expansion of n?

6. Generate all the permutations of a set with eight elements.

7. Generate all the 6-permutations of a set with nine elements.

8. Generate all combinations of a set with eight elements.

9. Generate all 5-combinations with repetition allowed of a set with seven elements.

Writing Projects

Respond to these with essays using outside sources.

1. Describe some of the earliest uses of the pigeonhole principle by Dirichlet and other mathematicians.
2. Discuss ways in which the current telephone numbering plan can be extended to accommodate the rapid demand for more telephone numbers. (See if you can find some of the proposals coming from the telecommunications industry.) For each new numbering plan you discuss, show how to find the number of different telephone numbers it supports.
3. Discuss the importance of combinatorial reasoning in gene sequencing and related problems involving genomes.
4. Many combinatorial identities are described in this book. Find some sources of such identities and describe important combinatorial identities besides those already introduced in this book. Give some representative proofs, including combinatorial ones, of some of these identities.
5. Describe the different models used to model the distribution of particles in statistical mechanics, including Maxwell–Boltzmann, Bose–Einstein, and Fermi–Dirac statistics. In each case, describe the counting techniques used in the model.
6. Define the Stirling numbers of the first kind and describe some of their properties and the identities they satisfy.
7. Describe some of the properties and the identities that Stirling numbers of the second kind satisfy, including the connection between Stirling numbers of the first and second kinds.
8. Describe the latest discoveries of values and bounds for Ramsey numbers.
9. Describe additional ways to generate all the permutations of a set with n elements besides those found in Section 6.6. Compare these algorithms and the algorithms described in the text and exercises of Section 6.6 in terms of their computational complexity.
10. Describe at least one way to generate all the partitions of a positive integer n. (See Exercise 49 in Section 5.3.)

CHAPTER

8 Advanced Counting Techniques

Many counting problems cannot be solved easily using the methods discussed in Chapter 6. One such problem is: How many bit strings of length n do not contain two consecutive zeros? To solve this problem, let a_n be the number of such strings of length n. An argument can be given that shows that the sequence $\{a_n\}$ satisfies the recurrence relation $a_{n+1} = a_n + a_{n-1}$ and the initial conditions $a_1 = 2$ and $a_2 = 3$. This recurrence relation and the initial conditions determine the sequence $\{a_n\}$. Moreover, an explicit formula can be found for a_n from the equation relating the terms of the sequence. As we will see, a similar technique can be used to solve many different types of counting problems.

We will discuss two ways that recurrence relations play important roles in the study of algorithms. First, we will introduce an important algorithmic paradigm known as dynamic programming. Algorithms that follow this paradigm break down a problem into overlapping subproblems. The solution to the problem is then found from the solutions to the subproblems through the use of a recurrence relation. Second, we will study another important algorithmic paradigm, divide-and-conquer. Algorithms that follow this paradigm can be used to solve a problem by recursively breaking it into a fixed number of nonoverlapping subproblems until these problems can be solved directly. The complexity of such algorithms can be analyzed using a special type of recurrence relation. In this chapter we will discuss a variety of divide-and-conquer algorithms and analyze their complexity using recurrence relations.

We will also see that many counting problems can be solved using formal power series, called generating functions, where the coefficients of powers of x represent terms of the sequence we are interested in. Besides solving counting problems, we will also be able to use generating functions to solve recurrence relations and to prove combinatorial identities.

Many other kinds of counting problems cannot be solved using the techniques discussed in Chapter 6, such as: How many ways are there to assign seven jobs to three employees so that each employee is assigned at least one job? How many primes are there less than 1000? Both of these problems can be solved by counting the number of elements in the union of sets. We will develop a technique, called the principle of inclusion–exclusion, that counts the number of elements in a union of sets, and we will show how this principle can be used to solve counting problems.

The techniques studied in this chapter, together with the basic techniques of Chapter 6, can be used to solve many counting problems.

8.1 Applications of Recurrence Relations

8.1.1 Introduction

Recall from Chapter 2 that a recursive definition of a sequence specifies one or more initial terms and a rule for determining subsequent terms from those that precede them. Also, recall that a rule of the latter sort (whether or not it is part of a recursive definition) is called a **recurrence relation** and that a sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

In this section we will show that such relations can be used to study and to solve counting problems. For example, suppose that the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in n hours? To solve this problem, let a_n be the number of bacteria at the end of n hours. Because the number of bacteria doubles every hour, the relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer. This recurrence relation, together with the initial condition $a_0 = 5$, uniquely determines a_n for all nonnegative integers n. We can find a formula for a_n using the iterative approach followed in Chapter 2, namely that $a_n = 5 \cdot 2^n$ for all nonnegative integers n.

Some of the counting problems that cannot be solved using the techniques discussed in Chapter 6 can be solved by finding recurrence relations involving the terms of a sequence, as was done in the problem involving bacteria. In this section we will study a variety of counting problems that can be modeled using recurrence relations. In Chapter 2 we developed methods for solving certain recurrence relation. In Section 8.2 we will study methods for finding explicit formulae for the terms of sequences that satisfy certain types of recurrence relations.

We conclude this section by introducing the algorithmic paradigm of dynamic programming. After explaining how this paradigm works, we will illustrate its use with an example.

8.1.2 Modeling With Recurrence Relations

Assessment

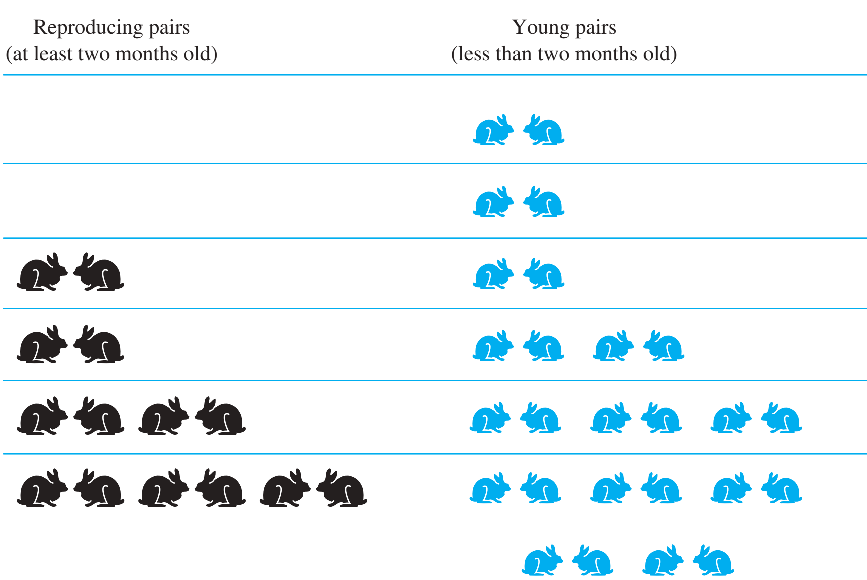
We can use recurrence relations to model a wide variety of problems, such as finding compound interest (see Example 11 in Section 2.4), counting rabbits on an island, determining the number of moves in the Tower of Hanoi puzzle, and counting bit strings with certain properties.

Extra Examples

Example 1 shows how the population of rabbits on an island can be modeled using a recurrence relation.

EXAMPLE 1 **Rabbits and the Fibonacci Numbers** Consider this problem, which was originally posed by Leonardo Pisano, also known as Fibonacci, in the thirteenth century in his book *Liber abaci*. A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month, as shown in Figure 1. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.

Links



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

FIGURE 1 **Rabbits on an island.**

Solution: Denote by f_n the number of pairs of rabbits after n months. We will show that f_n, $n = 1, 2, 3, \ldots$, are the terms of the Fibonacci sequence.

The rabbit population can be modeled using a recurrence relation. At the end of the first month, the number of pairs of rabbits on the island is $f_1 = 1$. Because this pair does not breed during the second month, $f_2 = 1$ also. To find the number of pairs after n months, add the number on the island the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least 2 months old.

The Fibonacci numbers appear in many other places in nature, including the number of petals on flowers and the number of spirals on seedheads.

Consequently, the sequence $\{f_n\}$ satisfies the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n \geq 3$ together with the initial conditions $f_1 = 1$ and $f_2 = 1$. Because this recurrence relation and the initial conditions uniquely determine this sequence, the number of pairs of rabbits on the island after n months is given by the nth Fibonacci number. ◀

Demo

Example 2 involves a famous puzzle.

EXAMPLE 2 **The Tower of Hanoi Puzzle** A popular puzzle of the late nineteenth century invented by the French mathematician Édouard Lucas, called the Tower of Hanoi, consists of three pegs mounted on a board together with disks of different sizes. Initially these disks are placed on the first peg in order of size, with the largest on the bottom (as shown in Figure 2). The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest on the bottom.

Links

Let H_n denote the number of moves needed to solve the Tower of Hanoi puzzle with n disks. Set up a recurrence relation for the sequence $\{H_n\}$.

Solution: Begin with n disks on peg 1. We can transfer the top $n - 1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves (see Figure 3 for an illustration of the pegs and disks at this point). We keep the largest disk fixed during these moves. Then, we use one move to transfer the largest disk to the second peg. Finally, we transfer the $n - 1$ disks on peg 3 to peg 2 using H_{n-1} moves, placing them on top of the largest disk, which always stays fixed on the bottom of peg 2. This shows that we can solve the Tower of Hano puzzle for n disks using $2H_{n-1} + 1$ moves.

Schemes for efficiently backing up computer files on multiple tapes or other media are based on the moves used to solve the Tower of Hanoi puzzle.

We now show that we cannot solve the puzzle for n disks using fewer that $2H_{n-1} + 1$ moves. Note that when we move the largest disk, we must have already moved the $n - 1$ smaller disks onto a peg other than peg 1. Doing so requires at least H_{n-1} moves. Another move is needed to

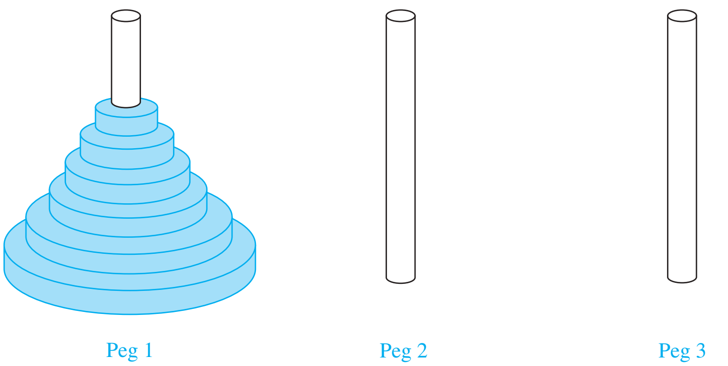


FIGURE 2 **The initial position in the Tower of Hanoi.**

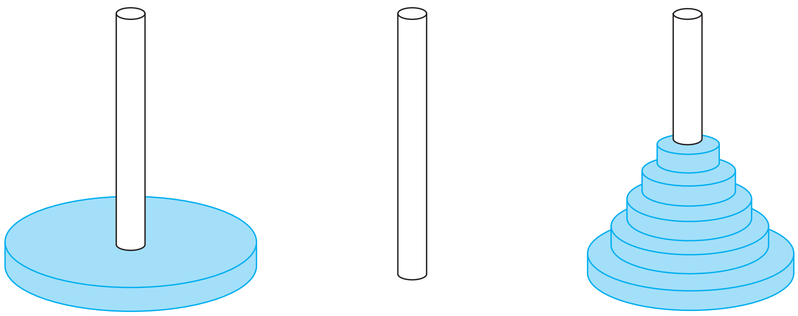


FIGURE 3 **An intermediate position in the Tower of Hanoi.**

transfer the largest disk. Finally, at least H_{n-1} more moves are needed to put the $n-1$ smallest disks back on top of the largest disk. Adding the number of moves required gives us the desired lower bound.

We conclude that

$$H_n = 2H_{n-1} + 1.$$

The initial condition is $H_1 = 1$, because one disk can be transferred from peg 1 to peg 2, according to the rules of the puzzle, in one move.

We can use an iterative approach to solve this recurrence relation. Note that

$$\begin{aligned} H_n &= 2H_{n-1} + 1 \\ &= 2(2H_{n-2} + 1) + 1 = 2^2H_{n-2} + 2 + 1 \\ &= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3H_{n-3} + 2^2 + 2 + 1 \\ &\vdots \\ &= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\ &= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\ &= 2^n - 1. \end{aligned}$$

We have used the recurrence relation repeatedly to express H_n in terms of previous terms of the sequence. In the next to last equality, the initial condition $H_1 = 1$ has been used. The last equality is based on the formula for the sum of the terms of a geometric series, which can be found in Theorem 1 in Section 2.4.

The iterative approach has produced the solution to the recurrence relation $H_n = 2H_{n-1} + 1$ with the initial condition $H_1 = 1$. This formula can be proved using mathematical induction. This is left for the reader as Exercise 1.

A myth created to accompany the puzzle tells of a tower in Hanoi where monks are transferring 64 gold disks from one peg to another, according to the rules of the puzzle. The myth says that the world will end when they finish the puzzle. How long after the monks started will the world end if the monks take one second to move a disk?

From the explicit formula, the monks require

$$2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$$

moves to transfer the disks. Making one move per second, it will take them more than 500 billion years to complete the transfer, so the world should survive a while longer than it already has. ◀

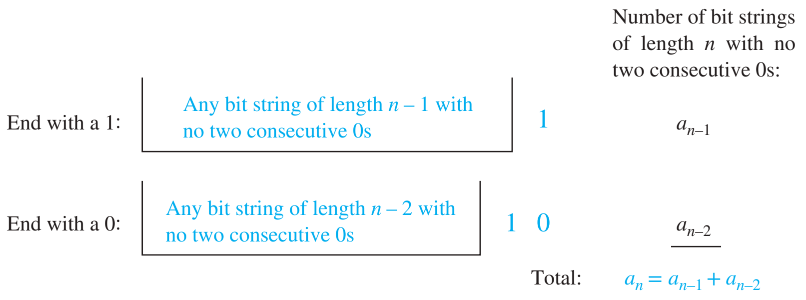


FIGURE 4 **Counting bit strings of length n with no two consecutive 0s.**

Links

Remark: Many people have studied variations of the original Tower of Hanoi puzzle discussed in Example 2. Some variations use more pegs, some allow disks to be of the same size, and some restrict the types of allowable disk moves. One of the oldest and most interesting variations is the **Reve's puzzle**,* proposed in 1907 by Henry Dudeney in his book *The Canterbury Puzzles*. The Reve's puzzle involves pilgrims challenged by the Reve to move a stack of cheese wheels of varying sizes from the first of four stools to another stool without ever placing a cheese wheel on one of smaller diameter. The Reve's puzzle, expressed in terms of pegs and disks, follows the same rules as the Tower of Hanoi puzzle, except that four pegs are used. Similarly, we can generalize the Tower of Hanoi puzzle where there are p pegs, where p is an integer greater than three. You may find it surprising that no one has been able to establish the minimum number of moves required to solve the generalization of this puzzle for p pegs. (Note that there have been some published claims that this problem has been solved, but these are not accepted by experts.) However, in 2014 Thierry Bousch showed that the minimum number of moves required when there are four pegs equals the number of moves used by an algorithm invented by Frame and Stewart in 1939. (See Exercises 38–45 and [St94] and [Bo14] for more information.)

Example 3 illustrates how recurrence relations can be used to count bit strings of a specified length that have a certain property.

EXAMPLE 3 Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length five?

Solution: Let a_n denote the number of bit strings of length n that do not have two consecutive 0s. We assume that $n \geq 3$, so that the bit string has at least three bits. Strings of this sort of length n can be divided into those that end in 1 and those that end in 0. The bit strings of length n ending with 1 that do not have two consecutive 0s are precisely the bit strings of length $n-1$ with no two consecutive 0s with a 1 added at the end. Consequently, there are a_{n-1} such bit strings.

Bit strings of length n ending with a 0 that do not have two consecutive 0s must have 1 as their $(n-1)$st bit; otherwise they would end with a pair of 0s. Hence, the bit strings of length n ending with a 0 that have no two consecutive 0s are precisely the bit strings of length $n-2$ with no two consecutive 0s with 10 added at the end. Consequently, there are a_{n-2} such bit strings.

We conclude, as illustrated in Figure 4, that

$$a_n = a_{n-1} + a_{n-2}$$

for $n \geq 3$.

***Reve*, more commonly spelled *reeve*, is an archaic word for *governor*.

The initial conditions are $a_1 = 2$, because both bit strings of length one, 0 and 1 do not have consecutive 0s, and $a_2 = 3$, because the valid bit strings of length two are 01, 10, and 11. To obtain a_5, we use the recurrence relation three times to find that

$$\begin{aligned} a_3 &= a_2 + a_1 = 3 + 2 = 5, \\ a_4 &= a_3 + a_2 = 5 + 3 = 8, \\ a_5 &= a_4 + a_3 = 8 + 5 = 13. \end{aligned}$$

◀

Remark: Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Because $a_1 = f_3$ and $a_2 = f_4$ it follows that $a_n = f_{n+2}$.

Example 4 shows how a recurrence relation can be used to model the number of codewords that are allowable using certain validity checks.

EXAMPLE 4 **Codeword Enumeration** A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let a_n be the number of valid n-digit codewords. Find a recurrence relation for a_n.

Solution: Note that $a_1 = 9$ because there are 10 one-digit strings, and only one, namely, the string 0, is not valid. A recurrence relation can be derived for this sequence by considering how a valid n-digit string can be obtained from strings of $n - 1$ digits. There are two ways to form a valid string with n digits from a string with one fewer digit.

First, a valid string of n digits can be obtained by appending a valid string of $n - 1$ digits with a digit other than 0. This appending can be done in nine ways. Hence, a valid string with n digits can be formed in this manner in $9a_{n-1}$ ways.

Second, a valid string of n digits can be obtained by appending a 0 to a string of length $n - 1$ that is not valid. (This produces a string with an even number of 0 digits because the invalid string of length $n - 1$ has an odd number of 0 digits.) The number of ways that this can be done equals the number of invalid $(n - 1)$-digit strings. Because there are 10^{n-1} strings of length $n - 1$, and a_{n-1} are valid, there are $10^{n-1} - a_{n-1}$ valid n-digit strings obtained by appending an invalid string of length $n - 1$ with a 0.

Because all valid strings of length n are produced in one of these two ways, it follows that there are

$$\begin{aligned} a_n &= 9a_{n-1} + (10^{n-1} - a_{n-1}) \\ &= 8a_{n-1} + 10^{n-1} \end{aligned}$$

valid strings of length n. ◀

Example 5 establishes a recurrence relation that appears in many different contexts.

EXAMPLE 5 Find a recurrence relation for C_n, the number of ways to parenthesize the product of $n + 1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, to specify the order of multiplication. For example, $C_3 = 5$ because there are five ways to parenthesize $x_0 \cdot x_1 \cdot x_2 \cdot x_3$ to determine the order of multiplication:

$$\begin{array}{lll} ((x_0 \cdot x_1) \cdot x_2) \cdot x_3 & (x_0 \cdot (x_1 \cdot x_2)) \cdot x_3 & (x_0 \cdot x_1) \cdot (x_2 \cdot x_3) \\ x_0 \cdot ((x_1 \cdot x_2) \cdot x_3) & x_0 \cdot (x_1 \cdot (x_2 \cdot x_3)). & \end{array}$$

Solution: To develop a recurrence relation for C_n, we note that however we insert parentheses in the product $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, one "$\cdot$" operator remains outside all parentheses, namely, the operator for the final multiplication to be performed. [For example, in $(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$, it is the final "$\cdot$", while in $(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$ it is the second "$\cdot$".] This final operator appears between two of the $n+1$ numbers, say, x_k and x_{k+1}. There are $C_k C_{n-k-1}$ ways to insert parentheses to determine the order of the $n+1$ numbers to be multiplied when the final operator appears between x_k and x_{k+1}, because there are C_k ways to insert parentheses in the product $x_0 \cdot x_1 \cdot \cdots \cdot x_k$ to determine the order in which these $k+1$ numbers are to be multiplied and C_{n-k-1} ways to insert parentheses in the product $x_{k+1} \cdot x_{k+2} \cdot \cdots \cdot x_n$ to determine the order in which these $n-k$ numbers are to be multiplied. Because this final operator can appear between any two of the $n+1$ numbers, it follows that

$$\begin{aligned} C_n &= C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1} C_0 \\ &= \sum_{k=0}^{n-1} C_k C_{n-k-1}. \end{aligned}$$

Note that the initial conditions are $C_0 = 1$ and $C_1 = 1$. ◀

The recurrence relation in Example 5 can be solved using the method of generating functions, which will be discussed in Section 8.4. It can be shown that $C_n = C(2n, n)/(n+1)$ (see Exercise 43 in Section 8.4) and that $C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$ (see [GrKnPa94]). The sequence $\{C_n\}$ is the sequence of **Catalan numbers**, named after Eugène Charles Catalan. This sequence appears as the solution of many different counting problems besides the one considered here (see the chapter on Catalan numbers in [MiRo91] or [RoTe03] for details).

Links

8.1.3 Algorithms and Recurrence Relations

Recurrence relations play an important role in many aspects of the study of algorithms and their complexity. In Section 8.3, we will show how recurrence relations can be used to analyze the complexity of divide-and-conquer algorithms, such as the merge sort algorithm introduced in Section 5.4. As we will see in Section 8.3, divide-and-conquer algorithms recursively divide a problem into a fixed number of nonoverlapping subproblems until they become simple enough to solve directly. We conclude this section by introducing another algorithmic paradigm known as **dynamic programming**, which can be used to solve many optimization problems efficiently.

Links

An algorithm follows the dynamic programming paradigm when it recursively breaks down a problem into simpler overlapping subproblems, and computes the solution using the solutions of the subproblems. Generally, recurrence relations are used to find the overall solution from the solutions of the subproblems. Dynamic programming has been used to solve important problems in such diverse areas as economics, computer vision, speech recognition, artificial intelligence, computer graphics, and bioinformatics. In this section we will illustrate the use of dynamic programming by constructing an algorithm for solving a scheduling problem. Before doing so, we will relate the amusing origin of the name *dynamic programming*, which was introduced by the mathematician Richard Bellman in the 1950s. Bellman was working at the RAND Corporation on projects for the U.S. military, and at that time, the U.S. Secretary of Defense was hostile to mathematical research. Bellman decided that to ensure funding, he needed a name not containing the word mathematics for his method for solving scheduling and planning problems. He decided to use the adjective *dynamic* because, as he said "it's impossible to use the word dynamic in a pejorative sense" and he thought that dynamic programming was "something not even a Congressman could object to."

AN EXAMPLE OF DYNAMIC PROGRAMMING The problem we use to illustrate dynamic programming is related to the problem studied in Example 7 in Section 3.1. In that problem our goal was to schedule as many talks as possible in a single lecture hall. These talks have preset start and end times; once a talk starts, it continues until it ends; no two talks can proceed at the same time; and a talk can begin at the same time another one ends. We developed a greedy algorithm that always produces an optimal schedule, as we proved in Example 12 in Section 5.1. Now suppose that our goal is not to schedule the most talks possible, but rather to have the largest possible combined attendance of the scheduled talks.

We formalize this problem by supposing that we have n talks, where talk j begins at time t_j, ends at time e_j, and will be attended by w_j students. We want a schedule that maximizes the total number of student attendees. That is, we wish to schedule a subset of talks to maximize the sum of w_j over all scheduled talks. (Note that when a student attends more than one talk, this student is counted according to the number of talks attended.) We denote by $T(j)$ the maximum number of total attendees for an optimal schedule from the first j talks, so $T(n)$ is the maximal number of total attendees for an optimal schedule for all n talks.

We first sort the talks in order of increasing end time. After doing this, we renumber the talks so that $e_1 \leq e_2 \leq \cdots \leq e_n$. We say that two talks are **compatible** if they can be part of the same schedule, that is, if the times they are scheduled do not overlap (other than the possibility one ends and the other starts at the same time). We define $p(j)$ to be largest integer i, $i < j$, for which $e_i \leq s_j$, if such an integer exists, and $p(j) = 0$ otherwise. That is, talk $p(j)$ is the talk ending latest among talks compatible with talk j that end before talk j ends, if such a talk exists, and $p(j) = 0$ if there are no such talks.

EXAMPLE 6 Consider seven talks with these start times and end times, as illustrated in Figure 5.

Talk 1: start 8 A.M., end 10 A.M.
Talk 2: start 9 A.M., end 11 A.M.
Talk 3: start 10:30 A.M., end 12 noon
Talk 4: start 9:30 A.M., end 1 P.M.
Talk 5: start 8:30 A.M., end 2 P.M.
Talk 6: start 11 A.M., end 2 P.M.
Talk 7: start 1 P.M., end 2 P.M.

Find $p(j)$ for $j = 1, 2, \ldots, 7$.

Solution: We have $p(1) = 0$ and $p(2) = 0$, because no talks end before either of the first two talks begin. We have $p(3) = 1$ because talk 3 and talk 1 are compatible, but talk 3 and talk 2 are not compatible; $p(4) = 0$ because talk 4 is not compatible with any of talks 1, 2, and 3; $p(5) = 0$

Links

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EUGÈNE CHARLES CATALAN (1814–1894) Eugène Catalan was born in Bruges, then part of France. His father became a successful architect in Paris while Eugène was a boy. Catalan attended a Parisian school for design hoping to follow in his father's footsteps. At 15, he won the job of teaching geometry to his design school classmates. After graduating, Catalan attended a school for the fine arts, but because of his mathematical aptitude his instructors recommended that he enter the École Polytechnique. He became a student there, but after his first year, he was expelled because of his politics. However, he was readmitted, and in 1835, he graduated and won a position at the Collège de Châlons sur Marne.

In 1838, Catalan returned to Paris where he founded a preparatory school with two other mathematicians, Sturm and Liouville. After teaching there for a short time, he was appointed to a position at the École Polytechnique. He received his doctorate from the École Polytechnique in 1841, but his political activity in favor of the French Republic hurt his career prospects. In 1846 Catalan held a position at the Collège de Charlemagne; he was appointed to the Lycée Saint Louis in 1849. However, when Catalan would not take a required oath of allegiance to the new Emperor Louis-Napoleon Bonaparte, he lost his job. For 13 years he held no permanent position. Finally, in 1865 he was appointed to a chair of mathematics at the University of Liège, Belgium, a position he held until his 1884 retirement.

Catalan made many contributions to number theory and to the related subject of continued fractions. He defined what are now known as the Catalan numbers when he solved the problem of dissecting a polygon into triangles using non-intersecting diagonals. Catalan is also well known for formulating what was known as the *Catalan conjecture*. This asserted that 8 and 9 are the only consecutive powers of integers, a conjecture not solved until 2003. Catalan wrote many textbooks, including several that became quite popular and appeared in as many as 12 editions. Perhaps this textbook will have a 12th edition someday!

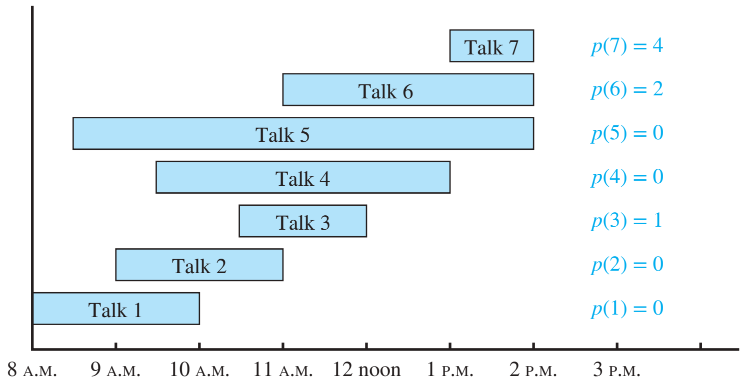

FIGURE 5 **A schedule of lectures with the values of $p(n)$ shown.**

because talk 5 is not compatible with any of talks 1, 2, 3, and 4; and $p(6) = 2$ because talk 6 and talk 2 are compatible, but talk 6 is not compatible with any of talks 3, 4, and 5. Finally, $p(7) = 4$, because talk 7 and talk 4 are compatible, but talk 7 is not compatible with either of talks 5 or 6. ◀

To develop a dynamic programming algorithm for this problem, we first develop a key recurrence relation. To do this, first note that if $j \leq n$, there are two possibilities for an optimal schedule of the first j talks (recall that we are assuming that the n talks are ordered by increasing end time): *(i)* talk j belongs to the optimal schedule or *(ii)* it does not.

Case (i): We know that talks $p(j) + 1, \ldots, j - 1$ do not belong to this schedule, for none of these other talks are compatible with talk j. Furthermore, the other talks in this optimal schedule must comprise an optimal schedule for talks $1, 2, \ldots, p(j)$. For if there were a better schedule for talks

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RICHARD BELLMAN (1920–1984) Richard Bellman, born in Brooklyn, where his father was a grocer, spent many hours in the museums and libraries of New York as a child. After graduating high school, he studied mathematics at Brooklyn College and graduated in 1941. He began postgraduate work at Johns Hopkins University, but because of the war, left to teach electronics at the University of Wisconsin. He was able to continue his mathematics studies at Wisconsin, and in 1943 he received his masters degree there. Later, Bellman entered Princeton University, teaching in a special U.S. Army program. In late 1944, he was drafted into the army. He was assigned to the Manhattan Project at Los Alamos where he worked in theoretical physics. After the war, he returned to Princeton and received his Ph.D. in 1946.

After briefly teaching at Princeton, he moved to Stanford University, where he attained tenure. At Stanford he pursued his fascination with number theory. However, Bellman decided to focus on mathematical questions arising from real-world problems. In 1952, he joined the RAND Corporation, working on multistage decision processes, operations research problems, and applications to the social sciences and medicine. He worked on many military projects while at RAND. In 1965 he left RAND to become professor of mathematics, electrical and biomedical engineering and medicine at the University of Southern California.

In the 1950s Bellman pioneered the use of dynamic programming, a technique invented earlier, in a wide range of settings. He is also known for his work on stochastic control processes, in which he introduced what is now called the Bellman equation. He coined the term *curse of dimensionality* to describe problems caused by the exponential increase in volume associated with adding extra dimensions to a space. He wrote an amazing number of books and research papers with many coauthors, including many on industrial production and economic systems. His work led to the application of computing techniques in a wide variety of areas ranging from the design of guidance systems for space vehicles, to network optimization, and even to pest control.

Tragically, in 1973 Bellman was diagnosed with a brain tumor. Although it was removed successfully, complications left him severely disabled. Fortunately, he managed to continue his research and writing during his remaining ten years of life. Bellman received many prizes and awards, including the first Norbert Wiener Prize in Applied Mathematics and the IEEE Gold Medal of Honor. He was elected to the National Academy of Sciences. He was held in high regard for his achievements, courage, and admirable qualities. Bellman was the father of two children.

$1, 2, \ldots, p(j)$, by adding talk j, we will have a schedule better than the overall optimal schedule. Consequently, in case *(i)*, we have $T(j) = w_j + T(p(j))$.

Case (ii): When talk j does not belong to an optimal schedule, it follows that an optimal schedule from talks $1, 2, \ldots, j$ is the same as an optimal schedule from talks $1, 2, \ldots, j - 1$. Consequently, in case *(ii)*, we have $T(j) = T(j - 1)$. Combining cases *(i)* and *(ii)* leads us to the recurrence relation

$$T(j) = \max(w_j + T(p(j)), T(j - 1)).$$

Now that we have developed this recurrence relation, we can construct an efficient algorithm, Algorithm 1, for computing the maximum total number of attendees. We ensure that the algorithm is efficient by storing the value of each $T(j)$ after we compute it. This allows us to compute $T(j)$ only once. If we did not do this, the algorithm would have exponential worst-case complexity. The process of storing the values as each is computed is known as **memoization** and is an important technique for making recursive algorithms efficient.

ALGORITHM 1 Dynamic Programming Algorithm for Scheduling Talks.

procedure *Maximum Attendees* ($s_1, s_2, \ldots, s_n$: start times of talks;
  $e_1, e_2, \ldots, e_n$: end times of talks; $w_1, w_2, \ldots, w_n$: number of attendees to talks)
  sort talks by end time and relabel so that $e_1 \leq e_2 \leq \cdots \leq e_n$
for $j := 1$ **to** n
    if no job i with $i < j$ is compatible with job j
        $p(j) = 0$
    else $p(j) := \max\{i — i < j$ and job i is compatible with job $j\}$
    $T(0) := 0$
for $j := 1$ **to** n
    $T(j) := \max(w_j + T(p(j)), T(j - 1))$
return $T(n)\{T(n)$ is the maximum number of attendees$\}$

In Algorithm 1 we determine the maximum number of attendees that can be achieved by a schedule of talks, but we do not find a schedule that achieves this maximum. To find talks we need to schedule, we use the fact that talk j belongs to an optimal solution for the first j talks if and only if $w_j + T(p(j)) \geq T(j - 1)$. We leave it as Exercise 53 to construct an algorithm based on this observation that determines which talks should be scheduled to achieve the maximum total number of attendees.

Algorithm 1 is a good example of dynamic programming as the maximum total attendance is found using the optimal solutions of the overlapping subproblems, each of which determines the maximum total attendance of the first j talks for some j with $1 \leq j \leq n - 1$. See Exercises 56 and 57 and Supplementary Exercises 14 and 17 for other examples of dynamic programming.

Exercises

1. Use mathematical induction to verify the formula derived in Example 2 for the number of moves required to complete the Tower of Hanoi puzzle.

2. a) Find a recurrence relation for the number of permutations of a set with n elements.

b) Use this recurrence relation to find the number of permutations of a set with n elements using iteration.

3. A vending machine dispensing books of stamps accepts only one-dollar coins, \$1 bills, and \$5 bills.

a) Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.

b) What are the initial conditions?

c) How many ways are there to deposit \$10 for a book of stamps?

4. A country uses as currency coins with values of 1 peso, 2 pesos, 5 pesos, and 10 pesos and bills with values of 5 pesos, 10 pesos, 20 pesos, 50 pesos, and 100 pesos. Find a recurrence relation for the number of ways to pay a bill of n pesos if the order in which the coins and bills are paid matters.

5. How many ways are there to pay a bill of 17 pesos using the currency described in Exercise 4, where the order in which coins and bills are paid matters?

***6. a)** Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and n as their last term, where n is a positive integer. That is, sequences $a_1, a_2, \ldots, a_k$, where $a_1 = 1$, $a_k = n$, and $a_j < a_{j+1}$ for $j = 1, 2, \ldots, k - 1$.

b) What are the initial conditions?

c) How many sequences of the type described in (a) are there when n is an integer with $n \geq 2$?

7. a) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven contain two consecutive 0s?

8. a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven contain three consecutive 0s?

9. a) Find a recurrence relation for the number of bit strings of length n that do not contain three consecutive 0s.

b) What are the initial conditions?

c) How many bit strings of length seven do not contain three consecutive 0s?

***10. a)** Find a recurrence relation for the number of bit strings of length n that contain the string 01.

b) What are the initial conditions?

c) How many bit strings of length seven contain the string 01?

11. a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

b) What are the initial conditions?

c) In how many ways can this person climb a flight of eight stairs?

12. a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two, or three stairs at a time.

b) What are the initial conditions?

c) In how many ways can this person climb a flight of eight stairs?

A string that contains only 0s, 1s, and 2s is called a **ternary string**.

13. a) Find a recurrence relation for the number of ternary strings of length n that do not contain two consecutive 0s.

b) What are the initial conditions?

c) How many ternary strings of length six do not contain two consecutive 0s?

14. a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive 0s.

b) What are the initial conditions?

c) How many ternary strings of length six contain two consecutive 0s?

***15. a)** Find a recurrence relation for the number of ternary strings of length n that do not contain two consecutive 0s or two consecutive 1s.

b) What are the initial conditions?

c) How many ternary strings of length six do not contain two consecutive 0s or two consecutive 1s?

***16. a)** Find a recurrence relation for the number of ternary strings of length n that contain either two consecutive 0s or two consecutive 1s.

b) What are the initial conditions?

c) How many ternary strings of length six contain two consecutive 0s or two consecutive 1s?

***17. a)** Find a recurrence relation for the number of ternary strings of length n that do not contain consecutive symbols that are the same.

b) What are the initial conditions?

c) How many ternary strings of length six do not contain consecutive symbols that are the same?

****18. a)** Find a recurrence relation for the number of ternary strings of length n that contain two consecutive symbols that are the same.

b) What are the initial conditions?

c) How many ternary strings of length six contain consecutive symbols that are the same?

19. Messages are transmitted over a communications channel using two signals. The transmittal of one signal requires 1 microsecond, and the transmittal of the other signal requires 2 microseconds.

a) Find a recurrence relation for the number of different messages consisting of sequences of these two signals, where each signal in the message is immediately followed by the next signal, that can be sent in n microseconds.

b) What are the initial conditions?

c) How many different messages can be sent in 10 microseconds using these two signals?

20. A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.

a) Find a recurrence relation for the number of different ways the bus driver can pay a toll of n cents (where the order in which the coins are used matters).

b) In how many different ways can the driver pay a toll of 45 cents?

21. a) Find the recurrence relation satisfied by R_n, where R_n is the number of regions that a plane is divided into by n lines, if no two of the lines are parallel and no three of the lines go through the same point.

b) Find R_n using iteration.

***22.** **a)** Find the recurrence relation satisfied by R_n, where R_n is the number of regions into which the surface of a sphere is divided by n great circles (which are the intersections of the sphere and planes passing through the center of the sphere), if no three of the great circles go through the same point.
b) Find R_n using iteration.

***23.** **a)** Find the recurrence relation satisfied by S_n, where S_n is the number of regions into which three-dimensional space is divided by n planes if every three of the planes meet in one point, but no four of the planes go through the same point.
b) Find S_n using iteration.

24. Find a recurrence relation for the number of bit sequences of length n with an even number of 0s.

25. How many bit sequences of length seven contain an even number of 0s?

26. **a)** Find a recurrence relation for the number of ways to completely cover a $2 \times n$ checkerboard with 1×2 dominoes. [*Hint:* Consider separately the coverings where the position in the top right corner of the checkerboard is covered by a domino positioned horizontally and where it is covered by a domino positioned vertically.]
b) What are the initial conditions for the recurrence relation in part (a)?
c) How many ways are there to completely cover a 2×17 checkerboard with 1×2 dominoes?

27. **a)** Find a recurrence relation for the number of ways to lay out a walkway with slate tiles if the tiles are red, green, or gray, so that no two red tiles are adjacent and tiles of the same color are considered indistinguishable.
b) What are the initial conditions for the recurrence relation in part (a)?
c) How many ways are there to lay out a path of seven tiles as described in part (a)?

28. Show that the Fibonacci numbers satisfy the recurrence relation $f_n = 5f_{n-4} + 3f_{n-5}$ for $n = 5, 6, 7, \ldots,$ together with the initial conditions $f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2,$ and $f_4 = 3$. Use this recurrence relation to show that f_{5n} is divisible by 5, for $n = 1, 2, 3, \ldots$.

***29.** Let $S(m, n)$ denote the number of onto functions from a set with m elements to a set with n elements. Show that $S(m, n)$ satisfies the recurrence relation

$$S(m, n) = n^m - \sum_{k=1}^{n-1} C(n, k)S(m, k)$$

whenever $m \geq n$ and $n > 1$, with the initial condition $S(m, 1) = 1$.

30. **a)** Write out all the ways the product $x_0 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4$ can be parenthesized to determine the order of multiplication.
b) Use the recurrence relation developed in Example 5 to calculate C_4, the number of ways to parenthesize the product of five numbers so as to determine the order of multiplication. Verify that you listed the correct number of ways in part (a).
c) Check your result in part (b) by finding C_4, using the closed formula for C_n mentioned in the solution of Example 5.

31. **a)** Use the recurrence relation developed in Example 5 to determine C_5, the number of ways to parenthesize the product of six numbers so as to determine the order of multiplication.
b) Check your result with the closed formula for C_5 mentioned in the solution of Example 5.

***32.** In the Tower of Hanoi puzzle, suppose our goal is to transfer all n disks from peg 1 to peg 3, but we cannot move a disk directly between pegs 1 and 3. Each move of a disk must be a move involving peg 2. As usual, we cannot place a disk on top of a smaller disk.
a) Find a recurrence relation for the number of moves required to solve the puzzle for n disks with this added restriction.
b) Solve this recurrence relation to find a formula for the number of moves required to solve the puzzle for n disks.
c) How many different arrangements are there of the n disks on three pegs so that no disk is on top of a smaller disk?
d) Show that every allowable arrangement of the n disks occurs in the solution of this variation of the puzzle.

Exercises 33–37 deal with a variation of the **Josephus problem** described by Graham, Knuth, and Patashnik in [GrKnPa94]. This problem is based on an account by the historian Flavius Josephus, who was part of a band of 41 Jewish rebels trapped in a cave by the Romans during the Jewish-Roman war of the first century. The rebels preferred suicide to capture; they decided to form a circle and to repeatedly count off around the circle, killing every third rebel left alive. However, Josephus and another rebel did not want to be killed this way; they determined the positions where they should stand to be the last two rebels remaining alive. The variation we consider begins with n people, numbered 1 to n, standing around a circle. In each stage, every second person still left alive is eliminated until only one survives. We denote the number of the survivor by $J(n)$.

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33. Determine the value of $J(n)$ for each integer n with $1 \leq n \leq 16$.

34. Use the values you found in Exercise 33 to conjecture a formula for $J(n)$. [*Hint:* Write $n = 2^m + k$, where m is a nonnegative integer and k is a nonnegative integer less than 2^m.]

35. Show that $J(n)$ satisfies the recurrence relation $J(2n) = 2J(n) - 1$ and $J(2n + 1) = 2J(n) + 1$, for $n \geq 1$, and $J(1) = 1$.

36. Use mathematical induction to prove the formula you conjectured in Exercise 34, making use of the recurrence relation from Exercise 35.

37. Determine $J(100)$, $J(1000)$, and $J(10{,}000)$ from your formula for $J(n)$.

Exercises 38–45 involve the Reve's puzzle, the variation of the Tower of Hanoi puzzle with four pegs and n disks. Before presenting these exercises, we describe the Frame–Stewart algorithm for moving the disks from peg 1 to peg 4 so that no disk is ever on top of a smaller one. This algorithm, given the number of disks n as input, depends on a choice of an integer k with $1 \leq k \leq n$. When there is only one disk, move it from peg 1 to peg 4 and stop. For $n > 1$, the algorithm proceeds recursively, using these three steps. Recursively move the stack of the $n - k$ smallest disks from peg 1 to peg 2, using all four pegs. Next move the stack of the k largest disks from peg 1 to peg 4, using the three-peg algorithm from the Tower of Hanoi puzzle without using the peg holding the $n - k$ smallest disks. Finally, recursively move the smallest $n - k$ disks to peg 4, using all four pegs. Frame and Stewart showed that to produce the fewest moves using their algorithm, k should be chosen to be the smallest integer such that n does not exceed $t_k = k(k+1)/2$, the kth triangular number, that is, $t_{k-1} < n \leq t_k$. The long-standing conjecture, known as **Frame's conjecture**, that this algorithm uses the fewest number of moves required to solve the puzzle, was proved by Thierry Bousch in 2014.

38. Show that the Reve's puzzle with three disks can be solved using five, and no fewer, moves.

39. Show that the Reve's puzzle with four disks can be solved using nine, and no fewer, moves.

40. Describe the moves made by the Frame–Stewart algorithm, with k chosen so that the fewest moves are required, for

a) 5 disks. **b)** 6 disks. **c)** 7 disks. **d)** 8 disks.

***41.** Show that if $R(n)$ is the number of moves used by the Frame–Stewart algorithm to solve the Reve's puzzle with n disks, where k is chosen to be the smallest integer with $n \leq k(k+1)/2$, then $R(n)$ satisfies the recurrence relation $R(n) = 2R(n-k) + 2^k - 1$, with $R(0) = 0$ and $R(1) = 1$.

***42.** Show that if k is as chosen in Exercise 41, then $R(n) - R(n-1) = 2^{k-1}$.

***43.** Show that if k is as chosen in Exercise 41, then $R(n) = \sum_{i=1}^{k} i2^{i-1} - (t_k - n)2^{k-1}$.

***44.** Use Exercise 43 to give an upper bound on the number of moves required to solve the Reve's puzzle for all integers n with $1 \leq n \leq 25$.

***45.** Show that $R(n)$ is $O(\sqrt{n}2^{\sqrt{2n}})$.

Let $\{a_n\}$ be a sequence of real numbers. The **backward differences** of this sequence are defined recursively as shown next. The **first difference** ∇a_n is

$$\nabla a_n = a_n - a_{n-1}.$$

The **$(k+1)$st difference** $\nabla^{k+1} a_n$ is obtained from $\nabla^k a_n$ by

$$\nabla^{k+1} a_n = \nabla^k a_n - \nabla^k a_{n-1}.$$

46. Find ∇a_n for the sequence $\{a_n\}$, where

a) $a_n = 4$. **b)** $a_n = 2n$.

c) $a_n = n^2$. **d)** $a_n = 2^n$.

47. Find $\nabla^2 a_n$ for the sequences in Exercise 46.

48. Show that $a_{n-1} = a_n - \nabla a_n$.

49. Show that $a_{n-2} = a_n - 2\nabla a_n + \nabla^2 a_n$.

***50.** Prove that a_{n-k} can be expressed in terms of a_n, ∇a_n, $\nabla^2 a_n, \ldots, \nabla^k a_n$.

51. Express the recurrence relation $a_n = a_{n-1} + a_{n-2}$ in terms of a_n, ∇a_n, and $\nabla^2 a_n$.

52. Show that any recurrence relation for the sequence $\{a_n\}$ can be written in terms of a_n, ∇a_n, $\nabla^2 a_n, \ldots$. The resulting equation involving the sequences and its differences is called a **difference equation**.

***53.** Construct the algorithm described in the text after Algorithm 1 for determining which talks should be scheduled to maximize the total number of attendees and not just the maximum total number of attendees determined by Algorithm 1.

54. Use Algorithm 1 to determine the maximum number of total attendees in the talks in Example 6 if w_i, the number of attendees of talk i, $i = 1, 2, \ldots, 7$, is

a) 20, 10, 50, 30, 15, 25, 40.

b) 100, 5, 10, 20, 25, 40, 30.

c) 2, 3, 8, 5, 4, 7, 10.

d) 10, 8, 7, 25, 20, 30, 5.

55. For each part of Exercise 54, use your algorithm from Exercise 53 to find the optimal schedule for talks so that the total number of attendees is maximized.

56. In this exercise we will develop a dynamic programming algorithm for finding the maximum sum of consecutive terms of a sequence of real numbers. That is, given a sequence of real numbers $a_1, a_2, \ldots, a_n$, the algorithm computes the maximum sum $\sum_{i=j}^{k} a_i$ where $1 \leq j \leq k \leq n$.

a) Show that if all terms of the sequence are nonnegative, this problem is solved by taking the sum of all terms. Then, give an example where the maximum sum of consecutive terms is not the sum of all terms.

b) Let $M(k)$ be the maximum of the sums of consecutive terms of the sequence ending at a_k. That is, $M(k) = \max_{1\leq j\leq k} \sum_{i=j}^{k} a_i$. Explain why the recurrence relation $M(k) = \max(M(k-1) + a_k, a_k)$ holds for $k = 2, \ldots, n$.

c) Use part (b) to develop a dynamic programming algorithm for solving this problem.

d) Show each step your algorithm from part (c) uses to find the maximum sum of consecutive terms of the sequence 2, −3, 4, 1, −2, 3.

e) Show that the worst-case complexity in terms of the number of additions and comparisons of your algorithm from part (c) is linear.

***57.** Dynamic programming can be used to develop an algorithm for solving the matrix-chain multiplication problem introduced in Section 3.3. This is the problem of determining how the product $\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_n$ can be computed using the fewest integer multiplications, where $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ are $m_1 \times m_2, m_2 \times m_3, \ldots, m_n \times m_{n+1}$ matrices, respectively, and each matrix has integer entries. Recall that by the associative law, the product does not depend on the order in which the matrices are multiplied.

a) Show that the brute-force method of determining the minimum number of integer multiplications needed to solve a matrix-chain multiplication problem has exponential worst-case complexity. [*Hint:* Do this by first showing that the order of multiplication of matrices is specified by parenthesizing the product. Then, use Example 5 and the result of part (c) of Exercise 43 in Section 8.4.]

b) Denote by $\mathbf{A}_{ij}$ the product $\mathbf{A}_i\mathbf{A}_{i+1} \ldots, \mathbf{A}_j$, and $M(i, j)$ the minimum number of integer multiplications required to find $\mathbf{A}_{ij}$. Show that if the least number of integer multiplications are used to compute $\mathbf{A}_{ij}$, where $i < j$, by splitting the product into the product of $\mathbf{A}_i$ through $\mathbf{A}_k$ and the product of $\mathbf{A}_{k+1}$ through $\mathbf{A}_j$, then the first k terms must be parenthesized so that $\mathbf{A}_{ik}$ is computed in the optimal way using $M(i, k)$ integer multiplications, and $\mathbf{A}_{k+1,j}$ must be parenthesized so that $\mathbf{A}_{k+1,j}$ is computed in the optimal way using $M(k + 1, j)$ integer multiplications.

c) Explain why part (b) leads to the recurrence relation $M(i, j) = \min_{i \le k < j}(M(i, k) + M(k + 1, j) + m_i m_{k+1} m_{j+1})$ if $1 \le i \le j < j \le n$.

d) Use the recurrence relation in part (c) to construct an efficient algorithm for determining the order the n matrices should be multiplied to use the minimum number of integer multiplications. Store the partial results $M(i, j)$ as you find them so that your algorithm will not have exponential complexity.

e) Show that your algorithm from part (d) has $O(n^3)$ worst-case complexity in terms of multiplications of integers.

8.2 Solving Linear Recurrence Relations

8.2.1 Introduction

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A wide variety of recurrence relations occur in models. Some of these recurrence relations can be solved using iteration or some other ad hoc technique. However, one important class of recurrence relations can be explicitly solved in a systematic way. These are recurrence relations that express the terms of a sequence as linear combinations of previous terms.

Definition 1

A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

The recurrence relation in the definition is **linear** because the right-hand side is a sum of previous terms of the sequence each multiplied by a function of n. The recurrence relation is **homogeneous** because no terms occur that are not multiples of the a_js. The coefficients of the terms of the sequence are all **constants**, rather than functions that depend on n. The **degree** is k because a_n is expressed in terms of the previous k terms of the sequence.

A consequence of the second principle of mathematical induction is that a sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the k initial conditions

$$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}.$$

EXAMPLE 1 The recurrence relation $P_n = (1.11)P_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five. ◀

To help clarify the definition of linear homogeneous recurrence relations with constant coefficients, we will now provide examples of recurrence relations each lacking one of the defining properties.

EXAMPLE 2 The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear. The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous. The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients. ◀

Linear homogeneous recurrence relations are studied for two reasons. First, they often occur in modeling of problems. Second, they can be systematically solved.

8.2.2 Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

Recurrence relations may be difficult to solve, but fortunately this is not the case for linear homogenous recurrence relations with constant coefficients. We can use two key ideas to find all their solutions. First, these recurrence relations have solutions of the form $a_n = r^n$, where r is a constant. To see this, observe that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ if and only if

$$r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_kr^{n-k}.$$

When both sides of this equation are divided by r^{n-k} (when $r \neq 0$) and the right-hand side is subtracted from the left, we obtain the equation

$$r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_{k-1}r - c_k = 0.$$

Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ where $r \neq 0$ is a solution if and only if r is a solution of this last equation. We call this the **characteristic equation** of the recurrence relation. The solutions of this equation are called the **characteristic roots** of the recurrence relation. As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

The other key observation is that a linear combination of two solutions of a linear homogeneous recurrence relation is also a solution. To see this, suppose that s_n and t_n are both solutions of $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$. Then

$$s_n = c_1s_{n-1} + c_2s_{n-2} + \cdots + c_ks_{n-k}$$

and

$$t_n = c_1t_{n-1} + c_2t_{n-2} + \cdots + c_kt_{n-k}.$$

Now suppose that b_1 and b_2 are real numbers. Then

$$\begin{aligned} b_1s_n + b_2t_n &= b_1(c_1s_{n-1} + c_2s_{n-2} + \cdots + c_ks_{n-k}) + b_2(c_1t_{n-1} + c_2t_{n-2} + \cdots + c_kt_{n-k}) \\ &= c_1(b_1s_{n-1} + b_2t_{n-1}) + c_2(b_1s_{n-2} + b_2t_{n-2}) + \cdots + c_k(b_1s_{n-k} + b_kt_{n-k}). \end{aligned}$$

This means that $b_1s_n + b_2t_n$ is also a solution of the same linear homogeneous recurrence relation.

Using these key observations, we will show how to solve linear homogeneous recurrence relations with constant coefficients.

THE DEGREE TWO CASE We now turn our attention to linear homogeneous recurrence relations of degree two. First, consider the case when there are two distinct characteristic roots.

THEOREM 1

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Proof: We must do two things to prove the theorem. First, it must be shown that if r_1 and r_2 are the roots of the characteristic equation, and α_1 and α_2 are constants, then the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation. Second, it must be shown that if the sequence $\{a_n\}$ is a solution, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some constants α_1 and α_2.

We now show that if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, then the sequence $\{a_n\}$ is a solution of the recurrence relation. Because r_1 and r_2 are roots of $r^2 - c_1 r - c_2 = 0$, it follows that $r_1^2 = c_1 r_1 + c_2$ and $r_2^2 = c_1 r_2 + c_2$.

From these equations, we see that

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\ &= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\ &= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\ &= \alpha_1 r_1^n + \alpha_2 r_2^n \\ &= a_n. \end{aligned}$$

This shows that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution of the recurrence relation.

To show that every solution $\{a_n\}$ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, for some constants α_1 and α_2, suppose that $\{a_n\}$ is a solution of the recurrence relation, and the initial conditions $a_0 = C_0$ and $a_1 = C_1$ hold. It will be shown that there are constants α_1 and α_2 such that the sequence $\{a_n\}$ with $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies these same initial conditions. This requires that

$$\begin{aligned} a_0 &= C_0 = \alpha_1 + \alpha_2, \\ a_1 &= C_1 = \alpha_1 r_1 + \alpha_2 r_2. \end{aligned}$$

We can solve these two equations for α_1 and α_2. From the first equation it follows that $\alpha_2 = C_0 - \alpha_1$. Inserting this expression into the second equation gives

$$C_1 = \alpha_1 r_1 + (C_0 - \alpha_1) r_2.$$

Hence,

$$C_1 = \alpha_1 (r_1 - r_2) + C_0 r_2.$$

This shows that

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$$

and

$$\alpha_2 = C_0 - \alpha_1 = C_0 - \frac{C_1 - C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2},$$

where these expressions for α_1 and α_2 depend on the fact that $r_1 \neq r_2$. (When $r_1 = r_2$, this theorem is not true.) Hence, with these values for α_1 and α_2, the sequence $\{a_n\}$ with $\alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the two initial conditions.

We know that $\{a_n\}$ and $\{\alpha_1 r_1^n + \alpha_2 r_2^n\}$ are both solutions of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and both satisfy the initial conditions when $n = 0$ and $n = 1$. Because there is a unique solution of a linear homogeneous recurrence relation of degree two with two initial conditions, it follows that the two solutions are the same, that is, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for all nonnegative integers n. We have completed the proof by showing that a solution of the linear homogeneous recurrence relation with constant coefficients of degree two must be of the form $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$, where α_1 and α_2 are constants. ◁

The characteristic roots of a linear homogeneous recurrence relation with constant coefficients may be complex numbers. Theorem 1 (and also subsequent theorems in this section) still applies in this case. Recurrence relations with complex characteristic roots will not be discussed in the text. Readers familiar with complex numbers may wish to solve Exercises 38 and 39.

Examples 3 and 4 show how to use Theorem 1 to solve recurrence relations.

EXAMPLE 3 What is the solution of the recurrence relation

Extra Examples

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Solution: Theorem 1 can be used to solve this problem. The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$. Its roots are $r = 2$ and $r = -1$. Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2(-1)^n,$$

for some constants α_1 and α_2. From the initial conditions, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2,$$
$$a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1).$$

Solving these two equations shows that $\alpha_1 = 3$ and $\alpha_2 = -1$. Hence, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$$a_n = 3 \cdot 2^n - (-1)^n.$$ ◀

EXAMPLE 4 Find an explicit formula for the Fibonacci numbers.

Solution: Recall that the sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ and also satisfies the initial conditions $f_0 = 0$ and $f_1 = 1$. The roots of the characteristic equation $r^2 - r - 1 = 0$ are $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$. Therefore, from Theorem 1 it follows that the Fibonacci numbers are given by

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

for some constants α_1 and α_2. The initial conditions $f_0 = 0$ and $f_1 = 1$ can be used to find these constants. We have

$$f_0 = \alpha_1 + \alpha_2 = 0,$$
$$f_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

The solution to these simultaneous equations for α_1 and α_2 is

$$\alpha_1 = 1/\sqrt{5}, \qquad \alpha_2 = -1/\sqrt{5}.$$

Consequently, the Fibonacci numbers are given by

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

◀

Theorem 1 does not apply when there is one characteristic root of multiplicity two. If this happens, then $a_n = nr_0^n$ is another solution of the recurrence relation when r_0 is a root of multiplicity two of the characteristic equation. Theorem 2 shows how to handle this case.

THEOREM 2 Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

The proof of Theorem 2 is left as Exercise 10. Example 5 illustrates the use of this theorem.

EXAMPLE 5 What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $a_0 = 1$ and $a_1 = 6$?

Solution: The only root of $r^2 - 6r + 9 = 0$ is $r = 3$. Hence, the solution to this recurrence relation is

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

for some constants α_1 and α_2. Using the initial conditions, it follows that

$$a_0 = 1 = \alpha_1,$$
$$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving these two equations shows that $\alpha_1 = 1$ and $\alpha_2 = 1$. Consequently, the solution to this recurrence relation and the initial conditions is

$$a_n = 3^n + n3^n.$$

◀

THE GENERAL CASE We will now state the general result about the solution of linear homogeneous recurrence relations with constant coefficients, where the degree may be greater than two, under the assumption that the characteristic equation has distinct roots. The proof of this result will be left as Exercise 16.

THEOREM 3 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots,$ where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

We illustrate the use of the theorem with Example 6.

EXAMPLE 6 Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution: The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6.$$

The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3)$. Hence, the solutions to this recurrence relation are of the form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

To find the constants α_1, α_2, and α_3, use the initial conditions. This gives

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3,$$
$$a_1 = 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3,$$
$$a_2 = 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9.$$

When these three simultaneous equations are solved for α_1, α_2, and α_3, we find that $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

◄

We now state the most general result about linear homogeneous recurrence relations with constant coefficients, allowing the characteristic equation to have multiple roots. The key point is that for each root r of the characteristic equation, the general solution has a summand of the form $P(n)r^n$, where $P(n)$ is a polynomial of degree $m - 1$, with m the multiplicity of this root. We leave the proof of this result as Exercise 51.

THEOREM 4 Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = \ & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Example 7 illustrates how Theorem 4 is used to find the general form of a solution of a linear homogeneous recurrence relation when the characteristic equation has several repeated roots.

EXAMPLE 7 Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9 (that is, there are three roots, the root 2 with multiplicity three, the root 5 with multiplicity two, and the root 9 with multiplicity one). What is the form of the general solution?

Solution: By Theorem 4, the general form of the solution is

$$(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2) 2^n + (\alpha_{2,0} + \alpha_{2,1} n) 5^n + \alpha_{3,0} 9^n.$$ ◀

We now illustrate the use of Theorem 4 to solve a linear homogeneous recurrence relation with constant coefficients when the characteristic equation has a root of multiplicity three.

EXAMPLE 8 Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

Solution: The characteristic equation of this recurrence relation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

Because $r^3 + 3r^2 + 3r + 1 = (r + 1)^3$, there is a single root $r = -1$ of multiplicity three of the characteristic equation. By Theorem 4 the solutions of this recurrence relation are of the form

$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1} n(-1)^n + \alpha_{1,2} n^2 (-1)^n.$$

To find the constants $\alpha_{1,0}$, $\alpha_{1,1}$, and $\alpha_{1,2}$, use the initial conditions. This gives

$$a_0 = 1 = \alpha_{1,0},$$
$$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2},$$
$$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}.$$

The simultaneous solution of these three equations is $\alpha_{1,0} = 1$, $\alpha_{1,1} = 3$, and $\alpha_{1,2} = -2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = (1 + 3n - 2n^2)(-1)^n.$$

◂

8.2.3 Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

We have seen how to solve linear homogeneous recurrence relations with constant coefficients. Is there a relatively simple technique for solving a linear, but not homogeneous, recurrence relation with constant coefficients, such as $a_n = 3a_{n-1} + 2n$? We will see that the answer is yes for certain families of such recurrence relations.

The recurrence relation $a_n = 3a_{n-1} + 2n$ is an example of a **linear nonhomogeneous recurrence relation with constant coefficients**, that is, a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically zero depending only on n. The recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$$

is called the **associated homogeneous recurrence relation**. It plays an important role in the solution of the nonhomogeneous recurrence relation.

EXAMPLE 9 Each of the recurrence relations $a_n = a_{n-1} + 2^n$, $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$, $a_n = 3a_{n-1} + n3^n$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$ is a linear nonhomogeneous recurrence relation with constant coefficients. The associated linear homogeneous recurrence relations are $a_n = a_{n-1}$, $a_n = a_{n-1} + a_{n-2}$, $a_n = 3a_{n-1}$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, respectively. ◂

The key fact about linear nonhomogeneous recurrence relations with constant coefficients is that every solution is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation, as Theorem 5 shows.

THEOREM 5 If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}.$$

Proof: Because $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous recurrence relation, we know that

$$a_n^{(p)} = c_1 a_{n-1}^{(p)} + c_2 a_{n-2}^{(p)} + \cdots + c_k a_{n-k}^{(p)} + F(n).$$

Now suppose that $\{b_n\}$ is a second solution of the nonhomogeneous recurrence relation, so that

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \cdots + c_k b_{n-k} + F(n).$$

Subtracting the first of these two equations from the second shows that

$$b_n - a_n^{(p)} = c_1(b_{n-1} - a_{n-1}^{(p)}) + c_2(b_{n-2} - a_{n-2}^{(p)}) + \cdots + c_k(b_{n-k} - a_{n-k}^{(p)}).$$

It follows that $\{b_n - a_n^p\}$ is a solution of the associated homogeneous linear recurrence, say, $\{a_n^{(h)}\}$. Consequently, $b_n = a_n^{(p)} + a_n^{(h)}$ for all n. ◁

By Theorem 5, we see that the key to solving nonhomogeneous recurrence relations with constant coefficients is finding a particular solution. Then every solution is a sum of this solution and a solution of the associated homogeneous recurrence relation. Although there is no general method for finding such a solution that works for every function $F(n)$, there are techniques that work for certain types of functions $F(n)$, such as polynomials and powers of constants. This is illustrated in Examples 10 and 11.

EXAMPLE 10 Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

Solution: To solve this linear nonhomogeneous recurrence relation with constant coefficients, we need to solve its associated linear homogeneous equation and to find a particular solution for the given nonhomogeneous equation. The associated linear homogeneous equation is $a_n = 3a_{n-1}$. Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

We now find a particular solution. Because $F(n) = 2n$ is a polynomial in n of degree one, a reasonable trial solution is a linear function in n, say, $p_n = cn + d$, where c and d are constants. To determine whether there are any solutions of this form, suppose that $p_n = cn + d$ is such a solution. Then the equation $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d) + 2n$. Simplifying and combining like terms gives $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d - 3c = 0$. This shows that $cn + d$ is a solution if and only if $c = -1$ and $d = -3/2$. Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

By Theorem 5 all solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha \cdot 3^n,$$

where α is a constant.

To find the solution with $a_1 = 3$, let $n = 1$ in the formula we obtained for the general solution. We find that $3 = -1 - 3/2 + 3\alpha$, which implies that $\alpha = 11/6$. The solution we seek is $a_n = -n - 3/2 + (11/6)3^n$. ◀

EXAMPLE 11 Find all solutions of the recurrence relation

Extra Examples ❯

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

Solution: This is a linear nonhomogeneous recurrence relation. The solutions of its associated homogeneous recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

are $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$, where α_1 and α_2 are constants. Because $F(n) = 7^n$, a reasonable trial solution is $a_n^{(p)} = C \cdot 7^n$, where C is a constant. Substituting the terms of this sequence into the recurrence relation implies that $C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$. Factoring out 7^{n-2}, this equation becomes $49C = 35C - 6C + 49$, which implies that $20C = 49$, or that $C = 49/20$. Hence, $a_n^{(p)} = (49/20)7^n$ is a particular solution. By Theorem 5, all solutions are of the form

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$$

◀

In Examples 10 and 11, we made an educated guess that there are solutions of a particular form. In both cases we were able to find particular solutions. This was not an accident. Whenever $F(n)$ is the product of a polynomial in n and the nth power of a constant, we know exactly what form a particular solution has, as stated in Theorem 6. We leave the proof of Theorem 6 as Exercise 52.

THEOREM 6 Suppose that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers. When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Note that in the case when s is a root of multiplicity m of the characteristic equation of the associated linear homogeneous recurrence relation, the factor n^m ensures that the proposed particular solution will not already be a solution of the associated linear homogeneous recurrence relation. We next provide Example 12 to illustrate the form of a particular solution provided by Theorem 6.

EXAMPLE 12 What form does a particular solution of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2 + 1)3^n$?

Solution: The associated linear homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$. Its characteristic equation, $r^2 - 6r + 9 = (r - 3)^2 = 0$, has a single root, 3, of multiplicity two. To apply Theorem 6, with $F(n)$ of the form $P(n)s^n$, where $P(n)$ is a polynomial and s is a constant, we need to ask whether s is a root of this characteristic equation.

Because $s = 3$ is a root with multiplicity $m = 2$ but $s = 2$ is not a root, Theorem 6 tells us that a particular solution has the form $p_0 n^2 3^n$ if $F(n) = 3^n$, the form $n^2(p_1 n + p_0)3^n$ if $F(n) =$

$n3^n$, the form $(p_2n^2 + p_1n + p_0)2^n$ if $F(n) = n^2 2^n$, and the form $n^2(p_2n^2 + p_1n + p_0)3^n$ if $F(n) = (n^2 + 1)3^n$. ◀

Care must be taken when $s = 1$ when solving recurrence relations of the type covered by Theorem 6. In particular, to apply this theorem with $F(n) = b_t n_t + b_{t-1}n_{t-1} + \cdots + b_1 n + b_0$, the parameter s takes the value $s = 1$ (even though the term 1^n does not explicitly appear). By the theorem, the form of the solution then depends on whether 1 is a root of the characteristic equation of the associated linear homogeneous recurrence relation. This is illustrated in Example 13, which shows how Theorem 6 can be used to find a formula for the sum of the first n positive integers.

EXAMPLE 13 Let a_n be the sum of the first n positive integers, so that

$$a_n = \sum_{k=1}^{n} k.$$

Note that a_n satisfies the linear nonhomogeneous recurrence relation

$$a_n = a_{n-1} + n.$$

(To obtain a_n, the sum of the first n positive integers, from a_{n-1}, the sum of the first $n - 1$ positive integers, we add n.) Note that the initial condition is $a_1 = 1$.

The associated linear homogeneous recurrence relation for a_n is

$$a_n = a_{n-1}.$$

The solutions of this homogeneous recurrence relation are given by $a_n^{(h)} = c(1)^n = c$, where c is a constant. To find all solutions of $a_n = a_{n-1} + n$, we need find only a single particular solution. By Theorem 6, because $F(n) = n = n \cdot (1)^n$ and $s = 1$ is a root of degree one of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form $n(p_1n + p_0) = p_1n^2 + p_0n$.

Inserting this into the recurrence relation gives $p_1n^2 + p_0n = p_1(n-1)^2 + p_0(n-1) + n$. Simplifying, we see that $n(2p_1 - 1) + (p_0 - p_1) = 0$, which means that $2p_1 - 1 = 0$ and $p_0 - p_1 = 0$, so $p_0 = p_1 = 1/2$. Hence,

$$a_n^{(p)} = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

is a particular solution. Hence, all solutions of the original recurrence relation $a_n = a_{n-1} + n$ are given by $a_n = a_n^{(h)} + a_n^{(p)} = c + n(n+1)/2$. Because $a_1 = 1$, we have $1 = a_1 = c + 1 \cdot 2/2 = c + 1$, so $c = 0$. It follows that $a_n = n(n+1)/2$. (This is the same formula given in Table 2 in Section 2.4 and derived previously.) ◀

Exercises

1. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-1} + 4a_{n-2} + 5a_{n-3}$
b) $a_n = 2na_{n-1} + a_{n-2}$
c) $a_n = a_{n-1} + a_{n-4}$
d) $a_n = a_{n-1} + 2$
e) $a_n = a_{n-1}^2 + a_{n-2}$
f) $a_n = a_{n-2}$
g) $a_n = a_{n-1} + n$

2. Determine which of these are linear homogeneous recurrence relations with constant coefficients. Also, find the degree of those that are.

a) $a_n = 3a_{n-2}$
b) $a_n = 3$
c) $a_n = a_{n-1}^2$
d) $a_n = a_{n-1} + 2a_{n-3}$
e) $a_n = a_{n-1}/n$
f) $a_n = a_{n-1} + a_{n-2} + n + 3$
g) $a_n = 4a_{n-2} + 5a_{n-4} + 9a_{n-7}$

3. Solve these recurrence relations together with the initial conditions given.

a) $a_n = 2a_{n-1}$ for $n \geq 1$, $a_0 = 3$
b) $a_n = a_{n-1}$ for $n \geq 1$, $a_0 = 2$
c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$
d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$
e) $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 1$
f) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 0$, $a_1 = 4$
g) $a_n = a_{n-2}/4$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

4. Solve these recurrence relations together with the initial conditions given.

a) $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$
b) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$
c) $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 10$
d) $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$
e) $a_n = a_{n-2}$ for $n \geq 2$, $a_0 = 5$, $a_1 = -1$
f) $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = -3$
g) $a_{n+2} = -4a_{n+1} + 5a_n$ for $n \geq 0$, $a_0 = 2$, $a_1 = 8$

5. How many different messages can be transmitted in n microseconds using the two signals described in Exercise 19 in Section 8.1?

6. How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

7. In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 2×2 pieces?

8. A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.

a) Find a recurrence relation for $\{L_n\}$, where L_n is the number of lobsters caught in year n, under the assumption for this model.
b) Find L_n if 100,000 lobsters were caught in year 1 and 300,000 were caught in year 2.

9. A deposit of \$100,000 is made to an investment fund at the beginning of a year. On the last day of each year two dividends are awarded. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.

a) Find a recurrence relation for $\{P_n\}$, where P_n is the amount in the account at the end of n years if no money is ever withdrawn.
b) How much is in the account after n years if no money has been withdrawn?

***10.** Prove Theorem 2.

11. The **Lucas numbers** satisfy the recurrence relation

Links
$$L_n = L_{n-1} + L_{n-2},$$

and the initial conditions $L_0 = 2$ and $L_1 = 1$.

a) Show that $L_n = f_{n-1} + f_{n+1}$ for $n = 2, 3, \ldots$, where f_n is the nth Fibonacci number.
b) Find an explicit formula for the Lucas numbers.

12. Find the solution to $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

13. Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

14. Find the solution to $a_n = 5a_{n-2} - 4a_{n-4}$ with $a_0 = 3$, $a_1 = 2$, $a_2 = 6$, and $a_3 = 8$.

15. Find the solution to $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.

***16.** Prove Theorem 3.

17. Prove this identity relating the Fibonacci numbers and the binomial coefficients:

$$f_{n+1} = C(n, 0) + C(n-1, 1) + \cdots + C(n-k, k),$$

where n is a positive integer and $k = \lfloor n/2 \rfloor$. [*Hint:* Let $a_n = C(n, 0) + C(n-1, 1) + \cdots + C(n-k, k)$. Show that the sequence $\{a_n\}$ satisfies the same recurrence relation and initial conditions satisfied by the sequence of Fibonacci numbers.]

18. Solve the recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$ with $a_0 = -5$, $a_1 = 4$, and $a_2 = 88$.

19. Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 5$, $a_1 = -9$, and $a_2 = 15$.

20. Find the general form of the solutions of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}$.

21. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1, −2, −2, −2, 3, 3, −4?

22. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has the roots −1, −1, −1, 2, 2, 5, 5, 7?

23. Consider the nonhomogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

a) Show that $a_n = -2^{n+1}$ is a solution of this recurrence relation.
b) Use Theorem 5 to find all solutions of this recurrence relation.
c) Find the solution with $a_0 = 1$.

24. Consider the nonhomogeneous linear recurrence relation $a_n = 2a_{n-1} + 2^n$.

a) Show that $a_n = n2^n$ is a solution of this recurrence relation.
b) Use Theorem 5 to find all solutions of this recurrence relation.
c) Find the solution with $a_0 = 2$.

25. a) Determine values of the constants A and B such that $a_n = An + B$ is a solution of recurrence relation $a_n = 2a_{n-1} + n + 5$.
b) Use Theorem 5 to find all solutions of this recurrence relation.
c) Find the solution of this recurrence relation with $a_0 = 4$.

26. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3} + F(n)$ if

a) $F(n) = n^2$? **b)** $F(n) = 2^n$?
c) $F(n) = n2^n$? **d)** $F(n) = (-2)^n$?
e) $F(n) = n^2 2^n$? **f)** $F(n) = n^3(-2)^n$?
g) $F(n) = 3$?

