LECTURE 2

2. Counting functions and relations

Definition 2.1. Let A and B be two sets. Then

$$A \times B = \{(a, b) \mid a \in A,\, b \in B\}$$

is called the *Cartesian product* (or *product set*) of A and B.

Example 2.2. (a) The set $\mathbb{R} \times \mathbb{R}$, also written $\mathbb{R}^2$, is the Cartesian product of $\mathbb{R}$ with itself.

(b) Let $A = \{1, 2, 3\}$ and $B = \{3, 5\}$. Then

$$A \times B = \{(1, 3), (2, 3), (3, 3), (1, 5), (2, 5), (3, 5)\}\}.$$

Definition 2.3. A *relation* $\mathcal{R}$ from A to B is a subset $\mathcal{R} \subseteq A \times B$. A *binary relation* $\mathcal{R}$ on A is a subset $\mathcal{R} \subseteq A \times A$.

Example 2.4. Set $A = B = \mathbb{Z}$ and $\mathcal{R} = \leq$. Then $2\mathcal{R}3$ and $4\mathcal{R}4$, but $(-1)\not\mathcal{R}(-3)$.

Theorem 2.5. *Let A and B be sets. If $|A| = m$ and $|B| = n$ then $|A \times B| = mn$ and the number of relations from A to B is 2^{mn}.*

Proof. It follows directly from the definition of $A \times B$ that $|A \times B| = mn$.

Next, each element of $A \times B$ is either in $\mathcal{R}$ or not. This gives 2^{mn} possibilities.

[An alternative proof for the last part: The number of subsets of $A \times B$ containing k elements is $\binom{mn}{k}$. This means that the total number of relations from A to B is

$$\sum_{k=0}^{mn} \binom{mn}{k} = \sum_{k=0}^{mn} \binom{mn}{k} 1^k 1^{(mn-k)} = (1+1)^{mn} = 2^{mn};$$

here we have used the Binomial Theorem.] □

Definition 2.6. A relation $\mathcal{R}$ from A to B is called a *function* from A to B if for each $a \in A$ there is a unique $b \in B$ such that $(a, b) \in \mathcal{R}$.

Note that one generally writes $f(a) = b$ in the above setting.

Example 2.7. All functions from Calculus.

Theorem 2.8. *Let A and B be sets with $|A| = m$ and $|B| = n$. Then there are n^m functions from A to B.*

Proof. For each $a \in A$ there are n choices for the image element in B. Hence, from the rule of product, we have n^m choices in total. □

Theorem 2.9. *Let A and B be sets with $|A| = m$ and $|B| = n$, where $m \leq n$. The number of injective functions from A to B is $P(n, m)$.*

Proof. Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$. The problem of determining an injective function $f : A \to B$ is equivalent to determining an m-permutation of B. Hence the total number is $P(n, m)$.

[An alternative phrasing: There are n possibilities for choosing $f(a_1)$. As we have already used up one element for the image of $f(a_1)$, we are left with $n-1$ possibilities for choosing $f(a_2)$. Continuing in this manner, we see that we have $n-2$ possibilities for choosing $f(a_3)$, etc, with $n-m+1$ possibilities for choosing $f(a_m)$. Therefore, in total we have

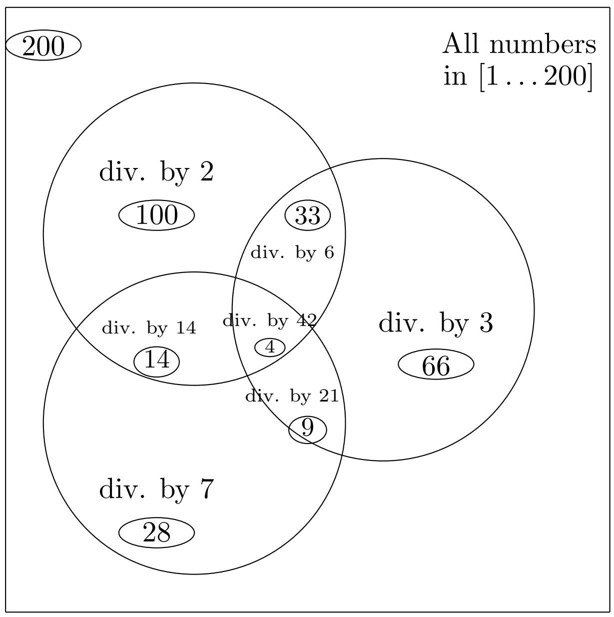
$$n(n-1)\cdots(n-m+1) = \frac{n!}{(n-m)!},$$

as required.] $\square$

Next, in order to compute the number of surjective functions from A to B, we first need the principle of inclusion and exclusion.

Example 2.10. How many numbers in $[1 \dots 200]$ are not divisible by any of the primes 2, 3, and 7?

