Mathematical Modelling of Revenue Forecasting: Nike and Adidas

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Abstract

In this report, we modelled the revenues of Adidas and Nike with non-linear autonomous ordinary differential equations, inspired from Lotka-Volterra model, to forecast revenues for these 2 companies. Furthermore, we analyzed the behaviour of our system of ordinary differential equations. The model has the same properties as economic market, which are unpredictable and sensitive to changes (chaotic). Thus, the model is able to forecast the revenues in short term period, specifically one year ahead of given data set.

Contents

1 Literature Review

1.1 Introduction

In the business world, competition has been widely believed to be pivotal in stimulating economic efficiency and productivity. In a paper by Overseas Development Institute (ODI), a research was conducted to test how business competition will affect the market of agricultural and technology markets across Africa and Asia from 1999–2009 [1]. The analysis on the ownership, competition authority, government regulations and trade policies were done to determine the degree of competition in each country and how it will affect the market outcomes. It was concluded that markets with more competitions, more players, and more intense rivalry for customers tend to deliver better market outcomes. These outcomes consist of lower prices, better access to service customers, and more internationally competitive productions.

Due to the pivotal role of business competition to the economy sector, in this paper, we will analyze how the business competition acts on the two largest sports' apparel brand industries: Nike and Adidas. However, instead of just analyzing the relationship between these companies, we will also discuss further on how the relationship will affect their revenues in the future.

1.2 Literature Review

There are numerous ways to model business competition. One of them is Panzar-Rosse approach, which was applied in a paper by Aktan and Masood to measure the competition between 17 significant banks in Turkey from 1998-2008 [2]. Panzar-Rosse methodology is a statistical approach which examines the relationship between input prices and equilibrium gross revenue of the competitive firms. This analysis will aim to observe the competition patterns in the long run equilibrium. It applies H-statistic testing, where the H-statistic value will determine whether the competition pattern is either monopolistic or perfect competition.

Besides the statistical approach to analyze business competition, a mathematical model inspired by predator-prey symbiosis in nature called Lotka-Volterra is also widely applied. This method was used in a paper by Addison et al. to analyze the stock market by assuming that the competing company acted as a prey, while the investing company acted as a predator [3]. It was concluded that stock market was indeed similar to ecosystem.

In another paper about Korean stocks market, a similar principle had been utilized to model the rivalry between two most famous stocks, Korean Stock Exchange (KSE) and Korean Securities Dealers Automated Quotation (KASDAQ) between 1997-2001 [4]. It was showed that the relation between the two stocks was dynamic over time: it changed from pure predator-prey relationship to mutualism symbiosis and finally to pure competition relationship. It was also concluded that equilibrium point existed in 2001. This competition pattern would be useful to observe the trend of the market and forecast which stock would yield better outcome in the following year. This insight may bring light for investors in allocating their capital budget to these stocks.

Henceforth, aligning with this objective, we will analyze the competition between Nike and Adidas based on their revenues annually. The analogy is based on a modified Lotka-Volterra model. An algorithm will be run to determine their relationship and investing capability instead of determining which company is the predator or prey. This rivalry pattern will forecast their revenues in 2019.

1.3 Advanced Theory of Ordinary Differential Equation

For the advanced mathematical analysis, we will use Lyapunov exponent, Lyapunov spectrum, and Kaplan-Yorke dimension.

1.3.1 Lyapunov Exponent

Lyapunov exponent is defined as [5]:

$$\lambda = \frac{1}{t - t_0} \ln \left| \frac{d}{d_0} \right| \tag{1}$$

where the variable d is the displacement between the points after some time $t > t_0$.

It is important to calculate the sensitivity of a dynamical system when there is a small change in initial condition. A system is chaotic when a small change in initial condition causes a big difference in the trajectory. This measurement of chaotic system is determined by the sign of maximum Lyapunov exponent. Positive maximum Lyapunov exponent value usually implies that the system is chaotic [6].

1.3.2 Lyapunov Spectrum

Lyapunov characteristic exponent is used to measure how chaotic a system is [7]. It is defined by the eigenvalues of Λ:

$$\Lambda = \lim_{t\to\infty} \frac{1}{2t} log(Y(t)Y^T(t)) \tag{2}$$

and the value of $\dot{Y} = JY$ and $J_{ij}(t) = \frac{df_i(x)}{dx_j}$.