27. What is the general form of the particular solution guaranteed to exist by Theorem 6 of the linear nonhomogeneous recurrence relation $a_n = 8a_{n-2} - 16a_{n-4} + F(n)$ if

a) $F(n) = n^3$? **b)** $F(n) = (-2)^n$?
c) $F(n) = n2^n$? **d)** $F(n) = n^2 4^n$?
e) $F(n) = (n^2 - 2)(-2)^n$? **f)** $F(n) = n^4 2^n$?
g) $F(n) = 2$?

28. a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.
b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 4$.

29. a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 3^n$.
b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 5$.

30. a) Find all solutions of the recurrence relation $a_n = -5a_{n-1} - 6a_{n-2} + 42 \cdot 4^n$.
b) Find the solution of this recurrence relation with $a_1 = 56$ and $a_2 = 278$.

31. Find all solutions of the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 3n$. [*Hint:* Look for a particular solution of the form $qn2^n + p_1 n + p_2$, where q, p_1, and p_2 are constants.]

32. Find the solution of the recurrence relation $a_n = 2a_{n-1} + 3 \cdot 2^n$.

33. Find all solutions of the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n$.

34. Find all solutions of the recurrence relation $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$ with $a_0 = -2$, $a_1 = 0$, and $a_2 = 5$.

35. Find the solution of the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$ with $a_0 = 1$ and $a_1 = 4$.

36. Let a_n be the sum of the first n perfect squares, that is, $a_n = \sum_{k=1}^{n} k^2$. Show that the sequence $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation $a_n = a_{n-1} + n^2$ and the initial condition $a_1 = 1$. Use Theorem 6 to determine a formula for a_n by solving this recurrence relation.

37. Let a_n be the sum of the first n triangular numbers, that is, $a_n = \sum_{k=1}^{n} t_k$, where $t_k = k(k+1)/2$. Show that $\{a_n\}$ satisfies the linear nonhomogeneous recurrence relation $a_n = a_{n-1} + n(n+1)/2$ and the initial condition $a_1 = 1$. Use Theorem 6 to determine a formula for a_n by solving this recurrence relation.

38. a) Find the characteristic roots of the linear homogeneous recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$. [*Note:* These are complex numbers.]
b) Find the solution of the recurrence relation in part (a) with $a_0 = 1$ and $a_1 = 2$.

***39. a)** Find the characteristic roots of the linear homogeneous recurrence relation $a_n = a_{n-4}$. [*Note:* These include complex numbers.]
b) Find the solution of the recurrence relation in part (a) with $a_0 = 1$, $a_1 = 0$, $a_2 = -1$, and $a_3 = 1$.

***40.** Solve the simultaneous recurrence relations

$$a_n = 3a_{n-1} + 2b_{n-1}$$
$$b_n = a_{n-1} + 2b_{n-1}$$

with $a_0 = 1$ and $b_0 = 2$.

***41. a)** Use the formula found in Example 4 for f_n, the nth Fibonacci number, to show that f_n is the integer closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

b) Determine for which n f_n is greater than

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

and for which n f_n is less than

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

42. Show that if $a_n = a_{n-1} + a_{n-2}$, $a_0 = s$ and $a_1 = t$, where s and t are constants, then $a_n = sf_{n-1} + tf_n$ for all positive integers n.

43. Express the solution of the linear nonhomogenous recurrence relation $a_n = a_{n-1} + a_{n-2} + 1$ for $n \geq 2$ where $a_0 = 0$ and $a_1 = 1$ in terms of the Fibonacci numbers. [*Hint:* Let $b_n = a_n + 1$ and apply Exercise 42 to the sequence b_n.]

***44.** (*Linear algebra required*) Let $\mathbf{A}_n$ be the $n \times n$ matrix with 2s on its main diagonal, 1s in all positions next to a diagonal element, and 0s everywhere else. Find a recurrence relation for d_n, the determinant of $\mathbf{A}_n$. Solve this recurrence relation to find a formula for d_n.

45. Suppose that each pair of a genetically engineered species of rabbits left on an island produces two new pairs of rabbits at the age of 1 month and six new pairs of rabbits at the age of 2 months and every month afterward. None of the rabbits ever die or leave the island.

a) Find a recurrence relation for the number of pairs of rabbits on the island n months after one newborn pair is left on the island.
b) By solving the recurrence relation in (a) determine the number of pairs of rabbits on the island n months after one pair is left on the island.

46. Suppose that there are two goats on an island initially. The number of goats on the island doubles every year by natural reproduction, and some goats are either added or removed each year.

a) Construct a recurrence relation for the number of goats on the island at the start of the nth year, assuming that during each year an extra 100 goats are put on the island.

b) Solve the recurrence relation from part (a) to find the number of goats on the island at the start of the nth year.

c) Construct a recurrence relation for the number of goats on the island at the start of the nth year, assuming that n goats are removed during the nth year for each $n \geq 3$.

d) Solve the recurrence relation in part (c) for the number of goats on the island at the start of the nth year.

47. A new employee at an exciting new software company starts with a salary of \$50,000 and is promised that at the end of each year her salary will be double her salary of the previous year, with an extra increment of \$10,000 for each year she has been with the company.

a) Construct a recurrence relation for her salary for her nth year of employment.

b) Solve this recurrence relation to find her salary for her nth year of employment.

Some linear recurrence relations that do not have constant coefficients can be systematically solved. This is the case for recurrence relations of the form $f(n)a_n = g(n)a_{n-1} + h(n)$. Exercises 48–50 illustrate this.

***48.** a) Show that the recurrence relation

$$f(n)a_n = g(n)a_{n-1} + h(n),$$

for $n \geq 1$, and with $a_0 = C$, can be reduced to a recurrence relation of the form

$$b_n = b_{n-1} + Q(n)h(n),$$

where $b_n = g(n+1)Q(n+1)a_n$, with

$$Q(n) = (f(1)f(2)\cdots f(n-1))/(g(1)g(2)\cdots g(n)).$$

b) Use part (a) to solve the original recurrence relation to obtain

$$a_n = \frac{C + \sum_{i=1}^{n} Q(i)h(i)}{g(n+1)Q(n+1)}.$$

***49.** Use Exercise 48 to solve the recurrence relation $(n+1)a_n = (n+3)a_{n-1} + n$, for $n \geq 1$, with $a_0 = 1$.

50. It can be shown that C_n, the average number of comparisons made by the quick sort algorithm (described in preamble to Exercise 50 in Section 5.4), when sorting n elements in random order, satisfies the recurrence relation

$$C_n = n + 1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k$$

for $n = 1, 2, \ldots$, with initial condition $C_0 = 0$.

a) Show that $\{C_n\}$ also satisfies the recurrence relation $nC_n = (n+1)C_{n-1} + 2n$ for $n = 1, 2, \ldots$.

b) Use Exercise 48 to solve the recurrence relation in part (a) to find an explicit formula for C_n.

****51.** Prove Theorem 4.

****52.** Prove Theorem 6.

53. Solve the recurrence relation $T(n) = nT^2(n/2)$ with initial condition $T(1) = 6$ when $n = 2^k$ for some integer k. [*Hint:* Let $n = 2^k$ and then make the substitution $a_k = \log T(2^k)$ to obtain a linear nonhomogeneous recurrence relation.]

8.3 Divide-and-Conquer Algorithms and Recurrence Relations

8.3.1 Introduction

Links

Many recursive algorithms take a problem with a given input and divide it into one or more smaller problems. This reduction is successively applied until the solutions of the smaller problems can be found quickly. For instance, we perform a binary search by reducing the search for an element in a list to the search for this element in a list half as long. We successively apply this reduction until one element is left. When we sort a list of integers using the merge sort, we split the list into two halves of equal size and sort each half separately. We then merge the two sorted halves. Another example of this type of recursive algorithm is a procedure for multiplying integers that reduces the problem of the multiplication of two integers to three multiplications of pairs of integers with half as many bits. This reduction is successively applied until integers with one bit are obtained. There procedures follow an important algorithmic paradigm known as **divide-and-conquer**, and are called **divide-and-conquer algorithms**, because they *divide* a problem into one or more instances of the same problem of smaller size and they *conquer* the problem by using the solutions of the smaller problems to find a solution of the original problem, perhaps with some additional work.

"Divide et impera" (translation: *"Divide and conquer"*) —Julius Caesar

In this section we will show how recurrence relations can be used to analyze the computational complexity of divide-and-conquer algorithms. We will use these recurrence relations to estimate the number of operations used by many different divide-and-conquer algorithms, including several that we introduce in this section.

8.3.2 Divide-and-Conquer Recurrence Relations

Suppose that a recursive algorithm divides a problem of size n into a subproblems, where each subproblem is of size n/b (for simplicity, assume that n is a multiple of b; in reality, the smaller problems are often of size equal to the nearest integers either less than or equal to, or greater than or equal to, n/b). Also, suppose that a total of $g(n)$ extra operations are required in the conquer step of the algorithm to combine the solutions of the subproblems into a solution of the original problem. Then, if $f(n)$ represents the number of operations required to solve the problem of size n, it follows that f satisfies the recurrence relation

$$f(n) = af(n/b) + g(n).$$

This is called a **divide-and-conquer recurrence relation**.

We will first set up the divide-and-conquer recurrence relations that can be used to study the complexity of some important algorithms. Then we will show how to use these divide-and-conquer recurrence relations to estimate the complexity of these algorithms.

EXAMPLE 1 **Binary Search** We introduced a binary search algorithm in Section 3.1. This binary search algorithm reduces the search for an element in a search sequence of size n to the binary search for this element in a search sequence of size $n/2$, when n is even. (Hence, the problem of size n has been reduced to *one* problem of size $n/2$.) Two comparisons are needed to implement this reduction (one to determine which half of the list to use and the other to determine whether any terms of the list remain). Hence, if $f(n)$ is the number of comparisons required to search for an element in a search sequence of size n, then

Extra Examples

$$f(n) = f(n/2) + 2$$

when n is even. ◀

EXAMPLE 2 **Finding the Maximum and Minimum of a Sequence** Consider the following algorithm for locating the maximum and minimum elements of a sequence $a_1, a_2, \dots, a_n$. If $n = 1$, then a_1 is the maximum and the minimum. If $n > 1$, split the sequence into two sequences, either where both have the same number of elements or where one of the sequences has one more element than the other. The problem is reduced to finding the maximum and minimum of each of the two smaller sequences. The solution to the original problem results from the comparison of the separate maxima and minima of the two smaller sequences to obtain the overall maximum and minimum.

Let $f(n)$ be the total number of comparisons needed to find the maximum and minimum elements of the sequence with n elements. We have shown that a problem of size n can be reduced into two problems of size $n/2$, when n is even, using two comparisons, one to compare the maxima of the two sequences and the other to compare the minima of the two sequences. This gives the recurrence relation

$$f(n) = 2f(n/2) + 2$$

when n is even. ◀

EXAMPLE 3 **Merge Sort** The merge sort algorithm (introduced in Section 5.4) splits a list to be sorted with n items, where n is even, into two lists with $n/2$ elements each, and uses fewer than n

comparisons to merge the two sorted lists of $n/2$ items each into one sorted list. Consequently, the number of comparisons used by the merge sort to sort a list of n elements is less than $M(n)$, where the function $M(n)$ satisfies the divide-and-conquer recurrence relation

$$M(n) = 2M(n/2) + n.$$

◀

EXAMPLE 4 Links

Fast Multiplication of Integers Surprisingly, there are more efficient algorithms than the conventional algorithm (described in Section 4.2) for multiplying integers. One of these algorithms, which uses a divide-and-conquer technique, will be described here. This fast multiplication algorithm proceeds by splitting each of two $2n$-bit integers into two blocks, each with n bits. Then, the original multiplication is reduced from the multiplication of two $2n$-bit integers to three multiplications of n-bit integers, plus shifts and additions.

Suppose that a and b are integers with binary expansions of length $2n$ (add initial bits of zero in these expansions if necessary to make them the same length). Let

$$a = (a_{2n-1}a_{2n-2} \cdots a_1a_0)_2 \qquad \text{and} \qquad b = (b_{2n-1}b_{2n-2} \cdots b_1b_0)_2.$$

Let

$$a = 2^nA_1 + A_0, \quad b = 2^nB_1 + B_0,$$

where

$$A_1 = (a_{2n-1} \cdots a_{n+1}a_n)_2, \qquad A_0 = (a_{n-1} \cdots a_1a_0)_2,$$
$$B_1 = (b_{2n-1} \cdots b_{n+1}b_n)_2, \qquad B_0 = (b_{n-1} \cdots b_1b_0)_2.$$

The algorithm for fast multiplication of integers is based on the fact that ab can be rewritten as

$$ab = (2^{2n} + 2^n)A_1B_1 + 2^n(A_1 - A_0)(B_0 - B_1) + (2^n + 1)A_0B_0.$$

The important fact about this identity is that it shows that the multiplication of two $2n$-bit integers can be carried out using three multiplications of n-bit integers, together with additions, subtractions, and shifts. This shows that if $f(n)$ is the total number of bit operations needed to multiply two n-bit integers, then

$$f(2n) = 3f(n) + Cn.$$

The reasoning behind this equation is as follows. The three multiplications of n-bit integers are carried out using $3f(n)$-bit operations. Each of the additions, subtractions, and shifts uses a constant multiple of n-bit operations, and Cn represents the total number of bit operations used by these operations. ◀

EXAMPLE 5 Links

Fast Matrix Multiplication In Example 7 of Section 3.3 we showed that multiplying two $n \times n$ matrices using the definition of matrix multiplication required n^3 multiplications and $n^2(n-1)$ additions. Consequently, computing the product of two $n \times n$ matrices in this way requires $O(n^3)$ operations (multiplications and additions). Surprisingly, there are more efficient divide-and-conquer algorithms for multiplying two $n \times n$ matrices. Such an algorithm, invented by Volker Strassen in 1969, reduces the multiplication of two $n \times n$ matrices, when n is even, to seven multiplications of two $(n/2) \times (n/2)$ matrices and 15 additions of $(n/2) \times (n/2)$ matrices.

(See [CoLeRiSt09] for the details of this algorithm.) Hence, if $f(n)$ is the number of operations (multiplications and additions) used, it follows that

$$f(n) = 7f(n/2) + 15n^2/4$$

when n is even. ◀

As Examples 1–5 show, recurrence relations of the form $f(n) = af(n/b) + g(n)$ arise in many different situations. It is possible to derive estimates of the size of functions that satisfy such recurrence relations. Suppose that f satisfies this recurrence relation whenever n is divisible by b. Let $n = b^k$, where k is a positive integer. Then

$$\begin{aligned} f(n) &= af(n/b) + g(n) \\ &= a^2f(n/b^2) + ag(n/b) + g(n) \\ &= a^3f(n/b^3) + a^2g(n/b^2) + ag(n/b) + g(n) \\ &\ \vdots \\ &= a^kf(n/b^k) + \sum_{j=0}^{k-1} a^j g(n/b^j). \end{aligned}$$

Because $n/b^k = 1$, it follows that

$$f(n) = a^kf(1) + \sum_{j=0}^{k-1} a^j g(n/b^j).$$

We can use this equation for $f(n)$ to estimate the size of functions that satisfy divide-and-conquer relations.

THEOREM 1 Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + c$$

whenever n is divisible by b, where $a \geq 1$, b is an integer greater than 1, and c is a positive real number. Then

$$f(n) \text{ is } \begin{cases} O(n^{\log_b a}) & \text{if } a > 1, \\ O(\log n) & \text{if } a = 1. \end{cases}$$

Furthermore, when $n = b^k$ and $a \neq 1$, where k is a positive integer,

$$f(n) = C_1 n^{\log_b a} + C_2,$$

where $C_1 = f(1) + c/(a-1)$ and $C_2 = -c/(a-1)$.

Proof: First let $n = b^k$. From the expression for $f(n)$ obtained in the discussion preceding the theorem, with $g(n) = c$, we have

$$f(n) = a^kf(1) + \sum_{j=0}^{k-1} a^j c = a^kf(1) + c\sum_{j=0}^{k-1} a^j.$$

When $a = 1$ we have

$$f(n) = f(1) + ck.$$

Because $n = b^k$, we have $k = \log_b n$. Hence,

$$f(n) = f(1) + c \log_b n.$$

When n is not a power of b, we have $b^k < n < b^{k+1}$, for a positive integer k. Because f is increasing, it follows that $f(n) \le f(b^{k+1}) = f(1) + c(k+1) = (f(1)+c) + ck \le (f(1)+c) + c \log_b n$. Therefore, in both cases, $f(n)$ is $O(\log n)$ when $a = 1$.

Now suppose that $a > 1$. First assume that $n = b^k$, where k is a positive integer. From the formula for the sum of terms of a geometric progression (Theorem 1 in Section 2.4), it follows that

$$\begin{aligned} f(n) &= a^k f(1) + c(a^k - 1)/(a-1) \\ &= a^k[f(1) + c/(a-1)] - c/(a-1) \\ &= C_1 n^{\log_b a} + C_2, \end{aligned}$$

because $a^k = a^{\log_b n} = n^{\log_b a}$ (see Exercise 4 in Appendix 2), where $C_1 = f(1) + c/(a-1)$ and $C_2 = -c/(a-1)$.

Now suppose that n is not a power of b. Then $b^k < n < b^{k+1}$, where k is a nonnegative integer. Because f is increasing,

$$\begin{aligned} f(n) &\le f(b^{k+1}) = C_1 a^{k+1} + C_2 \\ &\le (C_1 a) a^{\log_b n} + C_2 \\ &= (C_1 a) n^{\log_b a} + C_2, \end{aligned}$$

because $k \le \log_b n < k+1$.

Hence, we have $f(n)$ is $O(n^{\log_b a})$. ◁

Examples 6–9 illustrate how Theorem 1 is used.

EXAMPLE 6 *Extra Examples* Let $f(n) = 5f(n/2) + 3$ and $f(1) = 7$. Find $f(2^k)$, where k is a positive integer. Also, estimate $f(n)$ if f is an increasing function.

Solution: From the proof of Theorem 1, with $a = 5$, $b = 2$, and $c = 3$, we see that if $n = 2^k$, then

$$\begin{aligned} f(n) &= a^k[f(1) + c/(a-1)] + [-c/(a-1)] \\ &= 5^k[7 + (3/4)] - 3/4 \\ &= 5^k(31/4) - 3/4. \end{aligned}$$

Also, if $f(n)$ is increasing, Theorem 1 shows that $f(n)$ is $O(n^{\log_b a}) = O(n^{\log 5})$. ◀

We can use Theorem 1 to estimate the computational complexity of the binary search algorithm and the algorithm given in Example 2 for locating the minimum and maximum of a sequence.

EXAMPLE 7 Give a big-O estimate for the number of comparisons used by a binary search.

Solution: In Example 1 it was shown that $f(n) = f(n/2) + 2$ when n is even, where f is the number of comparisons required to perform a binary search on a sequence of size n. Hence, from Theorem 1, it follows that $f(n)$ is $O(\log n)$. ◀

EXAMPLE 8 Give a big-O estimate for the number of comparisons used to locate the maximum and minimum elements in a sequence using the algorithm given in Example 2.

Solution: In Example 2 we showed that $f(n) = 2f(n/2) + 2$, when n is even, where f is the number of comparisons needed by this algorithm. Hence, from Theorem 1, it follows that $f(n)$ is $O(n^{\log 2}) = O(n)$. ◀

We now state a more general, and more complicated, theorem, which has Theorem 1 as a special case. This theorem (or more powerful versions, including big-Theta estimates) is sometimes known as the master theorem because it is useful in analyzing the complexity of many important divide-and-conquer algorithms.

THEOREM 2 **MASTER THEOREM** Let f be an increasing function that satisfies the recurrence relation

$$f(n) = af(n/b) + cn^d$$

whenever $n = b^k$, where k is a positive integer, $a \geq 1$, b is an integer greater than 1, and c and d are real numbers with c positive and d nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d, \\ O(n^d \log n) & \text{if } a = b^d, \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

The proof of Theorem 2 is left for the reader as Exercises 29–33.

EXAMPLE 9 **Complexity of Merge Sort** In Example 3 we explained that the number of comparisons used by the merge sort to sort a list of n elements is less than $M(n)$, where $M(n) = 2M(n/2) + n$. By the master theorem (Theorem 2) we find that $M(n)$ is $O(n \log n)$, which agrees with the estimate found in Section 5.4. ◀

EXAMPLE 10 Give a big-O estimate for the number of bit operations needed to multiply two n-bit integers using the fast multiplication algorithm described in Example 4.

Solution: Example 4 shows that $f(n) = 3f(n/2) + Cn$, when n is even, where $f(n)$ is the number of bit operations required to multiply two n-bit integers using the fast multiplication algorithm. Hence, from the master theorem (Theorem 2), it follows that $f(n)$ is $O(n^{\log 3})$. Note that log 3 $\sim$ 1.6. Because the conventional algorithm for multiplication uses $O(n^2)$ bit operations, the fast multiplication algorithm is a substantial improvement over the conventional algorithm in terms of time complexity for sufficiently large integers, including large integers that occur in practical applications. ◀

EXAMPLE 11 Give a big-O estimate for the number of multiplications and additions required to multiply two $n \times n$ matrices using the matrix multiplication algorithm referred to in Example 5.

Solution: Let $f(n)$ denote the number of additions and multiplications used by the algorithm mentioned in Example 5 to multiply two $n \times n$ matrices. We have $f(n) = 7f(n/2) + 15n^2/4$, when n is even. Hence, from the master theorem (Theorem 2), it follows that $f(n)$ is $O(n^{\log 7})$. Note that $\log 7 \sim 2.8$. Because the conventional algorithm for multiplying two $n \times n$ matrices uses $O(n^3)$ additions and multiplications, it follows that for sufficiently large integers n, including those that occur in many practical applications, this algorithm is substantially more efficient in time complexity than the conventional algorithm. ◀

THE CLOSEST-PAIR PROBLEM We conclude this section by introducing a divide-and-conquer algorithm from computational geometry, the part of discrete mathematics devoted to algorithms that solve geometric problems.

EXAMPLE 12 **The Closest-Pair Problem** Consider the problem of determining the closest pair of points in a set of n points $(x_1, y_1), \ldots, (x_n, y_n)$ in the plane, where the distance between two points (x_i, y_i) and (x_j, y_j) is the usual Euclidean distance $\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$. This problem arises in many applications such as determining the closest pair of airplanes in the air space at a particular altitude being managed by an air traffic controller. How can this closest pair of points be found in an efficient way?

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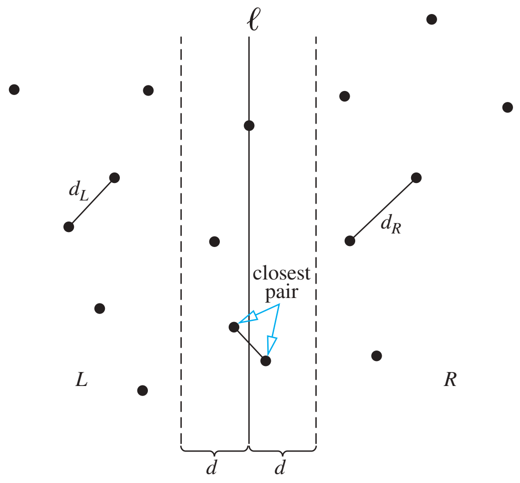
Solution: To solve this problem we can first determine the distance between every pair of points and then find the smallest of these distances. However, this approach requires $O(n^2)$ computations of distances and comparisons because there are $C(n, 2) = n(n-1)/2$ pairs of points. Surprisingly, there is an elegant divide-and-conquer algorithm that can solve the closest-pair problem for n points using $O(n \log n)$ computations of distances and comparisons. The algorithm we describe here is due to Michael Samos (see [PrSa85]).

It took researchers more than 10 years to find an algorithm with $O(n \log n)$ complexity that locates the closest pair of points among n points.

For simplicity, we assume that $n = 2^k$, where k is a positive integer. (We avoid some technical considerations that are needed when n is not a power of 2.) When $n = 2$, we have only one pair of points; the distance between these two points is the minimum distance. At the start of the algorithm we use the merge sort twice, once to sort the points in order of increasing x coordinates, and once to sort the points in order of increasing y coordinates. Each of these sorts requires $O(n \log n)$ operations. We will use these sorted lists in each recursive step.

The recursive part of the algorithm divides the problem into two subproblems, each involving half as many points. Using the sorted list of the points by their x coordinates, we construct a vertical line ℓ dividing the n points into two parts, a left part and a right part of equal size, each containing $n/2$ points, as shown in Figure 1. (If any points fall on the dividing line ℓ, we divide them among the two parts if necessary.) At subsequent steps of the recursion we need not sort on x coordinates again, because we can select the corresponding sorted subset of all the points. This selection is a task that can be done with $O(n)$ comparisons.

There are three possibilities concerning the positions of the closest points: (1) they are both in the left region L, (2) they are both in the right region R, or (3) one point is in the left region and the other is in the right region. Apply the algorithm recursively to compute d_L and d_R, where d_L is the minimum distance between points in the left region and d_R is the minimum distance between points in the right region. Let $d = \min(d_L, d_R)$. To successfully divide the problem of finding the closest two points in the original set into the two problems of finding the



In this illustration the problem of finding the closest pair in a set of 16 points is reduced to two problems of finding the closest pair in a set of eight points *and* the problem of determining whether there are points closer than $d = \min(d_L, d_R)$ within the strip of width $2d$ centered at ℓ.

FIGURE 1 The recursive step of the algorithm for solving the closest-pair problem.

shortest distances between points in the two regions separately, we have to handle the conquer part of the algorithm, which requires that we consider the case where the closest points lie in different regions, that is, one point is in L and the other in R. Because there is a pair of points at distance d where both points lie in R or both points lie in L, for the closest points to lie in different regions requires that they must be a distance less than d apart.

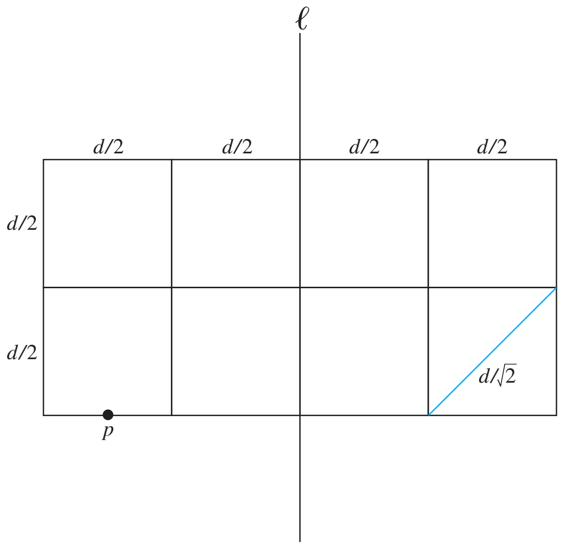
For a point in the left region and a point in the right region to lie at a distance less than d apart, these points must lie in the vertical strip of width $2d$ that has the line ℓ as its center. (Otherwise, the distance between these points is greater than the difference in their x coordinates, which exceeds d.) To examine the points within this strip, we sort the points so that they are listed in order of increasing y coordinates, using the sorted list of the points by their y coordinates. At each recursive step, we form a subset of the points in the region sorted by their y coordinates from the already sorted set of all points sorted by their y coordinates, which can be done with $O(n)$ comparisons.

Beginning with a point in the strip with the smallest y coordinate, we successively examine each point in the strip, computing the distance between this point and all other points in the strip that have larger y coordinates that could lie at a distance less than d from this point. Note that to examine a point p, we need only consider the distances between p and points in the set that lie within the rectangle of height d and width $2d$ with p on its base and with vertical sides at distance d from ℓ.

We can show that there are at most eight points from the set, including p, in or on this $2d \times d$ rectangle. To see this, note that there can be at most one point in each of the eight $d/2 \times d/2$ squares shown in Figure 2. This follows because the farthest apart points can be on or within one of these squares is the diagonal length $d/\sqrt{2}$ (which can be found using the Pythagorean theorem), which is less than d, and each of these $d/2 \times d/2$ squares lies entirely within the left region or the right region. This means that at this stage we need only compare at most seven distances, the distances between p and the seven or fewer other points in or on the rectangle, with d.

Because the total number of points in the strip of width $2d$ does not exceed n (the total number of points in the set), at most $7n$ distances need to be compared with d to find the minimum distance between points. That is, there are only $7n$ possible distances that could be less than d. Consequently, once the merge sort has been used to sort the pairs according to their x coordinates and according to their y coordinates, we find that the increasing function $f(n)$ satisfying the recurrence relation

$$f(n) = 2f(n/2) + 7n,$$



At most eight points, including p, can lie in or on the $2d \times d$ rectangle centered at ℓ because at most one point can lie in or on each of the eight $(d/2) \times (d/2)$ squares.

FIGURE 2 Showing that there are at most seven other points to consider for each point in the strip.

where $f(2) = 1$, exceeds the number of comparisons needed to solve the closest-pair problem for n points. By the master theorem (Theorem 2), it follows that $f(n)$ is $O(n \log n)$. The two sorts of points by their x coordinates and by their y coordinates each can be done using $O(n \log n)$ comparisons, by using the merge sort, and the sorted subsets of these coordinates at each of the $O(\log n)$ steps of the algorithm can be done using $O(n)$ comparisons each. Thus, we find that the closest-pair problem can be solved using $O(n \log n)$ comparisons. ◀

Exercises

1. How many comparisons are needed for a binary search in a set of 64 elements?
2. How many comparisons are needed to locate the maximum and minimum elements in a sequence with 128 elements using the algorithm in Example 2?
3. Multiply $(1110)_2$ and $(1010)_2$ using the fast multiplication algorithm.
4. Express the fast multiplication algorithm in pseudocode.
5. Determine a value for the constant C in Example 4 and use it to estimate the number of bit operations needed to multiply two 64-bit integers using the fast multiplication algorithm.
6. How many operations are needed to multiply two 32×32 matrices using the algorithm referred to in Example 5?
7. Suppose that $f(n) = f(n/3) + 1$ when n is a positive integer divisible by 3, and $f(1) = 1$. Find
   a) $f(3)$. **b)** $f(27)$. **c)** $f(729)$.
8. Suppose that $f(n) = 2f(n/2) + 3$ when n is an even positive integer, and $f(1) = 5$. Find
   a) $f(2)$. **b)** $f(8)$. **c)** $f(64)$. **d)** $f(1024)$.
9. Suppose that $f(n) = f(n/5) + 3n^2$ when n is a positive integer divisible by 5, and $f(1) = 4$. Find
   a) $f(5)$. **b)** $f(125)$. **c)** $f(3125)$.
10. Find $f(n)$ when $n = 2^k$, where f satisfies the recurrence relation $f(n) = f(n/2) + 1$ with $f(1) = 1$.
11. Give a big-O estimate for the function f in Exercise 10 if f is an increasing function.
12. Find $f(n)$ when $n = 3^k$, where f satisfies the recurrence relation $f(n) = 2f(n/3) + 4$ with $f(1) = 1$.
13. Give a big-O estimate for the function f in Exercise 12 if f is an increasing function.
14. Suppose that there are $n = 2^k$ teams in an elimination tournament, where there are $n/2$ games in the first round, with the $n/2 = 2^{k-1}$ winners playing in the second round, and so on. Develop a recurrence relation for the number of rounds in the tournament.
15. How many rounds are in the elimination tournament described in Exercise 14 when there are 32 teams?
16. Solve the recurrence relation for the number of rounds in the tournament described in Exercise 14.
17. Suppose that the votes of n people for different candidates (where there can be more than two candidates) for a particular office are the elements of a sequence. A person wins the election if this person receives a majority of the votes.

a) Devise a divide-and-conquer algorithm that determines whether a candidate received a majority and, if so, determine who this candidate is. [*Hint:* Assume that n is even and split the sequence of votes into two sequences, each with $n/2$ elements. Note that a candidate could not have received a majority of votes without receiving a majority of votes in at least one of the two halves.]

b) Use the master theorem to give a big-O estimate for the number of comparisons needed by the algorithm you devised in part (a).

18. Suppose that each person in a group of n people votes for exactly two people from a slate of candidates to fill two positions on a committee. The top two finishers both win positions as long as each receives more than $n/2$ votes.

a) Devise a divide-and-conquer algorithm that determines whether the two candidates who received the most votes each received at least $n/2$ votes and, if so, determine who these two candidates are.

b) Use the master theorem to give a big-O estimate for the number of comparisons needed by the algorithm you devised in part (a).

19. a) Set up a divide-and-conquer recurrence relation for the number of multiplications required to compute x^n, where x is a real number and n is a positive integer, using the recursive algorithm from Exercise 26 in Section 5.4.

b) Use the recurrence relation you found in part (a) to construct a big-O estimate for the number of multiplications used to compute x^n using the recursive algorithm.

20. a) Set up a divide-and-conquer recurrence relation for the number of modular multiplications required to compute a^n **mod** m, where a, m, and n are positive integers, using the recursive algorithms from Example 4 in Section 5.4.

b) Use the recurrence relation you found in part (a) to construct a big-O estimate for the number of modular multiplications used to compute a^n **mod** m using the recursive algorithm.

21. Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + 1$ whenever n is a perfect square greater than 1 and $f(2) = 1$.

a) Find $f(16)$.

b) Give a big-O estimate for $f(n)$. [*Hint:* Make the substitution $m = \log n$.]

22. Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square greater than 1 and $f(2) = 1$.

a) Find $f(16)$.

b) Find a big-O estimate for $f(n)$. [*Hint:* Make the substitution $m = \log n$.]

****23.** This exercise deals with the problem of finding the largest sum of consecutive terms of a sequence of n real numbers. When all terms are positive, the sum of all terms provides the answer, but the situation is more complicated when some terms are negative. For example, the maximum sum of consecutive terms of the sequence $-2, 3, -1, 6, -7, 4$ is $3 + (-1) + 6 = 8$. (This exercise is based on [Be86].) Recall that in Exercise 56 in Section 8.1 we developed a dynamic programming algorithm for solving this problem. Here, we first look at the brute-force algorithm for solving this problem; then we develop a divide-and-conquer algorithm for solving it.

a) Use pseudocode to describe an algorithm that solves this problem by finding the sums of consecutive terms starting with the first term, the sums of consecutive terms starting with the second term, and so on, keeping track of the maximum sum found so far as the algorithm proceeds.

b) Determine the computational complexity of the algorithm in part (a) in terms of the number of sums computed and the number of comparisons made.

c) Devise a divide-and-conquer algorithm to solve this problem. [*Hint:* Assume that there are an even number of terms in the sequence and split the sequence into two halves. Explain how to handle the case when the maximum sum of consecutive terms includes terms in both halves.]

d) Use the algorithm from part (c) to find the maximum sum of consecutive terms of each of the sequences: $-2, 4, -1, 3, 5, -6, 1, 2$; $4, 1, -3, 7, -1, -5, 3, -2$; and $-1, 6, 3, -4, -5, 8, -1, 7$.

e) Find a recurrence relation for the number of sums and comparisons used by the divide-and-conquer algorithm from part (c).

f) Use the master theorem to estimate the computational complexity of the divide-and-conquer algorithm. How does it compare in terms of computational complexity with the algorithm from part (a)?

24. Apply the algorithm described in Example 12 for finding the closest pair of points, using the Euclidean distance between points, to find the closest pair of the points $(1, 3)$, $(1, 7)$, $(2, 4)$, $(2, 9)$, $(3, 1)$, $(3, 5)$, $(4, 3)$, and $(4, 7)$.

25. Apply the algorithm described in Example 12 for finding the closest pair of points, using the Euclidean distance between points, to find the closest pair of the points $(1, 2)$, $(1, 6)$, $(2, 4)$, $(2, 8)$, $(3, 1)$, $(3, 6)$, $(3, 10)$, $(4, 3)$, $(5, 1)$, $(5, 5)$, $(5, 9)$, $(6, 7)$, $(7, 1)$, $(7, 4)$, $(7, 9)$, and $(8, 6)$.

***26.** Use pseudocode to describe the recursive algorithm for solving the closest-pair problem as described in Example 12.

27. Construct a variation of the algorithm described in Example 12 along with justifications of the steps used by the algorithm to find the smallest distance between two points if the distance between two points is defined to be $d((x_i, y_i), (x_j, y_j)) = \max(|x_i - x_j|, |y_i - y_j|)$.

***28.** Suppose someone picks a number x from a set of n numbers. A second person tries to guess the number by successively selecting subsets of the n numbers and asking the first person whether x is in each set. The first person answers either "yes" or "no." When the first

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person answers each query truthfully, we can find x using $\log n$ queries by successively splitting the sets used in each query in half. Ulam's problem, proposed by Stanislaw Ulam in 1976, asks for the number of queries required to find x, supposing that the first person is allowed to lie exactly once.

a) Show that by asking each question twice, given a number x and a set with n elements, and asking one more question when we find the lie, Ulam's problem can be solved using $2 \log n + 1$ queries.

b) Show that by dividing the initial set of n elements into four parts, each with $n/4$ elements, 1/4 of the elements can be eliminated using two queries. [*Hint:* Use two queries, where each of the queries asks whether the element is in the union of two of the subsets with $n/4$ elements and where one of the subsets of $n/4$ elements is used in both queries.]

c) Show from part (b) that if $f(n)$ equals the number of queries used to solve Ulam's problem using the method from part (b) and n is divisible by 4, then $f(n) = f(3n/4) + 2$.

d) Solve the recurrence relation in part (c) for $f(n)$.

e) Is the naive way to solve Ulam's problem by asking each question twice or the divide-and-conquer method based on part (b) more efficient? The most efficient way to solve Ulam's problem has been determined by A. Pelc [Pe87].

In Exercises 29–33, assume that f is an increasing function satisfying the recurrence relation $f(n) = af(n/b) + cn^d$, where $a \geq 1$, b is an integer greater than 1, and c and d are positive real numbers. These exercises supply a proof of Theorem 2.

***29.** Show that if $a = b^d$ and n is a power of b, then $f(n) = f(1)n^d + cn^d \log_b n$.

30. Use Exercise 29 to show that if $a = b^d$, then $f(n)$ is $O(n^d \log n)$.

***31.** Show that if $a \neq b^d$ and n is a power of b, then $f(n) = C_1 n^d + C_2 n^{\log_b a}$, where $C_1 = b^d c/(b^d - a)$ and $C_2 = f(1) + b^d c/(a - b^d)$.

32. Use Exercise 31 to show that if $a < b^d$, then $f(n)$ is $O(n^d)$.

33. Use Exercise 31 to show that if $a > b^d$, then $f(n)$ is $O(n^{\log_b a})$.

34. Find $f(n)$ when $n = 4^k$, where f satisfies the recurrence relation $f(n) = 5f(n/4) + 6n$, with $f(1) = 1$.

35. Give a big-O estimate for the function f in Exercise 34 if f is an increasing function.

36. Find $f(n)$ when $n = 2^k$, where f satisfies the recurrence relation $f(n) = 8f(n/2) + n^2$ with $f(1) = 1$.

37. Give a big-O estimate for the function f in Exercise 36 if f is an increasing function.

8.4 Generating Functions

8.4.1 Introduction

Links

Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series. Generating functions can be used to solve many types of counting problems, such as the number of ways to select or distribute objects of different kinds, subject to a variety of constraints, and the number of ways to make change for a dollar using coins of different denominations. Generating functions can be used to solve recurrence relations by translating a recurrence relation for the terms of a sequence into an equation involving a generating function. This equation can then be solved to find a closed form for the generating function. From this closed form, the coefficients of the power series for the generating function can be found, solving the original recurrence relation. Generating functions can also be used to prove combinatorial identities by taking advantage of relatively simple relationships between functions that can be translated into identities involving the terms of sequences. Generating functions are a helpful tool for studying many properties of sequences besides those described in this section, such as their use for establishing asymptotic formulae for the terms of a sequence.

We begin with the definition of the generating function for a sequence.

Definition 1 The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k.$$

Remark: The generating function for $\{a_k\}$ given in Definition 1 is sometimes called the **ordinary generating function** of $\{a_k\}$ to distinguish it from other types of generating functions for this sequence.

EXAMPLE 1 The generating functions for the sequences $\{a_k\}$ with $a_k = 3, a_k = k + 1$, and $a_k = 2^k$ are $\sum_{k=0}^{\infty} 3x^k$, $\sum_{k=0}^{\infty}(k + 1)x^k$, and $\sum_{k=0}^{\infty} 2^k x^k$, respectively. ◀

Extra Examples

We can define generating functions for **finite** sequences of real numbers by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0, a_{n+2} = 0$, and so on. The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n = \sum_{k=0}^{n} a_k x^k$$

EXAMPLE 2 What is the generating function for the sequence 1, 1, 1, 1, 1, 1?

Solution: The generating function of 1, 1, 1, 1, 1, 1 is

$$1 + 1x + 1x^2 + 1x^3 + 1x^4 + 1x^5.$$

By Theorem 1 of Section 2.4 we have

$$(x^6 - 1)/(x - 1) = 1 + x + x^2 + x^3 + x^4 + x^5$$

when $x \neq 1$. Consequently, $G(x) = (x^6 - 1)/(x - 1)$ is the generating function of the sequence 1, 1, 1, 1, 1, 1. [Because the powers of x are only place holders for the terms of the sequence in a generating function, we do not need to worry that $G(1)$ is undefined.] ◀

EXAMPLE 3 Let m be a positive integer. Let $a_k = C(m, k)$, for $k = 0, 1, 2, \ldots, m$. What is the generating function for the sequence $a_0, a_1, \ldots, a_m$?

Solution: The generating function for this sequence is

$$G(x) = C(m, 0) + C(m, 1)x + C(m, 2)x^2 + \cdots + C(m, m)x^m.$$

The binomial theorem shows that $G(x) = (1 + x)^m$. ◀

8.4.2 Useful Facts About Power Series

When generating functions are used to solve counting problems, they are usually considered to be **formal power series**. As such, they are treated as algebraic objects; questions about their convergence are ignored. However, when formal power series are convergent, valid operations carry over to their use as formal power series. We will take advantage of the power series of particular functions around $x = 0$. These power series are unique and have a positive radius of convergence. Readers familiar with calculus can consult textbooks on this subject for details about power series, including the convergence of the series we use here.

We will now state some widely important facts about infinite series used when working with generating functions. These facts can be found in calculus texts.

EXAMPLE 4 The function $f(x) = 1/(1 - x)$ is the generating function of the sequence $1, 1, 1, 1, \ldots$, because

$$1/(1 - x) = 1 + x + x^2 + \cdots$$

for $|x| < 1$. ◀

EXAMPLE 5 The function $f(x) = 1/(1 - ax)$ is the generating function of the sequence $1, a, a^2, a^3, \ldots$, because

$$1/(1 - ax) = 1 + ax + a^2x^2 + \cdots$$

when $|ax| < 1$, or equivalently, for $|x| < 1/|a|$ for $a \neq 0$. ◀

We also will need some results on how to add and how to multiply two generating functions. Proofs of these results can be found in calculus texts.

THEOREM 1 Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k \quad \text{and} \quad f(x)g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k.$$

Remark: Theorem 1 is valid only for power series that converge in an interval, as all series considered in this section do. However, the theory of generating functions is not limited to such series. In the case of series that do not converge, the statements in Theorem 1 can be taken as definitions of addition and multiplication of generating functions.

We will illustrate how Theorem 1 can be used with Example 6.

EXAMPLE 6 Let $f(x) = 1/(1 - x)^2$. Use Example 4 to find the coefficients $a_0, a_1, a_2, \ldots$ in the expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

Solution: From Example 4 we see that

$$1/(1 - x) = 1 + x + x^2 + x^3 + \cdots.$$

Hence, from Theorem 1, we have

$$1/(1 - x)^2 = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} 1 \right) x^k = \sum_{k=0}^{\infty} (k + 1) x^k.$$

◀

Remark: This result also can be derived from Example 4 by differentiation. Taking derivatives is a useful technique for producing new identities from existing identities for generating functions.

To use generating functions to solve many important counting problems, we will need to apply the binomial theorem for exponents that are not positive integers. Before we state an extended version of the binomial theorem, we need to define extended binomial coefficients.

Definition 2 Let u be a real number and k a nonnegative integer. Then the *extended binomial coefficient* $\binom{u}{k}$ is defined by

$$\binom{u}{k} = \begin{cases} u(u-1)\cdots(u-k+1)/k! & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases}$$

EXAMPLE 7 Find the values of the extended binomial coefficients $\binom{-2}{3}$ and $\binom{1/2}{3}$.

Solution: Taking $u = -2$ and $k = 3$ in Definition 2 gives us

$$\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4.$$

Similarly, taking $u = 1/2$ and $k = 3$ gives us

$$\begin{aligned}\binom{1/2}{3} &= \frac{(1/2)(1/2-1)(1/2-2)}{3!} \\ &= (1/2)(-1/2)(-3/2)/6 \\ &= 1/16.\end{aligned}$$

◀

Example 8 provides a useful formula for extended binomial coefficients when the top parameter is a negative integer. It will be useful in our subsequent discussions.

EXAMPLE 8 When the top parameter is a negative integer, the extended binomial coefficient can be expressed in terms of an ordinary binomial coefficient. To see that this is the case, note that

$$\begin{aligned}\binom{-n}{r} &= \frac{(-n)(-n-1)\cdots(-n-r+1)}{r!} && \text{by definition of extended binomial coefficient} \\ &= \frac{(-1)^r n(n+1)\cdots(n+r-1)}{r!} && \text{factoring out } -1 \text{ from each term in the numerator} \\ &= \frac{(-1)^r (n+r-1)(n+r-2)\cdots n}{r!} && \text{by the commutative law for multiplication} \\ &= \frac{(-1)^r (n+r-1)!}{r!(n-1)!} && \text{multiplying both the numerator and denominator by } (n-1)! \\ &= (-1)^r \binom{n+r-1}{r} && \text{by the definition of binomial coefficients} \\ &= (-1)^r C(n+r-1, r) && \text{using alternative notation for binomial coefficients.}\end{aligned}$$

◀

We now state the extended binomial theorem.

THEOREM 2 **THE EXTENDED BINOMIAL THEOREM** Let x be a real number with $|x| < 1$ and let u be a real number. Then

$$(1 + x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k.$$

Theorem 2 can be proved using the theory of Maclaurin series. We leave its proof to the reader with a familiarity with this part of calculus.

Remark: When u is a positive integer, the extended binomial theorem reduces to the binomial theorem presented in Section 6.4, because in that case $\binom{u}{k} = 0$ if $k > u$.

Example 9 illustrates the use of Theorem 2 when the exponent is a negative integer.

EXAMPLE 9 Find the generating functions for $(1 + x)^{-n}$ and $(1 - x)^{-n}$, where n is a positive integer, using the extended binomial theorem.

Solution: By the extended binomial theorem, it follows that

$$(1 + x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k.$$

Using Example 8, which provides a simple formula for $\binom{-n}{k}$, we obtain

$$(1 + x)^{-n} = \sum_{k=0}^{\infty} (-1)^k C(n + k - 1, k)x^k.$$

Replacing x by $-x$, we find that

$$(1 - x)^{-n} = \sum_{k=0}^{\infty} C(n + k - 1, k)x^k.$$

◄

Table 1 presents a useful summary of some generating functions that arise frequently.

Remark: Note that the second and third formulae in this table can be deduced from the first formula by substituting ax and x^r for x, respectively. Similarly, the sixth and seventh formulae can be deduced from the fifth formula using the same substitutions. The tenth and eleventh can be deduced from the ninth formula by substituting $-x$ and ax for x, respectively. Also, some of the formulae in this table can be derived from other formulae using methods from calculus (such as differentiation and integration). Students are encouraged to know the core formulae in this table (that is, formulae from which the others can be derived, perhaps the first, fourth, fifth, eighth, ninth, twelfth, and thirteenth formulae) and understand how to derive the other formulae from these core formulae.

$1,2,\ a_{n=2}\ 1 \Longrightarrow f(x) = (x+1)^2 = \sum_{k=0}^{2}\binom{2}{k}x^k = 1 + 2x + x^2$

$a_0, a_1, a_2 \,|\, nat, div\ by\ 3:\ 1 + x^3 + x^6 + \cdots = \dfrac{1}{1-x^3}$

$\dfrac{1}{(1-x)^7} = (1+x+x^2+\cdots)^7 = \sum_{k=0}^{\infty}\binom{6+k}{k}x^k$

TABLE 1 Useful Generating Functions.

	$G(x)$	a_k
	$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$ $= \binom{n}{0}x^0 1 + \binom{n}{1}nx^1 + \binom{n}{2}x^2 + \cdots + \binom{n}{n-1}nx^{n-1} + \binom{n}{n}x^n$	$\binom{n}{k}_{k:\,0\to n} = \binom{n}{n} = 1$ $a_0\binom{n}{0}, a_1\binom{n}{1}, \ldots, a_n\binom{n}{k} = a_0 1, a_1 n, a_2\binom{n}{2}, \ldots, a_{n-1}n, a_n 1$
2	$(1+ax)^n = \sum_{k=0}^{n}\binom{n}{k}a^k x^k$ $= \binom{n}{0}a^0x^0 1 + \binom{n}{1}na^1x^1 + \binom{n}{2}a^2x^2 + \cdots + \binom{n}{n-1}na^{n-1}x^{n-1} + \binom{n}{n}a^n x^n$	$\binom{n}{k}a^k \underset{k:\,0\to n}{=} \binom{n}{n}a^n$ $\binom{n}{0}a^0, \binom{n}{1}na^1, \binom{n}{2}a^2, \ldots, \binom{n}{k-1}a^{k-1}, \binom{n}{k}a^k$
3	$(1+x^r)^n = \sum_{k=0}^{n}\binom{n}{k}x^{rk}$ $= 1 + nx^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n, k/r)$ if $r \mid k$; 0 otherwise
4	$\dfrac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n}x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise $a_0, \ldots, a_n = 1, \ldots, 1$
5	$\dfrac{1}{1-x} = \sum_{k=0}^{\infty}x^k = 1 + x + x^2 + \cdots$	$a_0, a_1, \ldots = 1, 1, \ldots$
6	$\dfrac{1}{1-ax} = \sum_{k=0}^{\infty}a^k x^k = 1 + ax + a^2x^2 + \cdots$	$a_k = a^k$ $a_0 a^0, a_1 a^1, a_2 a^2, \ldots$
7	$\dfrac{1}{1-x^r} = \sum_{k=0}^{\infty}x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
8	$\dfrac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$ $a_0, a_1, \ldots = 1, 2, \ldots$
9	$\dfrac{1}{(1-x)^n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k$ $= 1 + nx + \binom{n+1}{2}x^2 + \cdots = (1+x+x^2+\cdots)^n$	$\binom{n+k-1}{k}$ $a_0, a_1, \ldots = \binom{n-1}{0}, \binom{n}{1}, \ldots = 1, n, \binom{n+1}{2}, \ldots$
10	$\dfrac{1}{(1+x)^n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}(-1)^k x^k$ $= 1 - nx + \binom{n+1}{2}x^2 - \cdots$	$(-1)^k C(n+k-1, k) = (-1)^k C(n+k-1, n-1)$
11	$\dfrac{1}{(1-ax)^n} = \sum_{k=0}^{\infty}\binom{n+k-1}{k}a^k x^k$ $= 1 + nax + \binom{n+1}{2}a^2x^2 + \cdots$	$C(n+k-1, k)a^k = C(n+k-1, n-1)a^k$ $a_0, a_1, \ldots = \binom{n-1}{0}a^0, \binom{n}{1}a^1, \ldots = 1, na, \binom{n+1}{2}a^2, \ldots$
12	$e^x = \sum_{k=0}^{\infty}\dfrac{x^k}{k!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$	$1/k!$ $\quad a_0, a_1, \ldots = \dfrac{1}{0!}, \dfrac{1}{1!}, \ldots = 1, 1, \dfrac{1}{2}, \dfrac{1}{3!}, \ldots$
13	$\ln(1+x) = \sum_{k=1}^{\infty}\dfrac{(-1)^{k+1}}{k}x^k = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

$a_0, a_1, a_2, a_3 \,|\, nat, not\ div\ by\ 3:$

$x^1 + x^2 + x^4 + x^5 + \cdots = (x+x^2)(1+x^3+x^6+\cdots) = (x+x^2)\cdot\frac{1}{1-x^3}$

8.4.3 Counting Problems and Generating Functions

Generating functions can be used to solve a wide variety of counting problems. In particular, they can be used to count the number of combinations of various types. In Chapter 6 we developed techniques to count the r-combinations from a set with n elements when repetition is allowed and additional constraints may exist. Such problems are equivalent to counting the solutions to equations of the form

$$e_1 + e_2 + \cdots + e_n = C,$$

where C is a constant and each e_i is a nonnegative integer that may be subject to a specified constraint. Generating functions can also be used to solve counting problems of this type, as Examples 10–12 show.

EXAMPLE 10 Find the number of solutions of

Extra Examples

$$e_1 + e_2 + e_3 = 17,$$

where e_1, e_2, and e_3 are nonnegative integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.

Solution: The number of solutions with the indicated constraints is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^6 + x^7).$$

This follows because we obtain a term equal to x^{17} in the product by picking a term in the first sum x^{e_1}, a term in the second sum x^{e_2}, and a term in the third sum x^{e_3}, where the exponents e_1, e_2, and e_3 satisfy the equation $e_1 + e_2 + e_3 = 17$ and the given constraints.

It is not hard to see that the coefficient of x^{17} in this product is 3. Hence, there are three solutions. (Note that the calculating of this coefficient involves about as much work as enumerating all the solutions of the equation with the given constraints. However, the method that this illustrates often can be used to solve wide classes of counting problems with special formulae, as we will see. Furthermore, a computer algebra system can be used to do such computations.) ◀

EXAMPLE 11 In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

Solution: Because each child receives at least two but no more than four cookies, for each child there is a factor equal to

$$(x^2 + x^3 + x^4)$$

in the generating function for the sequence $\{c_n\}$, where c_n is the number of ways to distribute n cookies. Because there are three children, this generating function is

$$(x^2 + x^3 + x^4)^3.$$

We need the coefficient of x^8 in this product. The reason is that the x^8 terms in the expansion correspond to the ways that three terms can be selected, with one from each factor, that have exponents adding up to 8. Furthermore, the exponents of the term from the first, second, and third factors are the numbers of cookies the first, second, and third children receive, respectively. Computation shows that this coefficient equals 6. Hence, there are six ways to distribute the cookies so that each child receives at least two, but no more than four, cookies. ◀

EXAMPLE 12 Use generating functions to determine the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs r dollars in both the cases when the order in which the tokens are inserted does not matter and when the order does matter. (For example, there are two ways to pay for an item that costs \$3 when the order in which the tokens are inserted does not matter: inserting three \$1 tokens or one \$1 token and a \$2 token. When the order matters, there are three ways: inserting three \$1 tokens, inserting a \$1 token and then a \$2 token, or inserting a \$2 token and then a \$1 token.)

Solution: Consider the case when the order in which the tokens are inserted does not matter. Here, all we care about is the number of each token used to produce a total of r dollars. Because we can use any number of \$1 tokens, any number of \$2 tokens, and any number of \$5 tokens, the answer is the coefficient of x^r in the generating function

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots).$$

(The first factor in this product represents the \$1 tokens used, the second the \$2 tokens used, and the third the \$5 tokens used.) For example, the number of ways to pay for an item costing \$7 using \$1, \$2, and \$5 tokens is given by the coefficient of x^7 in this expansion, which equals 6.

When the order in which the tokens are inserted matters, the number of ways to insert exactly n tokens to produce a total of r dollars is the coefficient of x^r in

$$(x + x^2 + x^5)^n,$$

because each of the r tokens may be a \$1 token, a \$2 token, or a \$5 token. Because any number of tokens may be inserted, the number of ways to produce r dollars using \$1, \$2, or \$5 tokens, when the order in which the tokens are inserted matters, is the coefficient of x^r in

$$\begin{aligned} 1 + (x + x^2 + x^5) + (x + x^2 + x^5)^2 + \cdots &= \frac{1}{1 - (x + x^2 + x^5)} \\ &= \frac{1}{1 - x - x^2 - x^5}, \end{aligned}$$

where we have added the number of ways to insert 0 tokens, 1 token, 2 tokens, 3 tokens, and so on, and where we have used the identity $1/(1 - x) = 1 + x + x^2 + \cdots$ with x replaced with $x + x^2 + x^5$. For example, the number of ways to pay for an item costing \$7 using \$1, \$2, and \$5 tokens, when the order in which the tokens are used matters, is the coefficient of x^7 in this expansion, which equals 26. [*Hint:* To see that this coefficient equals 26 requires the addition of the coefficients of x^7 in the expansions $(x + x^2 + x^5)^k$ for $2 \le k \le 7$. This can be done by hand with considerable computation, or a computer algebra system can be used.] ◀

Example 13 shows the versatility of generating functions when used to solve problems with differing assumptions.

EXAMPLE 13 Use generating functions to find the number of k-combinations of a set with n elements. Assume that the binomial theorem has already been established.

Solution: Each of the n elements in the set contributes the term $(1 + x)$ to the generating function $f(x) = \sum_{k=0}^{n} a_k x^k$. Here $f(x)$ is the generating function for $\{a_k\}$, where a_k represents the number of k-combinations of a set with n elements. Hence,

$$f(x) = (1 + x)^n.$$

But by the binomial theorem, we have

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Hence, $C(n, k)$, the number of k-combinations of a set with n elements, is

$$\frac{n!}{k!(n-k)!}.$$

◀

Remark: We proved the binomial theorem in Section 6.4 using the formula for the number of r-combinations of a set with n elements. This example shows that the binomial theorem, which can be proved by mathematical induction, can be used to derive the formula for the number of r-combinations of a set with n elements.

EXAMPLE 14 Use generating functions to find the number of r-combinations from a set with n elements when repetition of elements is allowed.

Solution: Let $G(x)$ be the generating function for the sequence $\{a_r\}$, where a_r equals the number of r-combinations of a set with n elements with repetitions allowed. That is, $G(x) = \sum_{r=0}^{\infty} a_r x^r$. Because we can select any number of a particular member of the set with n elements when we form an r-combination with repetition allowed, each of the n elements contributes $(1 + x + x^2 + x^3 + \cdots)$ to a product expansion for $G(x)$. Each element contributes this factor because it may be selected zero times, one time, two times, three times, and so on, when an r-combination is formed (with a total of r elements selected). Because there are n elements in the set and each contributes this same factor to $G(x)$, we have

$$G(x) = (1 + x + x^2 + \cdots)^n.$$

As long as $|x| < 1$, we have $1 + x + x^2 + \cdots = 1/(1 - x)$, so

$$G(x) = 1/(1 - x)^n = (1 - x)^{-n}.$$

Applying the extended binomial theorem (Theorem 2), it follows that

$$(1 - x)^{-n} = (1 + (-x))^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r.$$

The number of r-combinations of a set with n elements with repetitions allowed, when r is a positive integer, is the coefficient a_r of x^r in this sum. Consequently, using Example 8 we find that a_r equals

$$\begin{aligned}\binom{-n}{r}(-1)^r &= (-1)^r C(n + r - 1, r) \cdot (-1)^r \\ &= C(n + r - 1, r).\end{aligned}$$

◀

Note that the result in Example 14 is the same result we stated as Theorem 2 in Section 6.5.

EXAMPLE 15 Use generating functions to find the number of ways to select r objects of n different kinds if we must select at least one object of each kind.

Solution: Because we need to select at least one object of each kind, each of the n kinds of objects contributes the factor $(x + x^2 + x^3 + \cdots)$ to the generating function $G(x)$ for the sequence $\{a_r\}$, where a_r is the number of ways to select r objects of n different kinds if we need at least one object of each kind. Hence,

$$G(x) = (x + x^2 + x^3 + \cdots)^n = x^n(1 + x + x^2 + \cdots)^n = x^n/(1 - x)^n.$$

Using the extended binomial theorem and Example 8, we have

$$\begin{aligned} G(x) &= x^n/(1-x)^n \\ &= x^n \cdot (1-x)^{-n} \\ &= x^n \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r \\ &= x^n \sum_{r=0}^{\infty} (-1)^r C(n+r-1, r)(-1)^r x^r \\ &= \sum_{r=0}^{\infty} C(n+r-1, r) x^{n+r} \\ &= \sum_{t=n}^{\infty} C(t-1, t-n) x^t \\ &= \sum_{r=n}^{\infty} C(r-1, r-n) x^r. \end{aligned}$$

We have shifted the summation in the next-to-last equality by setting $t = n + r$ so that $t = n$ when $r = 0$ and $n + r - 1 = t - 1$, and then we replaced t by r as the index of summation in the last equality to return to our original notation. Hence, there are $C(r - 1, r - n)$ ways to select r objects of n different kinds if we must select at least one object of each kind. ◀

8.4.4 Using Generating Functions to Solve Recurrence Relations

We can find the solution to a recurrence relation and its initial conditions by finding an explicit formula for the associated generating function. This is illustrated in Examples 16 and 17.

EXAMPLE 16 Solve the recurrence relation $a_k = 3a_{k-1}$ for $k = 1, 2, 3, \ldots$ and initial condition $a_0 = 2$.

Extra Examples

Solution: Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$. First note that

$$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k.$$

Using the recurrence relation, we see that

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1}x^k \\ &= a_0 + \sum_{k=1}^{\infty}(a_k - 3a_{k-1})x^k \\ &= 2, \end{aligned}$$

because $a_0 = 2$ and $a_k = 3a_{k-1}$. Thus,

$$G(x) - 3xG(x) = (1 - 3x)G(x) = 2.$$

Solving for $G(x)$ shows that $G(x) = 2/(1 - 3x)$. Using the identity $1/(1 - ax) = \sum_{k=0}^{\infty} a^k x^k$, from Table 1, we have

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k.$$

Consequently, $a_k = 2 \cdot 3^k$. ◀

EXAMPLE 17 Suppose that a valid codeword is an n-digit number in decimal notation containing an even number of 0s. Let a_n denote the number of valid codewords of length n. In Example 4 of Section 8.1 we showed that the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1}$$

and the initial condition $a_1 = 9$. Use generating functions to find an explicit formula for a_n.

Solution: To make our work with generating functions simpler, we extend this sequence by setting $a_0 = 1$; when we assign this value to a_0 and use the recurrence relation, we have $a_1 = 8a_0 + 10^0 = 8 + 1 = 9$, which is consistent with our original initial condition. (It also makes sense because there is one code word of length 0—the empty string.)

We multiply both sides of the recurrence relation by x^n to obtain

$$a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n.$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $a_0, a_1, a_2, \ldots$. We sum both sides of the last equation starting with $n = 1$, to find that

$$\begin{aligned} G(x) - 1 = \sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty}(8a_{n-1}x^n + 10^{n-1}x^n) \\ &= 8\sum_{n=1}^{\infty} a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n \\ &= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\ &= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n \\ &= 8xG(x) + x/(1 - 10x), \end{aligned}$$

where we have used Example 5 to evaluate the second summation. Therefore, we have

$$G(x) - 1 = 8xG(x) + x/(1 - 10x).$$

Solving for $G(x)$ shows that

$$G(x) = \frac{1 - 9x}{(1 - 8x)(1 - 10x)}.$$

Expanding the right-hand side of this equation into partial fractions (as is done in the integration of rational functions studied in calculus) gives

$$G(x) = \frac{1}{2}\left(\frac{1}{1 - 8x} + \frac{1}{1 - 10x}\right).$$

Using Example 5 twice (once with $a = 8$ and once with $a = 10$) gives

$$\begin{aligned} G(x) &= \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n. \end{aligned}$$

Consequently, we have shown that

$$a_n = \frac{1}{2}(8^n + 10^n).$$

◀

8.4.5 Proving Identities via Generating Functions

In Chapter 6 we saw how combinatorial identities could be established using combinatorial proofs. Here we will show that such identities, as well as identities for extended binomial coefficients, can be proved using generating functions. Sometimes the generating function approach is simpler than other approaches, especially when it is simpler to work with the closed form of a generating function than with the terms of the sequence themselves. We illustrate how generating functions can be used to prove identities with Example 18.

EXAMPLE 18 Use generating functions to show that

$$\sum_{k=0}^{n} C(n, k)^2 = C(2n, n)$$

whenever n is a positive integer.

Solution: First note that by the binomial theorem $C(2n, n)$ is the coefficient of x^n in $(1 + x)^{2n}$. However, we also have

$$\begin{aligned} (1 + x)^{2n} &= [(1 + x)^n]^2 \\ &= [C(n, 0) + C(n, 1)x + C(n, 2)x^2 + \cdots + C(n, n)x^n]^2. \end{aligned}$$

The coefficient of x^n in this expression is

$$C(n, 0)C(n, n) + C(n, 1)C(n, n-1) + C(n, 2)C(n, n-2) + \cdots + C(n, n)C(n, 0).$$

This equals $\sum_{k=0}^{n} C(n, k)^2$, because $C(n, n-k) = C(n, k)$. Because both $C(2n, n)$ and $\sum_{k=0}^{n} C(n, k)^2$ represent the coefficient of x^n in $(1 + x)^{2n}$, they must be equal. ◀

Exercises 44 and 45 ask that Pascal's identity and Vandermonde's identity be proved using generating functions.

Exercises

1. Find the generating function for the finite sequence 2, 2, 2, 2, 2, 2.

2. Find the generating function for the finite sequence 1, 4, 16, 64, 256.

In Exercises 3–8, by a **closed form** we mean an algebraic expression not involving a summation over a range of values or the use of ellipses.

3. Find a closed form for the generating function for each of these sequences. (For each sequence, use the most obvious choice of a sequence that follows the pattern of the initial terms listed.)

a) $0, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0, 0, \ldots$
b) $0, 0, 0, 1, 1, 1, 1, 1, 1, \ldots$
c) $0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots$
d) $2, 4, 8, 16, 32, 64, 128, 256, \ldots$
e) $\binom{7}{0}, \binom{7}{1}, \binom{7}{2}, \ldots, \binom{7}{7}, 0, 0, 0, 0, 0, \ldots$
f) $2, -2, 2, -2, 2, -2, 2, -2, \ldots$
g) $1, 1, 0, 1, 1, 1, 1, 1, 1, 1, \ldots$
h) $0, 0, 0, 1, 2, 3, 4, \ldots$

4. Find a closed form for the generating function for each of these sequences. (Assume a general form for the terms of the sequence, using the most obvious choice of such a sequence.)

a) $-1, -1, -1, -1, -1, -1, -1, 0, 0, 0, 0, 0, 0, \ldots$
b) $1, 3, 9, 27, 81, 243, 729, \ldots$
c) $0, 0, 3, -3, 3, -3, 3, -3, \ldots$
d) $1, 2, 1, 1, 1, 1, 1, 1, 1, \ldots$
e) $\binom{7}{0}, 2\binom{7}{1}, 2^2\binom{7}{2}, \ldots, 2^7\binom{7}{7}, 0, 0, 0, 0, \ldots$
f) $-3, 3, -3, 3, -3, 3, \ldots$
g) $0, 1, -2, 4, -8, 16, -32, 64, \ldots$
h) $1, 0, 1, 0, 1, 0, 1, 0, \ldots$

5. Find a closed form for the generating function for the sequence $\{a_n\}$, where

a) $a_n = 5$ for all $n = 0, 1, 2, \ldots$.
b) $a_n = 3^n$ for all $n = 0, 1, 2, \ldots$.
c) $a_n = 2$ for $n = 3, 4, 5, \ldots$ and $a_0 = a_1 = a_2 = 0$.
d) $a_n = 2n + 3$ for all $n = 0, 1, 2, \ldots$.
e) $a_n = \binom{8}{n}$ for all $n = 0, 1, 2, \ldots$.
f) $a_n = \binom{n+4}{n}$ for all $n = 0, 1, 2, \ldots$.

6. Find a closed form for the generating function for the sequence $\{a_n\}$, where

a) $a_n = -1$ for all $n = 0, 1, 2, \ldots$.
b) $a_n = 2^n$ for $n = 1, 2, 3, 4, \ldots$ and $a_0 = 0$.
c) $a_n = n - 1$ for $n = 0, 1, 2, \ldots$.
d) $a_n = 1/(n + 1)!$ for $n = 0, 1, 2, \ldots$.
e) $a_n = \binom{n}{2}$ for $n = 0, 1, 2, \ldots$.
f) $a_n = \binom{10}{n+1}$ for $n = 0, 1, 2, \ldots$.

7. For each of these generating functions, provide a closed formula for the sequence it determines.

a) $(3x - 4)^3$
b) $(x^3 + 1)^3$
c) $1/(1 - 5x)$
d) $x^3/(1 + 3x)$
e) $x^2 + 3x + 7 + (1/(1 - x^2))$
f) $(x^4/(1 - x^4)) - x^3 - x^2 - x - 1$
g) $x^2/(1 - x)^2$
h) $2e^{2x}$

8. For each of these generating functions, provide a closed formula for the sequence it determines.

a) $(x^2 + 1)^3$
b) $(3x - 1)^3$
c) $1/(1 - 2x^2)$
d) $x^2/(1 - x)^3$
e) $x - 1 + (1/(1 - 3x))$
f) $(1 + x^3)/(1 + x)^3$
***g)** $x/(1 + x + x^2)$
h) $e^{3x^2} - 1$

9. Find the coefficient of x^{10} in the power series of each of these functions.

a) $(1 + x^5 + x^{10} + x^{15} + \cdots)^3$
b) $(x^3 + x^4 + x^5 + x^6 + x^7 + \cdots)^3$
c) $(x^4 + x^5 + x^6)(x^3 + x^4 + x^5 + x^6 + x^7)(1 + x + x^2 + x^3 + x^4 + \cdots)$
d) $(x^2 + x^4 + x^6 + x^8 + \cdots)(x^3 + x^6 + x^9 + \cdots)(x^4 + x^8 + x^{12} + \cdots)$
e) $(1 + x^2 + x^4 + x^6 + x^8 + \cdots)(1 + x^4 + x^8 + x^{12} + \cdots)(1 + x^6 + x^{12} + x^{18} + \cdots)$

10. Find the coefficient of x^9 in the power series of each of these functions.

a) $(1 + x^3 + x^6 + x^9 + \cdots)^3$
b) $(x^2 + x^3 + x^4 + x^5 + x^6 + \cdots)^3$
c) $(x^3 + x^5 + x^6)(x^3 + x^4)(x + x^2 + x^3 + x^4 + \cdots)$
d) $(x + x^4 + x^7 + x^{10} + \cdots)(x^2 + x^4 + x^6 + x^8 + \cdots)$
e) $(1 + x + x^2)^3$

11. Find the coefficient of x^{10} in the power series of each of these functions.

a) $1/(1-2x)$
b) $1/(1+x)^2$
c) $1/(1-x)^3$
d) $1/(1+2x)^4$
e) $x^4/(1-3x)^3$

12. Find the coefficient of x^{12} in the power series of each of these functions.

a) $1/(1+3x)$
b) $1/(1-2x)^2$
c) $1/(1+x)^8$
d) $1/(1-4x)^3$
e) $x^3/(1+4x)^2$

13. Use generating functions to determine the number of different ways 10 identical balloons can be given to four children if each child receives at least two balloons.

14. Use generating functions to determine the number of different ways 12 identical action figures can be given to five children so that each child receives at most three action figures.

15. Use generating functions to determine the number of different ways 15 identical stuffed animals can be given to six children so that each child receives at least one but no more than three stuffed animals.

16. Use generating functions to find the number of ways to choose a dozen bagels from three varieties—egg, salty, and plain—if at least two bagels of each kind but no more than three salty bagels are chosen.

17. In how many ways can 25 identical donuts be distributed to four police officers so that each officer gets at least three but no more than seven donuts?

18. Use generating functions to find the number of ways to select 14 balls from a jar containing 100 red balls, 100 blue balls, and 100 green balls so that no fewer than 3 and no more than 10 blue balls are selected. Assume that the order in which the balls are drawn does not matter.

19. What is the generating function for the sequence $\{c_k\}$, where c_k is the number of ways to make change for k dollars using \$1 bills, \$2 bills, \$5 bills, and \$10 bills?

20. What is the generating function for the sequence $\{c_k\}$, where c_k represents the number of ways to make change for k pesos using bills worth 10 pesos, 20 pesos, 50 pesos, and 100 pesos?

21. Give a combinatorial interpretation of the coefficient of x^4 in the expansion $(1+x+x^2+x^3+\cdots)^3$. Use this interpretation to find this number.

22. Give a combinatorial interpretation of the coefficient of x^6 in the expansion $(1+x+x^2+x^3+\cdots)^n$. Use this interpretation to find this number.

23. a) What is the generating function for $\{a_k\}$, where a_k is the number of solutions of $x_1+x_2+x_3=k$ when $x_1, x_2,$ and x_3 are integers with $x_1 \geq 2, 0 \leq x_2 \leq 3,$ and $2 \leq x_3 \leq 5$?
b) Use your answer to part (a) to find a_6.

24. a) What is the generating function for $\{a_k\}$, where a_k is the number of solutions of $x_1+x_2+x_3+x_4=k$ when x_1, x_2, x_3, and x_4 are integers with $x_1 \geq 3$, $1 \leq x_2 \leq 5$, $0 \leq x_3 \leq 4$, and $x_4 \geq 1$?
b) Use your answer to part (a) to find a_7.

25. Explain how generating functions can be used to find the number of ways in which postage of r cents can be pasted on an envelope using 3-cent, 4-cent, and 20-cent stamps.

a) Assume that the order the stamps are pasted on does not matter.
b) Assume that the order in which the stamps are pasted on matters.
c) Use your answer to part (a) to determine the number of ways 46 cents of postage can be pasted on an envelope using 3-cent, 4-cent, and 20-cent stamps when the order the stamps are pasted on does not matter. (Use of a computer algebra program is advised.)
d) Use your answer to part (b) to determine the number of ways 46 cents of postage can be pasted in a row on an envelope using 3-cent, 4-cent, and 20-cent stamps when the order in which the stamps are pasted on matters. (Use of a computer algebra program is advised.)

