Exercises in Fourier Analysis

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Exercises for Chapter 1 Series

Introductory Definitions and Examples

101. Determine for each of the series

$$\text{(a)} \sum_{k=0}^{\infty}(1+x^2)^{-k}\,, \quad \text{(b)} \sum_{k=0}^{\infty} e^{kx}\,, \quad \text{(c)} \sum_{k=1}^{\infty} xe^{-kx}\,, \quad \text{(d)} \sum_{k=0}^{\infty} \frac{3^k+4^k}{x^k}$$

the real numbers x for which it is convergent. Compute the sum for those x.

102. Let $\sum_{k=0}^{\infty} a_k$ be a convergent series with sum s. Show that $\sum_{k=0}^{\infty}(a_k + a_{k+1})$ is convergent and determine its sum.

Positive Series

103. Which of the following series are convergent?

$$\text{(a)} \sum_{k=1}^{\infty} \sin(k^{-2})\,, \quad \text{(b)} \sum_{k=1}^{\infty} \cos(k^{-2})\,, \quad \text{(c)} \sum_{k=1}^{\infty} \tan(1/k)\,,$$

$$\text{(d)} \sum_{k=1}^{\infty} \frac{k}{1+k^3}\,, \quad \text{(e)} \sum_{k=1}^{\infty} \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}}\,.$$

104. Check the convergence of the series

$$\text{(a)} \sum_{k=1}^{\infty} \frac{k^3}{2^k}\,, \quad \text{(b)} \sum_{k=1}^{\infty} \frac{k!}{k^k}\,, \quad \text{(c)} \sum_{k=1}^{\infty} \frac{(2k)!}{k^k}\,, \quad \text{(d)} \sum_{k=0}^{\infty} 2^{-\sqrt{k}}\,.$$

105. (a) For which positive real numbers a is $\sum_{k=1}^{\infty} \frac{2^{\sin k}}{k^a}$ convergent?

Answer the same question for the series

$${}^*\text{(b)} \sum_{k=1}^{\infty} k^a a^k\,, \quad \text{(c)} \sum_{k=1}^{\infty} \frac{1}{k^a + a^{-k}}\,, \quad \text{(d)} \sum_{k=1}^{\infty} \frac{a^{k^2}}{k!}\,,$$

$$\text{(e)} \sum_{k=1}^{\infty} \left(a - \frac{1}{k}\right)^k\,, \quad \text{(f)} \sum_{k=1}^{\infty} k^{a^2} a^{k^2}\,.$$

106. Compute $\sum_{j=2}^{\infty} \left(\sum_{k=2}^{\infty} k^{-j} \right)$.

107. Set $a_{jk} = x_k^j - x_{k+1}^j - x_k^{j+1} + x_{k+1}^{j+1}$ where $x_k = \frac{k}{k+1}$. Show that

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} a_{jk}\right) = \frac{1}{2} \quad \text{and} \quad \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{jk}\right) = -\frac{1}{2}.$$

Hint: First show that

$$\sum_{k=1}^{\infty} a_{jk} = x_1^j - x_1^{j+1} \quad \text{and} \quad \sum_{j=1}^{\infty} a_{jk} = x_k - x_{k+1}.$$

108. In the harmonic series $\sum\limits_{1}^{\infty} \frac{1}{k}$, all terms for which the integer k contains the *digit* 9 are deleted. Show that the resulting series is convergent.

Hint: Show that the number of terms $\frac{1}{k}$ for which k contains no nines and $10^{p-1} \le k < 10^p$ is less than 9^p.

Absolutely Convergent Series

109. Which of the following series are absolutely convergent?

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k^2}}{k^2}$, (b) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos k}{1+k^2}$,

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1+(-1)^k k}{k^2}$, (d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n^2\pi/3}{n^2 - \arctan n}$.

110. Show the following inequality for absolutely convergent series.

$$\left|\sum_{k=1}^{\infty} u_k\right| \le \sum_{k=1}^{\infty} |u_k|.$$

111. Set $t_k = \sum\limits_{j=2}^{\infty} \frac{1}{j^k}$ and $u_k = \sum\limits_{j=2}^{\infty} \frac{(-1)^j}{j^k}$. Compute

(a) $\displaystyle\sum_{k=1}^{\infty} t_{2k}$, (b) $\displaystyle\sum_{k=1}^{\infty} u_{2k}$.

Conditionally Convergent Series

112. For which values of α is the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{\alpha}}$ convergent?

113. Test the series $\sum_{k=1}^{\infty} \sin \pi(k + \frac{1}{k})$ for absolute convergence and convergence.

114. Find a number N such that $|s - s_n| \leq 10^{-3}$ when $n \geq N$ for each of the series

$$\text{(a)} \ \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\,, \qquad \text{(b)} \ \sum_{k=1}^{\infty} \frac{1}{k^2}\,.$$

Series with Complex Terms

115. Which of the following series are absolutely convergent?

$$\text{(a)} \ \sum_{k=1}^{\infty} \frac{e^{ik}}{k^2}\,, \qquad \text{(b)} \ \sum_{k=1}^{\infty} \frac{1}{k+i}\,.$$

116. For which complex numbers z is the series $\sum_{k=0}^{\infty} \frac{2k+i}{k+2i} z^k$ convergent?

Uniform Convergence

117. Compute $||f_n||_E$ if $f_n(x) = \frac{nx}{1+n^2x^2}$ and (a) $E = [0, 1]$, (b) $E = [1, \infty)$. In which of these cases is f_n uniformly convergent to 0?

118. Determine the limit function $f(x)$, when $x \geq 0$, of the function sequence

$$f_n(x) = \frac{x - nx^3}{1 + nx^2}$$

and compute $||f_n - f||_E$ to find out if f_n is uniformly convergent to f in the interval $E = [0, \infty)$.

119. Check if the function sequence $f_n(x) = \arctan nx$, $n = 1, 2, 3, \ldots$, is uniformly convergent in each of the intervals (a) $[-1, 1]$, (b) $[1, 2]$ and (c) $[2, \infty)$.

*120. Set

$$f_n(x) = (\sin x)^{1/n}\,, \quad 0 \leq x \leq \pi\,, \quad n = 1, 2, \ldots\,.$$

(a) Compute $\lim_{n\to\infty} f_n(x)$.

(b) Check if the convergence is uniform in $[0, \pi]$.

121. Set, for $x > 0$, $f_n(x) = \sqrt{nx}\arctan\frac{1}{nx}$, $n = 1, 2, 3, \ldots$.

(a) Compute $\lim_{n\to\infty} f_n(x)$.

(b) Is the convergence uniform in the interval $x > 0$?

122. Determine whether the function sequence

$$f_n(x) = \left(\frac{x}{1+x}\right)^n \frac{1}{1+x}, \quad n \geq 1,$$

is uniformly convergent or not in the interval $x \geq 0$.

*123. Let $s(x) = \sum_{k=0}^{\infty} \frac{x}{(1+x)^{k+1}}$, $x \geq 0$.

(a) Show that the series is pointwise convergent for all $x \geq 0$.

(b) Show that the series is uniformly convergent in each interval $x \geq d$, $d > 0$.

(c) Show that the convergence is not uniform in the interval $x \geq 0$.

124. Consider the series $\sum_{k=0}^{\infty} x\,e^{-kx}$.

(a) Determine the set E of real numbers x for which the series is convergent.

(b) Compute the sum $s(x)$ of the series for $x \in E$.

(c) Check if the series is uniformly convergent in E.

125. Show that

(a) $\sum_{k=0}^{\infty}(1-x)^2 x^k$ is uniformly convergent in the interval $0 \leq x \leq 1$,

(b) $\sum_{k=0}^{\infty}(1-x)x^k$ is not uniformly convergent in the interval $0 \leq x \leq 1$.

*126. Show that the function series $\sum_{k=1}^{\infty} \frac{x^{k^2}}{k^2}$ is uniformly convergent in the interval $|x| \leq 1$.

127. Show that the following function series are uniformly convergent in the intervals stated.

(a) $\sum_{0}^{\infty} \dfrac{x^k}{1+x^k}, \quad -\dfrac{1}{2} \le x \le \dfrac{1}{2},$ (b) $\sum_{0}^{\infty} \dfrac{\sin 3^k x}{2^k}, \quad x \in \mathbb{R},$

(c) $\sum_{1}^{\infty} \dfrac{1}{k^x}, \quad x \ge d > 1,$ (d) $\sum_{1}^{\infty} \dfrac{k^x}{x^k}, \quad 2 \le x \le 3.$

128. (a) Let $c \ge 1$ be a constant. Compute

$$\sup_{0 \le x \le 1} (1-x)x^c .$$

(b) Show that the function series

$$\sum_{k=1}^{\infty} (1-x)x^{k^2}$$

is uniformly convergent in the interval $0 \le x \le 1$.

129. (a) Show that the series $s(x) = \sum_{k=1}^{\infty} \frac{\arctan kx}{1+k^2x^2}$ is uniformly convergent in the interval $x \ge c$ where $c > 0$.

(b) Show that the sum $s(x)$ is continuous for $x > 0$.

*130. Show that the sum of the series

$$\sum_{k=0}^{\infty} \frac{1+\sqrt{k}x^2}{1+k^2\sqrt{x}}$$

is continuous when $x > 0$.

131. Is the sum

$$s(x) = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+x} \right)$$

continuous when $x \ge 0$?

132. Set $s(x) = \sum_{k=0}^{\infty} \frac{\cos kx}{2^k}$ and show that $s(x)$ is differentiable for all real x.

133. Show that the series $\sum_{k=0}^{\infty} 2^k \sin 3^{-k} x$ is absolutely convergent in $\mathbb{R}$ and that its sum $s(x)$ is a differentiable function. Also evaluate $s'(0)$.

134. Show that

$$\frac{d}{dx}\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k^2x}\right) = -\frac{1}{x}\sum_{k=1}^{\infty}\frac{1}{1+k^2x}, \qquad x>0\,.$$

135. Compute the sum of the series

$$\sum_{p=0}^{\infty}\frac{(-1)^p}{2p+1}\,.$$

*136. Show that a power series ansatz $f(x)=\sum_0^\infty a_kx^k$ for the problem

$$f(x) = (1-x)f(x^2) \quad \text{and} \quad f(0) = 1$$

leads to $a_0 = 1$, $a_{2k} = a_k$ and $a_{2k+1} = -a_k$ when $k \geq 0$. This determines the coefficients a_k of the power series uniquely. What is its radius of convergence, i.e. the largest value of R for which the series converges for all x with $|x| < R$?

137. Show that the function $s(x) = \sum_{k=0}^{\infty}\frac{e^{-kx}}{1+k^2}$, $x > 0$, is a solution of the differential equation

$$y''(x) + y(x) = (1-e^{-x})^{-1}, \qquad x>0\,.$$

138. Let $f(x) = \sum_{k=1}^{\infty}\frac{\arctan(kx)}{k^2}$.

(a) Show that f is continuous in $\mathbb{R}$.

(b) Show that f is differentiable when $x \neq 0$.

(c) Show that f is not differentiable at 0.

139. (a) Show that

$$s(x) = 1-\frac{x^3}{3!}+\frac{x^6}{6!}-\cdots = \sum_{k=0}^{\infty}\frac{(-1)^kx^{3k}}{(3k)!}, \qquad x\in\mathbb{R}\,,$$

is a solution of the differential equations

$$u''(x)-u'(x)+u(x) = e^{-x} \quad \text{and} \quad u'''(x)+u(x) = 0\,.$$

(b) Compute the sum $s(x)$ of the series for all x.

140. Show that the series

$$\sum_{k=1}^{\infty} \frac{x^k}{1-x^k}$$

defines an infinitely differentiable function f when $|x| < 1$ and that $\frac{1}{n!} f^{(n)}(0)$ equals the number of positive integers dividing n.

141. (a) Show that the function

$$s(x) = \sum_{k=1}^{\infty} \frac{x}{2+k^3 x}$$

is continuous for $x \geq 0$.

(b) Is the function $s(x)$ continuously differentiable for $x > 0$?

(4 Jan. 85)

142. Set $s(x) = \sum_{k=1}^{\infty} \frac{x^k - x^{3k}}{k}$.

(a) Evaluate $s(x)$ when $|x| < 1$, $x = 1$ and $x = -1$.

(b) Is the convergence uniform in the interval $0 \leq x \leq 1$?

(c) Can the result of (a) be used to infer that the convergence is uniform in the interval $-1 \leq x \leq 0$?

(4 April 85)

143. For which real values of a is the following series convergent?

$$\sum_{k=0}^{\infty} \frac{1}{3^k + a^2}\, a^k .$$

(2 Nov. 85)

144. Set $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{3^k + x^2}$, $|x| < 3$. Show that f is a continuous function.

(4 Jan. 86)

145. (a) For which positive values of x is the following series convergent?

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{x^k + 4^k} .$$

(b) Is the sum $s(x)$ of the series continuous for these values of x?

(3 April 86)

146. Show, for example by considering the cases $|x| < 3$ and $|x| > 2$ separately or by computing the suprema over $x \in \mathbb{R}$ of the terms, or otherwise, that the sum

$$s(x) = \sum_{k=1}^{\infty} \frac{x^k}{x^{2k} + 4^k}$$

is continuous for all real x. (4 May 86)

Exercises for Chapter 2 Fourier Series

*201. The function u has period 2π and satisfies

$$u(x) = \pi - x \quad \text{when} \quad 0 \le x < 2\pi\,.$$

(a) Sketch the graph of u in the interval $|x| \le 6\pi$.

(b) Compute the Fourier coefficients of u.

202. Compute the Fourier coefficients of u where $u(x)$ is

*(a) $\cos^2 x$, (b) $\sin x$, (c) $\cos^4 x$, (d) $\sin^2 x \cos^2 x$.

203. Let u be a function with the Fourier series $\sum_{-\infty}^{\infty} c_n e^{inx}$. Determine c_6 if $u(x) = \cos^{20} x$.

204. Compute the Fourier coefficients of the 2π-periodic function u given by

$$u(x) = |x| \quad \text{when} \quad -\pi \le x \le \pi\,.$$

205. The function u has period 2π and satisfies

$$u(x) = \begin{cases} 1 & \text{when } 0 \le x < \pi \\ 0 & \text{when } \pi \le x < 2\pi\,. \end{cases}$$

(a) Determine the Fourier series of u.

(b) Compute the sum of the Fourier series at $x = 0$ and $x = \pi$.

206. The function u has period 2π and is integrable over an interval of length 2π. Show the following statements for the Fourier coefficients of u.

(a) $c_{-n} = c_n$ if u is even, i.e. $u(-x) = u(x)$ for all x.

(b) $c_{-n} = -c_n$ if u is odd, i.e. $u(-x) = -u(x)$ for all x.

(c) c_n are real if $u(-x) = \overline{u(x)}$ for all x.

(d) c_n are real and $c_{-n} = c_n$ if u is real and even.

207. Consider the functions

$$R_0(x) = 1, \quad 0 \le x < 2\pi\,,$$

$$R_1(x) = \begin{cases} 1, & 0 \le x < \pi\,, \\ -1, & \pi \le x < 2\pi\,, \end{cases}$$

$$R_2(x) = \begin{cases} 1, & 0 \le x < \frac{\pi}{2}\,, \quad \pi \le x < \frac{3\pi}{2}\,, \\ -1, & \frac{\pi}{2} \le x < \pi\,, \quad \frac{3\pi}{2} \le x < 2\pi\,. \end{cases}$$

Show that R_0, R_1, R_2 constitute an orthonormal system with respect to the scalar product

$$(u|v) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(x)\overline{v(x)}\,dx\,.$$

208. Show that

$$||u+v||^2 \;+\; ||u-v||^2 \;=\; 2(||u||^2 + ||v||^2)\,,$$

where

$$||u||^2 \;=\; (u|u) \;=\; \frac{1}{2\pi}\int_0^{2\pi} |u(x)|^2\,dx\,.$$

*209. Determine all continuous differentiable functions with period 2π such that

$$2u'(x) \;=\; u(x+\frac{\pi}{2}) \;-\; u(x-\frac{\pi}{2}) \quad \text{for all } x\,.$$

210. Determine all twice continuously differentiable 2π-periodic functions such that

$$u''(x) \;=\; u(x+\pi)\,.$$

211. Which 2π-periodic C^1 function has the Fourier coefficients

$$c_n \;=\; \begin{cases} ne^{-n}\,, & n \ge 0 \\ 0\,, & n < 0\,? \end{cases}$$

212. Let u be a function that is piecewise C^1, 2π-periodic but not necessarily continuous. Denote its discontinuity points in the interval $[0, 2\pi]$ by $x_1, x_2, \ldots, x_k$ and the jumps at these points by $s(x_j)$, i.e.

$$s(x_j) \;=\; \lim_{\epsilon\to+0} u(x_j+\epsilon) \;-\; \lim_{\epsilon\to+0} u(x_j-\epsilon)\,.$$

Show that the Fourier coefficients of u' are

$$inc_n - \frac{1}{2\pi}\sum_{j=1}^{k} s(x_j)e^{-inx_j}$$

where c_n are the Fourier coefficients of u.

Hint: Partition $[0, 2\pi]$ into subintervals as in the proof of Theorem 2.10 and copy this proof. Observe that the boundary terms in the partial integrations no longer cancel each other but instead add to the sum in the expression above.

