

1a By the Beppo-Levi theorem we have

$$\int_{(0,\infty)} \sum_{n=1}^{\infty} \frac{e^{-nx}}{n} d\lambda = \sum_{n=1}^{\infty} \int_{(0,\infty)} \frac{e^{-nx}}{n} d\lambda. \text{ For finite } R$$

$$\text{we also have } \lim_{R \rightarrow \infty} \int_{(0,R)} \frac{e^{-nx}}{n} d\lambda = \lim_{R \rightarrow \infty} \int_0^R \frac{e^{-nx}}{n} dx = \lim_{R \rightarrow \infty} \left[-\frac{e^{-nx}}{n^2} \right]_0^R = \frac{1}{n^2}$$

so another application of MCT gives

$$\int_{(0,\infty)} \frac{e^{-nx}}{n} d\lambda = \frac{1}{n^2}. \text{ Thus } \int_{(0,\infty)} \sum_{n=1}^{\infty} \frac{e^{-nx}}{n} d\lambda = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$b) \sum_{n=1}^{\infty} (-1)^{n+1} e^{-nx} = \sum_{k=1}^{\infty} (e^{-(2k-1)x} - e^{-2kx}) \geq 0 \text{ for } x > 0.$$

$$\text{Similarly } \sum_{n=1}^{\infty} (-1)^{n+1} e^{-nx} = e^{-x} \sum_{k=1}^{\infty} (e^{-2kx} - e^{-(2k-1)x}) \leq e^{-x}$$

Setting $f_N = \sum_{n=1}^N (-1)^{n+1} e^{-nx}$ the above computations are easily modified to give $0 \leq f_N \leq e^{-x}$.

Hence the DCT and linearity imply the desired identity

$$c) \sum_{n=1}^{\infty} \int_{(0,\infty)} (-1)^{n+1} e^{-nx} d\lambda = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ is seen as in 1a).}$$

$$\text{Also } \sum_{n=1}^{\infty} (-1)^{n+1} e^{-nx} = e^{-x} \sum_{n=0}^{\infty} (-e^{-x})^n = \frac{e^{-x}}{1+e^{-x}}$$

with primitive function $-\ln(1+e^{-x})$.

Integrating over $(0,\infty)$ gives $\ln 2$.

2a Yes; if E_n 's are mutually disjoint we have

$$\begin{aligned}\mu\left(\bigcup_{k=1}^{\infty} E_k\right) &= \int e^{-|x|} \chi_{\bigcup_{k=1}^{\infty} E_k}(x) d\lambda = \int \sum_{k=1}^{\infty} e^{-|x|} \chi_{E_k}(x) d\lambda \\ &= \sum_{k=1}^{\infty} \int e^{-|x|} \chi_{E_k}(x) d\lambda = \sum_{k=1}^{\infty} \mu(E_k), \text{ where we}\end{aligned}$$

used Beppo-Levi to swap \int and \sum

2b No. Consider eg $f_n(x) = e^{n+1} \chi_{[n, n+1]}$. $(f_n)_{n=1}^{\infty}$ converges pointwise to 0 but

$$\int f_n d\mu = \int_n^{n+1} \underbrace{e^{n+1-x}}_{\geq 1} d\lambda \geq 1.$$

2c Yes, the measure is finite so the statement follows by the Egoroff theorem.

3 $|f(x)| \geq t \chi_{\{x: |f(x)| \geq t\}}(x)$. Raising to power p and integrating gives $\int |f|^p d\mu \geq t^p \mu(\{x: |f(x)| \geq t\})$. The inequality, known as Chebyshev's inequality, easily follows.

$$4a) \int_0^1 \int_0^\infty e^{-xy} - 2e^{-2xy} dx dy = \int_0^1 \left[\frac{e^{-xy} - e^{-2xy}}{-y} \right]_0^\infty dy$$

$$= \int_0^1 \frac{e^{-y} - e^{-2y}}{y} dy \geq 0 \quad \text{since } e^{-y} < 1 \text{ so}$$

$e^{-2y} = (e^{-y})^2 < e^{-y}$. We don't bother motivating swapping between Lebesgue & Riemann integrals, since this was done in 1.