Moreover, by calculating the Lyapunov characteristic exponents, we can determine the stability of the system where

$$\lambda_i < 0 \iff \text{stable manifold}$$

$$\lambda_i > 0 \iff \text{unstable manifold}$$

Here, λ_i indicates the Lyapunov Characteristics Exponential (LCE) spectrum [6].

1.3.3 Kaplan-Yorke Dimension

The Kaplan-Yorke dimension is defined as [8]:

$$D_{KY} \equiv j + \frac{\sigma_1 + \cdots + \sigma_j}{|\sigma_{j+1}|} \tag{3}$$

where $\sigma_1 \geq \sigma_n$ are Lyapunov characteristic exponents and j is the largest integer for which $\lambda_1 + \cdots + \lambda_j \leq 0$.

2 Data Collection

The objective of our model is to forecast the revenue of Nike and Adidas in 2019, hence we collected the data of companies' revenues from annual income statements of Nike and Adidas in recent years (2005-2018) [9, 10]. The collected data is based on total global sales of goods of each company. The plot of annual revenues of Nike and Adidas is shown in Figure 1 below.

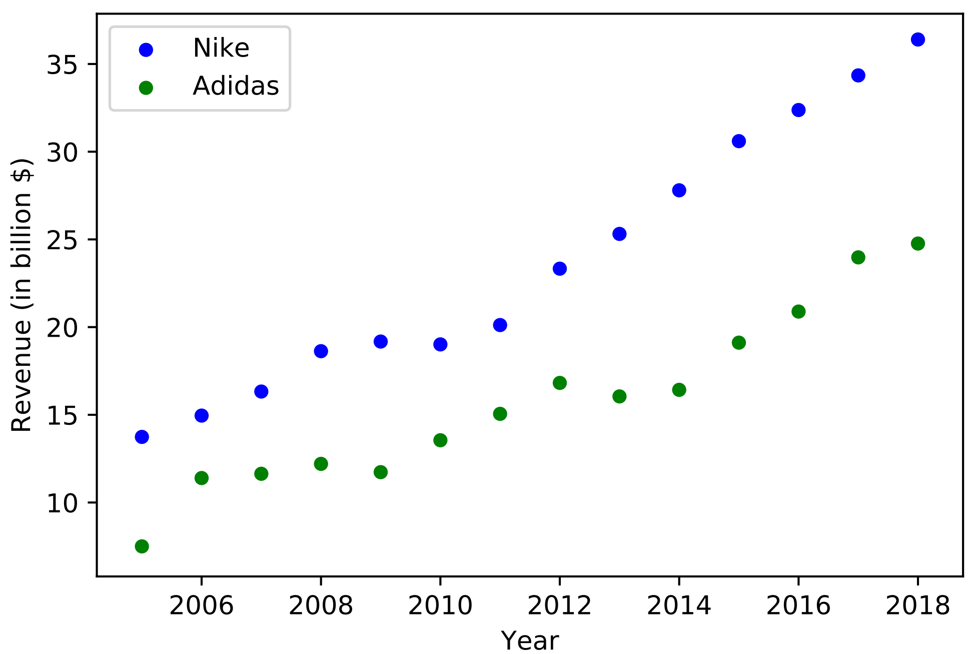

Figure 1: Annual revenue through 2005-2018 (in billion $)

3 Modelling

Our model is inspired by Lotka-Volterra model. The assumptions are as follows:

1. Nike and Adidas are the only existing companies in the sportswear industry.
2. The parameters in the equations are constant throughout calculations.

3.1 System of ODEs

The System of ODEs are:

$$\frac{dA}{dt} = \frac{r_1 AN + b_1 A}{A+N} = r_1 \cdot \left(\frac{N}{A+N}\right) \cdot A + b_1 \cdot \left(\frac{A}{A+N}\right) \tag{4}$$

$$\frac{dN}{dt} = \frac{r_2 AN + b_2 N}{A+N} = r_2 \cdot \left(\frac{A}{A+N}\right) \cdot N + b_2 \cdot \left(\frac{N}{A+N}\right) \tag{5}$$

where $A, N > 0$ and $r_1, r_2, b_1, b_2 \in [-1, 1]$.

3.2 Parameters

The parameters A and N are Adidas' and Nike's respective revenues in billions of dollars. Additionally, parameters r_1 and r_2 measure their influence on each other (relationship), while parameters b_1 and b_2 measure their investment rates.

