Answers exam Integration

Theory 21 March 2019

1 a) Countable additivity means that

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$ as long as the A_k's

are disjoint (and measurable). Since $\mu(\emptyset)=0$

we get $\mu(A\cup B)=\mu(A)+\mu(B)$ by

setting $A_1=A$, $A_2=B$ and $A_k=\emptyset$ for $k\geq 3$.

b We have $A\cup B=(A\setminus B)\cup(A\cap B)\cup(B\setminus A)$

$$\Rightarrow \mu(A\cup B)=\mu(A\setminus B)+\mu(A\cap B)+\mu(B\setminus A)$$

Also $B=(B\setminus A)\cup(A\cap B)$ and $A=(A\setminus B)\cup(A\cap B)$

so $\mu(A\setminus B)=\mu(A)-\mu(A\cap B)$ and

$\mu(B\setminus A)=\mu(B)-\mu(A\cap B)$.

This gives

$$\mu(A\cup B)=\mu(A)+\mu(B)-\mu(A\cap B).$$

2a) Its OK to use Riemann-integral formulas for continuous functions on bounded intervals. Thus $\int_0^A e^{-tx}\,d\lambda(x) =$

$$\int_0^A e^{-tx}\,dx = \left[\frac{e^{-tx}}{-t}\right]_0^A = \frac{1-e^{-tA}}{t}$$

By monotone convergence then we get

$$\int_0^\infty e^{-tx}\,d\lambda = \lim_{A\to\infty} \int e^{-tx}\chi_{[0,A]}(x)\,d\lambda(x) = \lim_{t\to\infty} \frac{1-e^{-tA}}{t}$$

$$= \frac{1}{t}.$$

b We are interested in $\lim_{h\to 0} \frac{\int e^{-(t_0+h)x} - e^{-t_0 x}\,dx}{h} = *$

where $t_0 > 0$. We can thus assume $h > \frac{t_0}{2}$.

The derivative of $t \to e^{-tx}$ is $-xe^{-tx}$ and

$|-xe^{-tx}| < |x|e^{-\frac{t_0 x}{2}}$ in the interval $t > \frac{t_0}{2}$.

Hence $\left|\frac{e^{-(t_0+h)x} - e^{-t_0 h}}{h}\right| \leq \frac{|x|e^{-\frac{t_0 x}{2}} \cdot |h|}{|h|} = |x|e^{-\frac{t_0 x}{2}}$

and the latter function is L^1. By DCT we get

$$* = \int_0^\infty \lim_{h\to 0} \dots\,dx = \int_0^\infty -xe^{-tx}\,dx.$$ Also $\frac{d}{dt}\frac{1}{t} = -\frac{1}{t^2}$.

2c This follows by induction. Assume that

$$(1) \int_0^\infty x^n e^{-tx}\,dx = \frac{n!}{t^{n+1}}. \quad \text{Then}$$

$$\frac{d}{dt}\int_0^\infty x^n e^{-tx}\,dx = \int_0^\infty \frac{d}{dt} x^n e^{-tx}\,dx = -\int_0^\infty x^{n+1} e^{-tx}\,dx$$

whereas $\frac{d}{dt}\frac{n!}{t^{n+1}} = -\frac{(n+1)!}{t^{n+2}}$. Thus (1) holds by induction. The desired identity follows by setting $t=1$.

3 a) If $\lim_{n\to\infty} f_n(x) = f(x)$ then by continuity we have $\lim_{n\to\infty} \phi(f_n(x)) = \phi(f(x))$. Since the first identity holds a.e. (by assumption), so does the second.

b) Given $\varepsilon > 0$ we have

$$\lambda\left(\left\{x : \left|\left(x+\tfrac{1}{n}\right)-x\right| > \varepsilon\right\}\right) = 0$$

as soon as $\frac{1}{n} < \varepsilon$. Thus $\lim_{n\to\infty} \lambda(\ldots) = 0$

3c) We have $\phi \circ f_n = e^{\frac{1}{n}} \cdot e^x$ & $\phi \circ f = e^x$.

