LECTURE 3

3. Generating functions

3.1. Introduction.

Example 3.1 (Motivating example)**.** In how many ways can eight dollar bills be distributed between three people A, B and C so that A gets an odd number of bills, person B gets at least \$1 and C gets either \$2 or \$3.

Solution: *Method I: List all possibilities.*

a) If $C = 2$, then $A + B = 6$. As A is odd, it follows that $B \geq 1$ is also odd. Therefore, $B \in \{1, 3, 5\}$. This gives 3 possibilities:

$$(A, B, C) \in \{(5, 1, 2),\ (3, 3, 2),\ (1, 5, 2)\}.$$

b) If $C = 3$, then $A + B = 5$. As A is odd, we must have $B \geq 1$ is even. Therefore, $B \in \{2, 4\}$. This gives 2 possibilities:

$$(A, B, C) \in \{(3, 2, 3),\ (1, 4, 3)\}.$$

In total, there are $3 + 2 = 5$ different distributions.

Method II: Generating functions. We turn the problem into algebra by letting

$$\begin{aligned}
A(x) &= x + x^3 + x^5 + x^7 \\
B(x) &= x + x^2 + x^3 + \cdots + x^8 \\
C(x) &= x^2 + x^3.
\end{aligned}$$

If we multiply them, that is, we consider $A(x)B(x)C(x)$, we get a sum of terms $x^a x^b x^c$ with

$$\begin{aligned}
a \ \text{ odd such that } &\ 1 \leq a \leq 7, \\
b \ \text{ such that } &\ 1 \leq b \leq 8, \\
c \ \text{ such that } &\ 2 \leq c \leq 3.
\end{aligned}$$

The distributions we want to count are those with $a + b + c = 8$ so the x^8 terms in $A(x)B(x)C(x)$. Let us compute it:

$$\begin{aligned}
&A(x)B(x)C(x) \\
&= \big(x + x^3 + x^5 + x^7\big)\big(x + x^2 + x^3 + \cdots + x^8\big)\big(x^2 + x^3\big) \\
&= \big(x + x^3 + x^5 + x^7\big)\Big((x^3 + x^4 + \cdots + x^{10}) + (x^4 + x^5 + \cdots + x^{11})\Big) \\
&= \big(x + x^3 + x^5 + x^7\big)\big(x^3 + 2x^4 + 2x^5 + \cdots + 2x^{10} + x^{11}\big).
\end{aligned}$$

Thus the coefficient of x^8 in the above product is

$$1 \text{ from } x^5 \cdot x^3 \ + \ 2 \text{ from } x^3 \cdot 2x^5 \ + \ 2 \text{ from } x \cdot 2x^7 = \ 5.$$

In our second method the calculations can be simplified by noting that we may add terms to $A(x)$, $B(x)$ or $C(x)$ that are of degree > 8 without affecting the coefficient of x^8 in $A(x)B(x)C(x)$. With this in mind, let

$$\begin{aligned}
\widetilde{A}(x) &= x + x^3 + x^5 + \cdots = \frac{x}{1-x^2}\\
\widetilde{B}(x) &= x + x^2 + x^3 + \cdots = x(1 + x + x^2 + x^3 + \cdots) = \frac{x}{1-x}\\
\widetilde{C}(x) &= C(x) = x^2 + x^3 = x^2(1+x).
\end{aligned}$$

(Here we have used the formula for the sum of a geometric series.) Now consider

$$\begin{aligned}
\widetilde{f}(x) &= \widetilde{A}(x)\widetilde{B}(x)\widetilde{C}(x)\\
&= \frac{x}{1-x^2}\frac{x}{1-x}x^2(1+x)\\
&= \frac{1}{(1-x)(1+x)}\frac{x^2}{1-x}x^2(1+x)\\
&= \frac{x^4}{(1-x)^2}.
\end{aligned}$$

The coefficient of x^8 in $\frac{x^4}{(1-x)^2}$ equals the coefficient of x^4 in $\frac{1}{(1-x)^2}$. We now use the Taylor expansion of $g(x) = (1-x)^{-2}$:

$$\begin{aligned}
g'(x) &= 2(1-x)^{-3}\\
g''(x) &= 2 \cdot 3(1-x)^{-4}\\
g^{(3)}(x) &= 2 \cdot 3 \cdot 4(1-x)^{-5}\\
g^{(4)}(x) &= 2 \cdot 3 \cdot 4 \cdot 5(1-x)^{-6}
\end{aligned}$$

Therefore the coefficient of x^4 is $\frac{g^{(4)}(0)}{4!} = 5$.

