

# Abstract Algebra - Homework 1

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## Problem 1

Let  $k = \mathbb{F}_7 = \mathbb{Z}/7\mathbb{Z}$ . We define  $\text{GL}_2(k)$  as the group of all invertible  $2 \times 2$  matrices with entries in the field  $k$ . In other words,

$$\text{GL}_2(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k) \mid ad - bc \neq 0 \right\}.$$

### $\text{GL}_2(k)$ Forms a Group

Let  $A, B \in \text{GL}_2(k)$ . Then the product  $AB$  is defined as:

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$

where all entries involve only addition and multiplication of elements in  $k$ . Since  $k$  is a field, it is closed under addition and multiplication, so all entries of  $AB$  lie in  $k$ . Furthermore,  $\det(AB) = \det(A)\det(B) \neq 0$ , so  $AB \in \text{GL}_2(k)$ . Thus,  $\text{GL}_2(k)$  is closed under matrix multiplication.

Since matrix multiplication is associative, the operation on  $\text{GL}_2(k)$  is associative.

The identity element in  $\text{GL}_2(k)$  is the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For any  $A \in \text{GL}_2(k)$ , we have  $AI = IA = A$ , and since the entries 1 and 0 are in  $k$ , we conclude  $I \in \text{GL}_2(k)$ .

Since every element of  $\text{GL}_2(k)$  is invertible by definition, the inverse of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(k)$$

is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since  $a, b, c, d \in k$ , and  $\det(A) = ad - bc \neq 0$ , we have  $\frac{1}{\det(A)} \in k$ , because  $k$  is a field. As  $k$  is closed under addition, subtraction, and multiplication, all entries of  $A^{-1}$  lie in  $k$ . Hence,  $A^{-1} \in \text{GL}_2(k)$ .

Therefore,  $\text{GL}_2(k)$  satisfies the group axioms under matrix multiplication and is a group.

### The Center of $\text{GL}_2(k)$

Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(k)$ . Suppose  $A \in Z(\text{GL}_2(k))$ . Then for all  $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \text{GL}_2(k)$ , we must have

$$AB = BA.$$

Compute both sides:

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}, \quad BA = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}.$$

For these to be equal for all  $e, f, g, h \in k$ , compare the corresponding entries:

$$\begin{aligned} ae + bg &= ea + fc & \Rightarrow & b = c = 0 \\ af + bh &= eb + fd & \Rightarrow & a = d, \text{ using } b = 0 \\ ce + dg &= ga + hc & \Rightarrow & c = b = 0 \\ cf + dh &= gb + hd & \Rightarrow & d = a \end{aligned}$$

Hence,  $A$  must be of the form:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I,$$

where  $\lambda \in k^\times = k \setminus \{0\}$  (since  $A$  must be invertible). Thus,

$$Z(\text{GL}_2(\mathbb{F}_7)) = \{\lambda I \mid \lambda \in k^\times\}.$$

This center consists of 6 elements and forms an abelian subgroup of  $\text{GL}_2(\mathbb{F}_7)$ .

## Group Action on $\mathbb{P}^1(k)$

We define the projective line over  $k$  as  $\mathbb{P}^1(k) = k \cup \{\infty\}$ . Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(k)$ . Then the action of  $\text{GL}_2(k)$  on  $\mathbb{P}^1(k)$  is given by the formula:

$$A \cdot z = \begin{cases} \frac{az + b}{cz + d}, & \text{if } z \in k \text{ and } cz + d \neq 0, \\ \infty, & \text{if } z \in k \text{ and } cz + d = 0, \\ \frac{a}{c}, & \text{if } z = \infty \text{ and } c \neq 0, \\ \infty, & \text{if } z = \infty \text{ and } c = 0. \end{cases}$$

We verify that this defines a group action.

Let  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Then for all  $z \in \mathbb{P}^1(k)$ ,

$$I \cdot z = \frac{1 \cdot z + 0}{0 \cdot z + 1} = \frac{z}{1} = z, \quad \text{and} \quad I \cdot \infty = \frac{1}{0} := \infty.$$

So the identity acts as the identity function.

Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$  be in  $\text{GL}_2(k)$ , and let  $z \in \mathbb{P}^1(k)$ .

Then:

$$B \cdot z = \frac{ez + f}{gz + h}, \quad \text{and} \quad A \cdot (B \cdot z) = \frac{a \cdot \left( \frac{ez+f}{gz+h} \right) + b}{c \cdot \left( \frac{ez+f}{gz+h} \right) + d}.$$

Simplifying this gives:

$$A \cdot (B \cdot z) = \frac{(aez + af + bgz + bh)}{(cez + cf + dgz + dh)} = \frac{(ae + bg)z + (af + bh)}{(ce + dg)z + (cf + dh)},$$

which is the action of the product matrix  $AB$  on  $z$ :

$$(AB) \cdot z.$$

Thus, the compatibility condition holds.

Therefore, the formula defines a group action of  $\text{GL}_2(k)$  on  $\mathbb{P}^1(k)$ .