26. Explain how generating functions can be used to find the number of ways in which postage of r cents can be pasted on an envelope using 2-cent, 7-cent, 13-cent, and 32-cent stamps.

a) Assume that the order the stamps are pasted on does not matter.
b) Assume that the order the stamps are pasted on matters.
c) Use your answer to part (a) to determine the number of ways 49 cents of postage can be pasted on an envelope using 2-cent, 7-cent, 13-cent, and 32-cent stamps when the order the stamps are pasted on does not matter. (Use of a computer algebra program is advised.)
d) Use your answer to part (b) to determine the number of ways 49 cents of postage can be pasted on an envelope using 2-cent, 7-cent, 13-cent, and 32-cent stamps when the order in which the stamps are pasted on matters. (Use of a computer algebra program is advised.)

27. Customers at a quirky tropical fruit stand can buy at most four mangos, at most two passion fruit, any even number of papayas, three or more coconuts, and carambolas in groups of five.

a) Explain how generating functions can be used to find the number of ways a customer can buy n pieces of these fruits, following the restrictions listed.
b) Use your answer in part (a) to determine the number of ways you can buy a dozen pieces of these fruits.

28. a) Show that $1/(1-x-x^2-x^3-x^4-x^5-x^6)$ is the generating function for the number of ways that the sum n can be obtained when a die is rolled repeatedly and the order of the rolls matters.
b) Use part (a) to find the number of ways to roll a total of 8 when a die is rolled repeatedly, and the order of the rolls matters. (Use of a computer algebra package is advised.)

29. Use generating functions (and a computer algebra package, if available) to find the number of ways to make change for \$1 using

a) dimes and quarters.
b) nickels, dimes, and quarters.
c) pennies, dimes, and quarters.
d) pennies, nickels, dimes, and quarters.

30. Use generating functions (and a computer algebra package, if available) to find the number of ways to make change for \$1 using pennies, nickels, dimes, and quarters with

a) no more than 10 pennies.
b) no more than 10 pennies and no more than 10 nickels.
***c)** no more than 10 coins.

31. Use generating functions to find the number of ways to make change for \$100 using

a) \$10, \$20, and \$50 bills.
b) \$5, \$10, \$20, and \$50 bills.
c) \$5, \$10, \$20, and \$50 bills if at least one bill of each denomination is used.
d) \$5, \$10, and \$20 bills if at least one and no more than four of each denomination is used.

32. If $G(x)$ is the generating function for the sequence $\{a_k\}$, what is the generating function for each of these sequences?

a) $2a_0, 2a_1, 2a_2, 2a_3, \ldots$
b) $0, a_0, a_1, a_2, a_3, \ldots$ (assuming that terms follow the pattern of all but the first term)
c) $0, 0, 0, 0, a_2, a_3, \ldots$ (assuming that terms follow the pattern of all but the first four terms)
d) $a_2, a_3, a_4, \ldots$
e) $a_1, 2a_2, 3a_3, 4a_4, \ldots$ [*Hint: Calculus required* here.]
f) $a_0^2,\ 2a_0a_1,\ a_1^2 + 2a_0a_2,\ 2a_0a_3 + 2a_1a_2,\ 2a_0a_4 + 2a_1a_3 + a_2^2, \ldots$

33. If $G(x)$ is the generating function for the sequence $\{a_k\}$, what is the generating function for each of these sequences?

a) $0, 0, 0, a_3, a_4, a_5, \ldots$ (assuming that terms follow the pattern of all but the first three terms)
b) $a_0, 0, a_1, 0, a_2, 0, \ldots$
c) $0, 0, 0, 0, a_0, a_1, a_2, \ldots$ (assuming that terms follow the pattern of all but the first four terms)
d) $a_0, 2a_1, 4a_2, 8a_3, 16a_4, \ldots$
e) $0, a_0, a_1/2, a_2/3, a_3/4, \ldots$ [*Hint: Calculus required* here.]
f) $a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \ldots$

34. Use generating functions to solve the recurrence relation $a_k = 7a_{k-1}$ with the initial condition $a_0 = 5$.

35. Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$.

36. Use generating functions to solve the recurrence relation $a_k = 3a_{k-1} + 4^{k-1}$ with the initial condition $a_0 = 1$.

37. Use generating functions to solve the recurrence relation $a_k = 5a_{k-1} - 6a_{k-2}$ with initial conditions $a_0 = 6$ and $a_1 = 30$.

38. Use generating functions to solve the recurrence relation $a_k = a_{k-1} + 2a_{k-2} + 2^k$ with initial conditions $a_0 = 4$ and $a_1 = 12$.

39. Use generating functions to solve the recurrence relation $a_k = 4a_{k-1} - 4a_{k-2} + k^2$ with initial conditions $a_0 = 2$ and $a_1 = 5$.

40. Use generating functions to solve the recurrence relation $a_k = 2a_{k-1} + 3a_{k-2} + 4^k + 6$ with initial conditions $a_0 = 20$, $a_1 = 60$.

41. Use generating functions to find an explicit formula for the Fibonacci numbers.

***42.** **a)** Show that if n is a positive integer, then

$$\binom{-1/2}{n} = \binom{2n}{n}\Big/(-4)^n.$$

b) Use the extended binomial theorem and part (a) to show that the coefficient of x^n in the expansion of $(1-4x)^{-1/2}$ is $\binom{2n}{n}$ for all nonnegative integers n.

***43.** (*Calculus required*) Let $\{C_n\}$ be the sequence of Catalan numbers, that is, the solution to the recurrence relation $C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$ with $C_0 = C_1 = 1$ (see Example 5 in Section 8.1).

a) Show that if $G(x)$ is the generating function for the sequence of Catalan numbers, then $xG(x)^2 - G(x) + 1 = 0$. Conclude (using the initial conditions) that $G(x) = (1 - \sqrt{1-4x})/(2x)$.

b) Use Exercise 42 to conclude that

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n,$$

so that

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

c) Show that $C_n \geq 2^{n-1}$ for all positive integers n.

44. Use generating functions to prove Pascal's identity: $C(n, r) = C(n-1, r) + C(n-1, r-1)$ when n and r are positive integers with $r < n$. [*Hint:* Use the identity $(1+x)^n = (1+x)^{n-1} + x(1+x)^{n-1}$.]

45. Use generating functions to prove Vandermonde's identity: $C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k)$, whenever m, n, and r are nonnegative integers with r not exceeding either m or n. [*Hint:* Look at the coefficient of x^r in both sides of $(1+x)^{m+n} = (1+x)^m(1+x)^n$.]

46. This exercise shows how to use generating functions to derive a formula for the sum of the first n squares.

a) Show that $(x^2 + x)/(1-x)^4$ is the generating function for the sequence $\{a_n\}$, where $a_n = 1^2 + 2^2 + \cdots + n^2$.

b) Use part (a) to find an explicit formula for the sum $1^2 + 2^2 + \cdots + n^2$.

The **exponential generating function** for the sequence $\{a_n\}$ is the series

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

For example, the exponential generating function for the sequence $1, 1, 1, \ldots$ is the function $\sum_{n=0}^{\infty} x^n/n! = e^x$. (You will find this particular series useful in these exercises.) Note that e^x is the (ordinary) generating function for the sequence $1, 1, 1/2!, 1/3!, 1/4!, \ldots$.

47. Find a closed form for the exponential generating function for the sequence $\{a_n\}$, where

a) $a_n = 2$. **b)** $a_n = (-1)^n$.
c) $a_n = 3^n$. **d)** $a_n = n + 1$.
e) $a_n = 1/(n + 1)$.

48. Find a closed form for the exponential generating function for the sequence $\{a_n\}$, where

a) $a_n = (-2)^n$. **b)** $a_n = -1$.
c) $a_n = n$. **d)** $a_n = n(n - 1)$.
e) $a_n = 1/((n + 1)(n + 2))$.

49. Find the sequence with each of these functions as its exponential generating function.

a) $f(x) = e^{-x}$ **b)** $f(x) = 3x^{2x}$
c) $f(x) = e^{3x} - 3e^{2x}$ **d)** $f(x) = (1 - x) + e^{-2x}$
e) $f(x) = e^{-2x} - (1/(1 - x))$
f) $f(x) = e^{-3x} - (1 + x) + (1/(1 - 2x))$
g) $f(x) = e^{x^2}$

50. Find the sequence with each of these functions as its exponential generating function.

a) $f(x) = e^{3x}$ **b)** $f(x) = 2e^{-3x+1}$
c) $f(x) = e^{4x} + e^{-4x}$ **d)** $f(x) = (1 + 2x) + e^{3x}$
e) $f(x) = e^x - (1/(1 + x))$
f) $f(x) = xe^x$ **g)** $f(x) = e^{x^3}$

51. A coding system encodes messages using strings of octal (base 8) digits. A codeword is considered valid if and only if it contains an even number of 7s.

a) Find a linear nonhomogeneous recurrence relation for the number of valid codewords of length n. What are the initial conditions?

b) Solve this recurrence relation using Theorem 6 in Section 8.2.

c) Solve this recurrence relation using generating functions.

***52.** A coding system encodes messages using strings of base 4 digits (that is, digits from the set $\{0, 1, 2, 3\}$). A codeword is valid if and only if it contains an even number of 0s and an even number of 1s. Let a_n equal the number of valid codewords of length n. Furthermore, let b_n, c_n, and d_n equal the number of strings of base 4 digits of length n with an even number of 0s and an odd number of 1s, with an odd number of 0s and an even number of 1s, and with an odd number of 0s and an odd number of 1s, respectively.

a) Show that $d_n = 4^n - a_n - b_n - c_n$. Use this to show that $a_{n+1} = 2a_n + b_n + c_n$, $b_{n+1} = b_n - c_n + 4^n$, and $c_{n+1} = c_n - b_n + 4^n$.

b) What are a_1, b_1, c_1, and d_1?

c) Use parts (a) and (b) to find a_3, b_3, c_3, and d_3.

d) Use the recurrence relations in part (a), together with the initial conditions in part (b), to set up three equations relating the generating functions $A(x)$, $B(x)$, and $C(x)$ for the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$, respectively.

e) Solve the system of equations from part (d) to get explicit formulae for $A(x)$, $B(x)$, and $C(x)$ and use these to get explicit formulae for a_n, b_n, c_n, and d_n.

Generating functions are useful in studying the number of different types of partitions of an integer n. A **partition** of a positive integer is a way to write this integer as the sum of positive integers where repetition is allowed and the order of the integers in the sum does not matter. For example, the partitions of 5 (with no restrictions) are $1 + 1 + 1 + 1 + 1$, $1 + 1 + 1 + 2$, $1 + 1 + 3$, $1 + 2 + 2$, $1 + 4$, $2 + 3$, and 5. Exercises 53–58 illustrate some of these uses.

53. Show that the coefficient $p(n)$ of x^n in the formal power series expansion of $1/((1-x)(1-x^2)(1-x^3)\cdots)$ equals the number of partitions of n.

54. Show that the coefficient $p_o(n)$ of x^n in the formal power series expansion of $1/((1-x)(1-x^3)(1-x^5)\cdots)$ equals the number of partitions of n into odd integers, that is, the number of ways to write n as the sum of odd positive integers, where the order does not matter and repetitions are allowed.

55. Show that the coefficient $p_d(n)$ of x^n in the formal power series expansion of $(1 + x)(1 + x^2)(1 + x^3)\cdots$ equals the number of partitions of n into distinct parts, that is, the number of ways to write n as the sum of positive integers, where the order does not matter but no repetitions are allowed.

56. Find $p_o(n)$, the number of partitions of n into odd parts with repetitions allowed, and $p_d(n)$, the number of partitions of n into distinct parts, for $1 \le n \le 8$, by writing each partition of each type for each integer.

57. Show that if n is a positive integer, then the number of partitions of n into distinct parts equals the number of partitions of n into odd parts with repetitions allowed; that is, $p_o(n) = p_d(n)$. [*Hint:* Show that the generating functions for $p_o(n)$ and $p_d(n)$ are equal.]

****58.** (*Requires calculus*) Use the generating function of $p(n)$ to show that $p(n) \le e^{C\sqrt{n}}$ for some constant C. [Hardy and Ramanujan showed that $p(n) \sim e^{\pi\sqrt{2/3}\sqrt{n}}/(4\sqrt{3}n)$, which means that the ratio of $p(n)$ and the right-hand side approaches 1 as n approaches infinity.]

Suppose that X is a random variable on a sample space S such that $X(s)$ is a nonnegative integer for all $s \in S$. The **probability generating function** for X is

Links

$$G_X(x) = \sum_{k=0}^{\infty} p(X(s) = k)x^k.$$

59. (*Requires calculus*) Show that if G_X is the probability generating function for a random variable X such that $X(s)$ is a nonnegative integer for all $s \in S$, then

a) $G_X(1) = 1$. **b)** $E(X) = G_X'(1)$.
c) $V(X) = G_X''(1) + G_X'(1) - G_X'(1)^2$.

60. Let X be the random variable whose value is n if the first success occurs on the nth trial when independent Bernoulli trials are performed, each with probability of success p.

a) Find a closed formula for the probability generating function G_X.

b) Find the expected value and the variance of X using Exercise 59 and the closed form for the probability generating function found in part (a).

61. Let m be a positive integer. Let X_m be the random variable whose value is n if the mth success occurs on the $(n+m)$th trial when independent Bernoulli trials are performed, each with probability of success p.

a) Using Exercise 32 in the Supplementary Exercises of Chapter 7, show that the probability generating function G_{X_m} is given by $G_{X_m}(x) = p^m/(1-qx)^m$, where $q = 1-p$.

b) Find the expected value and the variance of X_m using Exercise 59 and the closed form for the probability generating function in part (a).

62. Show that if X and Y are independent random variables on a sample space S such that $X(s)$ and $Y(s)$ are nonnegative integers for all $s \in S$, then $G_{X+Y}(x) = G_X(x)G_Y(x)$.

8.5 Inclusion–Exclusion

8.5.1 Introduction

A discrete mathematics class contains 30 women and 50 sophomores. How many students in the class are either women or sophomores? This question cannot be answered unless more information is provided. Adding the number of women in the class and the number of sophomores probably does not give the correct answer, because women sophomores are counted twice. This observation shows that the number of students in the class that are either sophomores or women is the sum of the number of women and the number of sophomores in the class minus the number of women sophomores. A technique for solving such counting problems was introduced in Section 6.1. In this section we will generalize the ideas introduced in that section to solve problems that require us to count the number of elements in the union of more than two sets.

8.5.2 The Principle of Inclusion–Exclusion

How many elements are in the union of two finite sets? In Section 2.2 we showed that the number of elements in the union of the two sets A and B is the sum of the numbers of elements in the sets minus the number of elements in their intersection. That is,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

As we showed in Section 6.1, the formula for the number of elements in the union of two sets is useful in counting problems. Examples 1–3 provide additional illustrations of the usefulness of this formula.

EXAMPLE 1 In a discrete mathematics class every student is a major in computer science or mathematics, or both. The number of students having computer science as a major (possibly along with mathematics) is 25; the number of students having mathematics as a major (possibly along with computer science) is 13; and the number of students majoring in both computer science and mathematics is 8. How many students are in this class?

Solution: Let A be the set of students in the class majoring in computer science and B be the set of students in the class majoring in mathematics. Then $A \cap B$ is the set of students in the class who are joint mathematics and computer science majors. Because every student in the class is majoring in either computer science or mathematics (or both), it follows that the number of students in the class is $|A \cup B|$. Therefore,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 25 + 13 - 8 = 30. \end{aligned}$$

Therefore, there are 30 students in the class. This computation is illustrated in Figure 1. ◀

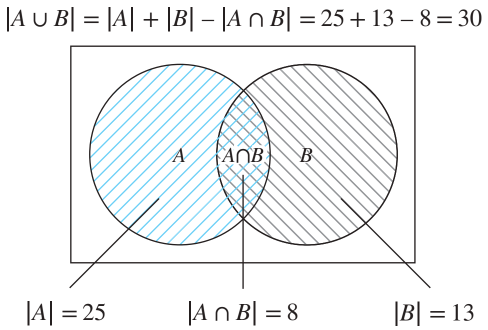


FIGURE 1 The set of students in a discrete mathematics class.

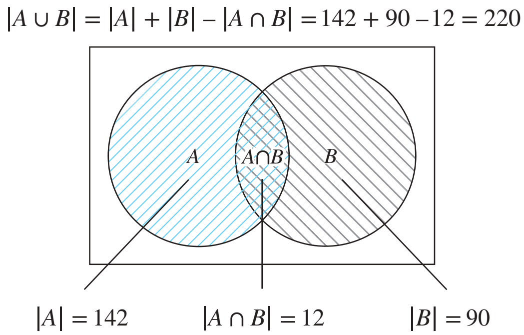


FIGURE 2 The set of positive integers not exceeding 1000 divisible by either 7 or 11.

EXAMPLE 2 How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution: Let A be the set of positive integers not exceeding 1000 that are divisible by 7, and let B be the set of positive integers not exceeding 1000 that are divisible by 11. Then $A \cup B$ is the set of integers not exceeding 1000 that are divisible by either 7 or 11, and $A \cap B$ is the set of integers not exceeding 1000 that are divisible by both 7 and 11. From Example 2 of Section 4.1, we know that among the positive integers not exceeding 1000 there are $\lfloor 1000/7 \rfloor$ integers divisible by 7 and $\lfloor 1000/11 \rfloor$ divisible by 11. Because 7 and 11 are relatively prime, the integers divisible by both 7 and 11 are those divisible by $7 \cdot 11$. Consequently, there are $\lfloor 1000/(11 \cdot 7) \rfloor$ positive integers not exceeding 1000 that are divisible by both 7 and 11. It follows that there are

$$\begin{aligned}
|A \cup B| &= |A| + |B| - |A \cap B| \\
&= \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{7 \cdot 11} \right\rfloor \\
&= 142 + 90 - 12 = 220
\end{aligned}$$

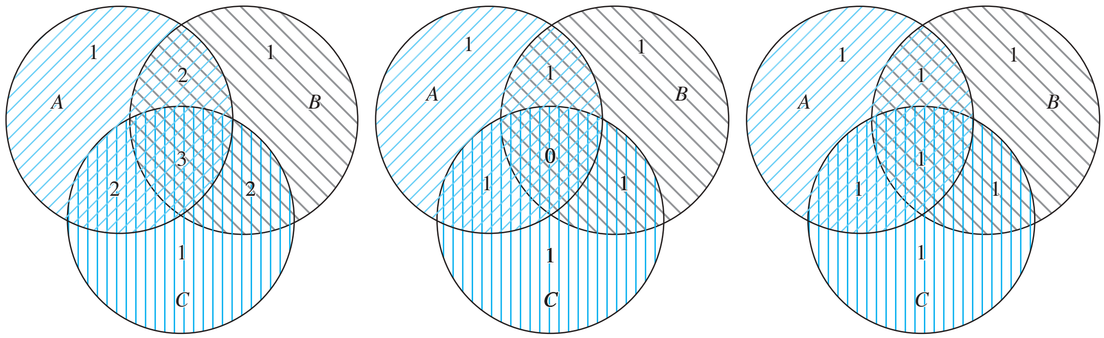
positive integers not exceeding 1000 that are divisible by either 7 or 11. This computation is illustrated in Figure 2. ◂

Example 3 shows how to find the number of elements in a finite universal set that are outside the union of two sets.

EXAMPLE 3 Suppose that there are 1807 freshmen at your school. Of these, 453 are taking a course in computer science, 567 are taking a course in mathematics, and 299 are taking courses in both computer science and mathematics. How many are not taking a course either in computer science or in mathematics?

Solution: To find the number of freshmen who are not taking a course in either mathematics or computer science, subtract the number that are taking a course in either of these subjects from the total number of freshmen. Let A be the set of all freshmen taking a course in computer science, and let B be the set of all freshmen taking a course in mathematics. It follows that $|A| = 453$, $|B| = 567$, and $|A \cap B| = 299$. The number of freshmen taking a course in either computer science or mathematics is

$$|A \cup B| = |A| + |B| - |A \cap B| = 453 + 567 - 299 = 721.$$



(a) Count of elements by $|A| + |B| + |C|$

(b) Count of elements by $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$

(c) Count of elements by $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

FIGURE 3 Finding a formula for the number of elements in the union of three sets.

Consequently, there are $1807 - 721 = 1086$ freshmen who are not taking a course in computer science or mathematics. ◀

We will now begin our development of a formula for the number of elements in the union of a finite number of sets. The formula we will develop is called the **principle of inclusion–exclusion**. For concreteness, before we consider unions of n sets, where n is any positive integer, we will derive a formula for the number of elements in the union of three sets A, B, and C. To construct this formula, we note that $|A| + |B| + |C|$ counts each element that is in exactly one of the three sets once, elements that are in exactly two of the sets twice, and elements in all three sets three times. This is illustrated in the first panel in Figure 3.

To remove the overcount of elements in more than one of the sets, we subtract the number of elements in the intersections of all pairs of the three sets. We obtain

$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|.$$

This expression still counts elements that occur in exactly one of the sets once. An element that occurs in exactly two of the sets is also counted exactly once, because this element will occur in one of the three intersections of sets taken two at a time. However, those elements that occur in all three sets will be counted zero times by this expression, because they occur in all three intersections of sets taken two at a time. This is illustrated in the second panel in Figure 3.

To remedy this undercount, we add the number of elements in the intersection of all three sets. This final expression counts each element once, whether it is in one, two, or three of the sets. Thus,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

This formula is illustrated in the third panel of Figure 3.

Example 4 illustrates how this formula can be used.

EXAMPLE 4 A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both

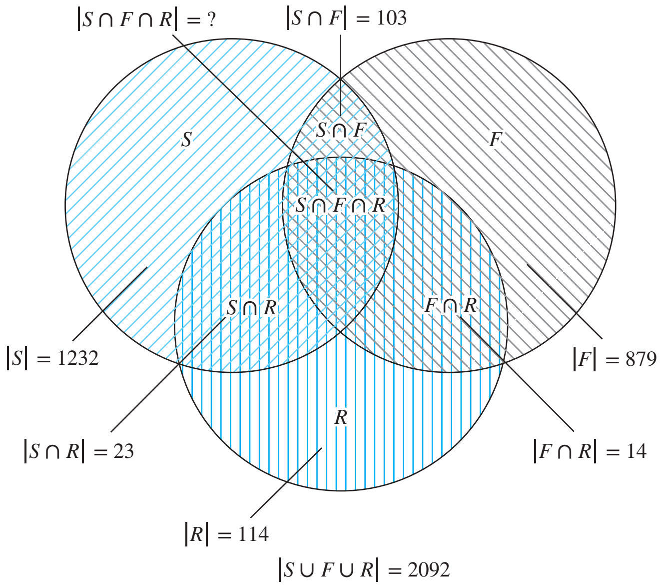


FIGURE 4 **The set of students who have taken courses in Spanish, French, and Russian.**

French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Solution: Let S be the set of students who have taken a course in Spanish, F the set of students who have taken a course in French, and R the set of students who have taken a course in Russian. Then

$$|S| = 1232, \quad |F| = 879, \quad |R| = 114,$$
$$|S \cap F| = 103, \ |S \cap R| = 23, \ |F \cap R| = 14,$$

and

$$|S \cup F \cup R| = 2092.$$

When we insert these quantities into the equation

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

we obtain

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|.$$

We now solve for $|S \cap F \cap R|$. We find that $|S \cap F \cap R| = 7$. Therefore, there are seven students who have taken courses in Spanish, French, and Russian. This is illustrated in Figure 4. ◀

We will now state and prove the **inclusion–exclusion principle** for n sets, where n is a positive integer. This priniciple tells us that we can count the elements in a union of n sets by adding the number of elements in the sets, then subtracting the sum of the number of elements in all intersections of two of these sets, then adding the number of elements in all intersections of three of these sets, and so on, until we reach the number of elements in the intersection of all the sets. It is added when there is an odd number of sets and added when there is an even number of sets.

THEOREM 1

THE PRINCIPLE OF INCLUSION–EXCLUSION Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

Proof: We will prove the formula by showing that an element in the union is counted exactly once by the right-hand side of the equation. Suppose that a is a member of exactly r of the sets $A_1, A_2, \ldots, A_n$ where $1 \le r \le n$. This element is counted $C(r, 1)$ times by $\Sigma|A_i|$. It is counted $C(r, 2)$ times by $\Sigma|A_i \cap A_j|$. In general, it is counted $C(r, m)$ times by the summation involving m of the sets A_i. Thus, this element is counted exactly

$$C(r, 1) - C(r, 2) + C(r, 3) - \cdots + (-1)^{r+1} C(r, r)$$

times by the expression on the right-hand side of this equation. Our goal is to evaluate this quantity. By Corollary 2 of Section 6.4, we have

$$C(r, 0) - C(r, 1) + C(r, 2) - \cdots + (-1)^r C(r, r) = 0.$$

Hence,

$$1 = C(r, 0) = C(r, 1) - C(r, 2) + \cdots + (-1)^{r+1} C(r, r).$$

Therefore, each element in the union is counted exactly once by the expression on the right-hand side of the equation. This proves the principle of inclusion–exclusion. ◁

The inclusion–exclusion principle gives a formula for the number of elements in the union of n sets for every positive integer n. There are terms in this formula for the number of elements in the intersection of every nonempty subset of the collection of the n sets. Hence, there are $2^n - 1$ terms in this formula.

EXAMPLE 5 Give a formula for the number of elements in the union of four sets.

Extra Examples

Solution: The inclusion–exclusion principle shows that

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & |A_1| + |A_2| + |A_3| + |A_4| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| \\ & - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| \\ & + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|.\end{aligned}$$

Note that this formula contains 15 different terms, one for each nonempty subset of $\{A_1, A_2, A_3, A_4\}$. ◀

Exercises

1. How many elements are in $A_1 \cup A_2$ if there are 12 elements in A_1, 18 elements in A_2, and
   a) $A_1 \cap A_2 = \emptyset$?
   b) $|A_1 \cap A_2| = 1$?
   c) $|A_1 \cap A_2| = 6$?
   d) $A_1 \subseteq A_2$?
2. There are 345 students at a college who have taken a course in calculus, 212 who have taken a course in discrete mathematics, and 188 who have taken courses in both calculus and discrete mathematics. How many students have taken a course in either calculus or discrete mathematics?
3. A survey of households in the United States reveals that 96% have at least one television set, 98% have telephone service, and 95% have telephone service and at least one television set. What percentage of households in the United States have neither telephone service nor a television set?
4. A marketing report concerning personal computers states that 650,000 owners will buy a printer for their machines next year and 1,250,000 will buy at least one software package. If the report states that 1,450,000 owners will buy either a printer or at least one software package, how many will buy both a printer and at least one software package?
5. Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in each set and if
   a) the sets are pairwise disjoint.
   b) there are 50 common elements in each pair of sets and no elements in all three sets.
   c) there are 50 common elements in each pair of sets and 25 elements in all three sets.
   d) the sets are equal.
6. Find the number of elements in $A_1 \cup A_2 \cup A_3$ if there are 100 elements in A_1, 1000 in A_2, and 10,000 in A_3 if
   a) $A_1 \subseteq A_2$ and $A_2 \subseteq A_3$.
   b) the sets are pairwise disjoint.
   c) there are two elements common to each pair of sets and one element in all three sets.
7. There are 2504 computer science students at a school. Of these, 1876 have taken a course in Java, 999 have taken a course in Linux, and 345 have taken a course in C. Further, 876 have taken courses in both Java and Linux, 231 have taken courses in both Linux and C, and 290 have taken courses in both Java and C. If 189 of these students have taken courses in Linux, Java, and C, how many of these 2504 students have not taken a course in any of these three programming languages?
8. In a survey of 270 college students, it is found that 64 like Brussels sprouts, 94 like broccoli, 58 like cauliflower, 26 like both Brussels sprouts and broccoli, 28 like both Brussels sprouts and cauliflower, 22 like both broccoli and cauliflower, and 14 like all three vegetables. How many of the 270 students do not like any of these vegetables?
9. How many students are enrolled in a course either in calculus, discrete mathematics, data structures, or programming languages at a school if there are 507, 292, 312, and 344 students in these courses, respectively; 14 in both calculus and data structures; 213 in both calculus and programming languages; 211 in both discrete mathematics and data structures; 43 in both discrete mathematics and programming languages; and no student may take calculus and discrete mathematics, or data structures and programming languages, concurrently?
10. Find the number of positive integers not exceeding 100 that are not divisible by 5 or by 7.
11. Find the number of positive integers not exceeding 1000 that are not divisible by 3, 17, or 35.
12. Find the number of positive integers not exceeding 10,000 that are not divisible by 3, 4, 7, or 11.
13. Find the number of positive integers not exceeding 100 that are either odd or the square of an integer.
14. Find the number of positive integers not exceeding 1000 that are either the square or the cube of an integer.
15. How many bit strings of length eight do not contain six consecutive 0s?

*16. How many permutations of the 26 letters of the English alphabet do not contain any of the strings *fish*, *rat* or *bird*?

17. How many permutations of the 10 digits either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth positions, or end with the 3 digits 123?
18. How many elements are in the union of four sets if each of the sets has 100 elements, each pair of the sets shares 50 elements, each three of the sets share 25 elements, and there are 5 elements in all four sets?
19. How many elements are in the union of four sets if the sets have 50, 60, 70, and 80 elements, respectively, each pair of the sets has 5 elements in common, each triple of the sets has 1 common element, and no element is in all four sets?
20. How many terms are there in the formula for the number of elements in the union of 10 sets given by the principle of inclusion–exclusion?
21. Write out the explicit formula given by the principle of inclusion–exclusion for the number of elements in the union of five sets.
22. How many elements are in the union of five sets if the sets contain 10,000 elements each, each pair of sets has 1000 common elements, each triple of sets has 100 common elements, every four of the sets have 10 common elements, and there is 1 element in all five sets?
23. Write out the explicit formula given by the principle of inclusion–exclusion for the number of elements in the union of six sets when it is known that no three of these sets have a common intersection.

***24.** Prove the principle of inclusion–exclusion using mathematical induction.

25. Let E_1, E_2, and E_3 be three events from a sample space S. Find a formula for the probability of $E_1 \cup E_2 \cup E_3$.

26. Find the probability that when a fair coin is flipped five times tails comes up exactly three times, the first and last flips come up tails, or the second and fourth flips come up heads.

27. Find the probability that when four numbers from 1 to 100, inclusive, are picked at random with no repetitions allowed, either all are odd, all are divisible by 3, or all are divisible by 5.

28. Find a formula for the probability of the union of four events in a sample space if no three of them can occur at the same time.

29. Find a formula for the probability of the union of five events in a sample space if no four of them can occur at the same time.

30. Find a formula for the probability of the union of n events in a sample space when no two of these events can occur at the same time.

31. Find a formula for the probability of the union of n events in a sample space.

8.6 Applications of Inclusion–Exclusion

8.6.1 Introduction

Many counting problems can be solved using the principle of inclusion–exclusion. For instance, we can use this principle to find the number of primes less than a positive integer. Many problems can be solved by counting the number of onto functions from one finite set to another. The inclusion–exclusion principle can be used to find the number of such functions. The well-known hatcheck problem can be solved using the principle of inclusion–exclusion. This problem asks for the probability that no person is given the correct hat back by a hatcheck person who gives the hats back randomly.

8.6.2 An Alternative Form of Inclusion–Exclusion

There is an alternative form of the principle of inclusion–exclusion that is useful in counting problems. In particular, this form can be used to solve problems that ask for the number of elements in a set that have none of n properties $P_1, P_2, \dots, P_n$.

Let A_i be the subset containing the elements that have property P_i. The number of elements with all the properties $P_{i_1}, P_{i_2}, \dots, P_{i_k}$ will be denoted by $N(P_{i_1}P_{i_2}\dots P_{i_k})$. Writing these quantities in terms of sets, we have

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = N(P_{i_1}P_{i_2}\dots P_{i_k}).$$

If the number of elements with none of the properties $P_1, P_2, \dots, P_n$ is denoted by $N(P'_1P'_2\dots P'_n)$ and the number of elements in the set is denoted by N, it follows that

$$N(P'_1P'_2\dots P'_n) = N - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

From the inclusion–exclusion principle, we see that

$$\begin{aligned} N(P'_1P'_2\dots P'_n) = N &- \sum_{1\le i\le n} N(P_i) + \sum_{1\le i<j\le n} N(P_iP_j) \\ &- \sum_{1\le i<j<k\le n} N(P_iP_jP_k) + \cdots + (-1)^n N(P_1P_2\dots P_n). \end{aligned}$$

Example 1 shows how the principle of inclusion–exclusion can be used to determine the number of solutions in integers of an equation with constraints.

EXAMPLE 1 How many solutions does

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers with $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$?

Solution: To apply the principle of inclusion–exclusion, let a solution have property P_1 if $x_1 > 3$, property P_2 if $x_2 > 4$, and property P_3 if $x_3 > 6$. The number of solutions satisfying the inequalities $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$ is

$$N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_1P_3) + N(P_2P_3) - N(P_1P_2P_3).$$

Using the same techniques as in Example 5 of Section 6.5, it follows that

- N = total number of solutions $= C(3 + 11 - 1, 11) = 78$,
- $N(P_1)$ = (number of solutions with $x_1 \geq 4$) $= C(3 + 7 - 1, 7) = C(9, 7) = 36$,
- $N(P_2)$ = (number of solutions with $x_2 \geq 5$) $= C(3 + 6 - 1, 6) = C(8, 6) = 28$,
- $N(P_3)$ = (number of solutions with $x_3 \geq 7$) $= C(3 + 4 - 1, 4) = C(6, 4) = 15$,
- $N(P_1P_2)$ = (number of solutions with $x_1 \geq 4$ and $x_2 \geq 5$) $= C(3 + 2 - 1, 2) = C(4, 2) = 6$,
- $N(P_1P_3)$ = (number of solutions with $x_1 \geq 4$ and $x_3 \geq 7$) $= C(3 + 0 - 1, 0) = 1$,
- $N(P_2P_3)$ = (number of solutions with $x_2 \geq 5$ and $x_3 \geq 7$) $= 0$,
- $N(P_1P_2P_3)$ = (number of solutions with $x_1 \geq 4$, $x_2 \geq 5$, and $x_3 \geq 7$) $= 0$.

Inserting these quantities into the formula for $N(P_1'P_2'P_3')$ shows that the number of solutions with $x_1 \leq 3$, $x_2 \leq 4$, and $x_3 \leq 6$ equals

$$N(P_1'P_2'P_3') = 78 - 36 - 28 - 15 + 6 + 1 + 0 - 0 = 6.$$ ◀

8.6.3 The Sieve of Eratosthenes

In Section 4.3 we showed how to use the sieve of Eratosthenes to find all primes less than a specified positive integer n. Using the principle of inclusion–exclusion, we can find the number of primes not exceeding a specified positive integer with the same reasoning as is used in the sieve of Eratosthenes. Recall that a composite integer is divisible by a prime not exceeding its square root. So, to find the number of primes not exceeding 100, first note that composite integers not exceeding 100 must have a prime factor not exceeding 10. Because the only primes not exceeding 10 are 2, 3, 5, and 7, the primes not exceeding 100 are these four primes and those positive integers greater than 1 and not exceeding 100 that are divisible by none of 2, 3, 5, or 7. To apply the principle of inclusion–exclusion, let P_1 be the property that an integer is divisible by 2, let P_2 be the property that an integer is divisible by 3, let P_3 be the property that an integer is divisible by 5, and let P_4 be the property that an integer is divisible by 7. Thus, the number of primes not exceeding 100 is

$$4 + N(P_1'P_2'P_3'P_4').$$

Because there are 99 positive integers greater than 1 and not exceeding 100, the principle of inclusion–exclusion shows that

$$\begin{aligned} N(P_1'P_2'P_3'P_4') = 99 &- N(P_1) - N(P_2) - N(P_3) - N(P_4) \\ &+ N(P_1P_2) + N(P_1P_3) + N(P_1P_4) + N(P_2P_3) + N(P_2P_4) + N(P_3P_4) \\ &- N(P_1P_2P_3) - N(P_1P_2P_4) - N(P_1P_3P_4) - N(P_2P_3P_4) \\ &+ N(P_1P_2P_3P_4). \end{aligned}$$

The number of integers not exceeding 100 (and greater than 1) that are divisible by all the primes in a subset of $\{2, 3, 5, 7\}$ is $\lfloor 100/N \rfloor$, where N is the product of the primes in this subset. (This follows because any two of these primes have no common factor.) Consequently,

$$\begin{aligned} N(P_1'P_2'P_3'P_4') = 99 &- \left\lfloor \frac{100}{2} \right\rfloor - \left\lfloor \frac{100}{3} \right\rfloor - \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{7} \right\rfloor \\ &+ \left\lfloor \frac{100}{2\cdot 3} \right\rfloor + \left\lfloor \frac{100}{2\cdot 5} \right\rfloor + \left\lfloor \frac{100}{2\cdot 7} \right\rfloor + \left\lfloor \frac{100}{3\cdot 5} \right\rfloor + \left\lfloor \frac{100}{3\cdot 7} \right\rfloor + \left\lfloor \frac{100}{5\cdot 7} \right\rfloor \\ &- \left\lfloor \frac{100}{2\cdot 3\cdot 5} \right\rfloor - \left\lfloor \frac{100}{2\cdot 3\cdot 7} \right\rfloor - \left\lfloor \frac{100}{2\cdot 5\cdot 7} \right\rfloor - \left\lfloor \frac{100}{3\cdot 5\cdot 7} \right\rfloor + \left\lfloor \frac{100}{2\cdot 3\cdot 5\cdot 7} \right\rfloor \\ &= 99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 1 - 0 + 0 \\ &= 21. \end{aligned}$$

Hence, there are $4 + 21 = 25$ primes not exceeding 100.

8.6.4 The Number of Onto Functions

The principle of inclusion–exclusion can also be used to determine the number of onto functions from a set with m elements to a set with n elements. First consider Example 2.

EXAMPLE 2 How many onto functions are there from a set with six elements to a set with three elements?

Solution: Suppose that the elements in the codomain are b_1, b_2, and b_3. Let P_1, P_2, and P_3 be the properties that b_1, b_2, and b_3 are not in the range of the function, respectively. Note that a function is onto if and only if it has none of the properties P_1, P_2, or P_3. By the inclusion–exclusion principle it follows that the number of onto functions from a set with six elements to a set with three elements is

$$\begin{aligned} N(P_1'P_2'P_3') = N &- [N(P_1) + N(P_2) + N(P_3)] \\ &+ [N(P_1P_2) + N(P_1P_3) + N(P_2P_3)] - N(P_1P_2P_3), \end{aligned}$$

where N is the total number of functions from a set with six elements to one with three elements. We will evaluate each of the terms on the right-hand side of this equation.

From Example 6 of Section 6.1, it follows that $N = 3^6$. Note that $N(P_i)$ is the number of functions that do not have b_i in their range. Hence, there are two choices for the value of the function at each element of the domain. Therefore, $N(P_i) = 2^6$. Furthermore, there are $C(3, 1)$ terms of this kind. Note that $N(P_iP_j)$ is the number of functions that do not have b_i and b_j in their range. Hence, there is only one choice for the value of the function at each element of the domain. Therefore, $N(P_iP_j) = 1^6 = 1$. Furthermore, there are $C(3, 2)$ terms of this kind. Also, note that $N(P_1P_2P_3) = 0$, because this term is the number of functions that have none of b_1, b_2, and b_3 in their range. Clearly, there are no such functions, so the number of onto functions from a set with six elements to one with three elements is

$$3^6 - C(3, 1)2^6 + C(3, 2)1^6 = 729 - 192 + 3 = 540.$$ ◀

The general result that tells us how many onto functions there are from a set with m elements to one with n elements will now be stated. The proof of this result is left as an exercise for the reader.

THEOREM 1 Let m and n be positive integers with $m \geq n$. Then, there are

$$n^m - C(n, 1)(n-1)^m + C(n, 2)(n-2)^m - \cdots + (-1)^{n-1}C(n, n-1) \cdot 1^m$$

onto functions from a set with m elements to a set with n elements.

Counting onto functions is much harder than counting one-to-one functions!

An onto function from a set with m elements to a set with n elements corresponds to a way to distribute the m elements in the domain to n indistinguishable boxes so that no box is empty, and then to associate each of the n elements of the codomain to a box. This means that the number of onto functions from a set with m elements to a set with n elements is the number of ways to distribute m distinguishable objects to n indistinguishable boxes so that no box is empty multiplied by the number of permutations of a set with n elements. Consequently, the number of onto functions from a set with m elements to a set with n elements equals $n!S(m, n)$, where $S(m, n)$ is a *Stirling number of the second kind* defined in Section 6.5. This means that we can use Theorem 1 to deduce the formula given in Section 6.5 for $S(m, n)$. (See Chapter 6 of [MiRo91] for more details about Stirling numbers of the second kind.)

One of the many different applications of Theorem 1 will now be described.

EXAMPLE 3 How many ways are there to assign five different jobs to four different employees if every employee is assigned at least one job?

Solution: Consider the assignment of jobs as a function from the set of five jobs to the set of four employees. An assignment where every employee gets at least one job is the same as an onto function from the set of jobs to the set of employees. Hence, by Theorem 1 it follows that there are

$$4^5 - C(4, 1)3^5 + C(4, 2)2^5 - C(4, 3)1^5 = 1024 - 972 + 192 - 4 = 240$$

ways to assign the jobs so that each employee is assigned at least one job. ◀

8.6.5 Derangements

The principle of inclusion–exclusion will be used to count the permutations of n objects that leave no objects in their original positions. Consider Example 4.

EXAMPLE 4 **The Hatcheck Problem** A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat? ◀

Remark: The answer is the number of ways the hats can be arranged so that there is no hat in its original position divided by $n!$, the number of permutations of n hats. We will return to this example after we find the number of permutations of n objects that leave no objects in their original position.

Links

A **derangement** is a permutation of objects that leaves no object in its original position. To solve the problem posed in Example 4 we will need to determine the number of derangements of a set of n objects.

EXAMPLE 5 The permutation 21453 is a derangement of 12345 because no number is left in its original position. However, 21543 is not a derangement of 12345, because this permutation leaves 4 fixed. ◀

Let D_n denote the number of derangements of n objects. For instance, $D_3 = 2$, because the derangements of 123 are 231 and 312. We will evaluate D_n, for all positive integers n, using the principle of inclusion–exclusion.

THEOREM 2 The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Proof: Let a permutation have property P_i if it fixes element i. The number of derangements is the number of permutations having none of the properties P_i for $i = 1, 2, \ldots, n$. This means that

$$D_n = N(P'_1 P'_2 \ldots P'_n).$$

Using the principle of inclusion–exclusion, it follows that

$$D_n = N - \sum_i N(P_i) + \sum_{i<j} N(P_i P_j) - \sum_{i<j<k} N(P_i P_j P_k) + \cdots + (-1)^n N(P_1 P_2 \ldots P_n),$$

where N is the number of permutations of n elements. This equation states that the number of permutations that fix no elements equals the total number of permutations, less the number that fix at least one element, plus the number that fix at least two elements, less the number that fix at least three elements, and so on. All the quantities that occur on the right-hand side of this equation will now be found.

First, note that $N = n!$, because N is simply the total number of permutations of n elements. Also, $N(P_i) = (n-1)!$. This follows from the product rule, because $N(P_i)$ is the number of permutations that fix element i, so the ith position of the permutation is determined, but each of the remaining positions can be filled arbitrarily. Similarly,

$$N(P_i P_j) = (n-2)!,$$

Links

HISTORICAL NOTE In *rencontres* (matches), an old French card game, the 52 cards in a deck are laid out in a row. The cards of a second deck are laid out with one card of the second deck on top of each card of the first deck. The score is determined by counting the number of matching cards in the two decks. In 1708 Pierre Raymond de Montmort (1678–1719) posed *le problème de rencontres:* What is the probability that no matches take place in the game of rencontres? The solution to Montmort's problem is the probability that a randomly selected permutation of 52 objects is a derangement, namely, $D_{52}/52!$, which, as we will see, is approximately $1/e$.

because this is the number of permutations that fix elements i and j, but where the other $n-2$ elements can be arranged arbitrarily. In general, note that

$$N(P_{i_1}P_{i_2}\dots P_{i_m}) = (n-m)!,$$

because this is the number of permutations that fix elements $i_1, i_2, \dots, i_m$, but where the other $n-m$ elements can be arranged arbitrarily. Because there are $C(n,m)$ ways to choose m elements from n, it follows that

$$\sum_{1\le i\le n} N(P_i) = C(n,1)(n-1)!,$$

$$\sum_{1\le i<j\le n} N(P_iP_j) = C(n,2)(n-2)!,$$

and in general,

$$\sum_{1\le i_1<i_2<\dots<i_m\le n} N(P_{i_1}P_{i_2}\dots P_{i_m}) = C(n,m)(n-m)!.$$

Consequently, inserting these quantities into our formula for D_n gives

$$\begin{aligned} D_n &= n! - C(n,1)(n-1)! + C(n,2)(n-2)! - \cdots + (-1)^n C(n,n)(n-n)! \\ &= n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n-2)! - \cdots + (-1)^n \frac{n!}{n!\,0!}0!. \end{aligned}$$

Simplifying this expression gives

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}\right].$$

◁

It is now straightforward to find D_n for a given positive integer n. For instance, using Theorem 2, it follows that

$$D_3 = 3!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right] = 6\left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right) = 2,$$

as we have previously remarked.

The solution of the problem in Example 4 can now be given.

Solution: The probability that no one receives the correct hat is $D_n/n!$. By Theorem 2, this probability is

$$\frac{D_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}.$$

The values of this probability for $2 \le n \le 7$ are displayed in Table 1.

TABLE 1 The Probability of a Derangement.

n	2	3	4	5	6	7
$D_n/n!$	0.50000	0.33333	0.37500	0.36667	0.36806	0.36786

By the identity $e^x = \sum_{j=0}^{\infty} x^j/j!$ for all real numbers x (from calculus), we know that

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!} + \cdots \approx 0.368.$$

Because this is an alternating series with terms tending to zero, it follows that as n grows without bound, the probability that no one receives the correct hat converges to $e^{-1} \approx 0.368$. In fact, this probability can be shown to be within $1/(n+1)!$ of e^{-1}. ◀

Exercises

1. Suppose that in a bushel of 100 apples there are 20 that have worms in them and 15 that have bruises. Only those apples with neither worms nor bruises can be sold. If there are 10 bruised apples that have worms in them, how many of the 100 apples can be sold?

2. Of 1000 applicants for a mountain-climbing trip in the Himalayas, 450 get altitude sickness, 622 are not in good enough shape, and 30 have allergies. An applicant qualifies if and only if this applicant does not get altitude sickness, is in good shape, and does not have allergies. If there are 111 applicants who get altitude sickness and are not in good enough shape, 14 who get altitude sickness and have allergies, 18 who are not in good enough shape and have allergies, and 9 who get altitude sickness, are not in good enough shape, and have allergies, how many applicants qualify?

3. How many solutions does the equation $x_1 + x_2 + x_3 = 13$ have where x_1, x_2, and x_3 are nonnegative integers less than 6?

4. Find the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 17$, where x_i, $i = 1, 2, 3, 4$, are nonnegative integers such that $x_1 \le 3$, $x_2 \le 4$, $x_3 \le 5$, and $x_4 \le 8$.

5. Find the number of primes less than 200 using the principle of inclusion–exclusion.

6. An integer is called **squarefree** if it is not divisible by the square of a positive integer greater than 1. Find the number of squarefree positive integers less than 100.

7. How many positive integers less than 10,000 are not the second or higher power of an integer?

8. How many onto functions are there from a set with seven elements to one with five elements?

9. How many ways are there to distribute six different toys to three different children such that each child gets at least one toy?

10. In how many ways can eight distinct balls be distributed into three distinct urns if each urn must contain at least one ball?

11. In how many ways can seven different jobs be assigned to four different employees so that each employee is assigned at least one job and the most difficult job is assigned to the best employee?

12. List all the derangements of $\{1, 2, 3, 4\}$.

13. How many derangements are there of a set with seven elements?

14. What is the probability that none of 10 people receives the correct hat if a hatcheck person hands their hats back randomly?

15. A machine that inserts letters into envelopes goes haywire and inserts letters randomly into envelopes. What is the probability that in a group of 100 letters

a) no letter is put into the correct envelope?
b) exactly one letter is put into the correct envelope?
c) exactly 98 letters are put into the correct envelopes?
d) exactly 99 letters are put into the correct envelopes?
e) all letters are put into the correct envelopes?

16. A group of n students is assigned seats for each of two classes in the same classroom. How many ways can these seats be assigned if no student is assigned the same seat for both classes?

***17.** How many ways can the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 be arranged so that no even digit is in its original position?

***18.** Use a combinatorial argument to show that the sequence $\{D_n\}$, where D_n denotes the number of derangements of n objects, satisfies the recurrence relation

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

for $n \ge 2$. [*Hint:* Note that there are $n - 1$ choices for the first element k of a derangement. Consider separately the derangements that start with k that do and do not have 1 in the kth position.]

***19.** Use Exercise 18 to show that

$$D_n = nD_{n-1} + (-1)^n$$

for $n \ge 1$.

20. Use Exercise 19 to find an explicit formula for D_n.

21. For which positive integers n is D_n, the number of derangements of n objects, even?

22. Suppose that p and q are distinct primes. Use the principle of inclusion–exclusion to find $\phi(pq)$, the number of positive integers not exceeding pq that are relatively prime to pq.

***23.** Use the principle of inclusion–exclusion to derive a formula for $\phi(n)$ when the prime factorization of n is

$$n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}.$$

*24. Show that if n is a positive integer, then

$$n! = C(n,0)D_n + C(n,1)D_{n-1} + \cdots + C(n,n-1)D_1 + C(n,n)D_0,$$

where D_k is the number of derangements of k objects.

25. How many derangements of $\{1, 2, 3, 4, 5, 6\}$ begin with the integers 1, 2, and 3, in some order?

26. How many derangements of $\{1, 2, 3, 4, 5, 6\}$ end with the integers 1, 2, and 3, in some order?

27. Prove Theorem 1.

Key Terms and Results

TERMS

recurrence relation: a formula expressing terms of a sequence, except for some initial terms, as a function of one or more previous terms of the sequence

initial conditions for a recurrence relation: the values of the terms of a sequence satisfying the recurrence relation before this relation takes effect

dynamic programming: an algorithmic paradigm that finds the solution to an optimization problem by recursively breaking down the problem into overlapping subproblems and combining their solutions with the help of a recurrence relation

linear homogeneous recurrence relation with constant coefficients: a recurrence relation that expresses the terms of a sequence, except initial terms, as a linear combination of previous terms

characteristic roots of a linear homogeneous recurrence relation with constant coefficients: the roots of the polynomial associated with a linear homogeneous recurrence relation with constant coefficients

linear nonhomogeneous recurrence relation with constant coefficients: a recurrence relation that expresses the terms of a sequence, except for initial terms, as a linear combination of previous terms plus a function that is not identically zero that depends only on the index

divide-and-conquer algorithm: an algorithm that solves a problem recursively by splitting it into a fixed number of smaller nonoverlapping subproblems of the same type

generating function of a sequence: the formal series that has the nth term of the sequence as the coefficient of x^n

sieve of Eratosthenes: a procedure for finding the primes less than a specified positive integer

derangement: a permutation of objects such that no object is in its original place

RESULTS

the formula for the number of elements in the union of two finite sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

the formula for the number of elements in the union of three finite sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

the principle of inclusion–exclusion:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

the number of onto functions from a set with *m* elements to a set with *n* elements:

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1} C(n,n-1) \cdot 1^m$$

the number of derangements of *n* objects:

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right]$$

Review Questions

1. a) What is a recurrence relation?
   b) Find a recurrence relation for the amount of money that will be in an account after n years if \$1,000,000 is deposited in an account yielding 9% annually.

2. Explain how the Fibonacci numbers are used to solve Fibonacci's problem about rabbits.

3. a) Find a recurrence relation for the number of steps needed to solve the Tower of Hanoi puzzle.
   b) Show how this recurrence relation can be solved using iteration.

4. a) Explain how to find a recurrence relation for the number of bit strings of length n not containing two consecutive 1s.
   b) Describe another counting problem that has a solution satisfying the same recurrence relation.

5. a) What is dynamic programming and how are recurrence relations used in algorithms that follow this paradigm?
   b) Explain how dynamic programming can be used to schedule talks in a lecture hall from a set of possible talks to maximize overall attendance.
6. Define a linear homogeneous recurrence relation of degree k.
7. a) Explain how to solve linear homogeneous recurrence relations of degree 2.
   b) Solve the recurrence relation $a_n = 13a_{n-1} - 22a_{n-2}$ for $n \geq 2$ if $a_0 = 3$ and $a_1 = 15$.
   c) Solve the recurrence relation $a_n = 14a_{n-1} - 49a_{n-2}$ for $n \geq 2$ if $a_0 = 3$ and $a_1 = 35$.
8. a) Explain how to find $f(b^k)$ where k is a positive integer if $f(n)$ satisfies the divide-and-conquer recurrence relation $f(n) = af(n/b) + g(n)$ whenever b divides the positive integer n.
   b) Find $f(256)$ if $f(n) = 3f(n/4) + 5n/4$ and $f(1) = 7$.
9. a) Derive a divide-and-conquer recurrence relation for the number of comparisons used to find a number in a list using a binary search.
   b) Give a big-O estimate for the number of comparisons used by a binary search from the divide-and-conquer recurrence relation you gave in (a) using Theorem 1 in Section 8.3.
10. a) Give a formula for the number of elements in the union of three sets.
    b) Explain why this formula is valid.
    c) Explain how to use the formula from (a) to find the number of integers not exceeding 1000 that are divisible by 6, 10, or 15.
    d) Explain how to use the formula from (a) to find the number of solutions in nonnegative integers to the equation $x_1 + x_2 + x_3 + x_4 = 22$ with $x_1 < 8$, $x_2 < 6$, and $x_3 < 5$.
11. a) Give a formula for the number of elements in the union of four sets and explain why it is valid.
    b) Suppose the sets A_1, A_2, A_3, and A_4 each contain 25 elements, the intersection of any two of these sets contains 5 elements, the intersection of any three of these sets contains 2 elements, and 1 element is in all four of the sets. How many elements are in the union of the four sets?
12. a) State the principle of inclusion–exclusion.
    b) Outline a proof of this principle.
13. Explain how the principle of inclusion–exclusion can be used to count the number of onto functions from a set with m elements to a set with n elements.
14. a) How can you count the number of ways to assign m jobs to n employees so that each employee is assigned at least one job?
    b) How many ways are there to assign seven jobs to three employees so that each employee is assigned at least one job?
15. Explain how the inclusion–exclusion principle can be used to count the number of primes not exceeding the positive integer n.
16. a) Define a derangement.
    b) Why is counting the number of ways a hatcheck person can return hats to n people, so that no one receives the correct hat, the same as counting the number of derangements of n objects?
    c) Explain how to count the number of derangements of n objects.

Supplementary Exercises

1. A group of 10 people begin a chain letter, with each person sending the letter to four other people. Each of these people sends the letter to four additional people.
   a) Find a recurrence relation for the number of letters sent at the nth stage of this chain letter, if no person ever receives more than one letter.
   b) What are the initial conditions for the recurrence relation in part (a)?
   c) How many letters are sent at the nth stage of the chain letter?
2. A nuclear reactor has created 18 grams of a particular radioactive isotope. Every hour 1% of this radioactive isotope decays.
   a) Set up a recurrence relation for the amount of this isotope left n hours after its creation.
   b) What are the initial conditions for the recurrence relation in part (a)?
   c) Solve this recurrence relation.
3. Every hour the U.S. government prints 10,000 more \$1 bills, 4000 more \$5 bills, 3000 more \$10 bills, 2500 more \$20 bills, 1000 more \$50 bills, and the same number of \$100 bills as it did the previous hour. In the initial hour 1000 of each bill were produced.
   a) Set up a recurrence relation for the amount of money produced in the nth hour.
   b) What are the initial conditions for the recurrence relation in part (a)?
   c) Solve the recurrence relation for the amount of money produced in the nth hour.
   d) Set up a recurrence relation for the total amount of money produced in the first n hours.
   e) Solve the recurrence relation for the total amount of money produced in the first n hours.
4. Suppose that every hour there are two new bacteria in a colony for each bacterium that was present the previous hour, and that all bacteria 2 hours old die. The colony starts with 100 new bacteria.

a) Set up a recurrence relation for the number of bacteria present after n hours.
b) What is the solution of this recurrence relation?
c) When will the colony contain more than 1 million bacteria?

5. Messages are sent over a communications channel using two different signals. One signal requires 2 microseconds for transmittal, and the other signal requires 3 microseconds for transmittal. Each signal of a message is followed immediately by the next signal.

a) Find a recurrence relation for the number of different signals that can be sent in n microseconds.
b) What are the initial conditions of the recurrence relation in part (a)?
c) How many different messages can be sent in 12 microseconds?

6. A small post office has only 4-cent stamps, 6-cent stamps, and 10-cent stamps. Find a recurrence relation for the number of ways to form postage of n cents with these stamps if the order that the stamps are used matters. What are the initial conditions for this recurrence relation?

7. How many ways are there to form these postages using the rules described in Exercise 6?

a) 12 cents   b) 14 cents
c) 18 cents   d) 22 cents

8. Find the solutions of the simultaneous system of recurrence relations

$$a_n = a_{n-1} + b_{n-1}$$
$$b_n = a_{n-1} - b_{n-1}$$

with $a_0 = 1$ and $b_0 = 2$.

9. Solve the recurrence relation $a_n = a_{n-1}^2/a_{n-2}$ if $a_0 = 1$ and $a_1 = 2$. [*Hint:* Take logarithms of both sides to obtain a recurrence relation for the sequence $\log a_n$, $n = 0, 1, 2, \ldots$.]

***10.** Solve the recurrence relation $a_n = a_{n-1}^3 a_{n-2}^2$ if $a_0 = 2$ and $a_1 = 2$. (See the hint for Exercise 9.)

11. Find the solution of the recurrence relation $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} + 1$ if $a_0 = 2$, $a_1 = 4$, and $a_2 = 8$.

12. Find the solution of the recurrence relation $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ if $a_0 = 2$, $a_1 = 2$, and $a_2 = 4$.

***13.** Suppose that in Example 1 of Section 8.1 a pair of rabbits leaves the island after reproducing twice. Find a recurrence relation for the number of rabbits on the island in the middle of the nth month.

***14.** In this exercise we construct a dynamic programming algorithm for solving the problem of finding a subset S of items chosen from a set of n items where item i has a weight w_i, which is a positive integer, so that the total weight of the items in S is a maximum but does not exceed a fixed weight limit W. Let $M(j, w)$ denote the maximum total weight of the items in a subset of the first j items such that this total weight does not exceed w. This problem is known as the **knapsack problem**.

a) Show that if $w_j > w$, then $M(j, w) = M(j-1, w)$.
b) Show that if $w_j \leq w$, then $M(j, w) = \max(M(j-1, w), w_j + M(j-1, w-w_j))$.
c) Use (a) and (b) to construct a dynamic programming algorithm for determining the maximum total weight of items so that this total weight does not exceed W. In your algorithm store the values $M(j, w)$ as they are found.
d) Explain how you can use the values $M(j, w)$ computed by the algorithm in part (c) to find a subset of items with maximum total weight not exceeding W.

In Exercises 15–18 we develop a dynamic programming algorithm for finding a longest common subsequence of two sequences $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$, an important problem in the comparison of DNA of different organisms.

15. Suppose that $c_1, c_2, \ldots, c_p$ is a longest common subsequence of the sequences $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$.

a) Show that if $a_m = b_n$, then $c_p = a_m = b_n$ and $c_1, c_2, \ldots, c_{p-1}$ is a longest common subsequence of $a_1, a_2, \ldots, a_{m-1}$ and $b_1, b_2, \ldots, b_{n-1}$ when $p > 1$.
b) Suppose that $a_m \neq b_n$. Show that if $c_p \neq a_m$, then $c_1, c_2, \ldots, c_p$ is a longest common subsequence of $a_1, a_2, \ldots, a_{m-1}$ and $b_1, b_2, \ldots, b_n$ and also show that if $c_p \neq b_n$, then $c_1, c_2, \ldots, c_p$ is a longest common subsequence of $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_{n-1}$.

16. Let $L(i, j)$ denote the length of a longest common subsequence of $a_1, a_2, \ldots, a_i$ and $b_1, b_2, \ldots, b_j$, where $0 \leq i \leq m$ and $0 \leq j \leq n$. Use parts (a) and (b) of Exercise 15 to show that $L(i, j)$ satisfies the recurrence relation $L(i, j) = L(i-1, j-1) + 1$ if both i and j are nonzero and $a_i = b_i$, and $L(i, j) = \max(L(i, j-1), L(i-1, j))$ if both i and j are nonzero and $a_i \neq b_i$, and the initial condition $L(i, j) = 0$ if $i = 0$ or $j = 0$.

17. Use Exercise 16 to construct a dynamic programming algorithm for computing the length of a longest common subsequence of two sequences $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$, storing the values of $L(i, j)$ as they are found.

18. Develop an algorithm for finding a longest common subsequence of two sequences $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ using the values $L(i, j)$ found by the algorithm in Exercise 17.

19. Find the solution to the recurrence relation $f(n) = f(n/2) + n^2$ for $n = 2^k$ where k is a positive integer and $f(1) = 1$.

20. Find the solution to the recurrence relation $f(n) = 3f(n/5) + 2n^4$, when n is divisible by 5, for $n = 5^k$, where k is a positive integer and $f(1) = 1$.

21. Give a big-O estimate for the size of f in Exercise 20 if f is an increasing function.

22. Find a recurrence relation that describes the number of comparisons used by the following algorithm: Find the largest and second largest elements of a sequence of n numbers recursively by splitting the sequence into two subsequences with an equal number of terms, or where there is one more term in one subsequence than in the other, at each stage. Stop when subsequences with two terms are reached.

23. Give a big-O estimate for the number of comparisons used by the algorithm described in Exercise 22.

24. A sequence $a_1, a_2, \ldots, a_n$ is **unimodal** if and only if there is an index m, $1 \le m \le n$, such that $a_i < a_{i+1}$ when $1 \le i < m$ and $a_i > a_{i+1}$ when $m \le i < n$. That is, the terms of the sequence strictly increase until the mth term and they strictly decrease after it, which implies that a_m is the largest term. In this exercise, a_m will always denote the largest term of the unimodal sequence $a_1, a_2, \ldots, a_n$.

a) Show that a_m is the unique term of the sequence that is greater than both the term immediately preceding it and the term immediately following it.

b) Show that if $a_i < a_{i+1}$ where $1 \le i < n$, then $i + 1 \le m \le n$.

c) Show that if $a_i > a_{i+1}$ where $1 \le i < n$, then $1 \le m \le i$.

d) Develop a divide-and-conquer algorithm for locating the index m. [*Hint:* Suppose that $i < m < j$. Use parts (a), (b), and (c) to determine whether $\lfloor (i+j)/2 \rfloor + 1 \le m \le n$, $1 \le m \le \lfloor (i+j)/2 \rfloor - 1$, or $m = \lfloor (i+j)/2 \rfloor$.]

25. Show that the algorithm from Exercise 24 has worst-case time complexity $O(\log n)$ in terms of the number of comparisons.

Let $\{a_n\}$ be a sequence of real numbers. The **forward differences** of this sequence are defined recursively as follows: The **first forward difference** is $\Delta a_n = a_{n+1} - a_n$; the **$(k+1)$st forward difference** $\Delta^{k+1} a_n$ is obtained from $\Delta^k a_n$ by $\Delta^{k+1} a_n = \Delta^k a_{n+1} - \Delta^k a_n$.

26. Find Δa_n, where

a) $a_n = 3$. **b)** $a_n = 4n+7$. **c)** $a_n = n^2+n+1$.

27. Let $a_n = 3n^3 + n + 2$. Find $\Delta^k a_n$, where k equals

a) 2. **b)** 3. **c)** 4.

***28.** Suppose that $a_n = P(n)$, where P is a polynomial of degree d. Prove that $\Delta^{d+1} a_n = 0$ for all nonnegative integers n.

29. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Show that

$$\Delta(a_n b_n) = a_{n+1}(\Delta b_n) + b_n(\Delta a_n).$$

30. Show that if $F(x)$ and $G(x)$ are the generating functions for the sequences $\{a_k\}$ and $\{b_k\}$, respectively, and c and d are real numbers, then $(cF + dG)(x)$ is the generating function for $\{ca_k + db_k\}$.

31. (*Requires calculus*) This exercise shows how generating functions can be used to solve the recurrence relation $(n+1)a_{n+1} = a_n + (1/n!)$ for $n \ge 0$ with initial condition $a_0 = 1$.

a) Let $G(x)$ be the generating function for $\{a_n\}$. Show that $G'(x) = G(x) + e^x$ and $G(0) = 1$.

b) Show from part (a) that $(e^{-x}G(x))' = 1$, and conclude that $G(x) = xe^x + e^x$.

c) Use part (b) to find a closed form for a_n.

32. Suppose that 14 students receive an A on the first exam in a discrete mathematics class, and 18 receive an A on the second exam. If 22 students received an A on either the first exam or the second exam, how many students received an A on both exams?

33. There are 323 farms in Monmouth County that have at least one of horses, cows, and sheep. If 224 have horses, 85 have cows, 57 have sheep, and 18 farms have all three types of animals, how many farms have exactly two of these three types of animals?

34. Queries to a database of student records at a college produced the following data: There are 2175 students at the college, 1675 of these are not freshmen, 1074 students have taken a course in calculus, 444 students have taken a course in discrete mathematics, 607 students are not freshmen and have taken calculus, 350 students have taken calculus and discrete mathematics, 201 students are not freshmen and have taken discrete mathematics, and 143 students are not freshmen and have taken both calculus and discrete mathematics. Can all the responses to the queries be correct?

35. Students in the school of mathematics at a university major in one or more of the following four areas: applied mathematics (AM), pure mathematics (PM), operations research (OR), and computer science (CS). How many students are in this school if (including joint majors) there are 23 students majoring in AM; 17 in PM; 44 in OR; 63 in CS; 5 in AM and PM; 8 in AM and CS; 4 in AM and OR; 6 in PM and CS; 5 in PM and OR; 14 in OR and CS; 2 in PM, OR, and CS; 2 in AM, OR, and CS; 1 in PM, AM, and OR; 1 in PM, AM, and CS; and 1 in all four fields.

36. How many terms are needed when the inclusion–exclusion principle is used to express the number of elements in the union of seven sets if no more than five of these sets have a common element?

37. How many solutions in positive integers are there to the equation $x_1 + x_2 + x_3 = 20$ with $2 < x_1 < 6$, $6 < x_2 < 10$, and $0 < x_3 < 5$?

38. How many positive integers less than 1,000,000 are

a) divisible by 2, 3, or 5?

b) not divisible by 7, 11, or 13?

c) divisible by 3 but not by 7?

39. How many positive integers less than 200 are

a) second or higher powers of integers?

b) either primes or second or higher powers of integers?

c) not divisible by the square of an integer greater than 1?

d) not divisible by the cube of an integer greater than 1?

e) not divisible by three or more primes?

*40. How many ways are there to assign six different jobs to three different employees if the hardest job is assigned to the most experienced employee and the easiest job is assigned to the least experienced employee?

41. What is the probability that exactly one person is given back the correct hat by a hatcheck person who gives n people their hats back at random?

42. How many bit strings of length six do not contain four consecutive 1s?

43. What is the probability that a bit string of length six chosen at random contains at least four 1s?

Computer Projects

Write programs with these input and output.

1. Given a positive integer n, list all the moves required in the Tower of Hanoi puzzle to move n disks from one peg to another according to the rules of the puzzle.
2. Given a positive integer n and an integer k with $1 \leq k \leq n$, list all the moves used by the Frame–Stewart algorithm (described in the preamble to Exercise 38 of Section 8.1) to move n disks from one peg to another using four pegs according to the rules of the puzzle.
3. Given a positive integer n, list all the bit sequences of length n that do not have a pair of consecutive 0s.
4. Given an integer n greater than 1, write out all ways to parenthesize the product of $n + 1$ variables.
5. Given a set of n talks, their start and end times, and the number of attendees at each talk, use dynamic programming to schedule a subset of these talks in a single lecture hall to maximize total attendance.
6. Given matrices $\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}$, with dimensions $m_1 \times m_2, m_2 \times m_3, \ldots, m_n \times m_{n+1}$, respectively, each with integer entries, use dynamic programming, as outlined in Exercise 57 in Section 8.1, to find the minimum number of multiplications of integers needed to compute $\mathbf{A_1 A_2} \cdots \mathbf{A_n}$.
7. Given a recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, where c_1 and c_2 are real numbers, initial conditions $a_0 = C_0$ and $a_1 = C_1$, and a positive integer k, find a_k using iteration.
8. Given a recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ and initial conditions $a_0 = C_0$ and $a_1 = C_1$, determine the unique solution.
9. Given a recurrence relation of the form $f(n) = af(n/b) + c$, where a is a real number, b is a positive integer, and c is a real number, and a positive integer k, find $f(b^k)$ using iteration.
10. Given the number of elements in the intersection of three sets, the number of elements in each pairwise intersection of these sets, and the number of elements in each set, find the number of elements in their union.
11. Given a positive integer n, produce the formula for the number of elements in the union of n sets.
12. Given positive integers m and n, find the number of onto functions from a set with m elements to a set with n elements.
13. Given a positive integer n, list all the derangements of the set $\{1, 2, 3, \ldots, n\}$.

Computations and Explorations

Use a computational program or programs you have written to do these exercises.

1. Find the exact value of f_{100}, f_{500}, and f_{1000}, where f_n is the nth Fibonacci number.
2. Find the smallest Fibonacci number greater than 1,000,000, greater than 1,000,000,000, and greater than 1,000,000,000,000.
3. Find as many prime Fibonacci numbers as you can. It is unknown whether there are infinitely many of these.
4. Write out all the moves required to solve the Tower of Hanoi puzzle with 10 disks.
5. Write out all the moves required to use the Frame–Stewart algorithm to move 20 disks from one peg to another peg using four pegs according to the rules of the Reve's puzzle.
6. Verify the Frame conjecture for solving the Reve's puzzle for n disks for as many integers n as possible by showing that the puzzle cannot be solved using fewer moves than are made by the Frame–Stewart algorithm with the optimal choice of k.
7. Compute the number of operations required to multiply two integers with n bits for various integers n including 16, 64, 256, and 1024 using the fast multiplication described in Example 4 of Section 8.3 and the standard algorithm for multiplying integers (Algorithm 3 in Section 4.2).
8. Compute the number of operations required to multiply two $n \times n$ matrices for various integers n including 4, 16, 64, and 128 using the fast matrix multiplication described

in Example 5 of Section 8.3 and the standard algorithm for multiplying matrices (Algorithm 1 in Section 3.3).

9. Find the number of primes not exceeding 10,000 using the method described in Section 8.6 to find the number of primes not exceeding 100.

10. List all the derangements of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

11. Compute the probability that a permutation of n objects is a derangement for all positive integers not exceeding 20 and determine how quickly these probabilities approach the number $1/e$.

Writing Projects

Respond to these with essays using outside sources.

1. Find the original source where Fibonacci presented his puzzle about modeling rabbit populations. Discuss this problem and other problems posed by Fibonacci and give some information about Fibonacci himself.
2. Explain how the Fibonacci numbers arise in a variety of applications, such as in phyllotaxis, the study of arrangement of leaves in plants, in the study of reflections by mirrors, and so on.
3. Describe different variations of the Tower of Hanoi puzzle, including those with more than three pegs (including the Reve's puzzle discussed in the text and exercises), those where disk moves are restricted, and those where disks may have the same size. Include what is known about the number of moves required to solve each variation.
4. Discuss as many different problems as possible where the Catalan numbers arise.
5. Discuss some of the problems in which Richard Bellman first used dynamic programming.
6. Describe the role dynamic programming algorithms play in bioinformatics including for DNA sequence comparison, gene comparison, and RNA structure prediction.
7. Describe the use of dynamic programming in economics including its use to study optimal consumption and saving.
8. Explain how dynamic programming can be used to solve the egg-dropping puzzle which determines which floors of a multistory building it is safe to drop eggs from without breaking.
9. Describe the solution of Ulam's problem (see Exercise 28 in Section 8.3) involving searching with one lie found by Andrzej Pelc.
10. Discuss variations of Ulam's problem (see Exercise 28 in Section 8.3) involving searching with more than one lie and what is known about this problem.
11. Define the convex hull of a set of points in the plane and describe three different algorithms, including a divide-and-conquer algorithm, for finding the convex hull of a set of points in the plane.
12. Describe how sieve methods are used in number theory. What kind of results have been established using such methods?
13. Look up the rules of the old French card game of *rencontres*. Describe these rules and describe the work of Pierre Raymond de Montmort on *le problème de rencontres*.
14. Describe how exponential generating functions can be used to solve a variety of counting problems.
15. Describe the Polyá theory of counting and the kind of counting problems that can be solved using this theory.
16. The *problème des ménages* (the problem of the households) asks for the number of ways to arrange n couples around a table so that the sexes alternate and no husband and wife are seated together. Explain the method used by E. Lucas to solve this problem.
17. Explain how *rook polynomials* can be used to solve counting problems.

