MEASURE AND INTEGRATION THEORY

(5 hours)

March 16, 2017

Please write on the front page your preferred time for the oral exam.

Note! Only students who are registered/re-registered on the course are entitled to take the examination.
OBS! Endast studenter som är registrerade alt. omregistrerade på kursen har rätt att tentera.

1. a) Let $(X, \mathcal{A}, \mu)$ be a measure space and let (f_n) be a sequence of real-valued measurable functions on X. Show that the set

$$\{x \in X : \ (f_n(x)) \text{ is divergent}\}$$

is measurable.
b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that its derivative f' is Lebesgue-measurable.

2. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set with $\lambda(E) > 0$.
a) Show that the function $f : [0, \infty) \to [0, \infty)$, defined by

$$f(x) = \lambda(E \cap [-x, x]),$$

is continuous.
b) Show that if $y \in [0, \lambda(E)]$, there exists a Lebesgue measurable subset F of E with $\lambda(F) = y$.

3. Compute the following limits whenever they exist:

$$\lim_{n\to\infty} \int_0^\infty \frac{n^2}{(1 + n^2x^2)(1 + x^2)} dx, \tag{1}$$

$$\lim_{n\to\infty} \int_0^\infty \frac{n}{1 + nx^3} e^{-\frac{x}{n}} dx. \tag{2}$$

Justify your answers!

4. a) Prove that $\mathcal{L}^7([0, 2], \mathcal{M}_\lambda, \lambda) \subset \mathcal{L}^6([0, 2], \mathcal{M}_\lambda, \lambda)$ and show by means of an example that the inclusion is strict.
b) Does any of the inclusions $\mathcal{L}^7([2, \infty), \mathcal{M}_\lambda, \lambda) \subset \mathcal{L}^6([2, \infty), \mathcal{M}_\lambda, \lambda)$, or $\mathcal{L}^6([2, \infty), \mathcal{M}_\lambda, \lambda) \subset \mathcal{L}^7([2, \infty), \mathcal{M}_\lambda, \lambda)$ hold? Justify your answer.

5. Let $1 < p < 2$ and $f \in L^p(\mathbb{R}, \lambda)$. Show that the function

$$h(y) = \int_{\mathbb{R}} \frac{f(x)}{\sqrt{|x| + |y| + 1}} \, dx$$

is well-defined and continuous on $\mathbb{R}$. Where is the function h differentiable? Justify your answer.

6. Let $(X, \mathcal{A}, \mu)$ be a measure space and let f, g be real-valued measurable functions on X. Prove that

$$\int_X |f-g| d\mu = \int_{\mathbb{R}} \mu(\{x \in X :\ g(x) \le y \le f(x)\}) d\lambda(y) + \int_{\mathbb{R}} \mu(\{x \in X :\ f(x) \le y < g(x)\}) d\lambda(y).$$

Hint: Start with the right hand side and think of Fubini-Tonelli

MEASURE AND INTEGRATION THEORY

(5 hours)

April 8, 2017

Please write on the front page your preferred time for the oral exam.

Note! Only students who are registered/re-registered on the course are entitled to take the examination.
OBS! Endast studenter som är registrerade alt. omregistrerade på kursen har rätt att tentera.

1. Let $(X, \mathcal{A}, \mu)$ be a measure space and let (A_n) be a sequence of sets in $\mathcal{A}$. Show that if

$$\sum_{n\geq 1} \mu(A_n) < \infty,$$

then

$$\lim_{n\to\infty} \mu(\cup_{n\geq 1} \cap_{k\geq n} A_k) = \lim_{n\to\infty} \mu(\cap_{n\geq 1} \cup_{k\geq n} A_k) = 0.$$

2. a) Let λ be the Lebesgue measure on $\mathbb{R}$, and let $\{r_n : n \geq 1\}$ be an enumeration of the set $\mathbb{Q}$ of rational real numbers (i.e. $\mathbb{Q}$ equals the set above). Let $f_n : \mathbb{R} \to \mathbb{R}$ be the characteristic function of $[r_n - \frac{1}{n}, r_n + \frac{1}{n}]$. Show that (f_n) does not converge pointwise on any λ-measurable subset of $\mathbb{R}$ with positive measure, but there exist subsequences of (f_n) which converge λ-almost everywhere on $\mathbb{R}$.
b) Let $g, g_n \in \mathcal{L}^1(\mathbb{R}, \lambda)$, $n \geq 1$, and assume that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |g_n - g| d\lambda = 0.$$

Show that there exists a subsequence (g_{n_k}) of (g_n) which converges λ-almost everywhere to g.

3. Let $f, g \in \mathcal{L}^1(\mathbb{R}, \lambda)$ Show if

$$\int_{\mathbb{R}} \left|\frac{g(x)}{x}\right| d\lambda(x) < \infty$$

then for λ almost every $y \in \mathbb{R}$, the function

$$h_y(x) = f(xy)g(x), \quad x \in \mathbb{R},$$

belongs to $\mathcal{L}^1(\mathbb{R}, \lambda)$, and that the function of two variables

$$H(x, y) = h_y(x) = f(xy)g(x),$$

belongs to $\mathcal{L}^1(\mathbb{R}^2, \lambda)$.

4. Compute the limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{n^2 \sin x}{(n^2 + |x|^3)(1 + x^4)} dx.$$

5. a) Let $f \in \mathcal{L}^2(\mathbb{R}, \lambda)$. Show that the function

$$g(x) = \int_0^{\infty} e^{-tx} f(t) d\lambda(t),$$

is well-defined and differentiable on $(0, \infty)$.
b) Find $f \in \mathcal{L}^2(\mathbb{R}, \lambda)$, such that the function g from a) satisfies

$$\lim_{x\to 0+} g(x) = +\infty.$$

6. a) Let μ be the counting measure on $\mathbb{N}$, i.e. for $A \subset \mathbb{N}$, $\mu(A)$ is the number of elements of A. Prove that if $1 < p < q < \infty$, then $\mathcal{L}^p(\mathbb{N}, \mu) \subset \mathcal{L}^q(\mathbb{N}, \mu)$.
b) Let λ be the Lebesgue measure on $[0, 1]$. Prove that if $1 < p < q < \infty$, then $\mathcal{L}^q([0, 1], \lambda) \subset \mathcal{L}^p([0, 1], \lambda)$.
c) Show by means of examples that the inclusions from a) and b) are strict.