Since $\lim_{x \to \infty} e^x = \infty$ & $e^{1/n} > 1$, we have

$$|e^{1/n} e^x - e^x| > \varepsilon$$ for all sufficiently large x. In other words

$$\lambda(\{x : |e^{1/n} e^x - e^x| > \varepsilon\}) = \infty$$

so the limit is also ∞.

3d Given $\varepsilon > 0$ pick δ s.t. $|\phi(x) - \phi(y)| < \varepsilon$ whenever $|x-y| < \delta$. Then

$$\{x : |\phi(f(x)) - \phi(f_n(x))| \geq \varepsilon\} \subseteq$$

$$\{x : |f(x) - f_n(x)| \geq \delta\}$$

so

$$\overline{\lim_{n \to \infty}} \mu(\{x : |\phi(f(x)) - \phi(f_n(x))| \geq \varepsilon\}) \leq$$

$$\overline{\lim_{n \to \infty}} \mu(\{x : |f(x) - f_n(x)| \geq \delta\}) = 0$$

Thus $\overline{\lim_{n \to \infty}} \dots = 0$ so therefore $\lim_{n \to \infty} \dots$ exists and equals 0 as well.

4/ a) If E is $\mathcal{P}(\mathbb{N})\times\mathcal{A}$-measurable then $E_k \in \mathcal{A}$ by a theorem. Conversely, if $E_k \in \mathcal{A}$ $\forall k \in \mathbb{N}$ then $\{k\}\times E_k$ is a product set, hence in $\mathcal{P}(\mathbb{N})\times\mathcal{A}$.

Since $E = \bigcup_{k=1}^{\infty} \{k\}\times E_k$, we see that E is measurable.

b) If $E = \{(k,x) : f(k,x) < t\}$ then

$$E_k = \{x : f_k(x) < t\}.$$ Now use a)

c) Since $\int g(k)\, d\sigma(k) = \sum_{k=1}^{\infty} g(k)$, the first statement is just Tonelli's theorem and the second is Fubini's theorem.

5a Pick a number α s.t. $p\alpha < 1$ & $r\alpha > 1$

Then $\chi_{[1,\infty)}(x) \cdot \frac{1}{x^\alpha}$ is in L^r but not L^p.

Similarly $\chi_{(0,1)}(x)\frac{1}{x^\alpha}$ is in L^p but not L^r.

b) The conditions for checking that $\|\cdot\|$ is a norm are immediately fulfilled, since $\|\cdot\|_p$ & $\|\cdot\|_r$ are norms.

To check completeness, let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence. Then it is also Cauchy in L^p so has a limit g, and

— " — L^r — " — h.

By a theorem, (f_n) converges in measure to g (& h), so a subsequence converges a.e. to g. A subsequence of that subsequence must by the same token converge to h a.e. But then $g = h$ a.e., so belong to the same equivalence class. Hence $f_n \to g$ in $L^p \cap L^r$.

5c) Set $E = \{x : |f(x)| > 1\}$, $f \in \mathcal{L}^p \cap \mathcal{L}^r$

Then $\lambda(E) \cdot 1 \leq \int |f(x)|^p \, d\lambda < \infty$, and

so $\int |f(x)|^q \chi_E(x) \, d\lambda \leq \| |f|^q \|_{r/q} \cdot \| \chi_E \|_{\frac{r/q}{r/q-1}}$

since $\frac{1}{(r/q)} + \frac{1}{\frac{r/q}{r/q-1}} = 1$. Since $f \in \mathcal{L}^r$ &

$\lambda(E) < \infty$, both numbers are finite.

For $x \in E^c$ we have $|f(x)|^q \leq |f(x)|^p$

since $q > p$, and therefore

$$\int |f(x)|^q \chi_{E^c}(x) \, d\lambda \leq \int |f(x)|^p \, d\lambda < \infty.$$

Thus $\int |f|^q (\chi_E + \chi_{E^c}) \, d\lambda < \infty$ Q.E.D.