Farzin Asadi

Engineering Mathematics with MATLAB® and Simulink®

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In loving memory of my father Moloud Asadi
and my mother Khorshid Tahmasebi,
always on my mind, forever in my heart.

Preface

The landscape of engineering education has undergone a significant transformation, driven by the increasing complexity of modern engineering problems and the rapid advancement of computational tools. This book provides a comprehensive guide to solving engineering mathematics problems, utilizing the power of MATLAB® and Simulink®.

By combining rigorous mathematical exposition with hands-on computational exercises, this book aims to equip readers with the essential skills to tackle a wide range of engineering challenges. Whether you are a student embarking on your engineering journey or a seasoned professional seeking to enhance your problem-solving abilities, this book offers a comprehensive and accessible guide.

The core objective of this book is to provide a solid foundation in engineering mathematics, covering topics such as linear algebra, differential equations, Fourier analysis, complex analysis, and probability and statistics. Each mathematical concept is illustrated with many examples, making the learning process engaging and relevant.

MATLAB® and Simulink® serve as powerful tools to reinforce these concepts. Throughout the book, readers will learn how to utilize these software packages to solve mathematical problems, analyze data, and simulate complex systems. By integrating theory with practice, this book fosters a deeper understanding of the underlying mathematical principles.

I hope that this book will serve as a valuable resource for students, researchers, and engineers alike. By mastering the concepts presented here and leveraging the capabilities of MATLAB® and Simulink®, readers will be well-prepared to address the challenges of the modern engineering world.

Ankara, Türkiye — Farzin Asadi

Competing Interests The author has no competing interests to declare that are relevant to the content of this manuscript.

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1 Functions

1.1 Introduction

The first part of this chapter reviews essential problem-solving techniques. The second part demonstrates MATLAB's application to the problems discussed.

1.2 Set of Numbers

Numbers, the fundamental units of mathematics, have evolved over centuries, shaping our understanding of quantity, measurement, and abstract concepts. From the simple counting of objects to the complex calculations of modern science, numbers underpin our world. Important number sets are:

1.3 Basic Number Sets

Natural Numbers ($\mathbb{N}$): Positive integers starting from 1: {1, 2, 3, ...}

Whole Numbers ($\mathbb{W}$): Natural numbers including 0: {0, 1, 2, 3, ...}

Integers ($\mathbb{Z}$): Positive and negative whole numbers, including zero: $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

Rational Numbers ($\mathbb{Q}$): Numbers that can be expressed as a fraction of two integers: $\{p/q | p, q \in Z, q \neq 0\}$

Irrational Numbers ($\mathbb{I}$): Numbers that cannot be expressed as a simple fraction, often represented by non-repeating, non-terminating decimals: $\left\{\sqrt{2}, \pi, e, \ldots\right\}$

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Real Numbers ($\mathbb{R}$): All rational and irrational numbers: This set encompasses all numbers on the number line.

Complex Numbers ($\mathbb{C}$): Numbers of the form a + bi, where a and b are real numbers, and i is the imaginary unit $\left(\sqrt{-1}\right)$: $\{a + bi | a, b \in \mathbb{R}\}$

1.4 Functions

A function in mathematics is a rule that assigns to each input exactly one output. Key components of a function:

Domain: The set of all possible input values.

Codomain: The set of all possible output values. It's like the entire menu at a restaurant, even if not all dishes are currently available.

Range: The subset of the codomain that consists of all actual output values. It's like the dishes that are actually served, based on the ingredients and the chef's choices.

Let's study some numeric examples.

Example 1.1 Value of $f(x) = x^2 + \sin(x)$ at $x = 0.5$ at $x = 0.5$ equals to $f(0.5) = 0.5^2 + \sin(0.5) = 0.7294$.

Example 1.2 Value of $z = f(x, y) = x^2 + y^2 + 3x^2y + \sin(y)$ at point $(1, 2)$ equals to $f(1, 2) = 1^2 + 2^2 + 3 \times 1^2 \times 2 + \sin(2) = 11.9093$.

Example 1.3 Domain of $f(x) = \frac{x+2}{x+3}$ is $\mathbb{R} - \{x : x + 3 = 0\} = \mathbb{R} - \{-3\}$.

Example 1.4 Domain of $f(x) = \ln\left(\frac{x+2}{x+3}\right)$ is $\mathbb{R} - [-3, -2] = (-\infty, -3) \cup (-2, \infty)$ (Fig. 1.1).

Fig. 1.1 Sign of $\frac{x+2}{x+3}$ for different values of x

	$-\infty$	-3		-2	$+\infty$
$x+2$	$-$		$-$	0	$+$
$x+3$	$-$	0	$+$		$+$
$\frac{x+2}{x+3}$	$+$	∞	$-$	0	$+$

1.5 Line Equation

Consider the line that passes through points $A = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $B = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ (Fig. 1.2).

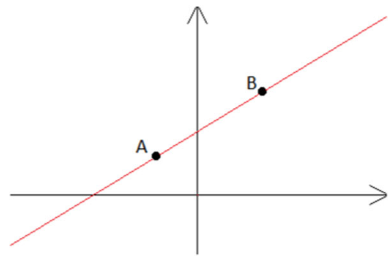


Fig. 1.2 Line passes through A and B

Equation of the line that passes through points $A = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $B = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ is:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

$$y - y_1 = m(x - x_1) \Rightarrow y = m(x - x_1) + y_1$$

or

$$y - y_2 = m(x - x_2) \Rightarrow y = m(x - x_2) + y_2$$

Example 1.5 Find the equation of the line passing through points $A = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

In this case $m = \frac{4-2}{3-(-1)} = \frac{2-4}{-1-3} = \frac{2}{4} = 0.5$.

$$y = 0.5(x - (-1)) + 2 = 0.5x + 2.5$$

$$y = 0.5(x - 3) + 4 = 0.5x + 2.5$$

Example 1.6 Find the equation of the line passing through point $A = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ with a slope of -5.

$$y - y_1 = m(x - x_1) \Rightarrow y = -5(x - (-1)) + 2 = -5x - 3$$

Example 1.7 Find the equation of the line shown in Fig. 1.3.

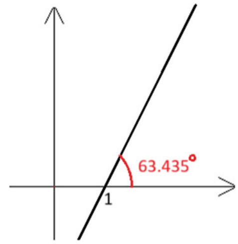


Fig. 1.3 Example line for equation determination

$$m = \tan(63.435) = 2 \Rightarrow y - y_1 = m(x - x_1) \Rightarrow y - 0 = 2(x - 1) \Rightarrow y = 2x - 2$$

Example 1.8 Find the equation of the line shown in Fig. 1.4.

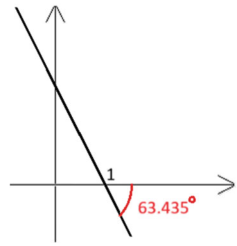


Fig. 1.4 Example line for equation determination

$$m = -\tan(63.435) = -2 \Rightarrow y - y_1 = m(x - x_1) \Rightarrow y - 0 = -2(x - 1) \Rightarrow y = -2x + 2$$

1.6 Angle Between Two Lines

The angle between two lines, $y_1 = m_1x + b_1$ and $y_2 = m_2x + b_2$ is $\alpha = \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_2m_1}\right|$.

Example 1.9 Angle between $y_1 = 2x$ and $y_2 = x$ is $\alpha = \tan^{-1}\left|\frac{1-2}{1+1\times 2}\right| = \tan^{-1}\left(\frac{1}{3}\right) = 0.3218\,\text{Rad} = 18.435°$ (Fig. 1.5).

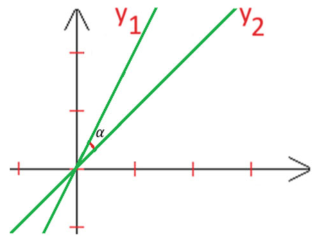


Fig. 1.5 Angle between lines y_1 and y_2

1.7 Distance Between Two Points

The distance between two points $A = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $B = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. The distance between two points $A = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $B = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

Example 1.10 The distance between points $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ (Fig. 1.6) is $d = \sqrt{(1 - (-1))^2 + (2 - (-3))^2} = \sqrt{29} = 5.38$.

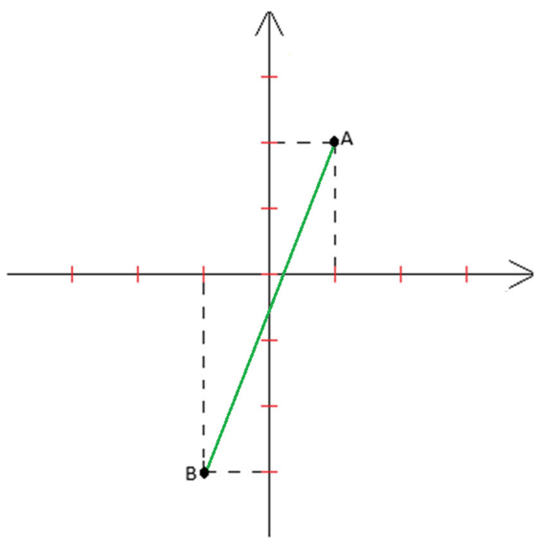


Fig. 1.6 Distance between points A and B

The distance between point $A = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and origin $O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is: $d = \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2} = \sqrt{x_1^2 + y_1^2}$. For instance, the distance between point $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the origin (Fig. 1.7) is: $d = \sqrt{1^2 + 2^2} = \sqrt{5} = 2.24$.

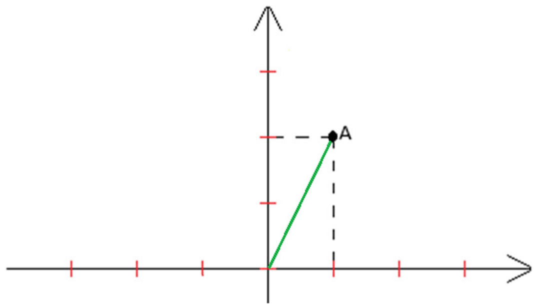


Fig. 1.7 Distance between point A and origin

1.8 Distance Between a Point and a Plane

Given a plane defined by the equation $Ax + By + Cz + D = 0$ and a point $P(x_0, y_0, z_0)$, the distance d between the point and the plane is given by the formula $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Example 1.11 Calculate the distance between point $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $x + 2y + z = 4$.

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \Rightarrow d = \frac{|1 + 2 \times 0 - 2 - 4|}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{5}{\sqrt{6}} = 2.0412$$

1.9 Exponential Function

An exponential function is a mathematical function of the form $f(x) = a^x$, where: x is a variable and a is a constant called the base, and it must be positive and not equal to 1. When the base of function is $e \approx 2.7182$, the function is called "natural exponential function". Important properties of the natural exponential function are studied in this section.

Consider $f(t) = e^{-pt}$ where $p > 0$ is a real numbers. $\int_0^\infty e^{-pt}dt$ equals to:

$$\int_0^\infty e^{-pt}dt = \frac{e^{-pt}}{-p}\bigg|_0^\infty = \frac{e^{-p\infty}}{-p} - \frac{e^{-p0}}{-p} = 0 - \frac{1}{-p} = \frac{1}{p}$$

$f(t) = e^{-pt}$ decays with the rate shown in Table 1.1.

Table 1.1 Rate of decay for $f(t) = e^{-pt}$

pt	e^{-pt}
0	1
1	0.37
2	0.13
3	0.05
4	0.02
5	0.006

For instance, graph of $f(t) = e^{-2t}$ is shown in Fig. 1.8. $f(t)$ is almost zero for $t \geq \frac{5}{2} = 2.5$.

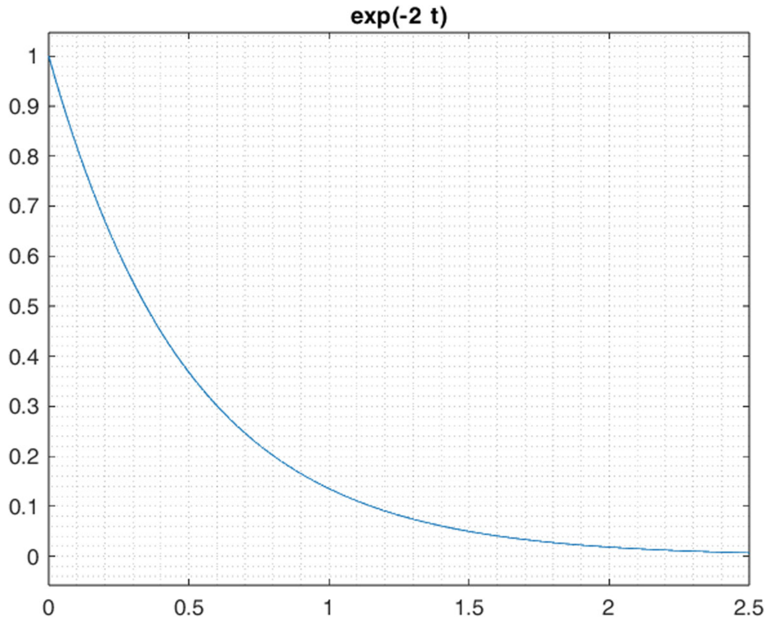


Fig. 1.8 Plot of e^{-2t}

If $a < 0$ then $\lim_{x\to\infty} e^{(a+jb)x} = 0$ (note that j shows the imaginary unit). For instance, $\lim_{x\to\infty} e^{(-3+j4)x}$, $\lim_{x\to\infty} e^{(-3-j4)x}$ or $\lim_{x\to\infty} e^{-6x}$ equals to zero.

If $a > 0$ then $\lim_{x\to-\infty} e^{(a+jb)x} = 0$ (note that j shows the imaginary unit). For instance, $\lim_{x\to-\infty} e^{(7+j2)x}$, $\lim_{x\to-\infty} e^{(7-j2)x}$ or $\lim_{x\to-\infty} e^{12x}$ equals to zero.

Let's study another important function: $f(t) = (I - F)e^{-pt} + F$. I and F are two real constants. Note that:

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}(I - F)e^{-pt} + F = I$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}(I - F)e^{-pt} + F = F$$

The time needed to travel from I to F is approximately $\frac{5}{p}$.

For instance, for $I = 3$, $F = 5$ and $p = 2$, $f(t) = (I - F)e^{-pt} + F = (3 - 5)e^{-2t} + 5 = 5 - 2e^{-2t}$. Note that $\lim_{t \to 0} f(t) = \lim_{t \to 0} 5 - 2e^{-2t} = 3$ and $\lim_{t \to \infty} f(t) = \lim_{t \to \infty} 5 - 2e^{-2t} = 5$. Graph of $f(t) = 5 - 2e^{-2t}$ is shown in Fig. 1.9. Travel from 3 to 5 takes around $\frac{5}{2} = 2.5$s.

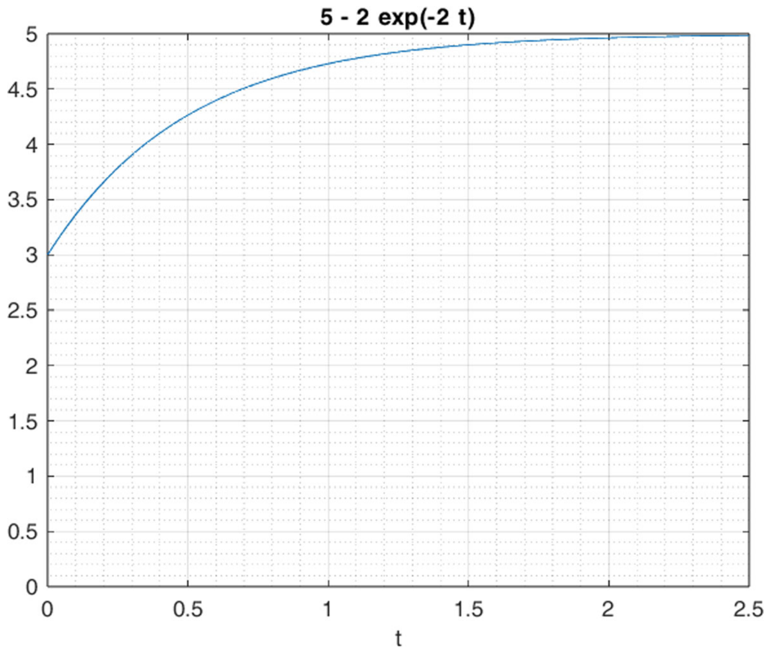


Fig. 1.9 Plot of $5 - 2e^{-2t}$

Let's study another numeric example. For $I = 5$, $F = 3$ and $p = 2$, $f(t) = (I - F)e^{-pt} + F = (5 - 3)e^{-2t} + 3 = 3 + 2e^{-2t}$. Note that $\lim_{t \to 0} f(t) = \lim_{t \to 0} 3 + 2e^{-2t} = 5$ and $\lim_{t \to \infty} f(t) = \lim_{t \to \infty} 3 + 2e^{-2t} = 3$. Graph of $f(t) = 3 + 2e^{-2t}$ is shown in Fig. 1.10. Travel from 5 to 3 takes around $\frac{5}{2} = 2.5$s.

