LUNDS UNIVERSITET

Matematikcentrum
Matematik NF

Topology, Spring 2026 **MATM36**

Extra problems 2

Let X be a vector space over $\mathbb{R}$. Recall that a norm $\|\cdot\|$ on X is a map $X \to \mathbb{R}$, $x \mapsto \|x\|$, such that

1. $\|x\| \geq 0$ for all $x \in X$, $\|x\| = 0 \Leftrightarrow x = 0$
2. $\|\alpha x\| = |\alpha|\|x\|$ for $\alpha \in \mathbb{R}$, $x \in X$
3. $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in X$.

An $\mathbb{R}$-vector space X together with a norm $\|\cdot\|$ is called a normed space. A norm always gives rise to a metric via $d(x, y) = \|x - y\|$. The metric topology generated by d is called the norm topology on X.

1. Recall from Linear Analysis the space of square summable sequences

$$\ell^2 = \{(x_j)_{j\geq 0} : x_j \in \mathbb{R}, \sum_{j=0}^{\infty} |x_j|^2 < \infty\}$$

This is a Hilbert space with the inner product

$$\langle \bar{x}, \bar{y} \rangle = \sum_{j=0}^{\infty} x_j y_j$$

and the norm

$$\|\bar{x}\| = \left(\sum_{j=0}^{\infty} |x_j|^2 \right)^{1/2}$$

for $\bar{x} = (x_j)_{j\geq 0}, \bar{y} = (y_j)_{j\geq 0} \in \ell^2$. We are given that the continuous linear maps $\phi : \ell^2 \to \mathbb{R}$ (where we mean continuous with respect to the norm topology $\mathcal{T}$ in ℓ^2 and the standard topology on $\mathbb{R}$) are of the form

$$\phi = \phi_{\bar{y}}, \text{ where } \bar{y} \in \ell^2, \phi_{\bar{y}}(\bar{x}) = \sum_{j=0}^{\infty} x_j y_j.$$

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Let $\mathcal{T}_w$ be the topology generated by the subbasis

$$\bigcup_{\bar{y}\in\ell^2} \{\phi_{\bar{y}}^{-1}(U) : U \subseteq \mathbb{R} \text{ open}\}.$$

$\mathcal{T}_w$ is the coarsest topology on ℓ^2 such that $\phi_{\bar{y}} : \ell^2 \to \mathbb{R}$ is continuous for each $\bar{y} \in \ell^2$. It is called the weak topology on ℓ^2.

a) Show that $\mathcal{T}_w \subseteq \mathcal{T}$.

b) Consider the standard orthonormal basis sequence $(\bar{e}_n)$ in ℓ^2, where $\bar{e}_n = (0, \dots, 0, 1, 0, \dots,)$, with a 1 at the nth position. Show that $\bar{e}_n \overset{n\to\infty}{\to} (0, 0, 0, \dots)$ with respect to $\mathcal{T}_w$, but $\bar{e}_n \overset{n\to\infty}{\not\to} (0, 0, 0, \dots)$ with respect to $\mathcal{T}$.

c) Conclude that $\mathcal{T}_w \subsetneq \mathcal{T}$.

d) Show that $(\bar{e}_n)$ has no convergent subsequence with respect to $\mathcal{T}$.

e) Conclude that the unit ball in ℓ^2, the set $\{\bar{x} \in \ell^2 : \|\bar{x}\| \leq 1\}$, is not compact in the norm topology.

2. We define the *Cantor set* $\mathcal{C}$:
Let $E_0 = [0, 1]$, $E_1 = [0, 1]\backslash(1/3, 2/3)$, $E_2 = E_1\backslash((1/9, 2/9) \cup (7/9, 8/9))$, ..., $E_{n+1} = E_n\backslash((1/3^{n+1}, 2/3^{n+1}) \cup (7/3^{n+1}, 8/3^{n+1}) \cup \cdots \cup (1 - 2/3^{n+1}, 1 - 1/3^{n+1}))$, and so on. In other words, start with the interval $[0, 1]$, remove the open middle third $(1/2, 2/3)$ - this gives E_1. E_1 consists of two intervals of length $1/3$. Remove from both the open middle third - this gives E_2. E_2 consists now of four intervals of length $1/9$. Remove from all four intervals the open middle third - this gives E_3, and so on. E_n then consists of 2^n intervals of length $1/3^n$.

Now let $\mathcal{C} = \bigcap_{n\in\mathbb{N}} E_n$. Show that with respect to the standard topology on $\mathbb{R}$,

a) Each E_n is compact.

b) $\mathcal{C}$ is compact.

c) The interior of $\mathcal{C}$ is empty.
(**Hint:** Use that each E_n consists of intervals of length $1/3^n$.)

d) $\mathcal{C}$ is totally disconnected.

Try to solve the problem no later than March 5. We'll try to discuss the solutions in class, during the seminars.