Definition 3.2. A function $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ is called a *generating function* for the sequence $a_0, a_1, a_2, a_3, \ldots$.

Example 3.3. If $n \in \mathbb{N}$ then $(1+x)^n$ gives the sequence

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}, 0, 0, \ldots$$

because

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

by the Binomial Theorem.

Example 3.4. For $n \in \mathbb{N}$, the function $\frac{1-x^{n+1}}{1-x}$ gives the sequence $1, \overset{n+1}{\ldots}, 1, 0, 0, \ldots$, because the geometric sum $1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$. One also observes that $\frac{1}{1-x}$ corresponds to the sequence $1, 1, \ldots$; compare the geometric series $1 + x + x^2 + \cdots = \frac{1}{1-x}$.

Example 3.5. Further examples of generating functions, and their corresponding sequences, include:

$$\frac{1}{1-4x} \longleftrightarrow 1, 4, 4^2, 4^3, \ldots$$
$$\frac{1}{1+2x} \longleftrightarrow 1, -2, 2^2, -2^3, 2^4, -2^5, \ldots$$

We will now show that the equality

$$(1+x)^n = \sum_{j=0}^{\infty} \binom{n}{j} x^j$$

can be generalised to the case $n \in \mathbb{R}$. The trick is to define the binomial coefficient $\binom{n}{j}$ in a useful way. [Note also that in the equation above, one would usually sum up to n, but one can sum to infinity by adding an infinite number of zeros at the end.]

For $n \in \mathbb{N}$ (and we assume $j \in \mathbb{N}$ throughout this argument) we have previously defined

$$\binom{n}{j} = \frac{n!}{(n-j)!j!} = \frac{n(n-1)(n-2)\cdots(n-j+1)}{j!}.$$

We now take the rightmost expression as the definition of $\binom{n}{j}$ for any $n \in \mathbb{R}$.

Using this, we next consider the Taylor expansion of $f(x) = (1+x)^n$, for $n \in \mathbb{R}$:

$$f'(x) = n(1+x)^{n-1} \implies \frac{f'(0)}{1!} = n = \binom{n}{1}$$
$$f''(x) = n(n-1)(1+x)^{n-2} \implies \frac{f''(0)}{2!} = \frac{n(n-1)}{2!} = \binom{n}{2}$$
$$f^{(3)}(x) = n(n-1)(n-2)(1+x)^{n-3} \implies \frac{f^{(3)}(0)}{3!} = \frac{n(n-1)(n-2)}{3!} = \binom{n}{3}$$
$$\vdots$$
$$f^{(j)}(x) = n(n-1)\cdots(n-j+1)(1+x)^{n-j} \implies \frac{f^{(j)}(0)}{j!} = \frac{n(n-1)\cdots(n-j+1)}{j!} = \binom{n}{j},$$

so

$$\begin{aligned}(1+x)^n = f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots \\ &= 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots \\ &= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots \\ &= \sum_{j=0}^{\infty} \binom{n}{j} x^j.\end{aligned}$$

Remark 3.6. In computations, the following relation will be useful:

$$\begin{aligned}\binom{-n}{j} &\overset{\text{def}}{=} \frac{(-n)(-n-1)\cdots(-n-j+1)}{j!}\\ &= (-1)^j\frac{n(n+1)\cdots(n+j-1)}{j!}\\ &= (-1)^j\binom{n+j-1}{j}.\end{aligned}$$

Example 3.7. Assume that we have three different boxes in the colours of red, blue, and green, and we want to distribute 12 identical marbles between them. In how many ways can we do this

(a) with at least 2 marbles in each box?
(b) with at least 2 and at most 5 marbles in each box?

Solution: (a) The exponents of

$$x^2+x^3+x^4+\cdots = x^2(1+x+x^2+\cdots) = \frac{x^2}{1-x}$$

keep track of the number of marbles in, for example, the red box. For the other two boxes we get the same function, so

$$f(x) = \left(\frac{x^2}{1-x}\right)^3 = x^6(1-x)^{-3}$$

is a generating function. To get the coefficient of x^{12} we expand $f(x) = x^6\sum_{j=0}^{\infty}\binom{-3}{j}x^j$. Using Remark 3.6, the sought coefficient is

$$\binom{-3}{6} = \binom{8}{6} = \frac{8\cdot 7}{2!} = 4\cdot 7 = 28.$$

(b) Here the number of marbles in a certain box corresponds to the exponents of

$$x^2+x^3+x^4+x^5 = x^2(1+x+x^2+x^3) = x^2\frac{1-x^4}{1-x}$$

and therefore

$$\begin{aligned}f(x) &= \left(\frac{x^2(1-x^4)}{(1-x)}\right)^3 = x^6(1-x^4)^3(1-x)^{-3}\\ &= x^6\big(1-3x^4+\text{ higher terms}\big)\sum_{j=0}^{\infty}\binom{-3}{j}x^j.\end{aligned}$$

