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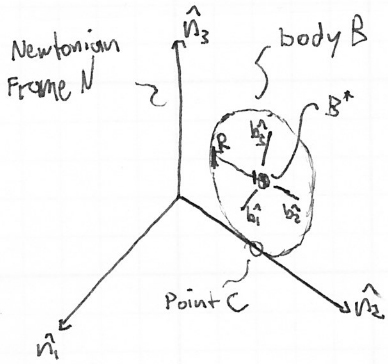
Kane's Method: Rolling Disc Example



Orientation of B in N: 3-2-1 or 3 simple rotations → introduce 3 generalized coordinates: q_1, q_2, q_3

$$N \xrightarrow[\hat{n}_3]{} B'' \xrightarrow[\hat{b}''_2]{} B' \xrightarrow[\hat{b}'_1]{} B$$

$$\begin{bmatrix}\hat{n}_1\\ \hat{n}_2\\ \hat{n}_3\end{bmatrix} = \begin{bmatrix}C_1 & -S_1 & 0\\ S_1 & C_1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}C_2 & 0 & S_2\\ 0 & 1 & 0\\ -S_2 & 0 & C_2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & C_3 & -S_3\\ 0 & S_3 & C_3\end{bmatrix}\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\\ \hat{b}_3\end{bmatrix}$$

where $C_i = \cos q_i$ & $S_i = \sin q_i$

This is a 3 D.O.F. system, so we introduce 3 generalized speeds: u_1, u_2, u_3 with body 321 rotations we have:

$$\boxed{{}^N\underline{\omega}^B = u_1\hat{b}_1 + u_2\hat{b}_2 + u_3\hat{b}_3}$$ which gives

$$\dot{q}_1 = (u_2 S_3 + u_3 C_3)/C_2 \qquad \dot{q}_2 = u_2 C_3 - u_3 S_3 \qquad \dot{q}_3 = u_1 + (u_2 S_3 + u_3 C_3)\, S_2/C_2$$

Two notes:

1) ${}^N\underline{\omega}^B$ means angular velocity (ω) of B, in N

2) $\dfrac{{}^N d\,\underline{a}}{dt} = \dfrac{{}^B d\,\underline{a}}{dt} + {}^N\underline{\omega}^B \times \underline{a}$, where $\underline{a}$ is "some vector"

With the no-slip condition for rolling at C, ${}^N\underline{v}^C = 0$

So ${}^N\underline{v}^{B^*} = {}^N\underline{\omega}^B \times \underline{r}^{CB^*} + \cancelto{0}{{}^N\underline{v}^C}$ where $\underline{r}^{CB^*} = R\hat{b}_3$

$${}^N\underline{v}^{B^*} = (u_1\hat{b}_1 + u_2\hat{b}_2 + u_3\hat{b}_3) \times R\hat{b}_3 = \boxed{{}^N\underline{v}^{B^*} = Ru_2\hat{b}_1 - Ru_1\hat{b}_2}$$

Now accelerations:

$${}^N\underline{\alpha}^B = \frac{{}^N d\, {}^N\underline{\omega}^B}{dt} = \frac{{}^B d\, {}^N\underline{\omega}^B}{dt} + \cancelto{0}{{}^N\underline{\omega}^B \times {}^N\underline{\omega}^B} = \boxed{{}^N\underline{\alpha}^B = \dot{u}_1\hat{b}_1 + \dot{u}_2\hat{b}_2 + \dot{u}_3\hat{b}_3}$$

$${}^N\underline{a}^{B^*} = \frac{{}^N d\, {}^N\underline{v}^{B^*}}{dt} = \frac{{}^B d\, {}^N\underline{v}^{B^*}}{dt} + {}^N\underline{\omega}^B \times {}^N\underline{v}^{B^*} = R\dot{u}_2\hat{b}_1 - R\dot{u}_1\hat{b}_2 + (u_1\hat{b}_1 + u_2\hat{b}_2 + u_3\hat{b}_3) \times (Ru_2\hat{b}_1 - Ru_1\hat{b}_2)$$