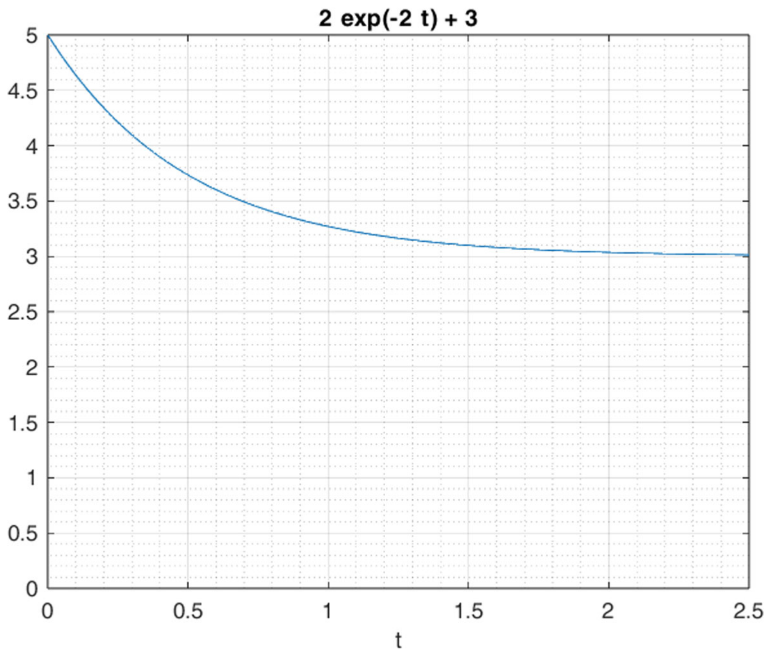


Fig. 1.10 Plot of $3 + 2e^{-2t}$

1.10 Logarithmic Function

A logarithm is essentially the inverse operation of exponentiation. It's a way to determine the exponent to which a base must be raised to produce a given number. For instance, $3^4 = 81 \Rightarrow \log_3 81 = 4$.

The following are the most important properties of logarithmic functions:

$$\log(a \times b) = \log(a) + \log(b)$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$$

$$\log\left(a^n\right) = n \times \log(a)$$

$$a^{log_a x} = x$$

Let's study some numeric example.

Example 1.12 Find solutions of $\log_2 x + \log_2(x-3) = 2$.

$$\begin{aligned}\log_2 x + \log_2 x - 3 = 2 \Rightarrow \log_2 x(x-3) = 2 \Rightarrow x(x-3) \\ = 2^2 \Rightarrow x^2 - 3x = 4 \Rightarrow x^2 - 3x - 4 \\ = 0 \Rightarrow x_1 = 4, x_2 = -1\end{aligned}$$

Example 1.13 Find solutions of $\log\left(x^2 - 11x + 130\right) = 2$.

$$x^2 - 11x + 130 = 10^2 \Rightarrow x^2 - 11x + 30 = 0 \Rightarrow x_1 = 5, x_1 = 6$$

Example 1.14 Calculate the value of $e^{3\ln(4)}$ and $10^{2\log(4)}$.

$$e^{3\ln(4)} = e^{\ln\left(4^3\right)} = 4^3 = 64$$

$$10^{2\log(4)} = 10^{\log\left(4^2\right)} = 4^2 = 16$$

1.11 Even and Odd Functions

A function $f(x)$ is even if $f(-x) = f(x)$ for all x in the domain. For instance, $f(x) = x^2, f(x) = \cos(x)$ or $f(x) = |x|$ are even function. The graph of an even function is symmetric about the y-axis. This means that if you reflect the graph across the y-axis, it will look exactly the same (Fig. 1.11).

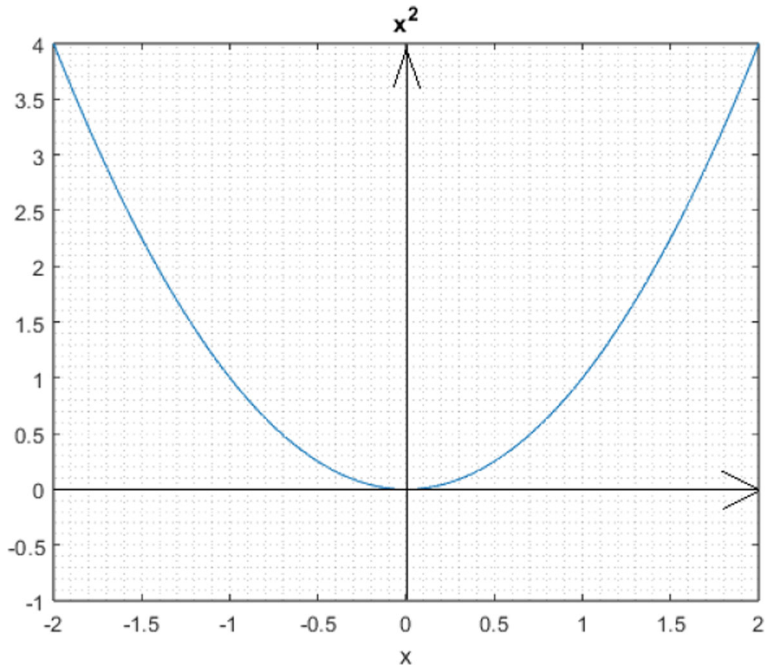


Fig. 1.11 Plot of $f(x) = x^2$

A function $f(x)$ is odd if $f(-x) = -f(x)$ for all x in the domain. For instance, $f(x) = x^3$, $f(x) = \sin(x)$ or $f(x) = x^3 - x$ are odd functions. The graph of an odd function is symmetric about the origin. If you draw a line from any point on the graph of an odd function to the origin, and then extend that line an equal distance on the other side of the origin, you will reach another point on the graph (Fig. 1.12).

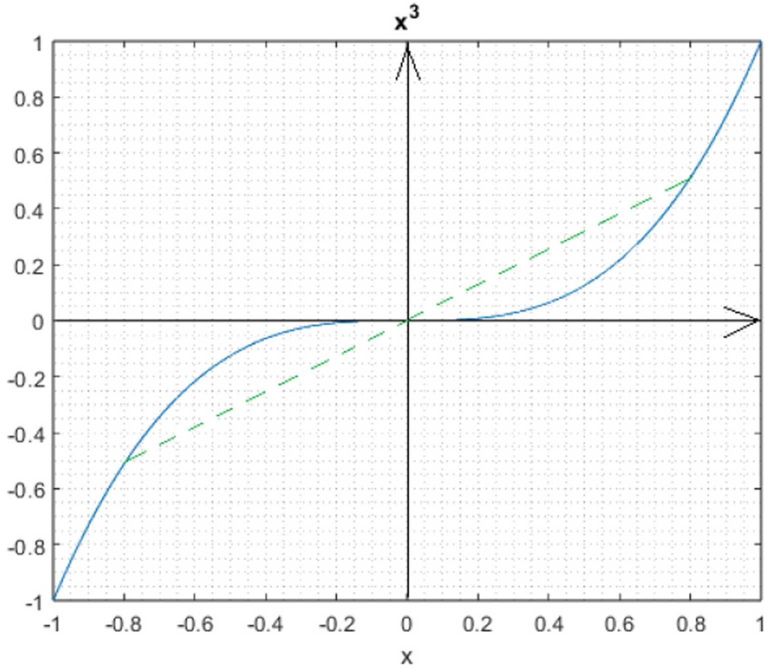


Fig. 1.12 Plot of $f(x) = x^3$

Not all functions are either even or odd. Many functions fall into the category of "neither even nor odd." The sum of two even functions is even. The sum of two odd functions is odd. The product of two even functions is even. The product of two odd functions is even.

For an even function $f(x)$ integrated over a symmetric interval $[-a, a]$, the integral can be simplified as $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.

For an odd function $f(x)$ integrated over a symmetric interval $[-a, a]$, the integral can be simplified as $\int_{-a}^{a} f(x)dx = 0$.

Exercise: Show that $\tan(kx)$ and $\text{cotan}(kx)$ are odd functions.

1.12 Linear Dependence of Functions

A set of functions $\{f_1(x), f_2(x), \ldots f_n(x)\}$ is said to be linearly dependent if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that $c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$ for all x in the domain of the functions. In simpler terms, a set of functions is linearly dependent if one of the functions can be expressed as a linear combination of the others. For instance, $f_1(x) = x^2$ and $f_2(x) = 2x^2$ are linearly dependent since $-2f_1(x) + f_2(x) = 0$.

A set of functions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ is said to be linearly independent if the only solution to the equation $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for all x in the domain of the functions, is the trivial solution $c_1 = c_2 = \cdots = c_n = 0$. For instance, $f_1(x) = x$ and $f_2(x) = 2x^2$ are linearly independent since $c_1 f_1(x) + c_2 f_2(x) = 0 \Rightarrow c_1 = c_2 = 0$.

In simpler terms, a set of functions is linearly independent if no function in the set can be expressed as a linear combination of the others.

The Wronskian is a determinant used to determine the linear independence of a set of functions. Given n functions, $f_1, f_2, \ldots, f_n$, that are $n-1$ times differentiable on an interval I, the Wronskian $W(f_1, f_2, \ldots, f_n)$ is defined as the determinant of the following matrix:

$$W(f_1(x), f_2(x), \ldots, f_n(x)) = \begin{vmatrix} f_1(x) & f_2(x) & \ldots & f_n(x) \\ f_1'(x) & f_2'(x) & \ldots & f_n'(x) \\ f_1''(x) & f_2''(x) & \ldots & f_2''(x) \\ \ldots & \ldots & \ldots & \ldots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \ldots & f_n^{(n-1)}(x) \end{vmatrix}$$

If the Wronskian is nonzero for at least one point in the interval I, then the functions $f_1, f_2, ..., f_n$ are linearly independent on I.

Example 1.15 Consider the functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Their Wronskian is:

$$W(\sin(x), \cos(x)) = \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} = -\sin^2(x) - \cos^2(x)$$
$$= -\left(\sin^2(x) + \cos^2(x)\right) = -1$$

Therefore, $\sin(x)$ and $\cos(x)$ are independent since $W(\sin(x), \cos(x)) = -1 \neq 0$.

Exercise: Investigate the linear dependence of the set of functions $\{e^x, e^{2x}, e^{3x}\}$.

1.13 Rotation

The rotation matrix $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ rotates a point by θ radians when multiplied to its coordinates.

Example 1.16 Rotate point $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by 90° in the counterclockwise direction.

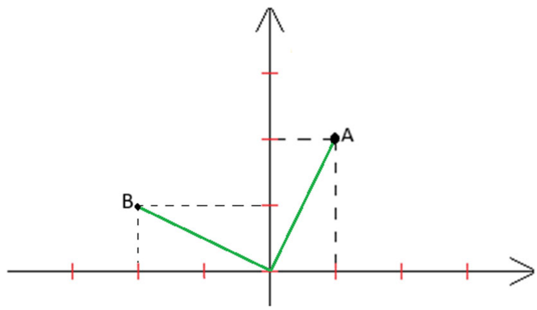
Rotation is counterclockwise, therefore, $\theta = +90°$.

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta)x - \sin(\theta)y \\ \sin(\theta)x + \cos(\theta)y \end{bmatrix}$$

$$B = \begin{bmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Figure 1.13 shows point A and the result of the rotation (point B).

Fig. 1.13 Point rotation



This problem can be solved with the aid of complex numbers as well.

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 + 2j \Rightarrow B = (1 + 2j) \times e^{j\frac{\pi}{2}} = -2 + j \Rightarrow B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Example 1.17 Rotate point $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by 63.435° in the clockwise direction.

Rotation is clockwise, therefore, $\theta = -63.435°$.

$$C = \begin{bmatrix} \cos(-63.435°) & -\sin(-63.435°) \\ \sin(-63.435°) & \cos(-63.435°) \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 0.4473 & 0.8944 \\ -0.8944 & 0.4473 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.2361 \\ 0 \end{bmatrix}$$

Figure 1.14 shows point A and the result of the rotation (point C).

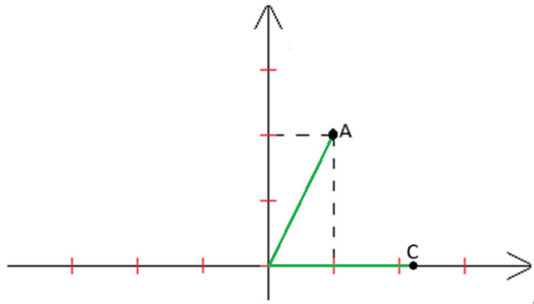


Fig. 1.14 Point rotation

This problem can be solved with the aid of complex numbers as well ($-63.435° = -1.1071\,\text{rad}$).

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 + 2j \Rightarrow C$$
$$= (1 + 2j) \times e^{-j1.1071}$$
$$= 2.2361 \Rightarrow C = \begin{bmatrix} 2.2361 \\ 0 \end{bmatrix}$$

1.14 Scalar Product of Two Vectors

The dot product or scalar product of two vectors, $\vec{A} = x_1 i + y_1 j$ and $\vec{B} = x_2 i + y_2 j$, is equal to the product of their magnitudes and the cosine of the angle α between them. The dot product of two vectors, $\vec{A}$ and $\vec{B}$, is denoted by $\vec{A}.\vec{B}$.

$$\vec{A} = x_1 i + y_1 j \Rightarrow \left|\vec{A}\right| = \sqrt{x_1^2 + y_1^2}$$

$$\vec{B} = x_2 i + y_2 j \Rightarrow \left|\vec{B}\right| = \sqrt{x_2^2 + y_2^2}$$

$$\vec{A}.\vec{B} = \left|\vec{A}\right| \times \left|\vec{B}\right| \times \cos(\alpha) = x_1 x_2 + y_1 y_2$$

$$\cos(\alpha) = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2} \times \sqrt{x_2^2 + y_2^2}} \Rightarrow \alpha = \cos^{-1}\left(\frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2} \times \sqrt{x_2^2 + y_2^2}}\right)$$

Note that here i and j represent unit vectors along the x and y axes, respectively, in a two-dimensional Cartesian coordinate system.

Example 1.18 Angle between $\vec{A} = i + 2j$ and $\vec{B} = i + j$ is $\alpha = \cos^{-1}\left(\frac{1+2}{\sqrt{2}\times\sqrt{5}}\right) = 18.435°$ (Fig. 1.15).

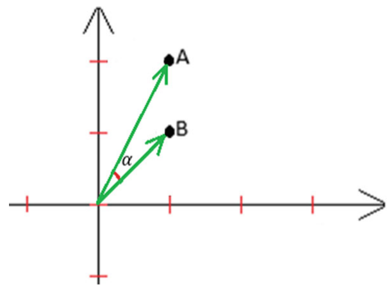


Fig. 1.15 Vectors $\vec{A}$ and $\vec{B}$

You can solve this problem with the aid of complex numbers as well:

$$\vec{A} = 1 + 2j = \sqrt{1^2 + 2^2}e^{j\tan^{-1}\left(\frac{2}{1}\right)} = \sqrt{5}e^{j1.1071}$$

$$\vec{B} = 1 + j = \sqrt{1^2 + 1^2}e^{j\tan^{-1}\left(\frac{1}{1}\right)} = \sqrt{2}e^{j\frac{\pi}{4}} = \sqrt{2}e^{j0.7854}$$

$$\alpha = 1.1071 - 0.7854 = 0.3217\,\text{rad} = 18.4321°$$

Very small deviations between obtained results come from rounding the numbers.

