

Topology: Exercises 1

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Problem 1

Proof. (\Rightarrow) Assume there exists a bijection $h : A \rightarrow B$. Since A is countable, there exists a set C and a bijection $f : A \rightarrow C$, where $C = \{1, \dots, n\}$ if A is finite and $C = \mathbb{N}$ if A is countably infinite. Define $g := f \circ h^{-1} : B \rightarrow C$. Since compositions of bijections are bijections, g is a bijection. Hence there exists a bijection from both A and B to the same set C , so they have the same cardinality.

(\Leftarrow) Assume A and B have the same cardinality. Then there exists a set $C \subseteq \mathbb{N}$ and bijections $f : A \rightarrow C$ and $g : B \rightarrow C$. Define $h := g^{-1} \circ f : A \rightarrow B$. Since f and g are bijections, h is a bijection. \square

Problem 2

Proof. Let $k \in \mathbb{N}$ and choose distinct primes p_1, \dots, p_k . Define

$$f : \mathbb{N}^k \rightarrow \mathbb{N}, \quad (n_1, \dots, n_k) \mapsto p_1^{n_1} \cdots p_k^{n_k}.$$

Let $x, y \in \mathbb{N}^k$ and suppose $f(x) = f(y)$. Then

$$p_1^{x_1} \cdots p_k^{x_k} = p_1^{y_1} \cdots p_k^{y_k}.$$

By uniqueness of prime factorization, we must have $x_i = y_i$ for each $i \in \{1, \dots, k\}$. Hence $x = y$, and f is injective. Since \mathbb{N} is countable and \mathbb{N}^k injects into \mathbb{N} , it follows by Theorem 7.1 that \mathbb{N}^k is countable. \square

Problem 3

Proof. \mathbb{Q} is infinite since the map $f : \mathbb{N} \rightarrow \mathbb{Q}$, $n \mapsto n$, is injective.

Define $g : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \rightarrow \mathbb{Q}$ by

$$g(m, n) = \frac{m}{n}.$$

Then g is surjective: for any $q \in \mathbb{Q}$, write $q = \frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus \{0\}$, so $q = g(m, n)$.

Since \mathbb{Z} and $\mathbb{Z} \setminus \{0\}$ are countable, $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable by Munkres Theorem 7.6. By Munkres Theorem 7.1, there exists a surjection $h : \mathbb{N} \rightarrow \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Then $g \circ h : \mathbb{N} \rightarrow \mathbb{Q}$ is a surjection, so \mathbb{Q} is countable. Therefore \mathbb{Q} is countably infinite. \square

Problem 4

a: Label all elements in $A = \{a_1, \dots, a_n\}$. Let $f : \{0, 1\}^n \rightarrow \mathcal{P}(A)$, $(x_1, \dots, x_n) \mapsto \{a_i \in A \mid x_i = 1\}$. Let $\mathbf{x}, \mathbf{y} \in \{0, 1\}^n$. Let $f(\mathbf{x}) = f(\mathbf{y}) = \{a_{i_1}, \dots, a_{i_m}\} \subseteq A$. This implies, for each $j \in \{1, \dots, n\}$:

$$x_j = y_j = \begin{cases} 1, & \text{if } j \in \{i_1, \dots, i_m\} \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

Hence $\mathbf{x} = \mathbf{y}$, and since they were arbitrary, f injective.

Let $B = \{a_{i_1}, \dots, a_{i_m}\} \subseteq A \implies m \leq n$. Define \mathbf{x} such that $x_j = \mathbf{1}_{\{j \in \{i_1, \dots, i_m\}\}}$. Then $f(\mathbf{x}) = B$, since B arbitrary f surjective and therefore bijective.

b: Let $A = \{a_1, \dots, a_n\}$. By (a) there is a bijection $f : \{0, 1\}^n \rightarrow \mathcal{P}(A)$. Define

$$g : \{0, 1\}^n \rightarrow \{1, \dots, 2^n\}, \quad (x_1, \dots, x_n) \mapsto 1 + \sum_{k=1}^n x_k 2^{k-1}.$$

Then g is a bijection. Hence $h := g \circ f^{-1} : \mathcal{P}(A) \rightarrow \{1, \dots, 2^n\}$ is a bijection, so $|\mathcal{P}(A)| = 2^n = 2^{|A|}$.

c: Define

$$f : \{0, 1\}^{\mathbb{N}} \rightarrow \mathcal{P}(\mathbb{N}), \quad f((x_k)_{k \in \mathbb{N}}) = \{k \in \mathbb{N} \mid x_k = 1\}.$$

To see f is injective, suppose $f(x) = f(y)$. Then for each $k \in \mathbb{N}$, we have $x_k = 1 \iff k \in f(x) = f(y) \iff y_k = 1$, hence $x_k = y_k$. So $x = y$. For surjectivity, let $B \subseteq \mathbb{N}$ and define $x \in \{0, 1\}^{\mathbb{N}}$ by $x_k = \mathbf{1}_{\{k \in B\}}$. Then $f(x) = B$. Thus f is bijective.

d: Suppose for contradiction that $\mathcal{P}(\mathbb{N})$ is countable. Then there is a list B_1, B_2, B_3, \dots of all subsets of \mathbb{N} . Define

$$D = \{n \in \mathbb{N} \mid n \notin B_n\}.$$

Then $D \subseteq \mathbb{N}$, so $D \in \mathcal{P}(\mathbb{N})$, hence $D = B_m$ for some m . But then

$$m \in D \iff m \notin B_m \iff m \notin D,$$

a contradiction. Therefore $\mathcal{P}(\mathbb{N})$ is uncountable.