Solution:



The answer is

$$200 - (100 + 66 + 28) + (14 + 33 + 9) - 4 = 58,$$

where we remove, from our 200 numbers, all those that are divisible by 2, divisible by 3 and divisible by 7 (that is, 100, 66 and 28 numbers respectively), then we add back those numbers that were removed twice in the previous step (i.e. 14, 33 and 9) and lastly, we remove those in the middle because they were counted $1-3+3=1$ time, and should be excluded.

Example 2.11. Let $A = \{x, y, z, w\}$ and $B = \{1, 2, 3\}$. How many surjective functions are there from A to B?

Solution. According to Theorem 2.8, the number of functions from A to B is 3^4. If $f : A \to B$ is not surjective then the image $f(A)$ is a proper subset of B. The proper subsets are

$$C_1 = \{1,2\}, \quad C_2 = \{2,3\}, \quad C_3 = \{1,3\}$$
$$D_1 = \{1\}, \quad D_2 = \{3\}, \quad D_3 = \{3\}, \quad E = \varnothing.$$

The number of functions from A to C_k is 2^4 (as for each element x, y, z, w in A, there are two choices for where they get mapped to), and similarly the number of functions from A to D_k

is 1^4. Therefore, as in the previous example, the total number of surjective functions from A to B is

$$3^4 - \binom{3}{2}2^4 + \binom{3}{1}1^4.$$

Theorem 2.12 (Inclusion-Exclusion Principle)**.** *Consider a set S, with $|S| = N$, and conditions c_i, for $1 \leq i \leq t$, each of which may be satisfied by some of the elements of S. The number of elements of S that satisfy none of the conditions c_i, $1 \leq i \leq t$, is denoted by $\overline{N} = N(\overline{c}_1\overline{c}_2\cdots\overline{c}_t)$ where*

$$\overline{N} = N - \sum_{1\leq i\leq t} N(c_i) + \sum_{1\leq i<j\leq t} N(c_ic_j) - \sum_{1\leq i<j<k\leq t} N(c_ic_jc_k) + \cdots + (-1)^t N(c_1c_2\cdots c_t). \tag{2.1}$$

Proof. For each $x \in S$ we will show that x contributes the same count, either 0 or 1, to each side of (2.1).

If x satisfies none of the conditions, then x is counted once in $\overline{N}$ and once in N, but not in any of the other terms in (2.1). Consequently x contributes a count of 1 to each side of the equation.

The other possibility is that x satisfies exactly r of the conditions where $1 \leq r \leq t$. In this case x contributes nothing to $\overline{N}$. However, on the right-hand side of (2.1), the element x is counted

- one time in N;
- r times in $\sum_{1\leq i\leq t} N(c_i)$ (once for each of the r conditions);
- $\binom{r}{2}$ times in $\sum_{1\leq i<j\leq t} N(c_ic_j)$ (once for each pair of conditions selected from the r conditions it satisfies);
- $\binom{r}{3}$ times in $\sum_{1\leq i<j<k\leq t} N(c_ic_jc_k)$;

$\vdots$

- $\binom{r}{r} = 1$ time in $\sum N(c_{i_1}c_{i_2}\cdots c_{i_r})$, where the summation is taken over all selections of size r from the t conditions.

Consequently, on the right-hand side of (2.1), the element x is counted

$$1 - r + \binom{r}{2} - \binom{r}{3} + \cdots + (-1)^r\binom{r}{r} = [1 + (-1)]^r = 0^r = 0 \text{ times},$$

by the Binomial Theorem. Therefore, the two sides of (2.1) count the same elements from S, and the equality holds. $\square$

Remark 2.13. For convenience, we will use the notation

$$S_k = \sum_{1\leq i_1<i_2<\cdots<i_k\leq t} N(c_{i_1}c_{i_2}\cdots c_{i_k}) \qquad \text{for } 1 \leq k \leq t,$$

and $S_0 = N$.

Example 2.14. How many arrangements of the letters in FRAGMENT do not contain any of the strings TAG, ME, or MAG?

Solution: Let S be the set of all permutations of the eight different letters. Then $S_0 = |S| = 8!$.

Let c_1 be the set of all permutations containing TAG, equivalently, all permutations of TAG, F, R, M, E, N. So $N(c_1) = 6!$.

Let c_2 be the set of all permutations containing ME. Here $N(c_2) = 7!$.

Finally, let c_3 be the set of all permutations containing MAG. Here $N(c_3) = 6!$.

So $S_1 = N(c_1) + N(c_2) + N(c_3) = 7! + 2 \cdot 6!$.

Further,

$$S_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) = 5! + 0 + 0;$$

indeed, here $N(c_1c_2)$ is the number of permutations of TAG, ME, F, R, N. Certainly as TAG and MAG cannot both occur, it follows that $N(c_1c_3)$ is zero, and similarly for $N(c_2c_3)$ since both ME and MAG cannot occur.

Similarly, note also that $S_3 = 0$.

