Topology: Extra Problems 1

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Problem 1

(a)

Proof. By definition $\varnothing \in \mathcal{T}_d$ and $\mathbb{R} \in \mathcal{T}_d$ since $\mathbb{R}\setminus\mathbb{R} = \varnothing$ is finite, hence $\mathcal{T}_a = \{\varnothing, \mathbb{R}\} \subset \mathcal{T}_d$. Let $U = \mathbb{R} \setminus \{0\}$; then $\mathbb{R} \setminus U = \{0\}$ is finite, so $U \in \mathcal{T}_d$ but $U \notin \mathcal{T}_a$, hence $\mathcal{T}_a \subsetneq \mathcal{T}_d$.

Let $U \in \mathcal{T}_d$. If $U = \varnothing$ then $U \in \mathcal{T}_c$. Otherwise $\mathbb{R} \setminus U$ is finite, hence closed in the standard topology, so $U = (\mathbb{R} \setminus U)^c$ is open and $U \in \mathcal{T}_c$. Thus $\mathcal{T}_d \subset \mathcal{T}_c$. However, $(0, 1) \in \mathcal{T}_c$ but $(0, 1) \notin \mathcal{T}_d$ since $\mathbb{R} \setminus (0, 1)$ is infinite, hence $\mathcal{T}_d \subsetneq \mathcal{T}_c$.

Since $\mathcal{T}_b = \mathcal{P}(\mathbb{R})$, every subset of $\mathbb{R}$ is in $\mathcal{T}_b$, in particular every $U \in \mathcal{T}_c$, hence $\mathcal{T}_c \subset \mathcal{T}_b$. Moreover, $\{0\} \in \mathcal{T}_b$ but $\{0\}$ closed $\implies \{0\} \notin \mathcal{T}_c \implies \mathcal{T}_c \subsetneq \mathcal{T}_b$. □

(b)

Proof. Since $\mathbb{R} \setminus U$ finite $\Rightarrow \mathbb{R} \setminus U$ countable, we have $\mathcal{T}_d \subset \mathcal{T}_e$. Let $U = \mathbb{R} \setminus \mathbb{Z}$. Then $\mathbb{R} \setminus U = \mathbb{Z}$ is countably infinite, so $U \in \mathcal{T}_e$, but $U \notin \mathcal{T}_d$; hence $\mathcal{T}_d \subsetneq \mathcal{T}_e$.

Since $\mathcal{T}_b = \mathcal{P}(\mathbb{R})$, every subset of $\mathbb{R}$ is in $\mathcal{T}_b$, in particular every $U \in \mathcal{T}_e$, hence $\mathcal{T}_e \subset \mathcal{T}_b$. Moreover, $\{0\} \in \mathcal{T}_b$ but $\{0\} \notin \mathcal{T}_e$ since $\mathbb{R} \setminus \{0\}$ is uncountable; therefore $\mathcal{T}_e \subsetneq \mathcal{T}_b$. □

Problem 2

(a)

Proof. $d_X \geq 0 \wedge d_Y \geq 0 \implies d \geq 0$. Assume $d((x_1, y_1), (x_2, y_2)) = 0$. Then

$$\max\{d_X(x_1, x_2),\ d_Y(y_1, y_2)\} = 0,$$

which implies $d_X(x_1, x_2) = d_Y(y_1, y_2) = 0$. Since d_X and d_Y are metrics, this gives $x_1 = x_2$ and $y_1 = y_2$, hence $(x_1, y_1) = (x_2, y_2)$. If $(x_1, y_1) = (x_2, y_2)$, then $d_X(x_1, x_2) = d_Y(y_1, y_2) = 0$, so $d((x_1, y_1), (x_2, y_2)) = 0$. Let $(x_i, y_i) \in X \times Y$,

$$d_X(x_1, x_3) \leq d_X(x_1, x_2) + d_X(x_2, x_3) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)),$$
$$d_Y(y_1, y_3) \leq d_Y(y_1, y_2) + d_Y(y_2, y_3) \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)),$$
$$d((x_1, y_1), (x_3, y_3)) = \max\{d_X(x_1, x_3),\ d_Y(y_1, y_3)\} \leq d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)).$$

□

(b)

Proof. Assume $(x', y') \in B_r^d((x,y))$. Then $d((x',y'),(x,y)) = \max\{d_X(x',x),\, d_Y(y',y)\} < r$, so $d_X(x',x) < r \implies x' \in B_r^{d_X}(x)$ and $d_Y(y',y) < r \implies y' \in B_r^{d_Y}(y)$. Therefore $(x',y') \in B_r^{d_X}(x) \times B_r^{d_Y}(y)$.