LUND UNIVERSITY

Written Examination
Integration Theory
Tuesday 15 August 2017
08:00–13:00

Centre for Mathematical Sciences
Mathematics, Faculty of Science

In order to sit the examination you must be enrolled in the course. No aids allowed. Use the papers provided by the department and write on one side of each sheet only. Fill in the cover completely and write legibly. Give concise and short arguments.

1. Let $(X, \mathscr{A}, \mu)$ be a measure space and $\{f_n\}$ a sequence of measurable functions on X. Show that the set

$$A = \{x \in X : \exists y \in \mathbb{R} \text{ such that } \lim_{n\to\infty} f_n(x) = y\}$$

is measurable.

2. Compute the following limits, if they exist.

a)

$$\lim_{n\to\infty} \int_1^\infty \frac{(1+nx)\sin\left(\frac{x}{n}\right) e^{-x}}{x^2}\, dx,$$

b)

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{n^2}{(1+nk)(1+nk^{\frac{1}{n}})}.$$

3. Let $(X, \mathscr{A}, \mu)$ be a measure space and let f, g, f_n, g_n, $n \geq 1$, be real-valued measurable functions on X such that

(i) $\{f_n\}$ converges to f in mean,

(ii) $|g_n| \leq 1$ for all n and $\{g_n\}$ converges to g μ-a.e.

Show that $\{f_n g_n\}$ converges to fg in mean.

4. Let λ_2 be the Lebesgue measure on $\mathbb{R}^2$.

a) Given $c \in \mathbb{R}$, find $\lambda_2(A)$ where $A = \{(x,y) \in \mathbb{R}^2 : x^2 - y = c\}$.

b) Find $\lambda_2(A)$, where A consists of all pairs (x,y) in $[0,\pi/2] \times [0,\pi/2]$ such that $\sin(x) \geq 1/2$ and $\cos(y)$ is irrational.

Please, turn over!

5. Let $f\colon [0,1] \to \mathbb{R}$ and $f_n\colon [0,1] \to \mathbb{R}$, $n = 1, 2, 3, \ldots$, be Borel measurable functions.

a) Suppose that $f_n \to f$ a.e. on $[0,1]$ and that there exists $M > 0$ with $\int_0^1 |f_n|\, d\lambda \le M$ for all $n \ge 1$. Prove that $f \in \mathscr{L}^1([0,1])$. However, show by means of a counterexample that we do not necessarily have that $\lim_{n\to\infty} \int_0^1 |f - f_n|\, d\lambda = 0$.

b) Prove that if $f_n \to f$ a.e. on $[0,1]$ and there exists $M > 0$ with $\int_0^1 f_n^2\, d\lambda \le M$ for all $n \ge 1$, then $\lim_{n\to\infty} \int_0^1 |f - f_n|\, d\lambda = 0$.

MEASURE AND INTEGRATION THEORY

(5 hours)

March 15, 2018

Please write on the front page your preferred time for the oral exam.

Note! Only students who are registered/re-registered on the course are entitled to take the examination.
OBS! Endast studenter som är registrerade alt. omregistrerade på kursen har rätt att tentera.

1. Let F_1, $F_2 : \mathbb{R} \to \mathbb{R}$ be nondecreasing, bounded, right-continuous functions with $\lim_{x\to-\infty} F_j(x) = 0$, $j = 1, 2$ and let μ_j, $j = 1, 2$, be the induced Borel measures, that is, the unique Borel measures on $\mathbb{R}$ with

$$\mu_j((-\infty, x]) = F_j(x), \quad j = 1, 2.$$

Show that the inequality

$$\mu_1(B) \leq \mu_2(B),$$

holds for all $B \in \mathcal{B}(\mathbb{R})$, if and only if $F_2 - F_1$ is a nondecreasing function on $\mathbb{R}$.

2. Let X be a nonvoid set, let $\mathcal{P}(x)$ be the collecttion of all subsets of X and let $A \subset X$ be a finite set.
a) Prove that the set-function $\mu_A : \mathcal{P}(X) \to [0, \infty)$ defined by

$$\mu_A(E) = \text{number of elements of } E \cap A,$$

is a finite measure.
b) Show that for $f : X \to \mathbb{R}$, we have

$$\int f d\mu_A = \sum_{x\in A} f(x).$$

c) By the general theory it follows that exists a unique measure $\mu_A \times \mu_A : X \times X \to [0, \infty)$ with $\mu_A \times \mu_A(E \times F) = \mu_A(E)\mu_A(F)$. You do not have to prove this statement. For $C \subset X \times X$, compute $\mu_A \times \mu_A(C)$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $g : X \to \mathbb{R}$ be measurable. Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}$.
a) Show that the set function $\mu_g : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ defined by

$$\mu_g(A) = \mu(g^{-1}(A)),$$

is a measure.
b) Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is Borel-measurable if and only if $f^{-1}(B) \subset \mathcal{B}(\mathbb{R})$, whenever $B \in \mathcal{B}(\mathbb{R})$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is Borel-measurable, then $f \circ g : X \to \mathbb{R}$ is measurable.
c) Show that if $f \geq 0$ is Borel-measurable on $\mathbb{R}$, then

$$\int f d\mu_g = \int f \circ g d\mu.$$

4. Compute the following limits whenever they exist:

$$\lim_{n\to\infty} n \int_0^3 \frac{\sqrt{x}}{1+nx^2} dx, \tag{1}$$

$$\lim_{n\to\infty} n \int_1^\infty \sqrt{x}(e^{\frac{1}{nx^2}} - 1) dx. \tag{2}$$

Justify your answers.

5. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $p_1, p_2 \in (1, \infty)$.
a) Show that if $f \in \mathcal{L}^{p_1}(X, \mathcal{A}, \mu), g \in \mathcal{L}^{p_2}(X, \mathcal{A}, \mu)$, then $fg \in \mathcal{L}^{\frac{p_1 p_2}{p_1+p_2}}(X, \mathcal{A}, \mu)$.
b) Prove that every function $h \in \mathcal{L}^{\frac{p_1 p_2}{p_1+p_2}}(X, \mathcal{A}, \mu)$ can be written in the form $h = fg$, with $f \in \mathcal{L}^{p_1}(X, \mathcal{A}, \mu), g \in \mathcal{L}^{p_2}(X, \mathcal{A}, \mu)$.

6. Let $f \in \mathcal{L}^1(\mathbb{R}, \lambda)$, and let

$$F(x) = \int_{\mathbb{R}} \sin(tx) f(t) d\lambda(t), \quad x \in \mathbb{R}.$$

a) Show that F is continuous on $\mathbb{R}$.
b) Assume that f does not vanish λ-a.e. on $\mathbb{R}$, and give a condition under which F is differentiable at 0. Justify your answer.

MEASURE AND INTEGRATION THEORY

(5 hours)

April 14, 2018

Please write on the front page your preferred time for the oral exam.

Note! Only students who are registered/re-registered on the course are entitled to take the examination.
OBS! Endast studenter som är registrerade alt. omregistrerade på kursen har rätt att tentera.

1. Let $\mathbb{N}$ be the set of positive integers, let $\mathcal{P}(\mathbb{N})$ be the collection of subsets of $\mathbb{N}$, and let $\mu : \mathcal{P}(X) \to [0, \infty]$, be the counting measure, i.e. $\mu(A) =$ number of elements of A, if A is finite and $\mu(A) = \infty$, if A is infinite.
a) Prove that if $f : X \to [0, \infty)$, then f is μ-measurable and

$$\int f d\mu = \sum_{n=1}^{\infty} f(n).$$

b) Given a measure space $(X; \mathcal{A}, \nu)$ with ν σ-finite and a sequence of (g_n) be a sequence of real-valued measurable functions on X. show that the function $h : \mathbb{N} \times X \to \mathbb{R}$, with $h(n, x) = g_n(x)$, is $\mu \times \nu$ measurable.
c) State the Beppo-Levi theorem and prove it using the iterated integral of the function h from b).

2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $p \in (1, \infty)$.
a) Let (f_n) be a sequence in $\mathcal{L}^p(X, \mathcal{A}, \mu)$ which converges μ-a.e. to the measurable function $f : X \to \mathbb{R}$. Assume that there exists $g \in \mathcal{L}^p(X, \mathcal{A}, \mu)$ such that $|f_n| \le |g|$, μ-a.e. for all n. Prove that $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$, and that

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

b) Assume that μ is σ-finite and let $g \in \mathcal{L}^p(X \times X, \mathcal{A} \times \mathcal{A}, \mu \times \mu)$. Show that $g_x : X \to \mathbb{R}$ (as usual $g_x(y) = g(x, y)$) belongs to $\mathcal{L}^p(X, \mathcal{A}, \mu)$ for μ-almost every $x \in X$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space, and for $n \in \mathbb{N}$ let $f_n : X \to [0, 1]$ be measurable functions, such that the sequence (f_n) converges in measure to the measurable function f.
a) Show that $f(x) \in [0, 1]$, μ-a.e..
b) Let $F : [0, 1] \to \mathbb{R}$ be a continuous function and define

$$F \circ f(x) = F(f(x)), \text{ if } f(x) \in [0, 1], \quad F \circ f(x) = 0, \text{ otherwise.}$$

Show that $(F \circ f_n)$ converges in measure to $F \circ f$.
Hint: F is actually uniformly continuous on $[0,1]$, i.e. for every $\varepsilon > 0$ there is $\delta > 0$ such that $|F(t) - F(s)| < \varepsilon$, whenever $|t - s| < \delta$.

4. Let $f : [0,\infty) \to \mathbb{R}$ be a bounded, Lebesgue measurable function such that $\lim_{x\to\infty} f(x) = L$. Show that if $g \in \mathcal{L}^1([0,\infty),\lambda)$, then

$$\lim_{n\to\infty} \int_{[0,\infty)} f(nx)g(x)d\lambda(x) = L \int_{[0,\infty)} g d\lambda.$$

5. Let $(X, \mathcal{A}, \mu)$ be a measure space, let $p \in (0,1)$ and $q = \frac{p}{p-1}$, i.e. $\frac{1}{p}+\frac{1}{q} = 1$ (Note that $q < 0$!). Let $f, g : X \to (0,\infty)$ be measurable such that

$$\int f^p d\mu, \int g^q d\mu < \infty.$$

Show that

$$\int fg d\mu \geq \left(\int f^p d\mu\right)^{1/p} \left(\int g^q d\mu\right)^{1/q}.$$

6. Let $g : [0,\infty) \to [0,\infty]$ be a λ-measurable function (λ = Lebesgue measure) which is Lebesgue-integrable on any interval $[0,a]$, $a > 0$. Set

$$G(x) = \int_{[0,x]} g d\lambda, \quad x \in [0,\infty).$$

a) Show that G is continuous on $[0,\infty)$.
b) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to [0,\infty)$ be $\mathcal{A}$-measurable. Prove that

$$\int G \circ f d\mu = \int_{[0,\infty)} g(t)\mu(\{x \in X : f(x) > t\})d\lambda(t).$$

MEASURE AND INTEGRATION THEORY

(5 hours)

August 27, 2018

Please write on the front page your preferred time for the oral exam.

Note! Only students who are registered/re-registered on the course are entitled to take the examination.
OBS! Endast studenter som är registrerade alt. omregistrerade på kursen har rätt att tentera.