## Transitivity and the Stabilizer of 0

Let  $z \in \mathbb{P}^1(k)$  be arbitrary. We want to find a matrix  $A \in \mathrm{GL}_2(k)$  such that  $A \cdot 0 = z$ . Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k).$$

Then the action on 0 is given by

$$A \cdot 0 = \frac{b}{d},$$

provided  $d \neq 0$ . For any  $z \in k$ , we can take  $b = z$  and  $d = 1$ , with  $a \in k^\times$ ,  $c \in k$  arbitrary, such that  $\det(A) = ad - bc \neq 0$ .

To obtain  $\infty$ , we require  $d = 0$ , in which case the formula becomes

$$A \cdot 0 = \frac{b}{d} = \infty,$$

provided  $b \neq 0$ . We can choose  $a, c \in k$  arbitrarily so that  $\det(A) = -bc \neq 0$ , ensuring  $A \in \mathrm{GL}_2(k)$ .

Hence, for any  $z \in \mathbb{P}^1(k)$ , there exists a matrix  $A \in \mathrm{GL}_2(k)$  such that  $A \cdot 0 = z$ . Therefore, the action is transitive.

We require  $A \cdot 0 = 0$ , so  $\frac{b}{d} = 0 \Rightarrow b = 0$ . Therefore, the stabilizer of 0 is the set of all invertible lower triangular matrices:

$$B = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k) \mid a, d \in k^\times, c \in k \right\}.$$

## Kernel of the Action

The kernel of the action can be defined as follows,

$$\ker = \{ A \in \mathrm{GL}_2(k) \mid A \cdot z = z \text{ for all } z \in \mathbb{P}^1(k) \}.$$

Then, for all  $z \in k$ ,

$$\begin{aligned} A \cdot z &= \frac{az + b}{cz + d} = z \\ \implies az + b &= z(cz + d) = cz^2 + dz \\ \implies cz^2 + (d - a)z - b &= 0 \\ \implies c &= 0, \quad d = a, \quad b = 0. \end{aligned}$$

Therefore,

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI, \quad \text{with } a \in k^\times.$$

These are exactly the scalar matrices which form the center. Hence, the kernel of the action is the center of  $\mathrm{GL}_2(k)$ .

## Intersection of Conjugates of the Stabilizer

Let  $B \subseteq \mathrm{GL}_2(k)$  be the stabilizer of  $0 \in \mathbb{P}^1(k)$ . We claim that

$$\bigcap_{g \in \mathrm{GL}_2(k)} gBg^{-1} = Z(\mathrm{GL}_2(k)).$$

This follows from the general fact that, if a group  $G$  acts transitively on a set  $X$ , then the intersection of all conjugates of the stabilizer  $G_x$  is equal to the kernel of the associated homomorphism:

$$\bigcap_{g \in G} gG_xg^{-1} = \ker(G \rightarrow \mathrm{Sym}(X)).$$

In our case,  $G = \mathrm{GL}_2(k)$ ,  $X = \mathbb{P}^1(k)$ . We have already shown that the action is transitive, and that the kernel of the action is the center:

$$\ker = Z(\mathrm{GL}_2(k)).$$

Therefore,

$$\bigcap_{g \in \mathrm{GL}_2(k)} gBg^{-1} = Z(\mathrm{GL}_2(k)).$$

## Problem 2

Let  $X$  be the set of all  $k$ -element subsets of  $\{1, \dots, n\}$ , and let  $S_n$  act on  $X$  by

$$\sigma \cdot E := \{\sigma(e) \mid e \in E\}.$$

Fix the subset  $A = \{1, \dots, k\} \in X$ . The stabilizer of  $A$  consists of all permutations in  $S_n$  that fix  $A$  setwise. These are exactly the permutations that act as an element of  $S_k$  on the set  $\{1, \dots, k\}$ , and as an element of  $S_{n-k}$  on its complement  $\{k+1, \dots, n\}$ , independently. Thus,

$$\mathrm{Stab}_{S_n}(A) \cong S_k \times S_{n-k}.$$

By the Orbit-Stabilizer Theorem, the number of  $k$ -element subsets is

$$|X| = [S_n : \text{Stab}_{S_n}(A)] = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

## Problem 3

### Group action

Let  $G$  be a finite group and  $H \subset G$  a subgroup. Define a map

$$\phi : H \times (G/H) \rightarrow G/H, \quad (h, gH) \mapsto (hg)H.$$

To verify that  $\phi$  defines a group action, we check the two axioms. For all  $gH \in G/H$ ,

$$e \cdot gH = (eg)H = gH.$$

For all  $h_1, h_2 \in H$  and  $gH \in G/H$ ,

$$h_1 \cdot (h_2 \cdot gH) = h_1 \cdot (h_2gH) = (h_1h_2g)H = ((h_1h_2) \cdot gH).$$

Hence, the identity and compatibility axioms are satisfied. Since we are viewing  $G/H$  purely as a set of left cosets (not as a quotient group), this confirms that  $\phi$  defines a group action of  $H$  on the set  $G/H$ .