1.15 Equation of Circle in Cartesian Coordinates

Equation of a circle with radius of R and center of $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ is: $(x - x_0)^2 + (y - y_0)^2 = R^2$. For instance, equation of circle shown in Fig. 1.16 is: $(x - 2)^2 + (y - 1)^2 = 1^2$.

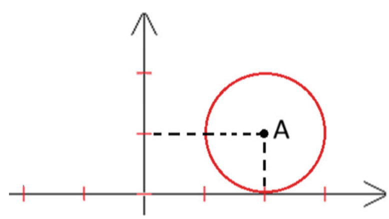


Fig. 1.16 Plot of $(x - 2)^2 + (y - 1)^2 = 1^2$

$(x-x_0)^2+(y-y_0)^2<R^2$ represents the points that fall inside a circle with a radius of R and a center at the point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ but excluding the circle itself. Such a region is called an opened disk.

$(x-x_0)^2+(y-y_0)^2\leq R^2$ describes the interior of a circle with a radius of R and a center at the point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and the circle itself. Such a region is called a closed disk.

For instance, equation of closed disk shown in Fig. 1.17 is: $(x-2)^2+(y-1)^2\leq 1^2$.

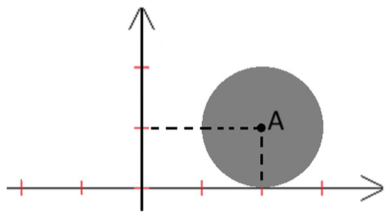


Fig. 1.17 Plot of $(x-2)^2+(y-1)^2\leq 1^2$

$(x-x_0)^2+(y-y_0)^2>R^2$ represents the points that fall outside a circle with a radius of R and a center at the point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ but excluding the circle itself.

$(x-x_0)^2+(y-y_0)^2\geq R^2$ describes the interior of a circle with a radius of R and a center at the point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and the circle itself.

For instance, Fig. 1.18 shows the points that satisfy the inequality $(x-2)^2+(y-1)^2>1^2$.

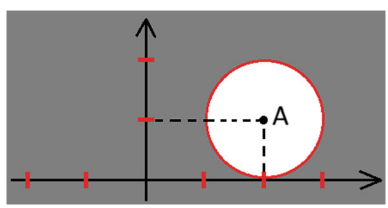


Fig. 1.18 Plot of $(x-2)^2+(y-1)^2>1^2$

1.16 Equation of Ellipse in Cartesian Coordinates

An ellipse is a plane curve surrounding two focal points, such that for all points on the curve, the sum of the two distances to the focal points is a constant (Fig. 1.19). It generalizes a circle, which is the special type of ellipse in which the two focal points are the same.

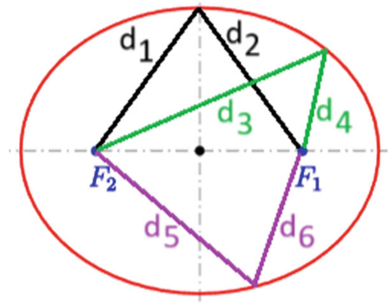


Fig. 1.19 An ellipse ($d_1 + d_2 = d_3 + d_4 = d_5 + d_6$)

Equation of the ellipse shown in Fig. 1.20 is $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$. Note that $a > b$.

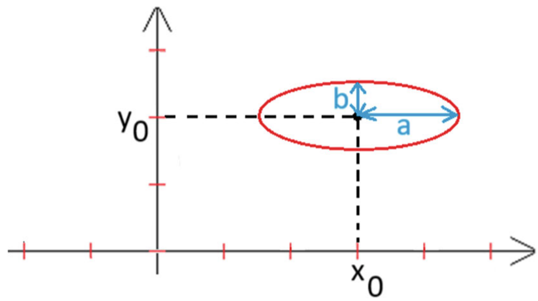


Fig. 1.20 A horizontal ellipse ($a > b$)

Equation of the ellipse shown in Fig. 1.21 is $\frac{(x-x_0)^2}{b^2} + \frac{(y-y_0)^2}{a^2} = 1$. Note that $a > b$.

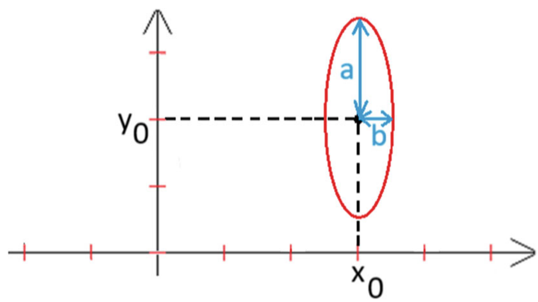


Fig. 1.21 A vertical ellipse ($a > b$)

Example 1.19 Equation of ellipse shown in Fig. 1.22 is $\frac{(x-0)^2}{2^2} + \frac{(y-2)^2}{1^2} = 1 \Rightarrow \frac{x^2}{4} + (y - 2)^2 = 1$.

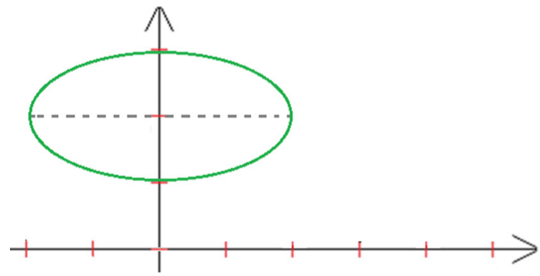


Fig. 1.22 Plot of $\frac{x^2}{4} + (y-2)^2 = 1$

1.17 Graphical Approach to Function Transformations

The function $f(t) + c$ represents a vertical shift of the function $f(t)$. When you add a constant c to the function, you shift the entire graph of $f(t)$ upward by c units if c is positive, and downward by $|c|$ units if c is negative.

For $t_0 > 0$, the graph of $f(t - t_0)$ is a horizontal translation of the graph of $f(t)$ by t_0 units to the right. The graph of $f(t + t_0)$ is a horizontal translation of the graph of $f(t)$ by t_0 units to the left.

The function $f(-t)$ represents a reflection of the function $f(t)$ about the y-axis. This means that the graph of $f(-t)$ is the mirror image of the graph of $f(t)$ across the y-axis.

The function $-f(t)$ represents a reflection of the function $f(t)$ about the x-axis. This means that the graph of $-f(t)$ is the mirror image of the graph of $f(t)$ across the x-axis.

The function $f(at)$ represents a horizontal scaling of the function $f(t)$. If $a > 1$, the graph of $f(at)$ is compressed horizontally. If $0 < a < 1$, the graph of $f(at)$ is stretched horizontally.

Example 1.20 Consider the function $f(t)$ shown in Fig. 1.23.

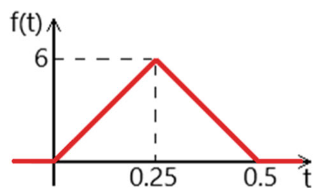
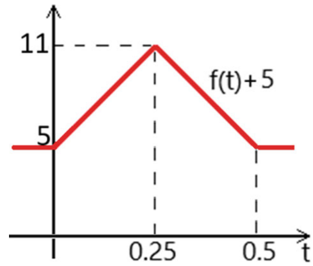


Fig. 1.23 Plot of $f(t)$

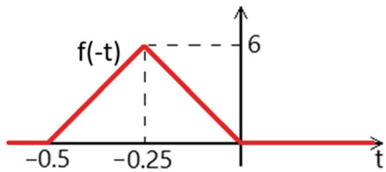
Graph of $f(t) + 5$ is shown in Fig. 1.24.

Fig. 1.24 Plot of $f(t) + 5$



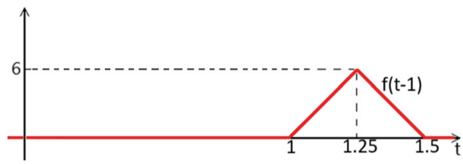
Graph of $f(-t)$ is shown in Fig. 1.25.

Fig. 1.25 Plot of $f(-t)$



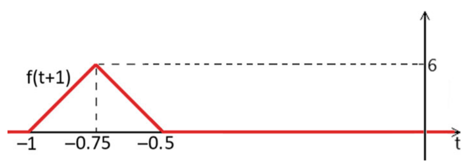
Graph of $f(t-1)$ is shown in Fig. 1.26.

Fig. 1.26 Plot of $f(t-1)$



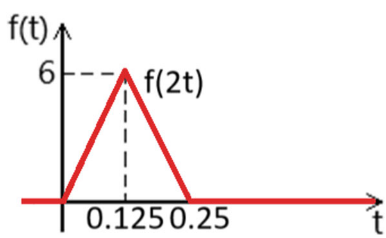
Graph of $f(t+1)$ is shown in Fig. 1.27.

Fig. 1.27 Plot of $f(t+1)$



Graph of $f(2t)$ is shown in Fig. 1.28.

Fig. 1.28 Plot of $f(2t)$



Graph of $f(0.5t)$ is shown in Fig. 1.29.

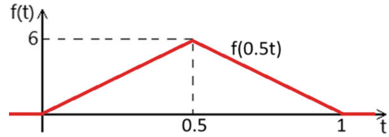


Fig. 1.29 Plot of $f(0.5t)$

Exercise: Draw the graph of $-f(-t+2)$.

1.18 Determining the Sign of Expressions

Figure 1.30 shows how to determine the sign of $f(x) = ax + b$.

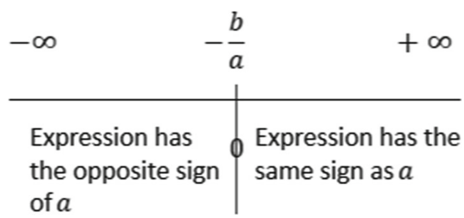


Fig. 1.30 Sign of $f(x) = ax + b$

Figure 1.31 shows how to determine the sign of $f(x) = ax^2 + bx + c$ when $ax^2 + bx + c$ has two real roots x_1 and x_2.

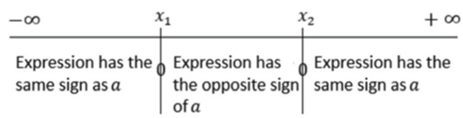


Fig. 1.31 Sign of $f(x) = ax^2 + bx + c$ (roots are real and different)

Figure 1.32 shows how to determine the sign of $f(x) = ax^2 + bx + c$ when $ax^2 + bx + c$ has repeated real roots.

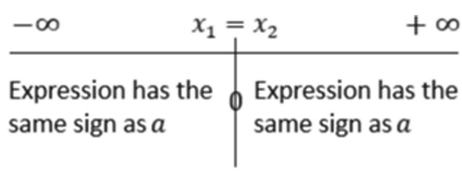


Fig. 1.32 Sign of $f(x) = ax^2 + bx + c$ (roots are real and repeated)

Figure 1.33 shows how to determine the sign of $f(x) = ax^2 + bx + c$ when $ax^2 + bx + c$ has no real roots.

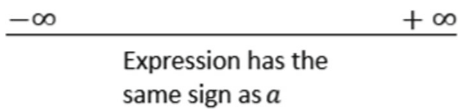


Fig. 1.33 Sign of $f(x) = ax^2 + bx + c$ (roots are complex)

Example 1.21 Determine the sign of $f(x) = x + 12$.

Sign of $f(x) = x + 12$ is shown in Fig. 1.34.

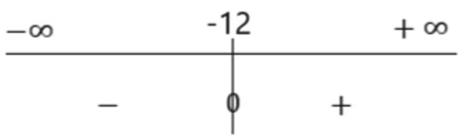


Fig. 1.34 Sign of $f(x) = x + 12$

Example 1.22 Determine the sign of $f(x) = \frac{x+1}{x+7}$.

Sign of $f(x) = \frac{x+1}{x+7}$ is shown in Fig. 1.35.

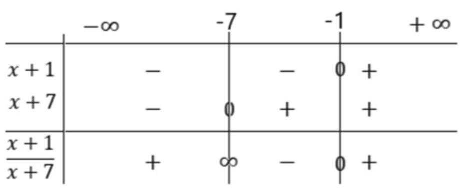


Fig. 1.35 Sign of $f(x) = \frac{x+1}{x+7}$

Graph of $f(x) = \frac{x+1}{x+7}$ is shown in Fig. 1.36, as well. Use this graph to verify the result shown in Fig. 1.35.

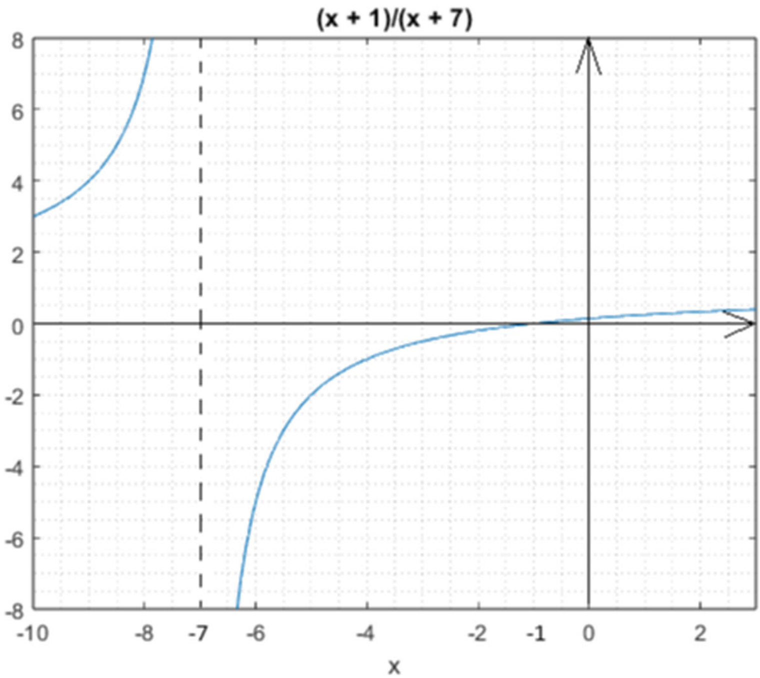


Fig. 1.36 Plot of $f(x) = \frac{x+1}{x+7}$

Example 1.23 Determine the sign of $f(x) = -x^2 + 4x - 10$.

$-x^2 + 4x - 10 = 0 \Rightarrow \Delta = 4^2 - 4(-1)(-10) = 16 - 40 = -24$. Therefore, this equation has no real roots. Sign of this expression is always negative since $a = -1 < 0$. Note that $f(x) = -x^2 + 4x - 10$ can be written as $f(x) = -(x-2)^2 - 6$. This expression clearly shows that $f(x)$ is always negative.

Example 1.24 Determine the sign of $f(x) = 2x^2 + 4x + 10$.

$2x^2 + 4x + 10 = 0 \Rightarrow \Delta = 4^2 - 4(2)(10) = 16 - 80 = -64$. Therefore, this equation has no real roots. Sign of this expression is always positive since $a = 2 > 0$. Note that $f(x) = 2x^2 + 4x + 10$ can be written as $f(x) = 2(x+1)^2 + 8$. This expression clearly shows that $f(x)$ is always positive.

Example 1.25 Determine the sign of $f(x) = \frac{x+1}{x^2+5x+6}$.

Sign of $f(x) = \frac{x+1}{x^2+5x+6} = \frac{x+1}{(x+2)(x+3)}$ is shown in Fig. 1.37.

