LUNDS UNIVERSITET

Tentamensskrivning
MATM19 Integration Theory
Thursday 21:st March 2019
Time: 14:00 - 19:00

Matematikcentrum
Matematik NF

You may only use pen, pencil and rubber. No calculators, graphical tools or similar is allowed. Use the paper provided at the exam. Only write on one side. Mark each page with the exercise number and write at most one solution per page. Complete the information on the cover. Write clearly and give clear but short motivations to your calculations. Use diagrams or pictures where suitable. Oral exam times can be obtained by writing to marcus.carlsson@math.lu.se

1. a) Let $(X, \mathcal{A}, \mu)$ be a measurable space. Using countable additivity of measures, prove that $\mu(A \cup B) = \mu(A) + \mu(B)$ as long as $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are disjoint. (1p)

b) If we remove the condition that A and B are disjoint, prove that

$$\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cup B). \quad (2p)$$

2. a) Let λ be the Lebesgue measure on the Borel σ-algebra. Show that

$$\int_0^\infty e^{-tx} d\lambda(x) = \frac{1}{t}, \quad t > 0.$$

If you use formulas from Riemann integration and/or limit arguments, make sure to explain why these are justified. (3p)

b) Use the dominated convergence theorem to show that you can differentiate inside the integral, and conclude that

$$\int_0^\infty xe^{-tx} d\lambda(x) = \frac{1}{t^2}. \quad (3p)$$

c) Assuming that it is ok to keep on differentiating under the integral, conclude that

$$\int_0^\infty x^n e^{-x} d\lambda(x) = n! \quad (1p)$$

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3. Let $(f_n)_{n=1}^{\infty}$ and f be Borel-measurable real-valued functions on $\mathbb{R}$, and let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let λ denote the Lebesgue measure on $\mathbb{R}$.

a) Prove that $f_n \to f$ λ-a.e. implies that $\phi \circ f_n \to \phi \circ f$ λ–a.e., as $n \to \infty$. (1p)

b) If now $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$, prove that $f_n \to f$ in measure, as $n \to \infty$. (1p)

c) With $\phi(x) = e^x$ and $f,\ f_n$ as in b), show that $(\phi \circ f_n)_{n=1}^{\infty}$ does not converge to $\phi \circ f$ in measure. (1p)

d) If ϕ is uniformly continuous, show that $f_n \to f$ in measure implies that $\phi \circ f_n \to \phi \circ f$ in measure, (as $n \to \infty$). (3p)

A function is uniformly continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space and let σ denote the counting measure on $\mathbb{N}$. Show that

a) $E \subset \mathbb{N} \times X$ is measurable if and only if $E_k = \{x : (k, x) \in E\}$ is in $\mathcal{A}$ for every $k \in \mathbb{N}$. (2p)

b) Show that $f : \mathbb{N} \times X \to \mathbb{R}$ is measurable if and only if $f_k(x)$ is for every $k \in \mathbb{N}$, where $f_k(x) = f(k, x)$. (1p)

c) Show that $\int |f|\ d(\sigma \times \mu) = \sum_{k=1}^{\infty} \int |f_k|\ d\mu$ and that, as long as this is finite, we have

$$\sum_{k=1}^{\infty} \int f_k\ d\mu = \int \sum_{k=1}^{\infty} f_k\ d\mu. \quad (3p)$$

5. Let $1 \leq p < q < r < \infty$ and write e.g. L^p in place of $L^p(\mathbb{R})$.

a) Show by examples that $L^p \not\subset L^r$ or $L^r \not\subset L^p$. (2p)

b) Show that $L^p \cap L^q$ is a Banach space with the norm $\|f\| = \|f\|_p + \|f\|_q$. (3p)

Hint: the main part consists of proving that a Cauchy sequence in $L^p \cap L^q$ has a limit in $L^p \cap L^q$.

c) Show that $L^p \cap L^r \subseteq L^q$. (3p)

Hint: consider separately the cases $|f(x)| > 1$ and $|f(x)| \leq 1$.