*213. The function u has period 2π and

$$u(x) = \frac{1}{2}(e^x + e^{-x}) \quad \text{when} \quad |x| \le \pi\,.$$

Expand u in Fourier series and compute

$$\sum_{k=1}^{\infty} \frac{1}{k^2+1}\,.$$

214. Let u be periodic with period 2π and assume that

$$u(x) = x^2 \quad \text{when} \quad |x| \le \pi\,.$$

(a) Show, by choosing x suitably in the Fourier series of u, that

$$\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots = \frac{\pi^2}{12}\,.$$

(b) What is the sum of the Fourier series at $x = 6$?

215. Expand the function

$$u(x) = \max(\cos x,\, 0)$$

in a Fourier series and compute

$$\sum_{1}^{\infty} \frac{(-1)^k}{4k^2-1} \quad \text{and} \quad \sum_{1}^{\infty} \frac{1}{4k^2-1}\,.$$

216. Show that

(a) $x \sin x \;=\; 1 - \frac{1}{2}\cos x \;-\; 2\sum\limits_{k=2}^{\infty}(-1)^k(k^2-1)^{-1}\cos kx\,,\;\; |x| \le \pi\,,$

(b) $|\sin x| \;=\; \frac{2}{\pi}\left(1 + \sum\limits_{k=1}^{\infty}\left(\frac{1}{2k+1} - \frac{1}{2k-1}\right)\cos 2kx\right)\,.$

217. The function u is periodic with period 2π and

$$u(x) \;=\; \cos\alpha x \quad \text{when} \quad |x| \le \pi$$

where α is a real number that is not an integer. Expand u in Fourier series and show that

$$\pi\cot\pi\alpha \;=\; \sum_{-\infty}^{\infty}\frac{\alpha}{\alpha^2 - n^2}\,.$$

218. Let u be the 2π-periodic function defined by

$$u(x) \;=\; \begin{cases} \frac{\sin x}{x} & \text{when } 0 < |x| \le \pi \\ 1 & \text{when } x = 0\,. \end{cases}$$

Show that the Fourier coefficients of u are given by

$$c_n \;=\; \frac{1}{2\pi}\int_{(n-1)\pi}^{(n+1)\pi}\frac{\sin x}{x}\,dx$$

and use this to compute the improper integral

$$\int_0^{\infty}\frac{\sin x}{x}\,dx\,.$$

219. Let u be a C^1 function with period 2π and Fourier coefficients c_n. Show that

(a) if u is in C^k, then $|n^k c_n| \le M_k$ where M_k is a constant independent of n,

(b) if $\sum\limits_{-\infty}^{\infty}|n^k c_n|$ is convergent, then u is in C^k,

(c) u is infinitely differentiable if and only if there exist constants M_k such that

$$|n^k c_n| \le M_k \quad \text{for all integers } n \text{ and } k \;=\; 0, 1, 2, \ldots\,.$$

(We see from this that the rate of convergence of the Fourier series increases with increasing regularity of the function and vice versa.)

220. Let g be a C^1 function such that the two series

$$\sum_{-\infty}^{\infty} g(x+2n\pi) \quad \text{and} \quad \sum_{-\infty}^{\infty} g'(x+2n\pi)$$

are uniformly convergent in the interval $0 \le x \le 2\pi$. Show *the Poisson summation formula:*

$$\sum_{n=-\infty}^{\infty} g(2n\pi) = \sum_{m=-\infty}^{\infty} \gamma_m$$

where

$$\gamma_m = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{-imx}\,dx$$

is assumed to be convergent.

Hint: The numbers γ_m are the Fourier coefficients of the 2π-periodic function $u(x) = \sum_{-\infty}^{\infty} g(x+2\pi n)$.

221. Let u be a piecewise continuous and 2π-periodic function with Fourier coefficients c_n. Set

$$U(x) = \int_0^x (u(t) - c_0)\,dt\,.$$

Show the following statements.

(a) U has the period 2π, is continuous and piecewise C^1.

(b) The Fourier series $\sum_{-\infty}^{\infty} d_n e^{inx}$ of U is absolutely convergent and has the sum $U(x)$ for each x.

(c) $d_n = -\frac{ic_n}{n}$ when $n \neq 0$.

(d) $d_0 = i\sum_{n\neq 0}\frac{c_n}{n}$.

Hint: Observe that $U(0) = 0 = \sum_{-\infty}^{\infty} d_n$.

(e)

$$\int_0^x u(t)\,dt = c_0 x + \sum_{n\neq 0} c_n \frac{e^{inx}-1}{in}\,.$$

(Hence, termwise integration of the Fourier series of u is permitted even when the series is not known to be convergent. The integrated series is absolutely convergent.)

222. When u is a function that has period 2π and is integrable in $[0, 2\pi]$, we define the symmetric partial sums s_N of the Fourier series by

$$s_N(x) = \sum_{-N}^{N} c_n e^{inx}$$

where c_n are the Fourier coefficients of u. Show that

(a)
$$s_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x-y) D_N(y)\, dy$$

where
$$D_N(y) = \sum_{-N}^{N} e^{iny} = \frac{\sin{(N+\frac{1}{2})y}}{\sin\frac{y}{2}},$$

(b)
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(y)\, dy = 1\,.$$

Hint to (b): Integrate the sum in the first expression for D_N termwise or apply the result of (a) to the function $u(x) \equiv 1$.

223. The arithmetic means σ_N of the symmetric partial sums in the previous exercise are defined by $\sigma_N = \frac{s_0+\cdots+s_N}{N+1}$.

Show the following statements.

(a)
$$\sigma_N(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x-y) K_N(y)\, dy$$

where
$$K_N(y) = \left(\frac{\sin{(N+1)\frac{y}{2}}}{\sin\frac{y}{2}}\right)^2 \cdot \frac{1}{N+1}\,.$$

(b)
$$\int_{-\pi}^{\pi} K_N(y)\, dy = 1\,.$$

Hint: Confer the second hint to 222 (b).

(c)
$$\int_{\delta<|y|<\pi} K_N(y)\, dy \to 0 \quad \text{as} \quad N \to \infty$$

for every fixed $\delta > 0$.

(d) $\sigma_N(x) \to u(x)$ as $N \to \infty$ for every fixed x if u is continuous.
Hint: Observe that (b) implies that

$$u(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) K_N(y)\, dy$$

and so

$$\sigma_N(x) - u(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (u(x-y)) - u(x)) K_N(y)\, dy\,.$$

Split the integral in the right-hand side into one integral over $|y| \le \delta$ and one over $\delta < |y| \le \pi$. Choose δ so that the first integral becomes small and then apply (c) to the second integral.

224. In a linear space endowed with a real scalar product we have that

$$||u-v|| = ||u+v|| \quad \text{if and only if} \quad (u|v) = 0.$$

What is the corresponding statement when the scalar product is complex?

225. Show that

(a) when the scalar product is real,

$$4(u|v) = ||u+v||^2 - ||u-v||^2\,,$$

(b) when the scalar product is complex,

$$4(u|v) = ||u+v||^2 - ||u-v||^2 + i\,||u+iv||^2 - i\,||u-iv||^2\,.$$

226. (a) Let e_n, $n = 0, \pm1, \pm2, \ldots$ be an orthonormal system in a linear space with a scalar product and set

$$f_n = \begin{cases} \frac{1}{\sqrt{2}}(e_n - e_{-n})\,, & n > 0 \\ e_0\,, & n = 0 \\ \frac{1}{\sqrt{2}}(e_n + e_{-n})\,, & n < 0\,. \end{cases}$$

Show that the vectors f_n also form an orthonormal system.

(b) Set

$$g_n(x) = \begin{cases} \sqrt{2}\sin nx\,, & n > 0 \\ 1\,, & n = 0 \\ \sqrt{2}\cos nx\,, & n < 0\,. \end{cases}$$

Show that the functions g_n form an orthonormal system with respect to the scalar product

$$(u|v) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\overline{v(x)}\,dx\,.$$

*227. Compute $\sum\limits_1^\infty \frac{1}{n^4}$ by means of Parseval's formula applied to

$$u(x) = x^2\,, \quad |x| \le \pi\,.$$

228. Compute $\int_0^{2\pi} \cos^4 x\,dx$ by means of Parseval's formula applied to

$$u(x) = \cos^2 x\,.$$

229. Compute $\sum\limits_1^\infty \frac{1}{k^2+1}$ by means of Parseval's formula applied to

$$u(x) = e^x\,, \quad 0 \le x < 2\pi\,.$$

230. Compute the Fourier coefficients of the 2π-periodic function u given by

$$u(x) = x\cos x \quad \text{when} \quad -\pi < x \le \pi\,.$$

Also determine

$$\sum_2^\infty \frac{n^2}{(n^2-1)^2}\,.$$

231. Show that

$$e^{i|x|} = \frac{2i}{\pi} + \cos x + \frac{4}{\pi i}\sum_1^\infty \frac{\cos 2kx}{4k^2-1}$$

for all real x with $|x| \le \pi$. Also compute

$$\sum_1^\infty \frac{1}{(4k^2-1)^2}\,.$$

232. (a) Compute the Fourier coefficients of the 2π-periodic function u given by

$$u(x) = x(x^2-\pi^2) \quad \text{when} \quad |x| \le \pi\,.$$

(b) Determine

$$\sum_{1}^{\infty} \frac{1}{n^6}.$$

233. The function u has period 2π and satisfies

$$u(x) = e^{iax} \quad \text{when} \quad 0 \le x < 2\pi$$

where a is real but not an integer.

(a) Determine the Fourier coefficients of u.

(b) Show that

$$\sum_{-\infty}^{\infty} \frac{(-1)^n}{a-n} = \frac{\pi}{\sin \pi a}.$$

(c) Show that

$$\sum_{-\infty}^{\infty} \frac{1}{(a-n)^2} = \frac{\pi^2}{\sin^2 \pi a}.$$

234. The Bessel functions $J_n(x)$ are the coefficients of the Fourier series expansion

$$e^{ix\sin t} = \sum_{-\infty}^{\infty} J_n(x) e^{int}.$$

Show that the sum $\sum_{-\infty}^{\infty} |J_n(x)|^2$ is independent of x and determine its value.

235. Expand the function

$$u(x) = \frac{1}{r - e^{ix}},$$

where $r > 1$, in a Fourier series and use this to compute the integral

$$\int_0^{2\pi} \frac{dx}{1 - 2r\cos x + r^2}.$$

236. Let u be a C^1 function in $[0, \pi]$ with $u(0) = u(\pi) = 0$. Show that

$$\int_0^{\pi} |u(x)|^2\, dx \le \int_0^{\pi} |u'(x)|^2\, dx$$

with equality if and only if u is a multiple of $\sin x$.

Hint: Extend u to an odd function.

237. Determine the Fourier coefficients of the function

$$u(x) = \cos^n x$$

where n is a positive integer. Use the result to show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Hint: Use Euler's formulae.

*238. Determine the numbers a, b and c such that the integral

$$\int_{-\pi}^{\pi} |x^2 - a - be^{ix} - ce^{-ix}|^2 \, dx$$

is as small as possible. Also state the minimum value.

239. Determine the trigonometric polynomial $p(x) = \sum_{|n|\le 4} a_n e^{inx}$ of degree at most 4 that minimises the integral

$$\int_{-\pi}^{\pi} |u(x) - p(x)|^2 \, dx$$

where $u(x) = (\sin x)^8$ and state the minimum value of the integral.

240. Let $u(x) = |x|$. Determine the second degree trigonometric polynomial $w(x) = \sum_{|n|\le 2} a_n e^{inx}$ that minimises the integral

$$\int_{-\pi}^{\pi} |u(x) - w(x)|^2 \, dx\,.$$

Also state the minimum value.

241. Set $u(x) = x^2$ when $-\pi \le x \le \pi$. What choices of a_0 minimise the following expressions?

(a) $|u(0) - a_0|$,
(b) $\int_{-\pi}^{\pi} |u(x) - a_0| \, dx$,
(c) $\int_{-\pi}^{\pi} |u(x) - a_0|^2 \, dx$,
(d) $\max_{[-\pi,\pi]} |u(x) - a_0|$.

Note that the four choices of a_0 are different although each gives the best approximation of u in some sense.

*242. Assume that u is a T-periodic function, i.e.

$$u(x+T) \;=\; u(x) \quad \text{for all } x\,.$$

The Fourier coefficients c_n of u are defined as the Fourier coefficients of the 2π-periodic function

$$v(x) \;=\; u\left(\frac{Tx}{2\pi}\right)\,.$$

(a) Show that

$$c_n \;=\; \frac{1}{T}\int_0^T u(x)\, e^{-2\pi i n x/T}\, dx\,.$$

(b) Show that

$$u(x) \;=\; \sum_{-\infty}^{\infty} c_n e^{2\pi i n x/T}$$

if u is continuous and piecewise C^1.

(c) Expand the 2-periodic function u given by $u(x) = 1 - |x|$ when $|x| \le 1$ in Fourier series.

(d) What does Parseval's formula look like for a T-periodic function?

(e) What is the result when Parseval's formula is applied to the function in (c)?

243. Compute the Fourier coefficients of the function u if

(a) $u(x) \;=\; x$ when $|x| < 1/2$ and u has period 1,

(b) $u(x) \;=\; x$ when $0 < x < T$ and u has period T?

*244. Expand $u(x) = x(\pi - x)$, $0 \le x \le \pi$, in a

(a) cosine series,

(b) sine series.

245. Expand $u(x) = \sin x$, $0 \le x \le \pi$, in a

(a) cosine series,

(b) sine series.

What is the sum of the cosine series for $-\pi \le x < 0$ and $\pi < x \le 2\pi$?

246. Expand $u(x) = x$, $0 \le x \le \pi$, in

(a) cosine series,

(b) sine series.

247. Expand $u(x) = \sin^3 x$, $0 \le x \le \pi$, in a sine series and use this to compute

$$\int_0^\pi \sin^6 x \, dx \,.$$

248. The function u is given in the interval $0 \le x \le \pi$ by

$$u(x) = \begin{cases} x(x - \frac{\pi}{2}) & \text{when } 0 \le x \le \frac{\pi}{2} \\ 0 & \text{when } \frac{\pi}{2} \le x \le \pi \,. \end{cases}$$

Expand u in a sine series and compute

$$\sum_0^\infty \frac{(-1)^k}{(2k+1)^3}$$

assuming the identity $\sum_0^\infty \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.

249. Show that

$$(x^2 - \pi^2)^2 = \frac{8}{15}\pi^4 - 48 \sum_{k=1}^\infty \frac{(-1)^k}{k^4} \cos kx$$

for all real x with $|x| \le \pi$ and compute

$$\text{(a)} \sum_{k=1}^\infty (-1)^k k^{-4} \,, \qquad \text{(b)} \sum_{k=1}^\infty k^{-4} \,, \qquad \text{(c)} \sum_{k=1}^\infty k^{-8} \,.$$

250. (a) Show that

$$\pi - |x| = \frac{\pi}{2} + \frac{4}{\pi} \sum_{k=0}^\infty \frac{\cos{(2k+1)x}}{(2k+1)^2}$$

for all real x with $|x| \le \pi$.

(b) Compute

$$\sum_0^\infty \frac{\sin{(2k+1)x}}{(2k+1)^3} \quad \text{when} \quad 0 \le x \le \pi$$

by means of (a).

251. Show that

(a) $(\pi^2 - 3x^2)/12 \;=\; \sum\limits_{k=1}^{\infty}(-1)^{k+1}k^{-2}\cos kx \quad \text{when} \quad |x| \le \pi\,,$

(b) $x(x^2 - \pi^2) \;=\; 12\sum\limits_{k=1}^{\infty}(-1)^{k}k^{-3}\sin kx \quad \text{when} \quad |x| \le \pi\,,$

(c) $(3x^2 - 6\pi x + 2\pi^2)/12 \;=\; \sum\limits_{k=1}^{\infty}k^{-2}\cos kx \quad \text{when} \quad 0 \le x \le 2\pi\,.$

252. Show that

$$\frac{\pi x}{4} \;=\; \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-2}\sin{(2k+1)x} \quad \text{when} \quad 0 \le x \le \frac{\pi}{2}\,.$$

253. In the discussion of the Gibbs phenomenon the function

$$w(t) \;=\; \frac{1}{2\tan\frac{t}{2}} \;-\; \frac{1}{t}$$

appears. Show that $w(t)$ and $w'(t)$ have limits as $t \to 0$ and evaluate these limits.

254. Determine the partial sums $S_N(x) \;=\; \sum\limits_{-N}^{N} c_n e^{inx}$ of the Fourier series of the 2π-periodic function given by

$$u(x) \;=\; \begin{cases} -1 & \text{when } -\pi \le x < 0 \\ 1 & \text{when } 0 \le x < \pi\,. \end{cases}$$

Evaluate $S_3(\pi/3)$, $S_4(\pi/4)$ and $S_6(\pi/6)$ and compare with the factor $g = \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}\,dt \approx 1.179$. Also evaluate $S_3(2\pi/3)$, $S_4(2\pi/4)$, $S_6(2\pi/6)$ and compare with the factor $\frac{2}{\pi}\int_0^{2\pi}\frac{\sin t}{t}\,dt \approx 0.903$.

