Topology: Exercises 1

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Problem 1

Proof. ($\Rightarrow$) Assume there exists a bijection $h : A \to B$. Since A is countable, there exists a set C and a bijection $f : A \to C$, where $C = \{1, \ldots, n\}$ if A is finite and $C = \mathbb{N}$ if A is countably infinite. Define $g := f \circ h^{-1} : B \to C$. Since compositions of bijections are bijections, g is a bijection. Hence there exisrs a bijection from both A and B to the same set C, so they have the same cardinality.

($\Leftarrow$) Assume A and B have the same cardinality. Then there exists a set $C \subseteq \mathbb{N}$ and bijections $f : A \to C$ and $g : B \to C$. Define $h := g^{-1} \circ f : A \to B$. Since f and g are bijections, h is a bijection. $\square$

Problem 2

Proof. Let $k \in \mathbb{N}$ and choose distinct primes $p_1, \ldots, p_k$. Define

$$f : \mathbb{N}^k \to \mathbb{N}, \qquad (n_1, \ldots, n_k) \mapsto p_1^{n_1} \cdots p_k^{n_k}.$$

Let $x, y \in \mathbb{N}^k$ and suppose $f(x) = f(y)$. Then

$$p_1^{x_1} \cdots p_k^{x_k} = p_1^{y_1} \cdots p_k^{y_k}.$$

By uniqueness of prime factorization, we must have $x_i = y_i$ for each $i \in \{1, \ldots, k\}$. Hence $x = y$, and f is injective. Since $\mathbb{N}$ is countable and $\mathbb{N}^k$ injects into $\mathbb{N}$, it follows by Theorem 7.1 that $\mathbb{N}^k$ is countable. $\square$

Problem 3

Proof. $\mathbb{Q}$ is infinite since the map $f : \mathbb{N} \to \mathbb{Q}$, $n \mapsto n$, is injective.

Define $g : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$ by

$$g(m, n) = \frac{m}{n}.$$

Then g is surjective: for any $q \in \mathbb{Q}$, write $q = \frac{m}{n}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z} \setminus \{0\}$, so $q = g(m, n)$.

Since $\mathbb{Z}$ and $\mathbb{Z} \setminus \{0\}$ are countable, $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable by Munkres Theorem 7.6. By Munkres Theorem 7.1, there exists a surjection $h : \mathbb{N} \to \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Then $g \circ h : \mathbb{N} \to \mathbb{Q}$ is a surjection, so $\mathbb{Q}$ is countable. Therefore $\mathbb{Q}$ is countably infinite. $\square$