Applying the Well-Ordering Principle: Integer Intervals for Real Numbers

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Introduction

This proof addresses **Exercise 1.122** from the textbook *Don't Panic: A Guide to MATA21 Analysis in One Variable* [Ols23, pg. 41].

Problem Statement

Assumptions: Let $x \in [0, \infty)$ and $n \in \mathbb{Z}$.
Statement: For every x, there exists an n such that $x \in [n, n+1)$.

The Well-Ordering Principle

Lemma 1 *Every non-empty subset of the natural numbers* $\mathbb{N}$ *has a least element [Ols23, pg. 41].*

Archimedean Property

The Archimedean Property states that for any real number $x \in \mathbb{R}$, there exists a natural number $n \in \mathbb{N}$ such that $n > x$ [Ols23, pg. 38].

Proof

Since $x \geq 0$, any integer k that satisfies $k > x$ must be positive or zero. Therefore, we restrict k to be in $\mathbb{N}$, the set of natural numbers. Now, consider the set

$$S = \{k \in \mathbb{N} \mid k > x\}.$$

By the Archimedean Property (see previous section), for any real number $y \geq 0$, there exists a natural number $m \in \mathbb{N}$ such that $m > y$. Therefore, the set S is non-empty. Since S is a non-empty subset of $\mathbb{N}$, the Well-Ordering Principle guarantees the existence of a least element. Denote this smallest element by k_0. By the definition of S, we know that

$$k_0 > x.$$

Since k_0 is the smallest natural number such that $k_0 > x$, it follows that $k_0 - 1$ cannot be greater than x. If $k_0 - 1 > x$, then $k_0 - 1$ would also belong to the set S, contradicting the fact that k_0 is the smallest element of S. Thus, we conclude that

$$k_0 - 1 \leq x.$$

Now let

$$n = k_0 - 1.$$

From the previous steps, we know that $k_0 > x$ and $k_0 - 1 \leq x$, therefore

$$n \leq x < n + 1.$$

Thus, for every $x \geq 0$, there exists an integer $n \in \mathbb{N}$ such that $n \leq x < n + 1$. Since $\mathbb{N} \subseteq \mathbb{Z}$, we can conclude that for every $x \geq 0$, there exists an integer $n \in \mathbb{Z}$ such that $x \in [n, n + 1)$, as required.

References

[Ols23] Jan-Fredrik Olsen. *Don't Panic: A Guide to MATA21 Analysis in One Variable*. Version: January 17, 2023. Exercise 1.122. 2023.