Nature of Relationship	**Value of r_1**	**Value of r_2**	**Remarks**
Neutral relationship	0	0	Both companies are independent as there are no interaction between them.
Mutualism	+	+	Both companies are beneficial for one another.
Predation	+ -	- +	The standard Predator-Prey relationship: Adidas will prey on Nike or Nike will prey on Adidas.
Competition	-	-	Both companies are competing with one another.
Commensalism	+ 0	0 +	Adidas benefits from Nike but Nike is not affected by Adidas. Nike benefits from Adidas but Adidas is not affected by Nike.

Table 1: Summary of Relationship between Nike and Adidas

Value of b	Remarks
+	Positive investment return rate of a company. The company is earning money from external investment.
-	Negative investment return rate of a company. The company is losing money from external investment.
0	The company is neither earning nor losing money from external investment.

Table 2: Investment Rates

3.3 Logic of Modelling

The equation can be separated into two terms. Here, $\frac{N}{A+N}$ and $\frac{A}{A+N}$ are the shares of Nike and Adidas respectively in their combined revenue. Multiplying r_1/r_2 with the competitor's proportion means that bigger company has more impact on smaller company while smaller company has less impact on bigger company.

Additionally, b_1/b_2 is multiplied with the company's own proportion. This implies that smaller company has less gain/loss from external investment while bigger company has more gain/loss from external investment.

4 Analysis

4.1 Model Fitting

We used Python as the language of programming for our calculations. Particle Swarm Optimization (PSO) from pyswarm library was used to do numerical simulation and parameters fitting to minimize error between the actual and the calculated values [11, 12]. The squared error formula for the calculation is defined as:

$$\sum_{t \epsilon T} \left[(\hat{x}_1(t) - x_1(t))^2 + (\hat{x}_2(t) - x_2(t))^2 \right] \tag{6}$$

where

- $\hat{x}_i(t)$ is the value from actual data ($i \in \{1, 2\}$).
- $x_i(t)$ is calculated value ($i \in \{1, 2\}$).
- $t = \{2005, 2006, 2007, \ldots, 2018\}$.

4.2 Results

The optimized parameters and squared error are shown in the table below.

r_1	r_2	b_1	b_2	Least Mean Square Error
0.15778102	0.21897772	−0.06277946	−0.22972655	44.15198663762149

Table 3: Values of Parameters

It is evident that Adidas and Nike are in mutual relationship, as both r_1 and r_2 are positive. In addition, both b_1 and b_2 are both negative, implying that without mutual support from each other, Adidas and Nike will have decreasing revenue rate.

The curves of both revenue predictions can be shown at Figure 2. In 2019, the revenue of Adidas is **$28.33B** and Nike's revenue is **$41.16B**.

