Abstract Algebra - Homework 1

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Problem 1

Let $k = \mathbb{F}_7 = \mathbb{Z}/7\mathbb{Z}$. We define $\mathrm{GL}_2(k)$ as the group of all invertible 2×2 matrices with entries in the field k. In other words,

$$\mathrm{GL}_2(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k) \;\middle|\; ad - bc \neq 0 \right\}.$$

$\mathrm{GL}_2(k)$ Forms a Group

Let $A, B \in \mathrm{GL}_2(k)$. Then the product AB is defined as:

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$

where all entries involve only addition and multiplication of elements in k. Since k is a field, it is closed under addition and multiplication, so all entries of AB lie in k. Furthermore, $\det(AB) = \det(A)\det(B) \neq 0$, so $AB \in \mathrm{GL}_2(k)$. Thus, $\mathrm{GL}_2(k)$ is closed under matrix multiplication.

Since matrix multiplication is associative, the operation on $\mathrm{GL}_2(k)$ is associative.

The identity element in $\mathrm{GL}_2(k)$ is the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For any $A \in \mathrm{GL}_2(k)$, we have $AI = IA = A$, and since the entries 1 and 0 are in k, we conclude $I \in \mathrm{GL}_2(k)$.

Since every element of $\mathrm{GL}_2(k)$ is invertible by definition, the inverse of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k)$$

is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since $a, b, c, d \in k$, and $\det(A) = ad - bc \neq 0$, we have $\frac{1}{\det(A)} \in k$, because k is a field. As k is closed under addition, subtraction, and multiplication, all entries of A^{-1} lie in k. Hence, $A^{-1} \in \mathrm{GL}_2(k)$.

Therefore, $\mathrm{GL}_2(k)$ satisfies the group axioms under matrix multiplication and is a group.

The Center of $\mathrm{GL}_2(k)$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k)$. Suppose $A \in Z(\mathrm{GL}_2(k))$. Then for all $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathrm{GL}_2(k)$, we must have

$$AB = BA.$$

Compute both sides:

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}, \quad BA = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}.$$

For these to be equal for all $e, f, g, h \in k$, compare the corresponding entries:

$$\begin{aligned} ae + bg &= ea + fc & &\Rightarrow \quad b = c = 0 \\ af + bh &= eb + fd & &\Rightarrow \quad a = d, \text{ using } b = 0 \\ ce + dg &= ga + hc & &\Rightarrow \quad c = b = 0 \\ cf + dh &= gb + hd & &\Rightarrow \quad d = a \end{aligned}$$

Hence, A must be of the form:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I,$$

where $\lambda \in k^\times = k \setminus \{0\}$ (since A must be invertible). Thus,

$$Z(\mathrm{GL}_2(\mathbb{F}_7)) = \left\{ \lambda I \mid \lambda \in k^\times \right\}.$$

This center consists of 6 elements and forms an abelian subgroup of $\mathrm{GL}_2(\mathbb{F}_7)$.

Group Action on $\mathbb{P}^1(k)$

We define the projective line over k as $\mathbb{P}^1(k) = k \cup \{\infty\}$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k)$. Then the action of $\mathrm{GL}_2(k)$ on $\mathbb{P}^1(k)$ is given by the formula:

$$A \cdot z = \begin{cases} \dfrac{az+b}{cz+d}, & \text{if } z \in k \text{ and } cz+d \neq 0, \\ \infty, & \text{if } z \in k \text{ and } cz+d = 0, \\ \dfrac{a}{c}, & \text{if } z = \infty \text{ and } c \neq 0, \\ \infty, & \text{if } z = \infty \text{ and } c = 0. \end{cases}$$

We verify that this defines a group action.

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then for all $z \in \mathbb{P}^1(k)$,

$$I \cdot z = \frac{1 \cdot z + 0}{0 \cdot z + 1} = \frac{z}{1} = z, \quad \text{and} \quad I \cdot \infty = \frac{1}{0} := \infty.$$

So the identity acts as the identity function.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ be in $\mathrm{GL}_2(k)$, and let $z \in \mathbb{P}^1(k)$. Then:

$$B \cdot z = \frac{ez+f}{gz+h}, \quad \text{and} \quad A \cdot (B \cdot z) = \frac{a \cdot \left(\frac{ez+f}{gz+h}\right) + b}{c \cdot \left(\frac{ez+f}{gz+h}\right) + d}.$$

Simplifying this gives:

$$A \cdot (B \cdot z) = \frac{(aez + af + bgz + bh)}{(cez + cf + dgz + dh)} = \frac{(ae+bg)z + (af+bh)}{(ce+dg)z + (cf+dh)},$$

which is the action of the product matrix AB on z:

$$(AB) \cdot z.$$

Thus, the compatibility condition holds.

Therefore, the formula defines a group action of $\mathrm{GL}_2(k)$ on $\mathbb{P}^1(k)$.

Transitivity and the Stabilizer of 0

Let $z \in \mathbb{P}^1(k)$ be arbitrary. We want to find a matrix $A \in \mathrm{GL}_2(k)$ such that $A \cdot 0 = z$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k).$$

Then the action on 0 is given by

$$A \cdot 0 = \frac{b}{d},$$

provided $d \neq 0$. For any $z \in k$, we can take $b = z$ and $d = 1$, with $a \in k^\times$, $c \in k$ arbitrary, such that $\det(A) = ad - bc \neq 0$.

