Topology: Exercises 1

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Problem 1

Proof. ($\Rightarrow$) Assume there exists a bijection $h : A \to B$. Since A is countable, there exists a set C and a bijection $f : A \to C$, where $C = \{1, \dots, n\}$ if A is finite and $C = \mathbb{N}$ if A is countably infinite. Define $g := f \circ h^{-1} : B \to C$. Since compositions of bijections are bijections, g is a bijection. Hence there exisrs a bijection from both A and B to the same set C, so they have the same cardinality.

($\Leftarrow$) Assume A and B have the same cardinality. Then there exists a set $C \subseteq \mathbb{N}$ and bijections $f : A \to C$ and $g : B \to C$. Define $h := g^{-1} \circ f : A \to B$. Since f and g are bijections, h is a bijection. $\square$

Problem 2

Proof. Let $k \in \mathbb{N}$, Choose $p_1, p_2, ..., p_k$ prime. Consider:

$$f : \mathbb{N}^k \to \mathbb{N}, \quad (n_1, ...n_k) \mapsto p_1^{n_1} ... p_k^{n_k}.$$

Let $x, y \in \mathbb{N}^k$ such that $f(x) = f(y) \implies p_1^{x_1} ... p_2^{x_k} = p_1^{y_1} ... p_2^{y_k}$. For each $i \in \{1, ..., k\}$, since p_i prime $p_i \mid p_j \Leftrightarrow i = j$. Which in turn implies $p_i^{x_i} = p_i^{y_i} \implies x_i = y_i$. Hence, $x = y$, since x, y arbitrary, f injective. By Theorem 7.1, Topology, Munkres, J.R, $\mathbb{N}^k$ countable. $\square$