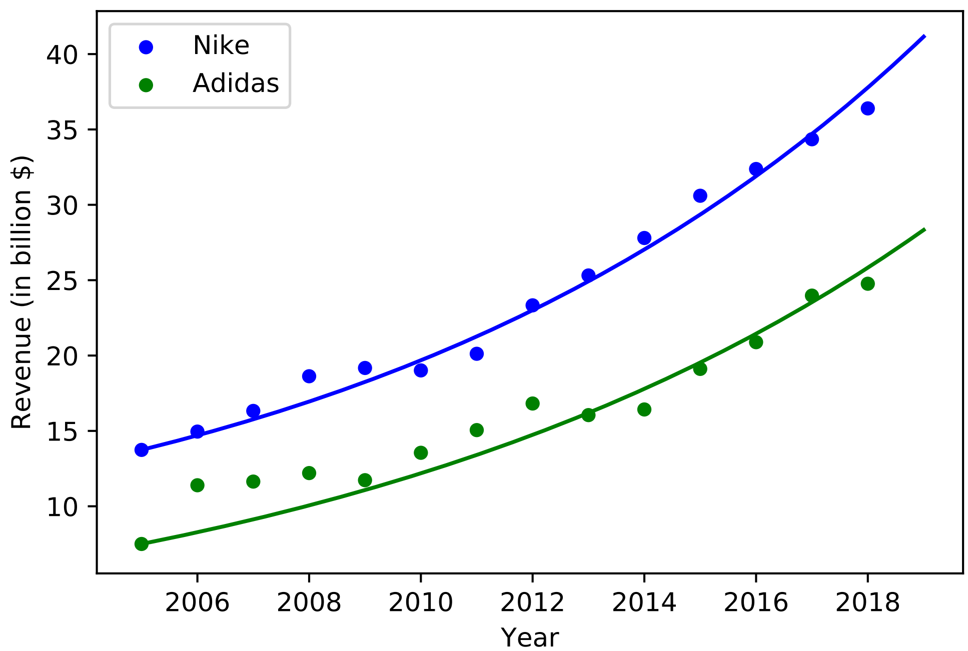

Figure 2: Model fitting through 2005-2019

4.3 Numerical Analysis

The system is re-written with obtained parameters. The equations are as follows:

$$\frac{dA}{dt} = 0.15778102 \cdot \left(\frac{N}{A+N}\right) \cdot A - 0.06277946 \cdot \left(\frac{A}{A+N}\right) \tag{7}$$

$$\frac{dN}{dt} = 0.21897772 \cdot \left(\frac{A}{A+N}\right) \cdot N - 0.22972655 \cdot \left(\frac{N}{A+N}\right) \tag{8}$$

Here, equating $\frac{dA}{dt}$ and $\frac{dN}{dt}$ to 0 yields 1 critical point: $(1.04909, 0.39789)$.

4.4 Critical Point and Stability

For critical point $(1.04909, 0.39789)$, the properties are as follows:

- $J(1.04909, 0.39789) = \begin{bmatrix} 5.710191161925615 \cdot 10^{-9} & 0.11439444117163498 \\ 0.060214259216616675 & 3.9396899750854253 \cdot 10^{-7} \end{bmatrix}$
- Eigenvalues = -0.08299495 & 0.08299527

Since determinant is less than 0, the critical point is a saddle point. The directions field around the critical point is shown in Figure 3 below.

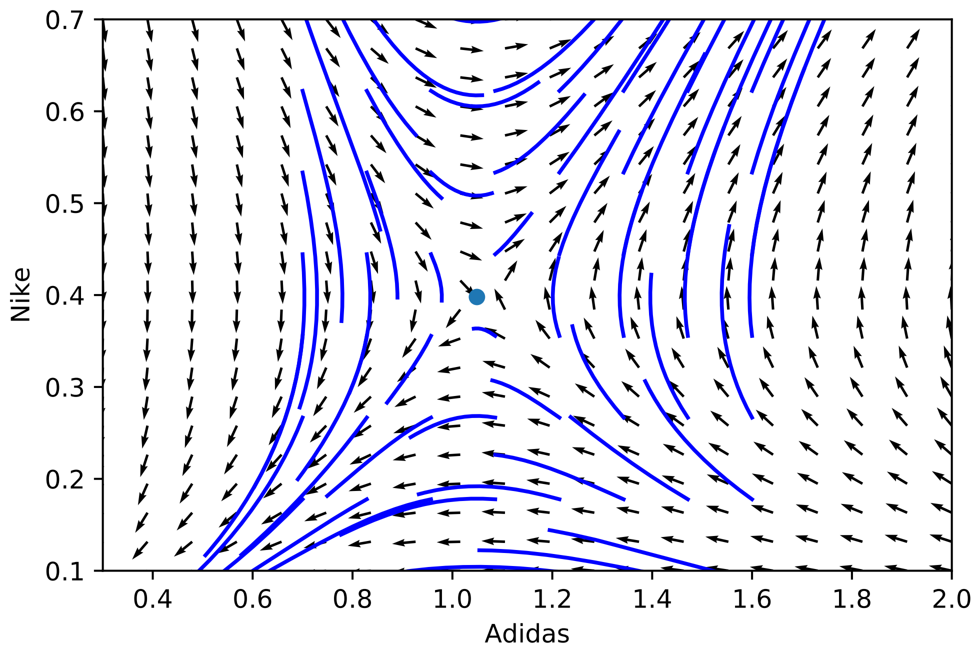

Figure 3: Direction Field and Solution Curve

The model does not have a stable point, which means both the revenues do not converge as $t \to \infty$ (as $t \to \infty$ both revenues will increase indefinitely, however this will not be proven in this report). This result supports the idea of unpredictable behaviour of revenue in companies. Having such property, our model is more suitable for short term predictions rather than long term predictions.

