LECTURE 1

1. Basic Combinatorics

Combinatorics is a very old subject and can be traced back to the 16th century BC to the ancient Egyptians. The term combinatorics is essentially just a fancy way of saying counting. There are many applications of combinatorics, such as to engineering, computer science, cryptography, biology, etc, some of which we will see in this course.

1.1. **Permutations.**

The rule of sum: If there are m choices for one action and n choices for another action, and the two actions cannot be done simultaneously, then there are $m+n$ ways to choose one of these actions.

The rule of product: If a procedure can be broken down into k stages and if there are n_i possible outcomes for stage i, then the total procedure can be carried out in the designated order in $n_1 \cdot n_2 \cdots n_{k-1} \cdot n_k$ ways.

Motivating question: In how many ways can one choose k objects from a collection of n different objects?

Example 1.1. Using only the letters A, B, C, D and E, how many strings of length 3 can you produce,

(a) if repetition is allowed?
(a) without repetition?

Solution: (a) $5 \cdot 5 \cdot 5 = 125$. (b) $5 \cdot 4 \cdot 3 = 60$.

In general, if we had n letters, the number of strings of length k would be n^k, if repetition is allowed, and the number of strings of length k without repetition is

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Definition 1.2. Let $\Omega = \{a_1, \dots, a_n\}$ be a given set with n elements (note that if $i \neq j$ then $a_i \neq a_j$). A *permutation* of Ω is an ordered set containing exactly the same elements as Ω. For $0 \leq k \leq n$, a *k-permutation* $(a_{i_1}, \dots, a_{i_k})$ of Ω is a permutation containing k elements from Ω.

Example 1.3. The ordered subsets $(1,6,4)$, $(1,2,4)$ and $(1,4,6)$ are three different 3-permutations of the set $\Omega = \{1,2,4,6,8,9\}$.

Theorem 1.4. *Let $n, k \in \mathbb{Z}_{\geq 0}$ be such that $0 \leq k \leq n$ and let $\Omega = \{a_1, \dots, a_n\}$ be a set of n elements. Then the number of all k-permutations of Ω is $\frac{n!}{(n-k)!}$.*

Proof. To determine a k-permutation of Ω, we must choose k elements from Ω in a given order.

The first element can be chosen in n ways.

The second element can be chosen in $n-1$ ways.

$\vdots$

The kth element can be chosen in $n-k+1$ ways.

According to the rule of product, we therefore have a total of

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!},$$

as required. $\square$

Definition 1.5. The quantity $\frac{n!}{(n-k)!}$, which is the number of k-permutations of n objects, is denoted $P(n,k)$.

Example 1.6. Consider the set $\Omega = \{1,2,4,6,8,9\}$. The total number of 3-permutations of Ω is $P(6,3) = \frac{6!}{3!} = 6\cdot 5\cdot 4 = 120$.

Example 1.7. How many different strings of length 5 can we create using the letters of $GREEN$?

Solution: Assume first that all letters are different, and we use the notation G, R, E_1, E_2, N. Then we obtain $P(5,5) = 5!$ strings.

If we identify strings with only a different indexing of the E's, such as $RE_1E_2GN \cong RE_2E_1GN$, then we get $\frac{5!}{2} = 60$ different strings.

Example 1.8. In how many ways can we arrange the letters in INTENSITIES?

Solution: Our multiset of letters is $\{I,I,I,N,N,T,T,E,E,S,S\}$. Hence

$$\begin{aligned}\frac{11!}{3!\,2!\,2!\,2!\,2!} &= \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(3\cdot 2)\cdot 2\cdot 2\cdot 2\cdot 2}\\ &= 415800.\end{aligned}$$

1.2. **Combinations.**

Example 1.9. In how many ways can James choose five different books to borrow from the science fiction shelf in the library? The science fiction shelf contains 27 different books.

Solution: There are $P(27,5) = 27\cdot 26\cdot 25\cdot 24\cdot 23$ ordered sets of five books that James can choose. Each unordered choice of five books can be ordered in 5! ways. This implies that there are

$$\frac{P(27,5)}{5!} = \frac{27!}{22!\,5!} = \frac{27\cdot 26\cdot 25\cdot 24\cdot 23}{120} = 80730$$

choices disregarding order.

Definition 1.10. For $n,k\in\mathbb{Z}$ such that $0\le k\le n$, we denote

$$C(n,k)\Big(\overset{\text{or}}{=}\binom{n}{k}\Big) := \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}.$$

Definition 1.11. Let $\Omega = \{a_1,\dots,a_n\}$ be a given set with n elements. A *combination* of Ω is a subset of Ω. For $0\le k\le n$, a *k-combination* of Ω is a combination of k elements from Ω.

Example 1.12. The subsets $\{1,2,3,4,5\}$ and $\{1,2,4\}$ are combinations of the set $\Omega = \{1,2,3,4,5,6,7,8,9\}$.