Exercises for Chapter 3 Heat Conduction and Music

*301. Determine a solution of

$$\begin{aligned} \partial_t u &= \partial_x^2 u\,, \quad 0 \le x \le \pi\,, \quad t > 0 \\ u(0,t) &= u(\pi,t) \;=\; 0\,, \quad t \ge 0 \\ u(x,0) &= \sin x \;+\; 2\sin 3x\,, \quad 0 \le x \le \pi\,. \end{aligned}$$

302. Solve the problem

$$\begin{aligned} \partial_t u &= \partial_x^2 u\,, \quad 0 \le x \le \pi\,, \quad t > 0 \\ u(0,t) &= u(\pi,t) \;=\; 0\,, \quad t \ge 0 \\ u(x,0) &= \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x \le \pi\,. \end{cases} \end{aligned}$$

303. Solve the heat transfer problem

$$\begin{aligned} \partial_t u &= 3\partial_x^2 u\,, \quad 0 \le x \le \pi\,, \quad t > 0 \\ u(0,t) &= u(\pi,t) \;=\; 0\,, \quad t \ge 0 \\ u(x,0) &= \sin x\, \cos 4x\,, \quad 0 \le x \le \pi\,. \end{aligned}$$

304. (a) Show that the substitution

$$\begin{aligned} v(x,t) &= u(x,t) \;-\; A\left(1 - \frac{x}{l}\right) \;-\; B\frac{x}{l} \\ g(x) &= f(x) \;-\; A\left(1 - \frac{x}{l}\right) \;-\; B\frac{x}{l} \end{aligned}$$

transforms the problem

$$\begin{aligned} \partial_t u &= a\partial_x^2 u\,, \quad 0 \le x \le l\,, \quad t > 0 \\ u(0,t) &= A \quad \text{och} \quad u(l,t) \;=\; B\,, \quad t \ge 0 \\ u(x,0) &= f(x)\,, \quad 0 \le x \le l \end{aligned}$$

into the equation

$$\partial_t v \;=\; a\partial_x^2 v\,, \quad 0 \le x \le l\,, \quad t > 0\,,$$

with boundary conditions

$$v(0,t) \;=\; v(l,t) \;=\; 0 \quad \text{when} \quad t \ge 0$$

and initial condition

$$v(x,0) = g(x) \quad \text{when} \quad 0 \le x \le l\,.$$

(Observe that the function $u(x) = A(1-\frac{x}{l}) + B\frac{x}{l}$ satisfies the condition $u''(x) = 0$. It is therefore a stationary solution of the heat equation, i.e. independent of time, which reflects the fact that the temperature is stabilised.)

(b) Determine the solution of the problem

$$\begin{aligned}
\partial_t u &= \partial_x^2 u\,, \quad 0 \le x \le \pi\,, \quad t > 0 \\
u(0,t) &= 1 \quad \text{och} \quad u(\pi,t) = 2\,, \quad t \ge 0 \\
u(x,0) &= x\sin x + 1 + \frac{x}{\pi}\,, \quad 0 \le x \le \pi\,.
\end{aligned}$$

305. A large (infinite) steak with thickness l and the thermal diffusivity a is put in the oven to grill at the temperature T_0. The steak is taken straight from the refrigerator where the temperature is $0°$ C. The temperature distribution in the steak is then described by

$$\begin{aligned}
\partial_t u &= a\partial_x^2 u\,, \quad 0 \le x \le l\,, \quad t > 0 \\
u(0,t) &= u(l,t) = T_0\,, \quad t > 0 \\
u(x,0) &= 0\,, \quad 0 \le x \le l\,.
\end{aligned}$$

(a) Determine the temperature distribution $u(x,t)$ in form of a series. Hint: Make use of Exercise 304 (a). The solution formula of Theorem 3.4 may then be used even though the initial and boundary values are incompatible at the points $(0,0)$ and $(l,0)$.

(b) Assume that t is so large that all the terms of the series except the first may be neglected. What is the cooking time for the steak if it is ready when the temperature everywhere exceeds $T_0/4$? How much is the cooking time increased if the thickness of the steak is doubled?

306. Assume that u is a real solution of the heat equation

$$\partial_t u = \partial_x^2 u \quad \text{for} \quad 0 \le x \le \pi\,, \quad t > 0$$

that fulfils the condition $u(0,t) = u(\pi,t) = 0$ for $t \ge 0$. Show that

$$\int_0^\pi (u(x,t))^2\,dx$$

is a decreasing function of t. The function u may be assumed to be C^2 for $0 \le x \le \pi$, $t > 0$ and continuous for $0 \le x \le \pi$, $t \ge 0$.

*307. Determine a solution of

$$\begin{aligned} \partial_t u &= \partial_x^2 u\,, \quad 0 \le x \le \pi\,, \quad t > 0 \\ \partial_x u(0,t) &= \partial_x u(\pi,t) \,=\, 0\,, \quad t \ge 0 \\ u(x,0) &= x(\pi - x)\,, \quad 0 \le x \le \pi\,. \end{aligned}$$

308. Determine a solution of

$$\begin{aligned} \partial_t u &= \partial_x^2 u\,, \quad 0 \le x \le \pi\,, \quad t > 0 \\ \partial_x u(0,t) &= \partial_x u(\pi,t) \,=\, 0\,, \quad t \ge 0 \\ u(x,0) &= \sin^2 x\, \cos^2 x\,, \quad 0 \le x \le \pi\,. \end{aligned}$$

309. Solve the heat conduction problem

$$\begin{aligned} \partial_t u &= \partial_x^2 u\,, \quad 0 \le x \le 1\,, \quad t > 0 \\ \partial_x u(0,t) &= \partial_x u(1,t) \,=\, 0\,, \quad t > 0 \end{aligned}$$

if $u(x,0)$ equals

(a) $10\,,$ (b) $9 + 2x\,,$

when $0 \le x \le 1$.

Also evaluate $\lim_{t\to\infty} u(x,t)$ in the two cases.

*310. Solve the problem

$$\begin{aligned} \partial_t u &= \partial_x^2 u\,, \quad 0 \le x \le \frac{\pi}{2}\,, \quad t > 0 \\ u(0,t) &= 0 \quad \text{och} \quad \partial_x u(\frac{\pi}{2},t) \,=\, 0\,, \quad t \ge 0 \\ u(x,0) &= \sin 3x\,, \quad 0 \le x \le \frac{\pi}{2}\,. \end{aligned}$$

311. Solve the differential equation

$$\partial_t u \,=\, 2\partial_x^2 u\,, \quad 0 \le x \le \frac{\pi}{2}\,, \quad t > 0\,,$$

with boundary values $u(0,t) = \partial_x u(\frac{\pi}{2},t) = 0$ for $t > 0$ and initial values $u(x,0) = x$ for $0 \le x \le \frac{\pi}{2}$.

312. Determine a function $u(x,t)$ such that

$$\begin{aligned} \partial_t u &= \partial_x^2 u\,, \quad 0 \le x \le \frac{\pi}{2}\,, \quad t > 0 \\ u(0,t) &= \partial_x u(\frac{\pi}{2},t) \,=\, 0\,, \quad t \ge 0 \\ u(x,0) &= x(\pi - x)\,, \quad 0 \le x \le \frac{\pi}{2}\,. \end{aligned}$$

313. (a) Solve the equation

$$\partial_t u = \partial_x^2 u - au, \quad 0 \le x \le \pi, \quad t > 0,$$

with boundary values $u(0,t) = u(\pi,t) = 0$ when $t \ge 0$ and initial values $u(x,0) = \sin^3 x$. (The equation mimics a rod with a heat-loss through the lateral surface proportional to the difference of the temperature of the rod and the temperature $0°$ of the surroundings.)

(b) For which values of the constant a is $\lim_{t\to\infty} u(x,t) = 0$?

Hint: Set $u(x,t) = \sum_1^\infty b_n(t) \sin nx$.

314. If heat sources occur, then the (one-dimensional) heat equation becomes

$$c\rho\partial_t u = \lambda\partial_x^2 u + q(x,t), \quad 0 \le x \le l, \quad t > 0,$$

where $q(x,t)$ is the increase in heat per unit time and unit length at the time t and the position x. (The function q, which is allowed to be negative, is assumed to be independent of the temperature u.) Solve the equation in the case where

$$\frac{q(x,t)}{c\rho} = e^{-4t}\cos 2x,$$

$l = \pi$, $\frac{\lambda}{c\rho} = 1$, and u satisfies the boundary conditions

$$\partial_x u(0,t) = \partial_x u(\pi,t) = 0 \quad \text{when} \quad t > 0$$

and the initial condition

$$u(x,0) = \cos x \quad \text{when} \quad 0 \le x \le \pi.$$

315. A circular wire (with an insulating lateral surface) has the radius 1. If the wire is thin enough, then we may assume that the temperature $u(x,t)$ is constant in every cross-section of the wire and hence that u satisfies a one-dimensional heat equation

$$\partial_t u = h\partial_x^2 u.$$

Here x is the arc length along the wire which is assumed to extend over the interval $0 \le x \le 2\pi$. Seek a 2π-periodic (in x) solution u of the equation that also satisfies the initial condition

$$u(x,0) = x(2\pi - x), \quad 0 \le x \le 2\pi.$$

Hint: Set $u(x,t) = \sum_{-\infty}^{\infty} c_n(t)\, e^{inx}$.

*316. Solve the wave equation

$$\begin{aligned} \partial_t^2 u &= 25\partial_x^2 u\,, \quad 0 \le x \le 1\,, \quad t > 0 \\ u(0,t) &= u(1,t) = 0\,, \quad t \ge 0 \\ u(x,0) &= \sin 2\pi x\,, \quad \partial_t u(x,0) = \sin \pi x - 2\sin 3\pi x\,, \quad 0 \le x \le 1\,. \end{aligned}$$

317. Determine a solution of the wave equation

$$\partial_t^2 u = 4\partial_x^2 u\,, \quad 0 \le x \le \pi\,, \quad t > 0\,,$$

that satisfies the boundary conditions

$$u(0,t) = u(\pi,t) = 0\,, \quad t \ge 0$$

and the initial conditions

$$u(x,0) = \sin x\,, \quad \partial_t u(x,0) = x(\pi - x)\,, \quad 0 \le x \le \pi\,.$$

318. Find a (weak) solution of the problem

$$\begin{aligned} \partial_t^2 u &= \partial_x^2 u\,, \quad 0 \le x \le \pi\,, \quad t > 0 \\ u(0,t) &= u(\pi,t) = 0\,, \quad t \ge 0 \\ u(x,0) &= 0\,, \quad 0 \le x \le \pi\,, \\ \partial_t u(x,0) &= x\,, \quad 0 \le x \le \pi\,. \end{aligned}$$

319. A string of length l is fixed at its ends. At the time $t = 0$, it is straight and at rest except in a short portion of length b. This portion is set in motion at a certain velocity v by the strike of a hammer in such a way that the initial conditions can be formulated as

$$\begin{aligned} u(x,0) &= 0 \quad \text{when} \quad 0 \le x \le l\,, \\ \partial_t u(x,0) &= \begin{cases} v & \text{when } |x-a| < \frac{b}{2} \\ 0 & \text{else when } 0 \le x \le l\,. \end{cases} \end{aligned}$$

(a) Determine the movement of the string.

(b) Where should the hammer hit the string in order that the ground tone be as strong as possible?

(c) Where should the hammer hit the string in order that the sixth overtone be as weak as possible?

320. Use the method of separation of variables in order to find a solution of the problem of the vibrating string:

$$\begin{aligned} \partial_t^2 u &= \partial_x^2 u\,, \quad 0 \le x \le \pi\,, \quad t > 0 \\ u(0,t) &= u(\pi,t) \,=\, 0\,, \quad t \ge 0 \\ u(x,0) &= f(x)\,, \quad \partial_t u(x,0) \,=\, g(x)\,, \quad 0 \le x \le \pi\,. \end{aligned}$$

321. Determine the function $u(x,t)$ that satisfies

$$\partial_t^2 u \,=\, 4\partial_x^2 u \quad \text{when} \quad t > 0,\ x \in \mathbb{R}$$

and the initial conditions

$$u(x,0) \,=\, x^2\,, \quad \partial_t u(x,0) \,=\, x\,.$$

322. Determine the solution of the wave equation $\partial_t^2 u = c^2 \partial_x^2 u$ in the half-plane $t > 0$ that has the initial values

$$u(x,0) \,=\, e^{-x^2}\,, \quad \partial_t u(x,0) \,=\, \frac{1}{1+x^2}\,.$$

323. The solution in Example 3.11 can be written as

$$u(x,t) \,=\, \begin{cases} \frac{2h}{l}x & \text{when } 0 \le x \le \frac{l}{2} - ct \\ h(1 - \frac{2ct}{l}) & \text{when } \frac{l}{2} - ct \le x \le \frac{l}{2} + ct \\ -\frac{2h}{l}(x-l) & \text{when } \frac{l}{2} + ct \le x \le l \end{cases}$$

when $0 \le t \le \frac{l}{2c}$ and satisfies

$$u(x, t + \frac{2l}{c}) \,=\, u(x,t)$$

and

$$u(x, \frac{l}{c} - t) \,=\, -\,u(x,t)\,.$$

Expand u in Fourier series and check that the solution is the same as that in Example 3.7.

324. One end of a string is firmly anchored at $x = 0$ while the other end is free to move along a line perpendicular to the x-axis at $x = \frac{\pi}{2}$. If

the initial velocity is zero, then we have the following initial-boundary problem.

$$\begin{aligned}
\partial_t^2 u &= c^2 \partial_x^2 u\,, \quad 0 \le x \le \frac{\pi}{2}\,, \quad t > 0\\
u(0,t) &= 0\,, \quad \partial_x u(\frac{\pi}{2}, t) = 0\,, \quad t > 0\\
u(x,0) &= f(x)\,, \quad \partial_t u(x,0) = 0\,, \quad 0 \le x \le \frac{\pi}{2}\,.
\end{aligned}$$

Solve this problem, e.g. by letting, for each fixed $t > 0$,

$$u(x,t) = \sum_{k=0}^{\infty} b_k(t) \sin{(2k+1)x}$$

where

$$b_k(t) = \frac{4}{\pi} \int_0^{\pi/2} u(x,t)\, \sin{(2k+1)x}\, dx\,.$$

325. Taking gravity into consideration, we get the following differential equation for the vibrating string:

$$\partial_t^2 u = c^2 \partial_x^2 u - g\,, \quad 0 \le x \le l\,, \quad t > 0\,.$$

Here g is the acceleration due to gravity. Solve the equation with boundary conditions

$$u(0,t) = u(l,t) = 0\,, \quad t \ge 0\,,$$

and initial conditions

$$u(x,0) = \begin{cases} \frac{x}{10} & \text{when } 0 \le x \le \frac{l}{2} \\ \\ \frac{l-x}{10} & \text{when } \frac{l}{2} \le x \le l\,. \end{cases}$$

Hint: Set $v(x,t) = u(x,t) - \frac{gx}{2c^2}(x-l)$ and try to figure out what problem is solved by v. (Note that $u(x,t) = \frac{gx}{2c^2}(x-l)$ (independently of t!) is a stationary solution of the equation.)

326. The vibration of a string is damped due to air resistance, which is assumed to be proportional to the velocity. This suggests the differential equation

$$\partial_t^2 u = c^2 \partial_x^2 u - 2h \partial_t u \quad \text{when} \quad 0 \le x \le l\,, \quad t > 0\,.$$

Assume that $0 < h < \frac{\pi c}{l}$, that

$$u(0,t) = u(l,t) = 0 \quad \text{when} \quad t \geq 0\,,$$

and that

$$u(x,0) = f(x)\,, \quad \partial_t u(x,0) = 0 \quad \text{when} \quad 0 \leq x \leq l\,.$$

Show, e.g. by substituting $u(x,t) = e^{-ht}v(x,t)$, that

$$u(x,t) = e^{-ht}\sum_1^\infty \frac{A_n}{\cos\beta_n}\cos(\omega_n t - \beta_n)\sin\frac{n\pi x}{l}$$

where $A_n = \frac{2}{\pi}\int_0^\pi f(\frac{lx}{\pi})\,\sin nx\,dx$, $\omega_n = \sqrt{(\frac{n\pi c}{l})^2 - h^2}$ and $\tan\beta_n = \frac{h}{\omega_n}$. (We see that the air resistance damps the amplitude exponentially, reduces the frequencies and causes a phase shift.)

327. A string, firmly fixed at $x = 0$ but elastically fastened at $x = l$, gives rise to an eigenvalue problem of the form

$$v''(x) = \lambda v(x) \quad \text{when} \quad 0 \leq x \leq l\,, \quad v(0) = 0\,, \quad v'(l) + hv(l) = 0$$

where h is a positive constant.

(a) Show that all eigenvalues are negative. Hint: Multiply the equation by $v(x)$ and integrate from 0 to l.

(b) Show that λ is an eigenvalue if and only if $\mu = \sqrt{-\lambda}$ satisfies $\mu = -h\tan\mu l$.

328. Let λ_k be the eigenvalues of the problem

$$v''(x) - Bv^{(4)}(x) = \lambda v(x)\,, \quad v(0) = v''(0) = v(\pi) = v''(\pi) = 0$$

where $B > 0$ and prove the following statements.

(a) $\lambda_k < 0$. Hint: Cf. Exercise 327 (a).

(b) $\lambda_k = -k^2(1 + Bk^2)$, $k = 1, 2, 3, \ldots$, and the corresponding eigenfunctions are $v_k(x) = \sin kx$.

