Topology: Exercises 1

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Problem 1

Proof. ($\Rightarrow$) Assume there exists a bijection $h : A \to B$. Since A is countable, there exists a set C and a bijection $f : A \to C$, where $C = \{1, \ldots, n\}$ if A is finite and $C = \mathbb{N}$ if A is countably infinite. Define $g := f \circ h^{-1} : B \to C$. Since compositions of bijections are bijections, g is a bijection. Hence there exisrs a bijection from both A and B to the same set C, so they have the same cardinality.

($\Leftarrow$) Assume A and B have the same cardinality. Then there exists a set $C \subseteq \mathbb{N}$ and bijections $f : A \to C$ and $g : B \to C$. Define $h := g^{-1} \circ f : A \to B$. Since f and g are bijections, h is a bijection. $\square$

Problem 2

Proof. Let $k \in \mathbb{N}$ and choose distinct primes $p_1, \ldots, p_k$. Define

$$f : \mathbb{N}^k \to \mathbb{N}, \qquad (n_1, \ldots, n_k) \mapsto p_1^{n_1} \cdots p_k^{n_k}.$$

Let $x, y \in \mathbb{N}^k$ and suppose $f(x) = f(y)$. Then

$$p_1^{x_1} \cdots p_k^{x_k} = p_1^{y_1} \cdots p_k^{y_k}.$$

By uniqueness of prime factorization, we must have $x_i = y_i$ for each $i \in \{1, \ldots, k\}$. Hence $x = y$, and f is injective. Since $\mathbb{N}$ is countable and $\mathbb{N}^k$ injects into $\mathbb{N}$, it follows by Theorem 7.1 that $\mathbb{N}^k$ is countable. $\square$

Problem 3

Proof. $\mathbb{Q}$ is infinite since the map $f : \mathbb{N} \to \mathbb{Q}$, $n \mapsto n$, is injective.

Define $g : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$ by

$$g(m, n) = \frac{m}{n}.$$

Then g is surjective: for any $q \in \mathbb{Q}$, write $q = \frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus \{0\}$, so $q = g(m, n)$.

Since $\mathbb{Z}$ and $\mathbb{Z} \setminus \{0\}$ are countable, $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable by Munkres Theorem 7.6. By Munkres Theorem 7.1, there exists a surjection $h : \mathbb{N} \to \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Then $g \circ h : \mathbb{N} \to \mathbb{Q}$ is a surjection, so $\mathbb{Q}$ is countable. Therefore $\mathbb{Q}$ is countably infinite. $\square$

Problem 4

a: Label all elements in $A = \{a_1, \cdots, a_n\}$. Let $f : \{0,1\}^n \to \mathcal{P}(\mathcal{A})$, $(x_1, \cdots, x_n) \mapsto \{a_i \in A \mid x_i = 1\}$. Let $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$. Let $f(\mathbf{x}) = f(\mathbf{y}) = \{a_{i_1}, \cdots, a_{i_m}\} \subseteq A$. This implies, for each $j \in \{1, \cdots, n\}$:

$$x_j = y_j = \begin{cases} 1, & \text{if } j \in \{i_1, \cdots, i_m\} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Hence $\mathbf{x} = \mathbf{y}$, and since they were arbitrary, f injective.
Let $B = \{a_{i_1}, \cdots, a_{i_m}\} \subseteq A \implies m \leq n$. Define $\mathbf{x}$ such that $x_j = \mathbf{1}_{\{j \in \{i_1, \cdot, i_m\}\}}$. Then $f(\mathbf{x}) = B$, since B arbitrary f surjective and therefore bijective.

b: Let $A = \{a_1, \ldots, a_n\}$. By (a) there is a bijection $f : \{0,1\}^n \to \mathcal{P}(A)$. Define

$$g : \{0,1\}^n \to \{1, \ldots, 2^n\}, \qquad (x_1, \ldots, x_n) \mapsto 1 + \sum_{k=1}^{n} x_k 2^{k-1}.$$

Then g is a bijection. Hence $h := g \circ f^{-1} : \mathcal{P}(A) \to \{1, \ldots, 2^n\}$ is a bijection, so $|\mathcal{P}(A)| = 2^n = 2^{|A|}$.

c: Define

$$f : \{0,1\}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N}), \qquad f((x_k)_{k \in \mathbb{N}}) = \{\, k \in \mathbb{N} \mid x_k = 1 \,\}.$$

To see f is injective, suppose $f(x) = f(y)$. Then for each $k \in \mathbb{N}$, we have $x_k = 1 \iff k \in f(x) = f(y) \iff y_k = 1$, hence $x_k = y_k$. So $x = y$. For surjectivity, let $B \subseteq \mathbb{N}$ and define $x \in \{0,1\}^{\mathbb{N}}$ by $x_k = \mathbf{1}_{\{k \in B\}}$. Then $f(x) = B$. Thus f is bijective.