1. a) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Show that the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = f(-x)$, $x \in \mathbb{R}$ is also Lebesgue measurable.
b) Let $f_n : \mathbb{R} \to \mathbb{R}$, $n \geq 1$ be Lebesgue measurable functions. Show that the set

$$\{x \in \mathbb{R} : \ (f_n(x)), \text{ or } (f_n(-x)) \text{ is divergent } \}$$

is measurable
c) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. If $h : \mathbb{R} \to \mathbb{R}$ satisfies $|h(x)| = |f(x)|$, $x \in \mathbb{R}$, must h be Lebesgue measurable?(Justify your answer)

2. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be a μ-measurable function. Show that the set-function $\mu_f : \mathcal{A} \to [0, \infty]$ with

$$\mu_f(A) = \int_A f d\mu = \int \chi_A f d\mu$$

is a measure.

3. Let λ be the Lebesgue measure on $\mathbb{R}$, and let $Y \subset \mathbb{R}$ be a λ-measurable set. Let $\nu : \mathcal{P}(Y) \to [0, \infty]$ be the counting measure on Y, that is, for $A \subset Y$, $\nu(A)$ equals the number of elements of A, if A is finite, and $\nu(A) = +\infty$ otherwise. Let $\Delta = \{(y, y) : \ y \in Y\} \subset \mathbb{R} \times Y$.
a) Show that Δ_x is ν-measurable, Δ^y is λ-meaurable and that $x \to \nu(\Delta_x)$ is λ-measurable, $y \to \lambda(\Delta^y)$ is ν- measurable.
b) Show that if Y is countable , then Δ is $\lambda \times \nu$-measurable with $\lambda \times \nu(\Delta) = 0$
c) Show that if $\lambda(Y) > 0$, then

$$\int \lambda(\Delta^y) d\nu \neq \int \nu(\Delta_x) d\lambda.$$

d) Give at least one reason why c) does not contradict Tonelli's theorem.

4. Show that:

$$\lim_{x\to\infty}\sum_{n=1}^{\infty}\frac{e^{-\frac{x}{n}}}{n^2}=0, \tag{1}$$

$$\lim_{n\to\infty}\int_0^{\infty}\frac{n}{1+nx^2}e^{-\frac{x^2}{n}}dx=+\infty. \tag{2}$$

5. Let $1 \le p < q$. a) Show that if $X = [0,1]$ and μ denotes the Lebesgue measure on $[0,1]$, then

$$\mathcal{L}^p(X,\mu) \supset \mathcal{L}^q(X,\mu)$$

and the inclusion is strict.
b) Give an example of X and μ such that the reverse inclusion holds and is strict.
c) Give an example such that neither inclusion holds.

6. Let λ be the Lebesgue measure on the real line, and let $f \in \mathcal{L}^1([0,\infty),\lambda)$. Consider the function $F : [0,\infty) \to \mathbb{R}$ defined by

$$F(x) = \int_{[0,\infty)} e^{-xt}f(t)d\lambda(t).$$

a) Prove that $F \in \mathcal{L}^1([0,\infty),\lambda)$.
b) Prove that F has continuous derivatives of any order on $(0,\infty)$ and compute $F^{(n)}(x)$, $x > 0$.
c) Show that for $f(x) = \frac{1}{1+x^2}$, $x > 0$, the corresponding function F is not differentiable (from the right) at $x = 0$.

LUNDS UNIVERSITET

Matematikcentrum

Matematik NF

Tentamensskrivning
MATM19 Integration Theory
Thursday 21:st March 2019
Time: 14:00 - 19:00

You may only use pen, pencil and rubber. No calculators, graphical tools or similar is allowed. Use the paper provided at the exam. Only write on one side. Mark each page with the exercise number and write at most one solution per page. Complete the information on the cover. Write clearly and give clear but short motivations to your calculations. Use diagrams or pictures where suitable. Oral exam times can be obtained by writing to marcus.carlsson@math.lu.se

1. a) Let $(X, \mathcal{A}, \mu)$ be a measurable space. Using countable additivity of measures, prove that $\mu(A \cup B) = \mu(A) + \mu(B)$ as long as $A \in \mathcal{A}$ and $B \in \mathcal{A}$ are disjoint. (1p)
   b) If we remove the condition that A and B are disjoint, prove that
$$\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cup B). \quad (2p)$$

2. a) Let λ be the Lebesgue measure on the Borel σ-algebra. Show that
$$\int_0^\infty e^{-tx} d\lambda(x) = \frac{1}{t}, \quad t > 0.$$
If you use formulas from Riemann integration and/or limit arguments, make sure to explain why these are justified. (3p)
   b) Use the dominated convergence theorem to show that you can differentiate inside the integral, and conclude that
$$\int_0^\infty xe^{-tx} d\lambda(x) = \frac{1}{t^2}. \quad (3p)$$
   c) Assuming that it is ok to keep on differentiating under the integral, conclude that
$$\int_0^\infty x^n e^{-x} d\lambda(x) = n! \quad (1p)$$

Var god vänd!

3. Let $(f_n)_{n=1}^{\infty}$ and f be Borel-measurable real-valued functions on $\mathbb{R}$, and let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let λ denote the Lebesgue measure on $\mathbb{R}$.

a) Prove that $f_n \to f$ λ-a.e. implies that $\phi \circ f_n \to \phi \circ f$ λ-a.e., as $n \to \infty$. (1p)

b) If now $f_n(x) = x + \frac{1}{n}$, prove that $f_n \to f$ in measure, as $n \to \infty$. (1p)

c) With $\phi(x) = e^x$, show that $(\phi \circ f_n)_{n=1}^{\infty}$ does not converge to $\phi \circ f$ in measure. (1p)

d) If ϕ is uniformly continuous, show that $f_n \to f$ in measure implies that $\phi \circ f_n \to \phi \circ f$ in measure, (as $n \to \infty$). (3p)