$$= R\dot{u}_2\hat{b}_1 - R\dot{u}_1\hat{b}_2 + (-Ru_1^2\hat{b}_3 - Ru_2^2\hat{b}_3 + Ru_2u_3\hat{b}_2 + Ru_1u_3\hat{b}_1$$

$$\boxed{{}^N\underline{a}^{B^*} = (R\dot{u}_2 + Ru_1u_3)\hat{b}_1 + (-R\dot{u}_1 + Ru_2u_3)\hat{b}_2 + (-Ru_1^2 - Ru_2^2)\hat{b}_3}$$

Masses & Inertias:

$$I^{B/B^*} = \begin{bmatrix}J & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}$$ mass of B = m

inertia of B around B^*

with $\hat{b}_1, \hat{b}_2, \hat{b}_3$ as principle axes

Now we form the partial velocities

	${}^N\underline{\omega}_r^B$	${}^N\underline{V}_r^{B^*}$
u_1	$\hat{b}_1$	$-R\hat{b}_2$
u_2	$\hat{b}_2$	$R\hat{b}_1$
u_3	$\hat{b}_3$	0

Now generalized active forces:

only gravity $\rightarrow$ $-mg\hat{n}_3$ @ B^*

so $F_r = -mg\hat{n}_3 \cdot {}^N\underline{V}_r^{B^*}$

generalized inertia forces:

$$F_r^* = {}^N\underline{\omega}_r^B \cdot T_B^* + {}^N\underline{V}_r^{B^*} \cdot R_B^*$$

$$R_B^* = -m\,{}^N\underline{a}^{B^*}$$

$$T_B^* = -[\alpha_1 I_1 - \omega_2\omega_3(I_2 - I_3)]\hat{b}_1 - [\alpha_2 I_2 - \omega_3\omega_1(I_3 - I_1)]\hat{b}_2 - [\alpha_3 I_3 - \omega_1\omega_2(I_1 - I_2)]\hat{b}_3$$

where $\alpha_i = {}^N\underline{\alpha}^B \cdot \hat{b}_i$, $\omega_i = {}^N\underline{\omega}^B \cdot \hat{b}_i$

and inertia tensor is $\begin{bmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{bmatrix}$ along $\hat{b}_1, \hat{b}_2, \hat{b}_3$

$F_r + F_r^* = 0$:

1) $-\dot{u}_1 J - m(R^2\dot{u}_1 - R^2 u_2 u_3) + mgR\,c_2 s_3 = 0$

2) $-\dot{u}_2 I + u_3 u_1(I - J) - m(R^2\dot{u}_2 + R^2 u_1 u_3) + mgR\,s_2 = 0$

3) $-\dot{u}_3 I + u_1 u_2(J - I) = 0$

$$\begin{bmatrix} -J - mR^2 & 0 & 0 \\ 0 & -I - mR^2 & 0 \\ 0 & 0 & -I \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \begin{bmatrix} -mR^2 u_2 u_3 - mgR\,c_2 s_3 \\ -u_1 u_3(I - J) + mR^2 u_1 u_3 - mgR\,s_2 \\ -u_1 u_2(J - I) \end{bmatrix}$$

Now let's multiply each side by ↑'s inverse

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \begin{bmatrix} (-mR^2 u_2 u_3 - mgR\,c_2 s_3)/(-J - mR^2) \\ (-u_1 u_3(I - J) + mR^2 u_1 u_3 - mgR\,s_2)/(-I - mR^2) \\ (-u_1 u_2(J - I))/-I \end{bmatrix}$$

if we say the disc is uniform density and flat: $J = mR^2$, $I = \frac{1}{2}mR^2$

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \begin{bmatrix} u_2 u_3/2 + g/2R\,c_2 s_3 \\ -u_1 u_3 + (2/3)\cdot(g/R)\cdot s_2 \\ u_1 u_2 \end{bmatrix}$$