CHAPTER

9 Relations

Relationships between elements of sets occur in many contexts. Every day we deal with relationships such as those between a business and its telephone number, an employee and his or her salary, a person and a relative, and so on. In mathematics we study relationships such as those between a positive integer and one that it divides, an integer and one that it is congruent to modulo 5, a real number and one that is larger than it, a real number x and the value $f(x)$ where f is a function, and so on. Relationships such as that between a program and a variable it uses, and that between a computer language and a valid statement in this language, often arise in computer science. Relationships between elements of two sets are represented using the structure called a binary relation, which is just a subset of the Cartesian product of the sets. Relations can be used to solve problems such as determining which pairs of cities are linked by airline flights in a network, or finding a viable order for the different phases of a complicated project. We will introduce a number of different properties binary relations may enjoy.

Relationships between elements of more than two sets arise in many contexts. These relationships can be represented by n-ary relations, which are collections of n-tuples. Such relations are the basis of the relational data model, the most common way to store information in computer databases. We will introduce the terminology used to study relational databases, define some important operations on them, and introduce the database query language SQL. We will conclude our brief study of n-ary relations and databases with an important application from data mining. In particular, we will show how databases of transactions, represented by n-ary relations, are used to measure the likelihood that someone buys a particular product from a store when they buy one or more other products.

Two methods for representing relations, using square matrices and using directed graphs, consisting of vertices and directed edges, will be introduced and used in later sections of the chapter. We will also study relationships that have certain collections of properties that relations may enjoy. For example, in some computer languages, only the first 31 characters of the name of a variable matter. The relation consisting of ordered pairs of strings in which the first string has the same initial 31 characters as the second string is an example of a special type of relation, known as an equivalence relation. Equivalence relations arise throughout mathematics and computer science. Finally, we will study relations called partial orderings, which generalize the notion of the less than or equal to relation. For example, the set of all pairs of strings of English letters in which the second string is the same as the first string or comes after the first in dictionary order is a partial ordering.

9.1 Relations and Their Properties

9.1.1 Introduction

Links

The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements. For this reason, sets of ordered pairs are called binary relations. In this section we introduce the basic terminology used to describe binary relations. Later in this chapter we will use relations to solve problems involving communications networks, project scheduling, and identifying elements in sets with common properties.

Definition 1 Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs, where the first element of each ordered pair comes from A and the second element comes from B. We use the notation $a\,R\,b$ to denote that $(a, b) \in R$ and $a \not R\, b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R, a is said to be **related to** b by R.

Binary relations represent relationships between the elements of two sets. We will introduce n-ary relations, which express relationships among elements of more than two sets, later in this chapter. We will omit the word *binary* when there is no danger of confusion.

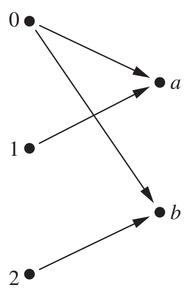
Examples 1–3 illustrate the notion of a relation.

EXAMPLE 1 Let A be the set of students in your school, and let B be the set of courses. Let R be the relation that consists of those pairs (a, b), where a is a student enrolled in course b. For instance, if Jason Goodfriend and Deborah Sherman are enrolled in CS518, the pairs (Jason Goodfriend, CS518) and (Deborah Sherman, CS518) belong to R. If Jason Goodfriend is also enrolled in CS510, then the pair (Jason Goodfriend, CS510) is also in R. However, if Deborah Sherman is not enrolled in CS510, then the pair (Deborah Sherman, CS510) is not in R.

Note that if a student is not currently enrolled in any courses there will be no pairs in R that have this student as the first element. Similarly, if a course is not currently being offered there will be no pairs in R that have this course as their second element. ◀

EXAMPLE 2 Let A be the set of cities in the U.S.A., and let B be the set of the 50 states in the U.S.A. Define the relation R by specifying that (a, b) belongs to R if a city with name a is in the state b. For instance, (Boulder, Colorado), (Bangor, Maine), (Ann Arbor, Michigan), (Middletown, New Jersey), (Middletown, New York), (Cupertino, California), and (Red Bank, New Jersey) are in R. ◀

EXAMPLE 3 Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that $0\,R\,a$, but that $1 \not R\, b$. Relations can be represented graphically, as shown in Figure 1, using arrows to represent ordered pairs. Another way to represent this relation is to use a table, which is also done in Figure 1. We will discuss representations of relations in more detail in Section 9.3. ◀



R	a	b
0	×	×
1	×	
2		×

FIGURE 1 Displaying the ordered pairs in the relation R from Example 3.

9.1.2 Functions as Relations

Recall that a function f from a set A to a set B (as defined in Section 2.3) assigns exactly one element of B to each element of A. The graph of f is the set of ordered pairs (a, b) such that $b = f(a)$. Because the graph of f is a subset of $A \times B$, it is a relation from A to B. Moreover, the graph of a function has the property that every element of A is the first element of exactly one ordered pair of the graph.

Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph. This can be done by assigning to an element a of A the unique element $b \in B$ such that $(a, b) \in R$. (Note that the relation R in Example 2 is not the graph of a function because Middletown occurs more than once as the first element of an ordered pair in R.)

A relation can be used to express a one-to-many relationship between the elements of the sets A and B (as in Example 2), where an element of A may be related to more than one element of B. A function represents a relation where exactly one element of B is related to each element of A.

Relations are a generalization of graphs of functions; they can be used to express a much wider class of relationships between sets. (Recall that the graph of the function f from A to B is the set of ordered pairs $(a, f(a))$ for $a \in A$.)

9.1.3 Relations on a Set

Relations from a set A to itself are of special interest.

Definition 2 A *relation on a set A* is a relation from A to A.

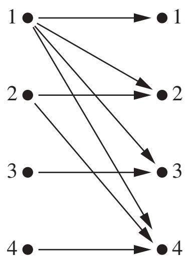
In other words, a relation on a set A is a subset of $A \times A$.

EXAMPLE 4 Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

The pairs in this relation are displayed both graphically and in tabular form in Figure 2. ◀



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

FIGURE 2 Displaying the ordered pairs in the relation R from Example 4.

Next, some examples of relations on the set of integers will be given in Example 5.

EXAMPLE 5 Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Remark: Unlike the relations in Examples 1–4, these are relations on an infinite set.

Solution: The pair (1, 1) is in R_1, R_3, R_4, and R_6; (1, 2) is in R_1 and R_6; (2, 1) is in R_2, R_5, and R_6; (1, −1) is in R_2, R_3, and R_6; and finally, (2, 2) is in R_1, R_3, and R_4. ◀

It is not hard to determine the number of relations on a finite set, because a relation on a set A is simply a subset of $A \times A$.

EXAMPLE 6 How many relations are there on a set with n elements?

Solution: A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements. For example, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{a, b, c\}$. ◀

9.1.4 Properties of Relations

There are several properties that are used to classify relations on a set. We will introduce the most important of these here. (You may find it instructive to study this material with the contents of Section 9.3. In that section, several methods for representing relations will be introduced that can help you understand each of the properties that we introduce here.)

In some relations an element is always related to itself. For instance, let R be the relation on the set of all people consisting of pairs (x, y) where x and y have the same mother and the same father. Then xRx for every person x.

Definition 3 A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

Remark: Using quantifiers we see that the relation R on the set A is reflexive if $\forall a((a, a) \in R)$, where the universe of discourse is the set of all elements in A.

We see that a relation on A is reflexive if every element of A is related to itself. Examples 7–9 illustrate the concept of a reflexive relation.

EXAMPLE 7 Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Solution: The relations R_3 and R_5 are reflexive because they both contain all pairs of the form (a, a), namely, $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$. The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R_1, R_2, R_4, and R_6 are not reflexive because $(3, 3)$ is not in any of these relations. ◀

EXAMPLE 8 Which of the relations from Example 5 are reflexive?

Solution: The reflexive relations from Example 5 are R_1 (because $a \leq a$ for every integer a), R_3, and R_4. For each of the other relations in this example it is easy to find a pair of the form (a, a) that is not in the relation. (This is left as an exercise for the reader.) ◀

EXAMPLE 9 Is the "divides" relation on the set of positive integers reflexive?

Solution: Because $a \mid a$ whenever a is a positive integer, the "divides" relation is reflexive. (Note that if we replace the set of positive integers with the set of all integers the relation is not reflexive because by definition 0 does not divide 0.) ◀

In some relations an element is related to a second element if and only if the second element is also related to the first element. The relation consisting of pairs (x, y), where x and y are students at your school with at least one common class has this property. Other relations have the property that if an element is related to a second element, then this second element is not related to the first. The relation consisting of the pairs (x, y), where x and y are students at your school, where x has a higher grade point average than y has this property.

Definition 4 A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

Remark: Using quantifiers, we see that the relation R on the set A is symmetric if $\forall a \forall b((a, b) \in R \rightarrow (b, a) \in R)$. Similarly, the relation R on the set A is antisymmetric if $\forall a \forall b(((a, b) \in R \land (b, a) \in R) \rightarrow (a = b))$.

In other words, a relation is symmetric if and only if a is related to b always implies that b is related to a. For instance, the equality relation is symmetric because $a = b$ if and only if $b = a$. A relation is antisymmetric if and only if there are no pairs of distinct elements a and b with a related to b and b related to a. That is, the only way to have a related to b and b related to a is for a and b to be the same element. For instance, the less than or equal to relation is antisymmetric. To see this, note that $a \leq b$ and $b \leq a$ implies that $a = b$. The terms *symmetric* and *antisymmetric* are not opposites, because a relation can have both of these properties or may lack both of them (see Exercise 10). A relation cannot be both symmetric and antisymmetric if it contains some pair of the form (a, b) in which $a \neq b$.

Remark: Although relatively few of the 2^{n^2} relations on a set with n elements are symmetric or antisymmetric, as counting arguments can show, many important relations have one of these properties. (See Exercise 49.)

EXAMPLE 10 Which of the relations from Example 7 are symmetric and which are antisymmetric?

Extra Examples

Solution: The relations R_2 and R_3 are symmetric, because in each case (b, a) belongs to the relation whenever (a, b) does. For R_2, the only thing to check is that both $(2, 1)$ and $(1, 2)$ are in the relation. For R_3, it is necessary to check that both $(1, 2)$ and $(2, 1)$ belong to the relation, and $(1, 4)$ and $(4, 1)$ belong to the relation. The reader should verify that none of the other relations is symmetric. This is done by finding a pair (a, b) such that it is in the relation but (b, a) is not.

R_4, R_5, and R_6 are all antisymmetric. For each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation. The reader should verify that none of the other relations is antisymmetric. This is done by finding a pair (a, b) with $a \neq b$ such that (a, b) and (b, a) are both in the relation. ◀

EXAMPLE 11 Which of the relations from Example 5 are symmetric and which are antisymmetric?

Solution: The relations R_3, R_4, and R_6 are symmetric. R_3 is symmetric, for if $a = b$ or $a = -b$, then $b = a$ or $b = -a$. R_4 is symmetric because $a = b$ implies that $b = a$. R_6 is symmetric because $a + b \leq 3$ implies that $b + a \leq 3$. The reader should verify that none of the other relations is symmetric.

The relations R_1, R_2, R_4, and R_5 are antisymmetric. R_1 is antisymmetric because the inequalities $a \leq b$ and $b \leq a$ imply that $a = b$. R_2 is antisymmetric because it is impossible that $a > b$ and $b > a$. R_4 is antisymmetric, because two elements are related with respect to R_4 if and only if they are equal. R_5 is antisymmetric because it is impossible that $a = b + 1$ and $b = a + 1$. The reader should verify that none of the other relations is antisymmetric. ◀

EXAMPLE 12 Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

Solution: This relation is not symmetric because $1 \mid 2$, but $2 \nmid 1$. However, it is antisymmetric. To see this, note that if a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$ (the verification of this is left as an exercise for the reader). ◀

Let R be the relation consisting of all pairs (x, y) of students at your school, where x has taken more credits than y. Suppose that x is related to y and y is related to z. This means that x has taken more credits than y and y has taken more credits than z. We can conclude that x has taken more credits than z, so that x is related to z. What we have shown is that R has the transitive property, which is defined as follows.

Definition 5 A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Remark: Using quantifiers we see that the relation R on a set A is transitive if we have $\forall a \forall b \forall c(((a, b) \in R \wedge (b, c) \in R) \to (a, c) \in R)$.

EXAMPLE 13 Which of the relations in Example 7 are transitive?

Extra Examples

Solution: R_4, R_5, and R_6 are transitive. For each of these relations, we can show that it is transitive by verifying that if (a, b) and (b, c) belong to this relation, then (a, c) also does. For instance, R_4 is transitive, because (3, 2) and (2, 1), (4, 2) and (2, 1), (4, 3) and (3, 1), and (4, 3) and (3, 2) are the only such sets of pairs, and (3, 1), (4, 1), and (4, 2) belong to R_4. The reader should verify that R_5 and R_6 are transitive.

R_1 is not transitive because (3, 4) and (4, 1) belong to R_1, but (3, 1) does not. R_2 is not transitive because (2, 1) and (1, 2) belong to R_2, but (2, 2) does not. R_3 is not transitive because (4, 1) and (1, 2) belong to R_3, but (4, 2) does not. ◀

EXAMPLE 14 Which of the relations in Example 5 are transitive?

Solution: The relations R_1, R_2, R_3, and R_4 are transitive. R_1 is transitive because $a \leq b$ and $b \leq c$ imply that $a \leq c$. R_2 is transitive because $a > b$ and $b > c$ imply that $a > c$. R_3 is transitive because $a = \pm b$ and $b = \pm c$ imply that $a = \pm c$. R_4 is clearly transitive, as the reader should verify. R_5 is not transitive because (2, 1) and (1, 0) belong to R_5, but (2, 0) does not. R_6 is not transitive because (2, 1) and (1, 2) belong to R_6, but (2, 2) does not. ◀

EXAMPLE 15 Is the "divides" relation on the set of positive integers transitive?

Solution: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. It follows that this relation is transitive. ◀

We can use counting techniques to determine the number of relations with specific properties. Finding the number of relations with a particular property provides information about how common this property is in the set of all relations on a set with n elements.

EXAMPLE 16 How many reflexive relations are there on a set with n elements?

Solution: A relation R on a set A is a subset of $A \times A$. Consequently, a relation is determined by specifying whether each of the n^2 ordered pairs in $A \times A$ is in R. However, if R is reflexive, each of the n ordered pairs (a, a) for $a \in A$ must be in R. Each of the other $n(n-1)$ ordered pairs of the form (a, b), where $a \neq b$, may or may not be in R. Hence, by the product rule for counting, there are $2^{n(n-1)}$ reflexive relations [this is the number of ways to choose whether each element (a, b), with $a \neq b$, belongs to R]. ◀

Formulas for the number of symmetric relations and the number of antisymmetric relations on a set with n elements can be found using reasoning similar to that in Example 16 (see Exercise 49). However, no general formula is known that counts the transitive relations on a set with n elements. Currently, $T(n)$, the number of transitive relations on a set with n elements, is known only for $0 \leq n \leq 18$. For example, $T(4) = 3{,}994$, $T(5) = 154{,}303$, and $T(6) = 9{,}415{,}189$. (The values of $T(n)$ for $n = 0, 1, 2, \ldots, 18$, are the terms of the sequence A006905 in the OEIS, which is discussed in Section 2.4.)

9.1.5 Combining Relations

Because relations from A to B are subsets of $A \times B$, two relations from A to B can be combined in any way two sets can be combined. Consider Examples 17–19.

EXAMPLE 17 Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relations $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_2 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$ can be combined to obtain

$$\begin{aligned}
R_1 \cup R_2 &= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (3, 3)\},\\
R_1 \cap R_2 &= \{(1, 1)\},\\
R_1 - R_2 &= \{(2, 2), (3, 3)\},\\
R_2 - R_1 &= \{(1, 2), (1, 3), (1, 4)\}.
\end{aligned}$$

◂

EXAMPLE 18 Let A and B be the set of all students and the set of all courses at a school, respectively. Suppose that R_1 consists of all ordered pairs (a, b), where a is a student who has taken course b, and R_2 consists of all ordered pairs (a, b), where a is a student who requires course b to graduate. What are the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$, and $R_2 - R_1$?

Solution: The relation $R_1 \cup R_2$ consists of all ordered pairs (a, b), where a is a student who either has taken course b or needs course b to graduate, and $R_1 \cap R_2$ is the set of all ordered pairs (a, b), where a is a student who has taken course b and needs this course to graduate. Also, $R_1 \oplus R_2$ consists of all ordered pairs (a, b), where student a has taken course b but does not need it to graduate or needs course b to graduate but has not taken it. $R_1 - R_2$ is the set of ordered pairs (a, b), where a has taken course b but does not need it to graduate; that is, b is an elective course that a has taken. $R_2 - R_1$ is the set of all ordered pairs (a, b), where b is a course that a needs to graduate but has not taken. ◂

EXAMPLE 19 Let R_1 be the less than relation on the set of real numbers and let R_2 be the greater than relation on the set of real numbers, that is, $R_1 = \{(x, y) \mid x < y\}$ and $R_2 = \{(x, y) \mid x > y\}$. What are $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, and $R_1 \oplus R_2$?

Solution: We note that $(x, y) \in R_1 \cup R_2$ if and only if $(x, y) \in R_1$ or $(x, y) \in R_2$. Hence, $(x, y) \in R_1 \cup R_2$ if and only if $x < y$ or $x > y$. Because the condition $x < y$ or $x > y$ is the same as the condition $x \neq y$, it follows that $R_1 \cup R_2 = \{(x, y) \mid x \neq y\}$. In other words, the union of the less than relation and the greater than relation is the not equals relation.

Next, note that it is impossible for a pair (x, y) to belong to both R_1 and R_2 because it is impossible that $x < y$ and $x > y$. It follows that $R_1 \cap R_2 = \emptyset$. We also see that $R_1 - R_2 = R_1$, $R_2 - R_1 = R_2$, and $R_1 \oplus R_2 = R_1 \cup R_2 - R_1 \cap R_2 = \{(x, y) \mid x \neq y\}$. ◂

There is another way that relations are combined that is analogous to the composition of functions.

Definition 6 Let R be a relation from a set A to a set B and S a relation from B to a set C. The *composite* of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.

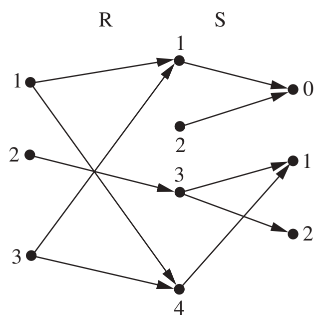
Computing the composite of two relations requires that we find elements that are the second element of ordered pairs in the first relation and the first element of ordered pairs in the second relation, as Examples 20 and 21 illustrate.

EXAMPLE 20 What is the composite of the relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Solution: $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S. For example, the ordered pairs $(2, 3)$ in R and $(3, 1)$ in S produce the ordered pair $(2, 1)$ in $S \circ R$. Computing all the ordered pairs in the composite, we find

$$S \circ R = \{(1, 0), (1, 1), (2, 1), (2, 2), (3, 0), (3, 1)\}.$$

Figure 3 illustrates how this composition is found. In the figure, we examine all paths that travel via two directed edges from the leftmost elements to the rightmost elements via an element in the middle.



$1 \to 1 \to 0$	$(1, 0)$
$1 \to 4 \to 1$	$(1, 1)$
$2 \to 3 \to 1$	$(2, 1)$
$2 \to 3 \to 2$	$(2, 2)$
$3 \to 1 \to 0$	$(3, 0)$
$3 \to 4 \to 1$	$(3, 1)$

FIGURE 3 **Constructing $S \circ R$.** ◀

EXAMPLE 21 **Composing the Parent Relation with Itself** Let R be the relation on the set of all people such that $(a, b) \in R$ if person a is a parent of person b. Then $(a, c) \in R \circ R$ if and only if there is a person b such that $(a, b) \in R$ and $(b, c) \in R$, that is, if and only if there is a person b such that a is a parent of b and b is a parent of c. In other words, $(a, c) \in R \circ R$ if and only if a is a grandparent of c. ◀

The powers of a relation R can be recursively defined from the definition of a composite of two relations.

Definition 7 Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by

$$R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R.$$

The definition shows that $R^2 = R \circ R$, $R^3 = R^2 \circ R = (R \circ R) \circ R$, and so on.

EXAMPLE 22 Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$. Find the powers R^n, $n = 2, 3, 4, \ldots$.

Solution: Because $R^2 = R \circ R$, we find that $R^2 = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$. Furthermore, because $R^3 = R^2 \circ R$, $R^3 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$. Additional computation shows that R^4

is the same as R^3, so $R^4 = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$. It also follows that $R^n = R^3$ for $n = 5, 6, 7, \ldots$. The reader should verify this. ◀

The following theorem shows that the powers of a transitive relation are subsets of this relation. It will be used in Section 9.4.

THEOREM 1 The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Proof: We first prove the "if" part of the theorem. We suppose that $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$. In particular, $R^2 \subseteq R$. To see that this implies R is transitive, note that if $(a, b) \in R$ and $(b, c) \in R$, then by the definition of composition, $(a, c) \in R^2$. Because $R^2 \subseteq R$, this means that $(a, c) \in R$. Hence, R is transitive.

We will use mathematical induction to prove the only if part of the theorem. Note that this part of the theorem is trivially true for $n = 1$.

Assume that $R^n \subseteq R$, where n is a positive integer. This is the inductive hypothesis. To complete the inductive step we must show that this implies that R^{n+1} is also a subset of R. To show this, assume that $(a, b) \in R^{n+1}$. Then, because $R^{n+1} = R^n \circ R$, there is an element x with $x \in A$ such that $(a, x) \in R$ and $(x, b) \in R^n$. The inductive hypothesis, namely, that $R^n \subseteq R$, implies that $(x, b) \in R$. Furthermore, because R is transitive, and $(a, x) \in R$ and $(x, b) \in R$, it follows that $(a, b) \in R$. This shows that $R^{n+1} \subseteq R$, completing the proof. ◁

Exercises

1. List the ordered pairs in the relation R from $A = \{0, 1, 2, 3, 4\}$ to $B = \{0, 1, 2, 3\}$, where $(a, b) \in R$ if and only if

a) $a = b$. **b)** $a + b = 4$.
c) $a > b$. **d)** $a \mid b$.
e) $\gcd(a, b) = 1$. **f)** $\operatorname{lcm}(a, b) = 2$.

2. a) List all the ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set $\{1, 2, 3, 4, 5, 6\}$.

b) Display this relation graphically, as was done in Example 4.

c) Display this relation in tabular form, as was done in Example 4.

3. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$
b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
c) $\{(2, 4), (4, 2)\}$
d) $\{(1, 2), (2, 3), (3, 4)\}$
e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

4. Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a) a is taller than b.
b) a and b were born on the same day.
c) a has the same first name as b.
d) a and b have a common grandparent.

5. Determine whether the relation R on the set of all Web pages is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

a) everyone who has visited Web page a has also visited Web page b.
b) there are no common links found on both Web page a and Web page b.
c) there is at least one common link on Web page a and Web page b.
d) there is a Web page that includes links to both Web page a and Web page b.

6. Determine whether the relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x + y = 0$. **b)** $x = \pm y$.
c) $x - y$ is a rational number.
d) $x = 2y$. **e)** $xy \geq 0$.
f) $xy = 0$. **g)** $x = 1$.
h) $x = 1$ or $y = 1$.

7. Determine whether the relation R on the set of all integers is reflexive, symmetric, antisymmetric, and/or transitive, where $(x, y) \in R$ if and only if

a) $x \neq y$. **b)** $xy \geq 1$.
c) $x = y + 1$ or $x = y - 1$.
d) $x \equiv y \pmod 7$. **e)** x is a multiple of y.
f) x and y are both negative or both nonnegative.
g) $x = y^2$. **h)** $x \geq y^2$.

8. Show that the relation $R = \emptyset$ on a nonempty set S is symmetric and transitive, but not reflexive.

9. Show that the relation $R = \emptyset$ on the empty set $S = \emptyset$ is reflexive, symmetric, and transitive.

10. Give an example of a relation on a set that is

a) both symmetric and antisymmetric.

b) neither symmetric nor antisymmetric.

A relation R on the set A is **irreflexive** if for every $a \in A, (a, a) \notin R$. That is, R is irreflexive if no element in A is related to itself.

11. Which relations in Exercise 3 are irreflexive?

12. Which relations in Exercise 4 are irreflexive?

13. Which relations in Exercise 5 are irreflexive?

14. Which relations in Exercise 6 are irreflexive?

15. Can a relation on a set be neither reflexive nor irreflexive?

16. Use quantifiers to express what it means for a relation to be irreflexive.

17. Give an example of an irreflexive relation on the set of all people.

A relation R is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$. Exercises 18–24 explore the notion of an asymmetric relation. Exercise 22 focuses on the difference between asymmetry and antisymmetry.

18. Which relations in Exercise 3 are asymmetric?

19. Which relations in Exercise 4 are asymmetric?

20. Which relations in Exercise 5 are asymmetric?

21. Which relations in Exercise 6 are asymmetric?

22. Must an asymmetric relation also be antisymmetric? Must an antisymmetric relation be asymmetric? Give reasons for your answers.

23. Use quantifiers to express what it means for a relation to be asymmetric.

24. Give an example of an asymmetric relation on the set of all people.

25. How many different relations are there from a set with m elements to a set with n elements?

☞ Let R be a relation from a set A to a set B. The **inverse relation** from B to A, denoted by R^{-1}, is the set of ordered pairs $\{(b, a) \mid (a, b) \in R\}$. The **complementary relation** $\overline{R}$ is the set of ordered pairs $\{(a, b) \mid (a, b) \notin R\}$.

26. Let R be the relation $R = \{(a, b) \mid a < b\}$ on the set of integers. Find

a) R^{-1}. **b)** $\overline{R}$.

27. Let R be the relation $R = \{(a, b) \mid a \text{ divides } b\}$ on the set of positive integers. Find

a) R^{-1}. **b)** $\overline{R}$.

28. Let R be the relation on the set of all states in the United States consisting of pairs (a, b) where state a borders state b. Find

a) R^{-1}. **b)** $\overline{R}$.

29. Suppose that the function f from A to B is a one-to-one correspondence. Let R be the relation that equals the graph of f. That is, $R = \{(a, f(a)) \mid a \in A\}$. What is the inverse relation R^{-1}?

30. Let $R_1 = \{(1, 2), (2, 3), (3, 4)\}$ and $R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (3, 4)\}$ be relations from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$. Find

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$.

c) $R_1 - R_2$. **d)** $R_2 - R_1$.

31. Let A be the set of students at your school and B the set of books in the school library. Let R_1 and R_2 be the relations consisting of all ordered pairs (a, b), where student a is required to read book b in a course, and where student a has read book b, respectively. Describe the ordered pairs in each of these relations.

a) $R_1 \cup R_2$ **b)** $R_1 \cap R_2$

c) $R_1 \oplus R_2$ **d)** $R_1 - R_2$

e) $R_2 - R_1$

32. Let R be the relation $\{(1, 2), (1, 3), (2, 3), (2, 4), (3, 1)\}$, and let S be the relation $\{(2, 1), (3, 1), (3, 2), (4, 2)\}$. Find $S \circ R$.

33. Let R be the relation on the set of people consisting of pairs (a, b), where a is a parent of b. Let S be the relation on the set of people consisting of pairs (a, b), where a and b are siblings (brothers or sisters). What are $S \circ R$ and $R \circ S$?

Exercises 34–38 deal with these relations on the set of real numbers:

$R_1 = \{(a, b) \in \mathbf{R}^2 \mid a > b\}$, the greater than relation,

$R_2 = \{(a, b) \in \mathbf{R}^2 \mid a \geq b\}$, the greater than or equal to relation,

$R_3 = \{(a, b) \in \mathbf{R}^2 \mid a < b\}$, the less than relation,

$R_4 = \{(a, b) \in \mathbf{R}^2 \mid a \leq b\}$, the less than or equal to relation,

$R_5 = \{(a, b) \in \mathbf{R}^2 \mid a = b\}$, the equal to relation,

$R_6 = \{(a, b) \in \mathbf{R}^2 \mid a \neq b\}$, the unequal to relation.

34. Find

a) $R_1 \cup R_3$. **b)** $R_1 \cup R_5$.

c) $R_2 \cap R_4$. **d)** $R_3 \cap R_5$.

e) $R_1 - R_2$. **f)** $R_2 - R_1$.

g) $R_1 \oplus R_3$. **h)** $R_2 \oplus R_4$.

35. Find

a) $R_2 \cup R_4$. **b)** $R_3 \cup R_6$.

c) $R_3 \cap R_6$. **d)** $R_4 \cap R_6$.

e) $R_3 - R_6$. **f)** $R_6 - R_3$.

g) $R_2 \oplus R_6$. **h)** $R_3 \oplus R_5$.

36. Find

a) $R_1 \circ R_1$. **b)** $R_1 \circ R_2$.

c) $R_1 \circ R_3$. **d)** $R_1 \circ R_4$.

e) $R_1 \circ R_5$. **f)** $R_1 \circ R_6$.

g) $R_2 \circ R_3$. **h)** $R_3 \circ R_3$.

37. Find

a) $R_2 \circ R_1$. **b)** $R_2 \circ R_2$.

c) $R_3 \circ R_5$. **d)** $R_4 \circ R_1$.

e) $R_5 \circ R_3$. **f)** $R_3 \circ R_6$.

g) $R_4 \circ R_6$. **h)** $R_6 \circ R_6$.

38. Find the relations R_i^2 for $i = 1, 2, 3, 4, 5, 6$.

39. Find the relations S_i^2 for $i = 1, 2, 3, 4, 5, 6$ where

$S_1 = \{(a, b) \in \mathbf{Z}^2 \mid a > b\}$, the greater than relation,
$S_2 = \{(a, b) \in \mathbf{Z}^2 \mid a \geq b\}$, the greater than or equal to relation,
$S_3 = \{(a, b) \in \mathbf{Z}^2 \mid a < b\}$, the less than relation,
$S_4 = \{(a, b) \in \mathbf{Z}^2 \mid a \leq b\}$, the less than or equal to relation,
$S_5 = \{(a, b) \in \mathbf{Z}^2 \mid a = b\}$, the equal to relation,
$S_6 = \{(a, b) \in \mathbf{Z}^2 \mid a \neq b\}$, the unequal to relation.

40. Let R be the parent relation on the set of all people (see Example 21). When is an ordered pair in the relation R^3?

41. Let R be the relation on the set of people with doctorates such that $(a, b) \in R$ if and only if a was the thesis advisor of b. When is an ordered pair (a, b) in R^2? When is an ordered pair (a, b) in R^n, when n is a positive integer? (Assume that every person with a doctorate has a thesis advisor.)

42. Let R_1 and R_2 be the "divides" and "is a multiple of" relations on the set of all positive integers, respectively. That is, $R_1 = \{(a, b) \mid a \text{ divides } b\}$ and $R_2 = \{(a, b) \mid a \text{ is a multiple of } b\}$. Find

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$.
c) $R_1 - R_2$. **d)** $R_2 - R_1$.
e) $R_1 \oplus R_2$.

43. Let R_1 and R_2 be the "congruent modulo 3" and the "congruent modulo 4" relations, respectively, on the set of integers. That is, $R_1 = \{(a, b) \mid a \equiv b \pmod 3\}$ and $R_2 = \{(a, b) \mid a \equiv b \pmod 4\}$. Find

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$.
c) $R_1 - R_2$. **d)** $R_2 - R_1$.
e) $R_1 \oplus R_2$.

44. List the 16 different relations on the set $\{0, 1\}$.

45. How many of the 16 different relations on $\{0, 1\}$ contain the pair $(0, 1)$?

46. Which of the 16 relations on $\{0, 1\}$, which you listed in Exercise 44, are

a) reflexive? **b)** irreflexive?
c) symmetric? **d)** antisymmetric?
e) asymmetric? **f)** transitive?

47. a) How many relations are there on the set $\{a, b, c, d\}$?
b) How many relations are there on the set $\{a, b, c, d\}$ that contain the pair (a, a)?

48. Let S be a set with n elements and let a and b be distinct elements of S. How many relations R are there on S such that

a) $(a, b) \in R$? **b)** $(a, b) \notin R$?
c) no ordered pair in R has a as its first element?
d) at least one ordered pair in R has a as its first element?
e) no ordered pair in R has a as its first element or b as its second element?
f) at least one ordered pair in R either has a as its first element or has b as its second element?

***49.** How many relations are there on a set with n elements that are

a) symmetric? **b)** antisymmetric?
c) asymmetric? **d)** irreflexive?
e) reflexive and symmetric?
f) neither reflexive nor irreflexive?

***50.** How many transitive relations are there on a set with n elements if

a) $n = 1$? **b)** $n = 2$? **c)** $n = 3$?

51. Find the error in the "proof" of the following "theorem."

"*Theorem*": Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.

"*Proof*": Let $a \in A$. Take an element $b \in A$ such that $(a, b) \in R$. Because R is symmetric, we also have $(b, a) \in R$. Now using the transitive property, we can conclude that $(a, a) \in R$ because $(a, b) \in R$ and $(b, a) \in R$.

52. Suppose that R and S are reflexive relations on a set A. Prove or disprove each of these statements.

a) $R \cup S$ is reflexive.
b) $R \cap S$ is reflexive.
c) $R \oplus S$ is irreflexive.
d) $R - S$ is irreflexive.
e) $S \circ R$ is reflexive.

53. Show that the relation R on a set A is symmetric if and only if $R = R^{-1}$, where R^{-1} is the inverse relation.

54. Show that the relation R on a set A is antisymmetric if and only if $R \cap R^{-1}$ is a subset of the diagonal relation $\Delta = \{(a, a) \mid a \in A\}$.

55. Show that the relation R on a set A is reflexive if and only if the inverse relation R^{-1} is reflexive.

56. Show that the relation R on a set A is reflexive if and only if the complementary relation $\overline{R}$ is irreflexive.

57. Let R be a relation that is reflexive and transitive. Prove that $R^n = R$ for all positive integers n.

58. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 3)$, $(2, 4)$, $(3, 1)$, $(3, 4)$, $(3, 5)$, $(4, 2)$, $(4, 5)$, $(5, 1)$, $(5, 2)$, and $(5, 4)$. Find

a) R^2. **b)** R^3. **c)** R^4. **d)** R^5.

59. Let R be a reflexive relation on a set A. Show that R^n is reflexive for all positive integers n.

***60.** Let R be a symmetric relation. Show that R^n is symmetric for all positive integers n.

61. Suppose that the relation R is irreflexive. Is R^2 necessarily irreflexive? Give a reason for your answer.

62. Derive a big-O estimate for the number of integer comparisons needed to count all transitive relations on a set with n elements using the brute force approach of checking every relation of this set for transitivity.

9.2 n-ary Relations and Their Applications

9.2.1 Introduction

Relationships among elements of more than two sets often arise. For instance, there is a relationship involving the name of a student, the student's major, and the student's grade point average. Similarly, there is a relationship involving the airline, flight number, starting point, destination, departure time, and arrival time of a flight. An example of such a relationship in mathematics involves three integers, where the first integer is larger than the second integer, which is larger than the third. Another example is the betweenness relationship involving points on a line, such that three points are related when the second point is between the first and the third.

We will study relationships among elements from more than two sets in this section. These relationships are called ***n*-ary relations**. These relations are used to represent computer databases. These representations help us answer queries about the information stored in databases, such as: Which flights land at O'Hare Airport between 3 A.M. and 4 A.M.? Which students at your school are sophomores majoring in mathematics or computer science and have greater than a 3.0 average? Which employees of a company have worked for the company less than 5 years and make more than $50,000?

9.2.2 n-ary Relations

We begin with the basic definition on which the theory of relational databases rests.

Definition 1 Let $A_1, A_2, \ldots, A_n$ be sets. An *n-ary relation* on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the *domains* of the relation, and n is called its *degree*.

EXAMPLE 1 Let R be the relation on $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ consisting of triples (a, b, c), where a, b, and c are integers with $a < b < c$. Then $(1, 2, 3) \in R$, but $(2, 4, 3) \notin R$. The degree of this relation is 3. Its domains are all equal to the set of natural numbers. ◀

EXAMPLE 2 Let R be the relation on $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$ consisting of all triples of integers (a, b, c) in which a, b, and c form an arithmetic progression. That is, $(a, b, c) \in R$ if and only if there is an integer k such that $b = a + k$ and $c = a + 2k$, or equivalently, such that $b - a = k$ and $c - b = k$. Note that $(1, 3, 5) \in R$ because $3 = 1 + 2$ and $5 = 1 + 2 \cdot 2$, but $(2, 5, 9) \notin R$ because $5 - 2 = 3$ while $9 - 5 = 4$. This relation has degree 3 and its domains are all equal to the set of integers. ◀

EXAMPLE 3 Let R be the relation on $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}^+$ consisting of triples (a, b, m), where a, b, and m are integers with $m \geq 1$ and $a \equiv b \pmod{m}$. Then $(8, 2, 3)$, $(-1, 9, 5)$, and $(14, 0, 7)$ all belong to R, but $(7, 2, 3)$, $(-2, -8, 5)$, and $(11, 0, 6)$ do not belong to R because $8 \equiv 2 \pmod{3}$, $-1 \equiv 9 \pmod{5}$, and $14 \equiv 0 \pmod{7}$, but $7 \not\equiv 2 \pmod{3}$, $-2 \not\equiv -8 \pmod{5}$, and $11 \not\equiv 0 \pmod{6}$. This relation has degree 3 and its first two domains are the set of all integers and its third domain is the set of positive integers. ◀

EXAMPLE 4 Let R be the relation consisting of 5-tuples (A, N, S, D, T) representing airplane flights, where A is the airline, N is the flight number, S is the starting point, D is the destination, and T is the departure time. For instance, if Nadir Express Airlines has flight 963 from Newark to Bangor at 15:00, then (Nadir, 963, Newark, Bangor, 15:00) belongs to R. The degree of this relation is 5, and its domains are the set of all airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times. ◀

9.2.3 Databases and Relations

Links

The time required to manipulate information in a database depends on how this information is stored. The operations of adding and deleting records, updating records, searching for records, and combining records from overlapping databases are performed millions of times each day in a large database. Because of the importance of these operations, various methods for representing databases have been developed. We will discuss one of these methods, called the **relational data model**, based on the concept of a relation.

A database consists of **records**, which are n-tuples, made up of **fields**. The fields are the entries of the n-tuples. For instance, a database of student records may be made up of fields containing the name, student number, major, and grade point average of the student. The relational data model represents a database of records as an n-ary relation. Thus, student records are represented as 4-tuples of the form (*Student_name, ID_number, Major, GPA*). A sample database of six such records is

(Ackermann, 231455, Computer Science, 3.88)
(Adams, 888323, Physics, 3.45)
(Chou, 102147, Computer Science, 3.49)
(Goodfriend, 453876, Mathematics, 3.45)
(Rao, 678543, Mathematics, 3.90)
(Stevens, 786576, Psychology, 2.99).

Relations used to represent databases are also called **tables**, because these relations are often displayed as tables. Each column of the table corresponds to an *attribute* of the database. For instance, the same database of students is displayed in Table 1. The attributes of this database are Student Name, ID Number, Major, and GPA.

A domain of an n-ary relation is called a **primary key** when the value of the n-tuple from this domain determines the n-tuple. That is, a domain is a primary key when no two n-tuples in the relation have the same value from this domain.

Records are often added to or deleted from databases. Because of this, the property that a domain is a primary key is time-dependent. Consequently, a primary key should be chosen that remains one whenever the database is changed. The current collection of n-tuples in a relation is called the **extension** of the relation. The more permanent part of a database, including the name and attributes of the database, is called its **intension**. When selecting a primary key, the goal should be to select a key that can serve as a primary key for all possible extensions of the database. To do this, it is necessary to examine the intension of the database to understand the set of possible n-tuples that can occur in an extension.

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

EXAMPLE 5 Which domains are primary keys for the n-ary relation displayed in Table 1, assuming that no n-tuples will be added in the future?

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Solution: Because there is only one 4-tuple in this table for each student name, the domain of student names is a primary key. Similarly, the ID numbers in this table are unique, so the domain of ID numbers is also a primary key. However, the domain of major fields of study is not a primary key, because more than one 4-tuple contains the same major field of study. The domain of grade point averages is also not a primary key, because there are two 4-tuples containing the same GPA. ◀

Combinations of domains can also uniquely identify n-tuples in an n-ary relation. When the values of a set of domains determine an n-tuple in a relation, the Cartesian product of these domains is called a **composite key**.

EXAMPLE 6 Is the Cartesian product of the domain of major fields of study and the domain of GPAs a composite key for the n-ary relation from Table 1, assuming that no n-tuples are ever added?

Solution: Because no two 4-tuples from this table have both the same major and the same GPA, this Cartesian product is a composite key. ◀

Because primary and composite keys are used to identify records uniquely in a database, it is important that keys remain valid when new records are added to the database. Hence, checks should be made to ensure that every new record has values that are different in the appropriate field, or fields, from all other records in this table. For instance, it makes sense to use the student identification number as a key for student records if no two students ever have the same student identification number. A university should not use the name field as a key, because two students may have the same name (such as John Smith).

9.2.4 Operations on n-ary Relations

There are a variety of operations on n-ary relations that can be used to form new n-ary relations. Applied together, these operations can answer queries on databases that ask for all n-tuples that satisfy certain conditions.

The most basic operation on an n-ary relation is determining all n-tuples in the n-ary relation that satisfy certain conditions. For example, we may want to find all the records of all computer science majors in a database of student records. We may want to find all students who have a grade point average above 3.5. We may want to find the records of all computer science majors who have a grade point average above 3.5. To perform such tasks we use the selection operator.

Definition 2 Let R be an n-ary relation and C a condition that elements in R may satisfy. Then the *selection operator* s_C maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy the condition C.

EXAMPLE 7 To find the records of computer science majors in the n-ary relation R shown in Table 1, we use the operator s_{C_1}, where C_1 is the condition Major = "Computer Science." The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Chou, 102147, Computer Science, 3.49). Similarly, to find the records of students who have a grade point average above 3.5 in this database, we use the operator s_{C_2}, where C_2 is the condition GPA > 3.5. The result is the two 4-tuples (Ackermann, 231455, Computer Science, 3.88) and (Rao, 678543, Mathematics,

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3.90). Finally, to find the records of computer science majors who have a GPA above 3.5, we use the operator s_{C_3}, where C_3 is the condition (Major = "Computer Science" $\wedge$ GPA > 3.5). The result consists of the single 4-tuple (Ackermann, 231455, Computer Science, 3.88). ◀

Projections are used to form new n-ary relations by deleting the same fields in every record of the relation.

Definition 3 The *projection* $P_{i_1 i_2,\ldots,i_m}$ where $i_1 < i_2 < \cdots < i_m$, maps the n-tuple $(a_1, a_2, \ldots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$, where $m \leq n$.

In other words, the projection $P_{i_1,i_2,\ldots,i_m}$ deletes $n - m$ of the components of an n-tuple, leaving the i_1th, i_2th, …, and i_mth components.

EXAMPLE 8 What results when the projection $P_{1,3}$ is applied to the 4-tuples (2, 3, 0, 4), (Jane Doe, 234111001, Geography, 3.14), and (a_1, a_2, a_3, a_4)?

Solution: The projection $P_{1,3}$ sends these 4-tuples to (2, 0), (Jane Doe, Geography), and (a_1, a_3), respectively. ◀

Example 9 illustrates how new relations are produced using projections.

EXAMPLE 9 What relation results when the projection $P_{1,4}$ is applied to the relation in Table 1?

Solution: When the projection $P_{1,4}$ is used, the second and third columns of the table are deleted, and pairs representing student names and grade point averages are obtained. Table 2 displays the results of this projection. ◀

Fewer rows may result when a projection is applied to the table for a relation. This happens when some of the n-tuples in the relation have identical values in each of the m components of the projection, and only disagree in components deleted by the projection. For instance, consider the following example.

EXAMPLE 10 What is the table obtained when the projection $P_{1,2}$ is applied to the relation in Table 3?

Solution: Table 4 displays the relation obtained when $P_{1,2}$ is applied to Table 3. Note that there are fewer rows after this projection is applied. ◀

TABLE 2 GPAs.

Student_name	*GPA*
Ackermann	3.88
Adams	3.45
Chou	3.49
Goodfriend	3.45
Rao	3.90
Stevens	2.99

TABLE 3 Enrollments.

Student	*Major*	*Course*
Glauser	Biology	BI 290
Glauser	Biology	MS 475
Glauser	Biology	PY 410
Marcus	Mathematics	MS 511
Marcus	Mathematics	MS 603
Marcus	Mathematics	CS 322
Miller	Computer Science	MS 575
Miller	Computer Science	CS 455

TABLE 4 Majors.

Student	*Major*
Glauser	Biology
Marcus	Mathematics
Miller	Computer Science

TABLE 5 Teaching_assignments.

Professor	*Department*	*Course_ number*
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE 6 Class_schedule.

Department	*Course_ number*	*Room*	*Time*
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A.M.
Zoology	335	A100	9:00 A.M.
Zoology	412	A100	8:00 A.M.

The **join** operation is used to combine two tables into one when these tables share some identical fields. For instance, a table containing fields for airline, flight number, and gate, and another table containing fields for flight number, gate, and departure time can be combined into a table containing fields for airline, flight number, gate, and departure time.

Definition 4

Let R be a relation of degree m and S a relation of degree n. The *join* $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$-tuples $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where the m-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S.

In other words, the join operator J_p produces a new relation from two relations by combining all m-tuples of the first relation with all n-tuples of the second relation, where the last p components of the m-tuples agree with the first p components of the n-tuples.

EXAMPLE 11 What relation results when the join operator J_2 is used to combine the relation displayed in Tables 5 and 6?

Solution: The join J_2 produces the relation shown in Table 7. ◀

There are other operators besides projections and joins that produce new relations from existing relations. A description of these operations can be found in books on database theory.

TABLE 7 Teaching_schedule.

Professor	*Department*	*Course_number*	*Room*	*Time*
Cruz	Zoology	335	A100	9:00 A.M.
Cruz	Zoology	412	A100	8:00 A.M.
Farber	Psychology	501	A100	3:00 P.M.
Farber	Psychology	617	A110	11:00 A.M.
Grammer	Physics	544	B505	4:00 P.M.
Rosen	Computer Science	518	N521	2:00 P.M.
Rosen	Mathematics	575	N502	3:00 P.M.

TABLE 8 Flights.

Airline	*Flight_number*	*Gate*	*Destination*	*Departure_time*
Nadir	122	34	Detroit	08:10
Acme	221	22	Denver	08:17
Acme	122	33	Anchorage	08:22
Acme	323	34	Honolulu	08:30
Nadir	199	13	Detroit	08:47
Acme	222	22	Denver	09:10
Nadir	322	34	Detroit	09:44

9.2.5 SQL

Links

The database query language SQL (short for Structured Query Language) can be used to carry out the operations we have described in this section. Example 12 illustrates how SQL commands are related to operations on n-ary relations.

EXAMPLE 12 We will illustrate how SQL is used to express queries by showing how SQL can be employed to make a query about airline flights using Table 8. The SQL statement

```
SELECT Departure_time
FROM Flights
WHERE Destination='Detroit'
```

is used to find the projection P_5 (on the Departure_time attribute) of the selection of 5-tuples in the Flights database that satisfy the condition: Destination = 'Detroit'. The output would be a list containing the times of flights that have Detroit as their destination, namely, 08:10, 08:47, and 09:44. SQL uses the FROM clause to identify the n-ary relation the query is applied to, the WHERE clause to specify the condition of the selection operation, and the SELECT clause to specify the projection operation that is to be applied. (*Beware:* SQL uses SELECT to represent a projection, rather than a selection operation. This is an unfortunate example of conflicting terminology.) ◀

Example 13 shows how SQL queries can be made involving more than one table.

EXAMPLE 13 The SQL statement

```
SELECT Professor, Time
FROM Teaching_assignments, Class_schedule
WHERE Department='Mathematics'
```

is used to find the projection $P_{1,5}$ of the 5-tuples in the database (shown in Table 7), which is the join J_2 of the Teaching_assignments and Class_schedule databases in Tables 5 and 6, respectively, which satisfy the condition: Department = Mathematics. The output would consist of the single 2-tuple (Rosen, 3:00 P.M.). The SQL FROM clause is used here to find the join of two different databases. ◀

We have only touched on the basic concepts of relational databases in this section. More information can be found in [AhUl95].

9.2.6 Association Rules from Data Mining

Links

We will now introduce concepts from **data mining**, the discipline with the goal of gaining useful information from data. In particular, we will discuss information that can be gleaned from databases of transactions. We will focus on supermarket transactions, but the ideas we present are relevant in a wide range of applications.

By a **transaction** we mean a set of items bought by a customer during a visit to the store, such as {milk, eggs, bread} or {orange juice, bananas, yogurt, cream}. Stores collect large databases of transactions that can be used to help them manage their businesses. We will discuss how these databases can be used to address the question: How likely is it that a customer buys a product given that they also buy a collection of one or more specified products?

We call each product in the store an **item**. A collection of items is known as an **itemset**. A ***k*-itemset** is an itemset that contains exactly k items. The terms **transaction** and **basket** are used synonymously with the word itemset. When a store has n items, $a_1, a_2, \ldots, a_n$, for sale, each transaction can be represented by an n-tuple $b_1, b_2, \ldots, b_n$, where b_i is a binary variable that tells us whether a_i occurs in this transaction. That is, $b_i = 1$ if a_i is in this transaction and $b_i = 0$ otherwise. (Note that we only care whether an item occurs in a transaction and not how many times it occurs.) We can represent a transaction by an $(n+1)$-tuple of the form (*transaction number*, $b_1, b_2, \ldots, b_n$). The collection of all these $(n+1)$-tuples forms a database of transactions.

We now define additional terms used in the study of questions relating to the purchase of particular itemsets.

Definition 5

The *count* of an itemset I, denoted by $\sigma(I)$, in a set of transactions $T = \{t_1, t_2, \ldots, t_k\}$, where k is a positive integer, is the number of transactions that contain this itemset. That is,

$$\sigma(I) = |\{t_i \in T \mid I \subseteq t_i\}|.$$

The *support* of an itemset I is the probability that I is included in a randomly selected transaction from T. That is,

$$\text{support}(I) = \frac{\sigma(I)}{|T|}.$$

The **support threshold** s is specified for a particular application. **Frequent itemset mining** is the process of finding itemsets I with support greater than or equal to s. Such itemsets are said to be **frequent**.

EXAMPLE 14

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The morning transactions at a market stand that sells apples, pears, cider, donuts, and mangos are shown in Table 9. In Table 10 we display the corresponding binary database, where each record is an $(n+1)$-tuple consisting of the transaction number followed by binary entries that represent this itemset. Because apples occurs in five of the eight transactions, we see that $\sigma(\{\text{apples}\}) = 5$ and $\text{support}(\{\text{apples}\}) = 5/8$. Similarly, because the itemset {apples, cider} is a subset of four of the eight transactions, we have $\sigma(\{\text{apples, cider}\}) = 4$ and $\text{support}(\{\text{cider}\}) = 4/8 = 1/2$.

If we set the support threshold to be 0.5, an item is frequent if it occurs in at least four of the eight transactions. Consequently, with this support threshold, apples, pears, donuts and cider are the frequent items. The itemset {apples, cider} is a frequent itemset, but the itemset {donuts pears} is not a frequent itemset. ◀

TABLE 9 A Set of Transactions.

Transaction Number	*Items*
1	{apples, pears, mangos}
2	{pears, cider}
3	{apples, cider, donuts, mangos}
4	{apples, pears, cider, donuts}
5	{apples, cider, donuts}
6	{pears, cider, donuts}
7	{pears, donuts}
8	{apples, pears, cider}

TABLE 10 Binary Database for the Transactions in Table 9.

Transaction Number	*Apples*	*Pears*	*Cider*	*Donuts*	*Mangos*
1	1	1	0	0	1
2	0	1	1	0	0
3	1	0	1	1	1
4	1	1	1	1	0
5	1	0	1	1	0
6	0	1	1	1	0
7	0	1	0	1	0
8	1	1	1	0	0

We can use sets of transactions to help us predict whether a customer will buy a particular item given that they also buy all the items in an itemset (which might just be one item). Before we address a question of this type, we introduce some terminology. If S is a set of items and T is a set of transactions, an **association rule** is an implication of the form $I \to J$, where I and J are disjoint subsets of S. Although this notation borrows the notation for logical implications, its meaning is not entirely analagous. The association rule $I \to J$ is not the statement that whenever I is a subset of a transaction, then J must also be one. Instead, the strength of an association rule is measured in terms of its **support**, the frequency of transactions that contain both I and J, and its **confidence**, the frequency of transactions that contain J when they also contain I.

Definition 6 If I and J are subsets of a set T of transactions, then

$$\text{support}(I \to J) = \frac{\sigma(I \cup J)}{|T|}$$

and

$$\text{confidence}(I \to J) = \frac{\sigma(I \cup J)}{\sigma(I)}.$$

The support of the association rule $I \to J$, the fraction of transactions that contain both I and J, is a useful measure because a low support value tells us that the basket containing all items in

I and all items in J is seldom purchased, whereas a high value tells us that they are purchased together in a large fraction of transactions. Note that the confidence of the association rule $I \to J$ is the conditional probability that a transaction will contain all the items in I and in J given that it contains all the items in I. So, the larger the confidence of $I \to J$, the more likely it is for J to be a subset of a transaction that contains I.

EXAMPLE 15 What are the support and the confidence of the association rule {cider, donuts} → {apples} for the set of transactions in Example 14?

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Solution: The support of this association rule is $\sigma(\{\text{cider, donuts}\} \cup \{\text{apples}\})/|T|$. Because $\sigma(\{\text{cider, donuts}\} \cup \{\text{apples}\}) = \sigma(\{\text{cider, donuts, apples}\}) = 3$ and $|T| = 8$, we see that the support of this rule is $3/8 = 0.375$.

The confidence of this rule is $\sigma(\{\text{cider, donuts}\} \cup \{\text{apples}\})/\sigma(\{\text{cider, donuts}\}) = 3/4 = 0.75$. ◀

An important problem in data mining is to find **strong association rules**, which have support greater than or equal to a minimum support level and confidence greater than or equal to a minimum confidence level. It is important to have efficient algorithms to find strong association rules because the number of available items can be extremely large. For instance, a supermarket may have tens of thousands, or even hundreds of thousands, of items in stock. The brute-force approach of finding association rules with sufficiently large support and confidence by computing the support and confidence of all possible association rules is infeasible because there are an exponential number of such association rules (see Exercise 41). Several widely used algorithms have been developed to solve this problem much more efficiently than brute force. Such algorithms first find frequent itemsets and then turn their attention to finding all the association rules with high confidence from the frequent itemsets that have been found. Consult data mining texts such as [Ag15] for details.

Although we have presented association rules in the context of market baskets, they are useful in a surprisingly wide variety of applications. For instance, association rules can be used to improve medical diagnoses, in which itemsets are collections of test results or symptoms and transactions are the collections of test results and symptoms found on patient records. Association rules, in which itemsets are baskets of key words and transactions are the collections of words on web pages, are used by search engines. Cases of plagiarism can be found using association rules, in which itemsets are collections of sentences and transactions are the contents of documents. Association rules also play helpful roles in various aspects of computer security, including intrusion detection, in which the itemsets are collections of patterns and transactions are the strings transmitted during network attacks. The interested reader will be able to find many more such applications by searching the web.

Exercises

1. List the triples in the relation $\{(a, b, c) \mid a, b,$ and c are integers with $0 < a < b < c < 5\}$.

2. Which 4-tuples are in the relation $\{(a, b, c, d) \mid a, b, c,$ and d are positive integers with $abcd = 6\}$?

3. List the 5-tuples in the relation in Table 8.

4. Assuming that no new n-tuples are added, find all the primary keys for the relations displayed in

a) Table 3. **b)** Table 5.

c) Table 6. **d)** Table 8.

5. Assuming that no new n-tuples are added, find a composite key with two fields containing the *Airline* field for the database in Table 8.

6. Assuming that no new n-tuples are added, find a composite key with two fields containing the Professor field for the database in Table 7.

7. The 3-tuples in a 3-ary relation represent the following attributes of a student database: student ID number, name, phone number.

a) Is student ID number likely to be a primary key?
b) Is name likely to be a primary key?
c) Is phone number likely to be a primary key?

8. The 4-tuples in a 4-ary relation represent these attributes of published books: title, ISBN, publication date, number of pages.

a) What is a likely primary key for this relation?
b) Under what conditions would (title, publication date) be a composite key?
c) Under what conditions would (title, number of pages) be a composite key?

9. The 5-tuples in a 5-ary relation represent these attributes of all people in the United States: name, Social Security number, street address, city, state.

a) Determine a primary key for this relation.
b) Under what conditions would (name, street address) be a composite key?
c) Under what conditions would (name, street address, city) be a composite key?

10. What do you obtain when you apply the selection operator s_C, where C is the condition Room = A100, to the database in Table 7?

11. What do you obtain when you apply the selection operator s_C, where C is the condition Destination = Detroit, to the database in Table 8?

12. What do you obtain when you apply the selection operator s_C, where C is the condition (Project = 2) $\wedge$ (Quantity $\geq$ 50), to the database in Table 10?

13. What do you obtain when you apply the selection operator s_C, where C is the condition (Airline = Nadir) $\vee$ (Destination = Denver), to the database in Table 8?

14. What do you obtain when you apply the projection $P_{2,3,5}$ to the 5-tuple (a, b, c, d, e)?

15. Which projection mapping is used to delete the first, second, and fourth components of a 6-tuple?

16. Display the table produced by applying the projection $P_{1,2,4}$ to Table 8.

17. Display the table produced by applying the projection $P_{1,4}$ to Table 8.

18. How many components are there in the n-tuples in the table obtained by applying the join operator J_3 to two tables with 5-tuples and 8-tuples, respectively?

19. Construct the table obtained by applying the join operator J_2 to the relations in Tables 11 and 12.

20. Show that if C_1 and C_2 are conditions that elements of the n-ary relation R may satisfy, then $s_{C_1 \wedge C_2}(R) = s_{C_1}(s_{C_2}(R))$.

21. Show that if C_1 and C_2 are conditions that elements of the n-ary relation R may satisfy, then $s_{C_1}(s_{C_2}(R)) = s_{C_2}(s_{C_1}(R))$.

22. Show that if C is a condition that elements of the n-ary relations R and S may satisfy, then $s_C(R \cup S) = s_C(R) \cup s_C(S)$.

23. Show that if C is a condition that elements of the n-ary relations R and S may satisfy, then $s_C(R \cap S) = s_C(R) \cap s_C(S)$.

24. Show that if C is a condition that elements of the n-ary relations R and S may satisfy, then $s_C(R - S) = s_C(R) - s_C(S)$.

25. Show that if R and S are both n-ary relations, then $P_{i_1,i_2,\ldots,i_m}(R \cup S) = P_{i_1,i_2,\ldots,i_m}(R) \cup P_{i_1,i_2,\ldots,i_m}(S)$.

26. Give an example to show that if R and S are both n-ary relations, then $P_{i_1,i_2,\ldots,i_m}(R \cap S)$ may be different from $P_{i_1,i_2,\ldots,i_m}(R) \cap P_{i_1,i_2,\ldots,i_m}(S)$.

27. Give an example to show that if R and S are both n-ary relations, then $P_{i_1,i_2,\ldots,i_m}(R - S)$ may be different from $P_{i_1,i_2,\ldots,i_m}(R) - P_{i_1,i_2,\ldots,i_m}(S)$.

28. **a)** What are the operations that correspond to the query expressed using this SQL statement?

```
SELECT Supplier
FROM Part_needs
WHERE 1000 ≤ Part_number ≤ 5000
```

b) What is the output of this query given the database in Table 11 as input?

29. **a)** What are the operations that correspond to the query expressed using this SQL statement?

```
SELECT Supplier, Project
FROM Part_needs, Parts_inventory
WHERE Quantity ≤ 10
```

b) What is the output of this query given the databases in Tables 11 and 12 as input?

30. Determine whether there is a primary key for the relation in Example 2.

31. Determine whether there is a primary key for the relation in Example 3.

32. Show that an n-ary relation with a primary key can be thought of as the graph of a function that maps values of the primary key to $(n - 1)$-tuples formed from values of the other domains.

33. Suppose that the transactions at a convenience store during an evening are {bread, milk, diapers, juice}, {bread, milk, diapers, eggs}, {milk, diapers, beer, eggs}, {bread, beer}, {milk, diapers, eggs, juice}, and {milk, diapers, beer}.

a) Find the count and support of diapers.
b) Find all frequent itemsets if the threshold level is 0.6.

34. Suppose that the key words on eight different web pages are {evolution, primate, Human, Neanderthal, DNA, fossil}, {evolution, Neanderthal, Denisovan, Human, DNA}, {cave, fossil, primate}, {Human, Neanderthal, Denisovan, evolution}, {DNA, genome, evolution, fossil}, {DNA, Human, Neanderthal, Denisovan, genome}, {evolution, primate, cave, fossil}, and {Human, Neanderthal, genome}.

a) Find the count and support of Neanderthal.
b) Find all frequent itemsets if the threshold level is 0.6.

35. Find the support and confidence of the association rule {beer} → {diapers} for the set of transactions in Exercise 33. (This association rule has played an important role in the development of the subject.)

36. Find the support and confidence of the association rule {human, DNA} → {Neanderthal} for the set of transactions in Exercise 34.

TABLE 11 Part_needs.

Supplier	*Part_number*	*Project*
23	1092	1
23	1101	3
23	9048	4
31	4975	3
31	3477	2
32	6984	4
32	9191	2
33	1001	1

TABLE 12 Parts_inventory.

Part_number	*Project*	*Quantity*	*Color_code*
1001	1	14	8
1092	1	2	2
1101	3	1	1
3477	2	25	2
4975	3	6	2
6984	4	10	1
9048	4	12	2
9191	2	80	4

37. Suppose that I is an itemset with positive count in a set of transactions. Find the confidence of the association rule $I \to \emptyset$.

38. Suppose that I, J, and K are itemsets. Show that the six association rules $\{I, J\} \to K$, $\{J, K\} \to I$, $\{I, K\} \to J$, $I \to \{J, K\}$, $J \to \{I, K\}$, and $K \to \{I, J\}$ all have the same support.

39. The **lift** of the association rule $I \to J$, where I and J are itemsets with positive support in a set of transactions, equals support$(I \cup J)$/(support(I)support(J)).

a) Show that the lift of $I \to J$, when support(I) and support(J) are both positive, equals 1 if and only if the occurrence of I in a transaction and the occurrence of J in a transaction are independent events.

b) Find the lift of the association rule $\{\text{beer}\} \to \{\text{diapers}\}$ for the set of transactions in Exercises 33.

c) Find the lift of the association rule $\{\text{evolution}\} \to \{\text{Neanderthals, Denisovans}\}$ for the set of transactions in Exercise 34.

40. Show that if an itemset is frequent in a set of transactions, then all its subsets are also frequent itemsets in this set of transactions.

41. Given n unique items, show that there are 3^n possible association rules of the form $I \to J$, where I and J are disjoint subsets of the set of all items. Be sure to allow the association rules where I or J, or both, are empty.

9.3 Representing Relations

9.3.1 Introduction

In this section, and in the remainder of this chapter, all relations we study will be binary relations. Because of this, in this section and in the rest of this chapter, the word relation will always refer to a binary relation. There are many ways to represent a relation between finite sets. As we have seen in Section 9.1, one way is to list its ordered pairs. Another way to represent a relation is to use a table, as we did in Example 3 in Section 9.1. In this section we will discuss two alternative methods for representing relations. One method uses zero–one matrices. The other method uses pictorial representations called directed graphs, which we will discuss later in this section.

Generally, matrices are appropriate for the representation of relations in computer programs. On the other hand, people often find the representation of relations using directed graphs useful for understanding the properties of these relations.

9.3.2 Representing Relations Using Matrices

A relation between finite sets can be represented using a zero–one matrix. Suppose that R is a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$. (Here the elements of the sets A and B have been listed in a particular, but arbitrary, order. Furthermore, when $A = B$ we use the same ordering for A and B.) The relation R can be represented by the matrix $\mathbf{M}_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

In other words, the zero–one matrix representing R has a 1 as its (i, j) entry when a_i is related to b_j, and a 0 in this position if a_i is not related to b_j. (Such a representation depends on the orderings used for A and B.)

The use of matrices to represent relations is illustrated in Examples 1–6.

EXAMPLE 1 Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R if $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$, and $b_1 = 1$ and $b_2 = 2$?

Solution: Because $R = \{(2, 1), (3, 1), (3, 2)\}$, the matrix for R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The 1s in $\mathbf{M}_R$ show that the pairs (2, 1), (3, 1), and (3, 2) belong to R. The 0s show that no other pairs belong to R. ◀

EXAMPLE 2 Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}?$$

Solution: Because R consists of those ordered pairs (a_i, b_j) with $m_{ij} = 1$, it follows that

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}.$$ ◀

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$$

FIGURE 1 The zero–one matrix for a reflexive relation. (Off diagonal elements can be 0 or 1.)

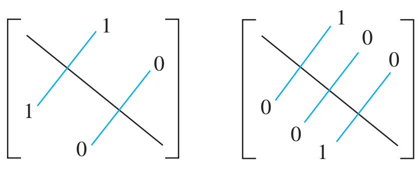
The matrix of a relation on a set, which is a square matrix, can be used to determine whether the relation has certain properties. Recall that a relation R on A is reflexive if $(a, a) \in R$ whenever $a \in A$. Thus, R is reflexive if and only if $(a_i, a_i) \in R$ for $i = 1, 2, \ldots, n$. Hence, R is reflexive if and only if $m_{ii} = 1$, for $i = 1, 2, \ldots, n$. In other words, R is reflexive if all the elements on the main diagonal of $\mathbf{M}_R$ are equal to 1, as shown in Figure 1. Note that the elements off the main diagonal can be either 0 or 1.

The relation R is symmetric if $(a, b) \in R$ implies that $(b, a) \in R$. Consequently, the relation R on the set $A = \{a_1, a_2, \ldots, a_n\}$ is symmetric if and only if $(a_j, a_i) \in R$ whenever $(a_i, a_j) \in R$. In terms of the entries of $\mathbf{M}_R$, R is symmetric if and only if $m_{ji} = 1$ whenever $m_{ij} = 1$. This also means $m_{ji} = 0$ whenever $m_{ij} = 0$. Consequently, R is symmetric if and only if $m_{ij} = m_{ji}$, for all pairs of integers i and j with $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Recalling the definition of the transpose of a matrix from Section 2.6, we see that R is symmetric if and only if

$$\mathbf{M}_R = (\mathbf{M}_R)^t,$$

that is, if $\mathbf{M}_R$ is a symmetric matrix. The form of the matrix for a symmetric relation is illustrated in Figure 2(a).

The relation R is antisymmetric if and only if $(a, b) \in R$ and $(b, a) \in R$ imply that $a = b$. Consequently, the matrix of an antisymmetric relation has the property that if $m_{ij} = 1$ with $i \neq j$, then $m_{ji} = 0$. Or, in other words, either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$. The form of the matrix for an antisymmetric relation is illustrated in Figure 2(b).



(a) Symmetric (b) Antisymmetric

FIGURE 2 The zero–one matrices for symmetric and antisymmetric relations.

EXAMPLE 3 Suppose that the relation R on a set is represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, because $\mathbf{M}_R$ is symmetric, it follows that R is symmetric. It is also easy to see that R is not antisymmetric. ◀

The Boolean operations join and meet (discussed in Section 2.6) can be used to find the matrices representing the union and the intersection of two relations. Suppose that R_1 and R_2 are relations on a set A represented by the matrices $\mathbf{M}_{R_1}$ and $\mathbf{M}_{R_2}$, respectively. The matrix representing the union of these relations has a 1 in the positions where either $\mathbf{M}_{R_1}$ or $\mathbf{M}_{R_2}$ has a 1. The matrix representing the intersection of these relations has a 1 in the positions where both $\mathbf{M}_{R_1}$ and $\mathbf{M}_{R_2}$ have a 1. Thus, the matrices representing the union and intersection of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \quad \text{and} \quad \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}.$$

EXAMPLE 4 Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Solution: The matrices of these relations are

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$\mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

◀

We now turn our attention to determining the matrix for the composite of relations. This matrix can be found using the Boolean product of the matrices (discussed in Section 2.6) for these relations. In particular, suppose that R is a relation from A to B and S is a relation from B to C. Suppose that A, B, and C have m, n, and p elements, respectively. Let the zero–one matrices for $S \circ R$, R, and S be $\mathbf{M}_{S \circ R} = [t_{ij}]$, $\mathbf{M}_R = [r_{ij}]$, and $\mathbf{M}_S = [s_{ij}]$, respectively (these matrices have sizes $m \times p$, $m \times n$, and $n \times p$, respectively). The ordered pair (a_i, c_j) belongs to $S \circ R$ if and only if there is an element b_k such that (a_i, b_k) belongs to R and (b_k, c_j) belongs to S. It follows that $t_{ij} = 1$ if and only if $r_{ik} = s_{kj} = 1$ for some k. From the definition of the Boolean product, this means that

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S.$$

EXAMPLE 5 Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Solution: The matrix for $S \circ R$ is

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

◀

The matrix representing the composite of two relations can be used to find the matrix for $\mathbf{M}_{R^n}$. In particular,

$$\mathbf{M}_{R^n} = \mathbf{M}_R^{[n]},$$

from the definition of Boolean powers. Exercise 35 asks for a proof of this formula.

EXAMPLE 6 Find the matrix representing the relation R^2, where the matrix representing R is

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Solution: The matrix for R^2 is

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

◀

9.3.3 Representing Relations Using Digraphs

We have shown that a relation can be represented by listing all of its ordered pairs or by using a zero–one matrix. There is another important way of representing a relation using a pictorial representation. Each element of the set is represented by a point, and each ordered pair is represented using an arc with its direction indicated by an arrow. We use such pictorial representations when we think of relations on a finite set as **directed graphs**, or **digraphs**.

Definition 1 A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge.

An edge of the form (a, a) is represented using an arc from the vertex a back to itself. Such an edge is called a **loop**.

EXAMPLE 7 The directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b), and (d, b) is displayed in Figure 3. ◀

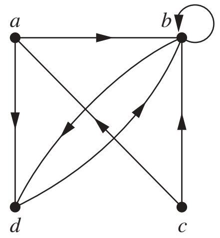


FIGURE 3
A directed graph.

The relation R on a set A is represented by the directed graph that has the elements of A as its vertices and the ordered pairs (a, b), where $(a, b) \in R$, as edges. This assignment sets up a one-to-one correspondence between the relations on a set A and the directed graphs with A as their set of vertices. Thus, every statement about relations corresponds to a statement about directed graphs, and vice versa. Directed graphs give a visual display of information about relations. As such, they are often used to study relations and their properties. (Note that relations from a set A to a set B can be represented by a directed graph where there is a vertex for each element of A and a vertex for each element of B, as shown in Section 9.1. However, when $A = B$, such representation provides much less insight than the digraph representations described here.) The use of directed graphs to represent relations on a set is illustrated in Examples 8–10.

EXAMPLE 8 The directed graph of the relation

$$R_1 = \{(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}$$

on the set $\{1, 2, 3, 4\}$ is shown in Figure 4. ◀

EXAMPLE 9 What are the ordered pairs in the relation R_2 represented by the directed graph shown in Figure 5?

Solution: The ordered pairs (x, y) in the relation are

$$R_2 = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3)\}.$$

Each of these pairs corresponds to an edge of the directed graph, with $(2, 2)$ and $(3, 3)$ corresponding to loops. ◀

We will study directed graphs extensively in Chapter 10.

The directed graph representing a relation can be used to determine whether the relation has various properties. For instance, a relation is reflexive if and only if there is a loop at every vertex of the directed graph, so that every ordered pair of the form (x, x) occurs in the relation. A relation is symmetric if and only if for every edge between distinct vertices in its digraph there is an edge in the opposite direction, so that (y, x) is in the relation whenever (x, y) is in the relation. Similarly, a relation is antisymmetric if and only if there are never two edges in opposite directions between distinct vertices. Finally, a relation is transitive if and only if whenever there is an edge from a vertex x to a vertex y and an edge from a vertex y to a vertex z, there is an edge from x to z (completing a triangle where each side is a directed edge with the correct direction).

Remark: Note that a symmetric relation can be represented by an undirected graph, which is a graph where edges do not have directions. We will study undirected graphs in Chapter 10.

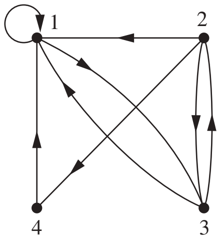


FIGURE 4 **The directed graph of the relation R_1.**

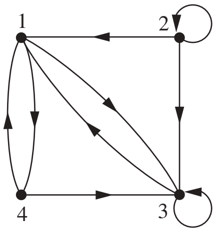
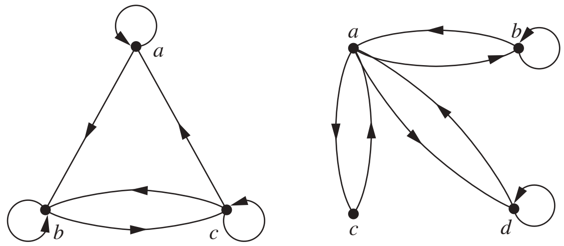


FIGURE 5 **The directed graph of the relation R_2.**



(a) Directed graph of S_1 (b) Directed graph of S_2

FIGURE 6 **The directed graphs of the relations S_1 and S_2.**

EXAMPLE 10 Determine whether the relations for the directed graphs shown in Figure 6 are reflexive, symmetric, antisymmetric, and/or transitive.

