LECTURE 4

3.3. **Exponential generating functions.** In the previous subsection, we considered identical objects being distributed among different places (with certain conditions). Here we will consider different objects being distributed among different places (with certain conditions).

Example 3.9 (Motivating example)**.** How many words of length 4 can we create using the letters A, A, A, B, B, C?

Solution: For each choice of a, b, c with $a+b+c=4$, and $0 \leq a \leq 3$, $0 \leq b \leq 2$, $0 \leq c \leq 1$, we can create $\frac{4!}{a!\,b!\,c!}$ words. This means that the answer equals $4!$ times the coefficient of x^4 in

$$f(x) = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)(1+x)\,.$$

[For example, one contribution to the coefficient of x^4 comes from $\frac{x^3}{3!} \cdot \frac{x}{1!} \cdot 1$, which corresponds to words using 3 A's, 1 B and 0 C's.]

Definition 3.10. A function of the form

$$f(x) = a_0 + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \cdots$$

is called an *exponential generating function* for the sequence $a_0, a_1, a_2, a_3, \ldots$.

Example 3.11. Find the exponential generating function for the sequences 1, -5, 5^2, -5^3, 5^4, $\ldots$ and $1, b^3, b^6, b^9, b^{12}, \ldots$.

Solution: For the first series:

$$\begin{aligned} f(x) &= 1 - \frac{5x}{1!} + \frac{5^2x^2}{2!} - \frac{5^3x^3}{3!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-5x)^k}{k!} \\ &= e^{-5x}. \end{aligned}$$

For the second series:

$$\begin{aligned} g(x) &= 1 + \frac{b^3x}{1!} + \frac{b^6x^2}{2!} + \frac{b^9x^3}{3!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(b^3x)^k}{k!} \\ &= e^{b^3x}. \end{aligned}$$

Example 3.12. Assume that we have 48 flags: 12 each of the colours red, white, blue and black.

(a) How many signals can we get by placing 12 of them on a pole if there should be an even number of red flags?
(b) How many signals can we get by using at least one flag of each colour?
(c) Relate (b) to counting onto functions.

Solution: (a) We have

$$\begin{aligned} f(x) &= \underbrace{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^3}_{\text{white, blue, black flags}} \underbrace{\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)}_{\text{red flags}} \\ &= (e^x)^3\left(\frac{e^x+e^{-x}}{2}\right) \\ &= \frac{1}{2}(e^{4x}+e^{2x}) \\ &= \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{(4x)^j}{j!}+\frac{(2x)^j}{j!}\right). \end{aligned}$$

Therefore the answer is 12! times the coefficient of x^{12}, which is

$$12!\cdot\frac{1}{2}\left(\frac{4^{12}}{12!}+\frac{2^{12}}{12!}\right)=2\cdot 4^{11}+2^{11}.$$

(b) We have

$$\begin{aligned} f(x) &= \left(\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^4 \\ &= (e^x-1)^4 \\ &= (e^x)^4+4(e^x)^3(-1)+6(e^x)^2(-1)^2+4e^x(-1)^3+(-1)^4 \\ &= e^{4x}-4e^{3x}+6e^{2x}-4e^x+1 \\ &= \sum_{j=0}^{\infty}\left(\frac{(4x)^j}{j!}-4\frac{(3x)^j}{j!}+6\frac{(2x)^j}{j!}-4\frac{x^j}{j!}\right)+1. \end{aligned}$$

As before the answer is 12! times the coefficient of x^{12}, which is

$$12!\cdot\left(\frac{4^{12}}{12!}-4\frac{3^{12}}{12!}+6\frac{2^{12}}{12!}-4\frac{1}{12!}\right)=4^{12}-4\cdot 3^{12}+6\cdot 2^{12}-4.$$

(c) This is also the number of onto functions $f : A \to B$, where $A = \{\text{red}, \text{blue}, \text{white}, \text{black}\}$ and $B = \{\text{position } 1, \text{position } 2, \text{position } 3, \ldots, \text{position } 12\}$.

3.4. **The summation operation.**

Theorem 3.13. *If $f(x)$ is the generating function corresponding to the sequence*

$$a_0, a_1, a_2, a_3, \ldots,$$

then $\dfrac{f(x)}{1-x}$ is the generating function corresponding to the sequence

$$a_0,\ a_0+a_1,\ a_0+a_1+a_2,\ a_0+a_1+a_2+a_3,\ \ldots\ ,$$

Proof. We observe that

$$\begin{aligned}
\frac{f(x)}{1-x} &= \sum_{i=0}^{\infty} a_i x^i \sum_{j=0}^{\infty} x^j \\
&= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots) \\
&= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + \cdots ,
\end{aligned}$$

hence the result. □

Example 3.14. Find a formula for $0^2 + 1^2 + \cdots + n^2$ (using generating functions).