329. The Dirichlet problem on the unit disc consists in finding a function u in the disc that satisfies the Laplace equation $\Delta u = 0$ and attains

prescribed values on the boundary of the disc. In polar coordinates the problem can be written as

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= 0 \\ u(1,\theta) &= f(\theta) \end{aligned}$$

where f is a given 2π-periodic continuous function of θ. The solution u is required to be C^2 in the inner of the disc and continuous in the closed disc $|x| \le 1$. (For instance, u may be interpreted as the stationary temperature distribution in a circular disc that leaks heat only through the boundary $r = 1$, when the distribution on the boundary is given by f.)

(a) Let $u(r,\theta) = \sum_{-\infty}^{\infty} c_n(r)e^{in\theta}$ be the Fourier series expansion of u for a fixed $r < 1$. Show that the differential equation above means that

$$c_n''(r) + \frac{1}{r}c_n'(r) - \frac{n^2}{r^2}c_n(r) = 0\,, \quad 0 < r < 1\,,$$

for each n.

(b) Show that $c_n(r) = C_n r^{|n|}$ where C_n is a constant. (Hint: Substitute $r = e^t$. Use that u is continuous at the origin.)

(c) Show that the boundary condition $u(1,\theta) = f(\theta)$ implies that C_n is the n^{th} Fourier coefficient of f.

Hence, a solution u of the problem must satisfy

$$u(x) = \sum_{-\infty}^{\infty} C_n r^{|n|} e^{in\theta}$$

where C_n, $n = 0\,, \pm 1\,, \dots$, are the Fourier coefficients of f, hence uniquely determined.

(d) Conversely, show that this function u really is a solution if f is C^1.

330. Assume that u is a harmonic function (i.e. $\Delta u = 0$) in the unit disc. Show that the mean value

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(r,\theta)\,d\theta,$$

where (r,θ) are polar coordinates, is independent of $r \in [0,1)$ and equal to the value of u at the origin.

Hint: Use 329 (a) and (b) for $n = 0$.

331. Show that separation of variables in the eigenvalue problem

$$\begin{aligned} \Delta u &= \lambda u \quad \text{in the rectangle } 0 < x < a\,, \quad 0 < y < b\,, \\ u &= 0 \quad \text{on the boundary of the rectangle}\,, \end{aligned}$$

leads to solutions of the form

$$u(x, y) = \sin \frac{nx}{a} \sin \frac{my}{b}$$

where

$$\lambda = -\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2\,, \quad n\,, m = 1, 2, 3, \ldots$$

are the corresponding eigenvalues.

Exercises for Chapter 4 The Fourier Transform

The Definition and the Inversion formula

*401. Compute the Fourier transform of

$$u(x) = \begin{cases} (1-|x|)^2\,, & |x| \le 1 \\ 0\,, & |x| > 1\,. \end{cases}$$

In particular, specify $\hat{u}(0)$.

*402. (a) Compute the Fourier transform of

$$u(x) = \begin{cases} \sin x\,, & 0 \le x \le 2\pi \\ 0\,, & x \notin [0, 2\pi]\,. \end{cases}$$

(b) Show that $|\hat{u}(\xi)| \le 4$.

*403. (a) Determine the Fourier transform of

$$u(x) = \max\,(1-|x|, 0)\,.$$

(b) Use (a) to show that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos\xi}{\xi^2}\cos x\xi\,d\xi = \begin{cases} 1-|x|\,, & |x| \le 1 \\ 0\,, & |x| > 1\,. \end{cases}$$

(c) Compute, by means of (b),

$$\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\,dt\,.$$

*404. The function u is continuous and integrable in $\mathbb{R}$. Show that the following statements hold for the Fourier transform $\hat{u}$.

(a) $\hat{u}$ is even if u is even.

(b) $\hat{u}$ is odd if u is odd.

(c) $\hat{u}$ is real-valued if $\overline{u(x)} = u(-x)$ for all $x \in \mathbb{R}$.

(d) $\hat{u}$ is even and real-valued if u is even and real-valued.

Also show that "if" may be replaced by "if and only if" in these statements provided that $\hat{u}$ is integrable. (Cf. Exercise 206.)

Rules of Calculation

When computing Fourier transforms, it is essential to know the transforms of some simple functions. At least you should learn that the Fourier transform of $e^{-|x|}$ is $\frac{2}{1+\xi^2}$ and that the Fourier transform of $e^{-x^2/2}$ is $\sqrt{2\pi}e^{-\xi^2/2}$. Knowing these transforms, it is easy to use the rule of scaling in order to derive the transforms of $e^{-a|x|}$ and e^{-ax^2} $(a > 0)$. Furthermore, using the inversion formula, one finds that $\frac{1}{1+x^2}$ has the transform $\pi e^{-|\xi|}$.

*405. Compute the Fourier transform of

$$\begin{aligned} &\text{(a)}\ u(x) = e^{-5|x|}\,, \\ &\text{(b)}\ u(x) = \tfrac{e^{ix}}{1+2x^2}\,, \\ &\text{(c)}\ u(x) = xe^{-x^2}\,. \end{aligned}$$

*406. Determine the continuous, integrable function that has the Fourier transform

$$\begin{aligned} &\text{(a)}\ \hat{u}(\xi) = e^{2i\xi-4|\xi|}\,, \\ &\text{(b)}\ \hat{u}(\xi) = \xi\, e^{-(\xi-3)^2}\,, \\ &\text{(c)}\ \hat{u}(\xi) = \xi^2\, e^{-\xi^2/4}\,. \end{aligned}$$

The convolution Theorem, the Plancherel formula

*407. Compute the Fourier transform $\hat{u}$ of the following function, which is a convolution.

$$u(x) = \int_{-\infty}^{\infty} e^{-|x-y|}e^{-|y|}\,dy\,.$$

*408. Find a solution u of the convolution equation

$$\int_{-\infty}^{\infty} u(x-y)e^{-2y^2}\,dy = e^{-x^2}\,, \quad x \in \mathbb{R}\,.$$

*409. State the Plancherel formula for the function $(1+x^2)^{-1}$ and use this to evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\,dx\,.$$

Applications

*410. Use a direct calculation to show that

$$E(x,t) \;=\; \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}$$

is a solution of the heat equation

$$\partial_t u \;=\; \partial_x^2 u$$

in the region $t > 0$, $x \in \mathbb{R}$.

411. Show that if u is a bounded solution of the heat equation

$$\partial_t u \;=\; k\,\partial_x^2 u\,, \quad t > 0\,, \quad x \in \mathbb{R}\,,$$

with initial condition $u(x,0) = \phi(x)$ where ϕ is a bounded, continuous and *odd* function, then u is an odd function of x.

412. (a) Show that the function $E_1(x,t)$, defined by

$$E_1(x,t) \;=\; \partial_x \left(\frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t} \right)\,,$$

solves the heat equation

$$\partial_t u \;=\; \partial_x^2 u\,, \quad t > 0\,, \quad x \in \mathbb{R}\,,$$

and satisfies the initial condition $u(x,0) = 0$, $x \in \mathbb{R}$, in the sense that for every fixed $x \in \mathbb{R}$, $E_1(x,t) \to 0$ as $t \to 0$, $t > 0$.

(b) Plainly, also $u = 0$ satisfies the heat equation and equals zero at $t = 0$. Both this function and the function E_1 are, therefore, solutions of the initial-value problem (at least if the initial-value condition is interpreted favourably) for the heat equation. Why does this not contradict the uniqueness in Theorem 4.15 ?

Mixed Exercises

413. Compute the Fourier transform of

$$u(x) \;=\; \begin{cases} \cos\frac{\pi x}{2}\,, & |x| \le 1 \\ 0\,, & |x| > 1\,. \end{cases}$$

Specify, in particular, $\hat{u}(\xi)$ when $\xi = \pm\frac{\pi}{2}$ and show that

$$|\hat{u}(\xi)| \le 4/\pi \quad \text{for all } \xi \in \mathbb{R}\,.$$

414. Compute the Fourier transform of

$$u(x) = \begin{cases} e^{i\omega_0 x}, & a \le x \le a+T \\ 0, & x \notin [a, a+T]. \end{cases}$$

What happens when $a = -T/2$?

415. Compute the Fourier transform of $u(x) = \max(1 - x^2, 0)$.

416. Compute the Fourier transform of

$$u(x) = \begin{cases} e^{-ax}, & x \ge 0 \\ e^{bx}, & x < 0 \end{cases}$$

where a and b are assumed to be positive, and show that

$$e^{-|x|} = \frac{2}{\pi} \int_0^\infty \frac{\cos x\xi}{1+\xi^2}\, d\xi$$

for all real x.

417. Determine the continuous, integrable function that has the Fourier transform $|\xi| e^{-|\xi|}$.

418. The function u is continuous and integrable over $\mathbb{R}$. Show the following statements.

(a) If u is even, then

$$\hat{u}(\xi) = 2 \int_0^\infty u(x) \cos x\xi \, dx.$$

(b) If u is odd, then

$$\hat{u}(\xi) = -2i \int_0^\infty u(x) \sin x\xi \, dx.$$

419. Let u be a continuous and integrable function in $\mathbb{R}$. Show that if $f(x) = u(x) \cos ax$, then

$$\hat{f}(\xi) = \frac{1}{2}(\hat{u}(\xi - a) + \hat{u}(\xi + a)).$$

420. Is there any continuous, integrable function u in $\mathbb{R}$ such that

$$\hat{u}(\xi) = 1 - \sin \xi\,?$$

421. Let u and u_1, u_2, $\ldots$ be continuous, integrable functions in $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty} |u_n(x) - u(x)|\, dx \to 0 \quad \text{as} \quad n \to \infty\,.$$

Show that $\hat{u}_n \to \hat{u}$ uniformly in $\mathbb{R}$.

422. Set, when u is a bounded, integrable and continuous function i $\mathbb{R}$,

$$U(x_1, x_2) \;=\; \begin{cases} \frac{1}{\pi}\int_{-\infty}^{\infty} u(t)\frac{x_2}{x_2^2+(x_1-t)^2}\, dt\,, & x_2 > 0 \\ u(x_1)\,, & x_2 = 0\,. \end{cases}$$

(a) Show that U is bounded and continuous in the region $x_2 \geq 0$.

(b) Show that U is harmonic in $x_2 > 0$, i.e.

$$\Delta u = (\partial_{x_1}^2 + \partial_{x_2}^2)u = 0\,.$$

(Hint: Differentiate under the integral sign.)

Remark. The function U is called the Poisson integral of u for the upper half-plane. For a given bounded, integrable and continuous function u in $\mathbb{R}$, U is the (unique) bounded and harmonic function in the upper half-plane that has continuous boundary values u.

423. Let u be a continuous, integrable and bounded function in $\mathbb{R}$. Show that

$$u(x) \;=\; \lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R}(1 - \frac{|\xi|}{R})\, \hat{u}(\xi)\, e^{ix\xi}\, d\xi\,.$$

(If $\hat{u}$ is integrable, this is an almost trivial consequence of the inversion formula.)

Hint:

(a) Set

$$S_R(x) \;=\; \frac{1}{2\pi}\int_{-R}^{R}(1 - \frac{|\xi|}{R})\, \hat{u}(\xi)\, e^{ix\xi}\, d\xi$$

and show that

$$S_R(x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-R}^{R} e^{iy\xi}(1 - \frac{|\xi|}{R})\, d\xi\right)\, u(x-y)\, dy\,.$$

(b) Show, using methods including partial integration, that the inner integral equals

$$\frac{2}{Ry^2}(1 - \cos Ry)\,.$$

(c) Then show that

$$S_R(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} u(x - \frac{s}{R}) \frac{1 - \cos s}{s^2}\, ds$$

and that $\lim_{R \to \infty} S_R(x)$ may be evaluated by taking the limit under the integral sign.

(d) Complete the proof using the fact that

$$\int_{-\infty}^{\infty} \frac{1 - \cos s}{s^2}\, ds = \int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt = \pi.$$

(Cf. Exercise 403.)

424. Compute the Fourier transform of

$$\frac{1}{x^2 + 2ax + a^2 + s^2}, \qquad a, s > 0,$$

using the fact that we know that $f_s(x) = e^{-s|x|}$ has the Fourier transform $\frac{2s}{s^2+\xi^2}$.

425. Compute the Fourier transform of

$$\text{(a) } xe^{-|x|}, \qquad \text{(b) } \frac{x}{(1+x^2)^2}, \qquad \text{(c) } \frac{x^2}{(1+x^2)^2}, \qquad \text{(d) } \frac{1}{(1+x^2)^2}.$$

426. Determine the continuous and integrable function u that satisfies

$$(\mathcal{F}(u'))(\xi) = \xi e^{-\xi^2/2}.$$

427. The Fourier transform of the function u is $\hat{u}$. Determine the Fourier transform of $u(x)\sin^2 2x$.

428. Determine the Fourier transform of

$$\text{(a) } e^{-|x|}\cos x, \qquad \text{(b) } e^{-|x|}\sin|x|, \qquad \text{(c) } \frac{1}{x^4+4}.$$

429. Let $u(x) = 0$ when $x < 0$ and $u(x) = x^k e^{-\alpha x}$ when $x \geq 0$. Here $\alpha > 0$ and k is a positive integer. Determine the Fourier transform of u when (a) $k = 1$, (b) $k = 2$, (c) k is arbitrary.

430. Determine the Fourier transform of (a) e^{-x^2-2x}, (b) $e^{-\alpha x^2 - \beta x}$ where $\alpha > 0$ and $\beta \in \mathbb{R}$.

431. Compute $u * u$, when $u(x) = \frac{1}{\pi(1+x^2)}$, making use of the Fourier transform of $\frac{1}{1+x^2}$.

432. Determine a function u such that

$$\int_{-\infty}^{\infty} u(x-y)\, u(y)\, dy = e^{-x^2}$$

for all real x.

433. Evaluate, for all real values of x, the integral

$$\int_{-\infty}^{\infty} \frac{dy}{(1+4(x-y)^2)(1+y^2)} .$$

434. Find a solution of the integral equation

$$\int_{-\infty}^{\infty} u(y)\, e^{-|x-y|}\, dy = e^{-x^2}, \quad x \in \mathbb{R} .$$

435. Show that if u is a continuous, bounded and integrable function such that

$$\int_{-\infty}^{\infty} |u(x+h) - u(x)|^2\, dx \le C h^{\alpha}, \quad 0 \le h \le 1,$$

where $\alpha > 1$, then $\hat{u}$ is integrable.

Hint:

(a) Set $v_h(x) = u(x+h) - u(x)$ and show that $\hat{v}_h(\xi) = (e^{ih\xi} - 1)\hat{u}(\xi)$.

(b) Show, using the Plancherel formula, that

$$4 \int_{-\infty}^{\infty} \sin^2 \frac{h\xi}{2}\, |\hat{u}(\xi)|^2\, d\xi \le 2\pi\, C h^{\alpha}, \quad 0 \le h \le 1 .$$

(c) Then let $h = 2^{-n}$, where n is a positive integer, and show that

$$\int_{2^{n-1} < |\xi| < 2^n} |\hat{u}(\xi)|^2\, d\xi \le C'\, 2^{-n\alpha} .$$

(d) Make use of the Cauchy–Schwarz inequality to conclude that

$$\int_{2^{n-1} < |\xi| < 2^n} |\hat{u}(\xi)|\, d\xi \le C''\, 2^{n(1-\alpha)/2} .$$

(e) Show that it follows from this that $\hat{u}$ is integrable.

436. Make use of the Fourier transform to find a solution of the ordinary differential equation

$$u''(x) - u(x) = -e^{-|x|}.$$

Hint: Use 425 (d) to recover u from $\hat{u}$.

437. Solve the heat conduction problem

$$\begin{aligned} \partial_t u &= \partial_x^2 u, \quad t > 0, \quad x \in \mathbb{R} \\ u(x,0) &= e^{-x^2}, \quad x \in \mathbb{R}. \end{aligned}$$

438. Solve the heat conduction problem

$$\begin{aligned} \partial_t u &= \partial_x^2 u, \quad t > 0, \quad x \in \mathbb{R} \\ u(x,0) &= E(x,s) = \frac{1}{\sqrt{4\pi s}} e^{-\frac{x^2}{4s}}, \quad x \in \mathbb{R} \end{aligned}$$

where $s > 0$ is a fixed constant. Interpret the result!

439. Solve the heat conduction problem

$$\begin{aligned} \partial_t u &= \partial_x^2 u, \quad t > 0, \quad x \in \mathbb{R} \\ u(x,0) &= \cos x, \quad x \in \mathbb{R}. \end{aligned}$$

440. Let $u(x,t)$ be the solution of the heat conduction problem

$$\begin{aligned} \partial_t u &= \partial_x^2 u, \quad t > 0, \quad x \in \mathbb{R} \\ u(x,0) &= f(x), \quad x \in \mathbb{R} \end{aligned}$$

where $f \in C_0(\mathbb{R})$. Show that

$$\int_{-\infty}^{\infty} u(y,t)\, dy = \int_{-\infty}^{\infty} f(x)\, dx.$$

What is the interpretation?

Answers to the Exercises

Solutions to exercises marked with an asterisk can be found after the answers.

Answers for Chapter 1

101. (a) $x \neq 0$, $1 + \frac{1}{x^2}$. (b) $x < 0$, $\frac{1}{1-e^x}$.
(c) $x \geq 0$, $\frac{x}{e^x-1}$ if $x > 0$, 0 if $x = 0$. (d) $|x| > 4$, $\frac{x}{x-3} + \frac{x}{x-4}$.

102. $2s - a_0$

103. (a) and (d) are convergent, (b), (c) and (e) are divergent.

104. (a), (b) and (d) are convergent, (c) is divergent.

