EXAM, INTEGRATION THEORY, MARCH 17 2022.

(Allowed tools: Pen or pencil, rubber.) The exam time is 5 hours. Cheating is not allowed.

Remember to explain all non-trivial identities with references to the appropriate theorems.

PROBLEM 1

- a) Compute $\int_{(0,\infty)} \sum_{n=1}^{\infty} \frac{e^{-nx}}{n} \, d\lambda$. Motivate any use of the Riemann integral.
- b) Use the Dominated Convergence Theorem to prove that

$$\int_{(0,\infty)} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-nx} \, d\lambda = \sum_{n=1}^{\infty} \int_{(0,\infty)} (-1)^{n+1} e^{-nx} \, d\lambda.$$

  Hint: Group the series in pairs of two
- c) Conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$.

PROBLEM 2

Let λ denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$ and define μ on $\mathcal{B}(\mathbb{R})$ via

$$\mu(E) = \int_E e^{-|x|} \, d\lambda.$$

- a) Is μ a measure? Justify your answer.
- b) Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions that converges a.e. to f. If each f_n is bounded a.e., does it follow that $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$?
- c) Given any $\epsilon > 0$, can one find a set E with $\mu(\mathbb{R}\backslash E) < \epsilon$, with the property that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on E?

PROBLEM 3

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f \in \mathcal{L}^p(X, \mathcal{A}, \mu)$. Prove that

$$\mu(\{x : |f(x)| > t\}) \leq \left(\frac{\|f\|_p}{t}\right)^p.$$

Hint: Consider a suitably chosen characteristic function.

Problem 4

- a) Show that

$$\int_{(0,1)} \int_{(1,\infty)} (e^{-xy} - 2e^{-2xy}) \, d\lambda(x) \, d\lambda(y) \neq \int_{(1,\infty)} \int_{(0,1)} (e^{-xy} - 2e^{-2xy}) \, d\lambda(y) \, d\lambda(x).$$

  Hint: It is not necessary to compute explicitly the left and right hand side in order to see that they are not equal.
- b) Is the function $(e^{-xy} - 2e^{-2xy})\chi_{(1,\infty)}(x)\chi_{(0,1)}(y)$ integrable?

Problem 5

Let λ be the Lebesgue measure on the real line, and let $f \in \mathcal{L}^1((0,\infty), \mathcal{B}((0,\infty)), \lambda)$ Consider the function $F : [0,\infty) \to \mathbb{R}$ defined by

$$F(r) = \int_{(0,\infty)} e^{-rx^2} f(x) d\lambda(x).$$

- a) Prove that $F \in \mathcal{L}^1([0,\infty))$ whenever f has compact support in $(0,\infty)$, and provide a bound of $\|F\|_1$ in terms of a certain integral involving f.
- b) Again assuming that f has compact support, prove that F has continuous derivatives of any order on $(0,\infty)$ and compute $F^{(n)}(r), r > 0$.
- c) Show that for $f(x) = \frac{1}{1+x^3}$, the corresponding function F is not differentiable (from the right) at $r = 0$.

Problem 6

Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let f_n be a sequence of measurable functions that converge to a measurable function f in measure. Assume that all involved functions are non-zero μ-a.e. Prove that $1/f_n$ converges to $1/f$ in measure.