Solution: If we had a generating function $g(x)$ for $0^2, 1^2, 2^2, \ldots$, we could then obtain our answer by taking the coefficient of x^n in $\frac{g(x)}{1-x}$; compare the previous theorem.

To find $g(x)$, we start from

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

and differentiate to obtain

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots ,$$

and then multiply by x to get

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots .$$

Differentiating again yields

$$\begin{aligned}
1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \cdots &= \frac{1 \cdot (1-x)^2 - x \cdot 2(1-x)(-1)}{(1-x)^4} \\
&= \frac{1 - 2x + x^2 + 2x - 2x^2}{(1-x)^4} \\
&= \frac{1 - x^2}{(1-x)^4} \\
&= \frac{1+x}{(1-x)^3}.
\end{aligned}$$

Multiplying by x one more time gives

$$\frac{x(1+x)}{(1-x)^3} = x + 2^2 x^2 + 3^2 x^3 + 4^2 x^4 + \cdots ,$$

hence we can take $g(x) = \frac{x(1+x)}{(1-x)^3}$ and then $\frac{g(x)}{1-x}$ generates

$$0^2,\ 0^2 + 1^2,\ 0^2 + 1^2 + 2^2,\ 0^2 + 1^2 + 2^2 + 3^2,\ \ldots .$$

The coefficient of x^n in

$$\begin{aligned}
\frac{g(x)}{1-x} &= \frac{x(1+x)}{(1-x)^4} \\
&= (x + x^2)(1-x)^{-4} \\
&= (x + x^2) \sum_{j=0}^{\infty} \binom{-4}{j} (-x)^j
\end{aligned}$$

is

$$\begin{aligned}\binom{-4}{n-1}(-1)^{n-1} + \binom{-4}{n-2}(-1)^{n-2} \\ &= \binom{n+2}{n-1}(-1)^{n-1}(-1)^{n-1} + \binom{n+1}{n-2}(-1)^{n-2}(-1)^{n-2} \\ &= \frac{(n+2)(n+1)n}{3!} + \frac{(n+1)n(n-1)}{3!} \\ &= \frac{n(n+1)}{6}(n+2+n-1) \\ &= \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

4. Recurrence relations

4.1. **Introduction.** Here we shall investigate functions $a : \mathbb{Z}_{\geq 0} \to \mathbb{C}$, where $a(n)$ depends on some prior terms $a(n-1), a(n-2), \ldots, a(1), a(0)$. This is the study of what we call recurrence relations (sometimes referred to as difference equations). Recurrence relations are likewise (cf. combinatorics) applied widely within the sciences, such as in biology and computer science, even in financial mathematics.

In this setting, one usually writes a_n for $a(n)$.

Definition 4.1. A *recurrence relation* of order k (also called a *kth order recurrence relation*) is an equality $a_{n+k} = f(n, a_n, a_{n+1}, \ldots, a_{n+k-1})$.

Furthermore, the recurrence relation is *linear*, if f is linear in $a_n, a_{n+1}, \ldots, a_{n+k-1}$.

Example 4.2. The first order recurrence relation $a_n = n \cdot a_{n-1}$ for $n > 0$, with the initial condition $0! = 1$, defines the factorial $n! = n(n-1)!$.

Example 4.3. The relations $a_{n+2} = a_{n+1} \cdot a_n$, $a_{n+4} = -2a_{n+3} - 5a_{n+2} + 2a_n^2$ and $a_{n+3} = 4a_{n+2} - 3a_{n+1} + 2a_n + n^2$ are recurrence relations of the orders 2, 4, and 3.

In the previous example, the last recurrence relation is linear with constant coefficients:

Definition 4.4. A *linear recurrence relation with constant coefficients* is a recurrence relation of the form

$$a_{n+k} + c_1 a_{n+k-1} + \cdots + c_k a_n = g(n) \tag{4.1}$$

where $c_1, \ldots, c_k$ are (real) constants and g is some function in n. Such a linear recurrence relation is furthermore called *homogeneous* if $g(n) \equiv 0$.