To obtain ∞, we require $d = 0$, in which case the formula becomes

$$A \cdot 0 = \frac{b}{d} = \infty,$$

provided $b \neq 0$. We can choose $a, c \in k$ arbitrarily so that $\det(A) = -bc \neq 0$, ensuring $A \in \mathrm{GL}_2(k)$.

Hence, for any $z \in \mathbb{P}^1(k)$, there exists a matrix $A \in \mathrm{GL}_2(k)$ such that $A \cdot 0 = z$. Therefore, the action is transitive.

We require $A \cdot 0 = 0$, so $\frac{b}{d} = 0 \Rightarrow b = 0$. Therefore, the stabilizer of 0 is the set of all invertible lower triangular matrices:

$$B = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k) \;\middle|\; a, d \in k^\times,\ c \in k \right\}.$$

Kernel of the Action

The kernel of the action can be defined as follows,

$$\ker = \left\{ A \in \mathrm{GL}_2(k) \mid A \cdot z = z \text{ for all } z \in \mathbb{P}^1(k) \right\}.$$

Then, for all $z \in k$,

$$\begin{aligned} A \cdot z = \frac{az+b}{cz+d} &= z \\ \implies az + b &= z(cz+d) = cz^2 + dz \\ \implies cz^2 + (d-a)z - b &= 0 \\ \implies \quad c = 0, \quad d &= a, \quad b = 0. \end{aligned}$$

Therefore,

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI, \quad \text{with } a \in k^\times.$$

These are exactly the scalar matrices which form the center. Hence, the kernel of the action is the center of $\mathrm{GL}_2(k)$.

Intersection of Conjugates of the Stabilizer

Let $B \subseteq \mathrm{GL}_2(k)$ be the stabilizer of $0 \in \mathbb{P}^1(k)$. We claim that

$$\bigcap_{g \in \mathrm{GL}_2(k)} gBg^{-1} = Z(\mathrm{GL}_2(k)).$$

This follows from the general fact that, if a group G acts transitively on a set X, then the intersection of all conjugates of the stabilizer G_x is equal to the kernel of the associated homomorphism:

$$\bigcap_{g \in G} gG_x g^{-1} = \ker(G \to \mathrm{Sym}(X)).$$

In our case, $G = \mathrm{GL}_2(k)$, $X = \mathbb{P}^1(k)$. We have already shown that the action is transitive, and that the kernel of the action is the center:

$$\ker = Z(\mathrm{GL}_2(k)).$$

Therefore,

$$\bigcap_{g \in \mathrm{GL}_2(k)} gBg^{-1} = Z(\mathrm{GL}_2(k)).$$

Problem 2

Let X be the set of all k-element subsets of $\{1, \ldots, n\}$, and let S_n act on X by

$$\sigma \cdot E := \{\sigma(e) \mid e \in E\}.$$

Fix the subset $A = \{1, \ldots, k\} \in X$. The stabilizer of A consists of all permutations in S_n that fix A setwise. These are exactly the permutations that act as an element of S_k on the set $\{1, \ldots, k\}$, and as an element of S_{n-k} on its complement $\{k+1, \ldots, n\}$, independently. Thus,

$$\mathrm{Stab}_{S_n}(A) \cong S_k \times S_{n-k}.$$

By the Orbit-Stabilizer Theorem, the number of k-element subsets is

$$|X| = [S_n : \mathrm{Stab}_{S_n}(A)] = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

Problem 3

Group action

Let G be a finite group and $H \subset G$ a subgroup. Define a map

$$\phi : H \times (G/H) \to G/H, \quad (h, gH) \mapsto (hg)H.$$

To verify that ϕ defines a group action, we check the two axioms. For all $gH \in G/H$,

$$e \cdot gH = (eg)H = gH.$$

For all $h_1, h_2 \in H$ and $gH \in G/H$,

$$h_1 \cdot (h_2 \cdot gH) = h_1 \cdot (h_2 gH) = (h_1 h_2 g)H = ((h_1 h_2) \cdot gH).$$

Hence, the identity and compatibility axioms are satisfied. Since we are viewing G/H purely as a set of left cosets (not as a quotient group), this confirms that ϕ defines a group action of H on the set G/H.

Trivial action implies normality

Suppose the action defined by

$$\phi : H \times (G/H) \to G/H, \quad (h, gH) \mapsto (hg)H$$

is trivial. That means for all $h \in H$ and $gH \in G/H$, we have:

$$\phi(h, gH) = gH \quad \Rightarrow \quad (hg)H = gH.$$

This implies:

$$hg \in gH \quad \Rightarrow \quad g^{-1}hg \in H \quad \text{for all } h \in H, g \in G.$$

Therefore, H is closed under conjugation by elements of G, i.e., $gHg^{-1} \subseteq H$. Since $g \in G$ was arbitrary, it follows that H is a normal subgroup of G.

The action is not transitive

Assume $[G : H] = p$, where p is the smallest prime dividing $|G|$.

Suppose, for contradiction, that the action is transitive. Then the orbit of any element $gH \in G/H$ under the action of H is the entire set G/H, which has p elements.

But the size of any orbit under a group action divides the order of the acting group and in this case, $|H| = |G|/p$.

Therefore, the size of the orbit must divide $|H| = |G|/p$, but it is equal to p. Since p is the smallest prime dividing $|G|$, it does not divide $|G|/p$, and thus does not divide $|H|$.

This is a contradiction. Hence, the action cannot be transitive.