Topology: Extra Problems 1

Simon Gustafsson

Problem 1

(a)

Proof. By definition $\varnothing \in \mathcal{T}_d$ and $\mathbb{R} \in \mathcal{T}_d$ since $\mathbb{R} \setminus \mathbb{R} = \varnothing$ is finite, hence $\mathcal{T}_a = \{\varnothing, \mathbb{R}\} \subset \mathcal{T}_d$. Let $U = \mathbb{R} \setminus \{0\}$; then $\mathbb{R} \setminus U = \{0\}$ is finite, so $U \in \mathcal{T}_d$ but $U \notin \mathcal{T}_a$, hence $\mathcal{T}_a \subsetneq \mathcal{T}_d$.

Let $U \in \mathcal{T}_d$. If $U = \varnothing$ then $U \in \mathcal{T}_c$. Otherwise $\mathbb{R} \setminus U$ is finite, hence closed in the standard topology, so $U = (\mathbb{R} \setminus U)^c$ is open and $U \in \mathcal{T}_c$. Thus $\mathcal{T}_d \subset \mathcal{T}_c$. However, $(0,1) \in \mathcal{T}_c$ but $(0,1) \notin \mathcal{T}_d$ since $\mathbb{R} \setminus (0,1)$ is infinite, hence $\mathcal{T}_d \subsetneq \mathcal{T}_c$.

Since $\mathcal{T}_b = \mathcal{P}(\mathbb{R})$, every subset of $\mathbb{R}$ is in $\mathcal{T}_b$, in particular every $U \in \mathcal{T}_c$, hence $\mathcal{T}_c \subset \mathcal{T}_b$. Moreover, $\{0\} \in \mathcal{T}_b$ but $\{0\}$ closed $\implies \{0\} \notin \mathcal{T}_c \implies \mathcal{T}_c \subsetneq \mathcal{T}_b$. $\qquad\square$

(b)

Proof. Since $\mathbb{R} \setminus U$ finite $\Rightarrow \mathbb{R} \setminus U$ countable, we have $\mathcal{T}_d \subset \mathcal{T}_e$. Let $U = \mathbb{R} \setminus \mathbb{Z}$. Then $\mathbb{R} \setminus U = \mathbb{Z}$ is countably infinite, so $U \in \mathcal{T}_e$, but $U \notin \mathcal{T}_d$; hence $\mathcal{T}_d \subsetneq \mathcal{T}_e$.

Since $\mathcal{T}_b = \mathcal{P}(\mathbb{R})$, every subset of $\mathbb{R}$ is in $\mathcal{T}_b$, in particular every $U \in \mathcal{T}_e$, hence $\mathcal{T}_e \subset \mathcal{T}_b$. Moreover, $\{0\} \in \mathcal{T}_b$ but $\{0\} \notin \mathcal{T}_e$ since $\mathbb{R} \setminus \{0\}$ is uncountable; therefore $\mathcal{T}_e \subsetneq \mathcal{T}_b$. $\qquad\square$

Problem 2

(a)

Proof. $d_X \geq 0 \wedge d_Y \geq 0 \implies d \geq 0$. Assume $d((x_1, y_1), (x_2, y_2)) = 0$. Then

$$\max\{d_X(x_1, x_2),\, d_Y(y_1, y_2)\} = 0,$$

which implies $d_X(x_1, x_2) = d_Y(y_1, y_2) = 0$. Since d_X and d_Y are metrics, this gives $x_1 = x_2$ and $y_1 = y_2$, hence $(x_1, y_1) = (x_2, y_2)$. If $(x_1, y_1) = (x_2, y_2)$, then $d_X(x_1, x_2) = d_Y(y_1, y_2) = 0$, so $d((x_1, y_1), (x_2, y_2)) = 0$. Let $(x_i, y_i) \in X \times Y$,

$$d_X(x_1, x_3) \leq d_X(x_1, x_2) + d_X(x_2, x_3) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)),$$
$$d_Y(y_1, y_3) \leq d_Y(y_1, y_2) + d_Y(y_2, y_3) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)),$$
$$d((x_1, y_1), (x_3, y_3)) = \max\{d_X(x_1, x_3),\, d_Y(y_1, y_3)\} \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)).$$

$\qquad\square$

(b)

Proof. Assume $(x', y') \in B_r^d((x, y))$. Then $d((x', y'), (x, y)) = \max\{d_X(x', x), d_Y(y', y)\} < r$, so $d_X(x', x) < r \implies x' \in B_r^{d_X}(x)$ and $d_Y(y', y) < r \implies y' \in B_r^{d_Y}(y)$. Therefore $(x', y') \in B_r^{d_X}(x) \times B_r^{d_Y}(y)$.

Assume $(x', y') \in B_r^{d_X}(x) \times B_r^{d_Y}(y)$. Then $d_X(x', x) = r_1 < r$ and $d_Y(y', y) = r_2 < r$, hence $d((x', y'), (x, y)) = \max\{r_1, r_2\} < r \implies (x', y') \in B_r^d((x, y))$ $\qquad\square$