Abstract Algebra - Homework 1

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1 Group Action on the Projective Line

Let $k = \mathbb{F}_7 = \mathbb{Z}/7\mathbb{Z}$. We define $\mathrm{GL}_2(k)$ as the group of all invertible 2×2 matrices with entries in the field k. In other words,

$$\mathrm{GL}_2(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k) \;\middle|\; ad - bc \neq 0 \right\}.$$

$\mathrm{GL}_2(k)$ Forms a Group

Let $A, B \in \mathrm{GL}_2(k)$. Then the product AB is defined as:

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$

where all entries involve only addition and multiplication of elements in k. Since k is a field, it is closed under addition and multiplication, so all entries of AB lie in k. Furthermore, $\det(AB) = \det(A)\det(B) \neq 0$, so $AB \in \mathrm{GL}_2(k)$. Thus, $\mathrm{GL}_2(k)$ is closed under matrix multiplication.

Since matrix multiplication is associative, the operation on $\mathrm{GL}_2(k)$ is associative.

The identity element in $\mathrm{GL}_2(k)$ is the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For any $A \in \mathrm{GL}_2(k)$, we have $AI = IA = A$, and since the entries 1 and 0 are in k, we conclude $I \in \mathrm{GL}_2(k)$.

Since every element of $\mathrm{GL}_2(k)$ is invertible by definition, the inverse of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k)$$

is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since $a, b, c, d \in k$, and $\det(A) = ad - bc \neq 0$, we have $\frac{1}{\det(A)} \in k$, because k is a field. As k is closed under addition, subtraction, and multiplication, all entries of A^{-1} lie in k. Hence, $A^{-1} \in \mathrm{GL}_2(k)$.

Therefore, $\mathrm{GL}_2(k)$ satisfies the group axioms under matrix multiplication and is a group.

The Center of $\mathrm{GL}_2(k)$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k)$. Suppose $A \in Z(\mathrm{GL}_2(k))$. Then for all $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \mathrm{GL}_2(k)$, we must have

$$AB = BA.$$

Compute both sides:

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}, \quad BA = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}.$$

For these to be equal for all $e, f, g, h \in k$, compare the corresponding entries:

$$\begin{aligned}
ae + bg &= ea + fc & \Rightarrow \quad & b = c = 0 \\
af + bh &= eb + fd & \Rightarrow \quad & a = d, \text{ using } b = 0 \\
ce + dg &= ga + hc & \Rightarrow \quad & c = b = 0 \\
cf + dh &= gb + hd & \Rightarrow \quad & d = a
\end{aligned}$$

Hence, A must be of the form:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I,$$

where $\lambda \in k^\times = k \setminus \{0\}$ (since A must be invertible). Thus,

$$Z(\mathrm{GL}_2(\mathbb{F}_7)) = \left\{ \lambda I \mid \lambda \in k^\times \right\}.$$

This center consists of 6 elements and forms an abelian subgroup of $\mathrm{GL}_2(\mathbb{F}_7)$.