Example 4.5 (Fibonacci numbers)**.** The Fibonacci sequence is defined by the linear homogeneous recurrence relation with constant coefficients as follows:

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions

$$F_0 = 0 \quad \text{and} \quad F_1 = 1.$$

This gives the well-known sequence of Fibonacci numbers:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Example 4.6 (Binomial coefficients)**.** The binomial coefficients can be defined by a so-called *multi-dimensional* recurrence relation as follows:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

with initial conditions

$$\binom{n}{0} = \binom{n}{n} = 1.$$

We will now concentrate on first and second order linear recurrence relations with constant coefficients, and we first focus on the homogeneous type since all solutions of (4.1) are sums of one particular solution and all solutions of the homogeneous equation (compare this with linear differential equations).

4.2. **Linear homogeneous recurrence relations with constant coefficients.**

4.2.1. *First order linear homogeneous recurrence relations with constant coefficients.* Consider $a_{n+1} + c_1 a_n = 0$. We have $a_n = -c_1 a_{n-1} = (-c_1)^2 a_{n-2} = \cdots = (-c_1)^n a_0$.

Example 4.7. Tim puts \$100 into a bank account with a 6% yearly interest rate. How much money will he have 10 years later?

Solution: We have $a_0 = 100$ and $a_{n+1} = a_n + 0.06a_n = 1.06a_n$, hence $a_n = (1.06)^n a_0 = 100(1.06)^n$. There $a_{10} = 100(1.06)^{10} \approx 1005, 60$, that is 1005 dollars and 60 cents.

4.2.2. *Second order linear homogeneous with constant coefficients.* We have

$$a_{n+2} + c_1 a_{n+1} + c_2 a_n = 0. \tag{4.2}$$

Using (4.2) repeatedly we obtain an expression of the form $a_n = g_n(c_1, c_2)a_1 + h_n(c_1, c_2)a_0$, so all solutions are linear combinations of the solutions $g_n(c_1, c_2)$ and $h_n(c_1, c_2)$ that we get in the cases $a_1 = 1$, $a_0 = 0$ and $a_1 = 0$, $a_0 = 1$ respectively.

This shows that we have exactly two linearly independent solutions to (4.2). Let us now try to find these solutions by trying with $a_n = Kr^n$:

$$\begin{aligned} Kr^{n+2} + c_1 Kr^{n+1} + c_2 Kr^n = 0 \quad &\Longleftrightarrow \quad Kr^n(r^2 + c_1 r + c_2) = 0 \\ &\overset{K,r\neq 0}{\Longleftrightarrow} \quad r^2 + c_1 r + c_2 = 0. \end{aligned}$$

The equation $r^2 + c_1 r + c_2 = 0$ is called the *characteristic equation* for (4.2) and we have shown that Kr^n is a solution if and only if r solves the characteristic equation.

Case I: If $r_1 \neq r_2$ are the roots of the characteristic equation then we get two linearly independent solutions r_1^n and r_2^n so all solutions to (4.2) are $a_n = K_1 r_1^n + K_2 r_2^n$ where K_1, K_2 are arbitrary complex constants.

Suppose r_1, r_2 are real. Then we get all real solutions as $a_n = K_1 r_1^n + K_2 r_2^n$ with $K_1, K_2 \in \mathbb{R}$.

Suppose r_1, r_2 are not real. Then $r_2 = \overline{r_1}$; recall we assume real coefficients in the characteristic equation. Therefore we can write $r_1 = re^{i\theta}$ and $r_2 = re^{-i\theta}$ where $r = |r_1|$, and all solutions are given by

$$\begin{aligned} a_n &= K_1(re^{i\theta})^n + K_2(re^{-i\theta})^n \\ &= K_1 r^n(\cos n\theta + i\sin n\theta) + K_2 r^n(\cos n\theta - i\sin n\theta) \\ &= r^n\left(\underbrace{(K_1+K_2)}_{=K_1'}\cos n\theta + \underbrace{i(K_1-K_2)}_{=K_2'}\sin n\theta\right) \\ &= r^n\left(K_1'\cos n\theta + K_2'\sin n\theta\right) \end{aligned}$$

and real solutions are obtained when K_1', K_2' are real.

Example 4.8. How many strings of length n in $0, 1, 2$ are there with no consecutive 1's and 2's?

Solution: Consider

$$a_n = b_n + c_n$$

where

$$\begin{aligned} a_n &= \text{number of such strings of length } n, \\ b_n &= \text{number of such strings of length } n \text{ with last symbol } 0, \\ c_n &= \text{number of such strings of length } n \text{ with last symbol } 1 \text{ or } 2. \end{aligned}$$

Observe that

$$a_n = 3b_{n-1} + 2c_{n-1}$$

since for the strings ending in 0 we can add anything at the end, and for the strings ending in 1 or 2, we have two possibilities for adding a symbol at the end.

