Topology: Extra Problems 1

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Problem 1

(a)

Proof. By definition $\varnothing \in \mathcal{T}_d$ and $\mathbb{R} \in \mathcal{T}_d$ since $\mathbb{R}\setminus\mathbb{R} = \varnothing$ is finite, hence $\mathcal{T}_a = \{\varnothing, \mathbb{R}\} \subset \mathcal{T}_d$. Let $U = \mathbb{R} \setminus \{0\}$; then $\mathbb{R} \setminus U = \{0\}$ is finite, so $U \in \mathcal{T}_d$ but $U \notin \mathcal{T}_a$, hence $\mathcal{T}_a \subsetneq \mathcal{T}_d$.

Let $U \in \mathcal{T}_d$. If $U = \varnothing$ then $U \in \mathcal{T}_c$. Otherwise $\mathbb{R} \setminus U$ is finite, hence closed in the standard topology, so $U = (\mathbb{R} \setminus U)^c$ is open and $U \in \mathcal{T}_c$. Thus $\mathcal{T}_d \subset \mathcal{T}_c$. However, $(0, 1) \in \mathcal{T}_c$ but $(0, 1) \notin \mathcal{T}_d$ since $\mathbb{R} \setminus (0, 1)$ is infinite, hence $\mathcal{T}_d \subsetneq \mathcal{T}_c$.

Since $\mathcal{T}_b = \mathcal{P}(\mathbb{R})$, every subset of $\mathbb{R}$ is in $\mathcal{T}_b$, in particular every $U \in \mathcal{T}_c$, hence $\mathcal{T}_c \subset \mathcal{T}_b$. Moreover, $\{0\} \in \mathcal{T}_b$ but $\{0\}$ closed $\implies \{0\} \notin \mathcal{T}_c \implies \mathcal{T}_c \subsetneq \mathcal{T}_b$. □

(b)

Proof. Since $\mathbb{R} \setminus U$ finite $\Rightarrow \mathbb{R} \setminus U$ countable, we have $\mathcal{T}_d \subset \mathcal{T}_e$. Let $U = \mathbb{R} \setminus \mathbb{Z}$. Then $\mathbb{R} \setminus U = \mathbb{Z}$ is countably infinite, so $U \in \mathcal{T}_e$, but $U \notin \mathcal{T}_d$; hence $\mathcal{T}_d \subsetneq \mathcal{T}_e$.

Since $\mathcal{T}_b = \mathcal{P}(\mathbb{R})$, every subset of $\mathbb{R}$ is in $\mathcal{T}_b$, in particular every $U \in \mathcal{T}_e$, hence $\mathcal{T}_e \subset \mathcal{T}_b$. Moreover, $\{0\} \in \mathcal{T}_b$ but $\{0\} \notin \mathcal{T}_e$ since $\mathbb{R} \setminus \{0\}$ is uncountable; therefore $\mathcal{T}_e \subsetneq \mathcal{T}_b$. □

Problem 2

(a)

Proof. $d_X \geq 0 \wedge d_Y \geq 0 \implies d \geq 0$. Assume $d((x_1, y_1), (x_2, y_2)) = 0$. Then

$$\max\{d_X(x_1, x_2),\ d_Y(y_1, y_2)\} = 0,$$

which implies $d_X(x_1, x_2) = d_Y(y_1, y_2) = 0$. Since d_X and d_Y are metrics, this gives $x_1 = x_2$ and $y_1 = y_2$, hence $(x_1, y_1) = (x_2, y_2)$. If $(x_1, y_1) = (x_2, y_2)$, then $d_X(x_1, x_2) = d_Y(y_1, y_2) = 0$, so $d((x_1, y_1), (x_2, y_2)) = 0$. Let $(x_i, y_i) \in X \times Y$,

$$\begin{aligned}
d_X(x_1, x_3) &\leq d_X(x_1, x_2) + d_X(x_2, x_3) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)),\\
d_Y(y_1, y_3) &\leq d_Y(y_1, y_2) + d_Y(y_2, y_3) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)),\\
d((x_1, y_1), (x_3, y_3)) &= \max\{d_X(x_1, x_3),\ d_Y(y_1, y_3)\} \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)).
\end{aligned}$$

□

(b)

Proof. Assume $(x', y') \in B_r^d((x,y))$. Then $d((x',y'),(x,y)) = \max\{d_X(x',x),\, d_Y(y',y)\} < r$, so $d_X(x',x) < r \implies x' \in B_r^{d_X}(x)$ and $d_Y(y',y) < r \implies y' \in B_r^{d_Y}(y)$. Therefore $(x',y') \in B_r^{d_X}(x) \times B_r^{d_Y}(y)$.

Assume $(x',y') \in B_r^{d_X}(x) \times B_r^{d_Y}(y)$. Then $d_X(x',x) = r_1 < r$ and $d_Y(y',y) = r_2 < r$, hence $d((x',y'),(x,y)) = \max\{r_1, r_2\} < r \implies (x',y') \in B_r^d((x,y))$ □

(c)

Proof. U open $\implies \exists r_1 > 0 : B_{r_1}^{d_X}(x) \subseteq U$, and since V is open, $\exists r_2 > 0 : B_{r_2}^{d_Y}(y) \subseteq V$. Let $\varepsilon = \min\{r_1, r_2\}$. Then

$$B_\varepsilon^d((x,y)) = B_\varepsilon^{d_X}(x) \times B_\varepsilon^{d_Y}(y) \subseteq B_{r_1}^{d_X}(x) \times B_{r_2}^{d_Y}(y) \subseteq U \times V.$$

Let $(x,y) \in X \times Y$ and $\varepsilon > 0$. Set $U := B_\varepsilon^{d_X}(x) \subseteq X, V := B_\varepsilon^{d_Y}(y) \subseteq Y$. Then U and V are open, $(x,y) \in U \times V$, and by previous

$$U \times V = B_\varepsilon^{d_X}(x) \times B_\varepsilon^{d_Y}(y) = B_\varepsilon^d((x,y)),$$

hence $(x,y) \in U \times V \subseteq B_\varepsilon^d((x,y))$.

□