Now

$$\int_1^\infty \int_0^1 e^{-xy} - 2e^{-2xy} dy dx = \int_1^\infty \left[\frac{e^{-xy} - e^{-2xy}}{-x} \right]_0^1 dx =$$

$$= \int_1^\infty \frac{e^{-2x} - e^{-x}}{x} dx < 0, \quad \text{for the same reasons}$$

as above. Hence the two are not equal.

b) If $(e^{-xy} - 2e^{-2xy})\chi_{(1,\infty)}(x)\chi_{(0,1)}(y)$ would be integrable, then its modulus has a finite integral, and then Fubini's theorem states that $\iint -dx dy = \iint -dy dx$. Hence this can not be the case.

$$5a) \|F\|_1 = \int_0^\infty \left| \int_0^\infty e^{-rx^2} f(x) dx \right| dr \leq \int_0^\infty \int_0^\infty e^{-rx^2} |f(x)| dx dr$$

by the triangle inequality for integrals.

Tonelli's theorem then gives that this is $\leq \int_0^\infty \int_0^\infty e^{-rx^2} |f(x)| dr dx = \int_0^\infty \frac{|f(x)|}{x^2} dx$.

The right integral is clearly finite if f has compact support in $(0, \infty)$.

5b Differentiating inside the integral gives

$$\int_0^\infty \frac{d^n}{dr^n} e^{-rx^2} f(x) dx = \int_0^\infty (-x^2)^n e^{-rx^2} f(x) dx$$

Suppose we have verified that $F^{(n)}(r) = \int_0^\infty (-x^2)^n e^{-rx^2} f(x) dx$. We then get (by induction)

$$F^{(n+1)}(r) = \int_0^\infty (-x^2)^{n+1} e^{-rx^2} f(x) dx \quad \text{if } \exists g(x)$$

such that $|(-x^2)^{n+1} e^{-rx^2} f(x)| \leq g(x) \quad \forall r > 0$.

Since f has support in $[c, R]$ for some $0 < c < R$

we can take $g(x) = R^{2n+2} |f(x)|$.

5c Set $g_r(x) = \frac{e^{-rx^2}}{r} \cdot \frac{1}{1+x^3}$. Clearly $g_r \geq 0$ on $(0, \infty)$ and $\lim_{r \rightarrow 0^+} g_r(x) = 0 \cdot e^{-rx^2} \cdot \frac{1}{1+x^3} = \frac{x^2}{1+x^3}$

Hence Fatou's lemma gives

$$\lim_{r \rightarrow 0^+} \int_0^{\infty} g_r(x) dx \geq \int_0^{\infty} \lim_{r \rightarrow 0^+} g_r(x) dx = \int_0^{\infty} \frac{x^2}{1+x^3} dx$$

$$= \infty. \text{ Since } \frac{F(r) - F(0)}{r} = - \int_0^{\infty} g_r dx,$$

we conclude that " $\frac{d}{dr^+} F(0) = -\infty$ ",

which is not right differentiable at 0.

6 Suppose not. Then $\exists \epsilon > 0$ and subsequence f_{n_k} st $\mu(\{x: |\frac{1}{f(x)} - \frac{1}{f_{n_k}(x)}| > \epsilon\})$ is bounded below by some $\delta > 0$.

Since $f_{n_k} \rightarrow f$ in measure there exists a subsequence $f_{n_{k_m}}$ that converge pointwise to f . But then

$$\frac{1}{f_{n_{k_m}}} \rightarrow \frac{1}{f} \text{ m.a.e.}, \text{ which (by another}$$

theorem in 31, using that μ is finite)

implies that $\frac{1}{f_{n_{k_m}}} \rightarrow \frac{1}{f}$ in measure.

This gives $\lim_{m \rightarrow \infty} \mu(\{x: |\frac{1}{f(x)} - \frac{1}{f_{n_{k_m}}(x)}| > \epsilon\}) = 0$ which is a contradiction.