Fig. 1.37 Sign of $f(x) = \frac{x+1}{x^2+5x+6} = \frac{x+1}{(x+2)(x+3)}$

	$-\infty$	-3		-2		-1	$+\infty$
$x+1$	$-$		$-$		$-$	0	$+$
$x+2$	$-$		$-$	0	$+$		$+$
$x+3$	$-$	0	$+$		$+$		$+$
$\frac{x+1}{(x+2)(x+3)}$	$-$	∞	$+$	∞	$-$	0	$+$

Graph of $f(x) = \frac{x+1}{x^2+5x+6}$ is shown in Fig. 1.38, as well. Use this graph to verify the result shown in Fig. 1.37.

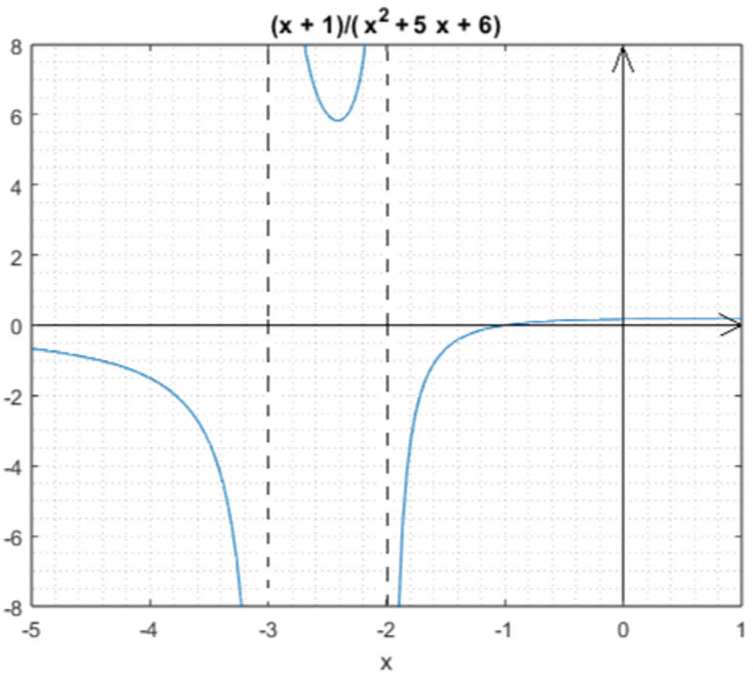


Fig. 1.38 Plot of $f(x) = \frac{x+1}{x^2+5x+6}$

1.19 Optimizing Single-Variable Functions

Single-variable optimization is a mathematical technique used to find the maximum or minimum value of a function of one variable within a specified domain.

To find the optimum point of a function of one variable, we typically calculate the derivative of the function, set it equal to zero, and solve for the critical points. Critical points are points where the derivative of the function is zero or undefined.

To classify these critical points as local maxima, minima, or points of inflection, we typically employ the second derivative test. By evaluating the second derivative at each critical point, we can determine the concavity of the function at that point. A positive second derivative indicates a local minimum, while a negative second derivative signifies a local maximum.

Example 1.26 Find the optimum point of $f(x) = x^2 + 3x + 5$.

$$f(x) = x^2 + 3x + 5 \Rightarrow f'(x) = 2x + 3 \Rightarrow 2x + 3 = 0 \Rightarrow x = -1.5$$

$$f''(x) = 2 > 0 \Rightarrow f''(-1.5) = 2 > 0$$

Therefore, the critical point $x = -1.5$ is a minimum (Fig. 1.39). Value of function at $x = -1.5$ is $f(-1.5) = (-1.5)^2 + 3 \times -1.5 + 5 = 2.7500$.

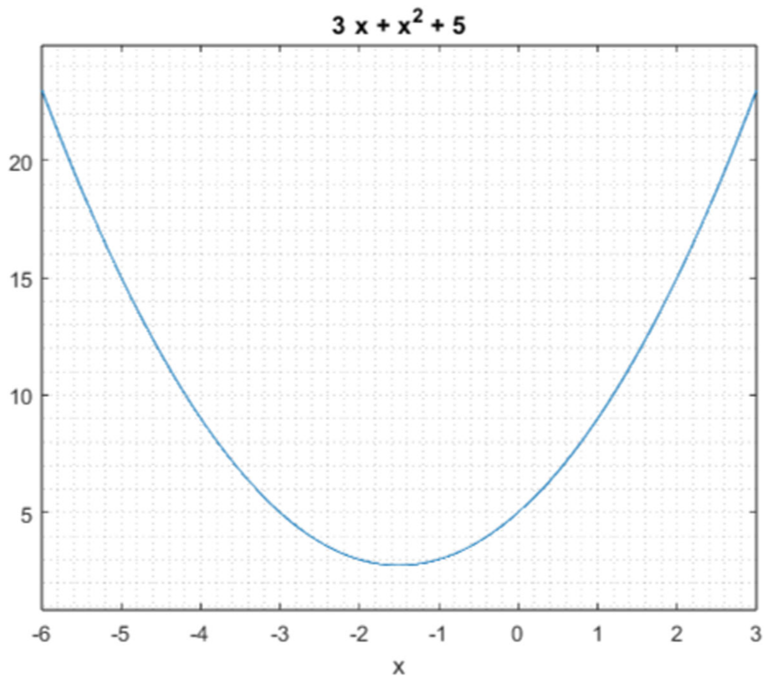


Fig. 1.39 Graph of $f(x) = x^2 + 3x + 5$

Example 1.27 Given the constraint $x + y = 7$, determine the maximum value of the $x \times y$.

$$f = x \cdot y = x \cdot (7 - x) = -x^2 + 7x \Rightarrow f'(x) = -2x + 7 = 0 \Rightarrow x = 3.5 \Rightarrow y = 3.5$$

$$x.y = 3.5 \times 3.5 = 12.25$$

Example 1.28 The circuit in Fig. 1.40, has a fixed resistor R_1 and a fixed voltage source V. Determine the optimal value of resistor R_2 to maximize the power dissipated across it.

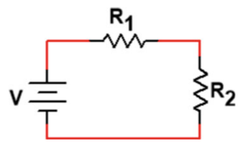


Fig. 1.40 A simple circuit

According to the principles of physics, power dissipation in the resistor R_2 is given by:

$$P_{R_2} = V_{R_2} \times I_{R_2} = \frac{R_2}{R_1 + R_2} V \times \frac{1}{R_1 + R_2} V = \frac{V^2 R_2}{(R_1 + R_2)^2}$$

We want to maximize the power dissipated in the resistor R_2.

$$\begin{aligned} \frac{dP_{R_2}}{dR_2} = 0 \Rightarrow & \frac{V^2(R_1 + R_2)^2 - 2(R_1 + R_2)(V^2 R_2)}{(R_1 + R_2)^4} = 0 \\ & \Rightarrow V^2(R_1 + R_2)^2 - 2(R_1 + R_2)(V^2 R_2) = 0 \\ & \Rightarrow V^2(R_1 + R_2) - 2V^2 R_2 = 0 \Rightarrow R_2 = R_1 \end{aligned}$$

$$R_2 = R_1 \Rightarrow P_{\max} = \frac{V^2 R_1}{(R_1 + R_1)^2} = \frac{V^2}{4R_1}$$

For $V = 10V$ and $R_1 = 2.5\Omega$, $P_{R_2} = \frac{100R_2}{(2.5+R_2)^2}$ (Fig. 1.41). When $R_2 = R_1 = 2.5\Omega$, $P_{\max} = \frac{V^2}{4R_1} = \frac{100}{4\times2.5} = 10\,\text{W}$ (Fig. 1.42).

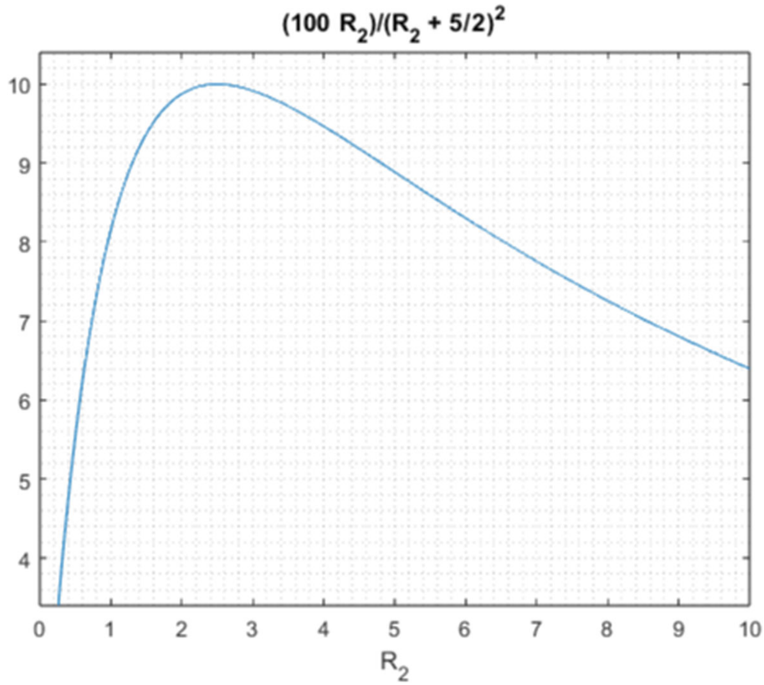


Fig. 1.41 Graph of $P_{R_2} = \frac{100R_2}{(2.5+R_2)^2}$

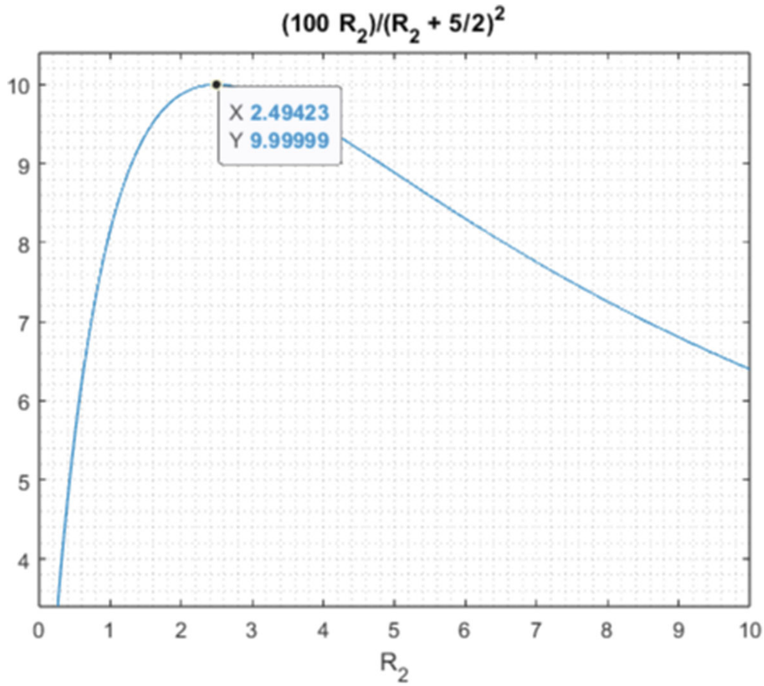


Fig. 1.42 The function $P_{R_2} = \frac{100R_2}{(2.5+R_2)^2}$ reaches its peak value of 10 around the point 2.5 on the x-axis

Example 1.29 Find the minimum value of $f(x, y) = (x + y - 3)^2 + (2x + 3y - 7)^2 + (x + y - 4)^2$.

Let's find the critical point of cost function:

$$\frac{\partial f}{\partial x} = 2(x + y - 3) + 4(2x + 3y - 7) + 2(x + y - 4) = 12x + 16y - 42$$

$$\frac{\partial f}{\partial y} = 2(x + y - 3) + 6(2x + 3y - 7) + 2(x + y - 4) = 16x + 22y - 56$$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \Rightarrow 12x + 16y - 42 = 0 \Rightarrow 6x + 8y = 21 \\ \frac{\partial f}{\partial y} = 0 \Rightarrow 16x + 22y - 56 = 0 \Rightarrow 8x + 11y = 28 \end{cases} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.5 \\ 0 \end{bmatrix}$$

The critical point is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.5 \\ 0 \end{bmatrix}$. Let's determine the type of obtained critical point. Remember that critical point is:

A. a local minimum if $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$,
B. a local maximum if $D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, and,
C. a saddle point if $D < 0$.

Note that D shows the discriminant and $D = \frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$. If $D = 0$, the test is inconclusive. For given cost function:

$$\frac{\partial^2 f}{\partial x^2} = 12$$

$$\frac{\partial^2 f}{\partial y^2} = 22$$

$$\frac{\partial^2 f}{\partial x \partial y} = 16$$

$$D = \frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 12 \times 22 - 16^2 = 8$$

Obtained critical point is global minimum since $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$. Value of cost function at critical point is $f(3.5, 0) = 0.5$.

Exercise: Calculate the minimum value of $\sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2}$.

1.20 Arithmetic Sequence

An arithmetic sequence is a sequence of numbers where the difference between any two consecutive terms is constant. This constant difference is known as the common difference. For instance, $2, 5, 8, 11, 14, 17, \ldots, 2 + (n-1) \times 3$ is an arithmetic sequence.

Consider the arithmetic sequence shown in below:

An arithmetic sequence can be represented as: $a_1, a_1 + d, a_1 + 2d, a_1 + 3d, a_1 + 4d, \ldots, a_1 + (n-1)d$ where a_1 is the first term, n is the position of the term and d is the common difference between consecutive terms.

The formula to calculate the sum of the first N terms of an arithmetic sequence is:

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + a_3 + \cdots + a_N = \frac{N(a_1 + a_N)}{2}$$
$$= \frac{N(a_1 + a_1 + (N-1)d)}{2} = \frac{N(2a_1 + (N-1)d)}{2}$$

For instance, for $2, 5, 8, 11, 14, 17, \ldots, 2 + (n-1) \times 3$:

$$S_5 = 2 + 5 + 8 + 11 + 14 = \frac{5(2 \times 2 + (5-1) \times 3)}{2} = \frac{5(4+12)}{2} = 40.$$

1.21 Geometric Sequence

A geometric sequence is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed, non-zero number called the common ratio. A geometric sequence can be represented as: $a_1, a_1 r, a_1 r^2, a_1 r^3, a_1 r^4, \ldots, a_1 r^{n-1}$ where a_1 is the first term and r is the common ratio. For instance, $2, 6, 18, 54, 162, 486, \ldots, 2 \times (3)^{n-1}$ is a geometric sequence.

The formula to calculate the sum of the first N terms of an arithmetic sequence is:

$$S_n = \sum_{n=1}^{N} a_n = \sum_{n=1}^{N} a_1 r^{n-1} = \frac{a_1\left(1 - r^N\right)}{1 - r}$$

For instance, for $2, 6, 18, 54, 162, 486, \ldots, 2 \times (3)^{n-1}$:

$$S_5 = 2 + 6 + 18 + 54 + 162 = \frac{2\left(1 - 3^5\right)}{1 - 3} = 242$$

The sum of an infinite geometric series with the first term a_1 and common ratio $|r| < 1$ is given by $\frac{a_1}{1-r}$. For instance, infinite sum of $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots$ is $\frac{\frac{1}{2}}{1-\frac{1}{3}} = \frac{\frac{1}{2}}{\frac{2}{3}} = \frac{3}{4} = 0.75$.

Example 1.30 Calculate the value of $1 + x + x^2 + x^3 + \ldots$ when $|x| < 1$.

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=1}^{\infty} 1 \times x^{n-1} = \frac{1}{1 - x} \quad |x| < 1$$

Example 1.31 Calculate the value of $1 + 2x + 3x^2 + \cdots$ when $|x| < 1$.

In the previous example we showed that: $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} |x| < 1$. Let's differentiate both sides of the equation:

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1 - x)^2} |x| < 1$$

1.22 Problem Solving with MATLAB®

This section provides examples of how MATLAB can be employed to solve the types of problems encountered in this chapter.

Example 1.32 In MATLAB, `ans` is a special variable that stores the result of the most recent calculation or expression that wasn't explicitly assigned to another variable. For instance, the code in Fig. 1.43 assigns 56 to `ans`.

