Topology: Exercises 1

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Problem 1

Proof. ($\Rightarrow$) Assume there exists a bijection $f : A \to B$. Since A is countable, there exists a countable set C and a bijection $g : A \to C$, where $C = \{1, \dots, n\}$ if A is finite and $C = \mathbb{N}$ if A is countably infinite. Define $h := g \circ f^{-1} : B \to C$. Since compositions of bijections are bijections, g is a bijection. Hence there exists a bijection from both A and B to the same set C, so they have the same cardinality.

($\Leftarrow$) Assume A and B have the same cardinality. Then there exists a set $C \subseteq \mathbb{N}$ and bijections $f : A \to C$ and $g : B \to C$. Define $h := g^{-1} \circ f : A \to B$. Since f and g are bijections, h is a bijection. □

Problem 2

Proof. Let $k \in \mathbb{N}$ and choose distinct primes $p_1, \dots, p_k$. Define

$$f : \mathbb{N}^k \to \mathbb{N}, \qquad (n_1, \dots, n_k) \mapsto p_1^{n_1} \cdots p_k^{n_k}.$$

Let $x, y \in \mathbb{N}^k$ and suppose $f(x) = f(y)$. Then

$$p_1^{x_1} \cdots p_k^{x_k} = p_1^{y_1} \cdots p_k^{y_k}.$$

By uniqueness of prime factorization, we must have $x_i = y_i$ for each $i \in \{1, \dots, k\}$. Hence $x = y$, and f is injective. It then follows immediately by Theorem 7.1 that $\mathbb{N}^k$ is countable. □

Problem 3

Proof. Firstly, the trivial map $f : \mathbb{N} \to \mathbb{Q}$ $n \mapsto n$ is injective $\implies$ $\mathbb{Q}$ is infinite.

Next consider

$$g : \mathbb{Z} \times \mathbb{Z}^* \to \mathbb{Q} \quad g(m, n) = \frac{m}{n}.$$

$\forall q \in \mathbb{Q}$, $\exists m \in \mathbb{Z} \wedge \exists n \in \mathbb{Z}^*$ such that $q = \frac{m}{n} \implies g(m, n) = q$ and hence g surjective.

Since $\mathbb{Z}$ and $\mathbb{Z}^*$ are both countable, $\mathbb{Z} \times \mathbb{Z}^*$ is countable by Theorem 7.6. By Theorem 7.1, there exists a surjection $h : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}^*$. Then $g \circ h : \mathbb{N} \to \mathbb{Q}$ is a surjection, so $\mathbb{Q}$ is countable. Therefore $\mathbb{Q}$ is countably infinite. □

Problem 4

a: Label all elements in $A = \{a_1, \cdots, a_n\}$. Let $f : \{0,1\}^n \to \mathcal{P}(\mathcal{A})$, $(x_1, \cdots, x_n) \mapsto \{a_i \in A \mid x_i = 1\}$. Let $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$. Let $f(\mathbf{x}) = f(\mathbf{y}) = \{a_{i_1}, \cdots, a_{i_m}\} \subseteq A$. This implies, for each $j \in \{1, \cdots, n\}$:

$$x_j = y_j = \begin{cases} 1, & \text{if } j \in \{i_1, \cdots, i_m\} \\ 0, & \text{otherwise} \end{cases}$$

Hence $\mathbf{x} = \mathbf{y}$, and since they were arbitrary, f injective.
Let $B = \{a_{i_1}, \cdots, a_{i_m}\} \subseteq A \implies m \leq n$. Define $\mathbf{x}$ such that $x_j = \mathbf{1}_{\{j \in \{i_1, \cdots, i_m\}\}}$. Then $f(\mathbf{x}) = B$, since B arbitrary f surjective and therefore bijective.

b: Let $A = \{a_1, \ldots, a_n\}$. By (a) there is a bijection $f : \{0,1\}^n \to \mathcal{P}(A)$. Consider

$$g : \{0,1\}^n \to \{1, \ldots, 2^n\}, \qquad (x_1, \ldots, x_n) \mapsto 1 + \sum_{k=1}^{n} x_k 2^{k-1}.$$

g is a bijection. Hence $h := g \circ f^{-1} : \mathcal{P}(A) \to \{1, \ldots, 2^n\}$ is a bijection, so $|\mathcal{P}(A)| = 2^n = 2^{|A|}$.

c: Define

$$f : \{0,1\}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N}), \qquad (x_1, x_2 \cdots) \mapsto \{\, k \in \mathbb{N} \mid x_k = 1 \,\}.$$

Let $\mathbf{x}, \mathbf{y} \in \{0,1\}^{\mathbb{N}} : f(\mathbf{x}) = f(\mathbf{y}) = C \subseteq \mathbb{N}$. Then

$$x_i = y_i = \begin{cases} 1, & \text{if } i \in C \\ 0, & \text{otherwise} \end{cases} \implies \mathbf{x} = \mathbf{y}$$

Hence f injective. Let $A \in \mathcal{P}(\mathbb{N})$, $\exists \mathbf{x} \in \{0,1\}^{\mathbb{N}} : x_k = \mathbf{1}_{\{k \in A\}}$. Then $f(\mathbf{x}) = A.$, Since A arbitrary, f surjective and therefore also bijective.

d: Suppose for contradiction that $\mathcal{P}(\mathbb{N})$ is countable. Then there is a list $B_1, B_2, B_3, \ldots$ of all subsets of $\mathbb{N}$. Define

$$D = \{\, n \in \mathbb{N} \mid n \notin B_n \,\}.$$

Then $D \subseteq \mathbb{N}$, so $D \in \mathcal{P}(\mathbb{N})$, hence $D = B_m$ for some m. But then

$$m \in D \iff m \notin B_m \iff m \notin D,$$

a contradiction. Therefore $\mathcal{P}(\mathbb{N})$ is uncountable.