Topology: Extra Problems 1

Simon Gustafsson

Problem 1

(a)

Proof. By definition $\varnothing \in \mathcal{T}_d$ and $\mathbb{R} \in \mathcal{T}_d$ since $\mathbb{R}\setminus\mathbb{R} = \varnothing$ is finite, hence $\mathcal{T}_a = \{\varnothing, \mathbb{R}\} \subset \mathcal{T}_d$. Let $U = \mathbb{R} \setminus \{0\}$; then $\mathbb{R} \setminus U = \{0\}$ is finite, so $U \in \mathcal{T}_d$ but $U \notin \mathcal{T}_a$, hence $\mathcal{T}_a \subsetneq \mathcal{T}_d$.

Let $U \in \mathcal{T}_d$. If $U = \varnothing$ then $U \in \mathcal{T}_c$. Otherwise $\mathbb{R} \setminus U$ is finite, hence closed in the standard topology, so $U = (\mathbb{R} \setminus U)^c$ is open and $U \in \mathcal{T}_c$. Thus $\mathcal{T}_d \subset \mathcal{T}_c$. However, $(0,1) \in \mathcal{T}_c$ but $(0,1) \notin \mathcal{T}_d$ since $\mathbb{R} \setminus (0,1)$ is infinite, hence $\mathcal{T}_d \subsetneq \mathcal{T}_c$.

Since $\mathcal{T}_b = \mathcal{P}(\mathbb{R})$, every subset of $\mathbb{R}$ is in $\mathcal{T}_b$, in particular every $U \in \mathcal{T}_c$, hence $\mathcal{T}_c \subset \mathcal{T}_b$. Moreover, $\{0\} \in \mathcal{T}_b$ but $\{0\}$ closed $\implies \{0\} \notin \mathcal{T}_c \implies \mathcal{T}_c \subsetneq \mathcal{T}_b$. $\square$

(b)

Proof. Since $\mathbb{R} \setminus U$ finite $\Rightarrow \mathbb{R} \setminus U$ countable, we have $\mathcal{T}_d \subset \mathcal{T}_e$. Let $U = \mathbb{R} \setminus \mathbb{Z}$. Then $\mathbb{R} \setminus U = \mathbb{Z}$ is countably infinite, so $U \in \mathcal{T}_e$, but $U \notin \mathcal{T}_d$; hence $\mathcal{T}_d \subsetneq \mathcal{T}_e$.

Since $\mathcal{T}_b = \mathcal{P}(\mathbb{R})$, every subset of $\mathbb{R}$ is in $\mathcal{T}_b$, in particular every $U \in \mathcal{T}_e$, hence $\mathcal{T}_e \subset \mathcal{T}_b$. Moreover, $\{0\} \in \mathcal{T}_b$ but $\{0\} \notin \mathcal{T}_e$ since $\mathbb{R} \setminus \{0\}$ is uncountable; therefore $\mathcal{T}_e \subsetneq \mathcal{T}_b$. $\square$