Solution: Because there are loops at every vertex of the directed graph of S_1, it is reflexive. The relation S_1 is neither symmetric nor antisymmetric because there is an edge from a to b but not one from b to a, but there are edges in both directions connecting b and c. Finally, S_1 is not transitive because there is an edge from a to b and an edge from b to c, but no edge from a to c.

Because loops are not present at all the vertices of the directed graph of S_2, this relation is not reflexive. It is symmetric and not antisymmetric, because every edge between distinct vertices is accompanied by an edge in the opposite direction. It is also not hard to see from the directed graph that S_2 is not transitive, because (c, a) and (a, b) belong to S_2, but (c, b) does not belong to S_2. ◀

Exercises

1. Represent each of these relations on $\{1, 2, 3\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1, 1), (1, 2), (1, 3)\}$
b) $\{(1, 2), (2, 1), (2, 2), (3, 3)\}$
c) $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$
d) $\{(1, 3), (3, 1)\}$

2. Represent each of these relations on $\{1, 2, 3, 4\}$ with a matrix (with the elements of this set listed in increasing order).

a) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
b) $\{(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)\}$
c) $\{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$
d) $\{(2, 4), (3, 1), (3, 2), (3, 4)\}$

3. List the ordered pairs in the relations on $\{1, 2, 3\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4. List the ordered pairs in the relations on $\{1, 2, 3, 4\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

5. How can the matrix representing a relation R on a set A be used to determine whether the relation is irreflexive?

6. How can the matrix representing a relation R on a set A be used to determine whether the relation is asymmetric?

7. Determine whether the relations represented by the matrices in Exercise 3 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

8. Determine whether the relations represented by the matrices in Exercise 4 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

9. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \ldots, 100\}$ consisting of the first 100 positive integers have if R is

a) $\{(a, b) \mid a > b\}$? **b)** $\{(a, b) \mid a \neq b\}$?
c) $\{(a, b) \mid a = b + 1\}$? **d)** $\{(a, b) \mid a = 1\}$?
e) $\{(a, b) \mid ab = 1\}$?

10. How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \ldots, 1000\}$ consisting of the first 1000 positive integers have if R is

a) $\{(a, b) \mid a \leq b\}$?
b) $\{(a, b) \mid a = b \pm 1\}$?
c) $\{(a, b) \mid a + b = 1000\}$?
d) $\{(a, b) \mid a + b \leq 1001\}$?
e) $\{(a, b) \mid a \neq 0\}$?

11. How can the matrix for $\overline{R}$, the complement of the relation R, be found from the matrix representing R, when R is a relation on a finite set A?

12. How can the matrix for R^{-1}, the inverse of the relation R, be found from the matrix representing R, when R is a relation on a finite set A?

13. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the matrix representing

a) R^{-1}. **b)** $\overline{R}$. **c)** R^2.

14. Let R_1 and R_2 be relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Find the matrices that represent

a) $R_1 \cup R_2$. **b)** $R_1 \cap R_2$. **c)** $R_2 \circ R_1$.
d) $R_1 \circ R_1$. **e)** $R_1 \oplus R_2$.

15. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find the matrices that represent

a) R^2. **b)** R^3. **c)** R^4.

16. Let R be a relation on a set A with n elements. If there are k nonzero entries in $\mathbf{M}_R$, the matrix representing R, how many nonzero entries are there in $\mathbf{M}_{R^{-1}}$, the matrix representing R^{-1}, the inverse of R?

17. Let R be a relation on a set A with n elements. If there are k nonzero entries in $\mathbf{M}_R$, the matrix representing R, how many nonzero entries are there in $\mathbf{M}_{\overline{R}}$, the matrix representing $\overline{R}$, the complement of R?

18. Draw the directed graphs representing each of the relations from Exercise 1.

19. Draw the directed graphs representing each of the relations from Exercise 2.

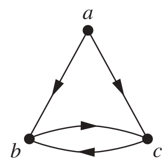
20. Draw the directed graph representing each of the relations from Exercise 3.

21. Draw the directed graph representing each of the relations from Exercise 4.

22. Draw the directed graph that represents the relation $\{(a, a), (a, b), (b, c), (c, b), (c, d), (d, a), (d, b)\}$.

In Exercises 23–28 list the ordered pairs in the relations represented by the directed graphs.

23.



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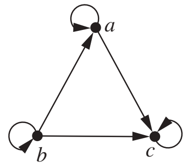

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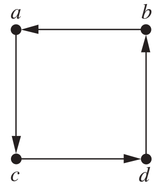

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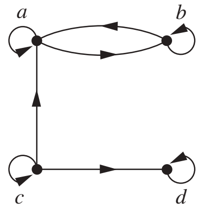

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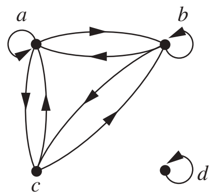
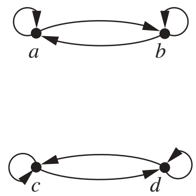


28.



29. How can the directed graph of a relation R on a finite set A be used to determine whether a relation is asymmetric?

30. How can the directed graph of a relation R on a finite set A be used to determine whether a relation is irreflexive?

31. Determine whether the relations represented by the directed graphs shown in Exercises 23–25 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

32. Determine whether the relations represented by the directed graphs shown in Exercises 26–28 are reflexive, irreflexive, symmetric, antisymmetric, asymmetric, and/or transitive.

33. Let R be a relation on a set A. Explain how to use the directed graph representing R to obtain the directed graph representing the inverse relation R^{-1}.

34. Let R be a relation on a set A. Explain how to use the directed graph representing R to obtain the directed graph representing the complementary relation $\overline{R}$.

35. Show that if $\mathbf{M}_R$ is the matrix representing the relation R, then $\mathbf{M}_R^{[n]}$ is the matrix representing the relation R^n.

36. Given the directed graphs representing two relations, how can the directed graph of the union, intersection, symmetric difference, difference, and composition of these relations be found?

9.4 Closures of Relations

9.4.1 Introduction

A computer network has data centers in Boston, Chicago, Denver, Detroit, New York, and San Diego. There are direct, one-way telephone lines from Boston to Chicago, from Boston to Detroit, from Chicago to Detroit, from Detroit to Denver, and from New York to San Diego. Let R be the relation containing (a, b) if there is a telephone line from the data center in a to that in b. How can we determine if there is some (possibly indirect) link composed of one or more telephone lines from one center to another? Because not all links are direct, such as the link from Boston to Denver that goes through Detroit, R cannot be used directly to answer this. In the language of relations, R is not transitive, so it does not contain all the pairs that can be linked. As we will show in this section, we can find all pairs of data centers that have a link by constructing a transitive relation S containing R such that S is a subset of every transitive relation containing R. Here, S is the smallest transitive relation that contains R. This relation is called the **transitive closure** of R.

9.4.2 Different Types of Closures

If R is a relation on a set A, it may or may not have some property $\mathbf{P}$, such as reflexivity, symmetry, or transitivity. When R does not enjoy property $\mathbf{P}$, we would like to find the smallest relation S on A with property $\mathbf{P}$ that contains R.

Definition 1 If R is a relation on a set A, then the **closure** of R with respect to $\mathbf{P}$, if it exists, is the relation S on A with property $\mathbf{P}$ that contains R and is a subset of every subset of $A \times A$ containing R with property $\mathbf{P}$.

If there is a relation S that is a subset of every relation containing R with property $\mathbf{P}$, it must be unique. To see this, suppose that relations S and T both have property $\mathbf{P}$ and are subsets of every relation with property $\mathbf{P}$ that contains R. Then, S and T are subsets of each other, and so are equal. Such a relation, if it exists, is the smallest relation with property $\mathbf{P}$ that contains R because it is a subset of every relation with property $\mathbf{P}$ that contains R.

We will show how reflexive, symmetric, and transitive closures of relations can be found. In Exercises 15 and 35 we give properties $\mathbf{P}$ for which the closure of a relation with respect to $\mathbf{P}$ may not exist.

The relation $R = \{(1, 1), (1, 2), (2, 1), (3, 2)\}$ on the set $A = \{1, 2, 3\}$ is not reflexive. How can we produce a reflexive relation containing R that is as small as possible? This can be done by adding $(2, 2)$ and $(3, 3)$ to R, because these are the only pairs of the form (a, a) that are not in R. This new relation contains R. Furthermore, *any* reflexive relation that contains R must also contain $(2, 2)$ and $(3, 3)$. Because this relation contains R, is reflexive, and is contained within every reflexive relation that contains R, it is called the **reflexive closure** of R.

As this example illustrates, given a relation R on a set A, the reflexive closure of R can be formed by adding to R all pairs of the form (a, a) with $a \in A$, not already in R. The addition of these pairs produces a new relation that is reflexive, contains R, and is contained within any reflexive relation containing R. We see that the reflexive closure of R equals $R \cup \Delta$, where $\Delta = \{(a, a) \mid a \in A\}$ is the **diagonal relation** on A. (The reader should verify this.)

EXAMPLE 1 What is the reflexive closure of the relation $R = \{(a, b) \mid a < b\}$ on the set of integers?

Solution: The reflexive closure of R is

$$R \cup \Delta = \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in \mathbf{Z}\} = \{(a, b) \mid a \leq b\}.$$

◀

The relation $\{(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 2)\}$ on $\{1, 2, 3\}$ is not symmetric. How can we produce a symmetric relation that is as small as possible and contains R? To do this, we need only add $(2, 1)$ and $(1, 3)$, because these are the only pairs of the form (b, a) with $(a, b) \in R$ that are not in R. This new relation is symmetric and contains R. Furthermore, *any* symmetric relation that contains R must contain this new relation, because a symmetric relation that contains R must contain $(2, 1)$ and $(1, 3)$. Consequently, this new relation is called the **symmetric closure** of R.

As this example illustrates, the symmetric closure of a relation R can be constructed by adding all ordered pairs of the form (b, a), where (a, b) is in the relation, that are not already present in R. Adding these pairs produces a relation that is symmetric, that contains R, and that is contained in any symmetric relation that contains R. The symmetric closure of a relation can be constructed by taking the union of a relation with its inverse (defined in the preamble of Exercise 26 in Section 9.1); that is, $R \cup R^{-1}$ is the symmetric closure of R, where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$. The reader should verify this statement.

EXAMPLE 2 What is the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers?

Extra Examples

Solution: The symmetric closure of R is the relation

$$R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}.$$

This last equality follows because R contains all ordered pairs of positive integers, where the first element is greater than the second element, and R^{-1} contains all ordered pairs of positive integers, where the first element is less than the second. ◀

Suppose that a relation R is not transitive. How can we produce a transitive relation that contains R such that this new relation is contained within any transitive relation that contains R? Can the transitive closure of a relation R be produced by adding all the pairs of the form (a, c), where (a, b) and (b, c) are already in the relation? Consider the relation $R = \{(1, 3), (1, 4), (2, 1), (3, 2)\}$ on the set $\{1, 2, 3, 4\}$. This relation is not transitive because it does not contain all pairs of the form (a, c) where (a, b) and (b, c) are in R. The pairs of this form not in R are $(1, 2)$, $(2, 3)$, $(2, 4)$, and $(3, 1)$. Adding these pairs does *not* produce a transitive relation, because the resulting relation contains $(3, 1)$ and $(1, 4)$ but does not contain $(3, 4)$. This shows that constructing the transitive closure of a relation is more complicated than constructing either the reflexive or symmetric closure. The rest of this section develops algorithms for constructing transitive closures. As will be shown later in this section, the transitive closure of a relation can be found by adding new ordered pairs that must be present and then repeating this process until no new ordered pairs are needed.

9.4.3 Paths in Directed Graphs

We will see that representing relations by directed graphs helps in the construction of transitive closures. We now introduce some terminology that we will use for this purpose.

A path in a directed graph is obtained by traversing along edges (in the same direction as indicated by the arrow on the edge).

Definition 2 A *path* from a to b in the directed graph G is a sequence of edges (x_0, x_1), (x_1, x_2), $(x_2, x_3), \ldots, (x_{n-1}, x_n)$ in G, where n is a nonnegative integer, and $x_0 = a$ and $x_n = b$, that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ and has *length* n. We view the empty set of edges as a path of length zero from a to a. A path of length $n \geq 1$ that begins and ends at the same vertex is called a *circuit* or *cycle*.

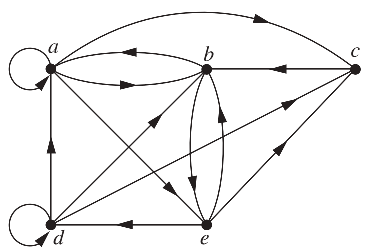


FIGURE 1 **A directed graph.**

A path in a directed graph can pass through a vertex more than once. Moreover, an edge in a directed graph can occur more than once in a path.

EXAMPLE 3 Which of the following are paths in the directed graph shown in Figure 1: a, b, e, d; a, e, c, d, b; b, a, c, b, a, a, b; d, c; c, b, a; e, b, a, b, a, b, e? What are the lengths of those that are paths? Which of the paths in this list are circuits?

Solution: Because each of (a, b), (b, e), and (e, d) is an edge, a, b, e, d is a path of length three. Because (c, d) is not an edge, a, e, c, d, b is not a path. Also, b, a, c, b, a, a, b is a path of length six because (b, a), (a, c), (c, b), (b, a), (a, a), and (a, b) are all edges. We see that d, c is a path of length one, because (d, c) is an edge. Also c, b, a is a path of length two, because (c, b) and (b, a) are edges. All of (e, b), (b, a), (a, b), (b, a), (a, b), and (b, e) are edges, so e, b, a, b, a, b, e is a path of length six.

The two paths b, a, c, b, a, a, b and e, b, a, b, a, b, e are circuits because they begin and end at the same vertex. The paths a, b, e, d; c, b, a; and d, c are not circuits. ◀

The term *path* also applies to relations. Carrying over the definition from directed graphs to relations, there is a **path** from a to b in R if there is a sequence of elements $a, x_1, x_2, \dots, x_{n-1}, b$ with $(a, x_1) \in R, (x_1, x_2) \in R, \dots,$ and $(x_{n-1}, b) \in R$. Theorem 1 can be obtained from the definition of a path in a relation.

THEOREM 1 Let R be a relation on a set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

Proof: We will use mathematical induction. By definition, there is a path from a to b of length one if and only if $(a, b) \in R$, so the theorem is true when $n = 1$.

Assume that the theorem is true for the positive integer n. This is the inductive hypothesis. There is a path of length $n + 1$ from a to b if and only if there is an element $c \in A$ such that there is a path of length one from a to c, so $(a, c) \in R$, and a path of length n from c to b, that is, $(c, b) \in R^n$. Consequently, by the inductive hypothesis, there is a path of length $n + 1$ from a to b if and only if there is an element c with $(a, c) \in R$ and $(c, b) \in R^n$. But there is such an element if and only if $(a, b) \in R^{n+1}$. Therefore, there is a path of length $n + 1$ from a to b if and only if $(a, b) \in R^{n+1}$. This completes the proof. ◁

9.4.4 Transitive Closures

We now show that finding the transitive closure of a relation is equivalent to determining which pairs of vertices in the associated directed graph are connected by a path. With this in mind, we define a new relation.

Definition 3 Let R be a relation on a set A. The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

Because R^n consists of the pairs (a, b) such that there is a path of length n from a to b, it follows that R^* is the union of all the sets R^n. In other words,

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

The connectivity relation is useful in many models.

EXAMPLE 4 Let R be the relation on the set of all people in the world that contains (a, b) if a has met b. What is R^n, where n is a positive integer greater than one? What is R^*?

Solution: The relation R^2 contains (a, b) if there is a person c such that $(a, c) \in R$ and $(c, b) \in R$, that is, if there is a person c such that a has met c and c has met b. Similarly, R^n consists of those pairs (a, b) such that there are people $x_1, x_2, \ldots, x_{n-1}$ such that a has met x_1, x_1 has met $x_2, \ldots,$ and x_{n-1} has met b.

The relation R^* contains (a, b) if there is a sequence of people, starting with a and ending with b, such that each person in the sequence has met the next person in the sequence. (There are many interesting conjectures about R^*. Do you think that this connectivity relation includes the pair with you as the first element and the president of Mongolia as the second element? We will use graphs to model this application in Chapter 10.) ◀

EXAMPLE 5 Let R be the relation on the set of all subway stops in New York City that contains (a, b) if it is possible to travel from stop a to stop b without changing trains. What is R^n when n is a positive integer? What is R^*?

Solution: The relation R^n contains (a, b) if it is possible to travel from stop a to stop b by making at most $n - 1$ changes of trains. The relation R^* consists of the ordered pairs (a, b) where it is possible to travel from stop a to stop b making as many changes of trains as necessary. (The reader should verify these statements.) ◀

EXAMPLE 6 Let R be the relation on the set of all states in the United States that contains (a, b) if state a and state b have a common border. What is R^n, where n is a positive integer? What is R^*?

Solution: The relation R^n consists of the pairs (a, b), where it is possible to go from state a to state b by crossing exactly n state borders. R^* consists of the ordered pairs (a, b), where it is possible to go from state a to state b crossing as many borders as necessary. (The reader should verify these statements.) The only ordered pairs not in R^* are those containing states that are not connected to the continental United States (that is, those pairs containing Alaska or Hawaii). ◀

Theorem 2 shows that the transitive closure of a relation and the associated connectivity relation are the same.

THEOREM 2 The transitive closure of a relation R equals the connectivity relation R^*.

Proof: Note that R^* contains R by definition. To show that R^* is the transitive closure of R we must also show that R^* is transitive and that $R^* \subseteq S$ whenever S is a transitive relation that contains R.

First, we show that R^* is transitive. If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from a to b and from b to c in R. We obtain a path from a to c by starting with the path from a to b and following it with the path from b to c. Hence, $(a, c) \in R^*$. It follows that R^* is transitive.

Now suppose that S is a transitive relation containing R. Because S is transitive, S^n also is transitive (the reader should verify this) and $S^n \subseteq S$ (by Theorem 1 of Section 9.1). Furthermore, because

$$S^* = \bigcup_{k=1}^{\infty} S^k$$

and $S^k \subseteq S$, it follows that $S^* \subseteq S$. Now note that if $R \subseteq S$, then $R^* \subseteq S^*$, because any path in R is also a path in S. Consequently, $R^* \subseteq S^* \subseteq S$. Thus, any transitive relation that contains R must also contain R^*. Therefore, R^* is the transitive closure of R. ◁

Now that we know that the transitive closure equals the connectivity relation, we turn our attention to the problem of computing this relation. We do not need to examine arbitrarily long paths to determine whether there is a path between two vertices in a finite directed graph. As Lemma 1 shows, it is sufficient to examine paths containing no more than n edges, where n is the number of elements in the set.

LEMMA 1 Let A be a set with n elements, and let R be a relation on A. If there is a path of length at least one in R from a to b, then there is such a path with length not exceeding n. Moreover, when $a \neq b$, if there is a path of length at least one in R from a to b, then there is such a path with length not exceeding $n - 1$.

Proof: Suppose there is a path from a to b in R. Let m be the length of the shortest such path. Suppose that $x_0, x_1, x_2, \ldots, x_{m-1}, x_m$, where $x_0 = a$ and $x_m = b$, is such a path.

Suppose that $a = b$ and that $m > n$, so that $m \geq n + 1$. By the pigeonhole principle, because there are n vertices in A, among the m vertices $x_0, x_1, \ldots, x_{m-1}$, at least two are equal (see Figure 2).

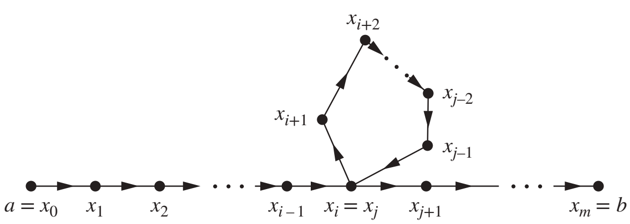


FIGURE 2 **Producing a path with length not exceeding *n*.**

Suppose that $x_i = x_j$ with $0 \le i < j \le m - 1$. Then the path contains a circuit from x_i to itself. This circuit can be deleted from the path from a to b, leaving a path, namely, $x_0, x_1, \dots, x_i, x_{j+1}, \dots, x_{m-1}, x_m$, from a to b of shorter length. Hence, the path of shortest length must have length less than or equal to n.

The case where $a \neq b$ is left as an exercise for the reader. ◁

From Lemma 1, we see that the transitive closure of R is the union of $R, R^2, R^3, \dots$, and R^n. This follows because there is a path in R^* between two vertices if and only if there is a path between these vertices in R^i, for some positive integer i with $i \le n$. Because

$$R^* = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$$

and the zero–one matrix representing a union of relations is the join of the zero–one matrices of these relations, the zero–one matrix for the transitive closure is the join of the zero–one matrices of the first n powers of the zero–one matrix of R.

THEOREM 3 Let $\mathbf{M}_R$ be the zero–one matrix of the relation R on a set with n elements. Then the zero–one matrix of the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

EXAMPLE 7 Find the zero–one matrix of the transitive closure of the relation R where

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: By Theorem 3, it follows that the zero–one matrix of R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}.$$

Because

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

it follows that

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

◀

Links

Theorem 3 can be used as a basis for an algorithm for computing the matrix of the relation R^*. To find this matrix, the successive Boolean powers of $\mathbf{M}_R$, up to the nth power, are computed. As each power is calculated, its join with the join of all smaller powers is formed. When this is done with the nth power, the matrix for R^* has been found. This procedure is displayed as Algorithm 1.

ALGORITHM 1 A Procedure for Computing the Transitive Closure.

procedure *transitive closure* ($\mathbf{M}_R$: zero–one $n \times n$ matrix)
$\mathbf{A} := \mathbf{M}_R$
$\mathbf{B} := \mathbf{A}$
for $i := 2$ **to** n
  $\mathbf{A} := \mathbf{A} \odot \mathbf{M}_R$
  $\mathbf{B} := \mathbf{B} \vee \mathbf{A}$
return $\mathbf{B}$\{$\mathbf{B}$ is the zero–one matrix for R^*\}

We can easily find the number of bit operations used by Algorithm 1 to determine the transitive closure of a relation. Computing the Boolean powers $\mathbf{M}_R, \mathbf{M}_R^{[2]}, \ldots, \mathbf{M}_R^{[n]}$ requires that $n-1$ Boolean products of $n \times n$ zero–one matrices be found. Each of these Boolean products can be found using $n^2(2n-1)$ bit operations. Hence, these products can be computed using $n^2(2n-1)(n-1)$ bit operations.

To find $\mathbf{M}_{R^*}$ from the n Boolean powers of $\mathbf{M}_R$, $n-1$ joins of zero–one matrices need to be found. Computing each of these joins uses n^2 bit operations. Hence, $(n-1)n^2$ bit operations are used in this part of the computation. Therefore, when Algorithm 1 is used, the matrix of the transitive closure of a relation on a set with n elements can be found using $n^2(2n-1)(n-1) + (n-1)n^2 = 2n^3(n-1)$, which is $O(n^4)$ bit operations. The remainder of this section describes a more efficient algorithm for finding transitive closures.

9.4.5 Warshall's Algorithm

Links

Warshall's algorithm, named after Stephen Warshall, who described this algorithm in 1960, is an efficient method for computing the transitive closure of a relation. Algorithm 1 can find the transitive closure of a relation on a set with n elements using $2n^3(n-1)$ bit operations. However, the transitive closure can be found by Warshall's algorithm using only $2n^3$ bit operations.

Remark: Warshall's algorithm is sometimes called the Roy–Warshall algorithm, because Bernard Roy described this algorithm in 1959.

Suppose that R is a relation on a set with n elements. Let $v_1, v_2, \ldots, v_n$ be an arbitrary listing of these n elements. The concept of the **interior vertices** of a path is used in Warshall's algorithm. If $a, x_1, x_2, \ldots, x_{m-1}, b$ is a path, its interior vertices are $x_1, x_2, \ldots, x_{m-1}$, that is, all the vertices of the path that occur somewhere other than as the first and last vertices in the path. For instance, the interior vertices of a path a, c, d, f, g, h, b, j in a directed graph are c, d, f, g, h, and b. The interior vertices of a, c, d, a, f, b are c, d, a, and f. (Note that the first vertex in the path is not an interior vertex unless it is visited again by the path, except as the last vertex. Similarly, the last vertex in the path is not an interior vertex unless it was visited previously by the path, except as the first vertex.)

Warshall's algorithm is based on the construction of a sequence of zero–one matrices. These matrices are $\mathbf{W}_0, \mathbf{W}_1, \ldots, \mathbf{W}_n$, where $\mathbf{W}_0 = \mathbf{M}_R$ is the zero–one matrix of this relation, and $\mathbf{W}_k = [w_{ij}^{(k)}]$, where $w_{ij}^{(k)} = 1$ if there is a path from v_i to v_j such that all the interior vertices of this path are in the set $\{v_1, v_2, \ldots, v_k\}$ (the first k vertices in the list) and is 0 otherwise. (The first and last vertices in the path may be outside the set of the first k vertices in the list.) Note that $\mathbf{W}_n = \mathbf{M}_{R^*}$, because the (i, j)th entry of $\mathbf{M}_{R^*}$ is 1 if and only if there is a path from v_i to v_j, with all interior vertices in the set $\{v_1, v_2, \dots, v_n\}$ (but these are the only vertices in the directed graph). Example 8 illustrates what the matrix $\mathbf{W}_k$ represents.

EXAMPLE 8 Let R be the relation with directed graph shown in Figure 3. Let a, b, c, d be a listing of the elements of the set. Find the matrices $\mathbf{W}_0$, $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, and $\mathbf{W}_4$. The matrix $\mathbf{W}_4$ is the transitive closure of R.

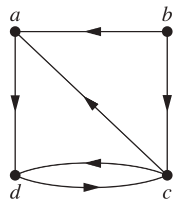


FIGURE 3 **The directed graph of the relation *R*.**

Solution: Let $v_1 = a$, $v_2 = b$, $v_3 = c$, and $v_4 = d$. $\mathbf{W}_0$ is the matrix of the relation. Hence,

$$\mathbf{W}_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$\mathbf{W}_1$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has only $v_1 = a$ as an interior vertex. Note that all paths of length one can still be used because they have no interior vertices. Also, there is now an allowable path from b to d, namely, b, a, d. Hence,

$$\mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$\mathbf{W}_2$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has only $v_1 = a$ and/or $v_2 = b$ as its interior vertices, if any. Because there are no edges that have b as a terminal vertex, no new paths are obtained when we permit b to be an interior vertex. Hence, $\mathbf{W}_2 = \mathbf{W}_1$.

$\mathbf{W}_3$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has only $v_1 = a$, $v_2 = b$, and/or $v_3 = c$ as its interior vertices, if any. We now have paths from d to a, namely, d, c, a, and from d to d, namely, d, c, d. Hence,

$$\mathbf{W}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Finally, $\mathbf{W}_4$ has 1 as its (i, j)th entry if there is a path from v_i to v_j that has $v_1 = a$, $v_2 = b$, $v_3 = c$, and/or $v_4 = d$ as interior vertices, if any. Because these are all the vertices of the graph, this entry is 1 if and only if there is a path from v_i to v_j. Hence,

$$\mathbf{W}_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

This last matrix, $\mathbf{W}_4$, is the matrix of the transitive closure. ◀

Warshall's algorithm computes $\mathbf{M}_{R^*}$ by efficiently computing $\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_n = \mathbf{M}_{R^*}$. This observation shows that we can compute $\mathbf{W}_k$ directly from $\mathbf{W}_{k-1}$: There is a path from v_i to v_j with no vertices other than $v_1, v_2, \dots, v_k$ as interior vertices if and only if either there is a path from v_i to v_j with its interior vertices among the first $k - 1$ vertices in the list, or there are paths from v_i to v_k and from v_k to v_j that have interior vertices only among the first $k - 1$ vertices in the list. That is, either a path from v_i to v_j already existed before v_k was permitted as an interior vertex, or allowing v_k as an interior vertex produces a path that goes from v_i to v_k and then from v_k to v_j. These two cases are shown in Figure 4.

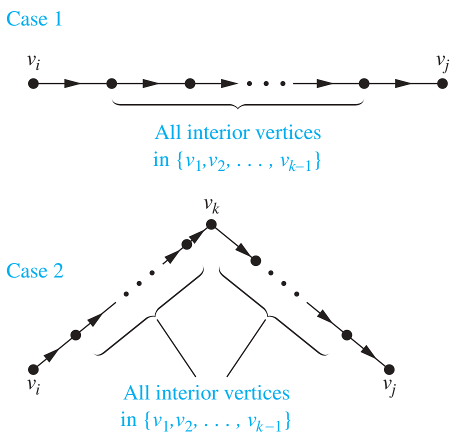


FIGURE 4 Adding v_k to the set of allowable interior vertices.

The first type of path exists if and only if $w_{ij}^{[k-1]} = 1$, and the second type of path exists if and only if both $w_{ik}^{[k-1]}$ and $w_{kj}^{[k-1]}$ are 1. Hence, $w_{ij}^{[k]}$ is 1 if and only if either $w_{ij}^{[k-1]}$ is 1 or both $w_{ik}^{[k-1]}$ and $w_{kj}^{[k-1]}$ are 1. This gives us Lemma 2.

LEMMA 2 Let $\mathbf{W}_k = [w_{ij}^{[k]}]$ be the zero–one matrix that has a 1 in its (i, j)th position if and only if there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \ldots, v_k\}$. Then

$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]}),$$

whenever i, j, and k are positive integers not exceeding n.

Lemma 2 gives us the means to compute efficiently the matrices $\mathbf{W}_k$, $k = 1, 2, \ldots, n$. We display the pseudocode for Warshall's algorithm, using Lemma 2, as Algorithm 2.

Links

Courtesy of Stephen Warshall

STEPHEN WARSHALL (1935–2006) Stephen Warshall, born in New York City, went to public school in Brooklyn. He attended Harvard University, receiving his degree in mathematics in 1956. He never received an advanced degree, because at that time no programs were available in his areas of interest. However, he took graduate courses at several different universities and contributed to the development of computer science and software engineering.

After graduating from Harvard, Warshall worked at ORO (Operation Research Office), which was set up by Johns Hopkins to do research and development for the U.S. Army. In 1958 he left ORO to take a position at a company called Technical Operations, where he helped build a research and development laboratory for military software projects. In 1961 he left Technical Operations to found Massachusetts Computer Associates. Later, this company became part of Applied Data Research (ADR). After the merger, Warshall sat on the board of directors of ADR and managed a variety of projects and organizations. He retired from ADR in 1982.

During his career Warshall carried out research and development in operating systems, compiler design, language design, and operations research. In the 1971–1972 academic year he presented lectures on software engineering at French universities. There is an interesting anecdote about his proof that the transitive closure algorithm, now known as Warshall's algorithm, is correct. He and a colleague at Technical Operations bet a bottle of rum on who first could determine whether this algorithm always works. Warshall came up with his proof overnight, winning the bet and the rum, which he shared with the loser of the bet. Because Warshall did not like sitting at a desk, he did much of his creative work in unconventional places, such as on a sailboat in the Indian Ocean or in a Greek lemon orchard.

ALGORITHM 2 Warshall Algorithm.

procedure *Warshall* ($\mathbf{M}_R$: $n \times n$ zero–one matrix)
$\mathbf{W} := \mathbf{M}_R$
for $k := 1$ **to** n
  for $i := 1$ **to** n
    for $j := 1$ **to** n
      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$
return $\mathbf{W}\{\mathbf{W} = [w_{ij}]$ is $\mathbf{M}_{R^*}\}$

The computational complexity of Warshall's algorithm can easily be computed in terms of bit operations. To find the entry $w_{ij}^{[k]}$ from the entries $w_{ij}^{[k-1]}$, $w_{ik}^{[k-1]}$, and $w_{kj}^{[k-1]}$ using Lemma 2 requires two bit operations. To find all n^2 entries of $\mathbf{W}_k$ from those of $\mathbf{W}_{k-1}$ requires $2n^2$ bit operations. Because Warshall's algorithm begins with $\mathbf{W}_0 = \mathbf{M}_R$ and computes the sequence of n zero–one matrices $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_n = \mathbf{M}_{R^*}$, the total number of bit operations used is $n \cdot 2n^2 = 2n^3$.

Exercises

1. Let R be the relation on the set $\{0, 1, 2, 3\}$ containing the ordered pairs $(0, 1)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 2)$, and $(3, 0)$. Find the

a) reflexive closure of R. **b)** symmetric closure of R.

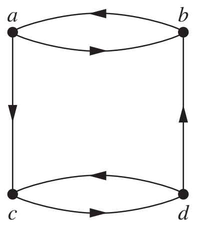
2. Let R be the relation $\{(a, b) \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R?

3. Let R be the relation $\{(a, b) \mid a$ divides $b\}$ on the set of integers. What is the symmetric closure of R?

4. How can the directed graph representing the reflexive closure of a relation on a finite set be constructed from the directed graph of the relation?

In Exercises 5–7 draw the directed graph of the reflexive closure of the relations with the directed graph shown.

5.



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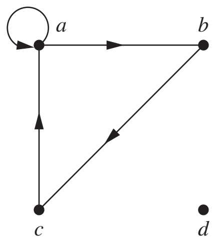

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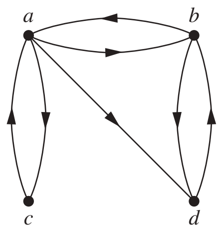


8. How can the directed graph representing the symmetric closure of a relation on a finite set be constructed from the directed graph for this relation?

9. Find the directed graphs of the symmetric closures of the relations with directed graphs shown in Exercises 5–7.

10. Find the smallest relation containing the relation in Example 2 that is both reflexive and symmetric.

11. Find the directed graph of the smallest relation that is both reflexive and symmetric that contains each of the relations with directed graphs shown in Exercises 5–7.

12. Suppose that the relation R on the finite set A is represented by the matrix $\mathbf{M}_R$. Show that the matrix that represents the reflexive closure of R is $\mathbf{M}_R \vee \mathbf{I}_n$.

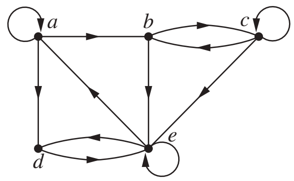
13. Suppose that the relation R on the finite set A is represented by the matrix $\mathbf{M}_R$. Show that the matrix that represents the symmetric closure of R is $\mathbf{M}_R \vee \mathbf{M}_R^t$.

14. Show that the closure of a relation R with respect to a property $\mathbf{P}$, if it exists, is the intersection of all the relations with property $\mathbf{P}$ that contain R.

15. When is it possible to define the "irreflexive closure" of a relation R, that is, a relation that contains R, is irreflexive, and is contained in every irreflexive relation that contains R?

16. Determine whether these sequences of vertices are paths in this directed graph.

a) a, b, c, e
b) b, e, c, b, e
c) a, a, b, e, d, e
d) b, c, e, d, a, a, b
e) b, c, c, b, e, d, e, d
f) $a, a, b, b, c, c, b, e, d$



17. Find all circuits of length three in the directed graph in Exercise 16.

18. Determine whether there is a path in the directed graph in Exercise 16 beginning at the first vertex given and ending at the second vertex given.

a) a, b **b)** b, a **c)** b, b
d) a, e **e)** b, d **f)** c, d
g) d, d **h)** e, a **i)** e, c

19. Let R be the relation on the set $\{1, 2, 3, 4, 5\}$ containing the ordered pairs $(1, 3)$, $(2, 4)$, $(3, 1)$, $(3, 5)$, $(4, 3)$, $(5, 1)$, $(5, 2)$, and $(5, 4)$. Find

a) R^2. **b)** R^3. **c)** R^4.
d) R^5. **e)** R^6. **f)** R^*.

20. Let R be the relation that contains the pair (a, b) if a and b are cities such that there is a direct nonstop airline flight from a to b. When is (a, b) in

a) R^2? **b)** R^3? **c)** R^*?

21. Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When is (a, b) in

a) R^2? **b)** R^3? **c)** R^*?

22. Suppose that the relation R is reflexive. Show that R^* is reflexive.

23. Suppose that the relation R is symmetric. Show that R^* is symmetric.

24. Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?

25. Use Algorithm 1 to find the transitive closures of these relations on $\{1, 2, 3, 4\}$.

a) $\{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$
b) $\{(2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3)\}$
c) $\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
d) $\{(1, 1), (1, 4), (2, 1), (2, 3), (3, 1), (3, 2), (3, 4), (4, 2)\}$

26. Use Algorithm 1 to find the transitive closures of these relations on $\{a, b, c, d, e\}$.

a) $\{(a, c), (b, d), (c, a), (d, b), (e, d)\}$
b) $\{(b, c), (b, e), (c, e), (d, a), (e, b), (e, c)\}$
c) $\{(a, b), (a, c), (a, e), (b, a), (b, c), (c, a), (c, b), (d, a), (e, d)\}$
d) $\{(a, e), (b, a), (b, d), (c, d), (d, a), (d, c), (e, a), (e, b), (e, c), (e, e)\}$

27. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 25.

28. Use Warshall's algorithm to find the transitive closures of the relations in Exercise 26.

29. Find the smallest relation containing the relation $\{(1, 2), (1, 4), (3, 3), (4, 1)\}$ that is

a) reflexive and transitive.
b) symmetric and transitive.
c) reflexive, symmetric, and transitive.

30. Finish the proof of the case when $a \neq b$ in Lemma 1.

31. Algorithms have been devised that use $O(n^{2.8})$ bit operations to compute the Boolean product of two $n \times n$ zero–one matrices. Assuming that these algorithms can be used, give big-O estimates for the number of bit operations using Algorithm 1 and using Warshall's algorithm to find the transitive closure of a relation on a set with n elements.

***32.** Devise an algorithm using the concept of interior vertices in a path to find the length of the shortest path between two vertices in a directed graph, if such a path exists.

33. Adapt Algorithm 1 to find the reflexive closure of the transitive closure of a relation on a set with n elements.

34. Adapt Warshall's algorithm to find the reflexive closure of the transitive closure of a relation on a set with n elements.

35. Show that the closure with respect to the property **P** of the relation $R = \{(0, 0), (0, 1), (1, 1), (2, 2)\}$ on the set $\{0, 1, 2\}$ does not exist if **P** is the property

a) "is not reflexive."
b) "has an odd number of elements."

36. Give an example of a relation R on the set $\{a, b, c\}$ such that the symmetric closure of the reflexive closure of the transitive closure of R is not transitive.

9.5 Equivalence Relations

9.5.1 Introduction

In some programming languages the names of variables can contain an unlimited number of characters. However, there is a limit on the number of characters that are checked when a compiler determines whether two variables are equal. For instance, in traditional C, only the first eight characters of a variable name are checked by the compiler. (These characters are uppercase or lowercase letters, digits, or underscores.) Consequently, the compiler considers strings longer than eight characters that agree in their first eight characters the same. Let R be the relation on the set of strings of characters such that sRt, where s and t are two strings, if s and t are at least eight characters long and the first eight characters of s and t agree, or $s = t$. It is easy to see that R is reflexive, symmetric, and transitive. Moreover, R divides the set of all strings into classes, where all strings in a particular class are considered the same by a compiler for traditional C.

The integers a and b are related by the "congruence modulo 4" relation when 4 divides $a - b$. We will show later that this relation is reflexive, symmetric, and transitive. It is not hard to see that a is related to b if and only if a and b have the same remainder when divided by 4. It follows that this relation splits the set of integers into four different classes.

When we care only what remainder an integer leaves when it is divided by 4, we need only know which class it is in, not its particular value.

These two relations, R and congruence modulo 4, are examples of equivalence relations, namely, relations that are reflexive, symmetric, and transitive. In this section we will show that such relations split sets into disjoint classes of equivalent elements. Equivalence relations arise whenever we care only whether an element of a set is in a certain class of elements, instead of caring about its particular identity.

9.5.2 Equivalence Relations

Links

In this section we will study relations with a particular combination of properties that allows them to be used to relate objects that are similar in some way.

Definition 1

A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Equivalence relations are important in every branch of mathematics!

Equivalence relations are important throughout mathematics and computer science. One reason for this is that in an equivalence relation, when two elements are related it makes sense to say they are equivalent.

Definition 2

Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

For the notion of equivalent elements to make sense, every element should be equivalent to itself, as the reflexive property guarantees for an equivalence relation. It makes sense to say that a and b are related (not just that a is related to b) by an equivalence relation, because when a is related to b, by the symmetric property, b is related to a. Furthermore, because an equivalence relation is transitive, if a and b are equivalent and b and c are equivalent, it follows that a and c are equivalent.

Examples 1–5 illustrate the notion of an equivalence relation.

EXAMPLE 1 Let R be the relation on the set of integers such that aRb if and only if $a = b$ or $a = -b$. In Section 9.1 we showed that R is reflexive, symmetric, and transitive. It follows that R is an equivalence relation. ◀

EXAMPLE 2 Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer. Is R an equivalence relation?

Extra Examples

Solution: Because $a - a = 0$ is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive. Now suppose that aRb. Then $a - b$ is an integer, so $b - a$ is also an integer. Hence, bRa. It follows that R is symmetric. If aRb and bRc, then $a - b$ and $b - c$ are integers. Therefore, $a - c = (a - b) + (b - c)$ is also an integer. Hence, aRc. Thus, R is transitive. Consequently, R is an equivalence relation. ◀

One of the most widely used equivalence relations is congruence modulo m, where m is an integer greater than 1.

EXAMPLE 3 **Congruence Modulo m** Let m be an integer with $m > 1$. Show that the relation

$$R = \{(a, b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Solution: Recall from Section 4.1 that $a \equiv b \pmod{m}$ if and only if m divides $a - b$. Note that $a - a = 0$ is divisible by m, because $0 = 0 \cdot m$. Hence, $a \equiv a \pmod{m}$, so congruence modulo m is reflexive. Now suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$. Hence, congruence modulo m is symmetric. Next, suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Therefore, there are integers k and l with $a - b = km$ and $b - c = lm$. Adding these two equations shows that $a - c = (a - b) + (b - c) = km + lm = (k + l)m$. Thus, $a \equiv c \pmod{m}$. Therefore, congruence modulo m is transitive. It follows that congruence modulo m is an equivalence relation. ◀

EXAMPLE 4 Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Because $l(a) = l(a)$, it follows that aRa whenever a is a string, so that R is reflexive. Next, suppose that aRb, so that $l(a) = l(b)$. Then bRa, because $l(b) = l(a)$. Hence, R is symmetric. Finally, suppose that aRb and bRc. Then $l(a) = l(b)$ and $l(b) = l(c)$. Hence, $l(a) = l(c)$, so aRc. Consequently, R is transitive. Because R is reflexive, symmetric, and transitive, it is an equivalence relation. ◀

EXAMPLE 5 Let n be a positive integer and S a set of strings. Suppose that R_n is the relation on S such that sR_nt if and only if $s = t$, or both s and t have at least n characters and the first n characters of s and t are the same. That is, a string of fewer than n characters is related only to itself; a string s with at least n characters is related to a string t if and only if t has at least n characters and t begins with the n characters at the start of s. For example, let $n = 3$ and let S be the set of all bit strings. Then sR_3t either when $s = t$ or both s and t are bit strings of length 3 or more that begin with the same three bits. For instance, $01R_3 01$ and $00111R_3 00101$, but $01 \not R_3 010$ and $01011 \not R_3 01110$.

Show that for every set S of strings and every positive integer n, R_n is an equivalence relation on S.

Solution: The relation R_n is reflexive because $s = s$, so that sR_ns whenever s is a string in S. If sR_nt, then either $s = t$ or s and t are both at least n characters long that begin with the same n characters. This means that tR_ns. We conclude that R_n is symmetric.

Now suppose that sR_nt and tR_nu. Then either $s = t$ or s and t are at least n characters long and s and t begin with the same n characters, and either $t = u$ or t and u are at least n characters long and t and u begin with the same n characters. From this, we can deduce that either $s = u$ or both s and u are n characters long and s and u begin with the same n characters (because in this case we know that s, t, and u are all at least n characters long and both s and u begin with the same n characters as t does). Consequently, R_n is transitive. It follows that R_n is an equivalence relation. ◀

In Examples 6 and 7 we look at two relations that are not equivalence relations.

EXAMPLE 6 Show that the "divides" relation on the set of positive integers in not an equivalence relation.

Solution: By Examples 9 and 15 in Section 9.1, we know that the "divides" relation is reflexive and transitive. However, by Example 12 in Section 9.1, we know that this relation is not symmetric (for instance, $2 \mid 4$ but $4 \nmid 2$). We conclude that the "divides" relation on the set of positive integers is not an equivalence relation. ◀

EXAMPLE 7 Let R be the relation on the set of real numbers such that xRy if and only if x and y are real numbers that differ by less than 1, that is, $|x - y| < 1$. Show that R is not an equivalence relation.

Solution: R is reflexive because $|x - x| = 0 < 1$ whenever $x \in \mathbf{R}$. R is symmetric, for if xRy, where x and y are real numbers, then $|x - y| < 1$, which tells us that $|y - x| = |x - y| < 1$, so that yRx. However, R is not an equivalence relation because it is not transitive. Take $x = 2.8$, $y = 1.9$, and $z = 1.1$, so that $|x - y| = |2.8 - 1.9| = 0.9 < 1$, $|y - z| = |1.9 - 1.1| = 0.8 < 1$, but $|x - z| = |2.8 - 1.1| = 1.7 > 1$. That is, $2.8\,R\,1.9$, $1.9\,R\,1.1$, but $2.8 \not\!R\, 1.1$. ◀

9.5.3 Equivalence Classes

Let A be the set of all students in your school who graduated from high school. Consider the relation R on A that consists of all pairs (x, y), where x and y graduated from the same high school. Given a student x, we can form the set of all students equivalent to x with respect to R. This set consists of all students who graduated from the same high school as x did. This subset of A is called an equivalence class of the relation.

Definition 3 Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$$[a]_R = \{s \mid (a, s) \in R\}.$$

If $b \in [a]_R$, then b is called a **representative** of this equivalence class. Any element of a class can be used as a representative of this class. That is, there is nothing special about the particular element chosen as the representative of the class.

EXAMPLE 8 What is the equivalence class of an integer for the equivalence relation of Example 1?

Solution: Because an integer is equivalent to itself and its negative in this equivalence relation, it follows that $[a] = \{-a, a\}$. This set contains two distinct integers unless $a = 0$. For instance, $[7] = \{-7, 7\}$, $[-5] = \{-5, 5\}$, and $[0] = \{0\}$. ◀

EXAMPLE 9 What are the equivalence classes of 0, 1, 2, and 3 for congruence modulo 4?

Extra Examples

Solution: The equivalence class of 0 contains all integers a such that $a \equiv 0 \pmod 4$. The integers in this class are those divisible by 4. Hence, the equivalence class of 0 for this relation is

$$[0] = \{\dots, -8, -4, 0, 4, 8, \dots\}.$$

The equivalence class of 1 contains all the integers a such that $a \equiv 1 \pmod 4$. The integers in this class are those that have a remainder of 1 when divided by 4. Hence, the equivalence class of 1 for this relation is

$$[1] = \{\dots, -7, -3, 1, 5, 9, \dots\}.$$

The equivalence class of 2 contains all the integers a such that $a \equiv 2 \pmod 4$. The integers in this class are those that have a remainder of 2 when divided by 4. Hence, the equivalence class of 2 for this relation is

$$[2] = \{\dots, -6, -2, 2, 6, 10, \dots\}.$$

The equivalence class of 3 contains all the integers a such that $a \equiv 3 \pmod 4$. The integers in this class are those that have a remainder of 3 when divided by 4. Hence, the equivalence class of 3 for this relation is

$$[3] = \{\dots, -5, -1, 3, 7, 11, \dots\}.$$

Note that every integer is in exactly one of the four equivalence classes and that the integer n is in the class containing n **mod** 4. ◀

In Example 9 the equivalence classes of 0, 1, 2, and 3 with respect to congruence modulo 4 were found. Example 9 can easily be generalized, replacing 4 with any positive integer m. The equivalence classes of the relation congruence modulo m are called the **congruence classes modulo** m. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\dots, a-2m, a-m, a, a+m, a+2m, \dots\}$. For instance, from Example 9 we have $[0]_4 = \{\dots, -8, -4, 0, 4, 8, \dots\}$, $[1]_4 = \{\dots, -7, -3, 1, 5, 9, \dots\}$, $[2]_4 = \{\dots, -6, -2, 2, 6, 10, \dots\}$, and $[3]_4 = \{\dots, -5, -1, 3, 7, 11, \dots\}$.

EXAMPLE 10 What is the equivalence class of the string 0111 with respect to the equivalence relation R_3 from Example 5 on the set of all bit strings? (Recall that sR_3t if and only if s and t are bit strings with $s = t$ or s and t are strings of at least three bits that start with the same three bits.)

Solution: The bit strings equivalent to 0111 are the bit strings with at least three bits that begin with 011. These are the bit strings 011, 0110, 0111, 01100, 01101, 01110, 01111, and so on. Consequently,

$$[011]_{R_3} = \{011, 0110, 0111, 01100, 01101, 01110, 01111, \dots\}.$$ ◀

EXAMPLE 11 **Identifiers in the C Programming Language** In the C programming language, an **identifier** is the name of a variable, a function, or another type of entity. Each identifier is a nonempty string of characters where each character is a lowercase or an uppercase English letter, a digit, or an underscore, and the first character is a lowercase or an uppercase English letter. Identifiers can be any length. This allows developers to use as many characters as they want to name an entity, such as a variable. However, for compilers for some versions of C, there is a limit on the number of characters checked when two names are compared to see whether they refer to the same thing. For example, Standard C compilers consider two identifiers the same when they agree in their first 31 characters. Consequently, developers must be careful not to use identifiers with the same initial 31 characters for different things. We see that two identifiers are considered the same when they are related by the relation R_{31} in Example 5. Using Example 5, we know that R_{31}, on the set of all identifiers in Standard C, is an equivalence relation.

What are the equivalence classes of each of the identifiers Number_of_tropical_storms, Number_of_named_tropical_storms, and Number_of_named_tropical_storms_in_the_Atlantic_in_2017?

Solution: Note that when an identifier is less than 31 characters long, by the definition of R_{31}, its equivalence class contains only itself. Because the identifier Number_of_tropical_storms is 25 characters long, its equivalence class contains exactly one element, namely, itself.

The identifier Number_of_named_tropical_storms is exactly 31 characters long. An identifier is equivalent to it when it starts with these same 31 characters. Consequently, every identifier at least 31 characters long that starts with Number_of_named_tropical_storms is equivalent to this identifier. It follows that the equivalence class of Number_of_named_tropical_storms is the set of all identifiers that begin with the 31 characters Number_of_named_tropical_storms.

An identifier is equivalent to the Number_of_named_tropical_storms_in_the_Atlantic_in_2017 if and only if it begins with its first 31 characters. Because these characters are Number_of_named_tropical_storms, we see that an identifier is equivalent to Number_of_named_tropical_storms_in_the_Atlantic_in_2017 if and only if it is equivalent to Number_of_named_tropical_storms. It follows that these last two identifiers have the same equivalence class. ◀

9.5.4 Equivalence Classes and Partitions

Let A be the set of students at your school who are majoring in exactly one subject, and let R be the relation on A consisting of pairs (x, y), where x and y are students with the same major. Then R is an equivalence relation, as the reader should verify. We can see that R splits all students in A into a collection of disjoint subsets, where each subset contains students with a specified major. For instance, one subset contains all students majoring (just) in computer science, and a second subset contains all students majoring in history. Furthermore, these subsets are equivalence classes of R. This example illustrates how the equivalence classes of an equivalence relation partition a set into disjoint, nonempty subsets. We will make these notions more precise in the following discussion.

Let R be a relation on the set A. Theorem 1 shows that the equivalence classes of two elements of A are either identical or disjoint.

THEOREM 1 Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) aRb $\qquad$ (ii) $[a] = [b]$ $\qquad$ (iii) $[a] \cap [b] \neq \emptyset$

Proof: We first show that (i) implies (ii). Assume that aRb. We will prove that $[a] = [b]$ by showing $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Suppose $c \in [a]$. Then aRc. Because aRb and R is symmetric, we know that bRa. Furthermore, because R is transitive and bRa and aRc, it follows that bRc. Hence, $c \in [b]$. This shows that $[a] \subseteq [b]$. The proof that $[b] \subseteq [a]$ is similar; it is left as an exercise for the reader.

Second, we will show that (ii) implies (iii). Assume that $[a] = [b]$. It follows that $[a] \cap [b] \neq \emptyset$ because $[a]$ is nonempty (because $a \in [a]$ because R is reflexive).

Next, we will show that (iii) implies (i). Suppose that $[a] \cap [b] \neq \emptyset$. Then there is an element c with $c \in [a]$ and $c \in [b]$. In other words, aRc and bRc. By the symmetric property, cRb. Then by transitivity, because aRc and cRb, we have aRb.

Because (i) implies (ii), (ii) implies (iii), and (iii) implies (i), the three statements, (i), (ii), and (iii), are equivalent. ◁

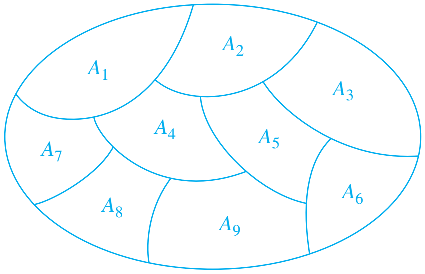


FIGURE 1 **A partition of a set.**

We are now in a position to show how an equivalence relation *partitions* a set. Let R be an equivalence relation on a set A. The union of the equivalence classes of R is all of A, because an element a of A is in its own equivalence class, namely, $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

In addition, from Theorem 1, it follows that these equivalence classes are either equal or disjoint, so

$$[a]_R \cap [b]_R = \emptyset,$$

when $[a]_R \neq [b]_R$.

Recall that an *index set* is a set whose members label, or index, the elements of a set.

These two observations show that the equivalence classes form a partition of A, because they split A into disjoint subsets. More precisely, a **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, $i \in I$ (where I is an index set) forms a partition of S if and only if

$$A_i \neq \emptyset \text{ for } i \in I,$$

$$A_i \cap A_j = \emptyset \text{ when } i \neq j,$$

and

$$\bigcup_{i \in I} A_i = S.$$

(Here the notation $\bigcup_{i \in I} A_i$ represents the union of the sets A_i for all $i \in I$.) Figure 1 illustrates the concept of a partition of a set.

EXAMPLE 12 Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ forms a partition of S, because these sets are disjoint and their union is S. ◀

We have seen that the equivalence classes of an equivalence relation on a set form a partition of the set. The subsets of S in this partition are the equivalence classes. Conversely, every partition of a set can be used to form an equivalence relation. Two elements are equivalent with respect to this relation if and only if they are in the same subset of S in the partition.

To see this, assume that $\{A_i \mid i \in I\}$ is a partition on S. Let R be the relation on S consisting of the pairs (x, y), where x and y belong to the same subset A_i in the partition. To show that R is an equivalence relation we must show that R is reflexive, symmetric, and transitive.

We see that $(a, a) \in R$ for every $a \in S$, because a is in the same subset of S as itself. Hence, R is reflexive. If $(a, b) \in R$, then b and a are in the same subset of S in the partition, so that

$(b, a) \in R$ as well. Hence, R is symmetric. If $(a, b) \in R$ and $(b, c) \in R$, then a and b are in the same subset X of S in the partition, and b and c are in the same subset Y of S of the partition. Because the subsets of S in the partition are disjoint and b belongs to X and Y, it follows that $X = Y$. Consequently, a and c belong to the same subset of S in the partition, so $(a, c) \in R$. Thus, R is transitive.

It follows that R is an equivalence relation. The equivalence classes of R consist of subsets of S containing related elements, and by the definition of R, these are the subsets of S in the partition. Theorem 2 summarizes the connections we have established between equivalence relations and partitions.

THEOREM 2 Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Example 13 shows how to construct an equivalence relation from a partition.

EXAMPLE 13 List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$, given in Example 12.

Solution: The subsets of S in the partition are the equivalence classes of R. The pair $(a, b) \in R$ if and only if a and b are in the same subset of the S in the partition. The pairs (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), and (3, 3) belong to R because $A_1 = \{1, 2, 3\}$ is an equivalence class; the pairs (4, 4), (4, 5), (5, 4), and (5, 5) belong to R because $A_2 = \{4, 5\}$ is an equivalence class; and finally the pair (6, 6) belongs to R because $\{6\}$ is an equivalence class. No pair other than those listed belongs to R. ◀

The congruence classes modulo m provide a useful illustration of Theorem 2. There are m different congruence classes modulo m, corresponding to the m different remainders possible when an integer is divided by m. These m congruence classes are denoted by $[0]_m, [1]_m, \ldots, [m-1]_m$. They form a partition of the set of integers.

EXAMPLE 14 What are the sets in the partition of the integers arising from congruence modulo 4?

Solution: In Example 9 we found the four congruence classes, $[0]_4$, $[1]_4$, $[2]_4$, and $[3]_4$. They are the sets

$$\begin{aligned}
[0]_4 &= \{\ldots, -8, -4, 0, 4, 8, \ldots\},\\
[1]_4 &= \{\ldots, -7, -3, 1, 5, 9, \ldots\},\\
[2]_4 &= \{\ldots, -6, -2, 2, 6, 10, \ldots\},\\
[3]_4 &= \{\ldots, -5, -1, 3, 7, 11, \ldots\}.
\end{aligned}$$

These congruence classes are disjoint, and every integer is in exactly one of them. In other words, as Theorem 2 says, these congruence classes form a partition. ◀

We now provide an example of a partition of the set of all strings arising from an equivalence relation on this set.

EXAMPLE 15 Let R_3 be the relation from Example 5. What are the sets in the partition of the set of all bit strings arising from the relation R_3 on the set of all bit strings? (Recall that sR_3t, where s and t are bit strings, if $s = t$ or s and t are bit strings with at least three bits that agree in their first three bits.)

Solution: Note that every bit string of length less than three is equivalent only to itself. Hence $[\lambda]_{R_3} = \{\lambda\}, [0]_{R_3} = \{0\}, [1]_{R_3} = \{1\}, [00]_{R_3} = \{00\}, [01]_{R_3} = \{01\}, [10]_{R_3} = \{10\}$, and $[11]_{R_3} = \{11\}$. Note that every bit string of length three or more is equivalent to one of the eight bit strings 000, 001, 010, 011, 100, 101, 110, and 111. We have

$$[000]_{R_3} = \{000, 0000, 0001, 00000, 00001, 00010, 00011, \ldots\},$$

$$[001]_{R_3} = \{001, 0010, 0011, 00100, 00101, 00110, 00111, \ldots\},$$

$$[010]_{R_3} = \{010, 0100, 0101, 01000, 01001, 01010, 01011, \ldots\},$$

$$[011]_{R_3} = \{011, 0110, 0111, 01100, 01101, 01110, 01111, \ldots\},$$

$$[100]_{R_3} = \{100, 1000, 1001, 10000, 10001, 10010, 10011, \ldots\},$$

$$[101]_{R_3} = \{101, 1010, 1011, 10100, 10101, 10110, 10111, \ldots\},$$

$$[110]_{R_3} = \{110, 1100, 1101, 11000, 11001, 11010, 11011, \ldots\},$$

$$[111]_{R_3} = \{111, 1110, 1111, 11100, 11101, 11110, 11111, \ldots\}.$$

These 15 equivalence classes are disjoint and every bit string is in exactly one of them. As Theorem 2 tells us, these equivalence classes partition the set of all bit strings. ◀

Exercises

1. Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$
b) $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$
c) $\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
d) $\{(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

2. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(a, b) \mid a$ and b are the same age$\}$
b) $\{(a, b) \mid a$ and b have the same parents$\}$
c) $\{(a, b) \mid a$ and b share a common parent$\}$
d) $\{(a, b) \mid a$ and b have met$\}$
e) $\{(a, b) \mid a$ and b speak a common language$\}$

3. Which of these relations on the set of all functions from $\mathbf{Z}$ to $\mathbf{Z}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(f, g) \mid f(1) = g(1)\}$
b) $\{(f, g) \mid f(0) = g(0)$ or $f(1) = g(1)\}$
c) $\{(f, g) \mid f(x) - g(x) = 1$ for all $x \in \mathbf{Z}\}$
d) $\{(f, g) \mid$ for some $C \in \mathbf{Z}$, for all $x \in \mathbf{Z}, f(x) - g(x) = C\}$
e) $\{(f, g) \mid f(0) = g(1)$ and $f(1) = g(0)\}$

4. Define three equivalence relations on the set of students in your discrete mathematics class different from the relations discussed in the text. Determine the equivalence classes for each of these equivalence relations.

5. Define three equivalence relations on the set of buildings on a college campus. Determine the equivalence classes for each of these equivalence relations.

6. Define three equivalence relations on the set of classes offered at your school. Determine the equivalence classes for each of these equivalence relations.

7. Show that the relation of logical equivalence on the set of all compound propositions is an equivalence relation. What are the equivalence classes of **F** and of **T**?

8. Let R be the relation on the set of all sets of real numbers such that $S\,R\,T$ if and only if S and T have the same cardinality. Show that R is an equivalence relation. What are the equivalence classes of the sets $\{0, 1, 2\}$ and $\mathbf{Z}$?

9. Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$.

a) Show that R is an equivalence relation on A.
b) What are the equivalence classes of R?

10. Suppose that A is a nonempty set and R is an equivalence relation on A. Show that there is a function f with A as its domain such that $(x, y) \in R$ if and only if $f(x) = f(y)$.

11. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

12. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings of length three or more that agree except perhaps in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

13. Show that the relation R consisting of all pairs (x, y) such that x and y are bit strings that agree in their first and third bits is an equivalence relation on the set of all bit strings of length three or more.

14. Let R be the relation consisting of all pairs (x, y) such that x and y are strings of uppercase and lowercase English letters with the property that for every positive integer n, the nth characters in x and y are the same letter, either uppercase or lowercase. Show that R is an equivalence relation.

15. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that R is an equivalence relation.

16. Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

17. (*Requires calculus*)

a) Show that the relation R on the set of all differentiable functions from $\mathbf{R}$ to $\mathbf{R}$ consisting of all pairs (f, g) such that $f'(x) = g'(x)$ for all real numbers x is an equivalence relation.

b) Which functions are in the same equivalence class as the function $f(x) = x^2$?

18. (*Requires calculus*)

a) Let n be a positive integer. Show that the relation R on the set of all polynomials with real-valued coefficients consisting of all pairs (f, g) such that $f^{(n)}(x) = g^{(n)}(x)$ is an equivalence relation. [Here $f^{(n)}(x)$ is the nth derivative of $f(x)$.]

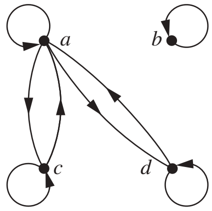
b) Which functions are in the same equivalence class as the function $f(x) = x^4$, where $n = 3$?

19. Let R be the relation on the set of all URLs (or Web addresses) such that $x\,R\,y$ if and only if the Web page at x is the same as the Web page at y. Show that R is an equivalence relation.

20. Let R be the relation on the set of all people who have visited a particular Web page such that $x\,R\,y$ if and only if person x and person y have followed the same set of links starting at this Web page (going from Web page to Web page until they stop using the Web). Show that R is an equivalence relation.

In Exercises 21–23 determine whether the relation with the directed graph shown is an equivalence relation.

21.



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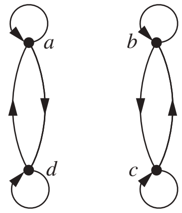
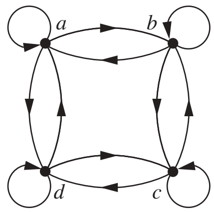


23.



24. Determine whether the relations represented by these zero–one matrices are equivalence relations.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

25. Show that the relation R on the set of all bit strings such that $s\,R\,t$ if and only if s and t contain the same number of 1s is an equivalence relation.

26. What are the equivalence classes of the equivalence relations in Exercise 1?

27. What are the equivalence classes of the equivalence relations in Exercise 2?

28. What are the equivalence classes of the equivalence relations in Exercise 3?

29. What is the equivalence class of the bit string 011 for the equivalence relation in Exercise 25?

30. What are the equivalence classes of these bit strings for the equivalence relation in Exercise 11?

a) 010 **b)** 1011 **c)** 11111 **d)** 01010101

31. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 12?

32. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 13?

33. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation R_4 from Example 5 on the set of all bit strings? (Recall that bit strings s and t are equivalent under R_4 if and only if they are equal or they are both at least four bits long and agree in their first four bits.)

34. What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation R_5 from Example 5 on the set of all bit strings? (Recall that bit strings s and t are equivalent under R_5 if and only if they are equal or they are both at least five bits long and agree in their first five bits.)

35. What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is

a) 2? **b)** 3? **c)** 6? **d)** −3?

36. What is the congruence class $[4]_m$ when m is

a) 2? **b)** 3? **c)** 6? **d)** 8?

37. Give a description of each of the congruence classes modulo 6.

38. What is the equivalence class of each of these strings with respect to the equivalence relation in Exercise 14?

a) *No* **b)** *Yes* **c)** *Help*

39. a) What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 15?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 15. [*Hint:* Look at the difference $a - b$ corresponding to (a, b).]

40. a) What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 16?

b) Give an interpretation of the equivalence classes for the equivalence relation R in Exercise 16. [*Hint:* Look at the ratio a/b corresponding to (a, b).]

41. Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$?

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$

b) $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$

c) $\{2, 4, 6\}, \{1, 3, 5\}$ **d)** $\{1, 4, 5\}, \{2, 6\}$

42. Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$?

a) $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$

b) $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$

c) $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$

d) $\{-3, -2, 2, 3\}, \{-1, 1\}$

43. Which of these collections of subsets are partitions of the set of bit strings of length 8?

a) the set of bit strings that begin with 1, the set of bit strings that begin with 00, and the set of bit strings that begin with 01

b) the set of bit strings that contain the string 00, the set of bit strings that contain the string 01, the set of bit strings that contain the string 10, and the set of bit strings that contain the string 11

c) the set of bit strings that end with 00, the set of bit strings that end with 01, the set of bit strings that end with 10, and the set of bit strings that end with 11

d) the set of bit strings that end with 111, the set of bit strings that end with 011, and the set of bit strings that end with 00

e) the set of bit strings that contain $3k$ ones for some nonnegative integer k, the set of bit strings that contain $3k + 1$ ones for some nonnegative integer k, and the set of bit strings that contain $3k + 2$ ones for some nonnegative integer k.

44. Which of these collections of subsets are partitions of the set of integers?

a) the set of even integers and the set of odd integers

b) the set of positive integers and the set of negative integers

c) the set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remainder of 2 when divided by 3

d) the set of integers less than −100, the set of integers with absolute value not exceeding 100, and the set of integers greater than 100

e) the set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6

45. Which of these are partitions of the set $\mathbf{Z} \times \mathbf{Z}$ of ordered pairs of integers?

a) the set of pairs (x, y), where x or y is odd; the set of pairs (x, y), where x is even; and the set of pairs (x, y), where y is even

b) the set of pairs (x, y), where both x and y are odd; the set of pairs (x, y), where exactly one of x and y is odd; and the set of pairs (x, y), where both x and y are even

c) the set of pairs (x, y), where x is positive; the set of pairs (x, y), where y is positive; and the set of pairs (x, y), where both x and y are negative

d) the set of pairs (x, y), where $3 \mid x$ and $3 \mid y$; the set of pairs (x, y), where $3 \mid x$ and $3 \nmid y$; the set of pairs (x, y), where $3 \nmid x$ and $3 \mid y$; and the set of pairs (x, y), where $3 \nmid x$ and $3 \nmid y$

e) the set of pairs (x, y), where $x > 0$ and $y > 0$; the set of pairs (x, y), where $x > 0$ and $y \leq 0$; the set of pairs (x, y), where $x \leq 0$ and $y > 0$; and the set of pairs (x, y), where $x \leq 0$ and $y \leq 0$

f) the set of pairs (x, y), where $x \neq 0$ and $y \neq 0$; the set of pairs (x, y), where $x = 0$ and $y \neq 0$; and the set of pairs (x, y), where $x \neq 0$ and $y = 0$

46. Which of these are partitions of the set of real numbers?

a) the negative real numbers, $\{0\}$, the positive real numbers

b) the set of irrational numbers, the set of rational numbers

c) the set of intervals $[k, k + 1]$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

d) the set of intervals $(k, k + 1)$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

e) the set of intervals $(k, k + 1]$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

f) the sets $\{x + n \mid n \in \mathbf{Z}\}$ for all $x \in [0, 1)$

47. List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.

a) $\{0\}, \{1, 2\}, \{3, 4, 5\}$

b) $\{0, 1\}, \{2, 3\}, \{4, 5\}$

c) $\{0, 1, 2\}, \{3, 4, 5\}$

d) $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

48. List the ordered pairs in the equivalence relations produced by these partitions of $\{a, b, c, d, e, f, g\}$.

a) $\{a, b\}, \{c, d\}, \{e, f, g\}$
b) $\{a\}, \{b\}, \{c, d\}, \{e, f\}, \{g\}$
c) $\{a, b, c, d\}, \{e, f, g\}$
d) $\{a, c, e, g\}, \{b, d\}, \{f\}$

A partition P_1 is called a **refinement** of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2.

49. Show that the partition formed from congruence classes modulo 6 is a refinement of the partition formed from congruence classes modulo 3.

50. Show that the partition of the set of people living in the United States consisting of subsets of people living in the same county (or parish) and same state is a refinement of the partition consisting of subsets of people living in the same state.

51. Show that the partition of the set of bit strings of length 16 formed by equivalence classes of bit strings that agree on the last eight bits is a refinement of the partition formed from the equivalence classes of bit strings that agree on the last four bits.

In Exercises 52 and 53, R_n refers to the family of equivalence relations defined in Example 5. Recall that $s\, R_n\, t$, where s and t are two strings if $s = t$ or s and t are strings with at least n characters that agree in their first n characters.

52. Show that the partition of the set of all bit strings formed by equivalence classes of bit strings with respect to the equivalence relation R_4 is a refinement of the partition formed by equivalence classes of bit strings with respect to the equivalence relation R_3.

53. Show that the partition of the set of all identifiers in C formed by the equivalence classes of identifiers with respect to the equivalence relation R_{31} is a refinement of the partition formed by equivalence classes of identifiers with respect to the equivalence relation R_8. (Compilers for "old" C consider identifiers the same when their names agree in their first eight characters, while compilers in standard C consider identifiers the same when their names agree in their first 31 characters.)

54. Suppose that R_1 and R_2 are equivalence relations on a set A. Let P_1 and P_2 be the partitions that correspond to R_1 and R_2, respectively. Show that $R_1 \subseteq R_2$ if and only if P_1 is a refinement of P_2.

55. Find the smallest equivalence relation on the set $\{a, b, c, d, e\}$ containing the relation $\{(a, b), (a, c), (d, e)\}$.

56. Suppose that R_1 and R_2 are equivalence relations on the set S. Determine whether each of these combinations of R_1 and R_2 must be an equivalence relation.

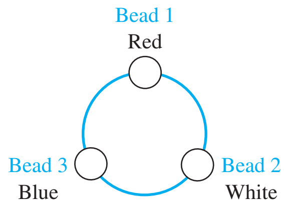
a) $R_1 \cup R_2$ **b)** $R_1 \cap R_2$ **c)** $R_1 \oplus R_2$

57. Consider the equivalence relation from Example 2, namely, $R = \{(x, y) \mid x - y \text{ is an integer}\}$.

a) What is the equivalence class of 1 for this equivalence relation?

b) What is the equivalence class of 1/2 for this equivalence relation?

***58.** Each bead on a bracelet with three beads is either red, white, or blue, as illustrated in the figure shown.



Define the relation R between bracelets as: (B_1, B_2), where B_1 and B_2 are bracelets, belongs to R if and only if B_2 can be obtained from B_1 by rotating it or rotating it and then reflecting it.

a) Show that R is an equivalence relation.

b) What are the equivalence classes of R?

***59.** Let R be the relation on the set of all colorings of the 2×2 checkerboard where each of the four squares is colored either red or blue so that (C_1, C_2), where C_1 and C_2 are 2×2 checkerboards with each of their four squares colored blue or red, belongs to R if and only if C_2 can be obtained from C_1 either by rotating the checkerboard or by rotating it and then reflecting it.

a) Show that R is an equivalence relation.

b) What are the equivalence classes of R?

60. **a)** Let R be the relation on the set of functions from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ such that (f, g) belongs to R if and only if f is $\Theta(g)$ (see Section 3.2). Show that R is an equivalence relation.

b) Describe the equivalence class containing $f(n) = n^2$ for the equivalence relation of part (a).

61. Determine the number of different equivalence relations on a set with three elements by listing them.

62. Determine the number of different equivalence relations on a set with four elements by listing them.

***63.** Do we necessarily get an equivalence relation when we form the transitive closure of the symmetric closure of the reflexive closure of a relation?

***64.** Do we necessarily get an equivalence relation when we form the symmetric closure of the reflexive closure of the transitive closure of a relation?

65. Suppose we use Theorem 2 to form a partition P from an equivalence relation R. What is the equivalence relation R' that results if we use Theorem 2 again to form an equivalence relation from P?

66. Suppose we use Theorem 2 to form an equivalence relation R from a partition P. What is the partition P' that results if we use Theorem 2 again to form a partition from R?