Also $b_{n-1} = a_{n-2}$, since we can just remove the final 0.

Rewriting gives

$$\begin{aligned} a_n &= 2(b_{n-1} + c_{n-1}) + b_{n-1} \\ &= 2a_{n-1} + a_{n-2} \end{aligned}$$

which gives the recurrence relation $a_n = 2a_{n-1} + a_{n-2}$. Our initial conditions are $a_0 = 1$ (empty string) and $a_1 = 3$.

The characteristic equation is $r^2 - 2r - 1 = 0$, which has roots $r = 1 \pm \sqrt{2}$. Therefore $a_n = C(1+\sqrt{2})^n + D(1-\sqrt{2})^n$. From the initial conditions

$$\begin{aligned} \begin{cases} 1 = a_0 = C + D \\ 3 = a_1 = C(1+\sqrt{2}) + D(1-\sqrt{2}) \end{cases} &\iff \begin{cases} C + D = 1 \\ \sqrt{2}(C - D) = 2 \end{cases} \\ &\iff \begin{cases} C + D = 1 \\ C - D = \sqrt{2} \end{cases} \\ &\iff \begin{cases} C = \frac{1+\sqrt{2}}{2} \\ D = \frac{1-\sqrt{2}}{2} \end{cases} \end{aligned}$$

and thus the answer is $a_n = \frac{(1+\sqrt{2})^{n+1}}{2} + \frac{(1-\sqrt{2})^{n+1}}{2}$.

Example 4.9. Find all real solutions to $a_{n+2} + 2a_n = 0$.

Solution: The characteristic equation is $r^2 + 2r = 0$, which gives $r = \pm\sqrt{2}i = \sqrt{2}e^{\pm\frac{i\pi}{2}}$. Hence

$$\begin{aligned}
a_n &= C(\sqrt{2}e^{\frac{i\pi}{2}})^n + D(\sqrt{2}e^{-\frac{i\pi}{2}})^n \\
&= \sqrt{2}^n[C(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}) + D(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2})] \\
&= \sqrt{2}^n[\underbrace{(C+D)}_{C'}\cos\frac{n\pi}{2} + \underbrace{i(C-D)}_{D'}\sin\frac{n\pi}{2}] \\
&= \sqrt{2}^n[C'\cos\frac{n\pi}{2} + D'\sin\frac{n\pi}{2}]
\end{aligned}$$

with C', D' real constants (these constants will of course depend on the initial conditions, which are not given in this example).

Case II: The characteristic equation has a real double root $r_1 = r_2$. In this case one can show that r_1^n and nr_1^n are (linearly independent) solutions, so all solutions are of the form $a_n = K_1 r_1^n + K_2 n r_1^n$.

Proof that nr_1^n is a solution:

We have that r_1 is a double root of $r^2 + c_1 r + c_2 = 0$. Thus $r_1^2 + c_1 r_1 + c_2 = 0$ and, from differentiating the previous equation with respect to r_1, we also have $2r_1 + c_1 = 0$. Plug in $a_n = nr_1^n$:

$$\begin{aligned}
a_{n+2} + c_1 a_{n+1} + c_2 a_n &= (n+2)r_1^{n+2} + c_1(n+1)r_1^{n+1} + c_1 n r_1^n \\
&= r_1^n\left((n+2)r_1^2 + c_1(n+1)r_1 + nc_2\right) \\
&= r_1^n\left(n(\underbrace{r_1^2 + c_1 r_1 + c_2}_{=0}) + (\underbrace{2r_1^2 + c_1 r_1}_{=0})\right) \\
&= 0
\end{aligned}$$

Example 4.10. Find the solution of $a_{n+2} + 2a_{n+1} + a_n = 0$ with $a_0 = 2$ and $a_1 = -1$.

Solution: The characteristic equation is $r^2 + 2r + 1 = 0$ with double root -1. Therefore $a_n = C(-1)^n + Dn(-1)^n$. The initial conditions now give us

$$\begin{cases} 2 = a_0 = C \\ -1 = a_1 = -C - D \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} C = 2 \\ D = 1 - C = -1 \end{cases}$$

and thus the answer is $a_n = 2(-1)^n - n(-1)^n = (2-n)(-1)^n$.