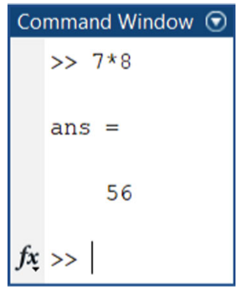


Fig. 1.43 Result is saved in `ans`

However, it's generally recommended to assign results to specific variables for better code readability and reusability. For instance, the code in Fig. 1.44 assigns the result of the calculation to the variable `A`.

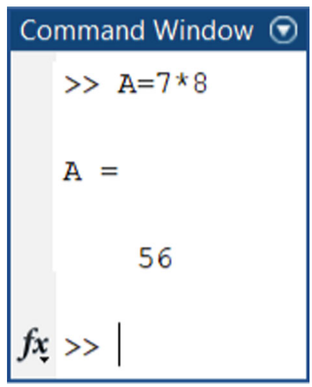


Fig. 1.44 Result is saved in A

When you place a semicolon at the end of a statement, MATLAB will execute the command but suppress the output to the command window (Fig. 1.45). This is useful when you want to perform calculations or assignments without cluttering the screen with intermediate results.

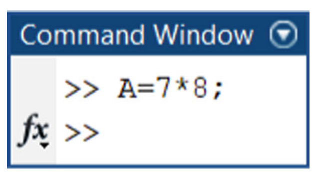


Fig. 1.45 Result is not shown

Example 1.33 The code in Fig. 1.46 draws the line passing through $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Output of this code is shown in Fig. 1.47.

Fig. 1.46 Sketching the line connecting points $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

```
Command Window
>> x=[-1 3];
>> y=[2 4];
>> plot(x,y),grid on
fx >>
```

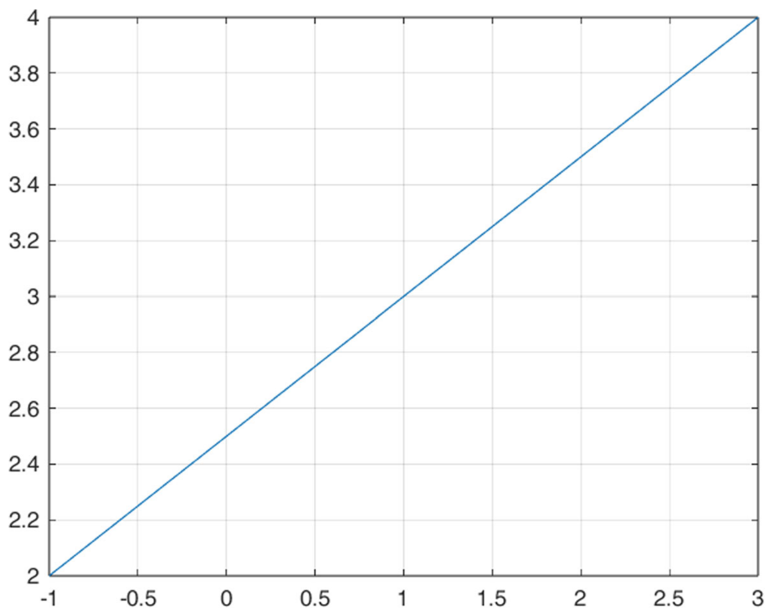


Fig. 1.47 Output of the code shown in Fig. 1.46

Example 1.34 The code in Figs. 1.48 or 1.49 determine the distance between the point $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the origin in the Cartesian plane.

Fig. 1.48 Calculating the distance between point $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the origin

```
Command Window
>> A=[1;2];
>> norm(A,2)

ans =

    2.2361

fx >>
```

Fig. 1.49 Calculation of $\sqrt{1^2 + 2^2}$

```
Command Window
>> sqrt(1^2+2^2)

ans =

    2.2361

fx >>
```

Example 1.35 The code in Figs. 1.50 and 1.51 determine the distance between the point $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$.

Fig. 1.50 Calculating the distance between the point $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$

```
Command Window
>> A=[1;2];
>> B=[-1;-3];
>> norm(A-B,2)

ans =

    5.3852

fx >>
```

Fig. 1.51 Calculation of $\sqrt{(1-(-1))^2+(2-(-3))^2}$

```
Command Window
>> sqrt((1-(-1))^2+(2-(-3))^2)

ans =

    5.3852

fx >>
```

Example 1.36 The code in Fig. 1.52 calculates $\int_0^\infty e^{-3t}dt$.

Fig. 1.52 Calculation of $\int_0^\infty e^{-3t}dt$

```
Command Window
>> syms t
>> int(exp(-3*t),t,0,inf)

ans =

1/3

fx >>
```

Example 1.37 The code in Fig. 1.53 generates a plot of the function e^{-2t} over the interval [0, 1.5]. Output of this code is shown in Fig. 1.54.

Fig. 1.53 Code of Example 1.37

```
Command Window
>> syms t
>> ezplot(exp(-2*t),[0 1.5]), grid on
>> xlabel("time (s)")
>> ylabel("voltage (V)")
fx >>
```

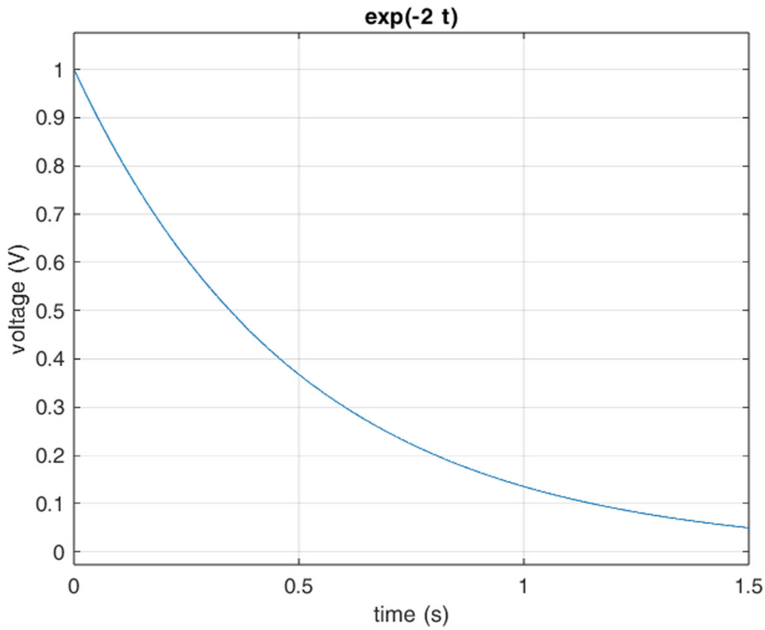
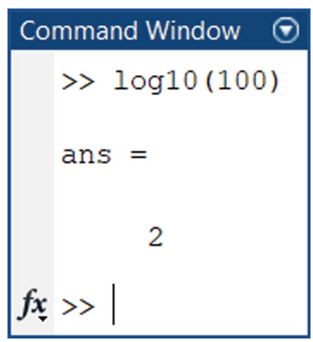


Fig. 1.54 Output of code shown in Fig. 1.53

Example 1.38 The code in Fig. 1.55 calculates log(100).

Fig. 1.55 Calculation of log(100)



Example 1.39 The code in Fig. 1.56 calculates ln(7).

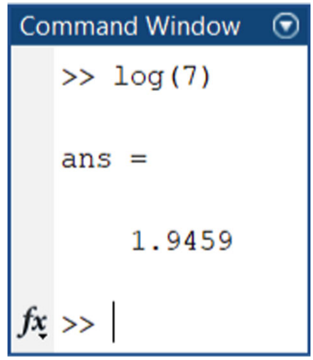


Fig. 1.56 Calculation of ln(7)

Example 1.40 The code in Fig. 1.57 calculates $\log_2 7$.

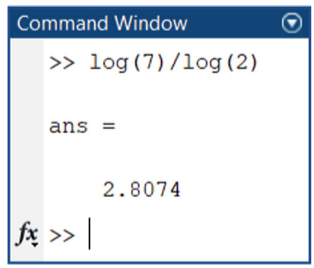


Fig. 1.57 Calculation of $\log_2 7$

Use the `format long` command to increase calculation precision (Fig. 1.58). Employ the `format short` command to reduce the number of decimal digits displayed.

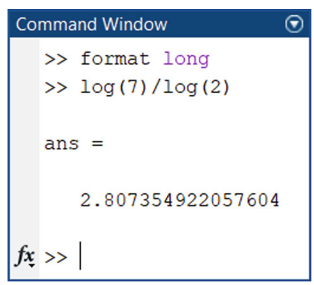


Fig. 1.58 Calculation of $\log_2 7$

Example 1.41 The code in Fig. 1.59 calculates the determinant of matrix $A = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 6 & -1 \\ 7 & 7 & 0 \end{bmatrix}$.

Fig. 1.59 Calculation of determinant of matrix $A = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 6 & -1 \\ 7 & 7 & 0 \end{bmatrix}$

```
Command Window
>> A=[1 2 7;3 6 -1;7 7 0];
>> det(A)

ans =

  -154

fx >> 
```

Example 1.42 The code in Fig. 1.60 calculates the Wronskian determinant for the functions e^t, e^{2t} and e^{3t}.

```
Command Window
>> syms t
>> y1=exp(t);
>> y2=exp(2*t);
>> y3=exp(3*t);
>> W=simplify(det([y1 y2 y3;diff(y1,t) diff(y2,t) diff(y3,t); diff(y1,t,2) diff(y2,t,2) diff(y3,t,2)]))

W =

2*exp(6*t)

fx >>
```

Fig. 1.60 Calculation of Wronskian determinant for the functions e^t, e^{2t} and e^{3t}

Example 1.43 The code in Fig. 1.61 converts $\frac{\pi}{6}$ radians to degrees.

Fig. 1.61 Conversion of $\frac{\pi}{6}$ radians to degrees

```
Command Window
>> rad2deg(pi/6)

ans =

   30.0000

fx >> 
```

Example 1.44 The code in Fig. 1.62 converts 60° degrees to radians.

Fig. 1.62 Conversion of 60° degrees to radians

```
Command Window
>> deg2rad(60)

ans =

    1.0472

fx >>
```

Example 1.45 The code in Figs. 1.63 and 1.64 rotates the point $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by 63.435° degrees clockwise.

Fig. 1.63 Rotation of point $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by 63.435° degrees clockwise

```
Command Window
>> A=[1;2];
>> theta=-63.435;
>> B=[cosd(theta) -sind(theta);sind(theta) cosd(theta)]*A

B =

    2.2361
   -0.0000

fx >>
```

Fig. 1.64 Rotation of point $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by 63.435° degrees clockwise

```
Command Window
>> A=[1;2];
>> theta=deg2rad(-63.435);
>> B=[cos(theta) -sin(theta);sin(theta) cos(theta)]*A

B =

    2.2361
   -0.0000

fx >>
```

Example 1.46 The code in Fig. 1.65 calculates the dot product of $\vec{A} = \hat{i} + 2\hat{j}$ and $\vec{B} = -7\hat{i} - 6\hat{j}$.

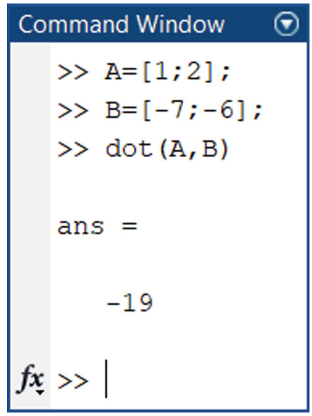

```
Command Window
>> A=[1;2];
>> B=[-7;-6];
>> dot(A,B)

ans =

   -19

fx >>
```

Fig. 1.65 Calculation of the dot product of the $\vec{A} = \hat{\imath} + 2\hat{\jmath}$ and $\vec{B} = -7\hat{\imath} - 6\hat{\jmath}$

Let's check the result shown in Fig. 1.65. $\vec{A} \cdot \vec{B} = \left(\hat{\imath} + 2\hat{\jmath}\right)\left(-7\hat{\imath} - 6\hat{\jmath}\right) = 1 \times -7 + 2 \times -6 = -7 - 12 = -19$.

Example 1.47 The code in Fig. 1.66 calculates the cross product of $\vec{A} = 1\hat{\imath} + 2\hat{\jmath} + 7\hat{k}$ and $\vec{B} = -7\hat{\imath} + 2\hat{\jmath} - 6\hat{k}$. According to Fig. 1.66, $\vec{A} \times \vec{B} = -26\hat{\imath} - 43\hat{\jmath} + 16\hat{k}$. Note that `cross` command calculates the cross product of two 3-dimensional vectors. It cannot be used with vectors of higher dimensions.

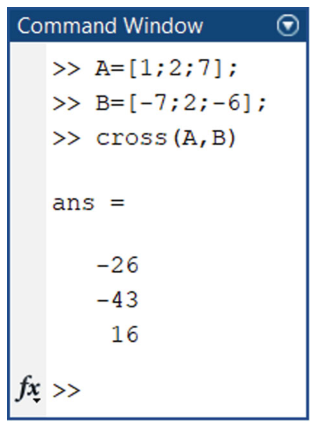

```
Command Window
>> A=[1;2;7];
>> B=[-7;2;-6];
>> cross(A,B)

ans =

   -26
   -43
    16

fx >>
```

Fig. 1.66 Calculation of the cross product of $\vec{A} = 1\hat{\imath} + 2\hat{\jmath} + 7\hat{k}$ and $\vec{B} = -7\hat{\imath} + 2\hat{\jmath} - 6\hat{k}$

Let's check the result shown in Fig. 1.66. $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 7 \\ -7 & 2 & -6 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 2 & 7 \\ 2 & -6 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 7 \\ -7 & -6 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ -7 & 2 \end{vmatrix} = -26\hat{\imath} - 43\hat{\jmath} + 16\hat{k}$.

Example 1.48 Figures. 1.67, 1.68 and 1.69 show different MATLAB codes for calculating the value of the function $f(x, y) = (x + y - 3)^2 + (2x + 3y - 7)^2 + (x + y - 4)^2$ at the point $(1, 7)$.

```
Command Window
>> f= @(x,y)(x+y-3)^2+(2*x+3*y-7)^2+(x+y-4)^2;
>> f(1,7)

ans =

   297

fx >>
```

Fig. 1.67 Calculation of $f(x, y) = (x + y - 3)^2 + (2x + 3y - 7)^2 + (x + y - 4)^2$ at $(1, 7)$

```
Command Window
>> syms x y
>> f=(x+y-3)^2+(2*x+3*y-7)^2+(x+y-4)^2;
>> subs(subs(f,y,7),x,1)

ans =

297

fx >>
```

Fig. 1.68 Calculation of $f(x, y) = (x + y - 3)^2 + (2x + 3y - 7)^2 + (x + y - 4)^2$ at $(1, 7)$

```
Command Window
>> syms x y
>> f=(x+y-3)^2+(2*x+3*y-7)^2+(x+y-4)^2;
>> subs(subs(f,x,1),y,7)

ans =

297

fx >>
```

Fig. 1.69 Calculation of $f(x, y) = (x + y - 3)^2 + (2x + 3y - 7)^2 + (x + y - 4)^2$ at $(1, 7)$

Example 1.49 The code in Fig. 1.70 calculates the critical points of the function $f(x) = x^2 + 3x + 5$ by finding the zeros of its derivative.

```
Command Window
>> syms x
>> y=x^2+3*x+5;
>> dydx=diff(y,x);
>> x_extremum=solve(dydx==0,x)

x_extremum =

-3/2

fx >>
```

Fig. 1.70 Critical point of $f(x) = x^2 + 3x + 5$

Example 1.50 The code in Fig. 1.71 calculates the critical points of the function $f(x, y) = (x + y - 3)^2 + (2x + 3y - 7)^2 + (x + y - 4)^2$ by finding the zeros of its partial derivatives.

```
Command Window
>> syms x y
>> f=(x+y-3)^2+(2*x+3*y-7)^2+(x+y-4)^2;
>> df_dx=diff(f,x);
>> df_dy=diff(f,y);
>> sol=solve([df_dx==0,df_dy==0],[x, y]);
>> disp(sol)
    x: 7/2
    y: 0

fx >>
```

Fig. 1.71 Critical point of $f(x, y) = (x + y - 3)^2 + (2x + 3y - 7)^2 + (x + y - 4)^2$

Example 1.51 The code in Fig. 1.72 demonstrates the computation of the global minimum of the function $f(x, y) = (x + y - 3)^2 + (2x + 3y - 7)^2 + (x + y - 4)^2$ and its associated coordinates.

```
Command Window
>> fun=@(x)  (x(1)+x(2)-3)^2+(2*x(1)+3*x(2)-7)^2+(x(1)+x(2)-4)^2;
>> x0=[0;0];
>> [x,fval]=fminsearch(fun,x0)

x =

    3.5000
   -0.0000


fval =

    0.5000

fx >>
```

Fig. 1.72 Global minimum of the function $f(x, y) = (x+y-3)^2 + (2x+3y-7)^2 + (x+y-4)^2$

Example 1.52 The code in Fig. 1.73 demonstrates the computation of the global minimum of the function $f(x, y) = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}$ and its associated coordinates.

```
Command Window
>> fun=@(x) sqrt((x(1)-1)^2+x(2)^2+(6-x(1)-2*x(2))^2);
>> x0=[0;0];
>> [x,fval]=fminsearch(fun,x0)

x =

    1.8333
    1.6666


fval =

    2.0412

fx >>
```

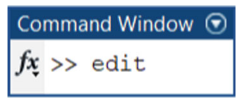
Fig. 1.73 Computation of the global minimum and its associated coordinates

Example 1.53 The following code finds the minimum value and its location for $f(x) = \frac{100x}{(2.5+x)^2}$ on $[0, 25]$ interval.

```
% Define the function
f = @(x) 100*x/(2.5+x)^2;

% Set the search interval
lowerBnd=0;
upperBnd=25;

% Find the maximum using fminbnd
[x_max, f_max] = fminbnd(@(x) -f(x), lowerBnd, upperBnd);

% The maximum value is the negative of the minimum
max_value = -f_max;

disp(['Maximum value of given function is:', num2str(max_value)]);
disp(['x value at maximum is:', num2str(x_max)]);
```

Use the `edit` command (Fig. 1.74) to open the Editor (Fig. 1.75).