67. Devise an algorithm to find the smallest equivalence relation containing a given relation.

*68. Let $p(n)$ denote the number of different equivalence relations on a set with n elements (and by Theorem 2 the number of partitions of a set with n elements). Show that $p(n)$ satisfies the recurrence relation $p(n) = \sum_{j=0}^{n-1} C(n-1, j)p(n-j-1)$ and the initial condition $p(0) = 1$. (*Note:* The numbers $p(n)$ are called **Bell numbers** after the American mathematician E. T. Bell.)

69. Use Exercise 68 to find the number of different equivalence relations on a set with n elements, where n is a positive integer not exceeding 10.

9.6 Partial Orderings

9.6.1 Introduction

Links

We often use relations to order some or all of the elements of sets. For instance, we order words using the relation containing pairs of words (x, y), where x comes before y in the dictionary. We schedule projects using the relation consisting of pairs (x, y), where x and y are tasks in a project such that x must be completed before y begins. We order the set of integers using the relation containing the pairs (x, y), where x is less than y. When we add all of the pairs of the form (x, x) to these relations, we obtain a relation that is reflexive, antisymmetric, and transitive. These are properties that characterize relations used to order the elements of sets.

Definition 1 A relation R on a set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R). Members of S are called *elements* of the poset.

We give examples of posets in Examples 1–3.

EXAMPLE 1 Show that the greater than or equal to relation ($\geq$) is a partial ordering on the set of integers.

Extra Examples

Solution: Because $a \geq a$ for every integer a, $\geq$ is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$. Hence, $\geq$ is antisymmetric. Finally, $\geq$ is transitive because $a \geq b$ and $b \geq c$ imply that $a \geq c$. It follows that $\geq$ is a partial ordering on the set of integers and $(\mathbf{Z}, \geq)$ is a poset. ◀

EXAMPLE 2 The divisibility relation | is a partial ordering on the set of positive integers, because it is reflexive, antisymmetric, and transitive, as was shown in Section 9.1. We see that $(\mathbf{Z}^+, |)$ is a poset. Recall that ($\mathbf{Z}^+$ denotes the set of positive integers.) ◀

EXAMPLE 3 Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution: Because $A \subseteq A$ whenever A is a subset of S, $\subseteq$ is reflexive. It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$. Finally, $\subseteq$ is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, $\subseteq$ is a partial ordering on $P(S)$, and $(P(S), \subseteq)$ is a poset. ◀

Example 4 illustrates a relation that is not a partial ordering.

EXAMPLE 4 Let R be the relation on the set of people such that xRy if x and y are people and x is older than y. Show that R is not a partial ordering.

Extra Examples

Solution: Note that R is antisymmetric because if a person x is older than a person y, then y is not older than x. That is, if xRy, then $y \not R x$. The relation R is transitive because if person x is older than person y and y is older than person z, then x is older than z. That is, if xRy

and yRz, then xRz. However, R is not reflexive, because no person is older than himself or herself. That is, $x \not R x$ for all people x. It follows that R is not a partial ordering. ◀

In different posets different symbols such as $\leq$, $\subseteq$, and $|$, are used for a partial ordering. However, we need a symbol that we can use when we discuss the ordering relation in an arbitrary poset. Customarily, the notation $a \preccurlyeq b$ is used to denote that $(a, b) \in R$ in an arbitrary poset (S, R). This notation is used because the less than or equal to relation on the set of real numbers is the most familiar example of a partial ordering and the symbol $\preccurlyeq$ is similar to the $\leq$ symbol. (Note that the symbol $\preccurlyeq$ is used to denote the relation in *any* poset, not just the less than or equal to relation.) The notation $a \prec b$ denotes that $a \preccurlyeq b$, but $a \neq b$. Also, we say "a is less than b" or "b is greater than a" if $a \prec b$.

When a and b are elements of the poset $(S, \preccurlyeq)$, it is not necessary that either $a \preccurlyeq b$ or $b \preccurlyeq a$. For instance, in $(P(\mathbf{Z}), \subseteq)$, $\{1, 2\}$ is not related to $\{1, 3\}$, and vice versa, because neither set is contained within the other. Similarly, in $(\mathbf{Z}^+, |)$, 2 is not related to 3 and 3 is not related to 2, because $2 \nmid 3$ and $3 \nmid 2$. This leads to Definition 2.

Definition 2 The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

EXAMPLE 5 In the poset $(\mathbf{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution: The integers 3 and 9 are comparable, because $3 \mid 9$. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$. ◀

The adjective "partial" is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a **total ordering**.

Definition 3 If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order.* A totally ordered set is also called a *chain*.

EXAMPLE 6 The poset $(\mathbf{Z}, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers. ◀

EXAMPLE 7 The poset $(\mathbf{Z}^+, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7. ◀

In Chapter 6 we noted that $(\mathbf{Z}^+, \leq)$ is well-ordered, where $\leq$ is the usual less than or equal to relation. We now define well-ordered sets.

Definition 4 $(S, \preccurlyeq)$ is a *well-ordered set* if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

EXAMPLE 8 The set of ordered pairs of positive integers, $\mathbf{Z}^+ \times \mathbf{Z}^+$, with $(a_1, a_2) \preccurlyeq (b_1, b_2)$ if $a_1 < b_1$, or if $a_1 = b_1$ and $a_2 \leq b_2$ (the lexicographic ordering), is a well-ordered set. The verification of this is left as Exercise 53. The set $\mathbf{Z}$, with the usual $\leq$ ordering, is not well-ordered because the set of negative integers, which is a subset of $\mathbf{Z}$, has no least element. ◀

At the end of Section 5.3 we showed how to use the principle of well-ordered induction (there called generalized induction) to prove results about a well-ordered set. We now state and prove that this proof technique is valid.

THEOREM 1 **THE PRINCIPLE OF WELL-ORDERED INDUCTION** Suppose that S is a well-ordered set. Then $P(x)$ is true for all $x \in S$, if

INDUCTIVE STEP: For every $y \in S$, if $P(x)$ is true for all $x \in S$ with $x \prec y$, then $P(y)$ is true.

Proof: Suppose it is not the case that $P(x)$ is true for all $x \in S$. Then there is an element $y \in S$ such that $P(y)$ is false. Consequently, the set $A = \{x \in S \mid P(x) \text{ is false}\}$ is nonempty. Because S is well-ordered, A has a least element a. By the choice of a as a least element of A, we know that $P(x)$ is true for all $x \in S$ with $x \prec a$. This implies by the inductive step $P(a)$ is true. This contradiction shows that $P(x)$ must be true for all $x \in S$. ◁

Remark: We do not need a basis step in a proof using the principle of well-ordered induction because if x_0 is the least element of a well-ordered set, the inductive step tells us that $P(x_0)$ is true. This follows because there are no elements $x \in S$ with $x \prec x_0$, so we know (using a vacuous proof) that $P(x)$ is true for all $x \in S$ with $x \prec x_0$.

The principle of well-ordered induction is a versatile technique for proving results about well-ordered sets. Even when it is possible to use mathematical induction for the set of positive integers to prove a theorem, it may be simpler to use the principle of well-ordered induction, as we saw in Examples 5 and 6 in Section 6.2, where we proved a result about the well-ordered set $(\mathbf{N} \times \mathbf{N}, \preccurlyeq)$ where $\preccurlyeq$ is lexicographic ordering on $\mathbf{N} \times \mathbf{N}$.

9.6.2 Lexicographic Order

The words in a dictionary are listed in alphabetic, or lexicographic, order, which is based on the ordering of the letters in the alphabet. This is a special case of an ordering of strings on a set constructed from a partial ordering on the set. We will show how this construction works in any poset.

First, we will show how to construct a partial ordering on the Cartesian product of two posets, $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$. The **lexicographic ordering** $\preccurlyeq$ on $A_1 \times A_2$ is defined by specifying that one pair is less than a second pair if the first entry of the first pair is less than (in A_1) the first entry of the second pair, or if the first entries are equal, but the second entry of this pair is less than (in A_2) the second entry of the second pair. In other words, (a_1, a_2) is less than (b_1, b_2), that is,

$$(a_1, a_2) \prec (b_1, b_2),$$

either if $a_1 \prec_1 b_1$ or if both $a_1 = b_1$ and $a_2 \prec_2 b_2$.

We obtain a partial ordering $\preccurlyeq$ by adding equality to the ordering $\prec$ on $A_1 \times A_2$. The verification of this is left as an exercise.

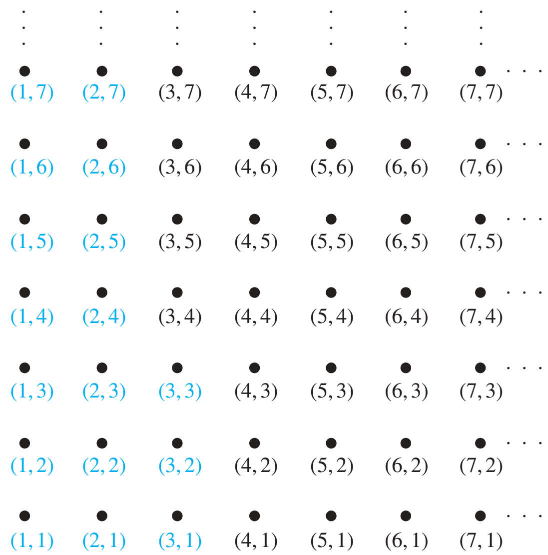


FIGURE 1 **The ordered pairs less than (3, 4) in lexicographic order.**

EXAMPLE 9 Determine whether $(3, 5) \prec (4, 8)$, whether $(3, 8) \prec (4, 5)$, and whether $(4, 9) \prec (4, 11)$ in the poset $(\mathbf{Z} \times \mathbf{Z}, \preccurlyeq)$, where $\preccurlyeq$ is the lexicographic ordering constructed from the usual $\leq$ relation on $\mathbf{Z}$.

Solution: Because $3 < 4$, it follows that $(3, 5) \prec (4, 8)$ and that $(3, 8) \prec (4, 5)$. We have $(4, 9) \prec (4, 11)$, because the first entries of $(4, 9)$ and $(4, 11)$ are the same but $9 < 11$. ◀

In Figure 1 the ordered pairs in $\mathbf{Z}^+ \times \mathbf{Z}^+$ that are less than $(3, 4)$ are highlighted. A lexicographic ordering can be defined on the Cartesian product of n posets $(A_1, \preccurlyeq_1)$, $(A_2, \preccurlyeq_2), \ldots, (A_n, \preccurlyeq_n)$. Define the partial ordering $\preccurlyeq$ on $A_1 \times A_2 \times \cdots \times A_n$ by

$$(a_1, a_2, \ldots, a_n) \prec (b_1, b_2, \ldots, b_n)$$

if $a_1 \prec_1 b_1$, or if there is an integer $i > 0$ such that $a_1 = b_1, \ldots, a_i = b_i$, and $a_{i+1} \prec_{i+1} b_{i+1}$. In other words, one n-tuple is less than a second n-tuple if the entry of the first n-tuple in the first position where the two n-tuples disagree is less than the entry in that position in the second n-tuple.

EXAMPLE 10 Note that $(1, 2, 3, 5) \prec (1, 2, 4, 3)$, because the entries in the first two positions of these 4-tuples agree, but in the third position the entry in the first 4-tuple, 3, is less than that in the second 4-tuple, 4. (Here the ordering on 4-tuples is the lexicographic ordering that comes from the usual less than or equal to relation on the set of integers.) ◀

We can now define lexicographic ordering of strings. Consider the strings $a_1a_2 \ldots a_m$ and $b_1b_2 \ldots b_n$ on a partially ordered set S. Suppose these strings are not equal. Let t be the minimum of m and n. The definition of lexicographic ordering is that the string $a_1a_2 \ldots a_m$ is less than $b_1b_2 \ldots b_n$ if and only if

$(a_1, a_2, \ldots, a_t) \prec (b_1, b_2, \ldots, b_t)$, or

$(a_1, a_2, \ldots, a_t) = (b_1, b_2, \ldots, b_t)$ and $m < n$,

where $\prec$ in this inequality represents the lexicographic ordering of S^t. In other words, to determine the ordering of two different strings, the longer string is truncated to the length of the shorter string, namely, to $t = \min(m, n)$ terms. Then the t-tuples made up of the first t terms of each string are compared using the lexicographic ordering on S^t. One string is less than another string if the t-tuple corresponding to the first string is less than the t-tuple of the second string, or if these two t-tuples are the same, but the second string is longer. The verification that this is a partial ordering is left as Exercise 38 for the reader.

EXAMPLE 11 Consider the set of strings of lowercase English letters. Using the ordering of letters in the alphabet, a lexicographic ordering on the set of strings can be constructed. A string is less than a second string if the letter in the first string in the first position where the strings differ comes before the letter in the second string in this position, or if the first string and the second string agree in all positions, but the second string has more letters. This ordering is the same as that used in dictionaries. For example,

$$\textit{discreet} \prec \textit{discrete},$$

because these strings differ first in the seventh position, and $e \prec t$. Also,

$$\textit{discreet} \prec \textit{discreetness},$$

because the first eight letters agree, but the second string is longer. Furthermore,

$$\textit{discrete} \prec \textit{discretion},$$

because

$$\textit{discrete} \prec \textit{discreti}.$$

◀

9.6.3 Hasse Diagrams

Many edges in the directed graph for a finite poset do not have to be shown because they must be present. For instance, consider the directed graph for the partial ordering $\{(a, b) \mid a \leq b\}$ on the set $\{1, 2, 3, 4\}$, shown in Figure 2(a). Because this relation is a partial ordering, it is reflexive, and its directed graph has loops at all vertices. Consequently, we do not have to show these loops because they must be present; in Figure 2(b) loops are not shown. Because a partial ordering is transitive, we do not have to show those edges that must be present because of transitivity. For example, in Figure 2(c) the edges (1, 3), (1, 4), and (2, 4) are not shown because they must be present. If we assume that all edges are pointed "upward" (as they are drawn in the figure), we do not have to show the directions of the edges; Figure 2(c) does not show directions.

In general, we can represent a finite poset $(S, \preccurlyeq)$ using this procedure: Start with the directed graph for this relation. Because a partial ordering is reflexive, a loop (a, a) is present at every vertex a. Remove these loops. Next, remove all edges that must be in the partial ordering because

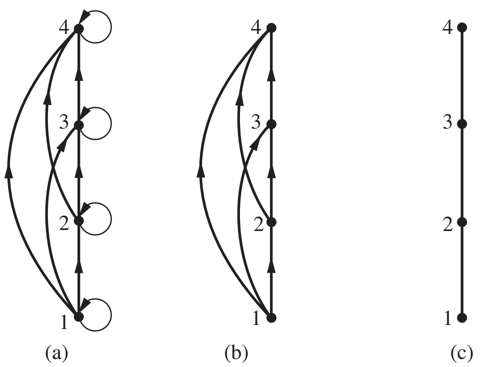


FIGURE 2 **Constructing the Hasse diagram for ({1, 2, 3, 4}, ≤).**

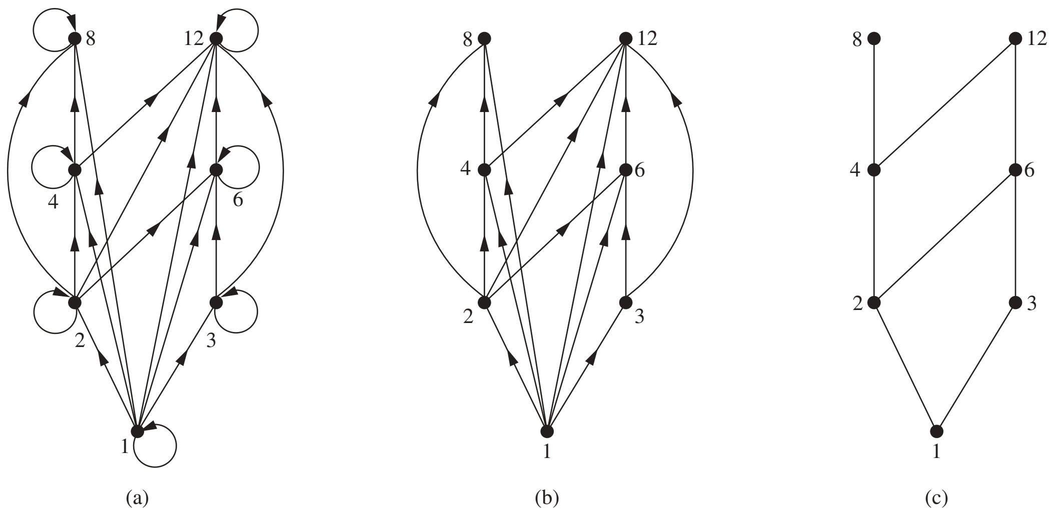


FIGURE 3 Constructing the Hasse diagram of ({1, 2, 3, 4, 6, 8, 12}, |).

of the presence of other edges and transitivity. That is, remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec x$. Finally, arrange each edge so that its initial vertex is below its terminal vertex (as it is drawn on paper). Remove all the arrows on the directed edges, because all edges point "upward" toward their terminal vertex.

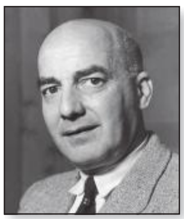
These steps are well defined, and only a finite number of steps need to be carried out for a finite poset. When all the steps have been taken, the resulting diagram contains sufficient information to find the partial ordering, as we will explain later. The resulting diagram is called the **Hasse diagram** of $(S, \preccurlyeq)$, named after the twentieth-century German mathematician Helmut Hasse who made extensive use of them.

Let $(S, \preccurlyeq)$ be a poset. We say that an element $y \in S$ **covers** an element $x \in S$ if $x \prec y$ and there is no element $z \in S$ such that $x \prec z \prec y$. The set of pairs (x, y) such that y covers x is called the **covering relation** of $(S, \preccurlyeq)$. From the description of the Hasse diagram of a poset, we see that the edges in the Hasse diagram of $(S, \preccurlyeq)$ are upwardly pointing edges corresponding to the pairs in the covering relation of $(S, \preccurlyeq)$. Furthermore, we can recover a poset from its covering relation, because it is the reflexive transitive closure of its covering relation. (Exercise 31 asks for a proof of this fact.) This tells us that we can construct a partial ordering from its Hasse diagram.

EXAMPLE 12 Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

Solution: Begin with the digraph for this partial order, as shown in Figure 3(a). Remove all loops, as shown in Figure 3(b). Then delete all the edges implied by the transitive property. These

Links



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HELMUT HASSE (1898–1979) Helmut Hasse was born in Kassel, Germany. He served in the German navy after high school. He began his university studies at Göttingen University in 1918, moving in 1920 to Marburg University to study under the number theorist Kurt Hensel. During this time, Hasse made fundamental contributions to algebraic number theory. He became Hensel's successor at Marburg, later becoming director of the renowned mathematical institute at Göttingen in 1934, and took a position at Hamburg University in 1950. Hasse served for 50 years as an editor of *Crelle's Journal*, an illustrious German mathematics periodical, taking over the job of chief editor in 1936 when the Nazis forced Hensel to resign. During World War II Hasse worked on applied mathematics research for the German navy. He was noted for the clarity and personal style of his lectures and was devoted both to number theory and to his students. (Hasse has been controversial for connections with the Nazi party. Investigations have shown he was a strong German nationalist but not an ardent Nazi.)

are (1, 4), (1, 6), (1, 8), (1, 12), (2, 8), (2, 12), and (3, 12). Arrange all edges to point upward, and delete all arrows to obtain the Hasse diagram. The resulting Hasse diagram is shown in Figure 3(c). ◀

EXAMPLE 13 Draw the Hasse diagram for the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set $P(S)$, where $S = \{a, b, c\}$.

Solution: The Hasse diagram for this partial ordering is obtained from the associated digraph by deleting all the loops and all the edges that occur from transitivity, namely, $(\emptyset, \{a, b\})$, $(\emptyset, \{a, c\})$, $(\emptyset, \{b, c\})$, $(\emptyset, \{a, b, c\})$, $(\{a\}, \{a, b, c\})$, $(\{b\}, \{a, b, c\})$, and $(\{c\}, \{a, b, c\})$. Finally, all edges point upward, and arrows are deleted. The resulting Hasse diagram is illustrated in Figure 4. ◀

9.6.4 Maximal and Minimal Elements

Elements of posets that have certain extremal properties are important for many applications. An element of a poset is called maximal if it is not less than any element of the poset. That is, a is **maximal** in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$. Similarly, an element of a poset is called minimal if it is not greater than any element of the poset. That is, a is **minimal** if there is no element $b \in S$ such that $b \prec a$. Maximal and minimal elements are easy to spot using a Hasse diagram. They are the "top" and "bottom" elements in the diagram.

EXAMPLE 14 Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, \mid)$ are maximal, and which are minimal?

Solution: The Hasse diagram in Figure 5 for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5. As this example shows, a poset can have more than one maximal element and more than one minimal element. ◀

Sometimes there is an element in a poset that is greater than every other element. Such an element is called the greatest element. That is, a is the **greatest element** of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$. The greatest element is unique when it exists [see Exercise 40(a)]. Likewise, an element is called the least element if it is less than all the other elements in the poset. That is, a is the **least element** of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$. The least element is unique when it exists [see Exercise 40(b)].

EXAMPLE 15 Determine whether the posets represented by each of the Hasse diagrams in Figure 6 have a greatest element and a least element.

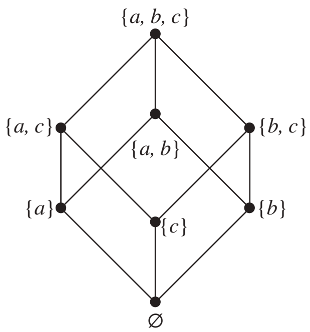


FIGURE 4 **The Hasse diagram of $(P(\{a, b, c\}), \subseteq)$.**

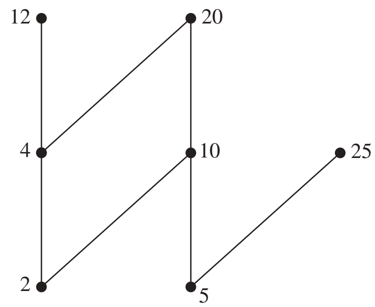


FIGURE 5 **The Hasse diagram of a poset.**

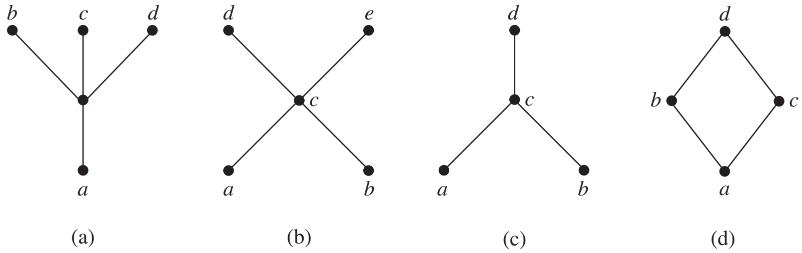


FIGURE 6 **Hasse diagrams of four posets.**

Solution: The least element of the poset with Hasse diagram (a) is a. This poset has no greatest element. The poset with Hasse diagram (b) has neither a least nor a greatest element. The poset with Hasse diagram (c) has no least element. Its greatest element is d. The poset with Hasse diagram (d) has least element a and greatest element d. ◀

EXAMPLE 16 Let S be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.

Solution: The least element is the empty set, because $\emptyset \subseteq T$ for any subset T of S. The set S is the greatest element in this poset, because $T \subseteq S$ whenever T is a subset of S. ◀

EXAMPLE 17 Is there a greatest element and a least element in the poset $(\mathbf{Z}^+, |)$?

Solution: The integer 1 is the least element because $1|n$ whenever n is a positive integer. Because there is no integer that is divisible by all positive integers, there is no greatest element. ◀

Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an **upper bound** of A. Likewise, there may be an element less than or equal to all the elements in A. If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a **lower bound** of A.

EXAMPLE 18 Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in Figure 7.

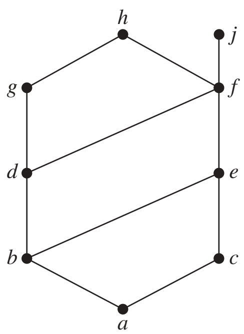


FIGURE 7 **The Hasse diagram of a poset.**

Solution: The upper bounds of $\{a, b, c\}$ are e, f, j, and h, and its only lower bound is a. There are no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e, and f. The upper bounds of $\{a, c, d, f\}$ are f, h, and j, and its lower bound is a. ◀

The element x is called the **least upper bound** of the subset A if x is an upper bound that is less than every other upper bound of A. Because there is only one such element, if it exists, it makes sense to call this element *the* least upper bound [see Exercise 42(a)]. That is, x is the least upper bound of A if $a \preccurlyeq x$ whenever $a \in A$, and $x \preccurlyeq z$ whenever z is an upper bound of A. Similarly, the element y is called the **greatest lower bound** of A if y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A. The greatest lower bound of A is unique if it exists [see Exercise 42(b)]. The greatest lower bound and least upper bound of a subset A are denoted by glb(A) and lub(A), respectively.

EXAMPLE 19 Find the greatest lower bound and the least upper bound of $\{b, d, g\}$, if they exist, in the poset shown in Figure 7.

Solution: The upper bounds of $\{b, d, g\}$ are g and h. Because $g \prec h$, g is the least upper bound. The lower bounds of $\{b, d, g\}$ are a and b. Because $a \prec b$, b is the greatest lower bound. ◀

EXAMPLE 20 Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

Extra Examples

Solution: An integer is a lower bound of $\{3, 9, 12\}$ if 3, 9, and 12 are divisible by this integer. The only such integers are 1 and 3. Because $1 \mid 3$, 3 is the greatest lower bound of $\{3, 9, 12\}$. The only lower bound for the set $\{1, 2, 4, 5, 10\}$ with respect to $|$ is the element 1. Hence, 1 is the greatest lower bound for $\{1, 2, 4, 5, 10\}$.

An integer is an upper bound for $\{3, 9, 12\}$ if and only if it is divisible by 3, 9, and 12. The integers with this property are those divisible by the least common multiple of 3, 9, and 12, which is 36. Hence, 36 is the least upper bound of $\{3, 9, 12\}$. A positive integer is an upper bound for the set $\{1, 2, 4, 5, 10\}$ if and only if it is divisible by 1, 2, 4, 5, and 10. The integers with this property are those integers divisible by the least common multiple of these integers, which is 20. Hence, 20 is the least upper bound of $\{1, 2, 4, 5, 10\}$. ◀

9.6.5 Lattices

A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a **lattice**. Lattices have many special properties. Furthermore, lattices are used in many different applications such as models of information flow and play an important role in Boolean algebra.

EXAMPLE 21 Determine whether the posets represented by each of the Hasse diagrams in Figure 8 are lattices.

Solution: The posets represented by the Hasse diagrams in (a) and (c) are both lattices because in each poset every pair of elements has both a least upper bound and a greatest lower bound, as the reader should verify. On the other hand, the poset with the Hasse diagram shown in (b) is not a lattice, because the elements b and c have no least upper bound. To see this, note that each of the elements d, e, and f is an upper bound, but none of these three elements precedes the other two with respect to the ordering of this poset. ◀

EXAMPLE 22 Is the poset $(\mathbf{Z}^+, |)$ a lattice?

Solution: Let a and b be two positive integers. The least upper bound and greatest lower bound of these two integers are the least common multiple and the greatest common divisor of these integers, respectively, as the reader should verify. It follows that this poset is a lattice. ◀

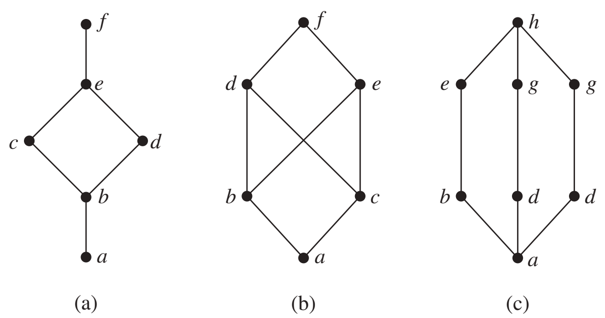


FIGURE 8 **Hasse diagrams of three posets.**

EXAMPLE 23 Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Solution: Because 2 and 3 have no upper bounds in $(\{1, 2, 3, 4, 5\}, |)$, they certainly do not have a least upper bound. Hence, the first poset is not a lattice.

Every two elements of the second poset have both a least upper bound and a greatest lower bound. The least upper bound of two elements in this poset is the larger of the elements and the greatest lower bound of two elements is the smaller of the elements, as the reader should verify. Hence, this second poset is a lattice. ◀

EXAMPLE 24 Determine whether $(P(S), \subseteq)$ is a lattice where S is a set.

Solution: Let A and B be two subsets of S. The least upper bound and the greatest lower bound of A and B are $A \cup B$ and $A \cap B$, respectively, as the reader can show. Hence, $(P(S), \subseteq)$ is a lattice. ◀

EXAMPLE 25 **The Lattice Model of Information Flow** In many settings the flow of information from one person or computer program to another is restricted via security clearances. We can use a lattice model to represent different information flow policies. For example, one common information flow policy is the *multilevel security policy* used in government and military systems. Each piece of information is assigned to a security class, and each security class is represented by a pair (A, C) where A is an *authority level* and C is a *category*. People and computer programs are then allowed access to information from a specific restricted set of security classes.

Links

There are billions of pages of classified U.S. government documents.

The typical authority levels used in the U.S. government are unclassified (0), confidential (1), secret (2), and top secret (3). (Information is said to be classified if it is confidential, secret, or top secret.) Categories used in security classes are the subsets of a set of all *compartments* relevant to a particular area of interest. Each compartment represents a particular subject area. For example, if the set of compartments is $\{$*spies, moles, double agents*$\}$, then there are eight different categories, one for each of the eight subsets of the set of compartments, such as $\{$*spies, moles*$\}$.

We can order security classes by specifying that $(A_1, C_1) \preccurlyeq (A_2, C_2)$ if and only if $A_1 \le A_2$ and $C_1 \subseteq C_2$. Information is permitted to flow from security class (A_1, C_1) into security class (A_2, C_2) if and only if $(A_1, C_1) \preccurlyeq (A_2, C_2)$. For example, information is permitted to flow from the security class (*secret*, $\{$*spies, moles*$\}$) into the security class (*top secret*, $\{$*spies, moles, double agents*$\}$), whereas information is not allowed to flow from the security class (*top secret*, $\{$*spies, moles*$\}$) into either of the security classes (*secret*, $\{$*spies, moles, double agents*$\}$) or (*top secret*, $\{$*spies*$\}$).

We leave it to the reader (see Exercise 48) to show that the set of all security classes with the ordering defined in this example forms a lattice. ◀

9.6.6 Topological Sorting

Links

Suppose that a project is made up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks? To model this problem we set up a partial order on the set of tasks so that $a \prec b$ if and only if a and b are tasks where b cannot be started until a has been completed. To produce a schedule for the project, we need to produce an order for all 20 tasks that is compatible with this partial order. We will show how this can be done.

We begin with a definition. A total ordering $\preccurlyeq$ is said to be **compatible** with the partial ordering R if $a \preccurlyeq b$ whenever aRb. Constructing a compatible total ordering from a partial ordering is called **topological sorting**.* We will need to use Lemma 1.

LEMMA 1 Every finite nonempty poset $(S, \preccurlyeq)$ has at least one minimal element.

Proof: Choose an element a_0 of S. If a_0 is not minimal, then there is an element a_1 with $a_1 \prec a_0$. If a_1 is not minimal, there is an element a_2 with $a_2 \prec a_1$. Continue this process, so that if a_n is not minimal, there is an element a_{n+1} with $a_{n+1} \prec a_n$. Because there are only a finite number of elements in the poset, this process must end with a minimal element a_n. ◁

The topological sorting algorithm we will describe works for any finite nonempty poset. To define a total ordering on the poset $(A, \preccurlyeq)$, first choose a minimal element a_1; such an element exists by Lemma 1. Next, note that $(A - \{a_1\}, \preccurlyeq)$ is also a poset, as the reader should verify. (Here by $\preccurlyeq$ we mean the restriction of the original relation $\preccurlyeq$ on A to $A - \{a_1\}$.) If it is nonempty, choose a minimal element a_2 of this poset. Then remove a_2 as well, and if there are additional elements left, choose a minimal element a_3 in $A - \{a_1, a_2\}$. Continue this process by choosing a_{k+1} to be a minimal element in $A - \{a_1, a_2, \ldots, a_k\}$, as long as elements remain.

Because A is a finite set, this process must terminate. The end product is a sequence of elements $a_1, a_2, \ldots, a_n$. The desired total ordering $\preccurlyeq_t$ is defined by

$$a_1 \prec_t a_2 \prec_t \cdots \prec_t a_n.$$

This total ordering is compatible with the original partial ordering. To see this, note that if $b \prec c$ in the original partial ordering, c is chosen as the minimal element at a phase of the algorithm where b has already been removed, for otherwise c would not be a minimal element. Pseudocode for this topological sorting algorithm is shown in Algorithm 1.

ALGORITHM 1 Topological Sorting.

procedure *topological sort* $((S, \preccurlyeq)$: finite poset)
$k := 1$
while $S \neq \emptyset$
    $a_k :=$ a minimal element of S {such an element exists by Lemma 1}
    $S := S - \{a_k\}$
    $k := k + 1$
return $a_1, a_2, \ldots, a_n$ $\{a_1, a_2, \ldots, a_n$ is a compatible total ordering of $S\}$

EXAMPLE 26 Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |)$.

Solution: The first step is to choose a minimal element. This must be 1, because it is the only minimal element. Next, select a minimal element of $(\{2, 4, 5, 12, 20\}, |)$. There are two minimal elements in this poset, namely, 2 and 5. We select 5. The remaining elements are $\{2, 4, 12, 20\}$. The only minimal element at this stage is 2. Next, 4 is chosen because it is the only minimal

*"Topological sorting" is terminology used by computer scientists; mathematicians use the terminology "linearization of a partial ordering" for the same thing. In mathematics, topology is the branch of geometry dealing with properties of geometric figures that hold for all figures that can be transformed into one another by continuous bijections. In computer science, a topology is any arrangement of objects that can be connected with edges.

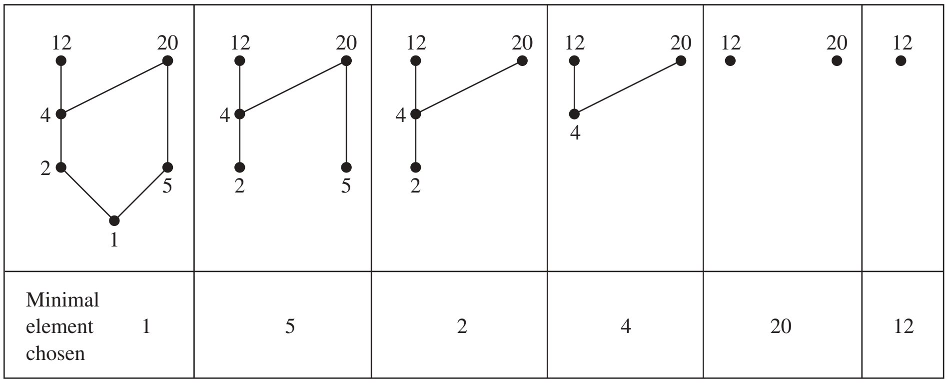


FIGURE 9 **A topological sort of ({1, 2, 4, 5, 12, 20}, |).**

element of ({4, 12, 20}, |). Because both 12 and 20 are minimal elements of ({12, 20}, |), either can be chosen next. We select 20, which leaves 12 as the last element left. This produces the total ordering

$$1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12.$$

The steps used by this sorting algorithm are displayed in Figure 9. ◀

Topological sorting has an application to the scheduling of projects.

EXAMPLE 27 A development project at a computer company requires the completion of seven tasks. Some of these tasks can be started only after other tasks are finished. A partial ordering on tasks is set up by considering task $X \prec$ task Y if task Y cannot be started until task X has been completed. The Hasse diagram for the seven tasks, with respect to this partial ordering, is shown in Figure 10. Find an order in which these tasks can be carried out to complete the project.

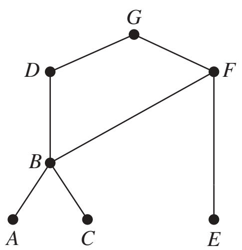


FIGURE 10 **The Hasse diagram for seven tasks.**

Solution: An ordering of the seven tasks can be obtained by performing a topological sort. The steps of a sort are illustrated in Figure 11. The result of this sort, $A \prec C \prec B \prec E \prec F \prec D \prec G$, gives one possible order for the tasks. ◀

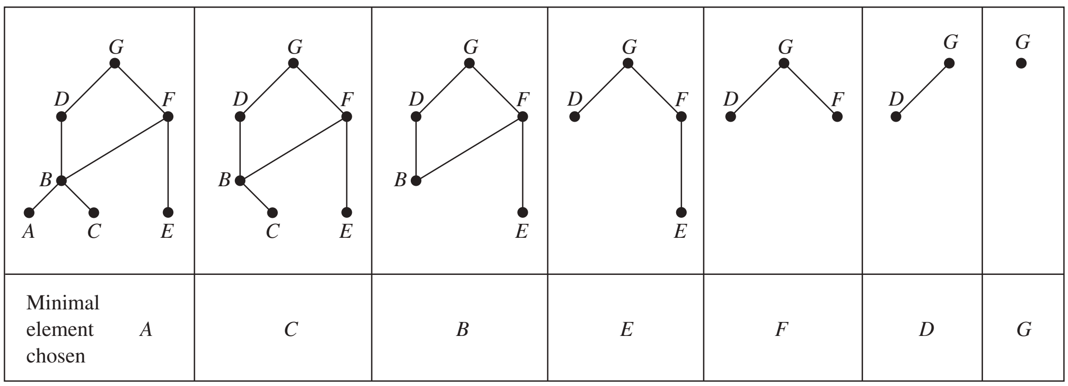


FIGURE 11 **A topological sort of the tasks.**

Exercises

1. Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 3)\}$

d) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

e) $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3)\}$

2. Which of these relations on $\{0, 1, 2, 3\}$ are partial orderings? Determine the properties of a partial ordering that the others lack.

a) $\{(0, 0), (2, 2), (3, 3)\}$

b) $\{(0, 0), (1, 1), (2, 0), (2, 2), (2, 3), (3, 3)\}$

c) $\{(0, 0), (1, 1), (1, 2), (2, 2), (3, 1), (3, 3)\}$

d) $\{(0, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (2, 3), (3, 0), (3, 3)\}$

e) $\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 0), (2, 2), (3, 3)\}$

3. Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is taller than b?

b) a is not taller than b?

c) $a = b$ or a is an ancestor of b?

d) a and b have a common friend?

4. Is (S, R) a poset if S is the set of all people in the world and $(a, b) \in R$, where a and b are people, if

a) a is no shorter than b?

b) a weighs more than b?

c) $a = b$ or a is a descendant of b?

d) a and b do not have a common friend?

5. Which of these are posets?

a) $(\mathbf{Z}, =)$ **b)** $(\mathbf{Z}, \neq)$ **c)** $(\mathbf{Z}, \geq)$ **d)** $(\mathbf{Z}, \nmid)$

6. Which of these are posets?

a) $(\mathbf{R}, =)$ **b)** $(\mathbf{R}, <)$ **c)** $(\mathbf{R}, \leq)$ **d)** $(\mathbf{R}, \neq)$

7. Determine whether the relations represented by these zero–one matrices are partial orders.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

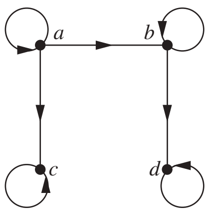
8. Determine whether the relations represented by these zero–one matrices are partial orders.

a) $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

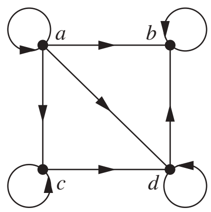
c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

In Exercises 9–11 determine whether the relation with the directed graph shown is a partial order.

9.



10.



11.

12. Let (S, R) be a poset. Show that (S, R^{-1}) is also a poset, where R^{-1} is the inverse of R. The poset (S, R^{-1}) is called the **dual** of (S, R).

13. Find the duals of these posets.

a) $(\{0, 1, 2\}, \leq)$ **b)** $(\mathbf{Z}, \geq)$

c) $(P(\mathbf{Z}), \supseteq)$ **d)** $(\mathbf{Z}^+, |)$

14. Which of these pairs of elements are comparable in the poset $(\mathbf{Z}^+, |)$?

a) 5, 15 **b)** 6, 9 **c)** 8, 16 **d)** 7, 7

15. Find two incomparable elements in these posets.

a) $(P(\{0, 1, 2\}), \subseteq)$ **b)** $(\{1, 2, 4, 6, 8\}, |)$

16. Let $S = \{1, 2, 3, 4\}$. With respect to the lexicographic order based on the usual less than elation,

a) find all pairs in $S \times S$ less than $(2, 3)$.

b) find all pairs in $S \times S$ greater than $(3, 1)$.

c) draw the Hasse diagram of the poset $(S \times S, \preccurlyeq)$.

17. Find the lexicographic ordering of these n-tuples:

a) $(1, 1, 2), (1, 2, 1)$ **b)** $(0, 1, 2, 3), (0, 1, 3, 2)$

c) $(1, 0, 1, 0, 1), (0, 1, 1, 1, 0)$

18. Find the lexicographic ordering of these strings of lowercase English letters:

a) *quack, quick, quicksilver, quicksand, quacking*

b) *open, opener, opera, operand, opened*

c) *zoo, zero, zoom, zoology, zoological*

19. Find the lexicographic ordering of the bit strings 0, 01, 11, 001, 010, 011, 0001, and 0101 based on the ordering $0 < 1$.

20. Draw the Hasse diagram for the greater than or equal to relation on $\{0, 1, 2, 3, 4, 5\}$.

21. Draw the Hasse diagram for the less than or equal to relation on $\{0, 2, 5, 10, 11, 15\}$.

22. Draw the Hasse diagram for divisibility on the set

a) $\{1, 2, 3, 4, 5, 6\}$. **b)** $\{3, 5, 7, 11, 13, 16, 17\}$.
c) $\{2, 3, 5, 10, 11, 15, 25\}$. **d)** $\{1, 3, 9, 27, 81, 243\}$.

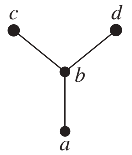
23. Draw the Hasse diagram for divisibility on the set

a) $\{1, 2, 3, 4, 5, 6, 7, 8\}$. **b)** $\{1, 2, 3, 5, 7, 11, 13\}$.
c) $\{1, 2, 3, 6, 12, 24, 36, 48\}$.
d) $\{1, 2, 4, 8, 16, 32, 64\}$.

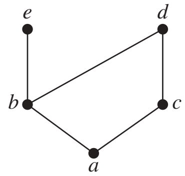
24. Draw the Hasse diagram for inclusion on the set $P(S)$, where $S = \{a, b, c, d\}$.

In Exercises 25–27 list all ordered pairs in the partial ordering with the accompanying Hasse diagram.

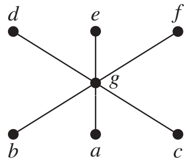
25.



26.



27.



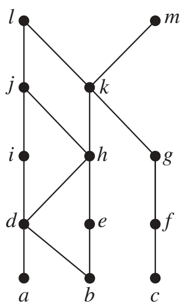
28. What is the covering relation of the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 12\}$?

29. What is the covering relation of the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set of S, where $S = \{a, b, c\}$?

30. What is the covering relation of the partial ordering for the poset of security classes defined in Example 25?

31. Show that a finite poset can be reconstructed from its covering relation. [*Hint:* Show that the poset is the reflexive transitive closure of its covering relation.]

32. Answer these questions for the partial order represented by this Hasse diagram.



a) Find the maximal elements.
b) Find the minimal elements.
c) Is there a greatest element?
d) Is there a least element?
e) Find all upper bounds of $\{a, b, c\}$.
f) Find the least upper bound of $\{a, b, c\}$, if it exists.
g) Find all lower bounds of $\{f, g, h\}$.
h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

33. Answer these questions for the poset $(\{3, 5, 9, 15, 24, 45\}, |)$.

a) Find the maximal elements.
b) Find the minimal elements.
c) Is there a greatest element?
d) Is there a least element?
e) Find all upper bounds of $\{3, 5\}$.
f) Find the least upper bound of $\{3, 5\}$, if it exists.
g) Find all lower bounds of $\{15, 45\}$.
h) Find the greatest lower bound of $\{15, 45\}$, if it exists.

34. Answer these questions for the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$.

a) Find the maximal elements.
b) Find the minimal elements.
c) Is there a greatest element?
d) Is there a least element?
e) Find all upper bounds of $\{2, 9\}$.
f) Find the least upper bound of $\{2, 9\}$, if it exists.
g) Find all lower bounds of $\{60, 72\}$.
h) Find the greatest lower bound of $\{60, 72\}$, if it exists.

35. Answer these questions for the poset $(\{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}, \subseteq)$.

a) Find the maximal elements.
b) Find the minimal elements.
c) Is there a greatest element?
d) Is there a least element?
e) Find all upper bounds of $\{\{2\}, \{4\}\}$.
f) Find the least upper bound of $\{\{2\}, \{4\}\}$, if it exists.
g) Find all lower bounds of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$.
h) Find the greatest lower bound of $\{\{1, 3, 4\}, \{2, 3, 4\}\}$, if it exists.

36. Give a poset that has

a) a minimal element but no maximal element.
b) a maximal element but no minimal element.
c) neither a maximal nor a minimal element.

37. Show that lexicographic order is a partial ordering on the Cartesian product of two posets.

38. Show that lexicographic order is a partial ordering on the set of strings from a poset.

39. Suppose that $(S, \preccurlyeq_1)$ and $(T, \preccurlyeq_2)$ are posets. Show that $(S \times T, \preccurlyeq)$ is a poset where $(s, t) \preccurlyeq (u, v)$ if and only if $s \preccurlyeq_1 u$ and $t \preccurlyeq_2 v$.

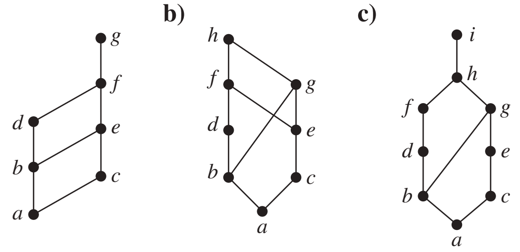
40. a) Show that there is exactly one greatest element of a poset, if such an element exists.
b) Show that there is exactly one least element of a poset, if such an element exists.

41. a) Show that there is exactly one maximal element in a poset with a greatest element.
b) Show that there is exactly one minimal element in a poset with a least element.

42. a) Show that the least upper bound of a set in a poset is unique if it exists.
b) Show that the greatest lower bound of a set in a poset is unique if it exists.

43. Determine whether the posets with these Hasse diagrams are lattices.

a) **b)** **c)**



44. Determine whether these posets are lattices.

a) $(\{1, 3, 6, 9, 12\}, |)$ **b)** $(\{1, 5, 25, 125\}, |)$

c) $(\mathbf{Z}, \geq)$

d) $(P(S), \supseteq)$, where $P(S)$ is the power set of a set S

45. Show that every nonempty finite subset of a lattice has a least upper bound and a greatest lower bound.

46. Show that if the poset (S, R) is a lattice then the dual poset (S, R^{-1}) is also a lattice.

47. In a company, the lattice model of information flow is used to control sensitive information with security classes represented by ordered pairs (A, C). Here A is an authority level, which may be nonproprietary (0), proprietary (1), restricted (2), or registered (3). A category C is a subset of the set of all projects {*Cheetah, Impala, Puma*}. (Names of animals are often used as code names for projects in companies.)

a) Is information permitted to flow from (*Proprietary*, {*Cheetah, Puma*}) into (*Restricted*, {*Puma*})?

b) Is information permitted to flow from (*Restricted*, {*Cheetah*}) into (*Registered*, {*Cheetah, Impala*})?

c) Into which classes is information from (*Proprietary*, {*Cheetah, Puma*}) permitted to flow?

d) From which classes is information permitted to flow into the security class (*Restricted*, {*Impala, Puma*})?

48. Show that the set S of security classes (A, C) is a lattice, where A is a positive integer representing an authority class and C is a subset of a finite set of compartments, with $(A_1, C_1) \preccurlyeq (A_2, C_2)$ if and only if $A_1 \leq A_2$ and $C_1 \subseteq C_2$. [*Hint:* First show that $(S, \preccurlyeq)$ is a poset and then show that the least upper bound and greatest lower bound of (A_1, C_1) and (A_2, C_2) are $(\max(A_1, A_2), C_1 \cup C_2)$ and $(\min(A_1, A_2), C_1 \cap C_2)$, respectively.]

*__49.__ Show that the set of all partitions of a set S with the relation $P_1 \preccurlyeq P_2$ if the partition P_1 is a refinement of the partition P_2 is a lattice. (See the preamble to Exercise 49 of Section 9.5.)

50. Show that every totally ordered set is a lattice.

51. Show that every finite lattice has a least element and a greatest element.

52. Give an example of an infinite lattice with

a) neither a least nor a greatest element.

b) a least but not a greatest element.

c) a greatest but not a least element.

d) both a least and a greatest element.

53. Verify that $(\mathbf{Z}^+ \times \mathbf{Z}^+, \preccurlyeq)$ is a well-ordered set, where $\preccurlyeq$ is lexicographic order, as claimed in Example 8.

54. Determine whether each of these posets is well-ordered.

a) $(S, \leq)$, where $S = \{10, 11, 12, \dots\}$

b) $(\mathbf{Q} \cap [0, 1], \leq)$ (the set of rational numbers between 0 and 1 inclusive)

c) $(S, \leq)$, where S is the set of positive rational numbers with denominators not exceeding 3

d) $(\mathbf{Z}^-, \geq)$, where $\mathbf{Z}^-$ is the set of negative integers

A poset $(R, \preccurlyeq)$ is **well-founded** if there is no infinite decreasing sequence of elements in the poset, that is, elements $x_1, x_2, \dots, x_n$ such that $\cdots \prec x_n \prec \cdots \prec x_2 \prec x_1$. A poset $(R, \preccurlyeq)$ is **dense** if for all $x \in S$ and $y \in S$ with $x \prec y$, there is an element $z \in R$ such that $x \prec z \prec y$.

55. Show that the poset $(\mathbf{Z}, \preccurlyeq)$, where $x \prec y$ if and only if $|x| < |y|$ is well-founded but is not a totally ordered set.

56. Show that a dense poset with at least two elements that are comparable is not well-founded.

57. Show that the poset of rational numbers with the usual less than or equal to relation, $(\mathbf{Q}, \leq)$, is a dense poset.

*__58.__ Show that the set of strings of lowercase English letters with lexicographic order is neither well-founded nor dense.

59. Show that a poset is well-ordered if and only if it is totally ordered and well-founded.

60. Show that a finite nonempty poset has a maximal element.

61. Find a compatible total order for the poset with the Hasse diagram shown in Exercise 32.

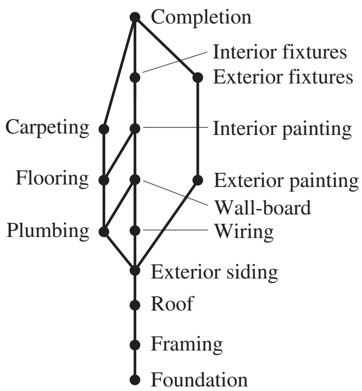
62. Find a compatible total order for the divisibility relation on the set $\{1, 2, 3, 6, 8, 12, 24, 36\}$.

63. Find all compatible total orderings for the poset $(\{1, 2, 4, 5, 12, 20\}, |\}$ from Example 26.

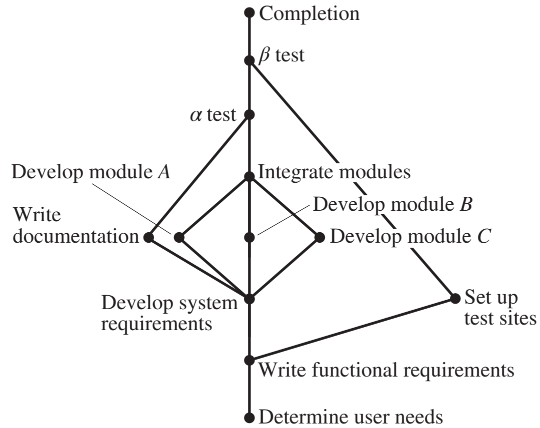
64. Find all compatible total orderings for the poset with the Hasse diagram in Exercise 27.

65. Find all possible orders for completing the tasks in the development project in Example 27.

66. Schedule the tasks needed to build a house, by specifying their order, if the Hasse diagram representing these tasks is as shown in the figure.



67. Find an ordering of the tasks of a software project if the Hasse diagram for the tasks of the project is as shown.



Key Terms and Results

TERMS

binary relation from A to B: a subset of $A \times B$

relation on A: a binary relation from A to itself (that is, a subset of $A \times A$)

$S \circ R$: composite of R and S

R^{-1}: inverse relation of R

R^n: nth power of R

reflexive: a relation R on A is reflexive if $(a, a) \in R$ for all $a \in A$

symmetric: a relation R on A is symmetric if $(b, a) \in R$ whenever $(a, b) \in R$

antisymmetric: a relation R on A is antisymmetric if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$

transitive: a relation R on A is transitive if $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$

n-ary relation on $A_1, A_2, \ldots, A_n$: a subset of $A_1 \times A_2 \times \cdots \times A_n$

relational data model: a model for representing databases using n-ary relations

primary key: a domain of an n-ary relation such that an n-tuple is uniquely determined by its value for this domain

composite key: the Cartesian product of domains of an n-ary relation such that an n-tuple is uniquely determined by its values in these domains

selection operator: a function that selects the n-tuples in an n-ary relation that satisfy a specified condition

projection: a function that produces relations of smaller degree from an n-ary relation by deleting fields

join: a function that combines n-ary relations that agree on certain fields

itemset: a collection of items

count of an itemset: the number of transactions that are supersets of the itemset

frequent itemset: an itemset with frequency greater than or equal to the support threshold

support of an itemset: the frequency of transactions that contain the itemset

association rule: an implication of the form $I \to J$, where I and J are itemsets

support of the association rule $I \to J$: the fraction of transactions that contain both the itemsets I and J

confidence of an association rule: the conditional probability that J is a subset of a transaction given that I is

directed graph or digraph: a set of elements called vertices and ordered pairs of these elements, called edges

loop: an edge of the form (a, a)

closure of a relation R with respect to a property P: the relation S (if it exists) that contains R, has property **P**, and is contained within any relation that contains R and has property **P**

path in a digraph: a sequence of edges $(a, x_1), (x_1, x_2), \ldots, (x_{n-2}, x_{n-1}), (x_{n-1}, b)$ such that the terminal vertex of each edge is the initial vertex of the succeeding edge in the sequence

circuit (or cycle) in a digraph: a path that begins and ends at the same vertex

R^* (connectivity relation): the relation consisting of those ordered pairs (a, b) such that there is a path from a to b

equivalence relation: a reflexive, symmetric, and transitive relation

equivalent: if R is an equivalence relation, a is equivalent to b if aRb

$[a]_R$ (equivalence class of a with respect to R): the set of all elements of A that are equivalent to a

$[a]_m$ (congruence class modulo m): the set of integers congruent to a modulo m

partition of a set S: a collection of pairwise disjoint nonempty subsets that have S as their union

partial ordering: a relation that is reflexive, antisymmetric, and transitive

poset (S, R): a set S and a partial ordering R on this set

comparable: the elements a and b in the poset $(A, \preccurlyeq)$ are comparable if $a \preccurlyeq b$ or $b \preccurlyeq a$

incomparable: elements in a poset that are not comparable

total (or linear) ordering: a partial ordering for which every pair of elements are comparable

totally (or linearly) ordered set: a poset with a total (or linear) ordering

well-ordered set: a poset $(S, \preccurlyeq)$, where $\preccurlyeq$ is a total order and every nonempty subset of S has a least element

lexicographic order: a partial ordering of Cartesian products or strings

Hasse diagram: a graphical representation of a poset where loops and all edges resulting from the transitive property are not shown, and the direction of the edges is indicated by the position of the vertices

maximal element: an element of a poset that is not less than any other element of the poset

minimal element: an element of a poset that is not greater than any other element of the poset

greatest element: an element of a poset greater than all other elements in this set

least element: an element of a poset less than all other elements in this set

upper bound of a set: an element in a poset greater than all other elements in the set

lower bound of a set: an element in a poset less than all other elements in the set

least upper bound of a set: an upper bound of the set that is less than all other upper bounds

greatest lower bound of a set: a lower bound of the set that is greater than all other lower bounds

lattice: a partially ordered set in which every two elements have a greatest lower bound and a least upper bound

compatible total ordering for a partial ordering: a total ordering that contains the given partial ordering

topological sort: the construction of a total ordering compatible with a given partial ordering

RESULTS

The reflexive closure of a relation R on the set A equals $R \cup \Delta$, where $\Delta = \{(a, a) \mid a \in A\}$.

The symmetric closure of a relation R on the set A equals $R \cup R^{-1}$, where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

The transitive closure of a relation equals the connectivity relation formed from this relation.

Warshall's algorithm for finding the transitive closure of a relation

Let R be an equivalence relation. Then the following three statements are equivalent: (1) $a\ R\ b$; (2) $[a]_R \cap [b]_R \neq \emptyset$; (3) $[a]_R = [b]_R$.

The equivalence classes of an equivalence relation on a set A form a partition of A. Conversely, an equivalence relation can be constructed from any partition so that the equivalence classes are the subsets in the partition.

The principle of well-ordered induction

The topological sorting algorithm

Review Questions

1. a) What is a relation on a set?
   b) How many relations are there on a set with n elements?
2. a) What is a reflexive relation?
   b) What is a symmetric relation?
   c) What is an antisymmetric relation?
   d) What is a transitive relation?
3. Give an example of a relation on the set $\{1, 2, 3, 4\}$ that is
   a) reflexive, symmetric, and not transitive.
   b) not reflexive, symmetric, and transitive.
   c) reflexive, antisymmetric, and not transitive.
   d) reflexive, symmetric, and transitive.
   e) reflexive, antisymmetric, and transitive.
4. a) How many reflexive relations are there on a set with n elements?
   b) How many symmetric relations are there on a set with n elements?
   c) How many antisymmetric relations are there on a set with n elements?
5. a) Explain how an n-ary relation can be used to represent information about students at a university.
   b) How can the 5-ary relation containing names of students, their addresses, telephone numbers, majors, and grade point averages be used to form a 3-ary relation containing the names of students, their majors, and their grade point averages?
   c) How can the 4-ary relation containing names of students, their addresses, telephone numbers, and majors and the 4-ary relation containing names of students, their student numbers, majors, and numbers of credit hours be combined into a single n-ary relation?
6. a) Explain how to use a zero–one matrix to represent a relation on a finite set.

b) Explain how to use the zero–one matrix representing a relation to determine whether the relation is reflexive, symmetric, and/or antisymmetric.

7. a) Explain how to use a directed graph to represent a relation on a finite set.

b) Explain how to use the directed graph representing a relation to determine whether a relation is reflexive, symmetric, and/or antisymmetric.

8. a) Define the reflexive closure and the symmetric closure of a relation.

b) How can you construct the reflexive closure of a relation?

c) How can you construct the symmetric closure of a relation?

d) Find the reflexive closure and the symmetric closure of the relation $\{(1, 2), (2, 3), (2, 4), (3, 1)\}$ on the set $\{1, 2, 3, 4\}$.

9. a) Define the transitive closure of a relation.

b) Can the transitive closure of a relation be obtained by including all pairs (a, c) such that (a, b) and (b, c) belong to the relation?

c) Describe two algorithms for finding the transitive closure of a relation.

d) Find the transitive closure of the relation $\{(1,1), (1,3), (2,1), (2,3), (2,4), (3,2), (3,4), (4,1)\}$.

10. a) Define an equivalence relation.

b) Which relations on the set $\{a, b, c, d\}$ are equivalence relations and contain (a, b) and (b, d)?

11. a) Show that congruence modulo m is an equivalence relation whenever m is a positive integer.

b) Show that the relation $\{(a, b) \mid a \equiv \pm b \pmod 7\}$ is an equivalence relation on the set of integers.

12. a) What are the equivalence classes of an equivalence relation?

b) What are the equivalence classes of the "congruent modulo 5" relation?

c) What are the equivalence classes of the equivalence relation in Question 11(b)?

13. Explain the relationship between equivalence relations on a set and partitions of this set.

14. a) Define a partial ordering.

b) Show that the divisibility relation on the set of positive integers is a partial order.

15. Explain how partial orderings on the sets A_1 and A_2 can be used to define a partial ordering on the set $A_1 \times A_2$.

16. a) Explain how to construct the Hasse diagram of a partial order on a finite set.

b) Draw the Hasse diagram of the divisibility relation on the set $\{2, 3, 5, 9, 12, 15, 18\}$.

17. a) Define a maximal element of a poset and the greatest element of a poset.

b) Give an example of a poset that has three maximal elements.

c) Give an example of a poset with a greatest element.

18. a) Define a lattice.

b) Give an example of a poset with five elements that is a lattice and an example of a poset with five elements that is not a lattice.

19. a) Show that every finite subset of a lattice has a greatest lower bound and a least upper bound.

b) Show that every lattice with a finite number of elements has a least element and a greatest element.

20. a) Define a well-ordered set.

b) Describe an algorithm for producing a totally ordered set compatible with a given partially ordered set.

c) Explain how the algorithm from (b) can be used to order the tasks in a project if tasks are done one at a time and each task can be done only after one or more of the other tasks have been completed.

Supplementary Exercises

1. Let S be the set of all strings of English letters. Determine whether these relations are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

a) $R_1 = \{(a, b) \mid a$ and b have no letters in common$\}$

b) $R_2 = \{(a, b) \mid a$ and b are not the same length$\}$

c) $R_3 = \{(a, b) \mid a$ is longer than $b\}$

2. Construct a relation on the set $\{a, b, c, d\}$ that is

a) reflexive, symmetric, but not transitive.

b) irreflexive, symmetric, and transitive.

c) irreflexive, antisymmetric, and not transitive.

d) reflexive, neither symmetric nor antisymmetric, and transitive.

e) neither reflexive, irreflexive, symmetric, antisymmetric, nor transitive.

3. Show that the relation R on $\mathbf{Z} \times \mathbf{Z}$ defined by $(a, b)\, R\, (c, d)$ if and only if $a + d = b + c$ is an equivalence relation.

4. Show that a subset of an antisymmetric relation is also antisymmetric.

5. Let R be a reflexive relation on a set A. Show that $R \subseteq R^2$.

6. Suppose that R_1 and R_2 are reflexive relations on a set A. Show that $R_1 \oplus R_2$ is irreflexive.

7. Suppose that R_1 and R_2 are reflexive relations on a set A. Is $R_1 \cap R_2$ also reflexive? Is $R_1 \cup R_2$ also reflexive?

8. Suppose that R is a symmetric relation on a set A. Is $\overline{R}$ also symmetric?

9. Let R_1 and R_2 be symmetric relations. Is $R_1 \cap R_2$ also symmetric? Is $R_1 \cup R_2$ also symmetric?

10. A relation R is called **circular** if aRb and bRc imply that cRa. Show that R is reflexive and circular if and only if it is an equivalence relation.

11. Show that a primary key in an n-ary relation is a primary key in any projection of this relation that contains this key as one of its fields.

12. Is the primary key in an n-ary relation also a primary key in a larger relation obtained by taking the join of this relation with a second relation?

13. Show that the reflexive closure of the symmetric closure of a relation is the same as the symmetric closure of its reflexive closure.

14. Let R be the relation on the set of all mathematicians that contains the ordered pair (a, b) if and only if a and b have written a published mathematical paper together.

a) Describe the relation R^2.

b) Describe the relation R^*.

c) The **Erdős number** of a mathematician is 1 if this mathematician wrote a paper with the prolific Hungarian mathematician Paul Erdős, it is 2 if this mathematician did not write a joint paper with Erdős but wrote a joint paper with someone who wrote a joint paper with Erdős, and so on (except that the Erdős number of Erdős himself is 0). Give a definition of the Erdős number in terms of paths in R.

15. a) Give an example to show that the transitive closure of the symmetric closure of a relation is not necessarily the same as the symmetric closure of the transitive closure of this relation.

b) Show, however, that the transitive closure of the symmetric closure of a relation must contain the symmetric closure of the transitive closure of this relation.

16. a) Let S be the set of subroutines of a computer program. Define the relation R by $\mathbf{P}\,R\,\mathbf{Q}$ if subroutine $\mathbf{P}$ calls subroutine $\mathbf{Q}$ during its execution. Describe the transitive closure of R.

b) For which subroutines $\mathbf{P}$ does $(\mathbf{P}, \mathbf{P})$ belong to the transitive closure of R?

c) Describe the reflexive closure of the transitive closure of R.

17. Suppose that R and S are relations on a set A with $R \subseteq S$ such that the closures of R and S with respect to a property $\mathbf{P}$ both exist. Show that the closure of R with respect to $\mathbf{P}$ is a subset of the closure of S with respect to $\mathbf{P}$.

18. Show that the symmetric closure of the union of two relations is the union of their symmetric closures.

***19.** Devise an algorithm, based on the concept of interior vertices, that finds the length of the longest path between two vertices in a directed graph, or determines that there are arbitrarily long paths between these vertices.

20. Which of these are equivalence relations on the set of all people?

a) $\{(x, y) \mid x$ and y have the same sign of the zodiac$\}$

b) $\{(x, y) \mid x$ and y were born in the same year$\}$

c) $\{(x, y) \mid x$ and y have been in the same city$\}$

***21.** How many different equivalence relations with exactly three different equivalence classes are there on a set with five elements?

22. Show that $\{(x, y) \mid x - y \in \mathbf{Q}\}$ is an equivalence relation on the set of real numbers, where $\mathbf{Q}$ denotes the set of rational numbers. What are $[1]$, $[\frac{1}{2}]$, and $[\pi]$?

Links


Courtesy of George Csicsery

PAUL ERDŐS (1913–1996) Paul Erdős, born in Budapest, Hungary, was the son of two high school mathematics teachers. He was a child prodigy; at age 3 he could multiply three-digit numbers in his head, and at 4 he discovered negative numbers on his own. Because his mother did not want to expose him to contagious diseases, he was mostly home-schooled. At 17 Erdős entered Eőtvős University, graduating four years later with a Ph.D. in mathematics. After graduating he spent four years at Manchester, England, on a postdoctoral fellowship. In 1938 he went to the United States because of the difficult political situation in Hungary, especially for Jews. He spent much of his time in the United States, except for 1954 to 1962, when he was banned as part of the paranoia of the McCarthy era. He also spent considerable time in Israel.

Erdős made many significant contributions to combinatorics and to number theory. One of the discoveries of which he was most proud is his elementary proof (in the sense that it does not use any complex analysis) of the prime number theorem, which provides an estimate for the number of primes not exceeding a fixed positive integer. He also participated in the modern development of the Ramsey theory.

Erdős traveled extensively throughout the world to work with other mathematicians, visiting conferences, universities, and research laboratories. He had no permanent home. He devoted himself almost entirely to mathematics, traveling from one mathematician to the next, proclaiming "My brain is open." Erdős was the author or coauthor of more than 1500 papers and had more than 500 coauthors. Copies of his articles are kept by Ron Graham, a famous discrete mathematician with whom he collaborated extensively and who took care of many of his worldly needs.

Erdős offered rewards, ranging from \$10 to \$10,000, for the solution of problems that he found particularly interesting, with the size of the reward depending on the difficulty of the problem. He paid out close to \$4000. Erdős had his own special language, using such terms as "epsilon" (child), "boss" (woman), "slave" (man), "captured" (married), "liberated" (divorced), "Supreme Fascist" (God), "Sam" (United States), and "Joe" (Soviet Union). Although he was curious about many things, he concentrated almost all his energy on mathematical research. He had no hobbies and no full-time job. He never married and apparently remained celibate. Erdős was extremely generous, donating much of the money he collected from prizes, awards, and stipends for scholarships and to worthwhile causes. He traveled extremely lightly and did not like having many material possessions.

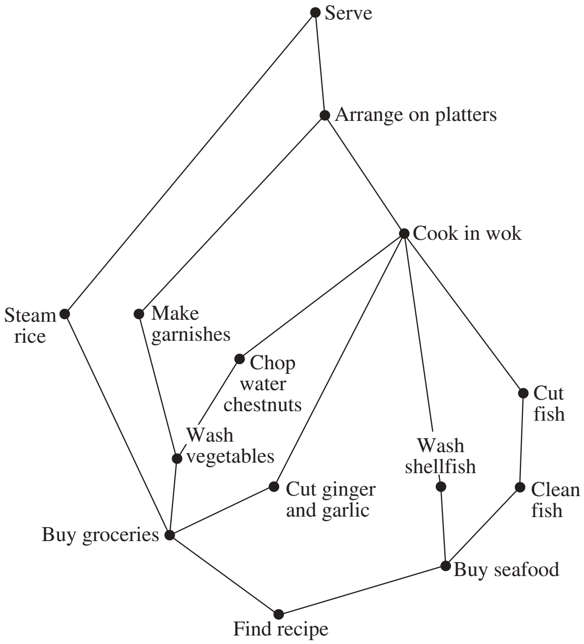
23. Suppose that $P_1 = \{A_1, A_2, \ldots, A_m\}$ and $P_2 = \{B_1, B_2, \ldots, B_n\}$ are both partitions of the set S. Show that the collection of nonempty subsets of the form $A_i \cap B_j$ is a partition of S that is a refinement of both P_1 and P_2 (see the preamble to Exercise 49 of Section 9.5).

***24.** Show that the transitive closure of the symmetric closure of the reflexive closure of a relation R is the smallest equivalence relation that contains R.

25. Let $\mathbf{R}(S)$ be the set of all relations on a set S. Define the relation $\preccurlyeq$ on $\mathbf{R}(S)$ by $R_1 \preccurlyeq R_2$ if $R_1 \subseteq R_2$, where R_1 and R_2 are relations on S. Show that $(\mathbf{R}(S), \preccurlyeq)$ is a poset.

26. Let $\mathbf{P}(S)$ be the set of all partitions of the set S. Define the relation $\preccurlyeq$ on $\mathbf{P}(S)$ by $P_1 \preccurlyeq P_2$ if P_1 is a refinement of P_2 (see Exercise 49 of Section 9.5). Show that $(\mathbf{P}(S), \preccurlyeq)$ is a poset.

27. Schedule the tasks needed to cook a Chinese meal by specifying their order, if the Hasse diagram representing these tasks is as shown here.



A subset of a poset such that every two elements of this subset are comparable is called a **chain**. A subset of a poset is called an **antichain** if every two elements of this subset are incomparable.

28. Find all chains in the posets with the Hasse diagrams shown in Exercises 25–27 in Section 9.6.

29. Find all antichains in the posets with the Hasse diagrams shown in Exercises 25–27 in Section 9.6.

30. Find an antichain with the greatest number of elements in the poset with the Hasse diagram of Exercise 32 in Section 9.6.

31. Show that every maximal chain in a finite poset $(S, \preccurlyeq)$ contains a minimal element of S. (A maximal chain is a chain that is not a subset of a larger chain.)

****32.** Show that every finite poset can be partitioned into k chains, where k is the largest number of elements in an antichain in this poset.

***33.** Show that in any group of $mn + 1$ people there is either a list of $m + 1$ people where a person in the list (except for the first person listed) is a descendant of the previous person on the list, or there are $n + 1$ people such that none of these people is a descendant of any of the other n people. [*Hint:* Use Exercise 32.]

Suppose that $(S, \preccurlyeq)$ is a well-founded partially ordered set. The *principle of well-founded induction* states that $P(x)$ is true for all $x \in S$ if $\forall x(\forall y(y \prec x \rightarrow P(y)) \rightarrow P(x))$.

34. Show that no separate basis case is needed for the principle of well-founded induction. That is, $P(u)$ is true for all minimal elements u in S if $\forall x(\forall y(y \prec x \rightarrow P(y)) \rightarrow P(x))$.

***35.** Show that the principle of well-founded induction is valid.

A relation R on a set A is a **quasi-ordering** on A if R is reflexive and transitive.

36. Let R be the relation on the set of all functions from $\mathbf{Z}^+$ to $\mathbf{Z}^+$ such that (f, g) belongs to R if and only if f is $O(g)$. Show that R is a quasi-ordering.

37. Let R be a quasi-ordering on a set A. Show that $R \cap R^{-1}$ is an equivalence relation.

***38.** Let R be a quasi-ordering and let S be the relation on the set of equivalence classes of $R \cap R^{-1}$ such that (C, D) belongs to S, where C and D are equivalence classes of R, if and only if there are elements c of C and d of D such that (c, d) belongs to R. Show that S is a partial ordering.

Let L be a lattice. Define the **meet** $(\wedge)$ and **join** $(\vee)$ operations by $x \wedge y = \text{glb}(x, y)$ and $x \vee y = \text{lub}(x, y)$.

39. Show that the following properties hold for all elements x, y, and z of a lattice L.

a) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ **(commutative laws)**

b) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$ **(associative laws)**

c) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ **(absorption laws)**

d) $x \wedge x = x$ and $x \vee x = x$ **(idempotent laws)**

40. Show that if x and y are elements of a lattice L, then $x \vee y = y$ if and only if $x \wedge y = x$.

A lattice L is **bounded** if it has both an **upper bound**, denoted by 1, such that $x \preccurlyeq 1$ for all $x \in L$ and a **lower bound**, denoted by 0, such that $0 \preccurlyeq x$ for all $x \in L$.