5 Extra Analysis

5.1 Lyapunov Characteristic Exponent (LCE)

In this section, we will determine the Lyapunov spectrum and maximum Lyapunov exponent in our model. As we have 1 critical point (saddle point), we will observe that the saddle point is not asymptotically stable. From the calculations for LCE spectrum, we obtained:

$$\lambda_1 = 0.083$$

$$\lambda_2 = -0.083$$

Since $\lambda_1 > 0$ and $\lambda_2 < 0$, there are one unstable manifold and one unstable manifold for the critical point (1.04909,0.39789) [6]. Thus, the critical point behaves as a saddle point which supports the result of that from Jacobian matrix. In 2-Dimensional system of ordinary differential equation, we do not have many classification for LCE spectrum. However, if we have 3-Dimensional system of ordinary differential equation, by the same technique of calculation in Appendix, we can obtain 3 different values for LCE spectrum and these results can be used to show the existence of various types of attractor. Types of attractor that exist in 1-Dimensional space to 3-Dimensional space can be summarized in the following table [6].

Dimension	**LCE Spectrum Sign**	**Attractor**
1	$(-)$	Fixed Point
2	$(-,-)$	Fixed Point
2	$(0,-)$	Limit Cycle
3	$(-,-,-)$	Fixed Point
3	$(0,-,-)$	Limit Cycle
3	$(0,0,-)$	Torus
3	$(+,0,-)$	Chaotic

Table 4: LCE Spectrum and Classification

Furthermore, the numerical approximation to calculate maximum Lyapunov exponent is $0.083474(> 0)$. In 2-Dimensional space, chaotic behaviour cannot be observed. Thus, positive maximum Lyapunov exponent does not mean a system is chaotic. On the other hand, in the case when the system of ordinary differential equation is in 3-Dimensional space, the maximum Lyapunov exponent can be approximated by using the same concept of numerical approximation. In 3-Dimensional space, a positive maximum Lyapunov exponent indicates that the attractor is chaotic. A chaotic attractor means that a small change in the system will lead to unpredictability of its trajectory. A chaotic behaviour of a trajectory can be repeatedly convergent and divergent [6].

5.2 Kaplan-Yorke Dimension

From our discoveries in 5.1, it can be concluded that the LCE are as follows:

$$\lambda_1 = 0.083$$

$$\lambda_2 = -0.083$$

Furthermore, by calculating Kaplan-Yorke Dimension, we obtained:

$$D = 1 + \frac{0.083}{|-0.083|} = 2$$

From Kaplan-Yorke Conjecture, this means that the attractor exists in 2-Dimensional space. However, we cannot classify the type of attractor, because 2-Dimensional space classification for attractor is only limited to limit cycle as shown in Table 4.

6 Conclusion

In this paper, we have modelled and analyzed the revenues for Adidas and Nike. Moreover, we have also forecast the revenues for Adidas ($28.33B) and Nike ($41.16B) in 2019. The values of parameters will hold for 1 year of forecasting only as refitting the graph using PSO again with new data set will be required. Our system of ordinary differential equations also yields one critical point, which is a saddle point. Furthermore, we have done some more advanced analysis and concluded that there exists a limit cycle attractor in the system . Further research can be done to analyze the relationship between companies in sportswear industry using similar system of ordinary differential equations for more than 2 companies in order to achieve better approximation of revenues in the real world.

References

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Calculation for Lyapunov Spectrum

We want to calculate the Lyapunov spectrum, by using the definition. Let J be the Jacobian matrix at the critical point and Y be a 2×2 matrix with function as its entries, then $\frac{dY}{dt} = JY$.

We solve the first column of ODE system with initial values $\begin{bmatrix} y_{11}(0) \\ y_{21}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then we proceed to the second column with initial value $\begin{bmatrix} y_{12}(0) \\ y_{22}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
So the first column for $Y' = JY$:

$$\frac{d}{dt}\begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} 5.7 \times 10^{-9} y_{11} + 1.14 \times 10^{-1} y_{21} \\ 6.02 \times 10^{-2} y_{11} + 3.156 \times 10^{-7} y_{21} \end{bmatrix}$$

Then, solving the corresponding initial value problem of differential equation, we get:

$$\begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{-0.083t} + \frac{1}{2}e^{0.083t} \\ \frac{1}{2}e^{-0.083t} - \frac{1}{2}e^{0.083t} \end{bmatrix}$$