105. (a) $a > 1$. (b) $a < 1$. (c) $a \neq 1$. (d) $a \leq 1$. (e) $a < 1$.

106. 1.

109. (a), (b) and (d).

111. (a) $3/4$. (b) $1/4$.

112. $\alpha > 0$.

113. Convergent but not absolutely convergent.

114. (a) E.g. $N = 31$. (b) E.g. $N = 1000$.

115. (a) Yes. (b) No.

116. $|z| < 1$.

117. (a) $||f_n||_E = f_n(1/n) = 1/2$. Not uniform convergence.
(b) $||f_n||_E = f_n(1) = \frac{n}{1+n^2}$. Uniform convergence.

118. $f(x) = -x$. Uniform convergence.

119. (a) No. (b) Yes. (c) Yes.

120. (a) 1 when $0 < x < \pi$, 0 when $x = 0$, $x = \pi$.
(b) Not uniform convergence.

121. (a) 0. (b) Not uniform convergence.

122. Uniform convergence.

124. (a) $x \geq 0$. (b) $x/(1-e^{-x})$ if $x \neq 0$, 0 if $x = 0$. (c) Not uniform convergence.

128. (a) $\frac{c^c}{(c+1)^{c+1}}$.

131. Yes.

133. $s'(0) = 3$.

135. $\pi/4$.

136. $R = 1$.

139. (b) $s(x) = \frac{1}{3}e^{-x} + \frac{2}{3}e^{x/2}\cos\frac{\sqrt{3}}{2}x$.

141. (b) Yes.

142. (a) $s(x) = \ln(1+x+x^2)$, $|x| < 1$, $s(1) = s(-1) = 0$.
(b) not uniformly convergent.
(c) no.

143. $|a| < 3$.

145. (a) $0 < x < 2$.
(b) $s(x)$ is continuous.

Answers for Chapter 2

201. (b) $c_0 = 0\,,\quad c_n = -\frac{i}{n}\quad$ if $n \neq 0$.

202. (a) $c_0 = \frac{1}{2}\,,\quad c_2 = c_{-2} = \frac{1}{4}\,,\quad$ the others equal 0.

(b) $c_n = 0$ when $n \neq \pm 1\,,\quad c_1 = -\frac{i}{2}\quad$ and $\quad c_{-1} = \frac{i}{2}\,.$

(c) $c_0 = \frac{3}{8}\,,\quad c_2 = c_{-2} = \frac{1}{4}\,,\quad c_4 = c_{-4} = \frac{1}{16}\,,\quad$ the others equal 0.

(d) $c_0 = \frac{1}{8},\quad c_4 = c_{-4} = -\frac{1}{16}\,,\quad$ the others equal 0.

203. $2^{-20}\binom{20}{7}\,.$

204. $c_n = \frac{(-1)^n - 1}{\pi n^2}$ if $n \neq 0,\quad c_0 = \frac{\pi}{2}\,.$

205. (a) $\frac{1}{2} + \sum\limits_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{e^{inx}}{i\pi n}\,.$

(b) $\frac{1}{2},\, \frac{1}{2}\,.$

209. $u(x) = c_0 + c_1 e^{ix} + c_{-1} e^{-ix}$ where c_0, c_1 and c_{-1} are arbitrary complex numbers.

210. $u(x) = c_1 e^{ix} + c_{-1} e^{-ix}$ where c_1 and c_{-1} are arbitrary.

211. $\frac{e^{ix-1}}{(1-e^{ix-1})^2}\,.$

213. (a) $u(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n}{1+n^2} e^{inx}\,.$

(b) $-\frac{1}{2} + \frac{\pi}{2}\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}\,.$

214. (b) $(6 - 2\pi)^2\,.$

215. (a)

$$\begin{aligned} u(x) &= \frac{1}{\pi} + \frac{1}{4}\left(e^{ix} + e^{-ix}\right) + \sum_{|n|\geq 2} \frac{\cos\frac{\pi n}{2}}{\pi(1-n^2)} e^{inx} \\ &= \frac{1}{\pi} + \frac{1}{2}\cos x + \frac{2}{\pi}\sum_{1}^{\infty} \frac{(-1)^k}{1-4k^2}\cos 2kx\,. \end{aligned}$$

(b) $\frac{1}{2} - \frac{\pi}{4}\,.$

(c) $\frac{1}{2}\,.$

217. $u(x) = \frac{\alpha \sin \pi\alpha}{\pi} \sum\limits_{-\infty}^{\infty} \frac{(-1)^n}{\alpha^2 - n^2}\, e^{inx}\,.$

218. The integral equals $\frac{\pi}{2}$.

224. $||u - v|| = ||u + v||$ if and only if $\operatorname{Re}(u|v) = 0$.

227. $\dfrac{\pi^4}{90}$.

228. $\dfrac{3\pi}{4}$.

229. $\frac{1}{2}\left(\frac{\pi(e^{2\pi}+1)}{e^{2\pi}-1} - 1\right)$.

230. $c_n = \frac{n(-1)^n}{i(n^2-1)}$ if $n \neq \pm 1$, $c_1 = \frac{i}{4}$, $c_{-1} = -\frac{i}{4}$. The sum equals $\frac{\pi^2}{12} + \frac{1}{16}$.

231. $\frac{\pi^2}{16} - \frac{1}{2}$.

232. (a) $c_n = -\frac{6i(-1)^n}{n^3}$ if $n \neq 0$, $\quad c_0 = 0$.
(b) $\frac{\pi^6}{945}$.

233. (a) $\frac{e^{2\pi i a}-1}{2\pi i(a-n)}$.

234. The constant is 1.

235. $u(x) = \sum\limits_{0}^{\infty} \frac{1}{r^{n+1}} e^{inx}$. The integral equals $\frac{2\pi}{r^2-1}$.

237. $c_{2k-n} = 2^{-n}\begin{pmatrix} n \\ k \end{pmatrix}$ when $k = 0, 1, 2, \ldots, n$, $c_j = 0$ for other values of j.

238. $a = \frac{\pi^2}{3}$, $b = c = -2$. The minimum value is $\frac{8\pi^5}{45} - 16\pi$.

239. $p(x) = \frac{1}{128}(14e^{4ix} - 28e^{2ix} + 35 - 28e^{-2ix} + 14e^{-4ix})$. The minimum value equals $\frac{65\pi}{2^{14}} = \frac{65\pi}{16384}$.

240. $w(x) = \frac{\pi}{2} - \frac{2}{\pi}(e^{ix} + e^{-ix})$. The minimum value is $\frac{\pi^3}{6} - \frac{16}{\pi}$.

241. (a) 0. (b) $\frac{\pi^2}{4}$. (c) $\frac{\pi^2}{3}$. (d) $\frac{\pi^2}{2}$.

242. (a) $u(x) = \frac{1}{2} + \frac{2}{\pi^2} \sum\limits_{-\infty}^{\infty} \frac{e^{\pi i(2k+1)x}}{(2k+1)^2}$.
(b) $\sum\limits_{-\infty}^{\infty} |c_n|^2 = \frac{1}{T}\int_{-T/2}^{T/2} |u(x)|^2\,dx$.
(c) $\sum\limits_{0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96}$.

243. (a) $c_0 = 0\,,\quad c_n \;=\; \frac{i(-1)^n}{2\pi n}$ when $n \neq 0$.
(b) $c_0 = \frac{T}{2}\,,\quad c_n \;=\; \frac{iT}{2\pi n}$ when $n \neq 0$.

244. (a) $u(x) \;=\; \frac{\pi^2}{6} - \sum_1^\infty \frac{\cos 2kx}{k^2}\,.$

(b) $u(x) \;=\; \frac{8}{\pi} \sum_0^\infty \frac{\sin(2k+1)x}{(2k+1)^3}\,.$

245. (a) $\frac{2}{\pi} - \frac{4}{\pi} \sum_1^\infty \frac{\cos 2kx}{4k^2-1}.$

(b) $\sin x$. The sum of the cosine series is $-\sin x$ when $-\pi \leq x < 0$ or $\pi < x \leq 2\pi$.

246. (a) $\frac{\pi}{2} - \frac{4}{\pi} \sum_0^\infty \frac{\cos(2k+1)x}{(2k+1)^2}\,.$

(b) $2 \sum_1^\infty \frac{(-1)^{n+1}}{n} \sin nx\,.$

247. $\frac{3}{4}\sin x \;-\; \frac{1}{4}\sin 3x$. The integral equals $\frac{5\pi}{16}$.

248. $u(x) \;=\; \sum_1^\infty b_n \sin nx\,,\quad b_n \;=\; \begin{cases} \frac{(-1)^k-1}{2\pi k^3} & \text{when } n = 2k \\ \frac{(-1)^k}{(2k+1)^2} \;-\; \frac{4}{\pi(2k+1)^3} & \text{when } n = 2k+1\,. \end{cases}$

The sum equals $\frac{\pi^3}{32}$.

249. (a) $-\dfrac{7\pi^4}{720}$. (b) $\dfrac{\pi^4}{90}$. (c) $\dfrac{\pi^8}{9450}$.

Remark: One might wonder if it is possible to check whether the answer in (c) is reasonable or not. Since $1 \leq \sum_{k=1}^\infty \frac{1}{k^8} \leq \sum_{k=1}^\infty \frac{1}{k^4}$, the answer must lie somewhere between 1 and $\pi^4/90$ (the answer in (b)). It can come in handy to know of the approximation $\pi^2 \approx 9.86 \approx 10$, with an error less than 2%. Using this approximation, we get

$$\frac{\pi^4}{90} \approx \frac{100}{90} \approx 1.11 \quad \text{and} \quad \frac{\pi^8}{9450} \approx \frac{10000}{9450} \approx 1.06\,,$$

which seems reasonable. It may also be worth to note that $\pi^2 \leq 10$ and hence these approximations yield upper bounds for the sums. A more accurate calculation shows that

$$\frac{\pi^4}{90} \approx 1.082 \quad \text{and} \quad \frac{\pi^8}{9450} \approx 1.00408\,,$$

whence in the latter case the first two terms $1 + 2^{-8} \approx 1.0039$ give a very good approximation of the sum of the series.

250. (b) $\dfrac{\pi x(\pi - x)}{8}$.

253. $w(t) \to 0$ and $w'(t) \to -\frac{1}{12}$ as $t \to 0$.

254. $S_{2K+1}(x) = \frac{4}{\pi}\sum_0^K \frac{\sin(2k+1)x}{2k+1}$, $S_{2K+2} = S_{2K+1}$, $K > 0$,

$$\begin{aligned}
S_3\left(\frac{\pi}{3}\right) &= \frac{2\sqrt{3}}{\pi} \approx 1.103\,,\\
S_4\left(\frac{\pi}{4}\right) &= \frac{8\sqrt{2}}{3\pi} \approx 1.200\,,\\
S_6\left(\frac{\pi}{6}\right) &= \frac{56}{15\pi} \approx 1.188\,,\\
S_3\left(\frac{2\pi}{3}\right) &= \frac{2\sqrt{3}}{\pi} \approx 1.103\,,\\
S_4\left(\frac{2\pi}{4}\right) &= \frac{8}{3\pi} \approx 0.849\,,\\
S_6\left(\frac{2\pi}{6}\right) &= \frac{8\sqrt{3}}{5\pi} \approx 0.882\,.
\end{aligned}$$

Answers for Chapter 3

301. $e^{-t}\sin x \,+\, 2e^{-9t}\sin 3x\,.$

302. $\sum\limits_0^\infty \frac{4(-1)^k}{\pi(2k+1)^2} e^{-(2k+1)^2 t}\sin(2k+1)x\,.$

303. $\dfrac{1}{2}\left(e^{-75t}\sin 5x \,-\, e^{-27t}\sin 3x\right).$

304. (b) $1+\frac{x}{\pi}+\frac{\pi}{2}e^{-t}\sin x - \frac{16}{\pi}\sum\limits_{k=1}^\infty \frac{k}{(4k^2-1)^2}e^{-4k^2 t}\sin 2kx\,.$

305. (a) $T_0\left(1-\frac{4}{\pi}\sum\limits_{k=0}^\infty \frac{1}{2k+1}e^{-\frac{(2k+1)^2\pi^2 a t}{l^2}}\sin(2k+1)\frac{\pi}{l}x\right).$

(b) The cooking time is approximately $\frac{l^2}{a\pi^2}\ln\frac{16}{3\pi}$. Note that it is proportional to the square of the thickness and inversely proportional to the thermal diffusivity.

307. $\frac{\pi^2}{6} \;-\; \sum\limits_1^\infty e^{-4k^2 t}\frac{\cos 2kx}{k^2}.$

308. $\frac{1}{8}(1-e^{-16t}\cos 4x).$

309. (a) $u(x,t) \,=\, 10\,.$

(b) $u(x,t) \,=\, 10 \,-\, \frac{8}{\pi^2}\sum\limits_0^\infty \frac{1}{(2k+1)^2}e^{-(2k+1)^2\pi^2 t}\cos(2k+1)\pi x\,.$

The limit equals 10 in both cases.

310. $e^{-9t}\sin 3x.$

311. $\dfrac{4}{\pi}\sum\limits_0^\infty \dfrac{(-1)^k}{(2k+1)^2}e^{-(2k+1)^2 2t}\sin(2k+1)x.$

312. $\dfrac{8}{\pi}\sum\limits_0^\infty \dfrac{1}{(2k+1)^3}e^{-(2k+1)^2 t}\sin(2k+1)x.$

313. (a) $\frac{3}{4}e^{-(1+a)t}\sin x \,-\, \frac{1}{4}e^{-(9+a)t}\sin 3x\,.$ (b) $a > -1.$

314. $e^{-t}\cos x \,+\, te^{-4t}\cos 2x\,.$

315. $\dfrac{2\pi^2}{3} \,-\, 4\sum\limits_1^\infty \dfrac{1}{n^2}e^{-hn^2 t}\cos nx\,.$

316. $\dfrac{1}{5}\sin 5\pi t \sin \pi x \;+\; \cos 10\pi t \sin 2\pi x \;-\; \dfrac{2}{15\pi}\sin 15\pi t \sin 3\pi x\,.$

317. $(\cos 2t + \frac{4}{\pi}\sin 2t)\,\sin x \;+\; \frac{4}{\pi}\sum\limits_{1}^{\infty} \frac{1}{(2k+1)^4}\,\sin\left(2(2k+1)t\right)\,\sin\,(2k+1)x\,.$

318. $2\displaystyle\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n^2}\sin nt \sin nx\,.$

319. (a) $\dfrac{4lv}{\pi^2 c}\displaystyle\sum_{1}^{\infty}\frac{1}{n^2}\,\sin\frac{n\pi b}{2l}\,\sin\frac{n\pi a}{l}\,\sin\frac{n\pi ct}{l}\,\sin\frac{n\pi x}{l}\,.$

(b) $a = \frac{l}{2}$

(c) $a \;=\; k\cdot\frac{l}{7}$, $k \;=\; 1, 2, \ldots, 6$, i.e. at a node point of the eigenmode that corresponds to the 6^{th} overtone.

321. $x^2 + 4t^2 + xt\,.$

322. $u(x,t)$ equals

$$\frac{1}{2}\left(e^{-(x+ct)^2} \;+\; e^{-(x-ct)^2} \;+\; \frac{1}{c}(\arctan\,(x+ct) \;-\; \arctan\,(x-ct))\right)\,.$$

324. $u(x,t)$ is equal to

$$\sum_{0}^{\infty} A_k\,\cos\left((2k+1)ct\right)\,\sin\,(2k+1)x$$

where $A_k \;=\; \frac{4}{\pi}\int_0^{\pi/2} f(x)\,\sin\,(2k+1)x\,dx\,.$

325. $u(x,t)$ is the function

$$\frac{gx(x-l)}{2c^2} \;+\; \sum_{0}^{\infty} A_k\,\cos\frac{(2k+1)\pi ct}{l}\,\sin\frac{(2k+1)\pi x}{l}$$

where $A_k \;=\; \dfrac{2l}{5\pi^2}\,\dfrac{(-1)^k}{(2k+1)^2} \;+\; \dfrac{4gl^2}{c^2\pi^3(2k+1)^3}\,.$

Answers for Chapter 4

401.

$$\hat{u}(\xi) = \begin{cases} \frac{4}{\xi^3}(\xi - \sin\xi) & \text{when } \xi \neq 0 \\ \frac{2}{3} & \text{when } \xi = 0\,. \end{cases}$$

402.

$$\hat{u}(\xi) = \begin{cases} \frac{1-e^{-i2\pi\xi}}{1-\xi^2} & \text{when } \xi \neq \pm 1 \\ \mp i\pi & \text{when } \xi = \pm 1\,. \end{cases}$$

403. (a)

$$\hat{u}(\xi) = \begin{cases} \frac{2(1-\cos\xi)}{\xi^2} & \text{when } \xi \neq 0 \\ 1 & \text{when } \xi = 0\,. \end{cases}$$

(b) See the solution.

(c) See the solution.