The coefficient of x^{12} is

$$\begin{aligned}1\cdot\binom{-3}{6} - 3\cdot\binom{-3}{2} &= \binom{8}{6} - 3\cdot\binom{4}{2}\\ &= 28-3\cdot 6\\ &= 28-18\\ &= 10.\end{aligned}$$

3.2. **Partition of integers.** Let $p(n)$ denote the number of ways to partition a positive integer n into positive summands (disregarding order).

Example 3.8. Find all partitions of 5.

Solution: If the maximal partition size is 5, then we only have the trivial partition consisting of all numbers.

If the maximal partition size is 4, then we have $4+1$.

If it is 3, then we have $3+2$ and $3+1+1$.

If it is 2, we have $2+2+1$ and $2+1+1+1+1$.

Finally if the maximal partition size is 1, then we only have $1+1+1+1+1$.

In total we have 7 partitions, therefore $p(5)=7$.

Now, using generating functions, let us count partitions of any n that consist only of the summands 1, 2 and 3. The relevant generating function is

$$f(x) = \underbrace{(1+x+x^2+x^3+\cdots)}_{\text{keeps track of no. of 1's}}\underbrace{(1+x^2+x^4+x^6+\cdots)}_{\text{keeps track of no. of 2's}}\underbrace{(1+x^3+x^6+x^9+\cdots)}_{\text{keeps track of no. of 3's}}$$

How can we get x^5?

$$\begin{array}{rcccl}
x^5\cdot 1\cdot 1 & \longleftrightarrow & 5\cdot 1+0\cdot 2+0\cdot 3 & \longleftrightarrow & 1+1+1+1+1\\
x^3\cdot x^2\cdot 1 & \longleftrightarrow & 3\cdot 1+1\cdot 2+0\cdot 3 & \longleftrightarrow & 1+1+1+2\\
x\cdot x^4\cdot 1 & \longleftrightarrow & 1\cdot 1+2\cdot 2+0\cdot 3 & \longleftrightarrow & 1+2+2\\
x^2\cdot 1\cdot x^3 & \longleftrightarrow & 2\cdot 1+0\cdot 2+1\cdot 3 & \longleftrightarrow & 1+1+3\\
1\cdot x^2\cdot x^3 & \longleftrightarrow & 0\cdot 1+1\cdot 2+1\cdot 3 & \longleftrightarrow & 2+3
\end{array}$$

Hence the number of partitions of 5 using summands 1, 2, 3 is the coefficient of x^5 in

$$f(x) = \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}.$$

Without restriction on the summands used, the generating function is

$$F(x) = \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^4}\cdots = \prod_{j=1}^{\infty}\frac{1}{1-x^j}.$$

Unfortunately it is hard to compute the coefficients of $F(x)$, or even $f(x)$, without the aid of computers. In the even simpler case $g(x)=\frac{1}{1-x}\cdot\frac{1}{1-x^2}$ using only 1's and 2's, one can use partial fraction decomposition. Write

$$\frac{1}{1-x}\cdot\frac{1}{1-x^2} = \frac{1}{(1-x)^2}\cdot\frac{1}{1+x} = \frac{A}{(1-x)^2}+\frac{B}{1-x}+\frac{C}{1+x},$$

which gives a system of equations:

$$\begin{cases} C-B &= 0\\ A-2C &= 0\\ A+B+C &= 0\end{cases} \iff \qquad A=\frac{1}{2},\quad B=C=\frac{1}{4}.$$

This implies that

$$\begin{aligned} g(x) &= \frac{1}{2}(1-x)^{-2} + \frac{1}{4}\cdot\frac{1}{1-x} + \frac{1}{4}\cdot\frac{1}{1+x} \\ &= \frac{1}{2}\sum_{j=0}^{\infty}\binom{-2}{j}x^j + \frac{1}{4}\sum_{j=0}^{\infty}x^j + \frac{1}{4}\sum_{j=0}^{\infty}(-x)^j \end{aligned}$$

and so the coefficient of x^n is

$$\begin{aligned} \frac{1}{2}\binom{-2}{n} + \frac{1}{4} + (-1)^n\frac{1}{4} &= \frac{1}{2}\binom{n+1}{n} + \frac{1}{4} + (-1)^n\frac{1}{4} \\ &= \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n+1}{2} + \frac{1}{2} = \frac{n+2}{2} & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$