

# Topology: Extra Problems 1

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## Problem 1

(a)

*Proof.* By definition  $\emptyset \in \mathcal{T}_d$  and  $\mathbb{R} \in \mathcal{T}_d$  since  $\mathbb{R} \setminus \mathbb{R} = \emptyset$  is finite, hence  $\mathcal{T}_a = \{\emptyset, \mathbb{R}\} \subset \mathcal{T}_d$ . Let  $U = \mathbb{R} \setminus \{0\}$ ; then  $\mathbb{R} \setminus U = \{0\}$  is finite, so  $U \in \mathcal{T}_d$  but  $U \notin \mathcal{T}_a$ , hence  $\mathcal{T}_a \subsetneq \mathcal{T}_d$ .

Let  $U \in \mathcal{T}_d$ . If  $U = \emptyset$  then  $U \in \mathcal{T}_c$ . Otherwise  $\mathbb{R} \setminus U$  is finite, hence closed in the standard topology, so  $U = (\mathbb{R} \setminus U)^c$  is open and  $U \in \mathcal{T}_c$ . Thus  $\mathcal{T}_d \subset \mathcal{T}_c$ . However,  $(0, 1) \in \mathcal{T}_c$  but  $(0, 1) \notin \mathcal{T}_d$  since  $\mathbb{R} \setminus (0, 1)$  is infinite, hence  $\mathcal{T}_d \subsetneq \mathcal{T}_c$ .

Since  $\mathcal{T}_b = \mathcal{P}(\mathbb{R})$ , every subset of  $\mathbb{R}$  is in  $\mathcal{T}_b$ , in particular every  $U \in \mathcal{T}_c$ , hence  $\mathcal{T}_c \subset \mathcal{T}_b$ . Moreover,  $\{0\} \in \mathcal{T}_b$  but  $\{0\}$  closed  $\implies \{0\} \notin \mathcal{T}_c \implies \mathcal{T}_c \subsetneq \mathcal{T}_b$ .  $\square$

(b)

*Proof.* Since  $\mathbb{R} \setminus U$  finite  $\implies \mathbb{R} \setminus U$  countable, we have  $\mathcal{T}_d \subset \mathcal{T}_e$ . Let  $U = \mathbb{R} \setminus \mathbb{Z}$ . Then  $\mathbb{R} \setminus U = \mathbb{Z}$  is countably infinite, so  $U \in \mathcal{T}_e$ , but  $U \notin \mathcal{T}_d$ ; hence  $\mathcal{T}_d \subsetneq \mathcal{T}_e$ .

Since  $\mathcal{T}_b = \mathcal{P}(\mathbb{R})$ , every subset of  $\mathbb{R}$  is in  $\mathcal{T}_b$ , in particular every  $U \in \mathcal{T}_e$ , hence  $\mathcal{T}_e \subset \mathcal{T}_b$ . Moreover,  $\{0\} \in \mathcal{T}_b$  but  $\{0\} \notin \mathcal{T}_e$  since  $\mathbb{R} \setminus \{0\}$  is uncountable; therefore  $\mathcal{T}_e \subsetneq \mathcal{T}_b$ .  $\square$

## Problem 2

(a)

*Proof.*  $d_X \geq 0 \wedge d_Y \geq 0 \implies d \geq 0$ . Assume  $d((x_1, y_1), (x_2, y_2)) = 0$ . Then

$$\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\} = 0,$$

which implies  $d_X(x_1, x_2) = d_Y(y_1, y_2) = 0$ . Since  $d_X$  and  $d_Y$  are metrics, this gives  $x_1 = x_2$  and  $y_1 = y_2$ , hence  $(x_1, y_1) = (x_2, y_2)$ . If  $(x_1, y_1) = (x_2, y_2)$ , then  $d_X(x_1, x_2) = d_Y(y_1, y_2) = 0$ , so  $d((x_1, y_1), (x_2, y_2)) = 0$ . Let  $(x_i, y_i) \in X \times Y$ ,

$$\begin{aligned} d_X(x_1, x_3) &\leq d_X(x_1, x_2) + d_X(x_2, x_3) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)), \\ d_Y(y_1, y_3) &\leq d_Y(y_1, y_2) + d_Y(y_2, y_3) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)), \\ d((x_1, y_1), (x_3, y_3)) &= \max\{d_X(x_1, x_3), d_Y(y_1, y_3)\} \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)). \end{aligned}$$

