

# Abstract Algebra - Homework 1

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## 1 Group Action on the Projective Line

Let  $k = \mathbb{F}_7 = \mathbb{Z}/7\mathbb{Z}$ . We define  $\mathrm{GL}_2(k)$  as the group of all invertible  $2 \times 2$  matrices with entries in the field  $k$ . In other words,

$$\mathrm{GL}_2(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k) \mid ad - bc \neq 0 \right\}.$$

### $\mathrm{GL}_2(k)$ Forms a Group

Let  $A, B \in \mathrm{GL}_2(k)$ . Then the product  $AB$  is defined as:

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix},$$

where all entries involve only addition and multiplication of elements in  $k$ . Since  $k$  is a field, it is closed under addition and multiplication, so all entries of  $AB$  lie in  $k$ . Furthermore,  $\det(AB) = \det(A)\det(B) \neq 0$ , so  $AB \in \mathrm{GL}_2(k)$ . Thus,  $\mathrm{GL}_2(k)$  is closed under matrix multiplication.

Since matrix multiplication is associative, the operation on  $\mathrm{GL}_2(k)$  is associative.

The identity element in  $\mathrm{GL}_2(k)$  is the identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For any  $A \in \mathrm{GL}_2(k)$ , we have  $AI = IA = A$ , and since the entries 1 and 0 are in  $k$ , we conclude  $I \in \mathrm{GL}_2(k)$ .

Since every element of  $\mathrm{GL}_2(k)$  is invertible by definition, the inverse of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k)$$

is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since  $a, b, c, d \in k$ , and  $\det(A) = ad - bc \neq 0$ , we have  $\frac{1}{\det(A)} \in k$ , because  $k$  is a field. As  $k$  is closed under addition, subtraction, and multiplication, all entries of  $A^{-1}$  lie in  $k$ . Hence,  $A^{-1} \in \text{GL}_2(k)$ .

Therefore,  $\text{GL}_2(k)$  satisfies the group axioms under matrix multiplication and is a group.