41. Show that if L is a bounded lattice with upper bound 1 and lower bound 0 then these properties hold for all elements $x \in L$.

a) $x \vee 1 = 1$ **b)** $x \wedge 1 = x$

c) $x \vee 0 = x$ **d)** $x \wedge 0 = 0$

42. Show that every finite lattice is bounded.

A lattice is called **distributive** if $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x, y, and z in L.

***43.** Give an example of a lattice that is not distributive.

44. Show that the lattice $(P(S), \subseteq)$ where $P(S)$ is the power set of a finite set S is distributive.

45. Is the lattice $(\mathbf{Z}^+, |)$ distributive?

The **complement** of an element a of a bounded lattice L with upper bound 1 and lower bound 0 is an element b such that $a \vee b = 1$ and $a \wedge b = 0$. Such a lattice is **complemented** if every element of the lattice has a complement.

46. Give an example of a finite lattice where at least one element has more than one complement and at least one element has no complement.

47. Show that the lattice $(P(S), \subseteq)$ where $P(S)$ is the power set of a finite set S is complemented.

***48.** Show that if L is a finite distributive lattice, then an element of L has at most one complement.

The game of Chomp, introduced in Example 12 in Section 1.8, can be generalized for play on any finite partially ordered set $(S, \preceq)$ with a least element a. In this game, a move consists of selecting an element x in S and removing x and all elements larger than it from S. The loser is the player who is forced to select the least element a.

49. Show that the game of Chomp with cookies arranged in an $m \times n$ rectangular grid, described in Example 12 in Section 1.8, is the same as the game of Chomp on the poset $(S, |)$, where S is the set of all positive integers that divide $p^{m-1}q^{n-1}$, where p and q are distinct primes.

50. Show that if $(S, \preceq)$ has a greatest element b, then a winning strategy for Chomp on this poset exists. [*Hint:* Generalize the argument in Example 12 in Section 1.8.]

Computer Projects

Write programs with these input and output.

1. Given the matrix representing a relation on a finite set, determine whether the relation is reflexive and/or irreflexive.

2. Given the matrix representing a relation on a finite set, determine whether the relation is symmetric and/or antisymmetric.

3. Given the matrix representing a relation on a finite set, determine whether the relation is transitive.

4. Given a positive integer n, display all the relations on a set with n elements.

***5.** Given a positive integer n, determine the number of transitive relations on a set with n elements.

***6.** Given a positive integer n, determine the number of equivalence relations on a set with n elements.

***7.** Given a positive integer n, display all the equivalence relations on the set of the n smallest positive integers.

8. Given an n-ary relation, find the projection of this relation when specified fields are deleted.

9. Given an m-ary relation and an n-ary relation, and a set of common fields, find the join of these relations with respect to these common fields.

10. Given the matrix representing a relation on a finite set, find the matrix representing the reflexive closure of this relation.

11. Given the matrix representing a relation on a finite set, find the matrix representing the symmetric closure of this relation.

12. Given the matrix representing a relation on a finite set, find the matrix representing the transitive closure of this relation by computing the join of the Boolean powers of the matrix representing the relation.

13. Given the matrix representing a relation on a finite set, find the matrix representing the transitive closure of this relation using Warshall's algorithm.

14. Given the matrix representing a relation on a finite set, find the matrix representing the smallest equivalence relation containing this relation.

15. Given a partial ordering on a finite set, find a total ordering compatible with it using topological sorting.

Computations and Explorations

Use a computational program or programs you have written to do these exercises.

1. Display all the different relations on a set with four elements.

2. Display all the different reflexive and symmetric relations on a set with six elements.

3. Display all the reflexive and transitive relations on a set with five elements.

***4.** Determine how many transitive relations there are on a set with n elements for all positive integers n with $n \leq 7$.

5. Find the transitive closure of a relation of your choice on a set with at least 20 elements. Either use a relation that corresponds to direct links in a particular transportation or communications network or use a randomly generated relation.

6. Compute the number of different equivalence relations on a set with n elements for all positive integers n not exceeding 20.
7. Display all the equivalence relations on a set with seven elements.

*8. Display all the partial orders on a set with five elements.

*9. Display all the lattices on a set with five elements.

Writing Projects

Respond to these with essays using outside sources.

1. Discuss the concept of a fuzzy relation. How are fuzzy relations used?
2. Describe the basic principles of relational databases, going beyond what was covered in Section 9.2. How widely used are relational databases as compared with other types of databases?
3. Explain how the Apriori algorithm is used to find frequent itemsets and strong association rules.
4. Describe some applications of association rules in detail.
5. Look up the original papers by Warshall and by Roy (in French) in which they develop algorithms for finding transitive closures. Discuss their approaches. Why do you suppose that what we call Warshall's algorithm was discovered independently by more than one person?
6. Describe how equivalence classes can be used to define the rational numbers as classes of pairs of integers and how the basic arithmetic operations on rational numbers can be defined following this approach. (See Exercise 40 in Section 9.5.)
7. Explain how Helmut Hasse used what we now call Hasse diagrams.
8. Describe some of the mechanisms used to enforce information flow policies in computer operating systems.
9. Discuss the use of the Program Evaluation and Review Technique (PERT) to schedule the tasks of a large complicated project. How widely is PERT used?
10. Discuss the use of the Critical Path Method (CPM) to find the shortest time for the completion of a project. How widely is CPM used?
11. Discuss the concept of *duality* in a lattice. Explain how duality can be used to establish new results.
12. Explain what is meant by a *modular lattice*. Describe some of the properties of modular lattices and describe how modular lattices arise in the study of projective geometry.

Answers to Odd-Numbered Exercises

CHAPTER 1

Section 1.1

1. a) Yes, T **b)** Yes, F **c)** Yes, T **d)** Yes, F **e)** No **f)** No **3. a)** Linda is not younger than Sanjay. **b)** Mei does not make more money than Isabella. **c)** Moshe is not taller than Monica. **d)** Abby is not richer than Ricardo. **5. a)** Mei does not have an MP3 player. **b)** There is pollution in New Jersey. **c)** $2 + 1 \neq 3$. **d)** The summer in Maine is not hot or it is not sunny. **7. a)** Steve does not have more than 100 GB free disk space on his laptop. **b)** Zach does not block e-mails from Jennifer, or he does not block texts from Jennifer. **c)** $7 \cdot 11 \cdot 13 \neq 999$. **d)** Diane did not ride her bike 100 miles on Sunday. **9. a)** F **b)** T **c)** T **d)** T **e)** T **11. a)** Sharks have not been spotted near the shore. **b)** Swimming at the New Jersey shore is allowed, and sharks have been spotted near the shore. **c)** Swimming at the New Jersey shore is not allowed, or sharks have been spotted near the shore. **d)** If swimming at the New Jersey shore is allowed, then sharks have not been spotted near the shore. **e)** If sharks have not been spotted near the shore, then swimming at the New Jersey shore is allowed. **f)** If swimming at the New Jersey shore is not allowed, then sharks have not been spotted near the shore. **g)** Swimming at the New Jersey shore is allowed if and only if sharks have not been spotted near the shore. **h)** Swimming at the New Jersey shore is not allowed, and either swimming at the New Jersey shore is allowed or sharks have not been spotted near the shore. (Note that we were able to incorporate the parentheses by using the word "either" in the second half of the sentence.) **13. a)** $p \wedge q$ **b)** $p \wedge \neg q$ **c)** $\neg p \wedge \neg q$ **d)** $p \vee q$ **e)** $p \rightarrow q$ **f)** $(p \vee q) \wedge (p \rightarrow \neg q)$ **g)** $q \leftrightarrow p$ **15. a)** $\neg p$ **b)** $p \wedge \neg q$ **c)** $p \rightarrow q$ **d)** $\neg p \rightarrow \neg q$ **e)** $p \rightarrow q$ **f)** $q \wedge \neg p$ **g)** $q \rightarrow p$ **17. a)** $r \wedge \neg p$ **b)** $\neg p \wedge q \wedge r$ **c)** $r \rightarrow (q \leftrightarrow \neg p)$ **d)** $\neg q \wedge \neg p \wedge r$ **e)** $(q \rightarrow (\neg r \wedge \neg p)) \wedge \neg((\neg r \wedge \neg p) \rightarrow q)$ **f)** $(p \wedge r) \rightarrow \neg q$ **19. a)** False **b)** True **c)** True **d)** True **21. a)** Exclusive or: You get only one beverage. **b)** Inclusive or: Long passwords can have any combination of symbols. **c)** Inclusive or: A student with both courses is even more qualified. **d)** Either interpretation possible; a traveler might wish to pay with a mixture of the two currencies, or the store may not allow that. **23. a)** Inclusive or: It is allowable to take discrete mathematics if you have had calculus or computer science, or both. Exclusive or: It is allowable to take discrete mathematics if you have had calculus or computer science, but not if you have had both. Most likely the inclusive or is intended. **b)** Inclusive or: You can take the rebate, or you can get a low-interest loan, or you can get both the rebate and a low-interest loan. Exclusive or: You can take the rebate, or you can get a low-interest loan, but you cannot get both the rebate and a low-interest loan. Most likely the exclusive or is intended. **c)** Inclusive or: You can order two items from column A and none from column B, or three items from column B and none from column A, or five items including two from column A and three from column B. Exclusive or: You can order two items from column A or three items from column B, but not both. Almost certainly the exclusive or is intended. **d)** Inclusive or: More than 2 feet of snow or windchill below -100 °F, or both, will close school. Exclusive or: More than 2 feet of snow or windchill below -100 °F, but not both, will close school. Certainly the inclusive or is intended. **25. a)** If the wind blows from the northeast, then it snows. **b)** If it stays warm for a week, then the apple trees will bloom. **c)** If the Pistons win the championship, then they beat the Lakers. **d)** If you get to the top of Long's Peak, then you must have walked 8 miles. **e)** If you are world famous, then you will get tenure as a professor. **f)** If you drive more than 400 miles, then you will need to buy gasoline. **g)** If your guarantee is good, then you must have bought your CD player less than 90 days ago. **h)** If the water is not too cold, then Jan will go swimming. **i)** If people believe in science, then we will have a future. **27. a)** You buy an ice cream cone if and only if it is hot outside. **b)** You win the contest if and only if you hold the only winning ticket. **c)** You get promoted if and only if you have connections. **d)** Your mind will decay if and only if you watch television. **e)** The train runs late if and only if it is a day I take the train. **29. a)** Converse: "I will ski tomorrow only if it snows today." Contrapositive: "If I do not ski tomorrow, then it will not have snowed today." Inverse: "If it does not snow today, then I will not ski tomorrow." **b)** Converse: "If I come to class, then there will be a quiz." Contrapositive: "If I do not come to class, then there will not be a quiz." Inverse: "If there is not going to be a quiz, then I don't come to class." **c)** Converse: "A positive integer is a prime if it has no divisors other than 1 and itself." Contrapositive: "If a positive integer has a divisor other than 1 and itself, then it is not prime." Inverse: "If a positive integer is not prime, then it has a divisor other than 1 and itself." **31. a)** 2 **b)** 16 **c)** 64 **d)** 16

33. a)

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

b)

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

c)

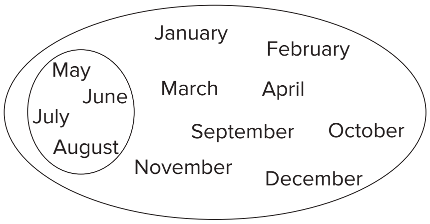
p	q	$\neg q$	$p \vee \neg q$	$(p \vee \neg q) \rightarrow q$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

CHAPTER 2

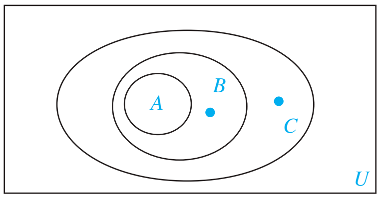
Section 2.1

1. a) $\{-1, 1\}$ **b)** $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ **c)** $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81\}$ **d)** $\emptyset$ **3. a)** $[0, 5), [0, 5]$ **b)** $(0, 5), (0, 5], [0, 5), [0, 5]$ **c)** $(0, 5), (0, 5], [0, 5), [0, 5], (1, 4], [2, 3]$ **d)** $(0, 5), (0, 5], [0, 5), [0, 5], (1, 4], [2, 3]$ **e)** $(0, 5), (0, 5], [0, 5), [0, 5], (1, 4]$ **f)** $(0, 5], [0, 5]$ **5. a)** The second is a subset of the first, but the first is not a subset of the second. **b)** Neither is a subset of the other. **c)** The first is a subset of the second, but the second is not a subset of the first. **7. a)** Yes **b)** No **c)** No **9. a)** Yes **b)** No **c)** Yes **d)** No **e)** No **f)** No **11. a)** False **b)** False **c)** False **d)** True **e)** False **f)** False **g)** True **13. a)** True **b)** True **c)** False **d)** True **e)** True **f)** False

15.



17. The dots in certain regions indicate that those regions are not empty.



19. Suppose that $x \in A$. Because $A \subseteq B$, this implies that $x \in B$. Because $B \subseteq C$, we see that $x \in C$. Because $x \in A$ implies that $x \in C$, it follows that $A \subseteq C$. **21. a)** 1 **b)** 1 **c)** 2 **d)** 3 **23. a)** $\{\emptyset, \{a\}\}$ **b)** $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ **c)** $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ **25. a)** 8 **b)** 16 **c)** 2 **27.** For the "if" part, given $A \subseteq B$, we want to show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, i.e., if $C \subseteq A$ then $C \subseteq B$. But this follows directly from Exercise 19. For the "only if" part, given that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we want to show that $A \subseteq B$. Suppose $a \in A$. Then $\{a\} \subseteq A$, so $\{a\} \in \mathcal{P}(A)$. Since $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, it follows that $\{a\} \in \mathcal{P}(B)$, which means that $\{a\} \subseteq B$. But this implies $a \in B$, as desired. **29. a)** $\{(a, y), (b, y), (c, y), (d, y), (a, z), (b, z), (c, z), (d, z)\}$ **b)** $\{(y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d)\}$ **31.** The set of triples (a, b, c), where a is an airline and b and c are cities. A useful subset of this set is the set of triples (a, b, c) for which a flies between b and c. **33.** $\emptyset \times A = \{(x, y) \mid x \in \emptyset \text{ and } y \in A\} = \emptyset = \{(x, y) \mid x \in A \text{ and } y \in \emptyset\} = A \times \emptyset$ **35. a)** $\{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (1, 3), (3, 0), (3, 1), (3, 3)\}$ **b)** $\{(1, 1), (1, 2), (1, a), (1, b), (2, 1), (2, 2), (2, a), (2, b), (a, 1), (a, 2), (a, a), (a, b), (b, 1), (b, 2), (b, a), (b, b)\}$ **37.** mn **39.** m^n **41.** The elements of $A \times B \times C$ consist of 3-tuples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$, whereas the elements of $(A \times B) \times C$ look like $((a, b), c)$—ordered pairs, the first coordinate of which is again an ordered pair. **43.** This is not true. The simplest counterexample is to let $A = B = \emptyset$. Then $A \times B = \emptyset$ and $\mathcal{P}(A \times B) = \{\emptyset\}$, whereas $\mathcal{P}(A) = \mathcal{P}(B) = \{\emptyset\}$ and $\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset)\}$. Thus, $\mathcal{P}(A \times B) \neq \mathcal{P}(A) \times \mathcal{P}(B)$. **45. a)** The square of a real number is never -1. True **b)** There exists an integer whose square is 2. False **c)** The square of every integer is positive. False **d)** There is a real number equal to its own square. True **47. a)** $\{-1, 0, 1\}$ **b)** $\mathbf{Z} - \{0, 1\}$ **c)** $\emptyset$ **49.** We must show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$. The "if" part is immediate. So assume these two sets are equal. First, consider the case when $a \neq b$. Then $\{\{a\}, \{a, b\}\}$ contains exactly two elements, one of which contains one element. Thus, $\{\{c\}, \{c, d\}\}$ must have the same property, so $c \neq d$ and $\{c\}$ is the element containing exactly one element. Hence, $\{a\} = \{c\}$, which implies that $a = c$. Also, the two-element sets $\{a, b\}$ and $\{c, d\}$ must be equal. Because $a = c$ and $a \neq b$, it follows that $b = d$. Second, suppose that $a = b$. Then $\{\{a\}, \{a, b\}\} = \{\{a\}\}$, a set with one element. Hence, $\{\{c\}, \{c, d\}\}$ has only one element, which can happen only when $c = d$, and the set is $\{\{c\}\}$. It then follows that $a = c$ and $b = d$. **51.** Let $S = \{a_1, a_2, \ldots, a_n\}$. Represent each subset of S with a bit string of length n, where the ith bit is 1 if and only if $a_i \in S$. To generate all subsets of S, list all 2^n bit strings of length n (for instance, in increasing order), and write down the corresponding subsets.

Section 2.2

1. a) The set of students who live within one mile of school and walk to classes **b)** The set of students who live within one mile of school or walk to classes (or do both) **c)** The set of students who live within one mile of school but do not walk to classes **d)** The set of students who walk to classes but live more than one mile away from school **3. a)** $\{0, 1, 2, 3, 4, 5, 6\}$ **b)** $\{3\}$ **c)** $\{1, 2, 4, 5\}$ **d)** $\{0, 6\}$ **5.** $\overline{\overline{A}} = \{x \mid \neg(x \in \overline{A})\} = \{x \mid \neg(\neg x \in A)\} = \{x \mid x \in A\} = A$ **7. a)** $A \cup U = \{x \mid x \in A \vee x \in U\} = \{x \mid x \in A \vee \mathbf{T}\} = \{x \mid \mathbf{T}\} = U$ **b)** $A \cap \emptyset = \{x \mid x \in A \wedge x \in \emptyset\} = \{x \mid x \in A \wedge \mathbf{F}\} = \{x \mid \mathbf{F}\} = \emptyset$ **9. a)** $A \cup \overline{A} = \{x \mid x \in A \vee x \notin A\} = U$ **b)** $A \cap \overline{A} = \{x \mid x \in A \wedge x \notin A\} = \emptyset$ **11. a)** $A \cup B = \{x \mid x \in A \vee x \in B\} = \{x \mid x \in B \vee x \in A\} = B \cup A$ **b)** $A \cap B = \{x \mid x \in A \wedge x \in B\} = \{x \mid x \in B \wedge x \in A\} = B \cap A$ **13.** Suppose $x \in A \cap (A \cup B)$. Then $x \in A$ and $x \in A \cup B$ by the definition of intersection. Because $x \in A$, we have proved that the left-hand side is a subset of the right-hand side. Conversely, let $x \in A$. Then by the definition of union, $x \in A \cup B$ as well. Therefore, $x \in A \cap (A \cup B)$ by the definition of intersection, so the right-hand side is a subset of the left-hand side. **15. a)** $x \in \overline{A \cup B} \equiv x \notin A \cup B \equiv \neg(x \in A \vee x \in B) \equiv \neg(x \in A) \wedge \neg(x \in B) \equiv x \notin A \wedge x \notin B \equiv x \in \overline{A} \wedge x \in \overline{B} \equiv x \in \overline{A} \cap \overline{B}$

b)

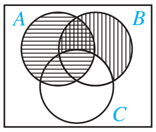
A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

17. Suppose $A \subseteq B$. We must show that every element x of U is an element of $\overline{A} \cup B$. Either $x \in \overline{A}$ or $x \in A$, and if $x \in A$ then $x \in B$. Thus, $x \in \overline{A} \cup B$ in all cases. Conversely, suppose that $\overline{A} \cup B = U$, and let $x \in A$. Then $x \notin \overline{A}$, so it must be that $x \in B$. This shows that $A \subseteq B$, and the proof is complete. **19. a)** $x \in \overline{A \cap B \cap C} \equiv x \notin A \cap B \cap C \equiv x \notin A \vee x \notin B \vee x \notin C \equiv x \in \overline{A} \vee x \in \overline{B} \vee x \in \overline{C} \equiv x \in \overline{A} \cup \overline{B} \cup \overline{C}$

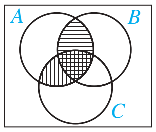
b)

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

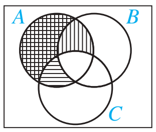
21. a) Both sides equal $\{x \mid x \in A \wedge x \notin B\}$. **b)** $A = A \cap U = A \cap (B \cup \overline{B}) = (A \cap B) \cup (A \cap \overline{B})$ **23.** $x \in A \cup (B \cup C) \equiv (x \in A) \vee (x \in (B \cup C)) \equiv (x \in A) \vee (x \in B \vee x \in C) \equiv (x \in A \vee x \in B) \vee (x \in C) \equiv x \in (A \cup B) \cup C$ **25.** $x \in A \cup (B \cap C) \equiv (x \in A) \vee (x \in (B \cap C)) \equiv (x \in A) \vee (x \in B \wedge x \in C) \equiv (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \equiv x \in (A \cup B) \cap (A \cup C)$ **27. a)** $\{4,6\}$ **b)** $\{0,1,2,3,4,5,6,7,8,9,10\}$ **c)** $\{4, 5, 6, 8, 10\}$ **d)** $\{0,2,4,5,6,7,8,9,10\}$ **29. a)** The double-shaded portion is the desired set.



b) The desired set is the entire shaded portion.



c) The desired set is the entire shaded portion.

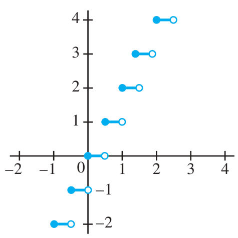


31. a) $B \subseteq A$ **b)** $A \subseteq B$ **c)** $A \cap B = \emptyset$ **d)** Nothing, because this is always true **e)** $A = B$ **33.** $A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B) \equiv \forall x(x \notin B \rightarrow x \notin A) \equiv \forall x(x \in \overline{B} \rightarrow x \in \overline{A}) \equiv \overline{B} \subseteq \overline{A}$ **35.** By De Morgan's law, the left-hand side equals $(\overline{A} \cap \overline{B}) \cap (\overline{B} \cap \overline{C}) \cap (\overline{A} \cap \overline{C})$. By the commutative, associative, and idempotent laws, this simplifies to the right-hand side. **37. a)** Let $(x, y) \in A \times (B - C)$, which means that $x \in A$ and y is an element of B but not C. Thus, $(x, y) \in A \times B$ and $(x, y) \notin A \times C$, so by the definition of set difference, $(x, y) \in (A \times B) - (A \times C)$. Conversely, let $(x, y) \in (A \times B) - (A \times C)$. Then $(x, y) \in A \times B$ and $(x, y) \notin A \times C$. Thus, $x \in A$ and $y \in B$, and because $x \in A$, it must be that $y \notin C$. This implies that $y \in B - C$, so indeed $(x, y) \in A \times (B - C)$. **b)** Note that the complement in the right-hand side must mean with respect to $U \times U$. This is not true. For example, let $U = \{a, b\}$, $A = \{a\}$, $B = \{b\}$, and $C = \emptyset$. Then the left-hand side is $\{(b, a)\}$, whereas the right-hand side is $\{(a, a), (b, a), (b, b)\}$. **39.** The set of students who are computer science majors but not mathematics majors or who are mathematics majors but not computer science majors **41.** An element is in $(A \cup B) - (A \cap B)$ if it is in the union of A and B but not in the intersection of A and B, which means that it is in either A or B but not in both A and B. This is exactly what it means for an element to belong to $A \oplus B$. **43. a)** $A \oplus A = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset$ **b)** $A \oplus \emptyset = (A - \emptyset) \cup (\emptyset - A) = A \cup \emptyset = A$ **c)** $A \oplus U = (A - U) \cup (U - A) = \emptyset \cup \overline{A} = \overline{A}$ **d)** $A \oplus \overline{A} = (A - \overline{A}) \cup (\overline{A} - A) = A \cup \overline{A} = U$ **45.** $B = \emptyset$ **47.** Yes **49.** Yes **51.** If $A \cup B$ were finite, then it would have n elements for some natural number n. But A already has more than n elements, because it is infinite, and $A \cup B$ has all the elements that A has, so $A \cup B$ has more than n elements. This contradiction shows that $A \cup B$ must be infinite. **53. a)** $\{1, 2, 3, \ldots, n\}$ **b)** $\{1\}$ **55. a)** A_n **b)** $\{0, 1\}$ **57. a)** $\mathbf{Z}$, $\{-1, 0, 1\}$ **b)** $\mathbf{Z} - \{0\}$, $\emptyset$ **c)** $\mathbf{R}$, $[-1, 1]$ **d)** $[1, \infty)$, $\emptyset$ **59. a)** $\{1, 2, 3, 4, 7, 8, 9, 10\}$ **b)** $\{2, 4, 5, 6, 7\}$ **c)** $\{1, 10\}$ **61.** The bit in the ith position of the bit string of the difference of two sets is 1 if the ith bit of the first string is 1 and the ith bit of the second string is 0, and is 0 otherwise. **63. a)** 11 1110 0000 0000 0000 0000 0000 $\vee$ 01 1100 1000 0000 0100 0101 0000 = 11 1110 1000 0000 0100 0101 0000, representing $\{a, b, c, d, e, g, p, t, v\}$ **b)** 11 1110 0000 0000 0000 0000 0000 $\wedge$ 01 1100 1000 0000 0100 0101 0000 = 01 1100 0000 0000 0000 0000 0000, representing $\{b, c, d\}$ **c)** (11 1110 0000 0000 0000 0000 0000 $\vee$ 00 0110 0110 0001 1000 0110 0110) $\wedge$ (01 1100 1000 0000 0100 0101 0000 $\vee$ 00 1010 0010 0000 1000 0010 0111) = 11 1110 0110 0001 1000 0110 0110 $\wedge$ 01 1110 1010 0000 1100 0111 0111 = 01 1110 0010 0000 1000 0110 0110, representing $\{b, c, d, e, i, o, t, u, x, y\}$ **d)** 11 1110 0000 0000 0000 0000 0000 $\vee$ 01 1100 1000 0000 0100 0101 0000 $\vee$ 00 1010 0010 0000 1000 0010 0111 $\vee$ 00 0110 0110 0001 1000 0110 0110 = 11 1110 1110 0001 1100 0111 0111, representing $\{a,b,c,d,e,g,h,i,n,o,p,t,u,v,x,y,z\}$ **65. a)** $\{1, 2, 3, \{1, 2, 3\}\}$ **b)** $\{\emptyset\}$ **c)** $\{\emptyset, \{\emptyset\}\}$ **d)** $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ **67. a)** $\{3 \cdot a, 3 \cdot b, 1 \cdot c, 4 \cdot d\}$ **b)** $\{2 \cdot a, 2 \cdot b\}$ **c)** $\{1 \cdot a, 1 \cdot c\}$ **d)** $\{1 \cdot b, 4 \cdot d\}$ **e)** $\{5 \cdot a, 5 \cdot b, 1 \cdot c, 4 \cdot d\}$ **69.** These all follow from the definitions of the multiset operations. **a)** False, because the union is actually $\{a, a, a\}$ **b)** False, because the union is actually $\{a, a, a\}$ **c)** True **d)** False, because the correct intersection is as stated in part (c) **e)** True **71. a)** 0, 1 **b)** $1/3, 2/3$ **c)** 1, 0 **d)** $1/6, 5/6$ **73.** $\overline{F}$ = {0.4 Alice, 0.1 Brian, 0.6 Fred, 0.9 Oscar, 0.5 Rita}, $\overline{R}$ = {0.6 Alice, 0.2 Brian, 0.8 Fred, 0.1 Oscar, 0.3 Rita} **75.** {0.4 Alice, 0.8 Brian, 0.2 Fred, 0.1 Oscar, 0.5 Rita}

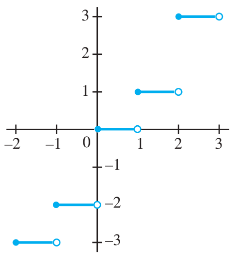
Section 2.3

1. a) $f(0)$ is not defined. **b)** $f(x)$ is not defined for $x < 0$. **c)** $f(x)$ is not well defined because there are two distinct values assigned to each x. **3. a)** Not a function **b)** A function **c)** Not a function **5. a)** Domain the set of bit strings; range the set of integers **b)** Domain the set of bit strings; range the set of even nonnegative integers **c)** Domain the set of bit strings; range the set of nonnegative integers not exceeding 7 **d)** Domain the set of positive integers; range the set of squares of positive integers $= \{1, 4, 9, 16, \dots\}$ **7. a)** Domain $\mathbf{Z}^+ \times \mathbf{Z}^+$; range $\mathbf{Z}^+$ **b)** Domain $\mathbf{Z}^+$; range $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ **c)** Domain the set of bit strings; range $\mathbf{N}$ **d)** Domain the set of bit strings; range $\mathbf{N}$ **9. a)** 1 **b)** 0 **c)** 0 **d)** −1 **e)** 3 **f)** −1 **g)** 2 **h)** 1 **11.** Only the function in part (a) **13.** Only the functions in parts (a) and (d) **15. a)** Onto **b)** Not onto **c)** Onto **d)** Not onto **e)** Onto **17. a)** Depends on whether teachers share offices **b)** One-to-one assuming only one teacher per bus **c)** Most likely not one-to-one, especially if salary is set by a collective bargaining agreement **d)** One-to-one **19.** Answers will vary. **a)** Set of offices at the school; probably not onto **b)** Set of buses going on the trip; onto, assuming every bus gets a teacher chaperone **c)** Set of real numbers; not onto **d)** Set of strings of nine digits with hyphens after third and fifth digits; not onto **21. a)** The function $f(x)$ with $f(x) = 3x + 1$ when $x \geq 0$ and $f(x) = -3x + 2$ when $x < 0$ **b)** $f(x) = |x| + 1$ **c)** The function $f(x)$ with $f(x) = 2x + 1$ when $x \geq 0$ and $f(x) = -2x$ when $x < 0$ **d)** $f(x) = x^2 + 1$ **23. a)** Yes **b)** No **c)** Yes **d)** No **25.** Suppose that f is strictly decreasing. This means that $f(x) > f(y)$ whenever $x < y$. To show that g is strictly increasing, suppose that $x < y$. Then $g(x) = 1/f(x) < 1/f(y) = g(y)$. Conversely, suppose that g is strictly increasing. This means that $g(x) < g(y)$ whenever $x < y$. To show that f is strictly decreasing, suppose that $x < y$. Then $f(x) = 1/g(x) > 1/g(y) = f(y)$. **27. a)** Let f be a given strictly decreasing function from $\mathbf{R}$ to itself. If $a < b$, then $f(a) > f(b)$; if $a > b$, then $f(a) < f(b)$. Thus, if $a \neq b$, then $f(a) \neq f(b)$. **b)** Answers will vary; for example, $f(x) = 0$ for $x < 0$ and $f(x) = -x$ for $x \geq 0$. **29.** The function is not one-to-one, so it is not invertible. On the restricted domain, the function is the identity function on the nonnegative real numbers, $f(x) = x$, so it is its own inverse. **31. a)** $f(S) = \{0, 1, 3\}$ **b)** $f(S) = \{0, 1, 3, 5, 8\}$ **c)** $f(S) = \{0, 8, 16, 40\}$ **d)** $f(S) = \{1, 12, 33, 65\}$ **33. a)** Let x and y be distinct elements of A. Because g is one-to-one, $g(x)$ and $g(y)$ are distinct elements of B. Because f is one-to-one, $f(g(x)) = (f \circ g)(x)$ and $f(g(y)) = (f \circ g)(y)$ are distinct elements of C. Hence, $f \circ g$ is one-to-one. **b)** Let $y \in C$. Because f is onto, $y = f(b)$ for some $b \in B$. Now because g is onto, $b = g(x)$ for some $x \in A$. Hence, $y = f(b) = f(g(x)) = (f \circ g)(x)$. It follows that $f \circ g$ is onto. **35.** Let $A = \{a\}$, $B = \{b_1, b_2\}$, $C = \{c\}$, $g(a) = b_1$, and $f(b_1) = f(b_2) = c$. **37.** No. For example, suppose that $A = \{a\}$, $B = \{b, c\}$, and $C = \{d\}$. Let $g(a) = b$, $f(b) = d$, and $f(c) = d$. Then f and $f \circ g$ are onto, but g is not. **39.** $(f + g)(x) = x^2 + x + 3$, $(fg)(x) = x^3 + 2x^2 + x + 2$ **41.** f is one-to-one because $f(x_1) = f(x_2) \to ax_1 + b = ax_2 + b \to ax_1 = ax_2 \to x_1 = x_2$. f is onto because $f((y - b)/a) = y$. $f^{-1}(y) = (y - b)/a$. **43. a)** $A = B = \mathbf{R}$, $S = \{x \mid x > 0\}$, $T = \{x \mid x < 0\}$, $f(x) = x^2$ **b)** It suffices to show that $f(S) \cap f(T) \subseteq f(S \cap T)$. Let $y \in B$ be an element of $f(S) \cap f(T)$. Then $y \in f(S)$, so $y = f(x_1)$ for some $x_1 \in S$. Similarly, $y = f(x_2)$ for some $x_2 \in T$. Because f is one-to-one, it follows that $x_1 = x_2$. Therefore, $x_1 \in S \cap T$, so $y \in f(S \cap T)$. **45. a)** $\{x \mid 0 \leq x < 1\}$ **b)** $\{x \mid -1 \leq x < 2\}$ **c)** $\emptyset$ **47.** $f^{-1}(\overline{S}) = \{x \in A \mid f(x) \notin S\} = \overline{\{x \in A \mid f(x) \in S\}} = \overline{f^{-1}(S)}$ **49.** Let $x = \lfloor x \rfloor + \epsilon$, where ϵ is a real number with $0 \leq \epsilon < 1$. If $\epsilon < \frac{1}{2}$, then $\lfloor x \rfloor - 1 < x - \frac{1}{2} < \lfloor x \rfloor$, so $\lceil x - \frac{1}{2} \rceil = \lfloor x \rfloor$ and this is the integer closest to x. If $\epsilon > \frac{1}{2}$, then $\lfloor x \rfloor < x - \frac{1}{2} < \lfloor x \rfloor + 1$, so $\lceil x - \frac{1}{2} \rceil = \lfloor x \rfloor + 1$ and this is the integer closest to x. If $\epsilon = \frac{1}{2}$, then $\lceil x - \frac{1}{2} \rceil = \lfloor x \rfloor$, which is the smaller of the two integers that surround x and are the same distance from x. **51.** Write the real number x as $\lfloor x \rfloor + \epsilon$, where ϵ is a real number with $0 \leq \epsilon < 1$. Because $\epsilon = x - \lfloor x \rfloor$, it follows that $0 \leq -\lfloor x \rfloor < 1$. The first two inequalities, $x - 1 < \lfloor x \rfloor$ and $\lfloor x \rfloor \leq x$, follow directly. For the other two inequalities, write $x = \lceil x \rceil - \epsilon'$, where $0 \leq \epsilon' < 1$. Then $0 \leq \lceil x \rceil - x < 1$, and the desired inequality follows. **53. a)** If $x < n$, because $\lfloor x \rfloor \leq x$, it follows that $\lfloor x \rfloor < n$. Suppose that $x \geq n$. By the definition of the floor function, it follows that $\lfloor x \rfloor \geq n$. This means that if $\lfloor x \rfloor < n$, then $x < n$. **b)** If $n < x$, then because $x \leq \lceil x \rceil$, it follows that $n \leq \lceil x \rceil$. Suppose that $n \geq x$. By the definition of the ceiling function, it follows that $\lceil x \rceil \leq n$. This means that if $n < \lceil x \rceil$, then $n < x$. **55.** If n is even, then $n = 2k$ for some integer k. Thus, $\lfloor n/2 \rfloor = \lfloor k \rfloor = k = n/2$. If n is odd, then $n = 2k + 1$ for some integer k. Thus, $\lfloor n/2 \rfloor = \lfloor k + \frac{1}{2} \rfloor = k = (n - 1)/2$. **57.** Assume that $x \geq 0$. The left-hand side is $\lceil -x \rceil$ and the right-hand side is $-\lfloor x \rfloor$. If x is an integer, then both sides equal $-x$. Otherwise, let $x = n + \epsilon$, where n is a natural number and ϵ is a real number with $0 \leq \epsilon < 1$. Then $\lceil -x \rceil = \lceil -n - \epsilon \rceil = -n$ and $-\lfloor x \rfloor = -\lfloor n + \epsilon \rfloor = -n$ also. When $x < 0$, the equation also holds because it can be obtained by substituting $-x$ for x. **59.** $\lceil b \rceil - \lfloor a \rfloor - 1$ **61. a)** 1 **b)** 3 **c)** 126 **d)** 3600 **63. a)** 100 **b)** 256 **c)** 1030 **d)** 30,200

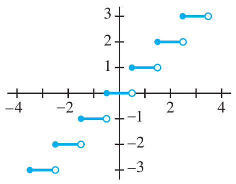
65.



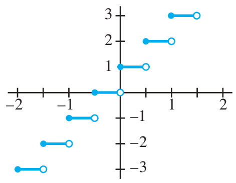
67.



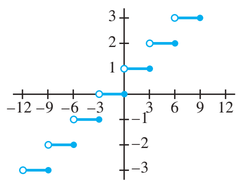
69. a)



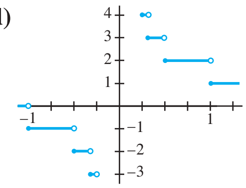
b)



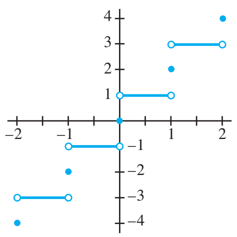
c)



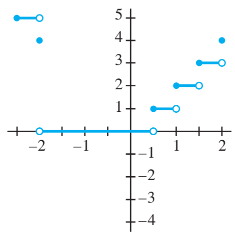
d)



e)



f)



g) See part (a). **71.** $f^{-1}(y) = (y-1)^{1/3}$ **73. a)** $f_{A\cap B}(x) = 1 \leftrightarrow x \in A \cap B \leftrightarrow x \in A$ and $x \in B \leftrightarrow f_A(x) = 1$ and $f_B(x) = 1 \leftrightarrow f_A(x)f_B(x) = 1$ **b)** $f_{A\cup B}(x) = 1 \leftrightarrow x \in A \cup B \leftrightarrow x \in A$ or $x \in B \leftrightarrow f_A(x) = 1$ or $f_B(x) = 1 \leftrightarrow f_A(x)+f_B(x)-f_A(x)f_B(x) = 1$ **c)** $f_{\overline{A}}(x) = 1 \leftrightarrow x \in \overline{A} \leftrightarrow x \notin A \leftrightarrow f_A(x) = 0 \leftrightarrow 1 - f_A(x) = 1$ **d)** $f_{A\oplus B}(x) = 1 \leftrightarrow x \in A \oplus B \leftrightarrow (x \in A$ and $x \notin B)$ or $(x \notin A$ and $x \in B) \leftrightarrow f_A(x) + f_B(x) - 2f_A(x)f_B(x) = 1$ **75. a)** True; because $\lfloor x \rfloor$ is already an integer, $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$. **b)** False; $x = \frac{1}{2}$ is a counterexample. **c)** True; if x or y is an integer, then by property 4b in Table 1, the difference is 0. If neither x nor y is an integer, then $x = n + \epsilon$ and $y = m + \delta$, where n and m are integers and ϵ and δ are positive real numbers less than 1. Then $m + n < x + y < m + n + 2$, so $\lceil x + y \rceil$ is either $m+n+1$ or $m+n+2$. Therefore, the given expression is either $(n + 1) + (m + 1) - (m + n + 1) = 1$ or $(n + 1) + (m + 1) - (m + n + 2) = 0$, as desired. **d)** False; $x = \frac{1}{4}$ and $y = 3$ is a counterexample. **e)** False; $x = \frac{1}{2}$ is a counterexample. **77. a)** If x is a positive integer, then the two sides are equal. So suppose that $x = n^2 + m + \epsilon$, where n^2 is the largest perfect square less than x, m is a nonnegative integer, and $0 < \epsilon \le 1$. Then both $\sqrt{x}$ and $\sqrt{\lfloor x \rfloor} = \sqrt{n^2 + m}$ are between n and $n + 1$, so both sides equal n. **b)** If x is a positive integer, then the two sides are equal. So suppose that $x = n^2 - m - \epsilon$, where n^2 is the smallest perfect square greater than x, m is a nonnegative integer, and ϵ is a real number with $0 < \epsilon \le 1$. Then both $\sqrt{x}$ and $\sqrt{\lceil x \rceil} = \sqrt{n^2 - m}$ are between $n - 1$ and n. Therefore, both sides of the equation equal n. **79. a)** Domain is $\mathbf{Z}$; codomain is $\mathbf{R}$; domain of definition is the set of nonzero integers; the set of values for which f is undefined is $\{0\}$; not a total function. **b)** Domain is $\mathbf{Z}$; codomain is $\mathbf{Z}$; domain of definition is $\mathbf{Z}$; set of values for which f is undefined is $\emptyset$; total function. **c)** Domain is $\mathbf{Z} \times \mathbf{Z}$; codomain is $\mathbf{Q}$; domain of definition is $\mathbf{Z} \times (\mathbf{Z} - \{0\})$; set of values for which f is undefined is $\mathbf{Z} \times \{0\}$; not a total function. **d)** Domain is $\mathbf{Z} \times \mathbf{Z}$; codomain is $\mathbf{Z}$; domain of definition is $\mathbf{Z} \times \mathbf{Z}$; set of values for which f is undefined is $\emptyset$; total function. **e)** Domain is $\mathbf{Z} \times \mathbf{Z}$; codomain is $\mathbf{Z}$; domain of definitions is $\{(m, n) \mid m > n\}$; set of values for which f is undefined is $\{(m, n) \mid m \le n\}$; not a total function. **81. a)** By definition, to say that S has cardinality m is to say that S has exactly m distinct elements. Therefore we can assign the first object to 1, the second to 2, and so on. This provides the one-to-one correspondence. **b)** By part (a), there is a bijection f from S to $\{1, 2, \ldots, m\}$ and a bijection g from T to $\{1, 2, \ldots, m\}$. Then the composition $g^{-1} \circ f$ is the desired bijection from S to T.

Section 2.4

1. a) 3 **b)** −1 **c)** 787 **d)** 2639 **3. a)** $a_0 = 2, a_1 = 3, a_2 = 5, a_3 = 9$ **b)** $a_0 = 1, a_1 = 4, a_2 = 27, a_3 = 256$ **c)** $a_0 = 0, a_1 = 0, a_2 = 1, a_3 = 1$ **d)** $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3$ **5. a)** 2, 5, 8, 11, 14, 17, 20, 23, 26, 29 **b)** 1, 1, 1, 2, 2, 2, 3, 3, 3, 4 **c)** 1, 1, 3, 3, 5, 5, 7, 7, 9, 9 **d)** −1, −2, −2, 8, 88, 656, 4912, 40064, 362368, 3627776 **e)** 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536 **f)** 2, 4, 6, 10, 16, 26, 42, 68, 110, 178 **g)** 1, 2, 2, 3, 3, 3, 3, 4, 4, 4 **h)** 3, 3, 5, 4, 4, 3, 5, 5, 4, 3 **7.** Each term could be twice the previous term; the nth term could be obtained from the previous term by adding $n - 1$; the terms could be the positive integers that are not multiples of 3; there are infinitely many other possibilities. **9. a)** 2, 12, 72, 432, 2592 **b)** 2, 4, 16, 256, 65, 536 **c)** 1, 2, 5, 11, 26 **d)** 1, 1, 6, 27, 204 **e)** 1, 2, 0, 1, 3 **11. a)** 6, 17, 49, 143, 421 **b)** $49 = 5 \cdot 17 - 6 \cdot 6$, $143 = 5 \cdot 49 - 6 \cdot 17$, $421 = 5 \cdot 143 - 6 \cdot 49$ **c)** $5a_{n-1} - 6a_{n-2} = 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2}) = 2^{n-2}(10 - 6) + 3^{n-2}(75 - 30) = 2^{n-2} \cdot 4 + 3^{n-2} \cdot 9 \cdot 5 = 2^n + 3^n \cdot 5 = a_n$ **13. a)** Yes **b)** No **c)** No **d)** Yes **e)** Yes **f)** Yes **g)** No **h)** No **15. a)** $a_{n-1} + 2a_{n-2} + 2n - 9 = -(n - 1) + 2 + 2[-(n - 2) + 2] + 2n - 9 = -n + 2 = a_n$ **b)** $a_{n-1} + 2a_{n-2} + 2n - 9 = 5(-1)^{n-1} - (n - 1) + 2 + 2[5(-1)^{n-2} - (n - 2) + 2] + 2n - 9 = 5(-1)^{n-2}(-1 + 2) - n + 2 = a_n$ **c)** $a_{n-1} + 2a_{n-2} + 2n - 9 = 3(-1)^{n-1} + 2^{n-1} - (n - 1) + 2 + 2[3(-1)^{n-2} + 2^{n-2} - (n - 2) + 2] + 2n - 9 = 3(-1)^{n-2}(-1 + 2) + 2^{n-2}(2 + 2) - n + 2 = a_n$ **d)** $a_{n-1} + 2a_{n-2} + 2n - 9 = 7 \cdot 2^{n-1} - (n - 1) + 2 + 2[7 \cdot 2^{n-2} - (n - 2) + 2] + 2n - 9 = 2^{n-2}(7 \cdot 2 + 2 \cdot 7) - n + 2 = a_n$ **17. a)** $a_n = 2 \cdot 3^n$ **b)** $a_n = 2n + 3$ **c)** $a_n = 1 + n(n + 1)/2$ **d)** $a_n = n^2 + 4n + 4$ **e)** $a_n = 1$ **f)** $a_n = (3^{n+1} - 1)/2$ **g)** $a_n = 5n!$ **h)** $a_n = 2^n n!$ **19. a)** $a_n = 3a_{n-1}$ **b)** 5,904,900 **21. a)** $a_n = n + a_{n-1}, a_0 = 0$ **b)** $a_{12} = 78$ **c)** $a_n = n(n + 1)/2$

23. $B(k) = [1 + (0.07/12)]B(k-1) - 100$, with $B(0) = 5000$ **25. a)** One 1 and one 0, followed by two 1s and two 0s, followed by three 1s and three 0s, and so on; 1, 1, 1 **b)** The positive integers are listed in increasing order with each even positive integer listed twice; 9, 10, 10. **c)** The terms in odd-numbered locations are the successive powers of 2; the terms in even-numbered locations are all 0; 32, 0, 64. **d)** $a_n = 3 \cdot 2^{n-1}$; 384, 768, 1536 **e)** $a_n = 15 - 7(n-1) = 22 - 7n$; -34, -41, -48 **f)** $a_n = (n^2 + n + 4)/2$; 57, 68, 80 **g)** $a_n = 2n^3$; 1024, 1458, 2000 **h)** $a_n = n! + 1$; 362881, 3628801, 39916801 **27.** Among the integers $1, 2, \ldots, a_n$, where a_n is the nth positive integer not a perfect square, the nonsquares are $a_1, a_2, \ldots, a_n$ and the squares are $1^2, 2^2, \ldots, k^2$, where k is the integer with $k^2 < n+k < (k+1)^2$. Consequently, $a_n = n + k$, where $k^2 < a_n < (k+1)^2$. To find k, first note that $k^2 < n + k < (k+1)^2$, so $k^2 + 1 \le n + k \le (k+1)^2 - 1$. Hence, $(k - \frac{1}{2})^2 + \frac{3}{4} = k^2 - k + 1 \le n \le k^2 + k = (k + \frac{1}{2})^2 - \frac{1}{4}$. It follows that $k - \frac{1}{2} < \sqrt{n} < k + \frac{1}{2}$, so $k = \{\sqrt{n}\}$ and $a_n = n + k = n + \{\sqrt{n}\}$. **29. a)** 20 **b)** 11 **c)** 30 **d)** 511 **31. a)** 1533 **b)** 510 **c)** 4923 **d)** 9842 **33. a)** 21 **b)** 78 **c)** 18 **d)** 18 **35.** $\sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - a_0$ **37. a)** n^2 **b)** $n(n+1)/2$ **39.** 15150 **41.** 34320 **43.** $\frac{n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2} + (n+1)(m - (n+1)^2 + 1)$, where $n = \lfloor \sqrt{m} \rfloor - 1$ **45. a)** 0 **b)** 1680 **c)** 1 **d)** 1024 **47.** 34

Section 2.5

1. a) Countably infinite, $-1, -2, -3, -4, \ldots$ **b)** Countably infinite, $0, 2, -2, 4, -4, \ldots$ **c)** Countably infinite, $99, 98, 97, \ldots$ **d)** Uncountable **e)** Finite **f)** Countably infinite, $0, 7, -7, 14, -14, \ldots$ **3. a)** Countable: match n with the string of n 1s. **b)** Countable. To find a correspondence, follow the path in Example 4, but omit fractions in the top three rows (as well as continuing to omit fractions not in lowest terms). **c)** Uncountable **d)** Uncountable **5.** Suppose m new guests arrive at the fully occupied hotel. Move the guest in Room n to Room $m + n$ for $n = 1, 2, 3, \ldots$; then the new guests can occupy rooms 1 to m. **7.** For $n = 1, 2, 3, \ldots$, put the guest currently in Room $2n$ into Room n, and the guest currently in Room $2n - 1$ into Room n of the new building. **9.** Move the guest currently in Room i to Room $2i + 1$ for $i = 1, 2, 3, \ldots$. Put the jth guest from the kth bus into Room $2^k(2j + 1)$. **11. a)** $A = [1, 2]$ (closed interval of real numbers from 1 to 2), $B = [3, 4]$ **b)** $A = [1, 2] \cup \mathbf{Z}^+$, $B = [3, 4] \cup \mathbf{Z}^+$ **c)** $A = [1, 3]$, $B = [2, 4]$ **13.** Suppose that A is countable. Then either A has cardinality n for some nonnegative integer n, in which case there is a one-to-one function from A to a subset of $\mathbf{Z}^+$ (the range is the first n positive integers), or there exists a one-to-one correspondence f from A to $\mathbf{Z}^+$; in either case we have satisfied Definition 2. Conversely, suppose that $|A| \le |\mathbf{Z}^+|$. By definition, this means that there is a one-to-one function from A to $\mathbf{Z}^+$, so A has the same cardinality as a subset of $\mathbf{Z}^+$ (namely the range of that function). By Exercise 16 we conclude that A is countable. **15.** Assume that B is countable. Then the elements of B can be listed as $b_1, b_2, b_3, \ldots$. Because A is a subset of B, taking the subsequence of $\{b_n\}$ that contains the terms that are in A gives a listing of the elements of A. Because A is uncountable, this is impossible. **17.** Assume that $A - B$ is countable. Then, because $A = (A - B) \cup (A \cap B)$, the elements of A can be listed in a sequence by alternating elements of $A - B$ and elements of $A \cap B$. This contradicts the uncountability of A. **19.** We are given bijections f from A to B and g from C to D. Then the function from $A \times C$ to $B \times D$ that sends (a, c) to $(f(a), g(c))$ is a bijection. **21.** By the definition of $|A| \le |B|$, there is a one-to-one function $f : A \to B$. Similarly, there is a one-to-one function $g : B \to C$. By Exercise 33 in Section 2.3, the composition $g \circ f : A \to C$ is one-to-one. Therefore, by definition $|A| \le |C|$. **23.** Using the Axiom of Choice from set theory, choose distinct elements $a_1, a_2, a_3, \ldots$ of A one at a time (this is possible because A is infinite). The resulting set $\{a_1, a_2, a_3, \ldots\}$ is the desired infinite subset of A. **25.** The set of finite strings of characters over a finite alphabet is countably infinite, because we can list these strings in alphabetical order by length. Therefore, the infinite set S can be identified with an infinite subset of this countable set, which by Exercise 16 is also countably infinite. **27.** Suppose that $A_1, A_2, A_3, \ldots$ are countable sets. Because A_i is countable, we can list its elements in a sequence as $a_{i1}, a_{i2}, a_{i3}, \ldots$. The elements of the set $\bigcup_{i=1}^{n} A_i$ can be listed by listing all terms a_{ij} with $i + j = 2$, then all terms a_{ij} with $i + j = 3$, then all terms a_{ij} with $i + j = 4$, and so on. **29.** There are a finite number of bit strings of length m, namely, 2^m. The set of all bit strings is the union of the sets of bit strings of length m for $m = 0, 1, 2, \ldots$. Because the union of a countable number of countable sets is countable (see Exercise 27), there are a countable number of bit strings. **31.** It is clear from the formula that the range of values the function takes on for a fixed value of $m + n$, say $m + n = x$, is $(x-2)(x-1)/2 + 1$ through $(x-2)(x-1)/2 + (x-1)$, because m can assume the values $1, 2, 3, \ldots, (x-1)$ under these conditions, and the first term in the formula is a fixed positive integer when $m + n$ is fixed. To show that this function is one-to-one and onto, we merely need to show that the range of values for $x + 1$ picks up precisely where the range of values for x left off, i.e., that $f(x-1, 1) + 1 = f(1, x)$. We have $f(x-1, 1) + 1 = \frac{(x-2)(x-1)}{2} + (x-1) + 1 = \frac{x^2 - x + 2}{2} = \frac{(x-1)x}{2} + 1 = f(1, x)$. **33.** By the Schröder-Bernstein theorem, it suffices to find one-to-one functions $f : (0, 1) \to [0, 1]$ and $g : [0, 1] \to (0, 1)$. Let $f(x) = x$ and $g(x) = (x+1)/3$. **35.** Each element A of the power set of the set of positive integers (i.e., $A \subseteq \mathbf{Z}^+$) can be represented uniquely by the bit string $a_1a_2a_3 \ldots$, where $a_i = 1$ if $i \in A$ and $a_i = 0$ if $i \notin A$. Assume there were a one-to-one correspondence $f : \mathbf{Z}^+ \to \mathcal{P}(\mathbf{Z}^+)$. Form a new bit string $s = s_1s_2s_3 \ldots$ by setting s_i to be 1 minus the ith bit of $f(i)$. Then because s differs in the i bit from $f(i)$, s is not in the range of f, a contradiction. **37.** For any finite alphabet there are a finite number of strings of length n, whenever n is a positive integer. It follows by the result of Exercise 27 that there are only a countable number of strings from any given finite alphabet. Because the set of all computer programs in a particular language is a subset of the set of all strings of a finite alphabet, which is a countable set by the result from Exercise 16, it is itself a countable set. **39.** Exercise 37 shows that there are only a countable number of computer programs. Consequently, there are only a countable number of computable functions. Because, as Exercise 38 shows, there are an uncountable number of functions, not all functions are computable. **41. a)** Note that if x is in the chain generated by y, then by the way the chains are generated, y is in the chain generated by x, so these two chains are the same. Thus, if x is in both the chain generated by y_1 and the chain generated by y_2, then the chain generated by y_1 and the chain generated by y_2 are both the same as the chain generated by x and are therefore the same chain. **b)** A moment's reflection will show that by the way the chains are constructed, this is true. **c)** Again, this is clear from the construction. **d)** Because the chains are disjoint and every element of A appears in exactly one chain and every element of B appears in exactly one chain, the function h cannot map two different elements of A to the same element of B. **e)** If an element b of B appears in a chain of types 1, 2, or 3, then it is preceded by an element of A, which maps to it under h. If b appears in a chain of type 4, then it is followed by an element of A, which maps to it under h.

Section 2.6

1. a) 3×4 **b)** $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ **c)** $\begin{bmatrix} 2 & 0 & 4 & 6 \end{bmatrix}$ **d)** 1 **e)** $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 4 & 3 \\ 3 & 6 & 7 \end{bmatrix}$ **3. a)** $\begin{bmatrix} 1 & 11 \\ 2 & 18 \end{bmatrix}$ **b)** $\begin{bmatrix} 2 & -2 & -3 \\ 1 & 0 & 2 \\ 9 & -4 & 4 \end{bmatrix}$ **c)** $\begin{bmatrix} -4 & 15 & -4 & 1 \\ -3 & 10 & 2 & -3 \\ 0 & 2 & -8 & 6 \\ 1 & -8 & 18 & -13 \end{bmatrix}$ **5.** $\begin{bmatrix} 9/5 & -6/5 \\ -1/5 & 4/5 \end{bmatrix}$
7. $\mathbf{0} + \mathbf{A} = [0 + a_{ij}] = [a_{ij} + 0] = \mathbf{0} + \mathbf{A}$ **9.** $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = [a_{ij} + (b_{ij} + c_{ij})] = [(a_{ij} + b_{ij}) + c_{ij}] = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ **11.** The number of rows of $\mathbf{A}$ equals the number of columns of $\mathbf{B}$, and the number of columns of $\mathbf{A}$ equals the number of rows of $\mathbf{B}$. **13.** $\mathbf{A}(\mathbf{BC}) = \left[\sum_q a_{iq} \left(\sum_r b_{qr} c_{rl}\right)\right] = \left[\sum_q \sum_r a_{iq} b_{qr} c_{rl}\right] = \left[\sum_r \sum_q a_{iq} b_{qr} c_{rl}\right] = \left[\sum_r \left(\sum_q a_{iq} b_{qr}\right) c_{rl}\right] = (\mathbf{AB})\mathbf{C}$ **15.** $\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ **17. a)** Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$. Then $\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$. We have $(\mathbf{A} + \mathbf{B})^t = [a_{ji} + b_{ji}] = [a_{ji}] + [b_{ji}] = \mathbf{A}^t + \mathbf{B}^t$. **b)** Using the same notation as in part (a), we have $\mathbf{B}^t\mathbf{A}^t = \left[\sum_q b_{qi} a_{jq}\right] = \left[\sum_q a_{jq} b_{qi}\right] = (\mathbf{AB})^t$, because the (i, j)th entry is the (j, i)th entry of $\mathbf{AB}$. **19.** The result follows because $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad - bc)\mathbf{I}_2 = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. **21.** $\mathbf{A}^n(\mathbf{A}^{-1})^n = \mathbf{A}(\mathbf{A} \cdots (\mathbf{A}(\mathbf{A}\mathbf{A}^{-1})\mathbf{A}^{-1}) \cdots \mathbf{A}^{-1})\mathbf{A}^{-1}$ by the associative law. Because $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, working from the inside shows that $\mathbf{A}^n(\mathbf{A}^{-1})^n = \mathbf{I}$. Similarly $(\mathbf{A}^{-1})^n\mathbf{A}^n = \mathbf{I}$. Therefore, $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$. **23.** The (i, j)th entry of $\mathbf{A} + \mathbf{A}^t$ is $a_{ij} + a_{ji}$, which equals $a_{ji} + a_{ij}$, the (j, i)th entry of $\mathbf{A} + \mathbf{A}^t$, so by definition $\mathbf{A} + \mathbf{A}^t$ is symmetric. **25.** $x_1 = 1, x_2 = -1, x_3 = -2$
27. a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$
29. a) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
31. a) $\mathbf{A} \vee \mathbf{B} = [a_{ij} \vee b_{ij}] = [b_{ij} \vee a_{ij}] = \mathbf{B} \vee \mathbf{A}$ **b)** $\mathbf{A} \wedge \mathbf{B} = [a_{ij} \wedge b_{ij}] = [b_{ij} \wedge a_{ij}] = \mathbf{B} \wedge \mathbf{A}$ **33. a)** $\mathbf{A} \vee (\mathbf{B} \wedge \mathbf{C}) = [a_{ij}] \vee [b_{ij} \wedge c_{ij}] = [a_{ij} \vee (b_{ij} \wedge c_{ij})] = [(a_{ij} \vee b_{ij}) \wedge (a_{ij} \vee c_{ij})] = [a_{ij} \vee b_{ij}] \wedge [a_{ij} \vee c_{ij}] = (\mathbf{A} \vee \mathbf{B}) \wedge (\mathbf{A} \vee \mathbf{C})$ **b)** $\mathbf{A} \wedge (\mathbf{B} \vee \mathbf{C}) = [a_{ij}] \wedge [b_{ij} \vee c_{ij}] = [a_{ij} \wedge (b_{ij} \vee c_{ij})] = [(a_{ij} \wedge b_{ij}) \vee (a_{ij} \wedge c_{ij})] = [a_{ij} \wedge b_{ij}] \vee [a_{ij} \wedge c_{ij}] = (\mathbf{A} \wedge \mathbf{B}) \vee (\mathbf{A} \wedge \mathbf{C})$ **35.** $\mathbf{A} \odot (\mathbf{B} \odot \mathbf{C}) = \left[\bigvee_q a_{iq} \wedge \left(\bigvee_r (b_{qr} \wedge c_{rl})\right)\right] = \left[\bigvee_q \bigvee_r (a_{iq} \wedge b_{qr} \wedge c_{rl})\right] = \left[\bigvee_r \bigvee_q (a_{iq} \wedge b_{qr} \wedge c_{rl})\right] = \left[\bigvee_r \left(\bigvee_q (a_{iq} \wedge b_{qr})\right) \wedge c_{rl}\right] = (\mathbf{A} \odot \mathbf{B}) \odot \mathbf{C}$

Supplementary Exercises

1. a) $\overline{A}$ **b)** $A \cap B$ **c)** $A - B$ **d)** $\overline{A} \cap \overline{B}$ **e)** $A \oplus B$
3. Yes **5.** $A - (A - B) = A - (A \cap \overline{B}) = A \cap (\overline{A \cap \overline{B}}) = A \cap (\overline{A} \cup B) = (A \cap \overline{A}) \cup (A \cap B) = \emptyset \cup (A \cap B) = A \cap B$ **7.** Let $A = \{1\}$, $B = \emptyset$, $C = \{1\}$. Then $(A - B) - C = \emptyset$, but $A - (B - C) = \{1\}$. **9.** No. For example, let $A = B = \{a, b\}$, $C = \emptyset$, and $D = \{a\}$. Then $(A - B) - (C - D) = \emptyset - \emptyset = \emptyset$, but $(A - C) - (B - D) = \{a, b\} - \{b\} = \{a\}$. **11. a)** $|\emptyset| \le |A \cap B| \le |A| \le |A \cup B| \le |U|$ **b)** $|\emptyset| \le |A - B| \le |A \oplus B| \le |A \cup B| \le |A| + |B|$ **13. a)** Yes, no **b)** Yes, no **c)** f has inverse with $f^{-1}(a) = 3$, $f^{-1}(b) = 4$, $f^{-1}(c) = 2$, $f^{-1}(d) = 1$; g has no inverse. **15.** If f is one-to-one, then f provides a bijection between S and $f(S)$, so they have the same cardinality. If f is not one-to-one, then there exist elements x and y in S such that $f(x) = f(y)$. Let $S = \{x, y\}$. Then $|S| = 2$ but $|f(S)| = 1$. **17.** Let $x \in A$. Then $S_f(\{x\}) = \{f(y) \mid y \in \{x\}\} = \{f(x)\}$. By the same reasoning, $S_g(\{x\}) = \{g(x)\}$. Because $S_f = S_g$, we can conclude that $\{f(x)\} = \{g(x)\}$, and so necessarily $f(x) = g(x)$. **19.** The equation is true if and only if the sum of the fractional parts of x and y is less than 1. **21.** The equation is true if and only if either both x and y are integers, or x is not an integer but the sum of the fractional parts of x and y is less than or equal to 1. **23.** If x is an integer, then $\lfloor x \rfloor + \lfloor m - x \rfloor = x + m - x = m$. Otherwise, write x in terms of its integer and fractional parts: $x = n + \epsilon$, where $n = \lfloor x \rfloor$ and $0 < \epsilon < 1$. In this case $\lfloor x \rfloor +$

is true because $f(1) = 1 = 1^2$, which follows from the definition of f. *Inductive step:* Assume $f(n) = n^2$. Then $f(n+1) = f((n+1)-1) + 2(n+1) - 1 = f(n) + 2n + 1 = n^2 + 2n + 1 = (n+1)^2$. **57. a)** λ, 0, 1, 00, 01, 11, 000, 001, 011, 111, 0000, 0001, 0011, 0111, 1111, 00000, 00001, 00011, 00111, 01111, 11111 **b)** $S = \{\alpha\beta \mid \alpha$ is a string of m 0s and β is a string of n 1s, $m \geq 0, n \geq 0\}$ **59.** Apply the first recursive step to λ to get () $\in B$. Apply the second recursive step to this string to get ()() $\in B$. Apply the first recursive step to this string to get (()()) $\in B$. By Exercise 62, (())) is not in B because the number of left parentheses does not equal the number of right parentheses. **61.** λ, (), (()), ()() **63. a)** 0 **b)** −2 **c)** 2 **d)** 0

65.

procedure *generate*(n: nonnegative integer)
if n is odd **then**
  $S := S(n-1)$ {the S constructed by *generate*$(n-1)$}
  $T := T(n-1)$ {the T constructed by *generate*$(n-1)$}
else if $n = 0$ **then**
  $S := \emptyset$
  $T := \{\lambda\}$
else
  $S' := S(n-2)$ {the S constructed by *generate*$(n-2)$}
  $T' := T(n-2)$ {the T constructed by *generate*$(n-2)$}
  $T := T' \cup \{(x) | x \in T' \cup S' \wedge \text{length}(x) = n-2\}$
  $S := S' \cup \{xy | x \in T' \wedge y \in T' \cup S' \wedge \text{length}(xy) = n\}$
{$T \cup S$ is the set of balanced strings of length at most n}

67. If $x \leq y$ initially, then $x := y$ is not executed, so $x \leq y$ is a true final assertion. If $x > y$ initially, then $x := y$ is executed, so $x \leq y$ is again a true final assertion.

69. procedure *zerocount*($a_1, a_2, \dots, a_n$: list of integers)
 if $n = 1$ **then**
  if $a_1 = 0$ **then return** 1
  else return 0
 else
  if $a_n = 0$ **then return** *zerocount*$(a_1, a_2, \dots, a_{n-1}) + 1$
  else return *zerocount*$(a_1, a_2, \dots, a_{n-1})$

71. We will prove that $a(n)$ is a natural number and $a(n) \leq n$. This is true for the base case $n = 0$ because $a(0) = 0$. Now assume that $a(n-1)$ is a natural number and $a(n-1) \leq n-1$. Then $a(a(n-1))$ is a applied to a natural number less than or equal to $n-1$. Hence, $a(a(n-1))$ is also a natural number less than or equal to $n-1$. Therefore, $n - a(a(n-1))$ is n minus some natural number less than or equal to $n-1$, which is a natural number less than or equal to n. **73.** From Exercise 72, $a(n) = \lfloor (n+1)\mu \rfloor$ and $a(n-1) = \lfloor n\mu \rfloor$. Because $\mu < 1$, these two values are equal or they differ by 1. First suppose that $\mu n - \lfloor \mu n \rfloor < 1 - \mu$. This is equivalent to $\mu(n+1) < 1 + \lfloor \mu n \rfloor$. If this is true, then $\lfloor \mu(n+1) \rfloor = \lfloor \mu n \rfloor$. On the other hand, if $\mu n - \lfloor \mu n \rfloor \geq 1 - \mu$, then $\mu(n+1) \geq 1 + \lfloor \mu n \rfloor$, so $\lfloor \mu(n+1) \rfloor = \lfloor \mu n \rfloor + 1$, as desired. **75.** $f(0) = 1, m(0) = 0$; $f(1) = 1, m(1) = 0$; $f(2) = 2, m(2) = 1$; $f(3) = 2, m(3) = 2$; $f(4) = 3, m(4) = 2$; $f(5) = 3, m(5) = 3$; $f(6) = 4, m(6) = 4$; $f(7) = 5, m(7) = 4$; $f(8) = 5, m(8) = 5$; $f(9) = 6, m(9) = 6$ **77.** The last occurrence of n is in the position for which the total number of 1s, 2s, …, ns all together is that position number. But because a_k is the number of occurrences of k, this is just $\sum_{k=1}^{n} a_k$, as desired. Because $f(n)$ is the sum of the first n terms of the sequence, $f(f(n))$ is the sum of the first $f(n)$ terms of the sequence. But because $f(n)$ is the last term whose value is n, this means that the sum is the sum of all terms of the sequence whose value is at most n. Because there are a_k terms of the sequence whose value is k, this sum is $\sum_{k=1}^{n} k \cdot a_k$, as desired.

CHAPTER 6

Section 6.1

1. a) 5850 **b)** 343 **3. a)** 4^{10} **b)** 5^{10} **5.** 42 **7.** 26^3 **9.** 676 **11.** 2^8 **13.** $n + 1$ (counting the empty string) **15.** 475,255 (counting the empty string) **17.** 1,321,368,961 **19. a)** 729 **b)** 256 **c)** 1024 **d)** 64 **21. a)** Seven: 56, 63, 70, 77, 84, 91, 98 **b)** Five: 55, 66, 77, 88, 99 **c)** One: 77 **23. a)** 128 **b)** 450 **c)** 9 **d)** 675 **e)** 450 **f)** 450 **g)** 225 **h)** 75 **25. a)** 990 **b)** 500 **c)** 27 **27.** 3^{50} **29.** 52,457,600 **31.** 20,077,200 **33. a)** 37,822,859,361 **b)** 8,204,716,800 **c)** 40,159,050,880 **d)** 12,113,640,000 **e)** 171,004,205,215 **f)** 72,043,541,640 **g)** 6,230,721,635 **h)** 223,149,655 **35. a)** 0 **b)** 120 **c)** 720 **d)** 2520 **37. a)** 2 if $n = 1$, 2 if $n = 2$, 0 if $n \geq 3$ **b)** 2^{n-2} for $n > 1$; 1 if $n = 1$ **c)** $2(n-1)$ **39.** $(n+1)^m$ **41.** If n is even, $2^{n/2}$; if n is odd, $2^{(n+1)/2}$ **43. a)** 175 **b)** 248 **c)** 232 **d)** 84 **45.** 40 **47.** 60 **49. a)** 240 **b)** 480 **c)** 360 **51.** 352 **53.** 147 **55.** 33 **57. a)** $9{,}920{,}671{,}339{,}261{,}325{,}541{,}376 \approx 9.9 \times 10^{21}$ **b)** $6{,}641{,}514{,}961{,}387{,}068{,}437{,}760 \approx 6.6 \times 10^{21}$ **c)** About 314,000 years **59.** $54(64^{65536} - 1)/63$ **61.** 7,104,000,000,000 **63.** $16^{10} + 16^{26} + 16^{58}$ **65.** 666,667 **67.** 18 **69.** 17 **71.** 22 **73.** $2n-2, 2n-1$ **75.** Let $P(m)$ be the sum rule for m tasks. For the basis case take $m = 2$. This is just the sum rule for two tasks. Now assume that $P(m)$ is true. Consider $m+1$ tasks, $T_1, T_2, \dots, T_m, T_{m+1}$, which can be done in $n_1, n_2, \dots, n_m, n_{m+1}$ ways, respectively, such that no two of these tasks can be done at the same time. To do one of these tasks, we can either do one of the first m of these or do task T_{m+1}. By the sum rule for two tasks, the number of ways to do this is the sum of the number of ways to do one of the first m tasks, plus n_{m+1}. By the inductive hypothesis, this is $n_1 + n_2 + \cdots + n_m + n_{m+1}$, as desired. **77.** $n(n-3)/2$

Section 6.2

1. Because there are six classes, but only five weekdays, the pigeonhole principle shows that at least two classes must be held on the same day. **3. a)** 3 **b)** 14 **5.** 85 **7.** Because there are four possible remainders when an integer is divided by 4, the pigeonhole principle implies that given five integers, at least two have the same remainder. **9.** Let $a, a+1, \dots,$

$a + n - 1$ be the integers in the sequence. The integers $(a + i)$ **mod** n, $i = 0, 1, 2, \ldots, n - 1$, are distinct, because $0 < (a + j) - (a + k) < n$ whenever $0 \le k < j \le n - 1$. Because there are n possible values for $(a + i)$ **mod** n and there are n different integers in the set, each of these values is taken on exactly once. It follows that there is exactly one integer in the sequence that is divisible by n. **11.** 4951 **13.** The midpoint of the segment joining the points (a, b, c) and (d, e, f) is $((a + d)/2, (b + e)/2, (c + f)/2)$. It has integer coefficients if and only if a and d have the same parity, b and e have the same parity, and c and f have the same parity. Because there are eight possible triples of parity [such as (*even*, *odd*, *even*)], by the pigeonhole principle at least two of the nine points have the same triple of parities. The midpoint of the segment joining two such points has integer coefficients. **15. a)** Group the first eight positive integers into four subsets of two integers each so that the integers of each subset add up to 9: $\{1, 8\}$, $\{2, 7\}$, $\{3, 6\}$, and $\{4, 5\}$. If five integers are selected from the first eight positive integers, by the pigeonhole principle at least two of them come from the same subset. Two such integers have a sum of 9, as desired. **b)** No. Take $\{1, 2, 3, 4\}$, for example. **17.** 4 **19.** 21,251 **21. a)** If there were fewer than 9 freshmen, fewer than 9 sophomores, and fewer than 9 juniors in the class, there would be no more than 8 with each of these three class standings, for a total of at most 24 students, contradicting the fact that there are 25 students in the class. **b)** If there were fewer than 3 freshmen, fewer than 19 sophomores, and fewer than 5 juniors, then there would be at most 2 freshmen, at most 18 sophomores, and at most 4 juniors, for a total of at most 24 students. This contradicts the fact that there are 25 students in the class. **23.** 4, 3, 2, 1, 8, 7, 6, 5, 12, 11, 10, 9, 16, 15, 14, 13 **25.** Number the seats around the table from 1 to 50, and think of seat 50 as being adjacent to seat 1. There are 25 seats with odd numbers and 25 seats with even numbers. If no more than 12 boys occupied the odd-numbered seats, then at least 13 boys would occupy the even-numbered seats, and vice versa. Without loss of generality, assume that at least 13 boys occupy the 25 odd-numbered seats. Then at least two of those boys must be in consecutive odd-numbered seats, and the person sitting between them will have boys as both of his or her neighbors.