Assume $(x',y') \in B_r^{d_X}(x) \times B_r^{d_Y}(y)$. Then $d_X(x',x) = r_1 < r$ and $d_Y(y',y) = r_2 < r$, hence $d((x',y'),(x,y)) = \max\{r_1, r_2\} < r \implies (x',y') \in B_r^d((x,y))$ □

(c)

Proof. U open $\implies \exists r_1 > 0 : B_{r_1}^{d_X}(x) \subseteq U$, and since V is open, $\exists r_2 > 0 : B_{r_2}^{d_Y}(y) \subseteq V$. Let $\varepsilon = \min\{r_1, r_2\}$. Then

$$B_\varepsilon^d((x,y)) = B_\varepsilon^{d_X}(x) \times B_\varepsilon^{d_Y}(y) \subseteq B_{r_1}^{d_X}(x) \times B_{r_2}^{d_Y}(y) \subseteq U \times V.$$

Let $(x,y) \in X \times Y$ and $\varepsilon > 0$. Set $U := B_\varepsilon^{d_X}(x) \subseteq X, V := B_\varepsilon^{d_Y}(y) \subseteq Y$. Then U and V are open, $(x,y) \in U \times V$, and by previous

$$U \times V = B_\varepsilon^{d_X}(x) \times B_\varepsilon^{d_Y}(y) = B_\varepsilon^d((x,y)),$$

hence $(x,y) \in U \times V \subseteq B_\varepsilon^d((x,y))$.

□

(d)

Proof. Let $W \in \mathcal{T}'$. Then $W = \bigcup_\alpha (U_\alpha \times V_\alpha)$ with each U_α open in X and V_α open in Y. Fix α and $(x,y) \in U_\alpha \times V_\alpha$. By (c)(i) there exists $\varepsilon > 0$ such that

$$B_\varepsilon^d((x,y)) \subseteq U_\alpha \times V_\alpha.$$

Hence each $U_\alpha \times V_\alpha$ is d-open, so $U_\alpha \times V_\alpha \in \mathcal{T}$, and therefore $W \in \mathcal{T}$ (arbitrary unions of d-open sets are d-open).

Let $W \in \mathcal{T}$ and fix $(x,y) \in W$. Then there exists $\varepsilon > 0$ such that

$$B_\varepsilon^d((x,y)) \subseteq W.$$

By (c)(ii) there exist U open in X and V open in Y with

$$(x,y) \in U \times V \subseteq B_\varepsilon^d((x,y)) \subseteq W.$$

Thus every point of W lies in some basis element $U \times V \in \mathcal{B}$ contained in W, so W is a union of such basis elements and hence $W \in \mathcal{T}' \implies \mathcal{T} \subseteq \mathcal{T}'$. Therefore $\mathcal{T} = \mathcal{T}'$. □

(e)

Proof. Assume $X = Y = \mathbb{R}$ and $d_X = d_Y = |\cdot|$. Then the metric d on $\mathbb{R}^2$ from 2(a) is

$$d((x_1,y_1),(x_2,y_2)) = \max\{|x_1 - x_2|,\ |y_1 - y_2|\} = d_\infty\big((x_1,y_1),(x_2,y_2)\big).$$

By (d) the metric topology $\mathcal{T} = \mathcal{T}(d_\infty)$ equals the topology $\mathcal{T}'$ generated by the basis $\{U \times V :\ U, V \text{ open in } \mathbb{R}\}$. The metrics d_2 and d_∞ on $\mathbb{R}^2$ are uniformly equivalent, hence they induce the same topology. In conclusion $\mathcal{T}(d_2) = \mathcal{T}(d_\infty) = \mathcal{T} = \mathcal{T}'$ □