Then the second column for $Y' = JY$ is:

$$\frac{d}{dt}\begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} 5.7 \times 10^{-9} y_{12} + 1.14 \times 10^{-1} y_{22} \\ 6.02 \times 10^{-2} y_{12} + 3.156 \times 10^{-7} y_{22} \end{bmatrix}$$

Then, solving the corresponding initial value problem of differential equation, we get:

$$\begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{-0.083t} - \frac{1}{2}e^{0.083t} \\ \frac{1}{2}e^{-0.083t} + \frac{1}{2}e^{0.083t} \end{bmatrix}$$

Therefore,

$$Y = \begin{bmatrix} \frac{1}{2}e^{-0.083t} + \frac{1}{2}e^{0.083t} & \frac{1}{2}e^{-0.083t} - \frac{1}{2}e^{0.083t} \\ \frac{1}{2}e^{-0.083t} - \frac{1}{2}e^{0.083t} & \frac{1}{2}e^{-0.083t} + \frac{1}{2}e^{0.083t} \end{bmatrix}$$

$$Y(t)Y^T(t) = \begin{bmatrix} \frac{1}{2}e^{-0.166t} + \frac{1}{2}e^{0.166t} & \frac{1}{2}e^{-0.166t} - \frac{1}{2}e^{0.166t} \\ \frac{1}{2}e^{-0.166t} - \frac{1}{2}e^{0.166t} & \frac{1}{2}e^{-0.166t} + \frac{1}{2}e^{0.166t} \end{bmatrix}$$

$$Y(t)Y^T(t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-0.166t} & 0 \\ 0 & e^{0.166t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\ln(Y(t)Y^T(t)) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \ln e^{-0.166t} & 0 \\ 0 & \ln e^{0.166t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\ln(Y(t)Y^T(t)) = \begin{bmatrix} 0 & -0.166t \\ -0.166t & 0 \end{bmatrix}$$

$$\lim_{t\to\infty} \frac{1}{2t} \ln(Y(t)Y^T(t)) = \begin{bmatrix} 0 & -0.083 \\ -0.083 & 0 \end{bmatrix}$$

The Lyapunov spectrum are defined by the eigenvalues of this matrix which are 0.083 and -0.083.