Fig. 1.74 `edit` command



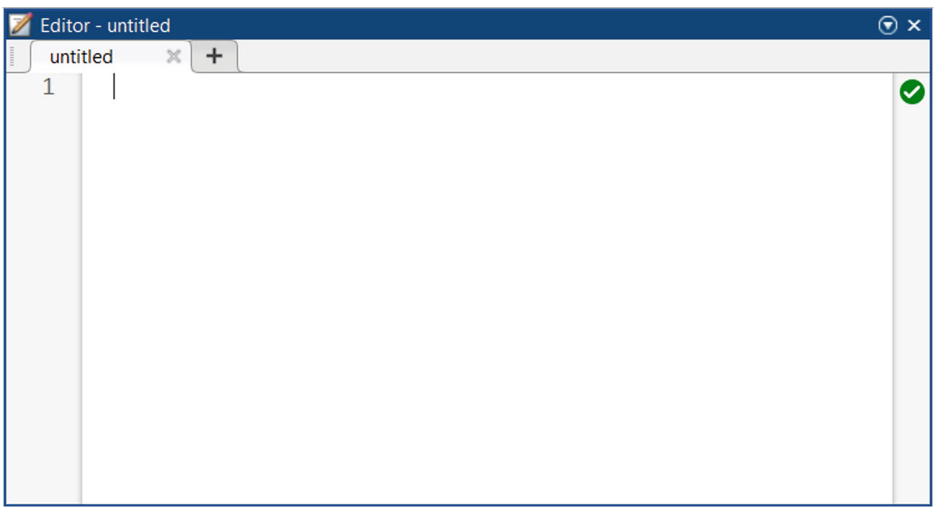


Fig. 1.75 Editor window

Type the given code (Fig. 1.76) and press the Ctrl + S to save it.

```
% Define the function
f = @(x) 100*x/(2.5+x)^2;

% Set the search interval
lowerBnd=0;
upperBnd=25;

% Find the maximum using fminbnd
[x_max, f_max] = fminbnd(@(x) -f(x), lowerBnd, upperBnd);

% The maximum value is the negative of the minimum
max_value = -f_max;

disp(['Maximum value of given function is:', num2str(max_value)]);
disp(['x value at maximum is:', num2str(x_max)]);
```

Fig. 1.76 Code of Example 1.53

Press the F5 key of your keyboard to execute the code or click the Run button (Fig. 1.77). Output of this code is shown in Fig. 1.78.

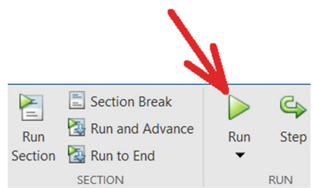


Fig. 1.77 The run button

```
Maximum value of given function is:10
x value at maximum is:2.5
>>
```

Fig. 1.78 Output of the code

Example 1.54 The following code maximize $f(x, y) = x.y$ subject to $\begin{cases} 2x + 3y = 7 \\ -4x + 3y \leq -1 \end{cases}$.

Note that the maximization of $f(x)$ is equivalent to the minimization of its negative counterpart, $-f(x)$.

```
% Objective function
fun = @(x) -x(1)*x(2); % Negative to find the maximum

% Linear inequality constraint
A = [-4 3];
b = -1;

% Linear equality constraint
Aeq = [2 3];
beq = 7;

% Initial guess
x0 = [1 1];

% Find the maximum
[x,fval] = fmincon(fun,x0,A,b,Aeq,beq);

% The maximum value is the negative of the minimum
max_value = -fval;

disp(['Maximum value:', num2str(max_value)]);
disp(['x and y values:', num2str(x(1)), ', ', num2str(x(2))]);
```

Output of this code is shown in Fig. 1.79.

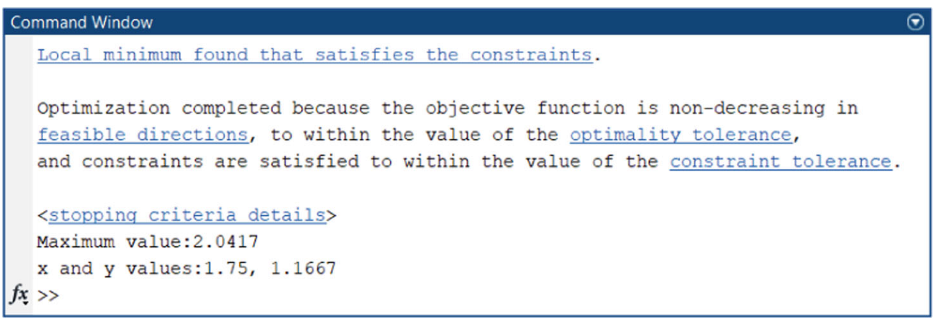


Fig. 1.79 Output of the code

Example 1.55 The following code minimize $f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)^2$ subject to $x^2 + y^2 + z^2 = 4$.

```
% Define the objective function
fun = @(x) (x(1)-3)^2 + (x(2)-1)^2 + (x(3)+1)^2;

% Initial guess
x0 = [1, 1, 1];

% Define the nonlinear constraint
function [c,ceq] = nonlcon(x)
    ceq = x(1)^2 + x(2)^2 + x(3)^2 - 4;
    c = []; % No nonlinear inequality constraints
end

% Solve the optimization problem
[x,min_value] = fmincon(fun,x0,[],[],[],[],[],[],@nonlcon,options)
```

Output of this code is shown in Fig. 1.80.

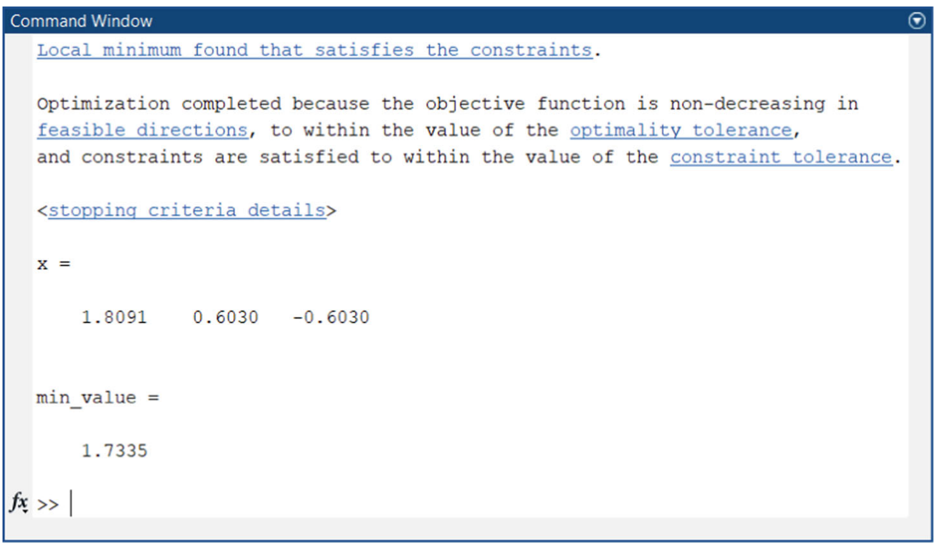


Fig. 1.80 Output of the code

Example 1.56 The following code calculates $\sum_{n=1}^{10} n^2$.

```
n = 10;
sum = 0;
for i = 1:n
    sum = sum + i^2;
end
disp(sum)
```

Enter the code to the Editor (Fig. 1.81) and execute it. Output of this code is 385 (Fig. 1.82).

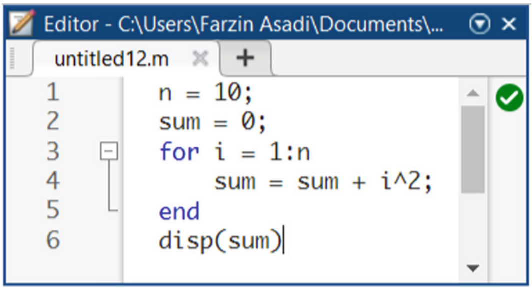


Fig. 1.81 Calculation of $\sum_{n=1}^{10} n^2$

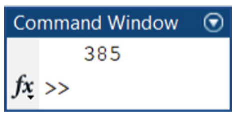


Fig. 1.82 Output of the code in Fig. 1.81

Example 1.57 The code in Fig. 1.83 calculates $\sum_{n=1}^{10} \frac{1}{n^2}$.

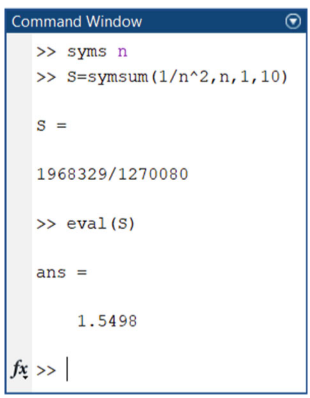


Fig. 1.83 Calculation of $\sum_{n=1}^{10} \frac{1}{n^2}$

Example 1.58 The code in Fig. 1.84 calculates $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

```
Command Window
>> syms n
>> S=symsum(1/n^2,n,1,Inf)

S =

pi^2/6

fx >>
```

Fig. 1.84 Calculation of $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$

Example 1.59 The code in Fig. 1.85 generates a plot of the function $e^{-t}\sin(2t)$ over the interval [0, 7]. Output of this code is shown in Fig. 1.86.

```
Command Window
>> syms t
>> ezplot(exp(-t)*sin(2*t),[0,7])
>> grid on
fx >>
```

Fig. 1.85 Code for generating the plot of $e^{-t}\sin(2t)$ over [0, 7] interval

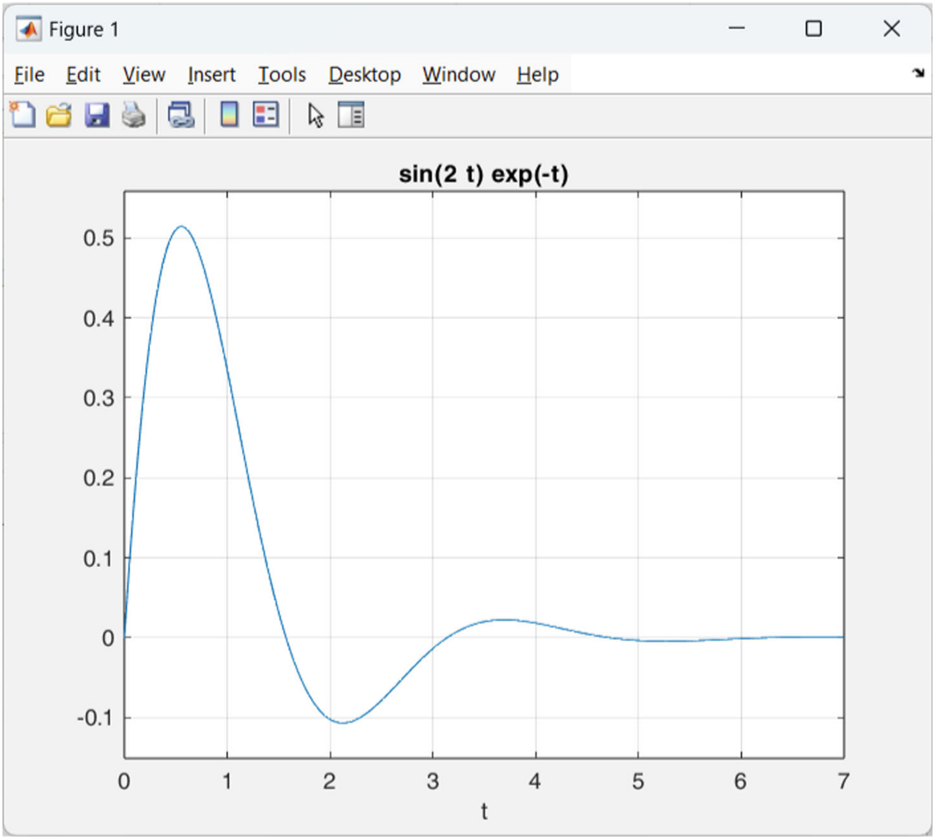


Fig. 1.86 Output of the code shown in Fig. 1.85

Use “File > Save As…” (Fig. 1.87) to export the plot as an image. Utilize the “Save as Type” dropdown menu (Fig. 1.88) to select the appropriate file format.

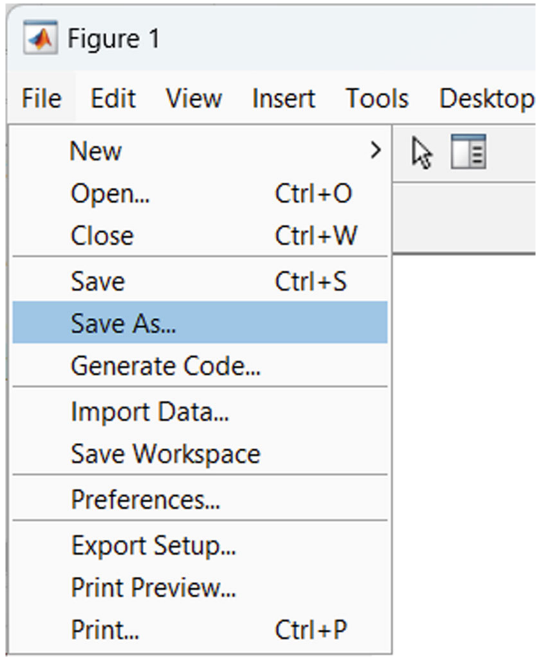


Fig. 1.87 File > Save As…

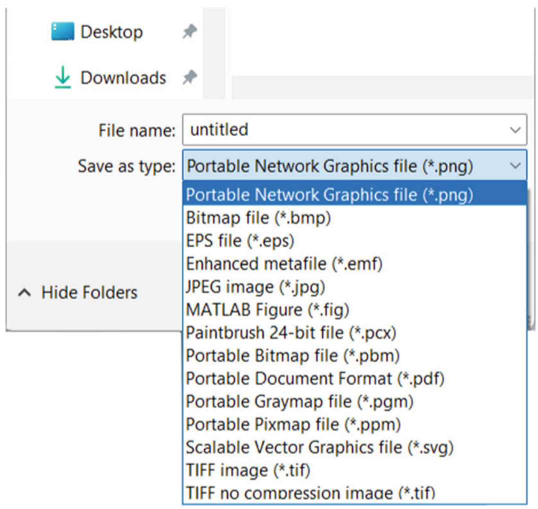


Fig. 1.88 Supported graphic formats

Use the magnifying glass icon (Fig. 1.89) to zoom in or out.

