LUNDS UNIVERSITET

Matematikcentrum
Matematik NF

Topology, Spring 2026 **MATM36**

Extra problems 1

1. Consider the topologies in parts from the Exercises for Seminar 2, denoting them by $\mathcal{T}_a, \dots, \mathcal{T}_e$. (You do not need to show again that these are topologies).

(a) The trivial topology, $\mathcal{T} = \{\emptyset, \mathbb{R}\}$.
(b) The discrete topology, $\mathcal{T} = \mathcal{P}(\mathbb{R})$.
(c) The standard topology $\mathcal{T} = \mathcal{T}_{\mathcal{B}}$, generated by the basis of r-balls, $\mathcal{B} = \{B_r(x) : x \in \mathbb{R}, r > 0\}$. Here, $B_r(x) = \{y \in \mathbb{R} : |x - y| < r\}$.
(d) The co-finite topology, $\mathcal{T} = \{U \subseteq \mathbb{R} : \mathbb{R}\backslash U \text{ is finite or } U = \emptyset\}$.
(e) The co-countable topology, $\mathcal{T} = \{U \subseteq \mathbb{R} : \mathbb{R}\backslash U \text{ is countable or } U = \emptyset\}$.

a) Show that $\mathcal{T}_a \subsetneq \mathcal{T}_d \subsetneq \mathcal{T}_c \subsetneq \mathcal{T}_b$ and $\mathcal{T}_d \subsetneq \mathcal{T}_e \subsetneq \mathcal{T}_b$.
b) Is it possible to compare $\mathcal{T}_c$ and $\mathcal{T}_e$, i.e. is one topology contained in the other?

2. Let (X, d_X) and (Y, d_Y) be two metric spaces. We define

$$d : X \times Y \to \mathbb{R}, \quad d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} \text{ for } (x_1, y_1), (x_2, y_2) \in X \times Y.$$

a) Show that d is a metric on $X \times Y$.
b) Show that for $x \in X$, $y \in Y$ and $r > 0$,

$$B_r^d((x, y)) = B_r^{d_X}(x) \times B_r^{d_Y}(y).$$

c) Show:
(i) Whenever U is open in (X, d_X) and V is open in (Y, d_Y), and $(x, y) \in U \times V$, then there exists $\varepsilon > 0$ with $B_\varepsilon^d((x, y)) \subseteq U \times V$.
(ii) Whenever $(x, y) \in X \times Y$ and $\varepsilon > 0$, there exists U open in (X, d_X) and V open in (Y, d_Y), such that $(x, y) \in U \times V \subseteq B_\varepsilon^d((x, y))$.

d) Let $\mathcal{T}$ be the metric topology for the metric d on $X \times Y$, and let $\mathcal{T}'$ be the topology on $X \times Y$ generated by the basis $\mathcal{B} = \{U \times V : U \text{ open in } (X, d_X), V \text{ open in } (Y, d_Y)\}$. Show that $\mathcal{T} = \mathcal{T}'$.

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e) Now let us assume that $X = Y = \mathbb{R}$, and that d_X and d_Y are both the standard metric on $\mathbb{R}$.
With $\mathcal{T}'$ defined as in d), conclude that the standard topology on $\mathbb{R}^2$ (that is, the metric topology with repect to the Euclidean metric d_2) is equal to the topology $\mathcal{T}'$.
Hint: You may use Question 1c of the Exercises for Seminar 2 for this.

Try to solve the problems no later than February 5. We will try to discuss the solutions in class, during the seminars.