Therefore, the answer is

$$N(\overline{c_1c_2c_3}) = S_0 - S_1 + S_2 - S_3 = 8! - (7! + 2 \cdot 6!) + 5! = 33960.$$

Remark. As seen in the above example, one can interchangeably think of c_i as a condition or a subset. Indeed, given a condition c_i, one can define the subset $C_i := \{x \in S \mid x \text{ satisfies } c_i\}$. Conversely, given a subset $C_i \subseteq S$, one can let c_i be the defining condition of the subset C_i. In this spirit, note that one can find (on the internet or in books) a version of the inclusion-exclusion principle that is phrased in terms of sets and their intersections.

Example 2.15. In how many ways can we arrange the numbers 1, 2, ..., 10 so that 1 is not in the first place, 2 is not in the second, 3 is not in the third, and so on? (These arrangements are called *derangements*.)

Solution: Let c_i be the set of permutations with the number i at place i.

We seek to compute

$$N(\overline{c_1}\,\overline{c_2}\cdots\overline{c_{10}}) = S_0 - S_1 + S_2 - S_3 + \cdots.$$

Here

$$\begin{aligned} S_0 &= 10! \\ S_1 &= N(c_1) + N(c_2) + \cdots + N(c_{10}) \\ S_2 &= \sum N(c_ic_j) = \binom{10}{2} 8!, \end{aligned}$$

where for S_2, we have $\binom{10}{2}$ terms in the sum, and each term has value $8!$. Similarly, we obtain

$$S_3 = \binom{10}{3} 7!,$$

and therefore

$$\begin{aligned} N(\overline{c_1}\,\overline{c_2}\cdots\overline{c_{10}}) &= 10! - 10 \cdot 9! + \binom{10}{2} 8! - \binom{10}{3} 7! + \cdots + \binom{10}{10} 0! \\ &= 10! - 10! + \frac{10!}{2!} - \frac{10!}{3!} + \cdots + 1 \\ &= 10!\big(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{1}{10!}\big) \\ &\approx 10! e^{-1} \end{aligned}$$

Example 2.16. One can use the principle of inclusion and exclusion to derive a formula for Euler's totient function. See Example 8.8 in Grimaldi's book.

Theorem 2.17. *Let A and B be sets with $|A| = m$ and $|B| = n$, where $m \geq n$. The number of surjective functions from A to B is*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

Proof. Let $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$ and S be the set of all functions $A \to B$. For $1 \leq i \leq n$ define c_i as follows: the function $f : A \to B$ satisfies c_i if and only if $b_i \notin f(A)$. Then a function $f : A \to B$ is surjective if and only if it does not satisfy any of the conditions $c_1, \ldots, c_n$. For different $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$, we have that $N(c_{i_1} c_{i_2} \cdots c_{i_k})$ is the number of functions $f : A \to B$ such that $b_{i_1}, \ldots, b_{i_k} \notin f(A)$, i.e. $f(A) \subseteq B \backslash \{b_{i_1}, \ldots, b_{i_k}\}$ which contains $(n-k)$ elements.

For each $D \subseteq B$ with $(n-k)$ elements, there exist $(n-k)^m$ functions $f : A \to D \subseteq B$ by Theorem 2.8, and we can choose D in $\binom{n}{n-k}$ different ways. This means that

$$S_k = \binom{n}{n-k} (n-k)^m = \binom{n}{k} (n-k)^m;$$

compare Remark 1.15. Then the principle of inclusion and exclusion gives the result. □

Definition 2.18. For $n, m \in \mathbb{Z}$ such that $0 < n \leq m$, we define the *Stirling number (of the second kind)*:

$$S(m, n) := \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

Theorem 2.19. *Let $m, n \in \mathbb{Z}$ be such that $0 < n \leq m$. Then $S(m, n)$ is the number of ways of distributing m different objects into n identical containers such that no container will be empty.*

Proof. Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$. The number of surjective functions $f : A \to B$ is the same as the number of ways of distributing m different objects into n different containers so that no container will be empty. The number of containers is n, so $n!$ such functions are associated with each distribution. Hence the total number that we are looking for is

$$\frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m = S(m, n),$$

as required. □

Theorem 2.20. *Let $m, n \in \mathbb{N}$ be such that $1 < n \leq m$. Then $S(m+1, n) = S(m, n-1) + nS(m, n)$.*

Proof. Let $A = \{a_1, \ldots, a_m, a_{m+1}\}$.

Suppose a_{m+1} is alone. This give $S(m, n-1)$ possibilities, as the elements in $\{a_1, \ldots, a_m\}$ have to be distributed among the $n-1$ containers, with no container left empty. The nth container contains only a_{m+1}.

Suppose a_{m+1} is not alone. We have that the number of ways of distributing $\{a_1, \ldots, a_m\}$ among the n containers, such that no container is empty, is $S(m, n)$, and we have n choices for where to place a_{m+1}. Hence we get

$$S(m+1, n) = S(m, n-1) + nS(m, n),$$

as required. □