Codes

```

# coding: utf-8

# In[ ]:

import pandas as pd
import matplotlib.pyplot as plt
import numpy as np
from scipy.integrate import odeint
from pyswarm import pso

df = pd.read_excel ('/Users/Downloads/revenue1.xlsx',header = 1)
df1 = pd.DataFrame(df,columns=['Year','Adidas','Nike'])


# In[ ]:

print(df1)
df2=pd.DataFrame(df,columns=['Adidas','Nike'])


# In[ ]:

arrayd=df2.as_matrix()
arrayd=np.transpose(arrayd)
print(arrayd)


# # CUSTOM

# In[ ]:

def A(x1,x2,b1,r1):
    return (r1*x1*x2+b1*x1)/(x1+x2)
def N(x1,x2,b2,r2):
    return (r2*x2*x1+b2*x2)/(x1+x2)

def test(B):
    def model(v,t):
        r1,r2,b1,b2 = B
        x1 = v[0] #adidas
        x2 = v[1] #nike

        dAdt = A(x1,x2,b1,r1)
        dNdt = N(x1,x2,b2,r2)

        return [dAdt,dNdt]
    return model

def solver(B,v0,t):
    f = test(B)
```

```
    n = odeint(f,v0,t)
    return n[:, 0], n[:, 1]

v0 = [7.499,13.740]
t = np.arange(2005,2018)

def f_(X):
    return np.sum((solver(X,v0,t)-arrayd)**2)

lb = [-1,-1,-1,-1] # lower bounds for params
ub = [1,1,1,1] # upper bounds for params

xopt, fopt = pso(f_, lb, ub, maxiter = 500)

print("Minimum parameters = ", xopt)
print("error = ", fopt)
#predator/total  <- our idea for this model


# In[ ]:

xopt = [r1,r2,b1,b2]
print(r1, r2, b1, b2)


# In[ ]:

df4=pd.DataFrame(df,columns=['Year','Adidas','Nike'])
t1 = np.linspace(2005,2019,10000)
Adidas, Nike = solver(xopt1,v0,t1)
ax = df4.plot.scatter(x='Year', y='Nike',color =
                      "blue",label = "Nike")
ax = df4.plot.scatter(x='Year', y='Adidas', color='Green',
                      label='Adidas', ax=ax)
ax.set_ylabel('Revenue (in billion $)')
plt.plot(t1,Nike,color = "blue")
plt.plot(t1,Adidas,color = "green")
plt.show()


# In[ ]:

fig1, ax1 = plt.subplots()


X1,X2 = np.meshgrid(np.linspace(-10,10,20),
                    (np.linspace(-10,10,20)))

U = A(X1,X2,b1,r1) / np.sqrt(A(X1,X2,b1,r1)**2
                             + N(X1,X2,b2,r2)**2);
V = N(X1,X2,b2,r2) / np.sqrt(A(X1,X2,b1,r1)**2
                             + N(X1,X2,b2,r2)**2)
ax1.quiver(X1, X2,U,V)
plt.scatter(1.04909,0.39789,s=10) #critical point (Saddle)
```

```
ax1.set_xlabel('Adidas')
ax1.set_ylabel('Nike')
ax1.set_aspect(aspect = "equal")
plt.show()


# In[ ]:

x= 1.04909
y= 0.39789
k = (0.15778102*y-0.06277946)/(x+y) -
(0.15778102*y*x-0.06277946*x)/(x+y)**2


# In[ ]:

l = (0.15778102*x)/(x+y) -
(0.15778102*y*x-0.06277946*x)/(x+y)**2


# In[ ]:

m = (0.218978*y)/(x+y) -
(0.218978*y*x-0.229727*y)/(x+y)**2


# In[ ]:

n = (0.218978*x-0.229727)/(x+y) -
(0.218978*y*x-0.229727*y)/(x+y)**2


# In[ ]:

mat = np.matrix([[k,l],[m,n]])


# eigenvalues : [-0.08299495,  0.08299527]
# eigenvector : [[-0.80941205, -0.80941101],
#               [ 0.58724112, -0.58724255]]

# In[ ]:

fig1, ax1 = plt.subplots()

X1,X2 = np.meshgrid(np.linspace(-10,30,21),(np.linspace(0,40,21)))

U = A(X1,X2,b1,r1);
#/ np.sqrt(A(X1,X2,b1,r1)**2 + N(X1,X2,b2,r2)**2);
V = N(X1,X2,b2,r2);
#/ np.sqrt(A(X1,X2,b1,r1)**2 + N(X1,X2,b2,r2)**2)
R = np.sqrt(A(X1,X2,b1,r1)**2 + N(X1,X2,b2,r2)**2);
U = U/R;
V = V/R;
```

```
ax1.quiver(X1, X2,U,V)
plt.scatter(1.04909,0.39789,s=10) #critical point (Saddle)
ax1.set_xlabel('Adidas')
ax1.set_ylabel('Nike')
plt.plot(Adidas, Nike)
ax1.set_aspect(aspect = "equal")
plt.show()


# # Lyapunov Exponent

# In[ ]:

#Maximum Lyapunov Exponent
 init = np.array(t0)
 deltat = 0.1
 t = np.arange(0, 2000 + deltat, deltat)
 n = len(t)
 ref = np.array(odeint(model, init, t, args=(r1, r2, b1, b2)))
 d0 = 10**(-4)
 p = init + (d0 / np.sqrt(2)) * np.array([1, 1])
 lamda = []
 result = []
 i = 0

 while i<1000:
   if len(result)>2 and abs(result[-1]-result[-2])<10**-8:
     break
   test = t[i : i + 2]
   soln = np.array(odeint(model, p, test, args=(r1, r2, b1, b2)))
   d1 = np.sqrt(np.sum((soln[1, : ] - ref[i + 1, : ]) ** 2))
   p = (d0 / d1) * (soln[1, : ] - ref[i + 1, : ]) +ref[i + 1, :]
   lamda.append(np.log(abs(d1/d0)) / deltat)
   result.append(sum(lamda)/len(lamda))
   i = i+1
 return result

crit = [1.04909,0.39789] #this is the critical point of our system
result = MLE(xopt1, crit)

print(result[-1])
plt.plot(result)
plt.title("Plot of Maximal Lyapunov Exponent")
plt.xlabel("no of iteration")
plt.ylabel("Maximum Lyapunov Exponent")
ax = plt.gca()
plt.show()
```