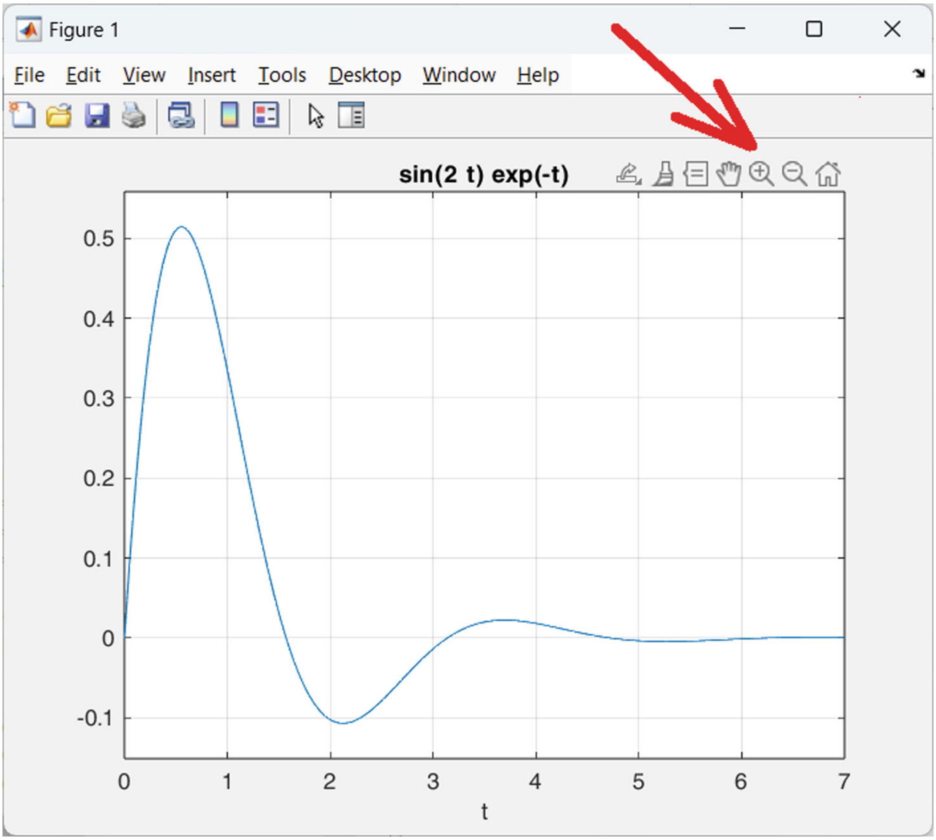
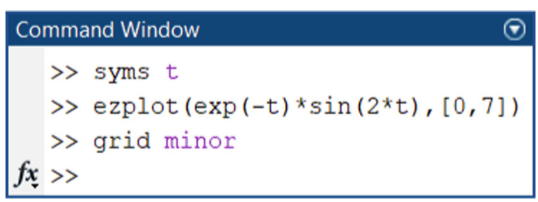


Fig. 1.89 Zoom in or out icons

Employ the `grid on` or `grid minor` command (Fig. 1.90) to overlay a grid on the plot (Fig. 1.91).

Fig. 1.90 Addition of grid to the plot



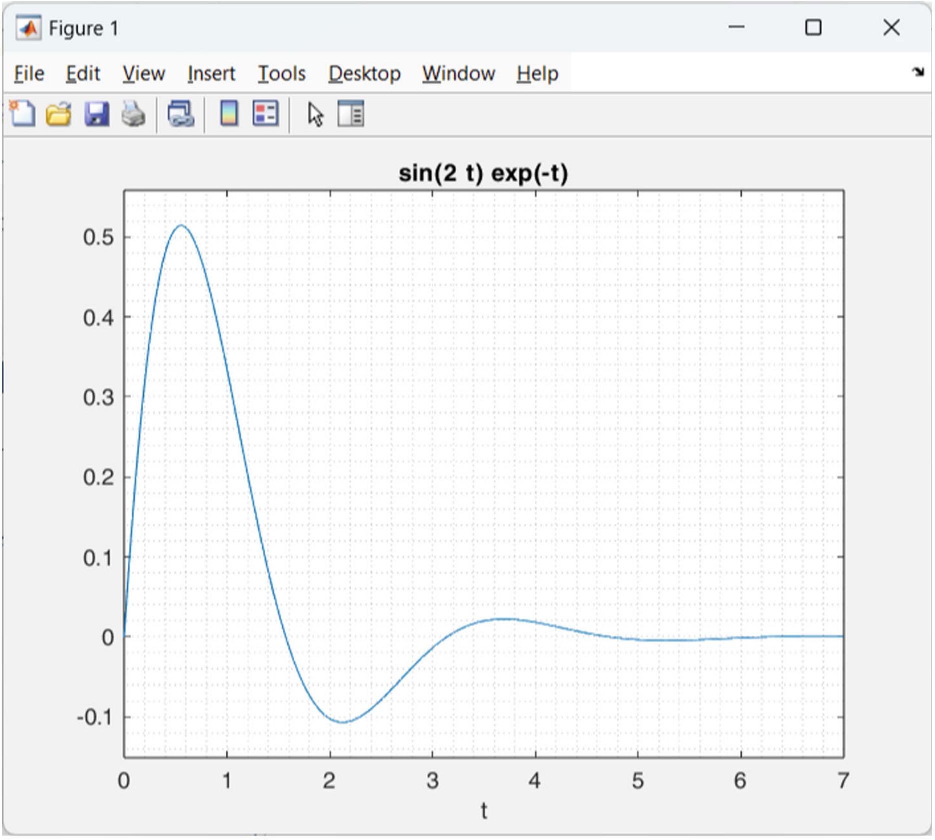


Fig. 1.91 Output of the code shown in Fig. 1.90

Utilize a mouse click on the graph to obtain the coordinates of a specific point. For example, Fig. 1.92 shows a maximum value of 0.514079 at $x \approx 0.546486$.

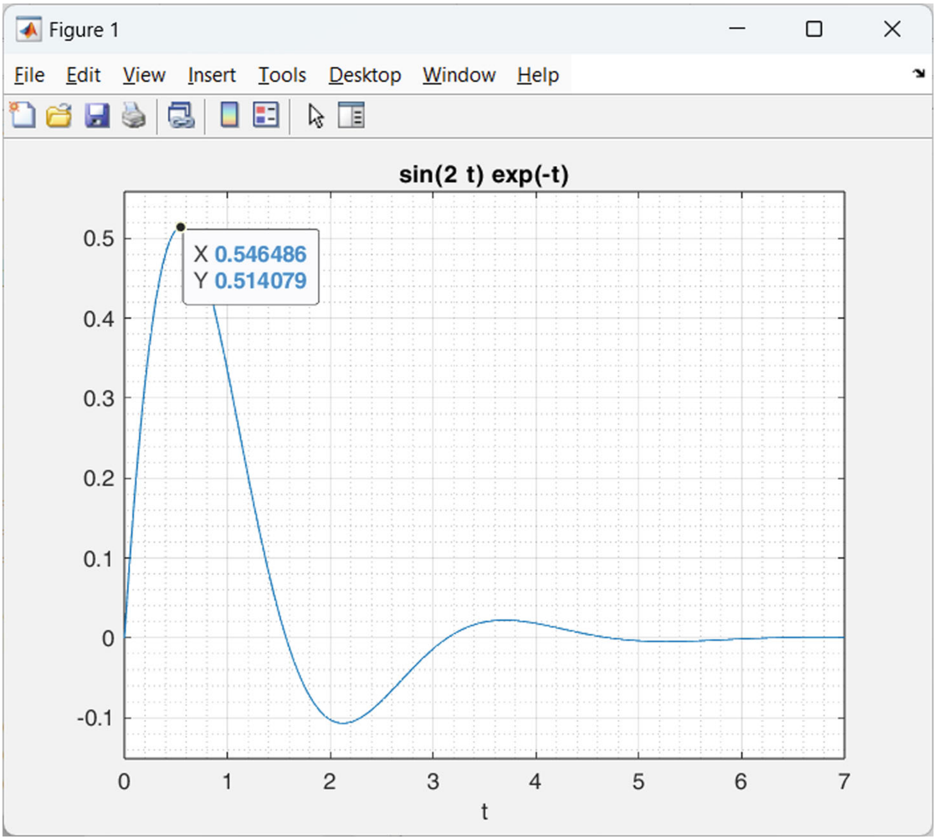
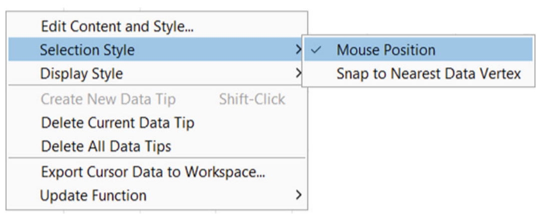


Fig. 1.92 Obtaining the coordinates of a point

Should you encounter difficulties in precisely positioning the cursor on the graph, right-click and select "Selection Style > Mouse Position" from the context menu (Fig. 1.93).

Fig. 1.93 Selection Style > Mouse Position



Example 1.60 The code in Fig. 1.94 plots $\sin(x)$ on $[0, 4\pi]$ with axis labels and a title. Output of this code is shown in Fig. 1.95.

Fig. 1.94 Plotting $\sin(x)$ on $[0, 4\pi]$

```
Command Window
>> syms t
>> ezplot(sin(t),[0,4*pi])
>> grid on
>> xlabel("time (s)")
>> ylabel("voltage (V)")
>> title("Capacitor voltage")
fx >>
```

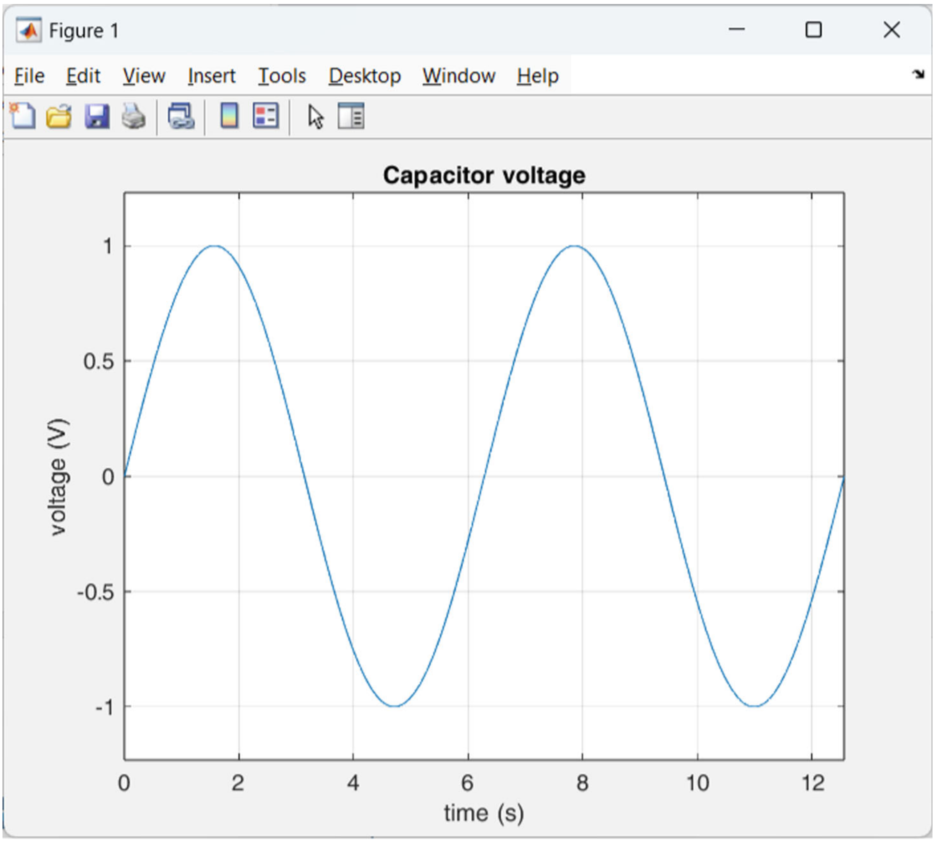


Fig. 1.95 Output of the code shown in Fig. 1.94

Example 1.61 The code in Fig. 1.96 plots the graph of $\begin{cases} x(t) = 3\cos(t) \\ y(t) = 3\sin(t) \end{cases}$ for $0 \le t \le 2\pi$. Output of this code is shown in Fig. 1.97.

Fig. 1.96 Plotting the graph of $\begin{cases} x(t) = 3\cos(t) \\ y(t) = 3\sin(t) \end{cases}$ for $0 \le t \le 2\pi$

```
Command Window
>> syms t
>> ezplot(3*cos(t),3*sin(t),[0,pi])
fx >>
```

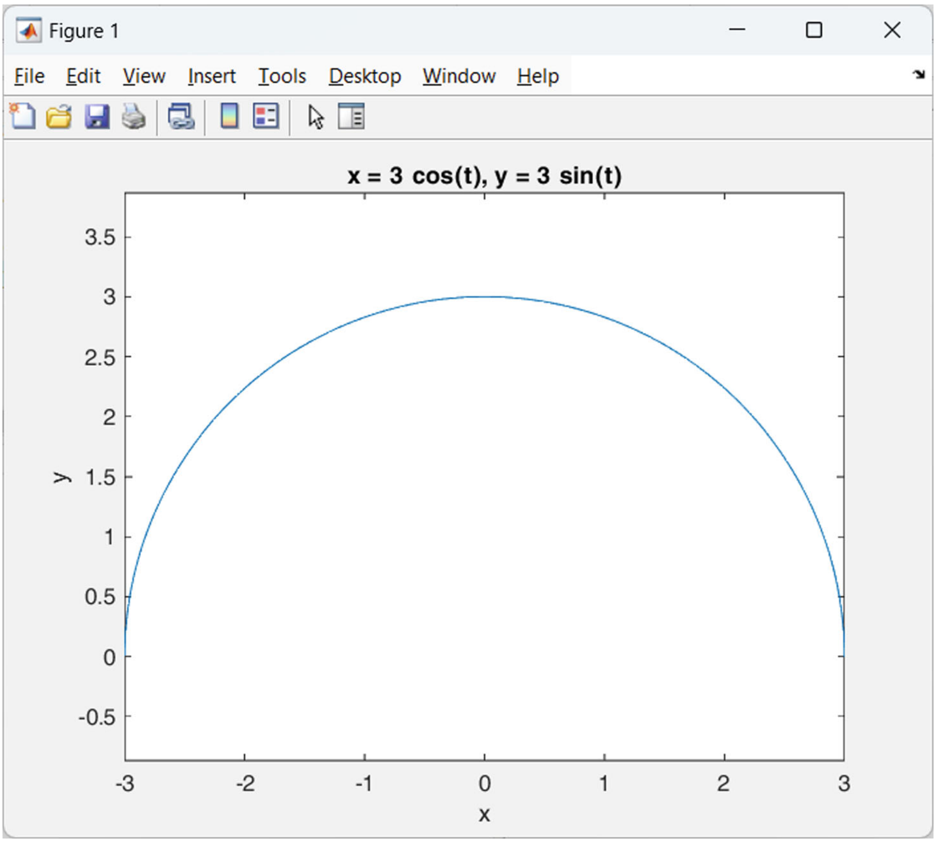


Fig. 1.97 Output of the code shown in Fig. 1.96

Example 1.62 The code in Fig. 1.98 generates a plot of the functions $e^{-t}\sin(2t)$ and $e^{-1.5t}\sin(3t)$ on the same coordinate system. Output of this code is shown in Fig. 1.99.

