EXAM, INTEGRATION THEORY, MARCH 20, 2023.

(Allowed tools: Pen or pencil, rubber.) The exam time is 5 hours. Cheating is not allowed.
Remember to explain all non-trivial identities with references to the appropriate theorems.

Problem 1

Consider the function $\mu^* : \mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}$ given by $\mu^*(A) = 0$ if A is countable, and $\mu(A) = 1$ else.

a) Show that μ^* is an outer measure.
b) What is the σ-algebra $\mathcal{M}_{\mu^*}$ generated by μ^*?
c) State the definition of the Lebesgue outer measure.

Problem 2

a) Evaluate $\sum_{k=0}^{\infty} \int_0^1 \frac{x^k}{k+1}\, dx$ in two different ways to conclude that
$$\int_0^1 \frac{-\ln(1-x)}{x} dx = \pi^2/6.$$
You may use the well known formula $\sum_{k=1}^{\infty} \frac{1}{k^2} = \pi^2/6$, but you are expected to motivate the switch from the Lebesgue integral $\int_{(0,1)} \frac{-\ln(1-x)}{x} d\lambda(x)$ to the above generalized Riemann integral. (You may use that $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = \ln(1+x)$ for all x with $|x| < 1$).
b) Show that
$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \int_{-1}^{1} \frac{-\ln(1-x)}{2x} dx,$$
using a similar approach.

Problem 3

Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $(f_n)_{n=1}^{\infty}$ a sequence of measurable functions that converge pointwise μ-a.e. to some function f. Moreover assume that the sequence is bounded in $\mathcal{L}^p(X, \mathcal{A}, \mu)$ for some $p > 1$.

a) Show that f belongs to $\mathcal{L}^p(X, \mathcal{A}, \mu)$.
b) Show that
$$\left| \int_A f d\mu \right| \leqslant (\mu(A))^{1/q} \|f\|_p$$
for all $A \in \mathcal{A}$ and a certain value $q > 1$.
c) Prove that $(f_n)_{n=1}^{\infty}$ converges to f in $\mathcal{L}^1(X, \mathcal{A}, \mu)$.

Problem 4

Let $(X, \mathcal{A}, \mu)$ be a measure space and f a non-negative measurable function.

a) Prove that
$$\int_X f d\mu = \lim_{n\to\infty} \sum_{k=1}^{n2^n} \left(\frac{k-1}{2^n}\right) \mu \left\{x : \frac{k-1}{2^n} \leqslant f(x) < \frac{k}{2^n}\right\},$$
(where $0 \cdot \infty$ is interpreted as 0).

b) Prove that the similar expression
$$\int_X f d\mu = \lim_{n\to\infty} \sum_{k=1}^{n2^n} \left(\frac{k}{2^n}\right) \mu \left\{x : \frac{k-1}{2^n} \leqslant f(x) < \frac{k}{2^n}\right\},$$
may be false.

c) Assume now that μ is a finite measure and that f is integrable (but not necessarily non-negative). Prove that
$$\int_X f d\mu = \lim_{n\to\infty} \sum_{k=-n2^n}^{n2^n} \left(\frac{k}{2^n}\right) \mu \left\{x : \frac{k-1}{2^n} \leqslant f(x) < \frac{k}{2^n}\right\}.$$

Problem 5

Let $(X_j, \mathcal{A}_j, \mu_j)$ be σ-finite measure spaces for $j = 1, 2, 3$.

a) Prove that $(\mathcal{A}_1 \times \mathcal{A}_2) \times \mathcal{A}_3$ equals $\mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3)$ and that both these equal the smallest σ-algebra containing all sets of the form $A_1 \times A_2 \times A_3$ for all $A_j \in \mathcal{A}_j$, $j = 1, 2, 3$.

b) Show that $(\mu_1 \times \mu_2) \times \mu_3 = \mu_1 \times (\mu_2 \times \mu_3)$.

Problem 6

The Gamma function (which frequently shows up in analysis, number theory and probability theory) is defined as
$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt,$$
and it is well-known that $\Gamma'(1)$ equals Euler's constant $\gamma \approx 0.58$. Prove that
$$\int_0^\infty \ln(t) e^{-t} \, dt = \gamma.$$