A function is uniformly continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space and let σ denote the counting measure on $\mathbb{N}$. Show that

a) $E \subset \mathbb{N} \times X$ is measurable if and only if $E_k = \{x : (k, x) \in E\}$ is in $\mathcal{A}$ for every $k \in \mathbb{N}$. (2p)

b) Show that $f : \mathbb{N} \times X \to \mathbb{R}$ is measurable if and only if $f_k(x)$ is for every $k \in \mathbb{N}$, where $f_k(x) = f(k, x)$. (1p)

c) Show that $\int |f| \, d(\sigma \times \mu) = \sum_{k=1}^{\infty} \int |f_k| \, d\mu$ and that, as long as this is finite, we have

$$\sum_{k=1}^{\infty} \int f_k \, d\mu = \int \sum_{k=1}^{\infty} f_k \, d\mu. \quad (3p)$$

5. Let $1 \le p < q < r < \infty$ and write e.g. L^p in place of $L^p(\mathbb{R})$.

a) Show by examples that $L^p \not\subset L^r$ or $L^r \not\subset L^p$. (2p)

b) Show that $L^p \cap L^q$ is a Banach space with the norm $\|f\| = \|f\|_p + \|f\|_q$. (3p)

Hint: the main part consists of proving that a Cauchy sequence in $L^p \cap L^q$ has a limit in $L^p \cap L^q$.

c) Show that $L^p \cap L^r \subseteq L^q$. (3p)

Hint: consider separately the cases $|f(x)| > 1$ and $|f(x)| \le 1$.

LUNDS UNIVERSITET

Tentamensskrivning
MATM19 Integration Theory
Saturday 13:th April 2019
Time: 8:00 - 13:00

Matematikcentrum
Matematik NF

You may only use pen, pencil and rubber. No calculators, graphical tools or similar is allowed. Use the paper provided at the exam. Only write on one side. Mark each page with the exercise number and write at most one solution per page. Complete the information on the cover. Write clearly and give clear but short motivations to your calculations. Use diagrams or pictures where suitable. Oral exam times can be obtained by writing to marcus.carlsson@math.lu.se

1. Consider the sequence of functions $f_n(x) = \frac{1}{x}(\sin(x))^n$ on $(0, \infty)$, equipped with the Borel $\sigma-$algebra and Lebesgue measure λ. Does it hold that $f_n \to 0$ as $n \to \infty$ for each of the following types of convergence:

- a) pointwise? (1p)
- a) pointwise a.e.? (1p)
- b) in measure? (2p)
- c) in L^p, $1 < p < \infty$? (2p)
- d) in L^∞? (1p)

Motivate your answers thoroughly.

2. a) Why is $S \times \{0\}$ Lebesgue measurable in $\mathbb{R}^2$, independent of $S \subset \mathbb{R}$? (1p)

b) Why does this not contradict the statement that sections $E^y = \{x : (x, y) \in E\}$ are always measurable as long as E belongs to the product σ-algebra? (1p)

c) Consider a set in $\mathbb{R}^2$ of the form $S \times [0, 1]$. Show, using the definition, that this is not Lebesgue measurable in $\mathbb{R}^2$ if S is not Lebesgue measurable in $\mathbb{R}$. (2p)
(*Hint: You may use that* $\lambda_1^*(B) = \lambda_2^*(B \times [0, 1])$ *for all* $B \subset \mathbb{R}$.)

d) Find an example of Lebesgue measurable sets $A, B \subset \mathbb{R}^2$ such that $A + B = \{x + y : x \in A, y \in B\}$ is not Lebesgue measurable. (1p)
Hint, you may use the result from a)

3. a) Show that $\int_0^1 \frac{e^{-x}}{x} dx = \infty$. Do this carefully and motivate any use of Riemann integral formulas. (3p)

b) Using Fatou's lemma, show that $\lim_{n\to\infty} \int_0^1 \frac{ne^{-x}+1}{nx+1}\, dx = \infty$. (3p)

4. Let $u \in L^1(\mathbb{R})$ and define $\tilde{u}(\xi) = \int u(x) \cos(x\xi)\, dx$.

a) Prove that $\tilde{u}$ is continuous. (1p) (In fact, it is uniformly continuous, but that is harder to show...)

b) Assuming that u has compact support, prove that we can differentiate under the integral, and derive an integral formula for $\frac{d}{d\xi}\tilde{u}$. (2p)

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c) Assuming that u is a step function, prove that $\lim_{\xi\to\infty} |\tilde{u}(\xi)| = 0$. (1p)
d) Prove that $\lim_{\xi\to\infty} \tilde{u}(\xi) = 0$ holds in general (i.e. for any function in $u \in L^1$). (2p)
You may use the result in c).

5. Let $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$. Prove

- a) $\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)dx = 0$ (1p)

Using the geometric series it is easy to see that $\sum_{n=1}^{\infty} f_n(x) = \frac{ae^{-ax}}{1-e^{-ax}} - \frac{be^{-bx}}{1-e^{-bx}}$. Prove that

- b) $\frac{ae^{-ax}}{1-e^{-ax}} \notin L^1((0,\infty),\lambda)$ (1p)

However, using a Taylor series expansion it is easy to see that $\frac{ae^{-ax}}{1-e^{-ax}} = \frac{1+O(x)}{x}$ where $O(x)$ represents any function h such that $\frac{h(x)}{x}$ is bounded near $x = 0$. Use this knowledge to show that

- c) $\sum_{n=1}^{\infty} f_n(x) \in L^1((0,\infty),\lambda)$. (1p)

Show that

- d) $\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)dx = \log(b/a)$ (1p).

You may use that $\lim_{\epsilon\to 0} \frac{1-e^{-a\epsilon}}{1-e^{-b\epsilon}} = \frac{a}{b}$, which is also easily established using Taylor's formula.

- e) Finally explain why it necessarily holds that $\sum_{n=1}^{\infty} \int_0^{\infty} |f_n(x)|dx = \infty$ (2p).

EXAM, INTEGRATION THEORY, 26 AUGUST 2019.

(Allowed tools: Pen or pencil, rubber.)

Remember to explain all non-trivial identities with references to the appropriate theorems.