27.
```
procedure long(a_1, ..., a_n: positive integers)
{first find longest increasing subsequence}
max := 0; set := 00 ... 00 {n bits}
for i := 1 to 2^n
  last := 0; count := 0, OK := true
  for j := 1 to n
    if set(j) = 1 then
      if a_j > last then last := a_j
      count := count + 1
    else OK := false
  if count > max then
    max := count
    best := set
  set := set + 1 (binary addition)
{max is length and best indicates the sequence}
{repeat for decreasing subsequence with only
   changes being a_j < last instead of a_j > last
   and last := ∞ instead of last := 0}
```

29. By symmetry we need prove only the first statement. Let A be one of the people. Either A has at least four friends, or A has at least six enemies among the other nine people (because $3 + 5 < 9$). Suppose, in the first case, that B, C, D, and E are all A's friends. If any two of these are friends with each other, then we have found three mutual friends. Otherwise $\{B, C, D, E\}$ is a set of four mutual enemies. In the second case, let $\{B, C, D, E, F, G\}$ be a set of enemies of A. By Example 13, among B, C, D, E, F, and G there are either three mutual friends or three mutual enemies, who form, with A, a set of four mutual enemies. **31.** We need to show two things: that if we have a group of n people, then among them we must find either a pair of friends or a subset of n of them all of whom are mutual enemies; and that there exists a group of $n - 1$ people for which this is not possible. For the first statement, if there is any pair of friends, then the condition is satisfied, and if not, then every pair of people are enemies, so the second condition is satisfied. For the second statement, if we have a group of $n - 1$ people all of whom are enemies of each other, then there is neither a pair of friends nor a subset of n of them all of whom are mutual enemies. **33.** There are 6,432,816 possibilities for the three initials and a birthday. So, by the generalized pigeonhole principle, there are at least $\lceil 39{,}000{,}000/6{,}432{,}816 \rceil = 7$ people who share the same initials and birthday. **35.** Because $800{,}001 > 200{,}000$, the pigeonhole principle guarantees that there are at least two Parisians with the same number of hairs on their heads. The generalized pigeonhole principle guarantees that there are at least $\lceil 800{,}001/200{,}000 \rceil = 5$ Parisians with the same number of hairs on their heads. **37.** 18 **39.** Because there are six computers, the number of other computers a computer is connected to is an integer between 0 and 5, inclusive. However, 0 and 5 cannot both occur. To see this, note that if some computer is connected to no others, then no computer is connected to all five others, and if some computer is connected to all five others, then no computer is connected to no others. Hence, by the pigeonhole principle, because there are at most five possibilities for the number of computers a computer is connected to, there are at least two computers in the set of six connected to the same number of others. **41.** Label the computers C_1 through C_{100}, and label the printers P_1 through P_{20}. If we connect C_k to P_k for $k = 1, 2, \ldots, 20$ and connect each of the computers C_{21} through C_{100} to *all* the printers, then we have used a total of $20 + 80 \cdot 20 = 1620$ cables. This is sufficient, because if computers C_1 through C_{20} need printers, then they can use the printers with the same subscripts, and if any computers with higher subscripts need a printer instead of one or more of these, then they can use the printers that are not being used, because they are connected to all the printers. Now we must show that 1619 cables is not enough. Because there are 1619 cables and 20 printers, the average number of computers per printer is 1619/20, which is less than 81. Therefore, some printer must be connected to fewer than 81 computers. That means it is connected to 80 or fewer computers, so there are 20 computers that are not connected to it. If those 20 computers all needed a printer simultaneously, then they would be out of luck, because they are connected to at most the 19 other printers. **43.** Let a_i be the number of matches completed by hour i. Then $1 \le a_1 < a_2 < \cdots < a_{75} \le 125$. Also $25 \le a_1 + 24 < a_2 + 24 < \cdots < a_{75} + 24 \le 149$. There are 150 numbers $a_1, \ldots, a_{75}, a_1 + 24, \ldots, a_{75} + 24$. By the pigeonhole principle, at least two are equal. Because all the a_is are distinct and all the $(a_i + 24)$s are distinct, it follows that $a_i = a_j + 24$ for some $i > j$. Thus, in the period from the $(j+1)$st to the ith hour, there are exactly 24 matches. **45.** Use the generalized pigeonhole principle, placing the $|S|$ objects $f(s)$ for $s \in S$ in $|T|$ boxes, one for each element of T. **47.** Let d_j be $jx - N(jx)$, where $N(jx)$ is the integer closest to jx for $1 \le j \le n$. Each d_j is an irrational number between $-1/2$ and $1/2$. We will assume that n is even; the case where n is odd is messier. Consider the n intervals $\{x \mid j/n < x < (j+1)/n\}$, $\{x \mid -(j+1)/n < x < -j/n\}$ for $j = 0, 1, \ldots, (n/2) - 1$. If d_j belongs to the interval $\{x \mid 0 < x < 1/n\}$ or to the interval $\{x \mid -1/n < x < 0\}$ for some j, we are done. If not, because there are $n-2$ intervals and n numbers d_j, the pigeonhole principle tells us that there is an interval $\{x \mid (k-1)/n < x < k/n\}$ containing d_r and d_s with $r < s$. The proof can be finished by showing that $(s - r)x$ is within $1/n$ of its nearest integer. **49. a)** Assume that $i_k \le n$ for all k. Then by the generalized pigeonhole principle, at least $\lceil (n^2 + 1)/n \rceil = n + 1$ of the numbers $i_1, i_2, \ldots, i_{n^2+1}$ are equal. **b)** If $a_{k_j} < a_{k_{j+1}}$, then the subsequence consisting of a_{k_j} followed by the increasing subsequence of length $i_{k_{j+1}}$ starting at $a_{k_{j+1}}$ contradicts the fact that $i_{k_j} = i_{k_{j+1}}$. Hence, $a_{k_j} > a_{k_{j+1}}$. **c)** If there is no increasing subsequence of length greater than n, then parts (a) and (b) apply. Therefore, we have $a_{k_{n+1}} > a_{k_n} > \cdots > a_{k_2} > a_{k_1}$, a decreasing sequence of length $n + 1$.

Section 6.3

1. abc, acb, bac, bca, cab, cba **3.** 720 **5. a)** 120 **b)** 720 **c)** 8 **d)** 6720 **e)** 40,320 **f)** 3,628,800 **7.** 15,120 **9.** 1320 **11. a)** 210 **b)** 386 **c)** 848 **d)** 252 **13.** $2(n!)^2$ **15.** 65,780 **17.** $2^{100} - 5051$ **19. a)** 1024 **b)** 45 **c)** 176 **d)** 252 **21. a)** 120 **b)** 24 **c)** 120 **d)** 24 **e)** 6 **f)** 0 **23.** 609,638,400 **25. a)** 17,280 **b)** 14,400 **27. a)** 94,109,400 **b)** 941,094 **c)** 3,764,376 **d)** 90,345,024 **e)** 114,072 **f)** 2328 **g)** 24 **h)** 79,727,040 **i)** 3,764,376 **j)** 109,440 **29. a)** 12,650 **b)** 303,600 **31. a)** 37,927 **b)** 18,915 **33. a)** 122,523,030 **b)** 72,930,375 **c)** 223,149,655 **d)** 100,626,625 **35.** 54,600 **37.** 45 **39.** 912 **41.** 11,232,000 **43.** $\frac{n!}{r(n-r)!}$ **45.** 13 **47.** 873

Section 6.4

1. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ **3.** $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$ **5.** 101 **7.** $-2^{10}\binom{19}{9} = -94{,}595{,}072$ **9.** $-2^{101}3^{99}\binom{200}{99}$ **11.** $\sum_{j=0}^{5}\binom{5}{j}(3x^4)^{5-j}(-2y^3)^j = 243x^{20} - 810x^{16}y^3 + 1080x^{12}y^6 - 720x^8y^9 + 240x^4y^{12} - 32y^{15}$ **13. a)** 71,680 **b)** 0 **c)** $-16{,}384$ **d)** $-35{,}840$ **e)** $-1{,}792$ **15.** $(-1)^{(200-k)/3}\binom{100}{(200-k)/3}$ if $k \equiv 2 \pmod 3$ and $-100 \le k \le 200$; 0 otherwise **17.** 1 9 36 84 126 126 84 36 9 1 **19.** The sum of *all* the positive numbers $\binom{n}{k}$, as k runs from 0 to n, is 2^n, so each one of them is no bigger than this sum. **21.** $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 2} \le \frac{n\cdot n\cdot\cdots\cdot n}{2\cdot 2\cdot\cdots\cdot 2} = n^k/2^{k-1}$ **23.** $\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} = \frac{n!}{k!(n-k+1)!} \cdot [k + (n - k + 1)] = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$ **25. a)** We show that each side counts the number of ways to choose from a set with n elements a subset with k elements and a distinguished element of that set. For the left-hand side, first choose the k-set (this can be done in $\binom{n}{k}$ ways) and then choose one of the k elements in this subset to be the distinguished element (this can be done in k ways). For the right-hand side, first choose the distinguished element out of the entire n-set (this can be done in n ways), and then choose the remaining $k - 1$ elements of the subset from the remaining $n - 1$ elements of the set (this can be done in $\binom{n-1}{k-1}$ ways). **b)** $k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}$ **27.** $\binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!} = \frac{(n+1)}{k}\frac{n!}{(k-1)![n-(k-1)]!} = (n + 1)\binom{n}{k-1}/k$. This identity together with $\binom{n}{0} = 1$ gives a recursive definition. **29.** $\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+1}{n+1} = \frac{1}{2}\left[\binom{2n+1}{n+1} + \binom{2n+1}{n+1}\right] = \frac{1}{2}\left[\binom{2n+1}{n+1} + \binom{2n+1}{n}\right] = \frac{1}{2}\binom{2n+2}{n+1}$ **31. a)** $\binom{n+r+1}{r}$ counts the number of ways to choose a sequence of r 0s and $n+1$ 1s by choosing the positions of the 0s. Alternately, suppose that the $(j + 1)$st term is the last term equal to 1, so that $n \le j \le n + r$. Once we have determined where the last 1 is, we decide where the 0s are to be placed in the j spaces before the last 1. There are n 1s and $j - n$ 0s in this range. By the sum rule it follows that there are $\sum_{j=n}^{n+r}\binom{j}{j-n} = \sum_{k=0}^{r}\binom{n+k}{k}$ ways to do this. **b)** Let $P(r)$ be the statement to be proved. The basis step is the equation $\binom{n}{0} = \binom{n+1}{0}$, which is just $1 = 1$. Assume that $P(r)$ is true. Then $\sum_{k=0}^{r+1}\binom{n+k}{k} = \sum_{k=0}^{r}\binom{n+k}{k} + \binom{n+r+1}{r+1} = \binom{n+r+1}{r} + \binom{n+r+1}{r+1} = \binom{n+r+2}{r+1}$, using the inductive hypothesis and Pascal's identity. **33.** We can choose the leader first in n different ways. We can then choose the rest of the committee in 2^{n-1} ways. Hence, there are $n2^{n-1}$ ways to choose the committee and its leader. Meanwhile, the number of ways to select a committee with k people is $\binom{n}{k}$. Once we have chosen a committee with k people, there are k ways to choose its leader. Hence, there are $\sum_{k=1}^{n} k\binom{n}{k}$ ways to choose the committee and its leader. Hence, $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$. **35.** Let the set have n elements. From Corollary 2 we have $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0$. It follows that $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$. The left-hand side gives the number of subsets with an even number of elements, and the right-hand side gives the number of subsets with an odd number of elements. **37. a)** A path of the desired type consists of m moves to the right and n moves up. Each such path can be represented by a bit string of length $m + n$ with m 0s and n 1s, where a 0 represents a move to the right and a 1 a move up. **b)** The number of bit strings of length $m + n$

containing exactly n 1s equals $\binom{m+n}{n} = \binom{m+n}{m}$ because such a string is determined by specifying the positions of the n 1s or by specifying the positions of the m 0s. **39.** By Exercise 37 the number of paths of length n of the type described in that exercise equals 2^n, the number of bit strings of length n. On the other hand, a path of length n of the type described in Exercise 37 must end at a point that has n as the sum of its coordinates, say $(n-k, k)$ for some k between 0 and n, inclusive. By Exercise 37, the number of such paths ending at $(n-k, k)$ equals $\binom{n-k+k}{k} = \binom{n}{k}$. Hence, $\sum_{k=0}^{n}\binom{n}{k} = 2^n$. **41.** By Exercise 37 the number of paths from (0, 0) to $(n+1, r)$ of the type described in that exercise equals $\binom{n+r+1}{r}$. But such a path starts by going j steps vertically for some j with $0 \le j \le r$. The number of these paths beginning with j vertical steps equals the number of paths of the type described in Exercise 37 that go from $(1, j)$ to $(n+1, r)$. This is the same as the number of such paths that go from (0, 0) to $(n, r-j)$, which by Exercise 37 equals $\binom{n+r-j}{r-j}$. Because $\sum_{j=0}^{r}\binom{n+r-j}{r-j} = \sum_{k=0}^{r}\binom{n+k}{k}$, it follows that $\sum_{k=1}^{r}\binom{n+k}{k} = \binom{n+r-1}{r}$. **43. a)** $\binom{n+1}{2}$ **b)** $\binom{n+2}{3}$ **c)** $\binom{2n-2}{n-1}$ **d)** $\binom{n-1}{\lfloor (n-1)/2 \rfloor}$ **e)** Largest odd entry in nth row of Pascal's triangle **f)** $\binom{3n-3}{n-1}$

Section 6.5

1. 243 **3.** 26^6 **5.** 125 **7.** 35 **9. a)** 1716 **b)** 50,388 **c)** 2,629,575 **d)** 330 **11.** 9 **13.** 4,504,501 **15. a)** 10,626 **b)** 1,365 **c)** 11,649 **d)** 106 **17.** 2,520 **19.** 302,702,400 **21. a)** 169 **b)** 156 **c)** 78 **d)** 91 **23.** 3003 **25.** 7,484,400 **27.** 30,492 **29.** $C(59, 50)$ **31.** 35 **33.** 83,160 **35.** 63 **37.** 19,635 **39.** 210 **41.** 27,720 **43.** $52!/(7!^5 17!)$ **45.** Approximately 6.5×10^{32} **47. a)** $C(k+n-1, n)$ **b)** $(k+n-1)!/(k-1)!$ **49.** There are $C(n, n_1)$ ways to choose n_1 objects for the first box. Once these objects are chosen, there are $C(n-n_1, n_2)$ ways to choose objects for the second box. Similarly, there are $C(n-n_1-n_2, n_3)$ ways to choose objects for the third box. Continue in this way until there is $C(n-n_1-n_2-\cdots-n_{k-1}, n_k) = C(n_k, n_k) = 1$ way to choose the objects for the last box (because $n_1+n_2+\cdots+n_k = n$). By the product rule, the number of ways to make the entire assignment is $C(n, n_1)C(n-n_1, n_2)C(n-n_1-n_2, n_3)\cdots C(n-n_1-n_2-\cdots-n_{k-1}, n_k)$, which equals $n!/(n_1!n_2!\cdots n_k!)$, as straightforward simplification shows. **51. a)** Because $x_1 \le x_2 \le \cdots \le x_r$, it follows that $x_1+0 < x_2+1 < \cdots < x_r+r-1$. The inequalities are strict because $x_j+j-1 < x_{j+1}+j$ as long as $x_j \le x_{j+1}$. Because $1 \le x_j \le n+r-1$, this sequence is made up of r distinct elements from T. **b)** Suppose that $1 \le x_1 < x_2 < \cdots < x_r \le n+r-1$. Let $y_k = x_k-(k-1)$. Then it is not hard to see that $y_k \le y_{k+1}$ for $k = 1, 2, \ldots, r-1$ and that $1 \le y_k \le n$ for $k = 1, 2, \ldots r$. It follows that $\{y_1, y_2, \ldots, y_r\}$ is an r-combination with repetitions allowed of S. **c)** From parts (a) and (b) it follows that there is a one-to-one correspondence of r-combinations with repetitions allowed of S and r-combinations of T, a set with $n+r-1$ elements. We conclude that there are $C(n+r-1, r)$ r-combinations with repetitions allowed of S. **53.** 65 **55.** 65 **57.** 2 **59.** 3 **61. a)** 150 **b)** 25 **c)** 6 **d)** 2 **63.** 90,720 **65.** The terms in the expansion are of the form $x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$, where $n_1+n_2+\cdots+n_m = n$. Such a term arises from choosing the x_1 in n_1 factors, the x_2 in n_2 factors, $\ldots$, and the x_m in n_m factors. This can be done in $C(n; n_1, n_2, \ldots, n_m)$ ways, because a choice is a permutation of n_1 labels "1," n_2 labels "2," $\ldots$, and n_m labels "m." **67.** 2520

Section 6.6

1. 14532, 15432, 21345, 23451, 23514, 31452, 31542, 43521, 45213, 45321 **3.** AAA1, AAA2, AAB1, AAB2, AAC1, AAC2, ABA1, ABA2, ABB1, ABB2, ABC1, ABC2, ACA1, ACA2, ACB1, ACB2, ACC1, ACC2, BAA1, BAA2, BAB1, BAB2, BAC1, BAC2, BBA1, BBA2, BBB1, BBB2, BBC1, BBC2, BCA1, BCA2, BCB1, BCB2, BCC1, BCC2, CAA1, CAA2, CAB1, CAB2, CAC1, CAC2, CBA1, CBA2, CBB1, CBB2, CBC1, CBC2, CCA1, CCA2, CCB1, CCB2, CCC1, CCC2 **5. a)** 2134 **b)** 54132 **c)** 12534 **d)** 45312 **7.** 1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321 **9.** {1, 2, 3}, {1, 2, 4}, {1, 2, 5}, {1, 3, 4}, {1, 3, 5}, {1, 4, 5}, {2, 3, 4}, {2, 3, 5}, {2, 4, 5}, {3, 4, 5} **11.** The bit string representing the next larger r-combination must differ from the bit string representing the original one in position i because positions $i+1, \ldots, r$ are occupied by the largest possible numbers. Also a_i+1 is the smallest possible number we can put in position i if we want a combination greater than the original one. Then $a_i+2, \ldots, a_i+r-i+1$ are the smallest allowable numbers for positions $i+1$ to r. Thus, we have produced the next r-combination. **13.** 123, 132, 213, 231, 312, 321, 124, 142, 214, 241, 412, 421, 125, 152, 215, 251, 512, 521, 134, 143, 314, 341, 413, 431, 135, 153, 315, 351, 513, 531, 145, 154, 415, 451, 514, 541, 234, 243, 324, 342, 423, 432, 235, 253, 325, 352, 523, 532, 245, 254, 425, 452, 524, 542, 345, 354, 435, 453, 534, 543 **15.** We will show that it is a bijection by showing that it has an inverse. Given a positive integer less than $n!$, let $a_1, a_2, \ldots, a_{n-1}$ be its Cantor digits. Put n in position $n-a_{n-1}$; then a_{n-1} is the number of integers less than n that follow n in the permutation. Then put $n-1$ in free position $(n-1)-a_{n-2}$, where we have numbered the free positions $1, 2, \ldots, n-1$ (excluding the position that n is already in). Continue until 1 is placed in the only free position left. Because we have constructed an inverse, the correspondence is a bijection.

17. procedure *Cantor permutation*(n, i: integers with $n \ge 1$ and $0 \le i < n!$)
$x := n$
for $j := 1$ **to** n
  $p_j := 0$
for $k := 1$ **to** $n-1$
  $c := \lfloor x/(n-k)! \rfloor$; $x := x - c(n-k)!$; $h := n$
  while $p_h \ne 0$

```
    h := h − 1
  for j := 1 to c
    h := h − 1
    while p_h ≠ 0
      h := h − 1
  p_h := n − k + 1
h := 1
while p_h ≠ 0
  h := h + 1
p_h := 1
{p_1 p_2 ... p_n is the permutation corresponding to i}
```

Supplementary Exercises

1. a) 151,200 **b)** 1,000,000 **c)** 210 **d)** 5005 **3.** 3^{100} **5.** 24,600 **7. a)** 4060 **b)** 2688 **c)** 25,009,600 **9. a)** 192 **b)** 301 **c)** 300 **d)** 300 **11.** 639 **13.** The maximum possible sum is 240, and the minimum possible sum is 15. So the number of possible sums is 226. Because there are 252 subsets with five elements of a set with 10 elements, by the pigeonhole principle it follows that at least two have the same sum. **15. a)** 50 **b)** 50 **c)** 14 **d)** 17 **17.** Let $a_1, a_2, \ldots, a_m$ be the integers, and let $d_i = \sum_{j=1}^{i} a_j$. If $d_i \equiv 0 \pmod{m}$ for some i, we are done. Otherwise d_1 **mod** m, d_2 **mod** $m, \ldots, d_m$ **mod** m are m integers with values in $\{1, 2, \ldots, m-1\}$. By the pigeonhole principle $d_k = d_l$ for some $1 \le k < l \le m$. Then $\sum_{j=k+1}^{l} a_j = d_l - d_k \equiv 0 \pmod{m}$. **19.** The decimal expansion of the rational number a/b can be obtained by division of b into a, where a is written with a decimal point and an arbitrarily long string of 0s following it. The basic step is finding the next digit of the quotient, namely, $\lfloor r/b \rfloor$, where r is the remainder with the next digit of the dividend brought down. The current remainder is obtained from the previous remainder by subtracting b times the previous digit of the quotient. Eventually the dividend has nothing but 0s to bring down. Furthermore, there are only b possible remainders. Thus, at some point, by the pigeonhole principle, we will have the same situation as had previously arisen. From that point onward, the calculation must follow the same pattern. In particular, the quotient will repeat. **21. a)** 125,970 **b)** 20 **c)** 141,120,525 **d)** 141,120,505 **e)** 177,100 **f)** 141,078,021 **23. a)** 10 **b)** 8 **c)** 7 **25.** 3^n **27.** $C(n+2, r+1) = C(n+1, r+1) + C(n+1, r) = 2C(n+1, r+1) - C(n+1, r+1) + C(n+1, r) = 2C(n+1, r+1) - (C(n, r+1) + C(n, r)) + (C(n, r) + C(n, r-1)) = 2C(n+1, r+1) - C(n, r+1) + C(n, r-1)$ **29.** Substitute $x = 1$ and $y = 3$ into the binomial theorem. **31.** Both sides count the number of ways to choose a subset of three distinct numbers $\{i, j, k\}$ with $i < j < k$ from $\{1, 2, \ldots, n\}$. **33.** $C(n+1, 5)$ **35.** 3,491,888,400 **37.** 5^{24} **39. a)** 45 **b)** 57 **c)** 12 **41. a)** 386 **b)** 56 **43.** 0 if $n < m$; $C(n-1, n-m)$ if $n \ge m$ **45. a)** 15,625 **b)** 202 **c)** 210 **d)** 10 **47. a)** 3 **b)** 11 **c)** 6 **d)** 10 **49.** There are two possibilities: three people seated at one table with everyone else sitting alone, which can be done in $2C(n, 3)$ ways (choose the three people and seat them in one of two arrangements), or two groups of two people seated together with everyone else sitting alone, which can be done in $3C(n, 4)$ ways (choose four people and then choose one of the three ways to pair them up). Both $2C(n, 3) + 3C(n, 4)$ and $(3n-1)C(n, 3)/4$ equal $n^4/8 - 5n^3/12 + 3n^2/8 - n/12$. **51.** The number of permutations of $2n$ objects of n different types, two of each type, is $(2n)!/2^n$. Because this must be an integer, the denominator must divide the numerator. **53.** CCGGUCCGAAAG **55.**

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procedure next permutation(n: positive integer,
    a_1, a_2, ..., a_r: positive integers not exceeding
    n with a_1 a_2 ... a_r ≠ nn ... n)
  i := r
  while a_i = n
    a_i := 1
    i := i − 1
  a_i := a_i + 1
  {a_1 a_2 ... a_r is the next permutation in lexicographic
    order}
```

57. We must show that if there are $R(m, n-1) + R(m-1, n)$ people at a party, then there must be at least m mutual friends or n mutual enemies. Consider one person; let's call him Jerry. Then there are $R(m-1, n) + R(m, n-1) - 1$ other people at the party, and by the pigeonhole principle there must be at least $R(m-1, n)$ friends of Jerry or $R(m, n-1)$ enemies of Jerry among these people. First let's suppose there are $R(m-1, n)$ friends of Jerry. By the definition of R, among these people we are guaranteed to find either $m-1$ mutual friends or n mutual enemies. In the former case, these $m-1$ mutual friends together with Jerry are a set of m mutual friends; and in the latter case, we have the desired set of n mutual enemies. The other situation is similar: Suppose there are $R(m, n-1)$ enemies of Jerry; we are guaranteed to find among them either m mutual friends or $n-1$ mutual enemies. In the former case, we have the desired set of m mutual friends, and in the latter case, these $n-1$ mutual enemies together with Jerry are a set of n mutual enemies.

CHAPTER 7

Section 7.1

1. 1/13 **3.** 1/2 **5.** 1/2 **7.** 1/64 **9.** 47/52 **11.** $1/C(52, 5)$ **13.** $1 - [C(48, 5)/C(52, 5)]$ **15.** $C(13, 2)C(4, 2)C(4, 2)C(44, 1)/C(52, 5)$ **17.** $10{,}240/C(52, 5)$ **19.** $1{,}302{,}540/C(52, 5)$ **21.** 1/64 **23.** 8/25 **25. a)** $1/C(50, 6) = 1/15{,}890{,}700$ **b)** $1/C(52, 6) = 1/20{,}358{,}520$ **c)** $1/C(56, 6) = 1/32{,}468{,}436$ **d)** $1/C(60, 6) = 1/50{,}063{,}860$ **27. a)** 139,128/319,865 **b)** 212,667/511,313 **c)** 151,340/386,529 **d)** 163,647/446,276 **29.** $1/C(100, 8)$ **31.** 3/100 **33. a)** 1/7,880,400 **b)** 1/8,000,000 **35. a)** 9/19 **b)** 81/361 **c)** 1/19 **d)** 1,889,568/2,476,099 **e)** 48/361 **37.** Three dice **39. a)** 4/756,438,375 **b)** 13/30,257,535 that we get each type, the probability of success on the next purchase (getting a new type) is $(n - j)/n$. **c)** This follows immediately from the definition of geometric distribution, the definition of X_j, and part (b). **d)** From part (c) it follows that $E(X_j) = n/(n - j)$. Thus, by the linearity of expectation and part (a), we have $E(X) = E(X_0) + E(X_1) + \cdots + E(X_{n-1}) = \frac{n}{n} + \frac{n}{n-1} + \cdots + \frac{n}{1} = n\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{1}\right)$. **e)** About 224.46 **35.** $24 \cdot 13^4/(52 \cdot 51 \cdot 50 \cdot 49)$

CHAPTER 8

Section 8.1

1. Let $P(n)$ be "$H_n = 2^n - 1$." *Basis step:* $P(1)$ is true because $H_1 = 1$. *Inductive step:* Assume that $H_n = 2^n - 1$. Then because $H_{n+1} = 2H_n + 1$, it follows that $H_{n+1} = 2(2^n - 1) + 1 = 2^{n+1} - 1$. **3. a)** $a_n = 2a_{n-1} + a_{n-5}$ for $n \geq 5$ **b)** $a_0 = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 = 8$, $a_4 = 16$ **c)** 1217 **5.** 9494 **7. a)** $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$ for $n \geq 2$ **b)** $a_0 = 0, a_1 = 0$ **c)** 94 **9. a)** $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$ **b)** $a_0 = 1, a_1 = 2, a_2 = 4$ **c)** 81 **11. a)** $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ **b)** $a_0 = 1, a_1 = 1$ **c)** 34 **13. a)** $a_n = 2a_{n-1} + 2a_{n-2}$ for $n \geq 2$ **b)** $a_0 = 1, a_1 = 3$ **c)** 448 **15. a)** $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 2$ **b)** $a_0 = 1, a_1 = 3$ **c)** 239 **17. a)** $a_n = 2a_{n-1}$ for $n \geq 2$ **b)** $a_1 = 3$ **c)** 96 **19. a)** $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ **b)** $a_0 = 1, a_1 = 1$ **c)** 89 **21. a)** $R_n = n + R_{n-1}, R_0 = 1$ **b)** $R_n = n(n+1)/2 + 1$ **23. a)** $S_n = S_{n-1} + (n^2 - n + 2)/2, S_0 = 1$ **b)** $S_n = (n^3 + 5n + 6)/6$ **25.** 64 **27. a)** $a_n = 2a_{n-1} + 2a_{n-2}$ **b)** $a_0 = 1, a_1 = 3$ **c)** 1224 **29.** Clearly, $S(m, 1) = 1$ for $m \geq 1$. If $m \geq n$, then a function that is not onto from the set with m elements to the set with n elements can be specified by picking the size of the range, which is an integer between 1 and $n - 1$ inclusive, picking the elements of the range, which can be done in $C(n, k)$ ways, and picking an onto function onto the range, which can be done in $S(m, k)$ ways. Hence, there are $\sum_{k=1}^{n-1} C(n, k)S(m, k)$ functions that are not onto. But there are n^m functions altogether, so $S(m, n) = n^m - \sum_{k=1}^{n-1} C(n, k)S(m, k)$. **31. a)** $C_5 = C_0C_4 + C_1C_3 + C_2C_2 + C_3C_1 + C_4C_0 = 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42$ **b)** $C(10, 5)/6 = 42$ **33.** $J(1) = 1$, $J(2) = 1$, $J(3) = 3$, $J(4) = 1$, $J(5) = 3$, $J(6) = 5$, $J(7) = 7$, $J(8) = 1$, $J(9) = 3$, $J(10) = 5$, $J(11) = 7$, $J(12) = 9$, $J(13) = 11$, $J(14) = 13$, $J(15) = 15$, $J(16) = 1$ **35.** First, suppose that the number of people is even, say $2n$. After going around the circle once and returning to the first person, because the people at locations with even numbers have been eliminated, there are exactly n people left and the person currently at location i is the person who was originally at location $2i - 1$. Therefore, the survivor [originally in location $J(2n)$] is now in location $J(n)$; this was the person who was at location $2J(n) - 1$. Hence, $J(2n) = 2J(n) - 1$. Similarly, when there are an odd number of people, say $2n + 1$, then after going around the circle once and then eliminating person 1, there are n people left and the person currently at location i is the person who was at location $2i + 1$. Therefore, the survivor will be the player currently occupying location $J(n)$, namely, the person who was originally at location $2J(n) + 1$. Hence, $J(2n + 1) = 2J(n) + 1$. The basis step is $J(1) = 1$. **37.** 73, 977, 3617 **39.** These nine moves solve the puzzle: Move disk 1 from peg 1 to peg 2; move disk 2 from peg 1 to peg 3; move disk 1 from peg 2 to peg 3; move disk 3 from peg 1 to peg 2; move disk 4 from peg 1 to peg 4; move disk 3 from peg 2 to peg 4; move disk 1 from peg 3 to peg 2; move disk 2 from peg 3 to peg 4; move disk 1 from peg 2 to peg 4. To see that at least nine moves are required, first note that at least seven moves are required no matter how many pegs are present: three to unstack the disks, one to move the largest disk 4, and three more moves to restack them. At least two other moves are needed, because to move disk 4 from peg 1 to peg 4 the other three disks must be on pegs 2 and 3, so at least one move is needed to restack them and one move to unstack them. **41.** The base cases are obvious. If $n > 1$, the algorithm consists of three stages. In the first stage, by the inductive hypothesis, $R(n - k)$ moves are used to transfer the smallest $n - k$ disks to peg 2. Then using the usual three-peg Tower of Hanoi algorithm, it takes $2^k - 1$ moves to transfer the rest of the disks (the largest k disks) to peg 4, avoiding peg 2. Then again by the inductive hypothesis, it takes $R(n - k)$ moves to transfer the smallest $n - k$ disks to peg 4; all the pegs are available for this, because the largest disks, now on peg 4, do not interfere. This establishes the recurrence relation. **43.** First note that $R(n) = \sum_{j=1}^{n}[R(j) - R(j-1)]$ [which follows because the sum is telescoping and $R(0) = 0$]. By Exercise 42, this is the sum of $2^{k'-1}$ for this range of values of j. Therefore, the sum is $\sum_{i=1}^{k} i2^{i-1}$, except that if n is not a triangular number, then the last few values when $i = k$ are missing, and that is what the final term in the given expression accounts for. **45.** By Exercise 43, $R(n)$ is no larger than $\sum_{i=1}^{k} i2^{i-1}$. It can be shown that this sum equals $(k + 1)2^k - 2^{k+1} + 1$, so it is no greater than $(k+1)2^k$. Because $n > k(k-1)/2$, the quadratic formula can be used to show that $k < 1 + \sqrt{2n}$ for all $n > 1$. Therefore, $R(n)$ is bounded above by $(1 + \sqrt{2n} + 1)2^{1+\sqrt{2n}} < 8\sqrt{n}2^{\sqrt{2n}}$ for all $n > 2$. Hence, $R(n)$ is $O(\sqrt{n}2^{\sqrt{2n}})$. **47. a)** 0 **b)** 0 **c)** 2 **d)** $2^{n-1} - 2^{n-2}$ **49.** $a_n - 2\nabla a_n + \nabla^2 a_n = a_n - 2(a_n - a_{n-1}) + (\nabla a_n - \nabla a_{n-1}) = -a_n + 2a_{n-1} + [(a_n - a_{n-1}) - (a_{n-1} - a_{n-2})] = -a_n + 2a_{n-1} + (a_n - 2a_{n-1} + a_{n-2}) = a_{n-2}$ **51.** $a_n = a_{n-1} + a_{n-2} = (a_n - \nabla a_n) + (a_n - 2\nabla a_n + \nabla^2 a_n) = 2a_n - 3\nabla a_n + \nabla^2 a_n$, or $a_n = 3\nabla a_n - \nabla^2 a_n$ **53.** Insert $S(0) := \emptyset$ after $T(0) := 0$ (where $S(j)$ will record the optimal set of talks among the first j talks), and replace the statement $T(j) := \max(w_j + T(p(j)), T(j-1))$ with the following code:

```
if w_j + T(p(j)) > T(j − 1) then
    T(j) := w_j + T(p(j))
    S(j) := S(p(j)) ∪ {j}
else
    T(j) := T(j − 1)
    S(j) := S(j − 1)
```

55. a) Talks 1, 3, and 7 **b)** Talks 1 and 6, or talks 1, 3, and 7 **c)** Talks 1, 3, and 7 **d)** Talks 1 and 6 **57. a)** This follows immediately from Example 5 and Exercise 43c in Section 8.4. **b)** The last step in computing $\mathbf{A}_{ij}$ is to multiply $\mathbf{A}_{ik}$ by $\mathbf{A}_{k+1,j}$ for some k between i and $j-1$ inclusive, which will require $m_i m_{k+1} m_{j+1}$ integer multiplications, independent of the manner in which $\mathbf{A}_{ik}$ and $\mathbf{A}_{k+1,j}$ are computed. Therefore, to minimize the total number of integer multiplications, each of those two factors must be computed in the most efficient manner. **c)** This follows immediately from part (b) and the definition of $M(i, j)$. **d)**

```
procedure matrix order(m_1, ..., m_{n+1}:
                positive integers)
  for i := 1 to n
    M(i, i) := 0
  for d := 1 to n − 1
    for i := 1 to n − d
      min := 0
      for k := i to i + d
        new := M(i, k) + M(k + 1, i + d) + m_i m_{k+1} m_{i+d+1}
        if new < min then
          min := new
          where(i, i + d) := k
      M(i, i + d) := min
```

e) The algorithm has three nested loops, each of which is indexed over at most n values.

Section 8.2

1. a) Degree 3 **b)** No **c)** Degree 4 **d)** No **e)** No **f)** Degree 2 **g)** No **3. a)** $a_n = 3 \cdot 2^n$ **b)** $a_n = 2$ **c)** $a_n = 3 \cdot 2^n - 2 \cdot 3^n$ **d)** $a_n = 6 \cdot 2^n - 2 \cdot n2^n$ **e)** $a_n = n(-2)^{n-1}$ **f)** $a_n = 2^n - (-2)^n$ **g)** $a_n = (1/2)^{n+1} - (-1/2)^{n+1}$ **5.** $a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$ **7.** $[2^{n+1} + (-1)^n]/3$ **9. a)** $P_n = 1.2P_{n-1} + 0.45P_{n-2}$, $P_0 = 100{,}000$, $P_1 = 120{,}000$ **b)** $P_n = (250{,}000/3)(3/2)^n + (50{,}000/3)(-3/10)^n$ **11. a)** *Basis step:* For $n = 1$ we have $1 = 0 + 1$, and for $n = 2$ we have $3 = 1 + 2$. *Inductive step:* Assume true for $k \le n$. Then $L_{n+1} = L_n + L_{n-1} = f_{n-1} + f_{n+1} + f_{n-2} + f_n = (f_{n-1} + f_{n-2}) + (f_{n+1} + f_n) = f_n + f_{n+2}$. **b)** $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$ **13.** $a_n = 8(-1)^n - 3(-2)^n + 4 \cdot 3^n$ **15.** $a_n = 5 + 3(-2)^n - 3^n$ **17.** Let $a_n = C(n, 0) + C(n-1, 1) + \cdots + C(n-k, k)$ where $k = \lfloor n/2 \rfloor$. First, assume that n is even, so that $k = n/2$, and the last term is $C(k, k)$. By Pascal's identity we have $a_n = 1 + C(n-2, 0) + C(n-2, 1) + C(n-3, 1) + C(n-3, 2) + \cdots + C(n-k, k-2) + C(n-k, k-1) + 1 = 1 + C(n-2, 1) + C(n-3, 2) + \cdots + C(n-k, k-1) + C(n-2, 0) + C(n-3, 1) + \cdots + C(n-k, k-2) + 1 = a_{n-1} + a_{n-2}$ because $\lfloor (n-1)/2 \rfloor = k - 1 = \lfloor (n-2)/2 \rfloor$. A similar calculation works when n is odd. Hence, $\{a_n\}$ satisfies the recurrence relation $a_n = a_{n-1} + a_{n-2}$ for all positive integers n, $n \ge 2$. Also, $a_1 = C(1, 0) = 1$ and $a_2 = C(2, 0) + C(1, 1) = 2$, which are f_2 and f_3. It follows that $a_n = f_{n+1}$ for all positive integers n. **19.** $a_n = (n^2 + 3n + 5)(-1)^n$ **21.** $(a_{1,0} + a_{1,1}n + a_{1,2}n^2 + a_{1,3}n^3) + (a_{2,0} + a_{2,1}n + a_{2,2}n^2)(-2)^n + (a_{3,0} + a_{3,1}n)3^n + a_{4,0}(-4)^n$ **23. a)** $3a_{n-1} + 2^n = 3(-2)^n + 2^n = 2^n(-3+1) = -2^{n+1} = a_n$ **b)** $a_n = \alpha 3^n - 2^{n+1}$ **c)** $a_n = 3^{n+1} - 2^{n+1}$ **25. a)** $A = -1$, $B = -7$ **b)** $a_n = \alpha 2^n - n - 7$ **c)** $a_n = 11 \cdot 2^n - n - 7$ **27. a)** $p_3 n^3 + p_2 n^2 + p_1 n + p_0$ **b)** $n^2 p_0 (-2)^n$ **c)** $n^2(p_1 n + p_0)2^n$ **d)** $(p_2 n^2 + p_1 n + p_0)4^n$ **e)** $n^2(p_2 n^2 + p_1 n + p_0)(-2)^n$ **f)** $n^2(p_4 n^4 + p_3 n^3 + p_2 n^2 + p_1 n + p_0)2^n$ **g)** p_0 **29. a)** $a_n = \alpha 2^n + 3^{n+1}$ **b)** $a_n = -2 \cdot 2^n + 3^{n+1}$ **31.** $a_n = \alpha 2^n + \beta 3^n - n \cdot 2^{n+1} + 3n/2 + 21/4$ **33.** $a_n = (\alpha + \beta n + n^2 + n^3/6)2^n$ **35.** $a_n = -4 \cdot 2^n - n^2/4 - 5n/2 + 1/8 + (39/8)3^n$ **37.** $a_n = n(n+1)(n+2)/6$ **39. a)** $1, -1, i, -i$ **b)** $a_n = \frac{1}{4} - \frac{1}{4}(-1)^n + \frac{2+i}{4}i^n + \frac{2-i}{4}(-i)^n$ **41. a)** Using the formula for f_n, we see that $\left| f_n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n \right| = \left| \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \right| < 1/\sqrt{5} < 1/2$. This means that f_n is the integer closest to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$. **b)** Less when n is even; greater when n is odd **43.** $a_n = f_{n-1} + 2f_n - 1$ **45. a)** $a_n = 3a_{n-1} + 4a_{n-2}$, $a_0 = 2$, $a_1 = 6$ **b)** $a_n = [4^{n+1} + (-1)^n]/5$ **47. a)** $a_n = 2a_{n+1} + (n-1)10{,}000$ **b)** $a_n = 70{,}000 \cdot 2^{n-1} - 10{,}000n - 10{,}000$ **49.** $a_n = 5n^2/12 + 13n/12 + 1$ **51.** See Chapter 11, Section 5 in [Ma93]. **53.** $6^n \cdot 4^{n-1}/n$

Section 8.3

1. 14 **3.** The first step is $(1110)_2(1010)_2 = (2^4 + 2^2)(11)_2\,(10)_2 + 2^2[(11)_2 - (10)_2][(10)_2 - (10)_2] + (2^2 + 1)(10)_2 \cdot (10)_2$. The product is $(10001100)_2$. **5.** $C = 50{,}665C + 729 = 33{,}979$ **7. a)** 2 **b)** 4 **c)** 7 **9. a)** 79 **b)** 48,829 **c)** 30,517,579 **11.** $O(\log n)$ **13.** $O(n^{\log_3 2})$ **15.** 5 **17. a)** *Basis step:* If the sequence has just one element, then the one person on the list is the winner. *Recursive step:* Divide the list into two parts—the first half and the second half—as equally as possible. Apply the algorithm recursively to each half to come up with at most two names. Then run through the entire list to count the number of occurrences of each of those names to decide which, if either, is the winner. **b)** $O(n \log n)$ **19. a)** $f(n) = f(n/2) + 2$ **b)** $O(\log n)$ **21. a)** 7 **b)** $O(\log n)$ **23. a)**

```
procedure largest sum(a_1, ..., a_n)
  best := 0 {empty subsequence has sum 0}
  for i := 1 to n
    sum := 0
    for j := i + 1 to n
      sum := sum + a_j
      if sum > best then best := sum
  {best is the maximum possible sum of numbers
    in the list}
```

b) $O(n^2)$ **c)** We divide the list into a first half and a second half and apply the algorithm recursively to find the largest sum of consecutive terms for each half. The largest sum of consecutive terms in the entire sequence is either one of these two numbers or the sum of a sequence of consecutive terms that crosses the middle of the list. To find the largest possible sum of a sequence of consecutive terms that crosses the middle of the list, we start at the middle and move forward to find the largest possible sum in the second half of the list, and move backward to find the largest possible sum in the first half of the list; the desired sum is the sum of these two quantities. The final answer is then the largest of this sum and the two answers obtained recursively. The base case is that the largest sum of a sequence of one term is the larger of that number and 0. **d)** 11, 9, 14 **e)** $S(n) = 2S(n/2) + n$, $C(n) = 2C(n/2)+n+2$, $S(1) = 0$, $C(1) = 1$ **f)** $O(n \log n)$, better than $O(n^2)$ **25.** (1, 6) and (3, 6) at distance 2 **27.** The algorithm is essentially the same as the algorithm given in Example 12. The central strip still has width $2d$ but we need to consider just two boxes of size $d \times d$ rather than eight boxes of size $(d/2) \times (d/2)$. The recurrence relation is the same as the recurrence relation in Example 12, except that the coefficient 7 is replaced by 1. **29.** With $k = \log_b n$, it follows that $f(n) = a^k f(1) + \sum_{j=0}^{k-1} a^j c(n/b^j)^d = a^k f(1) + \sum_{j=0}^{k-1} cn^d = a^k f(1) + kcn^d = a^{\log_b n} f(1) + c(\log_b n)n^d = n^{\log_b a} f(1) + cn^d \log_b n = n^d f(1) + cn^d \log_b n$. **31.** Let $k = \log_b n$ where n is a power of b. *Basis step:* If $n = 1$ and $k = 0$, then $c_1 n^d + c_2 n^{\log_b a} = c_1 + c_2 = b^d c/(b^d - a) + f(1) + b^d c/(a - b^d) = f(1)$. *Inductive step:* Assume true for k, where $n = b^k$. Then for $n = b^{k+1}$, $f(n) = af(n/b) + cn^d = a\{[b^d c/(b^d - a)](n/b)^d + [f(1) + b^d c/(a - b^d)] \cdot (n/b)^{\log_b a})\} + cn^d = b^d c/(b^d - a)n^d a/b^d + [f(1) + b^d c/(a - b^d)]n^{\log_b a} + cn^d = n^d[ac/(b^d - a) + c(b^d - a)/(b^d - a)] + [f(1) + b^d c/(a - b^d c)]n^{\log_b a} = [b^d c/(b^d - a)]n^d + [f(1) + b^d c/(a - b^d)]n^{\log_b a}$. **33.** If $a > b^d$, then $\log_b a > d$, so the second term dominates, giving $O(n^{\log_b a})$. **35.** $O(n^{\log_4 5})$ **37.** $O(n^3)$

Section 8.4

1. $f(x) = 2(x^6 - 1)/(x - 1)$ **3. a)** $f(x) = 2x(1 - x^6)/(1 - x)$ **b)** $x^3/(1 - x)$ **c)** $x/(1 - x^3)$ **d)** $2/(1 - 2x)$ **e)** $(1 + x)^7$ **f)** $2/(1+x)$ **g)** $[1/(1-x)] - x^2$ **h)** $x^3/(1-x)^2$ **5. a)** $5/(1-x)$ **b)** $1/(1-3x)$ **c)** $2x^3/(1-x)$ **d)** $(3-x)/(1-x)^2$ **e)** $(1+x)^8$ **7. a)** $a_0 = -64$, $a_1 = 144$, $a_2 = -108$, $a_3 = 27$, and $a_n = 0$ for all $n \geq 4$ **b)** The only nonzero coefficients are $a_0 = 1$, $a_3 = 3$, $a_6 = 3$, $a_9 = 1$. **c)** $a_n = 5^n$ **d)** $a_n = (-3)^{n-3}$ for $n \geq 3$, and $a_0 = a_1 = a_2 = 0$ **e)** $a_0 = 8$, $a_1 = 3$, $a_2 = 2$, $a_n = 0$ for odd n greater than 2 and $a_n = 1$ for even n greater than 2 **f)** $a_n = 1$ if n is a positive multiple 4, $a_n = -1$ if $n < 4$, and $a_n = 0$ otherwise **g)** $a_n = n - 1$ for $n \geq 2$ and $a_0 = a_1 = 0$ **h)** $a_n = 2^{n+1}/n!$ **9. a)** 6 **b)** 3 **c)** 9 **d)** 0 **e)** 5 **11. a)** 1024 **b)** 11 **c)** 66 **d)** 292,864 **e)** 20,412 **13.** 10 **15.** 50 **17.** 20 **19.** $f(x) = 1/[(1 - x)(1 - x^2)(1 - x^5)(1 - x^{10})]$ **21.** 15 **23. a)** $x^4(1 + x + x^2 + x^3)^2/(1-x)$ **b)** 6 **25. a)** The coefficient of x^r in the power series expansion of $1/[(1-x^3)(1-x^4)(1-x^{20})]$ **b)** $1/(1-x^3-x^4-x^{20})$ **c)** 7 **d)** 3224 **27. a)** The generating function is $(1+x+x^2+x^3+x^4)(1+x+x^2)(1+x^2+x^4+x^6+\cdots)(x^3+x^4+x^5+x^6+\cdots)(1+x^5+x^{10}+x^{15}+\cdots) = x^3(1+x+x^2+x^3+x^4)(1+x+x^2)/[(1-x^2)(1-x)(1-x^5)] = x^3+3x^4+7x^5+12x^6+19x^7+27x^8+37x^9+48x^{10}+61x^{11}+75x^{12}+\cdots$. The coefficient of x^n is the answer. **b)** 75 **29. a)** 3 **b)** 29 **c)** 29 **d)** 242 **31. a)** 10 **b)** 49 **c)** 2 **d)** 4 **33. a)** $G(x) - a_0 - a_1 x - a_2 x^2$ **b)** $G(x^2)$ **c)** $x^4 G(x)$ **d)** $G(2x)$ **e)** $\int_0^x G(t)dt$ **f)** $G(x)/(1 - x)$ **35.** $a_k = 2 \cdot 3^k - 1$ **37.** $a_k = 18 \cdot 3^k - 12 \cdot 2^k$ **39.** $a_k = k^2 + 8k + 20 + (6k - 18)2^k$ **41.** Let $G(x) = \sum_{k=0}^{\infty} f_k x^k$. After shifting indices of summation and adding series, we see that $G(x) - xG(x) - x^2 G(x) = f_0 + (f_1 - f_0)x + \sum_{k=2}^{\infty}(f_k - f_{k-1} - f_{k-2})x^k = 0 + x + \sum_{k=2}^{\infty} 0x^k$. Hence, $G(x) - xG(x) - x^2 G(x) = x$. Solving for $G(x)$ gives $G(x) = x/(1-x-x^2)$. By the method of partial fractions, it can be shown that $x/(1-x-x^2) = (1/\sqrt{5})[1/(1-\alpha x) - 1/(1-\beta x)]$, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Using the fact that $1/(1 - \alpha x) = \sum_{k=0}^{\infty} \alpha^k x^k$, it follows that $G(x) = (1/\sqrt{5}) \cdot \sum_{k=0}^{\infty}(\alpha^k - \beta^k)x^k$. Hence, $f_k = (1/\sqrt{5}) \cdot (\alpha^k - \beta^k)$. **43. a)** Let $G(x) = \sum_{n=0}^{\infty} C_n x^n$ be the generating function for $\{C_n\}$. Then $G(x)^2 = \sum_{n=0}^{\infty}(\sum_{k=0}^{n} C_k C_{n-k})x^n = \sum_{n=1}^{\infty}(\sum_{k=0}^{n-1} C_k C_{n-1-k})x^{n-1} = \sum_{n=1}^{\infty} C_n x^{n-1}$. Hence, $xG(x)^2 = \sum_{n=1}^{\infty} C_n x^n$, which implies that $xG(x)^2 - G(x) + 1 = 0$. Applying the quadratic formula shows that $G(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. We choose the minus sign in this formula because the choice of the plus sign leads to a division by zero. **b)** By Exercise 42, $(1 - 4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$. Integrating term by term (which is valid by a theorem from calculus) shows that $\int_0^x (1 - 4t)^{-1/2}dt = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^{n+1} = x\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n$. Because $\int_0^x (1 - 4t)^{-1/2}dt = \frac{1-\sqrt{1-4x}}{2} = xG(x)$, equating coefficients shows that $C_n = \frac{1}{n+1}\binom{2n}{n}$. **c)** Verify the basis step for $n = 1, 2, 3, 4, 5$. Assume the inductive hypothesis that $C_j \geq 2^{j-1}$ for $1 \leq j < n$, where $n \geq 6$. Then $C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1} \geq \sum_{k=1}^{n-2} C_k C_{n-k-1} \geq (n-2)2^{k-1}2^{n-k-2} = (n - 2)2^{n-1}/4 \geq 2^{n-1}$. **45.** Applying the binomial theorem to the equality $(1 + x)^{m+n} = (1 + x)^m(1 + x)^n$, shows that $\sum_{r=0}^{m+n} C(m + n, r)x^r = \sum_{r=0}^{m} C(m, r)x^r \cdot \sum_{r=0} C(n, r)x^r = \sum_{r=0}^{m+n} \left[\sum_{k=0}^{r} C(m, r - k)\, C(n, k)\right] x^r$. Comparing coefficients gives the desired identity. **47. a)** $2e^x$ **b)** e^{-x} **c)** e^{3x} **d)** $xe^x + e^x$ **49. a)** $a_n = (-1)^n$ **b)** $a_n = 3 \cdot 2^n$ **c)** $a_n = 3^n - 3 \cdot 2^n$ **d)** $a_n = (-2)^n$ for $n \geq 2$, $a_1 = -3$, $a_0 = 2$ **e)** $a_n = (-2)^n + n!$ **f)** $a_n = (-3)^n + n! \cdot 2^n$ for $n \geq 2$, $a_0 = 1$, $a_1 = -2$ **g)** $a_n = 0$ if n is odd and $a_n = n!/(n/2)!$ if n is even **51. a)** $a_n = 6a_{n-1} + 8^{n-1}$ for $n \geq 1$, $a_0 = 1$ **b)** The general solution of the associated linear homogeneous recurrence relation is $a_n^{(h)} = \alpha 6^n$. A particular solution is $a_n^{(p)} = \frac{1}{2} \cdot 8^n$. Hence, the general solution is $a_n = \alpha 6^n + \frac{1}{2} \cdot 8^n$. Using the initial condition, it follows that $\alpha = \frac{1}{2}$. Hence, $a_n = (6^n + 8^n)/2$. **c)** Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Using the recurrence relation for $\{a_k\}$, it can be shown that $G(x) - 6xG(x) = (1 - 7x)/(1 - 8x)$. Hence, $G(x) = (1 - 7x)/[(1 - 6x)(1 - 8x)]$. Using partial fractions, it follows that $G(x) = (1/2)/(1 - 6x) + (1/2)/(1 - 8x)$. With the help of Table 1, it follows that $a_n = (6^n + 8^n)/2$. **53.** $\frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots$ **55.** $(1+x)(1+x)^2(1+x)^3 \cdots$ **57.** The generating functions obtained in Exercises 54 and 55 are equal because $(1 + x)(1 + x^2)(1 + x^3) \cdots = \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdots = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots$. **59. a)** $G_X(1) = \sum_{k=0}^{\infty} p(X = k) \cdot 1^k = \sum_{k=0}^{\infty} P(X = k) = 1$ **b)** $G'_X(1) = \frac{d}{dx}\sum_{k=0}^{\infty} p(X = k) \cdot x^k|_{x=1} = \sum_{k=0}^{\infty} p(X = k) \cdot k \cdot x^{k-1}|_{x=1} = \sum_{k=0}^{\infty} p(X = k) \cdot k = E(X)$

c) $G''_X(1) = \frac{d^2}{dx^2} \sum_{k=0}^{\infty} p(X = k) \cdot x^k|_{x=1} = \sum_{k=0}^{\infty} p(X = k) \cdot k(k-1) \cdot x^{k-2}|_{x=1} = \sum_{k=0}^{\infty} p(X = k) \cdot (k^2 - k) = V(X) + E(X)^2 - E(X)$. Combining this with part (b) gives the desired results. **61. a)** $G(x) = p^m/(1-qx)^m$ **b)** $V(x) = mq/p^2$

Section 8.5

1. a) 30 **b)** 29 **c)** 24 **d)** 18 **3.** 1% **5. a)** 300 **b)** 150 **c)** 175 **d)** 100 **7.** 492 **9.** 974 **11.** 610 **13.** 55 **15.** 248 **17.** 50,138 **19.** 234 **21.** $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_1 \cap A_5| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_2 \cap A_5| - |A_3 \cap A_4| - |A_3 \cap A_5| - |A_4 \cap A_5| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_2 \cap A_5| + |A_1 \cap A_3 \cap A_4| + |A_1 \cap A_3 \cap A_5| + |A_1 \cap A_4 \cap A_5| + |A_2 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_5| + |A_2 \cap A_4 \cap A_5| + |A_3 \cap A_4 \cap A_5| - |A_1 \cap A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_5| - |A_1 \cap A_2 \cap A_4 \cap A_5| - |A_1 \cap A_3 \cap A_4 \cap A_5| - |A_2 \cap A_3 \cap A_4 \cap A_5| + |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|$ **23.** $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| + |A_6| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_1 \cap A_5| - |A_1 \cap A_6| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_2 \cap A_5| - |A_2 \cap A_6| - |A_3 \cap A_4| - |A_3 \cap A_5| - |A_3 \cap A_6| - |A_4 \cap A_5| - |A_4 \cap A_6| - |A_5 \cap A_6|$ **25.** $p(E_1 \cup E_2 \cup E_3) = p(E_1) + p(E_2) + p(E_3) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_2 \cap E_3) + p(E_1 \cap E_2 \cap E_3)$ **27.** 4972/71,295 **29.** $p(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5) = p(E_1) + p(E_2) + p(E_3) + p(E_4) + p(E_5) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_1 \cap E_4) - p(E_1 \cap E_5) - p(E_2 \cap E_3) - p(E_2 \cap E_4) - p(E_2 \cap E_5) - p(E_3 \cap E_4) - p(E_3 \cap E_5) - p(E_4 \cap E_5) + p(E_1 \cap E_2 \cap E_3) + p(E_1 \cap E_2 \cap E_4) + p(E_1 \cap E_2 \cap E_5) + p(E_1 \cap E_3 \cap E_4) + p(E_1 \cap E_3 \cap E_5) + p(E_1 \cap E_4 \cap E_5) + p(E_2 \cap E_3 \cap E_4) + P(E_2 \cap E_3 \cap E_5) + p(E_2 \cap E_4 \cap E_5) + p(E_3 \cap E_4 \cap E_5)$ **31.** $p\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{1 \le i \le n} p(E_i) - \sum_{1 \le i < j \le n} p(E_i \cap E_j) + \sum_{1 \le i < j < k \le n} p(E_i \cap E_j \cap E_k) - \cdots + (-1)^{n+1} p\left(\bigcap_{i=1}^{n} E_i\right)$

Section 8.6

1. 75 **3.** 6 **5.** 46 **7.** 9875 **9.** 540 **11.** 2100 **13.** 1854 **15. a)** $D_{100}/100!$ **b)** $100D_{99}/100!$ **c)** $C(100,2)/100!$ **d)** 0 **e)** $1/100!$ **17.** 2,170,680 **19.** By Exercise 18 we have $D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}]$. Iterating, we have $D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}] = -[-(D_{n-2} - (n-2)D_{n-3})] = D_{n-2} - (n-2)D_{n-3} = \cdots = (-1)^n(D_2 - 2D_1) = (-1)^n$ because $D_2 = 1$ and $D_1 = 0$. **21.** When n is odd **23.** $\phi(n) = n - \sum_{i=1}^{m} \frac{n}{p_i} + \sum_{1 \le i < j \le m} \frac{n}{p_i p_j} - \cdots \pm \frac{n}{p_1 p_2 \cdots p_m} = n \prod_{i=1}^{m} \left(a - \frac{1}{p_i}\right)$ **25.** 4 **27.** There are n^m functions from a set with m elements to a set with n elements, $C(n, 1)(n-1)^m$ functions from a set with m elements to a set with n elements that miss exactly one element, $C(n, 2)(n-2)^m$ functions from a set with m elements to a set with n elements that miss exactly two elements, and so on, with $C(n, n-1) \cdot 1^m$ functions from a set with m elements to a set with n elements that miss exactly $n-1$ elements. Hence, by the principle of inclusion–exclusion, there are $n^m - C(n, 1)(n-1)^m + C(n, 2)(n-2)^m - \cdots + (-1)^{n-1} C(n, n-1) \cdot 1^m$ onto functions.

Supplementary Exercises

1. a) $A_n = 4A_{n-1}$ **b)** $A_1 = 40$ **c)** $A_n = 10 \cdot 4^n$ **3. a)** $M_n = M_{n-1} + 160{,}000$ **b)** $M_1 = 186{,}000$ **c)** $M_n = 160{,}000n + 26{,}000$ **d)** $T_n = T_{n-1} + 160{,}000n + 26{,}000$ **e)** $T_n = 80{,}000n^2 + 106{,}000n$ **5. a)** $a_n = a_{n-2} + a_{n-3}$ **b)** $a_1 = 0$, $a_2 = 1$, $a_3 = 1$ **c)** $a_{12} = 12$ **7. a)** 2 **b)** 5 **c)** 8 **d)** 16 **9.** $a_n = 2^n$ **11.** $a_n = 2 + 4n/3 + n^2/2 + n^3/6$ **13.** $a_n = a_{n-2} + a_{n-3}$ **15. a)** Under the given conditions, one longest common subsequence ends at the last term in each sequence, so $a_m = b_n = c_p$. Furthermore, a longest common subsequence of what is left of the a-sequence and the b-sequence after those last terms are deleted has to form the beginning of a longest common subsequence of the original sequences. **b)** If $c_p \ne a_m$, then the longest common subsequence's appearance in the a-sequence must terminate before the end; therefore, the c-sequence must be a longest common subsequence of $a_1, a_2, \ldots, a_{m-1}$ and $b_1, b_2, \ldots, b_n$. The other half is similar.

17.
```
procedure howlong(a1, ..., am, b1, ..., bn: sequences)
for i := 1 to m
   L(i, 0) := 0
for j := 1 to n
   L(0, j) := 0
for i := 1 to m
   for j := 1 to n
      if ai = bj then L(i, j) := L(i − 1, j − 1) + 1
      else L(i, j) := max(L(i, j − 1), L(i − 1, j))
return L(m, n)
```

19. $f(n) = (4n^2 - 1)/3$ **21.** $O(n^4)$ **23.** $O(n)$ **25.** Using just two comparisons, the algorithm is able to narrow the search for m down to the first half or the second half of the original sequence. Since the length of the sequence is cut in half each time, only about $2 \log_2 n$ comparisons are needed in all. **27. a)** $18n + 18$ **b)** 18 **c)** 0 **29.** $\Delta(a_n b_n) = a_{n+1}b_{n+1} - a_n b_n = a_{n+1}(b_{n+1} - b_n) + b_n(a_{n+1} - a_n) = a_{n+1}\Delta b_n + b_n \Delta a_n$ **31. a)** Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $G'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$. Therefore, $G'(x) - G(x) = \sum_{n=0}^{\infty}[(n+1)a_{n+1} - a_n]x^n = \sum_{n=0} x^n/n! = e^x$, as desired. That $G(0) = a_0 = 1$ is given. **b)** We have $[e^{-x}G(x)]' = e^{-x}G'(x) - e^{-x}G(x) = e^{-x}[G'(x) - G(x)] = e^{-x} \cdot e^x = 1$. Hence, $e^{-x}G(x) = x + c$, where c is a constant. Consequently, $G(x) = xe^x + ce^x$. Because $G(0) = 1$, it follows that $c = 1$. **c)** We have $G(x) = \sum_{n=0}^{\infty} x^{n+1}/n! + \sum_{n=0}^{\infty} x^n/n! = \sum_{n=1}^{\infty} x^n/(n-1)! + \sum_{n=0}^{\infty} x^n/n!$. Therefore, $a_n = 1/(n-1)! + 1/n!$ for all $n \ge 1$, and $a_0 = 1$. **33.** 7 **35.** 110 **37.** 0 **39. a)** 19 **b)** 65 **c)** 122 **d)** 167 **e)** 168 **41.** $D_{n-1}/(n-1)!$ **43.** 11/32

LIST OF SYMBOLS

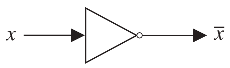
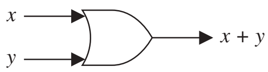
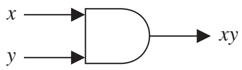
TOPIC	SYMBOL	MEANING	PAGE
LOGIC	$\neg p$	negation of p	3
	$p \wedge q$	conjunction of p and q	4
	$p \vee q$	disjunction of p and q	4
	$p \oplus q$	exclusive or of p and q	5
	$p \rightarrow q$	implication p implies q	6
	$p \leftrightarrow q$	biconditional of p and q	10
	$p \equiv q$	equivalence of p and q	27
	$\mathbf{T}$	tautology	29
	$\mathbf{F}$	contradiction	29
	$P(x_1, \ldots, x_n)$	propositional function	42
	$\forall x P(x)$	universal quantification of $P(x)$	44
	$\exists x P(x)$	existential quantification of $P(x)$	45
	$\exists! x P(x)$	uniqueness quantification of $P(x)$	46
	$\therefore$	therefore	73
	$p\{S\}q$	partial correctness of S	393
SETS	$x \in S$	x is a member of S	122
	$x \notin S$	x is not a member of S	122
	$\{a_1, \ldots, a_n\}$	list of elements of a set	122
	$\{x \mid P(x)\}$	set builder notation	122
	$\mathbf{N}$	set of natural numbers	122
	$\mathbf{Z}$	set of integers	122
	$\mathbf{Z}^+$	set of positive integers	122
	$\mathbf{Q}$	set of rational numbers	122
	$\mathbf{R}$	set of real numbers	122
	$[a, b]$, (a, b)	closed, open intervals	123
	$S = T$	set equality	123
	$\emptyset$	empty (or null) set	124
	$S \subseteq T$	S is a subset of T	125
	$S \subset T$	S is a proper subset of T	126
	$\|S\|$	cardinality of S	127
	$\mathcal{P}(S)$	power set of S	128
	$(a_1, \ldots, a_n)$	n-tuple	128
	(a, b)	ordered pair	128
	$A \times B$	Cartesian product of A and B	129
	$A \cup B$	union of A and B	133
	$A \cap B$	intersection of A and B	134
	$A - B$	difference of A and B	135
	$\overline{A}$	complement of A	135
	$\bigcup_{i=1}^{n} A_i$	union of A_i, $i = 1, 2, \ldots, n$	140
	$\bigcap_{i=1}^{n} A_i$	intersection of A_i, $i = 1, 2, \ldots, n$	140
	$A \oplus B$	symmetric difference of A and B	145
	$\aleph_0$	cardinality of a countable set	180
	$\mathfrak{c}$	cardinality of $\mathbf{R}$	185

TOPIC	SYMBOL	MEANING	PAGE
FUNCTIONS	$f(a)$	value of function f at a	147
	$f{:}A \to B$	function from A to B	147
	$f_1 + f_2$	sum of functions f_1 and f_2	149
	$f_1 f_2$	product of functions f_1 and f_2	149
	$f(S)$	image of set S under function f	149
	$\iota_A(s)$	identity function on A	153
	$f^{-1}(x)$	inverse of f	153
	$f \circ g$	composition of f and g	155
	$\lfloor x \rfloor$	floor function of x	157
	$\lceil x \rceil$	ceiling function of x	157
	a_n	term of $\{a_i\}$ with subscript n	165
	$\sum_{i=1}^{n} a_i$	sum of $a_1, a_2, \ldots, a_n$	172
	$\sum_{\alpha \in S} a_\alpha$	sum of a_α over $\alpha \in S$	175
	$\prod_{i=1}^{n} a_n$	product of $a_1, a_2, \ldots, a_n$	179
	$f(x)$ is $O(g(x))$	$f(x)$ is big-O of $g(x)$	217
	$n!$	n factorial	160
	$f(x)$ is $\Omega(g(x))$	$f(x)$ is big-Omega of $g(x)$	227
	$f(x)$ is $\Theta(g(x))$	$f(x)$ is big-Theta of $g(x)$	227
	$\sim$	asymptotic to	231
	$\min(x, y)$	minimum of x and y	281
	$\max(x, y)$	maximum of x and y	282
	$\approx$	approximately equal to	472
INTEGERS	$a \mid b$	a divides b	252
	$a \nmid b$	a does not divide b	252
	$a \ \mathbf{div}\ b$	quotient when a is divided by b	253
	$a \ \mathbf{mod}\ b$	remainder when a is divided by b	253
	$a \equiv b \pmod{m}$	a is congruent to b modulo m	254
	$a \not\equiv b \pmod{m}$	a is not congruent to b modulo m	254
	$\mathbf{Z}_m$	integers modulo m	257
	$(a_k a_{k-1} \ldots a_1 a_0)_b$	base b representation	260
	$\gcd(a, b)$	greatest common divisor of a and b	280
	$\operatorname{lcm}(a, b)$	least common multiple of a and b	282
MATRICES	$[a_{ij}]$	matrix with entries a_{ij}	188
	$\mathbf{A} + \mathbf{B}$	matrix sum of $\mathbf{A}$ and $\mathbf{B}$	189
	$\mathbf{AB}$	matrix product of $\mathbf{A}$ and $\mathbf{B}$	189
	$\mathbf{I}_n$	identity matrix of order n	190
	$\mathbf{A}^t$	transpose of $\mathbf{A}$	191
	$\mathbf{A} \vee \mathbf{B}$	join of $\mathbf{A}$ and $\mathbf{B}$	192
	$\mathbf{A} \wedge \mathbf{B}$	meet of $\mathbf{A}$ and $\mathbf{B}$	192
	$\mathbf{A} \odot \mathbf{B}$	Boolean product of $\mathbf{A}$ and $\mathbf{B}$	192
	$\mathbf{A}^{[n]}$	nth Boolean power of $\mathbf{A}$	193

(List of Symbols continued at back of book)

LIST OF SYMBOLS

TOPIC	SYMBOL	MEANING	PAGE
COUNTING AND PROBABILITY	$P(n, r)$	number of r-permutations of a set with n elements	429
	$C(n, r)$	number of r-combinations of a set with n elements	431
	$\binom{n}{r}$	binomial coefficient n choose r	431
	$C(n; n_1, n_2, \ldots, n_m)$	multinomial coefficient	457
	$p(E)$	probability of E	470
	$p(E \mid F)$	conditional probability of E given F	481
	$E(X)$	expected value of random variable X	503
	$V(X)$	variance of random variable X	513
	C_n	Catalan number	533
	$N(P_{i_1} \ldots P_{i_n})$	number of elements having all the properties $P_{i_j}, j = 1, \ldots, n$	585
	$N(P'_{i_1} \ldots P'_{i_n})$	number of elements having none of the properties $P_{i_j}, j = 1, \ldots, n$	585
	D_n	number of derangements of n objects	589
RELATIONS	$S \circ R$	composite of relations R and S	606
	R^n	nth power of relation R	607
	R^{-1}	inverse relation	609
	s_C	selection operator for condition C	613
	$P_{i_1, i_2, \ldots, i_m}$	projection	614
	$J_p(R, S)$	join	615
	Δ	diagonal relation	628
	R^*	connectivity relation of R	631
	$a \sim b$	a is equivalent to b	639
	$[a]_R$	equivalence class of a with respect to R	641
	$[a]_m$	congruence class modulo m	642
	(S, R)	poset consisting of set S and partial ordering R	650
	$a \prec b$	a is less than b	651
	$a \succ b$	a is greater than b	651
	$a \preccurlyeq b$	a is less than or equal to b	651
	$a \succcurlyeq b$	a is greater than or equal to b	651
GRAPHS AND TREES	(u, v)	directed edge	625
	$G = (V, E)$	graph with vertex set V and edge set E	673
	$\{u, v\}$	undirected edge	674
	$\deg(v)$	degree of vertex v	685
	$\deg^-(v)$	in-degree of vertex v	687
	$\deg^+(v)$	out-degree of vertex v	687
	K_n	complete graph on n vertices	688
	C_n	cycle of size n	688
	W_n	wheel of size n	689
	Q_n	n-cube	689
	$K_{m,n}$	complete bipartite graph of size m, n	691
	$G - e$	subgraph of G with edge e removed	697
	$G + e$	graph produced by adding edge e to graph G	697

TOPIC	SYMBOL	MEANING	PAGE
GRAPHS AND TREES (cont.)	$G_1 \cup G_2$	union of G_1 and G_2	699
	$a, x_1, \ldots, x_{n-1}, b$	path from a to b	714
	$a, x_1, \ldots, x_{n-1}, a$	circuit	714
	$\kappa(G)$	vertex connectivity of G	718
	$\lambda(G)$	edge connectivity of G	720
	r	number of regions of the plane	756
	$\deg(R)$	degree of region R	757
	$\chi(G)$	chromatic number of G	763
	m	greatest number of children of an internal vertex in a rooted tree	784
	n	number of vertices of a rooted tree	788
	i	number of internal vertices of a rooted tree	788
	l	number of leaves of a rooted tree	789
	h	height of a rooted tree	790
BOOLEAN ALGEBRA	$\overline{x}$	complement of Boolean variable x	847
	$x + y$	Boolean sum of x and y	847
	$x \cdot y$ (or xy)	Boolean product of x and y	847
	B	$\{0, 1\}$	848
	F^d	dual of F	852
	$x \mid y$	*x NAND y*	857
	$x \downarrow y$	*x NOR y*	857
		inverter	22, 859
		OR gate	22, 859
		AND gate	22, 859
LANGUAGES AND FINITE-STATE MACHINES	λ	empty string	166, 887
	xy	concatenation of x and y	371
	$l(x)$	length of string x	371
	w^R	reversal of w	380
	(V, T, S, P)	phrase-structure grammar	887
	S	start symbol	887
	$w \to w_1$	production	887
	$w_1 \Rightarrow w_2$	w_2 is directly derivable from w_1.	887
	$w_1 \stackrel{*}{\Rightarrow} w_2$	w_2 is derivable from w_1.	887
	$\langle A \rangle ::= \langle B \rangle c \mid d$	Backus–Naur form	893
	(S, I, O, f, g, s_0)	finite-state machine with output	898
	s_0	initial or start state	898
	AB	concatenation of sets A and B	904
	A^*	Kleene closure of A	905
	(S, I, f, s_0, F)	finite-state machine automaton with no output	905
	(S, I, f, s_0)	Turing machine	927