404. See the solution.

405. (a) $\hat{u}(\xi) = \frac{10}{25+\xi^2}\,.$

(b) $\hat{u}(\xi) = \frac{\pi}{\sqrt{2}} e^{-|\xi-1|/\sqrt{2}}\,.$

(c) $\hat{u}(\xi) = -\frac{1}{2}\, i\xi\sqrt{\pi}\, e^{-\xi^2/4}\,.$

406. (a) $u(x) = \frac{1}{2\pi}\hat{\hat{u}}(-x) = \frac{4}{\pi}(x^2+4x+20)^{-1}\,.$

(b) $u(x) = \frac{ix+6}{4\sqrt{\pi}}\, e^{3ix-\frac{x^2}{4}}\,.$

(c) $u(x) = \frac{2}{\sqrt{\pi}}(1-2x^2)\, e^{-x^2}\,.$

407. $\hat{u}(\xi) = \left(\frac{2}{1+\xi^2}\right)^2.$

408. $\frac{2}{\sqrt{\pi}}\, e^{-2x^2}\,.$

409. $\frac{\pi}{2}\,.$

410. See the solution.

411. See the remark after Example 4.20.

412. (a) $E_1(x,t) = \partial_x E(x,t)$ and so, for $t > 0$,

$$(\partial_t - \partial_x^2)E_1 = (\partial_t - \partial_x^2)\partial_x E = \partial_x(\partial_t - \partial_x^2)E = 0\,,$$

because $(\partial_t - \partial_x^2)E = 0$ when $t > 0$ by Exercise 410.

(b) On the curves where x^2/t is constant, $-E_1$ increases unboundedly as $t \to +0$. Hence, the function E_1 is not a bounded solution.

413. $\dfrac{4\pi\cos\xi}{\pi^2-4\xi^2}$ when $\xi \neq \pm\dfrac{\pi}{2}$, $\hat{u}(\pi/2)=\hat{u}(-\pi/2)=1$.

414. $\frac{1}{i(\omega_0-\xi)}\, e^{i(\omega_0-\xi)a}\left(e^{i(\omega_0-\xi)T}-1\right) = \frac{2}{\omega_0-\xi} e^{i(\omega_0-\xi)(a+T/2)} \sin\frac{(\omega_0-\xi)}{2}T$.

415. $\dfrac{4}{\xi^3}(\sin\xi - \xi\cos\xi)$ when $\xi\neq 0$, $\hat{u}(0)=4/3$.

416. $\dfrac{a+b}{(a+i\xi)(b-i\xi)}$.

417. $\dfrac{1}{\pi}\dfrac{1-x^2}{(1+x^2)^2}$.

420. No.

424. $\dfrac{\pi}{s}\, e^{-s|\xi|+i\xi a}$.

425. (a) $\dfrac{-4i\xi}{(1+\xi^2)^2}$.

(b) $\dfrac{-i\pi\xi}{2}\, e^{-|\xi|}$.

(c) $\dfrac{\pi}{2}\dfrac{d}{d\xi}(\xi e^{-|\xi|}) = \dfrac{\pi}{2}(1-|\xi|)\, e^{-|\xi|}$.

(d) $\dfrac{\pi}{2}(1+|\xi|)\, e^{-|\xi|}$.

426. $\dfrac{-i}{\sqrt{2\pi}}\, e^{-\xi^2/2}$.

427. $\dfrac{\hat{u}(\xi)}{2} - \dfrac{1}{4}(\hat{u}(\xi-4) + \hat{u}(\xi+4))$.

428. (a) $\dfrac{2(\xi^2+2)}{\xi^4+4}$. (b) $\dfrac{2(2-\xi^2)}{\xi^4+4}$. (c) $\dfrac{\pi}{4}(\cos\xi+\sin|\xi|)$.

429. (a) $\dfrac{1}{(\alpha+i\xi)^2}$.

(b) $\dfrac{2}{(\alpha+i\xi)^3}$.

(c) $\left(i\dfrac{d}{d\xi}\right)^{k-1}\dfrac{1}{(\alpha+i\xi)^2} = \dfrac{k!}{(\alpha+i\xi)^{k+1}}$.

430. (a) $\sqrt{\pi}\, e^{-(\xi-2i)^2/4}$. (b) $\sqrt{\dfrac{\pi}{\alpha}} e^{-(\xi-i\beta)^2/4\alpha}$.

431. $\dfrac{2}{\pi(4+x^2)}$.

432. $\pi^{-1/4}\, 2^{1/2}\, e^{-2x^2}$.

433. $\dfrac{3\pi}{9+4x^2}$.

434. $\dfrac{3-4x^2}{2}\, e^{-x^2}$.

436. $\dfrac{1+|x|}{2}\, e^{-|x|}$.

437. $(4t+1)^{-1/2}\, e^{-x^2/(4t+1)}$.

438. $u(x,t) = \frac{1}{\sqrt{4\pi(t+s)}} e^{-x^2/(4(t+s))} = E(x, t+s)$.

439. $e^{-t}\cos x$.

Solutions to Exercises Marked with an Asterisk

Solutions for Chapter 1

105. (b) We begin by applying the ratio test.

$$\frac{(k+1)^a a^{k+1}}{k^a a^k} = \left(1+\frac{1}{k}\right)^a a \to a \quad \text{as} \quad k \to \infty\,.$$

This shows that the series is convergent when $a < 1$ and divergent when $a > 1$. When $a = 1$, the test gives no information. However, for this value of a the series is $\sum_1^\infty k$ which is divergent because the terms do not tend to zero.

Answer: The series is convergent exactly when $a < 1$.

120. (a) When $0 < x < \pi$, $\sin x > 0$, and then we have that $(\sin x)^{1/n} \to 1$ as $n \to \infty$. When $x = 0$ or $x = \pi$, $\sin x = 0$, and then the limit of $(\sin x)^{1/n}$ is 0 as $n \to \infty$. In summary,

$$\lim_{n\to\infty} = \begin{cases} 1 & \text{when } 0 < x < \pi \\ 0 & \text{when } x = 0 \text{ or } x = \pi\,. \end{cases}$$

(b) Since all the functions f_n are continuous in the interval $[0, \pi]$ and the limit function is not, the convergence cannot be uniform in $[0, \pi]$.

Answer: (a) The limit function equals 1 when $0 < x < \pi$ and equals 0 when $x = 0$ or $x = \pi$. (b) The convergence is not uniform.

123. (a) For $x > 0$ we have that

$$s(x) = \frac{x}{1+x}\sum_{k=0}^{\infty}\left(\frac{1}{1+x}\right)^k = \frac{x}{1+x}\cdot\frac{1}{1-\frac{1}{1+x}} = 1$$

(it is a geometric series with the ratio $\frac{1}{1+x}$ and $0 < \frac{1}{1+x} < 1$). When $x = 0$, all the terms are zero and so $s(x) = 0$. Hence, the series is convergent for all $x \ge 0$.

(b) When $x \ge d > 0$, we have that

$$0 < \frac{x}{(1+x)^{k+1}} = \frac{x}{1+x}\,\frac{1}{(1+x)^k} < \frac{1}{(1+x)^k} \le \left(\frac{1}{1+d}\right)^k.$$

The series

$$\sum_{k=0}^{\infty}\left(\frac{1}{1+d}\right)^k$$

is a convergent geometric series. Hence, by the Weierstrass M-test, the given series is uniformly convergent in the interval $x \geq d$.

(c) The terms $\frac{x}{(1+x)^{k+1}}$ are continuous functions of x in the interval $x \geq 0$. Hence, if the convergence were uniform, then the sum $s(x)$ would also be continuous in this interval. That is not the case because, according to (a), $s(x) = 1$ when $x > 0$ but $s(0) = 0$. Hence, the series is not uniformly convergent in the interval $x \geq 0$.

Alternative solution: The n^{th} partial sum $s_n(x)$ of the series may be calculated using the formula for a geometric sum. When $x > 0$, we have

$$s_n(x) = \sum_{k=0}^{n}\frac{x}{(1+x)^{k+1}} = 1 - \left(\frac{1}{1+x}\right)^{n+1},$$

whereas $s_n(0) = 0$. With $s(x)$ from (a) above, we get that

$$|s_n(x) - s(x)| = \begin{cases} \left(\frac{1}{1+x}\right)^{n+1} & \text{when } x > 0 \\ 0 & \text{when } x = 0\,. \end{cases}$$

This yields that

$$\sup_{x\geq d}|s_n(x) - s(x)| = \left(\frac{1}{1+d}\right)^{n+1} \to 0 \quad \text{as} \quad n \to \infty$$

while

$$\sup_{x\geq 0}|s_n(x) - s(x)| = 1 \neq 0 \quad \text{for all } n\,.$$

Hence, according to the definition, the series is uniformly convergent in the interval $x \geq d$ but not in $x \geq 0$.

126. We have that

$$\left|\frac{x^{k^2}}{k^2}\right| \leq \frac{1}{k^2} \quad \text{when} \quad |x| \leq 1\,.$$

Since $\sum_1^{\infty} k^{-2}$ is convergent, the function series $\sum_1^{\infty} x^{k^2}/k^2$ is uniformly convergent in $|x| \leq 1$ by the Weierstrass M-test.

130. In order to show that the sum of the series is continuous when $x > 0$, it is enough to show that the series is uniformly convergent in each interval $[d, D]$ where $d > 0$. In such an interval is

$$0 < \frac{1+\sqrt{k}x^2}{1+k^2\sqrt{x}} \leq \frac{1+\sqrt{k}D^2}{k^2\sqrt{d}} \leq \frac{\sqrt{k}+\sqrt{k}D^2}{k^2\sqrt{d}} \leq \frac{1+D^2}{\sqrt{d}} \cdot \frac{1}{k^{3/2}}.$$

The uniform convergence therefore follows from the Weierstrass M-test and the convergence of $\sum_1^\infty k^{-3/2}$.

136. It follows from

$$f(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots + a_{2k}x^{2k} + a_{2k+1}x^{2k+1} + \cdots$$

that

$$f(x^2) = a_0 + a_1 x^2 + \cdots + a_k x^{2k} + \cdots$$

and hence,

$$(1-x)f(x^2) = a_0 - a_0 x + a_1 x^2 - a_1 x^3 + \cdots + a_k x^{2k} - a_k x^{2k+1} + \cdots .$$

The condition $f(x) = (1-x)f(x^2)$ therefore means that

$$a_{2k} = a_k, \quad a_{2k+1} = -a_k \quad \text{when} \quad k \geq 0.$$

Furthermore, $a_0 = f(0) = 1$ and hence

$$a_1 = -a_0 = -1,\ a_2 = a_1 = -1,\ a_3 = -a_1 = 1,\ a_4 = a_2 = -1,\ \ldots .$$

Therefore $a_k = 1$ or $a_k = -1$ for all $k \geq 0$, whence $|a_k| = 1$ and so

$$\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| = |x| \to |x| \quad \text{as} \quad k \to \infty .$$

By the ratio test, the series is therefore convergent when $|x| < 1$ and divergent when $|x| > 1$, i.e. the radius of convergence is 1.

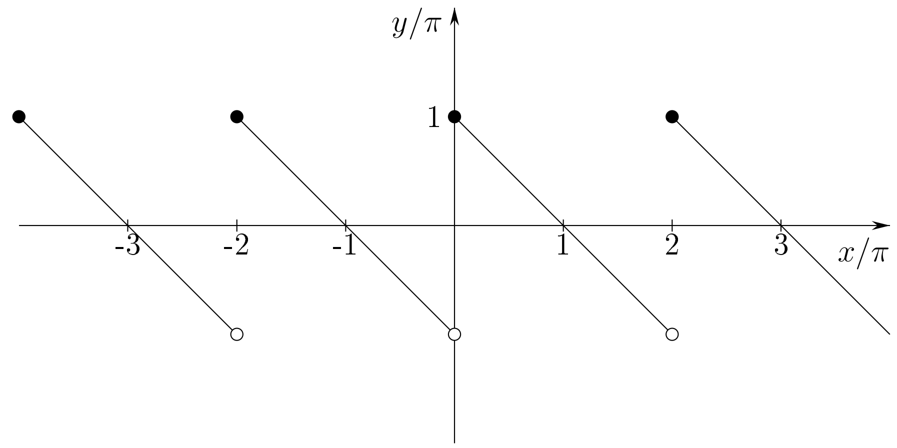
Answer: The radius of convergence is 1.

Remark: It can be shown that the sequence $\frac{a_{k+1}}{a_k}$ has no limit. The first twelve terms of the sequence are

$$-1,\quad 1,\ -1,\ -1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ -1.$$

Solutions for Chapter 2

201. (a) See the figure.



(b) If $n \neq 0$, then we have

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\, e^{-inx}\, dx \\ &= \frac{1}{2\pi}\left[(\pi - x)\frac{e^{-inx}}{-in}\right]_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} (-1)\frac{e^{-inx}}{-in}\, dx \\ &= \frac{1}{2\pi}\left(\frac{\pi}{in} + \frac{\pi}{in}\right) + 0 = -\frac{i}{n}, \end{aligned}$$

and $c_0 = \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\, dx = 0$, which can be seen directly from the figure.

202. (a) We wish to determine $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2 x\, e^{-inx}\, dx$ for each n. It is not particularly hard to evaluate these integrals directly but it is even simpler to rewrite $\cos^2 x$ using Euler's formulae:

$$\begin{aligned} \cos^2 x &= \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 = \frac{e^{2ix} + e^{-2ix} + 2}{4} \\ &= \frac{1}{4}e^{-2ix} + \frac{1}{2} + \frac{1}{4}e^{2ix}\,. \end{aligned}$$

Hence $c_{-2} = \frac{1}{4}$, $c_0 = \frac{1}{2}$, $c_2 = \frac{1}{4}$ and all other coefficients are 0. This follows from Theorem 2.2 that says that the coefficients c_n of a uniformly convergent series $\sum_{-\infty}^{\infty} c_n e^{inx}$ are equal to the Fourier coefficients of the sum of that series.

This is in particular true for a finite sum $s(x) = \sum_{k=-M}^{M} c_k e^{ikx}$ since the partial sums $s_N(x) = \sum_{k=-N}^{N} c_k e^{ikx}$ (where $c_k = 0$ when $|k| > M$) are identically equal to $s(x)$, and hence $||s_N - s||_{[0,2\pi]} = 0$, when $N > M$. Therefore, $||s_N - s||_{[0,2\pi]} \to 0$ as $N \to \infty$ and this shows that the convergence is uniform.

209. By (2.28), the Fourier coefficients of $u(x+\frac{\pi}{2})$ and $u(x-\frac{\pi}{2})$ are given by $e^{in\pi/2}c_n$ and $e^{-in\pi/2}c_n$, respectively. From Theorem 2.10 it follows that the Fourier coefficients of u' are inc_n. Hence the condition imposed on u implies that

$$2inc_n = e^{in\pi/2}c_n - e^{-in\pi/2}c_n .$$

We get that

$$nc_n = c_n \sin\frac{n\pi}{2}$$

or, equivalently, that

$$c_n\left(n - \sin\frac{n\pi}{2}\right) = 0 .$$

When $|n| > 1$, $|n - \sin\frac{n\pi}{2}| \geq |n| - 1 > 0$, and hence $c_n = 0$. When $n = 1$, $n - \sin\frac{n\pi}{2} = 1 - \sin\frac{\pi}{2} = 0$. Therefore c_1 may be chosen arbitrarily. In the same way we see that c_0 and c_{-1} are arbitrary because $n - \sin\frac{n\pi}{2} = 0$ when $n = 0$ or $n = -1$. In conclusion we see that u must be of the form

$$u(x) = c_0 + c_1 e^{ix} + c_{-1}e^{-ix} ,$$

where c_0, c_1 and c_{-1} are complex numbers.

Conversely, it is easy to check such functions u satisfy the condition.

213. The Fourier coefficients are equal to

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left(e^x + e^{-x}\right) e^{-inx}\,dx = \frac{1}{4\pi}\left[\frac{e^{x(1-in)}}{1-in} + \frac{e^{-x(1+in)}}{-(1+in)}\right]_{-\pi}^{\pi} \\
&= \frac{1}{4\pi}\left(\frac{e^{\pi}(-1)^n}{1-in} - \frac{e^{-\pi}(-1)^n}{1-in} - \frac{e^{\pi}(-1)^n}{1+in} + \frac{e^{\pi}(-1)^n}{1+in}\right) \\
&= \frac{(-1)^n}{4\pi}\left(e^{\pi}\left(\frac{1}{1-in} + \frac{1}{1+in}\right) - e^{-\pi}\left(\frac{1}{1-in} + \frac{1}{1+in}\right)\right) \\
&= \frac{(-1)^n}{4\pi}\cdot\frac{2}{1-(in)^2}\left(e^{\pi} - e^{-\pi}\right) = \frac{(-1)^n}{2\pi(1+n^2)}\left(e^{\pi} - e^{-\pi}\right) .
\end{aligned}$$

Since u is C^1, the Fourier inversion formula is valid for u, i.e.

$$u(x) = \frac{e^\pi - e^{-\pi}}{2\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n}{1+n^2}\, e^{inx}\,.$$

In particular, when $x = \pi$ we have that

$$u(\pi) = \frac{e^\pi - e^{-\pi}}{2\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n}{1+n^2}(-1)^n = \frac{e^\pi - e^{-\pi}}{2\pi} \sum_{-\infty}^{\infty} \frac{1}{n^2+1}\,.$$

Recalling that $u(\pi) = \frac{e^\pi + e^{-\pi}}{2}$, we see that

$$\sum_{-\infty}^{\infty} \frac{1}{n^2+1} = \pi \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}}\,.$$

Since the terms corresponding to $\pm n$ are equal, we have that

$$\sum_{-\infty}^{\infty} \frac{1}{n^2+1} = 1 + 2\sum_{1}^{\infty} \frac{1}{n^2+1}$$

and hence

$$\sum_{1}^{\infty} \frac{1}{n^2+1} = -\frac{1}{2} + \frac{\pi}{2}\, \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}}\,.$$

(Using the hyperbolic functions $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\coth x = \frac{\cosh x}{\sinh x}$ we can also write the answer as

$$\sum_{1}^{\infty} \frac{1}{n^2+1} = -\frac{1}{2} + \frac{\pi}{2} \coth \pi\,.)$$

Alternative computation: For the function $u(x) = \frac{e^x + e^{-x}}{2} = \cosh x$ we have that $u'(x) = \frac{e^x - e^{-x}}{2} = \sinh x$ and $u''(x) = \cosh x = u(x)$. (In particular, $v(x) = \sinh x$ is a primitive function of $u(x)$ and vice versa. In addition, u is an even function ($u(-x) = u(x)$) and v is an odd function ($v(-x) = -v(x)$).) These properties are easy to verify also without prior knowledge of the hyperbolic functions.