Problem 1

- a) Let $(X, \mathcal{A}, \mu)$ be a measure space. Explain the difference of $L^\infty(X, \mathcal{A}, \mu)$ and $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$.
- b) Is a function in $\mathcal{L}^\infty(X, \mathcal{A}, \mu)$ necessarily bounded?
- c) Give an example of a measure space such that $L^\infty(X, \mathcal{A}, \mu) = \mathcal{L}^\infty(X, \mathcal{A}, \mu)$.
- d) State, and prove, a condition on μ which is equivalent to the identity in c).

Problem 2

Let λ denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. Compute the following limits, whenever they exist. Motivate your calculations.

- a)
$$\lim_{n\to\infty} \int_1^\infty \frac{n^2}{(1+ne^x)(n+\ln(x))}\, d\lambda(x)$$
- b)
$$\lim_{n\to\infty} \int_0^\infty \frac{1+\sin(x)/n}{1+x^2}\, d\lambda(x), \quad (n>1)$$
- c)
$$\lim_{n\to\infty} \int_1^\infty \frac{n^2}{(n+\sin(x))(n+\cos(x))}\, d\lambda(x)$$
- d) Above you probably used formulas which you know for the Riemann integral. If you haven't done it already, explain in detail why this is allowed. (*Hint: Don't forget that you are integrating over an infinite domain.*)

Problem 3

Let λ denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$ and define μ on $\mathcal{B}(\mathbb{R})$ via

$$\mu(E) = \int_E e^{-|x|}\, d\mu.$$

- a) Explain why μ is a measure.
- b) Let $(f_n)_{n=1}^{\infty}$ be a sequence of $\mathcal{B}(\mathbb{R})$-measurable functions that converge pointwise to f. Is f $\mathcal{B}(\mathbb{R})$-measurable?
- c) Given any $\epsilon > 0$, can one find a set E with $\mu(\mathbb{R}\backslash E) < \epsilon$, with the property that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on E?

Problem 4

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f \in \mathcal{L}^2(X, \mathcal{A}, \mu, \mathbb{C})$ be an integrable function. Let $g : X \times X \to \mathbb{C}$ be defined by

$$g(x, y) = (f(x)f(y))^2.$$

- a) Prove that g is $\mathcal{A} \times \mathcal{A}$-measurable.
- b) Prove that $f^2 \in \mathcal{L}^1(X, \mathcal{A}, \mu, \mathbb{C})$
- c) Prove that $\int_{X\times X} g\, d(\mu \times \mu) = \left(\int_X f^2 d\mu\right)^2$.

Problem 5

Let $p_1, p_2, \ldots, p_n \in (1, \infty)$ be such that

$$\sum_{j=1}^{n} \frac{1}{p_j} = 1.$$

Show that if $f_1, f_2, \ldots, f_n$ are non-negative measurable functions on the measure space $(X, \mathcal{A}, \mu)$, then

$$\int f_1 f_2 \ldots f_n\, d\mu \leq \left(\int_X f_1^{p_1}\, d\mu\right)^{1/p_1} \left(\int_X f_2^{p_2}\, d\mu\right)^{1/p_2} \ldots \left(\int_X f_n^{p_n}\, d\mu\right)^{1/p_n}.$$

If you can do it for $n = 2$ but not in general, then you will still get some points for that.

MEASURE AND INTEGRATION THEORY

(5 hours)

2020-08-18

Please write on the front page your preferred time for the oral exam.

Note! Only students who are registered/re-registered on the course are entitled to take the examination.
OBS! Endast studenter som är registrerade alt. omregistrerade på kursen har rätt att tentera.

1. a) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Show that the function $g : \mathbb{R} \to \mathbb{R}$ with $g(x) = f(-x)$, $x \in \mathbb{R}$ is also Lebesgue measurable.
b) Let $f_n : \mathbb{R} \to \mathbb{R}$, $n \geq 1$ be Lebesgue measurable functions. Show that the set

$$\{x \in \mathbb{R} : \ (f_n(x)), \text{ or } (f_n(-x)) \text{ is divergent } \}$$

is measurable
c) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. If $h : \mathbb{R} \to \mathbb{R}$ satisfies $|h(x)| = |f(x)|$, $x \in \mathbb{R}$, must h be Lebesgue measurable?(Justify your answer)

2. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be a μ-measurable function. Show that the set-function $\mu_f : \mathcal{A} \to [0, \infty]$ with

$$\mu_f(A) = \int_A f d\mu = \int \chi_A f d\mu$$

is a measure.

3. Let λ be the Lebesgue measure on $\mathbb{R}$, and let $Y \subset \mathbb{R}$ be a λ-measurable set. Let $\nu : \mathcal{P}(Y) \to [0, \infty]$ be the counting measure on Y, that is, for $A \subset Y$, $\nu(A)$ equals the number of elements of A, if A is finite, and $\nu(A) = +\infty$ otherwise. Let $\Delta = \{(y, y) : \ y \in Y\} \subset \mathbb{R} \times Y$.
a) Show that Δ_x is ν-measurable, Δ^y is λ-meaurable and that $x \to \nu(\Delta_x)$ is λ-measurable, $y \to \lambda(\Delta^y)$ is ν- measurable.
b) Show that if Y is countable , then Δ is $\lambda \times \nu$-measurable with $\lambda \times \nu(\Delta) = 0$
c) Show that if $\lambda(Y) > 0$, then

$$\int \lambda(\Delta^y) d\nu \neq \int \nu(\Delta_x) d\lambda.$$

d) Give at least one reason why c) does not contradict Tonelli's theorem.

4. Show that:

(1)
$$\lim_{x\to\infty}\sum_{n=1}^{\infty}\frac{e^{-\frac{x}{n}}}{n^2}=0,$$

(2)
$$\lim_{n\to\infty}\int_0^{\infty}\frac{n}{1+nx^2}e^{-\frac{x^2}{n}}dx=+\infty.$$

5. Let $1\le p<q$. a) Show that if $X=[0,1]$ and μ denotes the Lebesgue measure on $[0,1]$, then

$$\mathcal{L}^p(X,\mu)\supset\mathcal{L}^q(X,\mu)$$

and the inclusion is strict.
b) Give an example of X and μ such that the reverse inclusion holds and is strict.
c) Give an example such that neither inclusion holds.

