LUNDS UNIVERSITET

Matematikcentrum
*Matematik NF*

**Topology, Spring 2026** **MATM36**

**Exercises for Seminar 2**

**1.** Gamelin, Section 1.1: 1,2,3,5,6,7,9,10,11,12,13,14, plus the following ones:

**2.** Let $X$ be a set, $X \neq \emptyset$, and let $d$, $\tilde{d}$ be equivalent metrics on $X$, that means, there exist $C > 0$ such that

$$\frac{1}{C}d(x,y) \leq \tilde{d}(x,y) \leq Cd(x,y) \text{ for all } x,y \in X$$

**a)** Show that $d$ and $\tilde{d}$ give rise to the same metric topology.

For $1 \leq p < \infty$, we define the metric on $\mathbb{R}^2$

$$d_p : \mathbb{R}^2 \times \mathbb{R}^2, \quad d_p(((x_1,x_2),(y_1,y_2)) = \left(|x_1 - y_1|^p + |x_2 - y_2|^p\right)^{1/p}.$$

We also define

$$d_\infty : \mathbb{R}^2 \times \mathbb{R}^2, \quad d_p(((x_1,x_2),(y_1,y_2)) = \max\left\{|x_1 - y_1|, |x_2 - y_2|\right\}$$

**b)** Show that $d_p$ is a metric on $\mathbb{R}^2$.
Hint: You may assume without proof the Minkowski inequality:

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p}$$

for $n \in \mathbb{N}$, $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$.

**c)** Show that all metrics $d_p$, $1 \leq p \leq \infty$, are equivalent.

Remark. The definition of equivalent metrics is not the same in all books, note that Gamelin has a different definition!

*Remark on the remark.* The above notion may more accurately be called "uniform equivalence" between the metrics $\tilde{d}$ and $d$, whereas Gamelin's definition, in Exercise 1.1.12, can be called "topological equivalence". These notions are not the same. However, the following implication is true, and we may consider the proof as an additional exercise:

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**3.** Show that if $d$ and $\tilde{d}$ are uniformly equivalent metrics on a set $X$, then they are topologically equivalent.

**4.** If time, we may also take a look at exercise 2 from “Extra problems 1”.

Remark. Probably I have listed too many exercises, but they are all useful. We will cover a selection in Seminar 2 but (to the extent there is a demand) we will have opportunity return to others later on.