Since u is even, we have by symmetry that

$$2\pi c_n = \int_{-\pi}^{\pi} u(x)(\cos nx - i \sin nx)\, dx = \int_{-\pi}^{\pi} \cosh x \cos nx\, dx.$$

We can now compute the left-hand side, which we denote by I_n, by means of two partial integrations. We have in fact that

$$\begin{aligned} I_n &= [\sinh x \cos nx]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \sinh x\,(-n \sin nx)\,dx \\ &= 2\sinh \pi \cos n\pi + [\cosh x(n \sin nx)]_{-\pi}^{\pi} - n^2 I_n \end{aligned}$$

from which it follows that

$$(1+n^2)I_n = 2\sinh \pi\,(-1)^n + 0\,.$$

Hence $c_n = \frac{I_n}{2\pi} = \frac{\sinh \pi}{\pi}\frac{(-1)^n}{1+n^2}$ which is the expression obtained above.

227. As in Example 2.4 we find that $c_0 = \frac{\pi^2}{3}$ and $c_n = \frac{2(-1)^n}{n^2}$ when $n \neq 0$. Parseval's formula yields

$$\sum_{-\infty}^{\infty} |c_n|^2 = \left(\frac{\pi^2}{3}\right)^2 + \sum_{n\neq 0} \frac{4}{n^4} = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{1}{\pi}\int_0^{\pi} x^4\,dx = \frac{\pi^4}{5}$$

and hence

$$2\sum_{1}^{\infty} \frac{4}{n^4} = \frac{\pi^4}{5} - \frac{\pi^4}{9} = \frac{4\pi^4}{45}\,,$$

i.e. $\sum\limits_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

238. The minimum value is attained for $a = c_0$, $b = c_1$ and $c = c_{-1}$ where c_n are the Fourier coefficients of $u(x) = x^2$, $|x| \leq \pi$. From the computation in Example 2.4 we have that $c_0 = \frac{\pi^2}{3}$, $c_1 = c_{-1} = -2\,$.

From (2.24) and the preceeding lines it follows that

$$\left|\left| u - \sum_{-N}^{N} c_n e_n \right|\right|^2 = ||u||^2 - \left|\left| \sum_{-N}^{N} c_n e_n \right|\right|^2 = ||u||^2 - \sum_{-N}^{N} |c_n|^2$$

and hence the minimum value is

$$\begin{aligned} &\int_{-\pi}^{\pi} x^4\,dx - 2\pi(|c_0|^2 + |c_1|^2 + |c_{-1}|^2) \\ &\qquad = 2\left[\frac{x^5}{5}\right]_0^{\pi} - 2\pi\left(\frac{\pi^4}{9} + 4 + 4\right) = \frac{8\pi^5}{45} - 16\pi\,. \end{aligned}$$

242. (a) Using the change of variables $x = \frac{Ty}{2\pi}$ we get

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_0^{2\pi} v(y)e^{-iny}\,dy = \frac{1}{2\pi}\int_0^{2\pi} u\left(\frac{Ty}{2\pi}\right) e^{-iny}\,dy \\ &= \frac{1}{T}\int_0^T u(x)\,e^{-2\pi inx/T}\,dx\,. \end{aligned}$$

(b) The function $v(x) = u\left(\frac{Tx}{2\pi}\right)$ satisfies the assumptions of the inversion formula. Hence

$$u\left(\frac{Tx}{2\pi}\right) = \sum_{-\infty}^{\infty} c_n e^{inx} \quad \text{for all } x\,,$$

i.e.

$$u(y) = \sum_{-\infty}^{\infty} c_n e^{2\pi iny/T}\,.$$

(c) The function $v(x) = u\left(\frac{x}{\pi}\right)$ has period 2π and the Fourier coefficients of u are by the definition equal to those of v. Hence

$$\begin{aligned} c_n(u) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} v(x)\,e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(\frac{x}{\pi}\right) e^{-inx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1-\frac{|x|}{\pi}\right) e^{-inx}\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1-\frac{|x|}{\pi}\right)(\cos nx - i\sin nx)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi}\left(1-\frac{x}{\pi}\right)\cos nx\,dx \\ &= \frac{1}{\pi}\left(\left[\left(1-\frac{x}{\pi}\right)\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\left(-\frac{1}{\pi}\right)\frac{\sin nx}{n}\right) \\ &= 0 + \frac{1}{\pi^2}\int_0^{\pi}\frac{\sin nx}{n}\,dx = \frac{1}{n^2}\left[-\frac{\cos nx}{n^2}\right]_0^{\pi} \\ &= \frac{-(-1)^n+1}{n^2\pi^2}\,. \end{aligned}$$

Here we assumed that $n \neq 0$. When $n = 0$ we have

$$c_n = c_0 = \frac{1}{\pi}\int_0^{\pi}(1-\frac{x}{\pi})\,dx = \frac{1}{\pi}\left[\frac{1}{2}(1-\frac{x}{\pi})^2(-\pi)\right]_0^{\pi} = \frac{1}{2}.$$

(Those who find the construction of the primitive function too complicated to be done in one sweep can first make the change of variables $y = 1 - x/\pi$ (or $x = \pi(1-y)$) and get

$$c_n = -\int_1^0 y\,dy \;=\; \left[\frac{y^2}{2}\right]_0^1 \;=\; \frac{1}{2}\,.)$$

In conclusion we have that $c_0 = \frac{1}{2}$, $c_n = 0$ if $n \neq 0$ is even, and $c_n = \frac{2}{\pi^2(2k+1)^2}$ if $n = 2k+1$. Hence

$$u(x) = v(\pi x) \;=\; \sum_{-\infty}^{\infty} c_n e^{\pi i n x} \;=\; \frac{1}{2} \;+\; \frac{2}{\pi^2}\sum_{-\infty}^{\infty} \frac{e^{\pi i(2k+1)x}}{(2k+1)^2}\,.$$

When the method above is used there is no need to know the formula

$$c_n(u) \;=\; \frac{1}{T}\int_0^T u(x)\,e^{-2\pi i n x/T}\,dx$$

for the Fourier coefficients of a T-periodic function by heart.

(d) With $v(x) \;=\; u\left(\frac{Tx}{2\pi}\right)$ we have

$$\begin{aligned}
\sum_{-\infty}^{\infty} |c_n(u)|^2 \;&=\; \sum_{-\infty}^{\infty} |c_n(v)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |v(x)|^2\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(\frac{Tx}{2\pi})|^2\,dx \;=\; \frac{1}{2\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} |u(y)|^2\,\frac{2\pi}{T}dy \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |u(y)|^2\,dy\,.
\end{aligned}$$

(e) By (c) and (d) we have that

$$\begin{aligned}
\frac{1}{4} \;+\; \frac{4}{\pi^4}\sum_{k=-\infty}^{\infty} \frac{1}{(2k+1)^4} \;&=\; \frac{1}{2}\int_{-1}^{1}(1-|y|)^2\,dy \\
&= \left[-\frac{1}{3}(1-y)^3\right]_0^1 \;=\; \frac{1}{3}\,.
\end{aligned}$$

Hence

$$\frac{4}{\pi^4}\sum_{k=-\infty}^{\infty}\frac{1}{(2k+1)^4} \;=\; \frac{8}{\pi^4}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} \;=\; \frac{1}{3} - \frac{1}{4} \;=\; \frac{1}{12}$$

and therefore

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} \;=\; \frac{\pi^4}{96}\,.$$

244. (a) Since u is continuous and piecewise C^1 in the interval $[0, \pi]$ we have, according to Theorem 2.23, that

$$u(x) = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nx$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi u(x)\, \cos nx\, dx\,.$$

We get

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi u(x)\, dx = \frac{2}{\pi}\int_0^\pi x(\pi - x)\, dx = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^\pi \\ &= \frac{2}{\pi}\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) = \frac{\pi^3}{3} \end{aligned}$$

and when $n \neq 0$, iterated partial integration yields

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\, \cos nx\, dx \\ &= \frac{2}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n}\right]_0^\pi - \frac{2}{\pi}\int_0^\pi (\pi - 2x)\frac{\sin nx}{n}\, dx \\ &= 0 - \frac{2}{\pi}\left[(\pi - 2x)\frac{\cos nx}{-n^2}\right]_0^\pi + \frac{2}{\pi}\int_0^\pi (-2)\frac{\cos nx}{-n^2}\, dx \\ &= 2\frac{\cos n\pi}{-n^2} + 2\frac{\cos 0}{-n^2} + \frac{4}{\pi n^3}[\sin nx]_0^\pi = -2\frac{(-1)^n + 1}{n^2} \\ &= \begin{cases} 0 & \text{when } n \text{ is odd} \\ -\frac{1}{k^2} & \text{when } n = 2k\,. \end{cases} \end{aligned}$$

Thus we have that

$$u(x) = \frac{1}{2}a_0 + \sum_1^\infty a_n \cos nx = \frac{\pi^2}{6} - \sum_1^\infty \frac{\cos 2kx}{k^2}$$

and hence, when $0 \leq x \leq \pi$, that

$$x(\pi - x) = \frac{\pi^2}{6} - \sum_1^\infty \frac{\cos 2kx}{k^2}\,.$$

(b) Since $u(0) = u(\pi) = 0$ we can expand u in a sine series. According to (2.50) and (2.48) we have

$$u(x) \;=\; \sum_1^\infty b_n \sin nx$$

where

$$b_n \;=\; \frac{2}{\pi}\int_0^\pi u(x)\,\sin nx\,dx\,.$$

Integrating by parts, we find that (note that n is always positive)

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi x(\pi-x)\,\sin nx\,dx \\
&= \frac{2}{\pi}\left[x(\pi-x)\frac{\cos nx}{-n}\right]_0^\pi - \frac{2}{\pi}\int_0^\pi(\pi-2x)\frac{\cos nx}{-n}\,dx \\
&= 0 - \frac{2}{\pi}\left[(\pi-2x)\frac{\sin nx}{-n^2}\right]_0^\pi + \frac{2}{\pi}\int_0^\pi(-2)\frac{\sin nx}{-n^2}\,dx \\
&= -\frac{4}{\pi}\left[\frac{\cos nx}{n^3}\right]_0^\pi = -\frac{4((-1)^n-1)}{\pi n^3} \\
&= \begin{cases} 0 & \text{when } n \text{ is even} \\ \frac{8}{\pi(2k+1)^3} & \text{when } n = 2k+1\,. \end{cases}
\end{aligned}$$

Hence

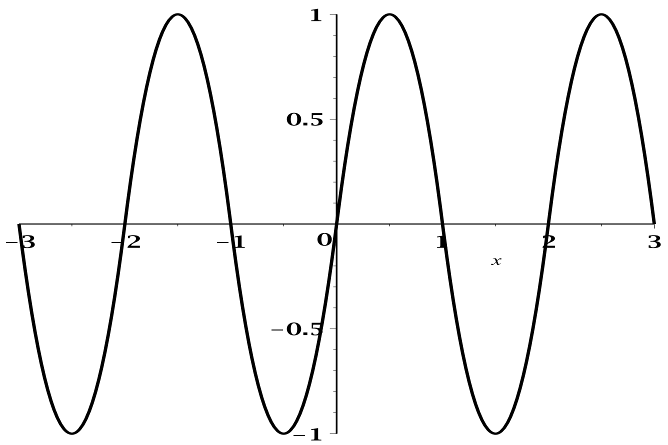
$$u(x) \;=\; \sum_{k=0}^\infty \frac{8}{\pi(2k+1)^3}\sin{(2k+1)x}$$

and therefore, when $0 \le x \le \pi$,

$$x(\pi-x) \;=\; \frac{8}{\pi}\sum_0^\infty \frac{\sin{(2k+1)x}}{(2k+1)^3}\,.$$

Remark: In Exercise 244 we used "ready-made formulae" for the cosine series and sine series expansions and for the coefficients a_n and b_n. Those who do not want to burden their memory with these formulae manage fine doing the simple derivation every time. We illustrate this for (b):

The graph of the extended 2π-periodic function appears in the figure (in which the x-axis is graded in multiples of π and the y-axis in multiples of $u(\frac{\pi}{2}) = \frac{\pi^2}{4}$):



(Here $u(x) = x(\pi + x)$ when $-\pi \le x \le 0$.) The Fourier coefficients of the extended function are

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\, e^{-inx}\, dx \\ &= \frac{1}{2\pi}\left(\int_{-\pi}^{0} u(x)e^{-inx}\, dx + \int_{0}^{\pi} u(x)e^{-inx}\, dx\right) \\ &= \frac{1}{2\pi}\int_{0}^{\pi} (u(-x)e^{inx} + u(x)e^{-inx})\, dx \\ &= \frac{1}{2\pi}\int_{0}^{\pi} u(x)(-e^{inx} + e^{-inx})\, dx \\ &= -\frac{i}{\pi}\int_{0}^{\pi} u(x)\, \sin nx\, dx = -\frac{i}{\pi}\int_{0}^{\pi} x(\pi - x)\, \sin nx\, dx \end{aligned}$$

which are equal to $\frac{2i((-1)^n-1)}{\pi n^3}$ when $n \neq 0$, as was shown above, and further $c_0 = 0$. When $0 \le x \le \pi$, we therefore get that

$$\begin{aligned} x(\pi - x) &= \sum_{n\neq 0} c_n e^{inx} = \sum_{1}^{\infty} (c_n e^{inx} + c_{-n} e^{-inx}) \\ &= \sum_{1}^{\infty} 2ic_n \sin nx = \frac{8}{\pi}\sum_{0}^{\infty} \frac{\sin{(2k+1)x}}{(2k+1)^3} \end{aligned}$$

using that $c_{-n} = -c_n$ and that $c_{2k} = 0$.

Another alternative is to use our knowledge of (finite-dimensional) linear algebra in order to reconstruct the formula for the expansion of u. If U is a vector in $\mathbb{R}^N$ and $V_1, \ldots, V_N$ is an orthogonal basis, then

$$U = \frac{(U, V_1)}{(V_1, V_1)}V_1 + \cdots + \frac{(U, V_N)}{(V_N, V_N)}V_N\,.$$

(If the basis elements are normalised, then all the denominators are equal to 1.) We know that the functions $v_k(x) = \sin kx$, $k = 1, 2, 3, \ldots$, form a complete orthogonal set, which is one of the possible generalisations of the concept of orthogonal basis in a finite-dimensional space, for the piecewise continuous functions with respect to the scalar product $(u, w) = \int_0^\pi u(x)\overline{w(x)}\, dx$. In analogy with the formula for U above, we have

$$u(x) \;=\; \sum_{k=1}^{\infty} \frac{(u, v_k)}{(v_k, v_k)} v_k(x) \;=\; \sum_{k=1}^{\infty} b_k \sin kx$$

where

$$b_k \;=\; \frac{(u, v_k)}{(v_k, v_k)} \;=\; \frac{\int_0^\pi u(x) \sin kx\, dx}{\int_0^\pi \sin^2 kx\, dx} \;=\; \frac{2}{\pi} \int_0^\pi u(x) \sin kx\, dx\, .$$

In the last step, we used that $\int_0^\pi \sin^2 kx\, dx \;=\; \frac{\pi}{2}$.

Solutions for Chapter 3

301. According to Theorem 3.4,

$$u(x,t) = \sum_1^\infty b_n e^{-n^2 t} \sin nx$$

is a solution of the problem. Here b_n are the coefficients of the expansion in sine series of the initial values $u(x,0) = f(x) = \sin x + 2\sin 3x$. Since the coefficients are uniquely determined, $b_1 = 1$, $b_3 = 2$ and all the other coefficients are zero. Hence,

$$u(x,t) = e^{-t}\sin x + 2e^{-9t}\sin 3x\,.$$

(The theorem also guarantees that there are no other solutions.)

Remark: In this case the initial values f were already given as a (finite) sine series and so we were able to read off the coefficients b_n. In general, a calculation is needed in order to find these coefficients.

307. According to Theorem 3.5, we can immediately write down the solution as soon as we have expanded $u(x,0) = x(\pi - x)$ in a cosine series. As in Exercise 244 (a) we get

$$x(\pi - x) = \frac{\pi^2}{6} - \sum_{k=1}^\infty \frac{\cos 2kx}{k^2} \quad \text{when} \quad 0 \le x \le \pi\,.$$

(This is where the actual work is done. Confer the remark to the solution to Exercise 301 above.) Hence,

$$u(x,t) = \frac{\pi^2}{6} - \sum_{k=1}^\infty e^{-4k^2 t}\frac{\cos 2kx}{k^2}$$

is a solution of the problem.

