ASTRACT ALGEBRA

HOMEWORK 1

Please hand in your solutions to the following problems by 2025-07-04. You must submit your homework solutions through the course website. You are encouraged to discuss the problems with your colleagues, but must hand in your own solutions. You are welcome to ask us questions, especially in connection to the discussion session.

Problem 1. (5 points)
Let k be a field (if you don't know what a field is, just take $k = \mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ according to your own preferences).

(1) Show that the set of invertible 2×2 matrices with coefficients in k, denoted $\mathbf{GL}_2(k)$, is a group. What is its center? **(1 point)**
(2) We define the projective line as $\mathbb{P}^1(k) := k \cup \{\infty\}$. Show that $\mathbf{GL}_2(k)$ acts on $\mathbb{P}^1(k)$ via, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(k)$ and $z \in \mathbb{P}^1(k)$,
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$
where $\frac{a\infty+b}{c\infty+d} = \frac{a}{c}$ and $\frac{1}{0} = \infty$, is an action. **(1 point)**
(3) Show that this action is transitive and find the stabilizer B of 0. **(1 point)**
(4) Show that the kernel of this action is the center of $\mathbf{GL}_2(k)$. **(1 point)**
(5) Show that the intersection of all conjugates of B is the center of $\mathbf{GL}_2$. **(1 point)**

Problem 2. (2 points)
Let $n \geq 1$ and $1 \leq k \leq n$ be an integer. We denote by X the set of all parts of $\{1, \cdots, n\}$ of cardinal k.

(1) Show that the stabilizer H of $\{1, \cdots, k\}$ for the action
$$S_n \times X \to X, \ (\sigma, E) \mapsto \sigma(E)$$
is isomorphic to $S_k \times S_{n-k}$. **(1 point)**
(2) Deduce from what precedes that $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$. **(1 point)**

Problem 3. (5 points)
Let G be a finite group and H be a subgroup of G. Then the order of H divides that of G and we call the index of H in G the number $\#G/\#H$. Let p be the least prime factor of the order of G, and let H be a subgroup of index p. We are going to show that H is normal in G.

(1) Show that the following map:
$$\begin{aligned} \phi : H \times (G/H) &\to G/H \\ (h, gH) &\mapsto (hg)H \end{aligned}$$
defines an action of H on G/H (be careful that so far G/H is only a set, not a group). **(1 point)**
(2) Show that if ϕ is trivial then H is a normal subgroup of G.**(1 point)**
(3) Show that the action ϕ is not transitive. **(1 point)**
(4) Write the analogue of the class equation for ϕ and deduce that H is normal in G. **(2 points)**

Problem 4. (3 points)
Let S_4 be the symmetric group of order 4.

(1) Show that $V_4 := \{\mathrm{Id}, (12)(34); (13)(24); (14)(23)\}$ is a normal subgroup in S_4. **(1 point)**
(2) Give an isomorphism $f : S_3 \to S_4/V_4$. **(1 point)**
(3) Find the normal subgroups of S_4 that contain V_4. **(1 point)**

Problem 5. (5 points)

Let E be a set of 3 elements.

(1) Let G be a group of order 35. How many group homomorphisms $\phi : G \to S_3$ are there? **(1 point)**
(2) Let G be a group of order 35. How many actions of G on E are there? **(1 point)**
(3) Let G be a group of even order which acts non transitively and non trivially on E, show that this action has two orbits and determine the cardinal of the stabilizer of an element of each orbit. Note: The action of G on E is trivial if $g \cdot x = x$ for any $g \in G$ and any $x \in E$. **(2 points)**
(4) If now G is of odd order, can we have a non transitive non trivial action of G on E? **(1 point)**