Theorem 1.13. *Let $n,k\in\mathbb{Z}_{\ge 0}$ be such that $0\le k\le n$ and let $\Omega=\{a_1,\dots,a_n\}$ be a set of n elements. Then the number of all k-combinations of Ω is $C(n,k)$.*

Proof. According to Theorem 1.4, the number of k-permutations is $P(n,k) = \frac{n!}{(n-k)!}$. Each k-combination is associated to $k!$ different k-permutations. Hence

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!},$$

is the number of k-combinations. $\square$

1.3. **Binomial Theorem.**

Theorem 1.14 (Binomial Theorem)**.** *For variables a, b and any positive integer n,*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Proof. Now

$$(a+b)^n = (a+b)(a+b)\cdots(a+b).$$

The coefficient of a^k equals the number of ways to choose k factors from which we use a. This is $\binom{n}{k}$, and the term involved is

$$\binom{n}{k} a^k b^{n-k}$$

where we note that we must choose b from the remaining $n-k$ factors. $\square$

Remark 1.15. Observe that $\binom{n}{k} = \binom{n}{n-k}$.

The above binomial theorem has a more general form:

Theorem 1.16 (Multinomial Theorem)**.** *For variables a, b, c and any positive integer n,*

$$(a+b+c)^n = \sum_{k_1+k_2+k_3=n} \binom{n}{k_1, k_2, k_3} a^{k_1} b^{k_2} c^{k_3}.$$

Here the multinomial coefficient $\binom{n}{k_1,k_2,k_3} = \frac{n!}{k_1!\,k_2!\,k_3!}$.

Example 1.17. In how many ways can we distribute ten (identical) apples among 4 children Carl, Theo, Sarah and Olga?

Solution: Each distribution corresponds to a string of ten x's and three |'s, for example

$$\underset{\text{Carl}}{\text{x x}} \,|\, \underset{\text{Theo}}{\text{x x x}} \,|\, \underset{\text{Sarah}}{\text{x x x x}} \,|\, \underset{\text{Olga}}{\text{x}}$$

which is equivalent to saying: 2 apples for Carl, 3 apples for Theo, 4 apples for Sarah and 1 apple for Olga. Another example is ||| x x x x x x x x x x, where Olga gets all the apples.

So the number of distributions of apples equals the number of words in ten x's and three |'s, which equals

$$\frac{13!}{10!\,3!} = \binom{13}{3}.$$

[Compare Examples 1.7 and 1.8.]

More generally, the number of ways to distribute k identical objects among n different containers is

$$\binom{k+n-1}{n-1} = \binom{n+k-1}{k};$$

compare Remark 1.15. In the above example, the apples are the identical objects, and the children are the different containers.

Example 1.18. How many integer solutions $(x_1, \ldots, x_6)$ does the following problem have?

$$(*) \begin{cases} x_1 + x_2 + \cdots + x_6 = 24, \\ 0 \leq x_1, x_2, x_3, \\ 1 \leq x_4, x_5, \\ 2 \leq x_6. \end{cases}$$

Solution. The problem $(*)$ has exactly as many solutions as the following:

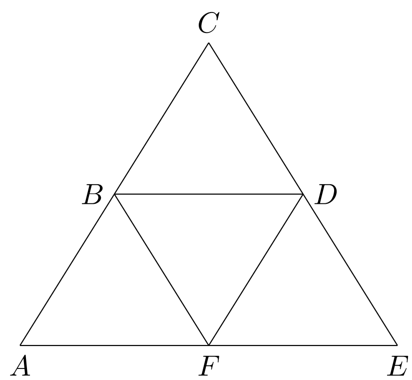
$$x_1 + x_2 + \cdots + x_6 = 24 + (-1) + (-1) + (-2) = 20,$$
$$0 \leq x_1, x_2, \ldots, x_6.$$

The number is the same as the number of ways of distributing 20 apples amongst 6 children:

$$\binom{20+6-1}{6-1} = \binom{25}{5} = 53130.$$

1.4. **The pigeonhole principle.** Let $m, n \in \mathbb{N}$ be such that $m > n$. If m objects are placed in n containers, then there exists at least one container containing at least two objects.

Example 1.19. Let ACE be an equivalent triangle in the plane such that $|AC| = 1$.



Let $P_1, P_2, \ldots, P_5$ be five points inside ACE. Then, by the pigeonhole principle, there exists i, j such that $|P_iP_j| < 1/2$.

Example 1.20. Let p be a prime number such that $p \neq 2, 5$. Show that p divides at least one of the numbers $1, 11, 111, 1111, \ldots$.

Solution. Write $a_k = 1\overset{k}{\ldots}1$. Let r_k be the remainder when we divide a_k with p, so

$$r_1, r_2, \ldots, r_k, \ldots \in \{0, 1, \ldots, p-1\}.$$

Thus, by the pigeonhole principle, there exist $r_i = r_j$ with $i \neq j$. Therefore p divides $a_i - a_j = 1\overset{i-j}{\ldots}1 \cdot 10^j$, and hence p divides a_{i-j}, since $p \neq 2, 5$.