310. This time we get the solution using Theorem 3.6:

$$u(x,t) = \sum_0^\infty b_k e^{-(2k+1)^2 t}\sin{(2k+1)x}$$

where b_k are the coefficients of the expansion

$$u(x,0) = \sum_0^\infty b_k \sin{(2k+1)x}\,, \quad 0 \le x \le \frac{\pi}{2}\,.$$

(Cf. Theorem 2.27.) In our case, $u(x,0) = \sin 3x$ is already of this form with $b_1 = 1$ and $b_k = 0$, $k \neq 1$, and so we have no use of the formula $b_k = \frac{4}{\pi}\int_0^\pi u(x,0)\sin(2k+1)x\,dx$. Hence, the solution is

$$u(x,t) = e^{-9t}\sin 3x\,.$$

316. Using the notation of (3.34)–(3.36), we have $c = 5$ and $l = 1$. Set, as in (3.38), $U(x,t) = u\left(\frac{x}{\pi}, \frac{t}{5\pi}\right)$. Then U solves the equation $\partial_t^2 U = \partial_x^2 U$ and satisfies

$$\begin{aligned} U(x,0) &= f\left(\frac{x}{\pi}\right) = \sin 2x\,,\\ \partial_t U(x,0) &= \frac{1}{5\pi} g\left(\frac{x}{\pi}\right) = \frac{1}{5\pi}(\sin x - 2\,\sin 3x)\,. \end{aligned}$$

These initial values are already expressed as sine series, and we get immediately (using the notation of Theorem 3.9) that $A_2 = 1$, $B_1 = \frac{1}{5\pi}$, $B_3 = -\frac{2}{5\pi}$ and that all other coefficients A_n and B_n are zero. The theorem now yields that

$$U(x,t) = \frac{1}{5\pi}\sin t \sin x + \cos 2t \sin 2x - \frac{2}{15\pi}\sin 3t \sin 3x$$

and hence

$$\begin{aligned} u(x,t) &= U(\pi x, 5\pi t)\\ &= \frac{1}{5\pi}\sin 5\pi t \sin x + \cos 10\pi t \sin 2\pi x - \frac{2}{15\pi}\sin 15\pi t \sin 3\pi x\,. \end{aligned}$$

Solutions for Chapter 4

401. We have that

$$\begin{aligned}\hat{u}(\xi) &= \int_{-1}^{1}(1-|x|)^2 e^{-ix\xi}\,dx = \int_0^1 (1-|x|)^2(e^{-ix\xi}+e^{ix\xi})\,dx \\ &= \int_0^1 (1-2x+x^2)\,2\cos x\xi\,dx\,.\end{aligned}$$

When $\xi \neq 0$, we integrate by parts and get

$$\begin{aligned}\hat{u}(\xi) &= \left[(1-2x+x^2)\,2\frac{\sin x\xi}{\xi}\right]_0^1 - \int_0^1 (-2+2x)\,2\frac{\sin x\xi}{\xi}\,dx \\ &= 0 - 0 - 4\left[(x-1)\,\frac{-\cos x\xi}{\xi^2}\right]_0^1 - 4\int_0^1 \frac{\cos x\xi}{\xi^2}\,dx \\ &= \frac{4}{\xi^2} - 4\left[\frac{\sin x\xi}{\xi^3}\right]_0^1 = \frac{4}{\xi^2} - \frac{4\sin x\xi}{\xi^3} = \frac{4}{\xi^3}(\xi - \sin\xi)\,.\end{aligned}$$

Furthermore, $\hat{u}(0) = 2\int_0^1 (1-x)^2\,dx = 2\left[-\frac{(1-x)^3}{3}\right]_0^1 = 2/3$.

Alternatively, we can evaluate $\hat{u}(0)$ using that $\hat{u}$ is continuous:

$$\begin{aligned}\hat{u}(0) &= \lim_{\xi\to 0}\hat{u}(\xi) = \lim_{\xi\to 0}\frac{4}{\xi^3}(\xi-\sin\xi) \\ &= \lim_{\xi\to 0}\frac{4}{\xi^3}(\xi - \xi + \xi^3/6 + \xi^5 B_5(\xi)) = \lim_{\xi\to 0}(\frac{2}{3} + \xi^2 B_5(\xi)).\end{aligned}$$

Here $B_5(\xi)$ is bounded in a neighbourhood of $\xi = 0$ and therefore $\hat{u}(0) = 2/3$.

402. (a)

$$\begin{aligned}\hat{u}(\xi) &= \int_0^{2\pi} \sin x\, e^{-ix\xi}\,dx = \int_0^{2\pi} \frac{e^{ix}-e^{-ix}}{2i}\, e^{-ix\xi}\,dx \\ &= \frac{1}{2i}\int_0^{2\pi}\left(e^{ix(1-\xi)} - e^{-ix(1+\xi)}\right)\,dx\,.\end{aligned}$$

When $\xi \neq \pm 1$, we have that

$$\begin{aligned}\hat{u}(\xi) &= \frac{1}{2i}\left[\frac{e^{ix(1-\xi)}}{i(1-\xi)} - \frac{e^{-ix(1+\xi)}}{-i(1+\xi)}\right]_0^{2\pi} \\ &= \frac{1}{2i}\left(\frac{e^{i2\pi(1-\xi)}-1}{i(1-\xi)} + \frac{e^{-i2\pi(1+\xi)}-1}{i(1+\xi)}\right) \\ &= -\frac{1}{2}\left(\frac{e^{-i2\pi\xi}-1}{1-\xi} + \frac{e^{-i2\pi\xi}-1}{1+\xi}\right) = \frac{1-e^{-i2\pi\xi}}{1-\xi^2}\,.\end{aligned}$$

In the next-to-last equality was used that $e^{\pm i2\pi} = 1$. Furthermore,

$$\hat{u}(1) = \frac{1}{2i}\int_0^{2\pi}(1 - e^{-2ix})\,dx = \frac{1}{2i}\left(2\pi - \left[\frac{e^{-2ix}}{-2i}\right]_0^{2\pi}\right)$$
$$= \frac{\pi}{i} = -i\pi\,.$$

We obtain in the same way that $\hat{u}(-1) = i\pi$. (Check this by evaluating $\lim_{\xi\to -1}\hat{u}(\xi)$!)

(b) Since

$$|\hat{u}(\xi)| \le \int_{-\infty}^{\infty}|u(x)|\,dx\,,$$

we get that

$$|\hat{u}(\xi)| \le \int_0^{2\pi}|\sin x|\,dx = 2\int_0^{\pi}\sin x\,dx = 4\,.$$

403. Since

$$1 - |x| \ge 0 \iff |x| \le 1\,,$$

we have that

$$\max(1 - |x|, 0) = \begin{cases} 1 - |x| & \text{when} \quad |x| \le 1 \\ 0 & \text{when} \quad |x| > 1\,. \end{cases}$$

(a) Since u is an even function, we get (when $\xi \neq 0$) that

$$\hat{u}(\xi) = \int_{-\infty}^{\infty} u(x)(\cos x\xi - i\sin x\xi)\,dx = 2\int_0^1 (1-x)\cos x\xi\,dx$$
$$= 2\left[(1-x)\frac{\sin x\xi}{\xi}\right]_0^1 + 2\int_0^1 \frac{\sin x\xi}{\xi}\,dx$$
$$= 2(0-0) + 2\left[-\frac{\cos x\xi}{\xi^2}\right]_0^1 = -\frac{2}{\xi^2}(\cos\xi - 1)$$
$$= \frac{2(1-\cos\xi)}{\xi^2}\,.$$

Finally we have that $\hat{u}(0) = \int_{-1}^{1} u(x)\,dx = 1$, since the integral expresses the area of a triangle with base 2 and height 1.

(b) The function $u(x)$ is continuous, bounded and integrable over $\mathbb{R}$. Furthermore, $\hat{u}$ is integrable over $\mathbb{R}$, since $\hat{u}(\xi)$ is continuous and

$$|\hat{u}(\xi)| \le 2(1 + |\cos\xi|)/\xi^2 \le 4/\xi^2$$

where the right-hand side is integrable over $|\xi| \geq 1$. Consequently, the assumptions of the inversion formula are fulfilled and we get

$$\begin{aligned} u(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi)\, e^{ix\xi}\, d\xi \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2(1-\cos\xi)}{\xi^2}\, e^{ix\xi}\, d\xi \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos\xi}{\xi^2}\cos x\xi\, d\xi \;+\; \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos\xi}{\xi^2}\sin x\xi\, d\xi\,. \end{aligned}$$

The integrand of the second integral is odd, hence that integral is zero. Therefore we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos\xi}{\xi^2}\cos x\xi\, d\xi \;=\; u(x) \;=\; \begin{cases} 1-|x|\,, & |x| \leq 1 \\ 0\,, & |x| > 1\,. \end{cases}$$

(c) Setting $x = 0$ in (b), we get

$$\begin{aligned} 1 &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos\xi}{\xi^2}\, d\xi \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\sin^2\frac{\xi}{2}}{\xi^2}\, d\xi \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2\sin^2 t}{(2t)^2}\, 2dt \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt\,. \end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt \;=\; \pi.$$

404. (a) We have that

$$\hat{u}(-\xi) \;=\; \int_{-\infty}^{\infty} u(x)\, e^{-ix(-\xi)}\, dx \;=\; \int_{-\infty}^{\infty} u(-y)\, e^{-iy\xi}\, dy\,.$$

In the last equality we made the change of variables $x = -y$. Using that u is even, we get that

$$\hat{u}(-\xi) \;=\; \int_{-\infty}^{\infty} u(y)e^{-iy\xi}\, dy \;=\; \hat{u}(\xi)\,,$$

i.e. $\hat{u}$ is even.

(b) Analogously, when u is odd we get

$$\hat{u}(-\xi) = \int_{-\infty}^{\infty} u(-y)e^{-iy\xi}\, dy = -\int_{-\infty}^{\infty} u(y)e^{-iy\xi}\, dy = -\hat{u}(\xi)\,,$$

i.e. also $\hat{u}$ is odd.

405. (a) With $f(x) = e^{-|x|}$ we have $u(x) = f(5x)$ and by 5. in Theorem 4.7 we get

$$\hat{u}(\xi) = \frac{1}{5}\hat{f}(\frac{\xi}{5}) = \frac{1}{5}\frac{2}{1+(\frac{\xi}{5})^2} = \frac{10}{25+\xi^2}.$$

(b) Set $g(x) = \frac{1}{1+x^2}$. Since $e^{-|x|}$ has the Fourier transform $\frac{2}{1+\xi^2}$, the Fourier transform of $\frac{2}{1+\xi^2}$ is $2\pi e^{-|-x|}$ by the inversion formula. By changing the roles of x and ξ, we get that

$$\hat{g}(\xi) = \frac{1}{2}2\pi\, e^{-|\xi|} = \pi\, e^{-|\xi|}.$$

Now, $u(x) = e^{ix} \cdot g(x\sqrt{2})$ and hence, by 4. and 5. in Theorem 4.7,

$$\hat{u}(\xi) = \frac{1}{\sqrt{2}}\hat{g}(\frac{\xi-1}{\sqrt{2}}) = \frac{\pi}{\sqrt{2}}e^{-|\xi-1|/\sqrt{2}}.$$

(c) With $h(x) = e^{-x^2}$ we have that $-iu(x) = -ixh(x)$. Hence, by 2. in Theorem 4.7,

$$-i\hat{u}(\xi) = \frac{d}{d\xi}\hat{h}(\xi) = \frac{d}{d\xi}\sqrt{\pi}\, e^{-\xi^2/4} = \sqrt{\pi}\,(-\frac{\xi}{2})e^{-\xi^2/4}.$$

We can also solve this exercise by observing that $u(x) = -\frac{1}{2}\frac{d}{dx}e^{-x^2}$ and using 1. in Theorem 4.7:

$$\hat{u}(\xi) = -\frac{1}{2}\, i\xi\, \sqrt{\pi}\, e^{-\xi^2/4}.$$

406. (a) Since the Fourier transform of $e^{-|\xi|}$ is $\frac{2}{1+x^2}$, 4. and 5. in Theorem 4.7 yield

$$\hat{\hat{u}}(x) = \frac{1}{4}\cdot\frac{2}{1+(\frac{x-2}{4})^2} = \frac{8}{16+(x-2)^2} = \frac{8}{x^2-4x+20}.$$

Hence, it follows from the inversion formula that

$$u(x) = \frac{1}{2\pi}\hat{\hat{u}}(-x) = \frac{4}{\pi}\frac{1}{x^2+4x+20}.$$

(b) The Fourier transform of $e^{-(\xi-3)^2}$ is

$$\sqrt{\pi}\, e^{-x^2/4} \cdot e^{-3ix}$$

by 3. in Theorem 4.7. From 2. in the same theorem it follows that

$$\hat{\hat{u}}(x) = i\,\frac{d}{dx}(\sqrt{\pi}\,e^{-3ix}\,e^{-x^2/4}) = i\sqrt{\pi}(-3i - \frac{x}{2})\,e^{-3ix-\frac{x^2}{4}}\,.$$

Hence, by the inversion formula,

$$u(x) = \frac{1}{2\pi}\,i\sqrt{\pi}\,e^{3ix-\frac{x^2}{4}}(-3i + \frac{x}{2}) = \frac{ix+6}{4\sqrt{\pi}}\,e^{3ix-\frac{x^2}{4}}\,.$$

(c) The Fourier transform of $e^{-\xi^2/4}$ is $2\sqrt{\pi}\,e^{-x^2}$ by Example 4.8. Using 2. in Theorem 4.7 twice, we get

$$\begin{aligned}\hat{\hat{u}}(x) &= i^2\frac{d^2}{dx^2}(2\sqrt{\pi}\,e^{-x^2}) = -2\sqrt{\pi}\frac{d}{dx}(-2x\,e^{-x^2})\\ &= 4\sqrt{\pi}(e^{-x^2} + x(-2x)e^{x^2}) = 4\sqrt{\pi}(1-2x^2)e^{-x^2}\,.\end{aligned}$$

Hence,

$$u(x) = \frac{1}{2\pi}\hat{\hat{u}}(-x) = \frac{2}{\sqrt{\pi}}(1-2x^2)e^{-x^2}\,.$$

407. With $v(x) = e^{-|x|}$ we have $u = v * v$. Theorem 4.10 yields

$$\hat{u}(\xi) = \hat{v}(\xi)\cdot\hat{v}(\xi) = (\frac{2}{1+\xi^2})^2\,.$$

408. Set $v(x) = e^{-2x^2}$. Then $\hat{v}(\xi) = \sqrt{\frac{\pi}{2}}\,e^{-\xi^2/8}$. The equation can be written as

$$(u * v)(x) = e^{-x^2}\,,$$

whence Fourier transformation gives

$$\widehat{u*v}(\xi) = \hat{u}(\xi)\cdot\hat{v}(\xi) = \hat{u}(\xi)\sqrt{\frac{\pi}{2}}\,e^{-\xi^2/8} = \sqrt{\pi}\,e^{-\xi^2/4}\,.$$

From this it follows that

$$\hat{u}(\xi) = \sqrt{2}\,e^{-\xi^2/8}\,.$$

Now we could use a Fourier transformation and the inversion formula once again in order to find u. Here, however, a simpler approach is to compare with the first line of this solution. We see that

$$\hat{u}(\xi) = \sqrt{2}\sqrt{\frac{2}{\pi}}\,\hat{v}(\xi) = \frac{2}{\sqrt{\pi}}\,\hat{v}(\xi)\,.$$

Hence,

$$u(x) = \frac{2}{\sqrt{\pi}}\,e^{-2x^2}\,.$$

409. The function $u(x) = \frac{1}{1+x^2}$ is continuous, integrable and bounded in $\mathbb{R}$. Hence, the assumptions of Theorem 4.11 are fulfilled. Furthermore $\hat{u}(\xi) = \pi\, e^{-|\xi|}$ and so

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} \;&=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \pi^2 e^{-2|\xi|}\, d\xi \;=\; \pi \int_0^{\infty} e^{-2\xi}\, d\xi \\ &=\; \pi \left[\frac{e^{-2\xi}}{-2}\right]_0^{\infty} \;=\; \frac{\pi}{2}\,.\end{aligned}$$

410. It follows from

$$E(x,t) = \frac{1}{\sqrt{4\pi}}(t^{-1/2}\, e^{-x^2/4t})$$

that

$$\begin{aligned}\partial_t E(x,t) \;&=\; \frac{1}{\sqrt{4\pi}}(-\frac{1}{2}t^{-3/2}\, e^{-x^2/4t} + t^{-1/2}(-\frac{x^2}{4}(-t^{-2}))\, e^{-x^2/4t}) \\ &=\; \frac{t^{-3/2}e^{-x^2/4t}}{\sqrt{4\pi}}\,(-\frac{1}{2} + \frac{x^2}{4t})\,, \\ \partial_x E(x,t) \;&=\; \frac{1}{\sqrt{4\pi}}(t^{-1/2}\, e^{-x^2/4t})\,(-\frac{2x}{4t}) \;=\; \frac{t^{-3/2}}{\sqrt{4\pi}}(-\frac{x}{2})\, e^{-x^4/4t}\,, \\ \partial_x^2 E(x,t) \;&=\; \frac{t^{-3/2}}{\sqrt{4\pi}}\,(-\frac{1}{2} - \frac{x}{2}(-\frac{x}{2t}))\, e^{-x^2/4t}\,,\end{aligned}$$

and we see that $\partial_t E \;=\; \partial_x^2 E\,.$