$\square$

(b)

*Proof.* Assume  $(x', y') \in B_r^d((x, y))$ . Then  $d((x', y'), (x, y)) = \max\{d_X(x', x), d_Y(y', y)\} < r$ , so  $d_X(x', x) < r \implies x' \in B_r^{d_X}(x)$  and  $d_Y(y', y) < r \implies y' \in B_r^{d_Y}(y)$ . Therefore  $(x', y') \in B_r^{d_X}(x) \times B_r^{d_Y}(y)$ .

Assume  $(x', y') \in B_r^{d_X}(x) \times B_r^{d_Y}(y)$ . Then  $d_X(x', x) = r_1 < r$  and  $d_Y(y', y) = r_2 < r$ , hence  $d((x', y'), (x, y)) = \max\{r_1, r_2\} < r \implies (x', y') \in B_r^d((x, y))$   $\square$

(c)

*Proof.*  $U$  open  $\implies \exists r_1 > 0 : B_{r_1}^{d_X}(x) \subseteq U$ , and since  $V$  is open,  $\exists r_2 > 0 : B_{r_2}^{d_Y}(y) \subseteq V$ . Let  $\varepsilon = \min\{r_1, r_2\}$ . Then

$$B_\varepsilon^d((x, y)) = B_\varepsilon^{d_X}(x) \times B_\varepsilon^{d_Y}(y) \subseteq B_{r_1}^{d_X}(x) \times B_{r_2}^{d_Y}(y) \subseteq U \times V.$$

Let  $(x, y) \in X \times Y$  and  $\varepsilon > 0$ . Set  $U := B_\varepsilon^{d_X}(x) \subseteq X, V := B_\varepsilon^{d_Y}(y) \subseteq Y$ . Then  $U$  and  $V$  are open,  $(x, y) \in U \times V$ , and by previous

$$U \times V = B_\varepsilon^{d_X}(x) \times B_\varepsilon^{d_Y}(y) = B_\varepsilon^d((x, y)),$$

hence  $(x, y) \in U \times V \subseteq B_\varepsilon^d((x, y))$ .  $\square$

(d)

*Proof.* Let  $W \in \mathcal{T}'$ . Then  $W = \bigcup_\alpha (U_\alpha \times V_\alpha)$  with each  $U_\alpha$  open in  $X$  and  $V_\alpha$  open in  $Y$ . Fix  $\alpha$  and  $(x, y) \in U_\alpha \times V_\alpha$ . By (c)(i) there exists  $\varepsilon > 0$  such that

$$B_\varepsilon^d((x, y)) \subseteq U_\alpha \times V_\alpha.$$

Hence each  $U_\alpha \times V_\alpha$  is  $d$ -open, so  $U_\alpha \times V_\alpha \in \mathcal{T}$ , and therefore  $W \in \mathcal{T}$  (arbitrary unions of  $d$ -open sets are  $d$ -open).

Let  $W \in \mathcal{T}$  and fix  $(x, y) \in W$ . Then there exists  $\varepsilon > 0$  such that

$$B_\varepsilon^d((x, y)) \subseteq W.$$

By (c)(ii) there exist  $U$  open in  $X$  and  $V$  open in  $Y$  with

$$(x, y) \in U \times V \subseteq B_\varepsilon^d((x, y)) \subseteq W.$$

Thus every point of  $W$  lies in some basis element  $U \times V \in \mathcal{B}$  contained in  $W$ , so  $W$  is a union of such basis elements and hence  $W \in \mathcal{T}' \text{ implies } \mathcal{T} \subseteq \mathcal{T}'$ . Therefore  $\mathcal{T} = \mathcal{T}'$ .  $\square$