6. Let λ be the Lebesgue measure on the real line, and let $f\in\mathcal{L}^1([0,\infty),\lambda)$. Consider the function $F:[0,\infty)\to\mathbb{R}$ defined by

$$F(x)=\int_{[0,\infty)}e^{-xt}f(t)d\lambda(t).$$

a) Prove that $F\in\mathcal{L}^1([0,\infty),\lambda)$, whenever $\int_{[0,\infty)}|f(t)|/t\ d\lambda(t)<\infty$
b) Prove that F has continuous derivatives of any order on $(0,\infty)$ and compute $F^{(n)}(x),\ x>0$.
c) Show that for $f(x)=\frac{1}{1+x^2}$, $x>0$, the corresponding function F is not differentiable (from the right) at $x=0$.

EXAM, INTEGRATION THEORY, 17 MARS 2021.

(Allowed tools: Pen or pencil, rubber.) The exam time is 5 hours. You will be monitored via Zoom. It is important that your camera is showing the workspace in front of you, mainly to show that you are not using the book. We may ask you to show around to verify lack of books or post-it-notes. At the end of the exam, you can take a photo of all relevant papers and upload on the canvas page under assignment. In case of poor photo quality you may be asked to scan the documents, but it is important that a decent quality photo is uploaded shortly after the end of the exam, which will be announced by me via Zoom. If you want to quit earlier, just upload documents earlier and you are free to go.

Remember to explain all non-trivial identities with references to the appropriate theorems.

Problem 1

Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Show that if $(f_n)_{n=1}^{\infty}$ converges to f in $\mathcal{L}^p(X, \mathcal{A}, \mu)$, then it also converges in measure.

What if X is σ-finite?

Problem 2

Prove that the function

$$f(x) = \frac{\sin(x)}{x^{3/2}}, \qquad x > 0,$$

is Lebesgue integrable (meaning integrable with respect to the Lebesgue measure on $(0, \infty)$).

Problem 3

Let $(f_n)_{n=1}^{\infty}$ be a decreasing sequence of non-negative Lebesgue integrable functions on a measure space $(X, \mathcal{A}, \mu)$. Show that if

$$\lim_{n\to\infty} \int f_n d\mu = 0,$$

then $(f_n)_{n=1}^{\infty}$ converges to 0 μ-a.e.

Problem 4

Let f, g be arbitrary functions in $\mathcal{L}^2(X, \mathcal{A}, \mu)$ where X is σ-finite. Show that

(a) Give a basic proof that $f \cdot g \in \mathcal{L}^1$, not relying on Cauchy-Schwarz or Hölder's inequalities.
(b) Show that $\int (|f(x)g(y)| + |f(y)g(x)|)^2 d\mu \times \mu < \infty$.
(c) Show that

$$\int |f|^2 d\mu \int |g|^2 d\mu - (\int fg d\mu)^2 = \frac{1}{2} \int (f(x)g(y) - f(y)g(x))^2 d\mu \times \mu$$

and use this to give an alternative proof for Cauchy-Schwarz inequality.

Problem 5

Suppose $1 < p < \infty$. The norm of a bounded linear functional F on $L^p(X, \mathcal{A}, \mu)$ is defined as $\|F\| = \sup_{f \in L^p;\ \|f\|_p \neq 0} \frac{|F(f)|}{\|f\|_p}$. Let q be the conjugate exponent and let $g \in L^q$ be given. Define $F_g : L^p \to \mathbb{R}$ by the formula

$$F_g(f) = \int fg \ d\mu.$$

(a) What does it mean to be a conjugate exponent?
(b) Show that $F_g(f)$ exists for all $f \in L^p$, and that it defines a bounded linear bounded functional.
(c) Prove that $\|F_g\| = \|g\|_q$.

Problem 6

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be Borel measurable, and $g : \mathbb{R}^2 \to \mathbb{R}^2$ Lebesgue measurable (in the sense that the preimage of Borel sets are Lebesgue).

- a) True or false: $f \circ g$ is necessarily Lebesgue measurable.
- b) True or false: $g \circ f$ is necessarily Lebesgue measurable.

EXAM, INTEGRATION THEORY, AUGUST 2021.

(Allowed tools: Pen or pencil, rubber.) The exam time is 5 hours. You will be monitored via Zoom. It is important that your camera is showing the workspace in front of you, mainly to show that you are not using the book. We may ask you to show around to verify lack of books or post-it-notes. At the end of the exam, you can take a photo of all relevant papers and upload on the canvas page under assignment. In case of poor photo quality you may be asked to scan the documents, but it is important that a decent quality photo is uploaded shortly after the end of the exam, which will be announced by me via Zoom. If you want to quit earlier, just upload documents earlier and you are free to go.

Remember to explain all non-trivial identities with references to the appropriate theorems.

Problem 1

Let λ denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$ and define μ on $\mathcal{B}(\mathbb{R})$ via

$$\mu(E) = \int_E \frac{1}{1+|x|}\, d\lambda.$$

- a) Is μ a measure? Verify your answer.
- b) Let $(f_n)_{n=1}^{\infty}$ be a sequence of $\mathcal{B}(\mathbb{R})$-measurable functions that converge pointwise to f. Is f $\mathcal{B}(\mathbb{R})$-measurable?
- c) Given any $\epsilon > 0$, can one find a set E with $\mu(\mathbb{R}\backslash E) < \epsilon$, with the property that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on E?

Problem 2

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f, g \in \mathcal{L}^4(X, \mathcal{A}, \mu, \mathbb{C})$ be integrable functions. Let $h : X \times X \to \mathbb{C}$ be defined by

$$h(x, y) = (f(x)g(y))^4.$$

- a) Prove that h is $\mathcal{A} \times \mathcal{A}$-measurable.
- b) Prove that $f^4 \in \mathcal{L}^1(X, \mathcal{A}, \mu, \mathbb{C})$
- c) Prove that $\int_{X\times X} h\; d(\mu \times \mu) = \int_X f^4 d\mu \int_X g^4 d\mu$.

Problem 3

a) Show that $\int_0^1 \frac{e^{-x}}{x} dx = \infty$. Do this carefully and motivate any use of Riemann integral formulas.

b) Using Fatou's lemma, show that $\lim_{n\to\infty} \int_0^1 \frac{ne^{-x}+1}{nx+1} dx = \infty$.

Problem 4

Let $(f_n)_{n=1}^{\infty}$ and f be Borel-measurable real-valued functions on $\mathbb{R}$, and let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let λ denote the Lebesgue measure on $\mathbb{R}$.

a) Prove that $f_n \to f$ λ-a.e. implies that $\phi \circ f_n \to \phi \circ f$ λ–a.e., as $n \to \infty$.

b) If now $f_n(x) = x + \frac{1}{n}$ and $f(x) = x$, prove that $f_n \to f$ in measure, as $n \to \infty$.

c) With $\phi(x) = e^x$ and f, f_n as in *b*), show that $(\phi \circ f_n)_{n=1}^{\infty}$ does not converge to $\phi \circ f$ in measure.

d) If ϕ is uniformly continuous, show that $f_n \to f$ in measure implies that $\phi \circ f_n \to \phi \circ f$ in measure, (as $n \to \infty$).

A function is uniformly continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.

Problem 5

Let λ be the Lebesgue measure on the real line, and let $f \in \mathcal{L}^1((0,\infty), \lambda)$ Consider the function $F : [0,\infty) \to \mathbb{R}$ defined by

$$F(r) = \int_{(0,\infty)} e^{-rx^2} f(x) d\lambda(x).$$

a) Prove that $F \in \mathcal{L}^1([0,\infty))$.

b) Prove that F has continuous derivatives of any order on $(0,\infty)$ and compute $F^{(n)}(r), r > 0$.

c) Show that for $f(x) = \frac{1}{1+x^3}$, the corresponding function F is not differentiable (from the right) at $r = 0$.

d) Prove that F is right differentiable at $r = 0$ if $\int_{(0,\infty)} \frac{|f(x)|}{x^2} d\lambda(x) < \infty$.

Problem 6

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $p_1, p_2 \in (1, \infty)$.

a) Show that, if $f \in \mathcal{L}^{p_1}(X, \mathcal{A}, \mu)$ and $g \in \mathcal{L}^{p_2}(X, \mathcal{A}, \mu)$ then

$$fg \in \mathcal{L}^{\frac{p_1 p_2}{p_1+p_2}}(X, \mathcal{A}, \mu).$$

b) Prove that every function $h \in \mathcal{L}^{\frac{p_1 p_2}{p_1+p_2}}(X, \mathcal{A}, \mu)$ can be written in the form fg for two functions as above.

EXAM, INTEGRATION THEORY, MARCH 17 2022.

(Allowed tools: Pen or pencil, rubber.) The exam time is 5 hours. Cheating is not allowed.
Remember to explain all non-trivial identities with references to the appropriate theorems.

Problem 1

- a) Compute $\int_{(0,\infty)} \sum_{n=1}^{\infty} \frac{e^{-nx}}{n}\, d\lambda$. Motivate any use of the Riemann integral.
- b) Use the Dominated Convergence Theorem to prove that

$$\int_{(0,\infty)} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-nx}\, d\lambda = \sum_{n=1}^{\infty} \int_{(0,\infty)} (-1)^{n+1} e^{-nx}\, d\lambda.$$

Hint: Group the series in pairs of two

- c) Conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$.

Problem 2

Let λ denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$ and define μ on $\mathcal{B}(\mathbb{R})$ via

$$\mu(E) = \int_E e^{-|x|}\, d\lambda.$$

- a) Is μ a measure? Verify your answer.
- b) If each f_n is bounded a.e., does it follow that $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$?
- c) Given any $\epsilon > 0$, can one find a set E with $\mu(\mathbb{R}\setminus E) < \epsilon$, with the property that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on E?

Problem 3

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$. Prove that

$$\mu(\{x : |f(x)| > t\}) \leqslant \left(\frac{\|f\|_p}{t}\right)^p.$$

Hint: Consider a suitably chosen characteristic function.

PROBLEM 4

- a) Show that
$$\int_{(0,1)}\int_{(1,\infty)}(e^{-xy}-2e^{-2xy})\,d\lambda(x)\,d\lambda(y)\neq\int_{(1,\infty)}\int_{(0,1)}(e^{-xy}-2e^{-2xy})\,d\lambda(y)\,d\lambda(x).$$
Hint: It is not necessary to compute explicitly the left and right hand side in order to see that they are not equal.
- b) Is the function $(e^{-xy}-2e^{-2xy})\chi_{(1,\infty)}(x)\chi_{(0,1)}(y)$ integrable?

PROBLEM 5

Let λ be the Lebesgue measure on the real line, and let $f\in\mathcal{L}^1((0,\infty),\mathcal{B}((0,\infty)),\lambda)$ Consider the function $F:[0,\infty)\to\mathbb{R}$ defined by
$$F(r)=\int_{(0,\infty)}e^{-rx^2}f(x)d\lambda(x).$$

- a) Prove that $F\in\mathcal{L}^1([0,\infty))$ whenever f has compact support in $(0,\infty)$, and provide a bound of $\|F\|_1$ in terms of a certain integral involving f.
- b) Again assuming that f has compact support, prove that F has continuous derivatives of any order on $(0,\infty)$ and compute $F^{(n)}(r), r>0$.
- c) Show that for $f(x)=\frac{1}{1+x^2}$, the corresponding function F is not differentiable (from the right) at $r=0$.

PROBLEM 6

Let $(X,\mathcal{A},\mu)$ be a finite measure space and let f_n be a sequence of measurable functions that converge to a measurable function f in measure. Assume that all involved functions are non-zero μ-a.e. Prove that $1/f_n$ converges to $1/f$ in measure.