Fig. 1.98 Plotting $e^{-t}\sin(2t)$ and $e^{-1.5t}\sin(3t)$ on the same coordinate system

```
Command Window
>> syms t
>> ezplot(exp(-t)*sin(2*t),[0,7,-.5,.6])
>> hold on
>> ezplot(exp(-1.5*t)*sin(3*t),[0,7,-.5,.6])
>> grid on
fx >>
```

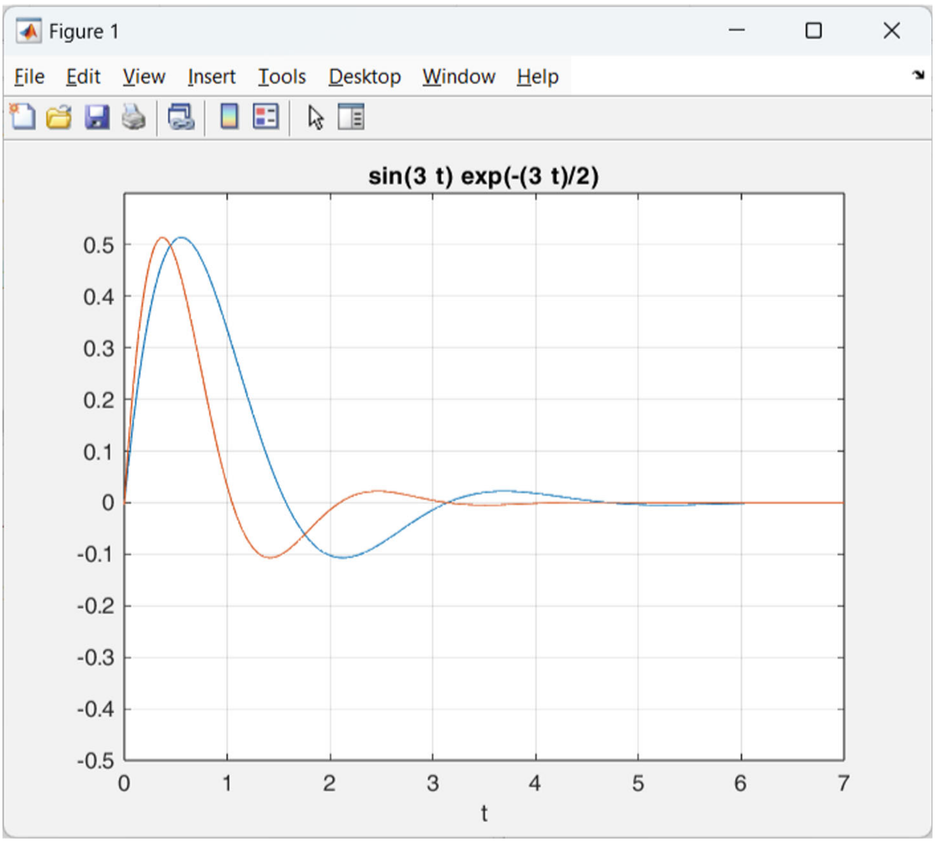


Fig. 1.99 Output of the code shown in Fig. 1.98

Example 1.63 In this example, we will visualize the data provided in Table 1.2 by generating a corresponding plot.

Table 1.2 Data points for Example 1.63

x	y
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100

The code in Fig. 1.100 generates the graph of Table 1.2. Output of this code is shown in Fig. 1.101.

```
Command Window
>> x=[1 2 3 4 5 6 7 8 9 10];
>> y=[1 4 9 16 25 36 49 64 81 100];
>> plot(x,y), grid on
>> xlabel("time (s)")
>> ylabel("voltage (V)")
>> title("Voltage vs. time")
fx >>
```

Fig. 1.100 Plotting the Table 1.2 data

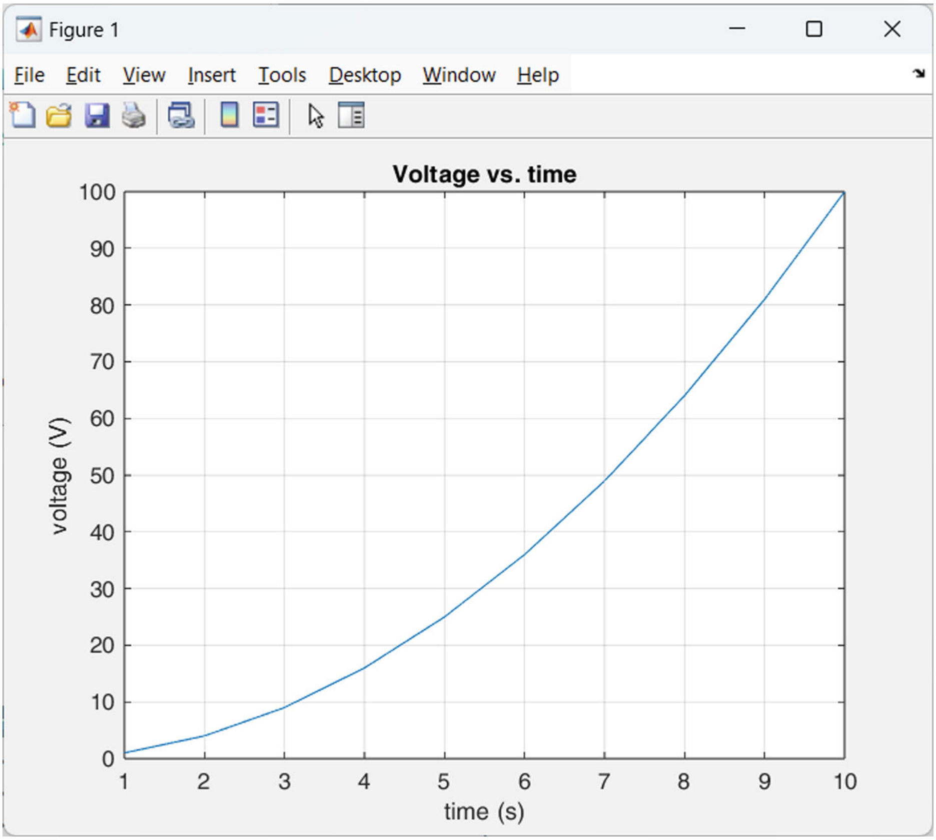


Fig. 1.101 Output of the code shown in Fig. 1.100

Example 1.64 The code in Fig. 1.102 generates a plot of the functions $\sin(x)$, $\sin\left(x - \frac{2\pi}{3}\right)$, and $\sin\left(x + \frac{2\pi}{3}\right)$ on the same set of axes. Output of this code is shown in Fig. 1.103.

```
Command Window
>> x=[0:2*pi/100:2*pi];
>> y1=sin(x);
>> y2=sin(x-2*pi/3);
>> y3=sin(x+2*pi/3);
>> plot(x,y1,x,y2,x,y3);
>> grid on
>> legend("y_1=sin(x)","y_2=sin(x-2\pi/3)","y_3=sin(x+2\pi/3)")
fx
```

Fig. 1.102 Code for Example 1.64

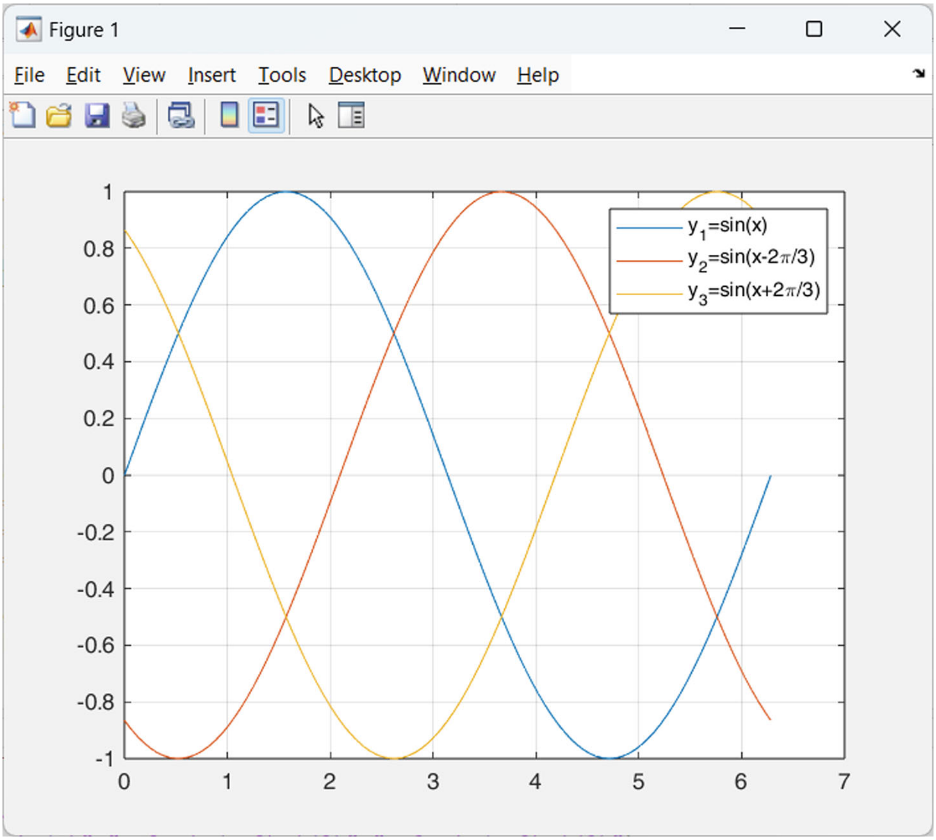


Fig. 1.103 Output of the code shown in Fig. 1.102

Utilize the mouse pointer to manually adjust the position of the legend box (Fig. 1.104).

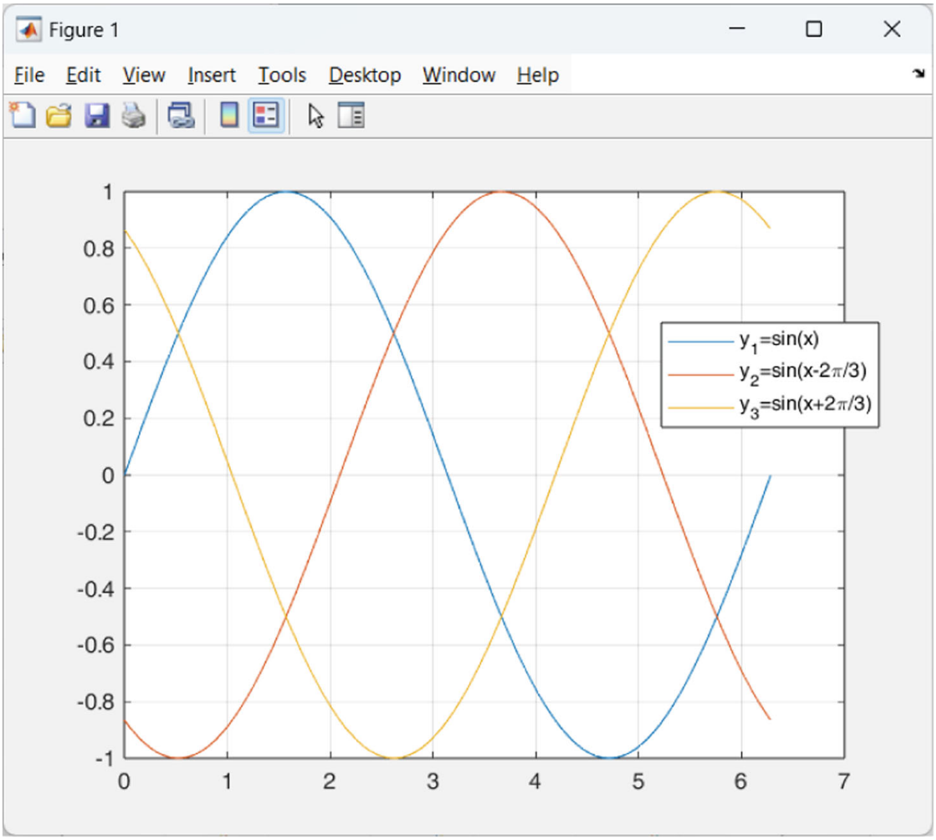


Fig. 1.104 Adjusting the position of the legend box

Example 1.65 The code in Fig. 1.105 generates a 3D plot of the function $z = x^2 - y^2$. Output of this code is shown in Fig. 1.106. Employ the mouse to manipulate the 3D plot's orientation.

Fig. 1.105 Plotting the $z = x^2 - y^2$ surface

```
Command Window
>> [x,y] = meshgrid(-2:0.2:2);
>> z=x.^2-y.^2;
>> surf(x,y,z)
>> xlabel('x'); ylabel('y'); zlabel('z');
fx >>
```

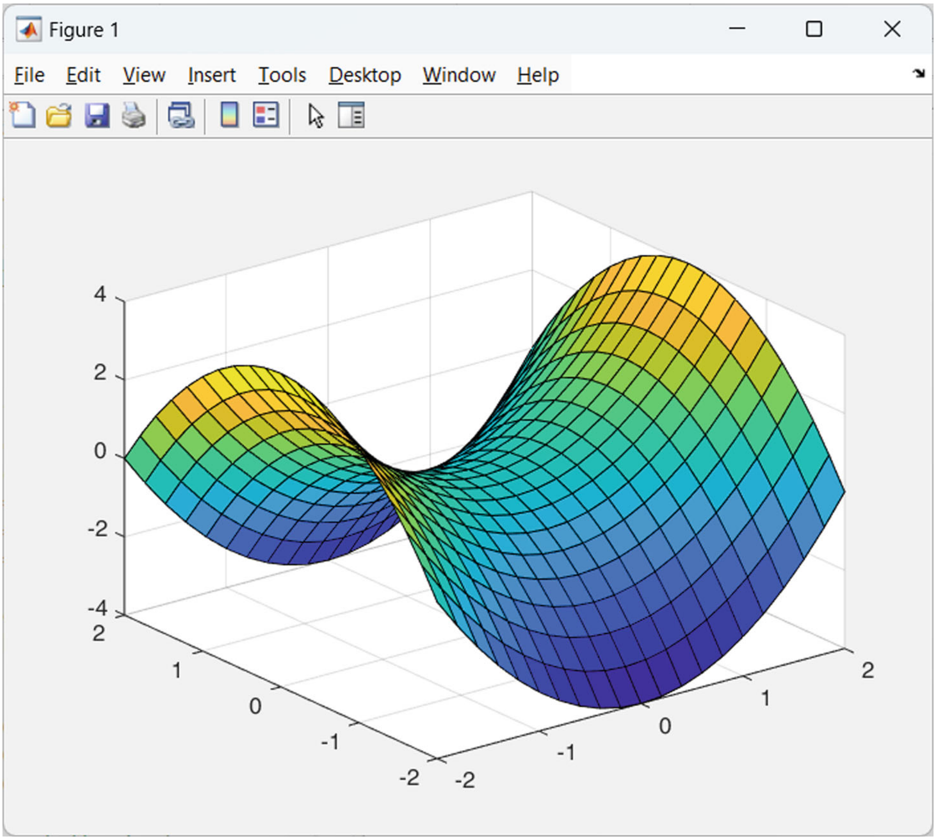
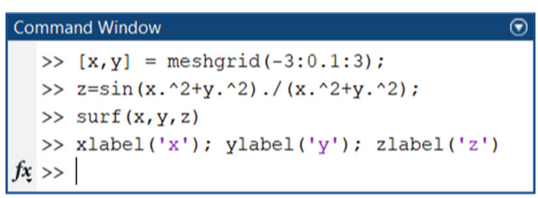


Fig. 1.106 Output of the code shown in Fig. 1.105

Example 1.66 The code in Fig. 1.107 generates a 3D plot of the function $z = \frac{\sin(x^2+y^2)}{x^2+y^2}$. Output of this code is shown in Fig. 1.108. Employ the mouse to manipulate the 3D plot's orientation.

Fig. 1.107 Plotting the $z = \frac{\sin(x^2+y^2)}{x^2+y^2}$ surface



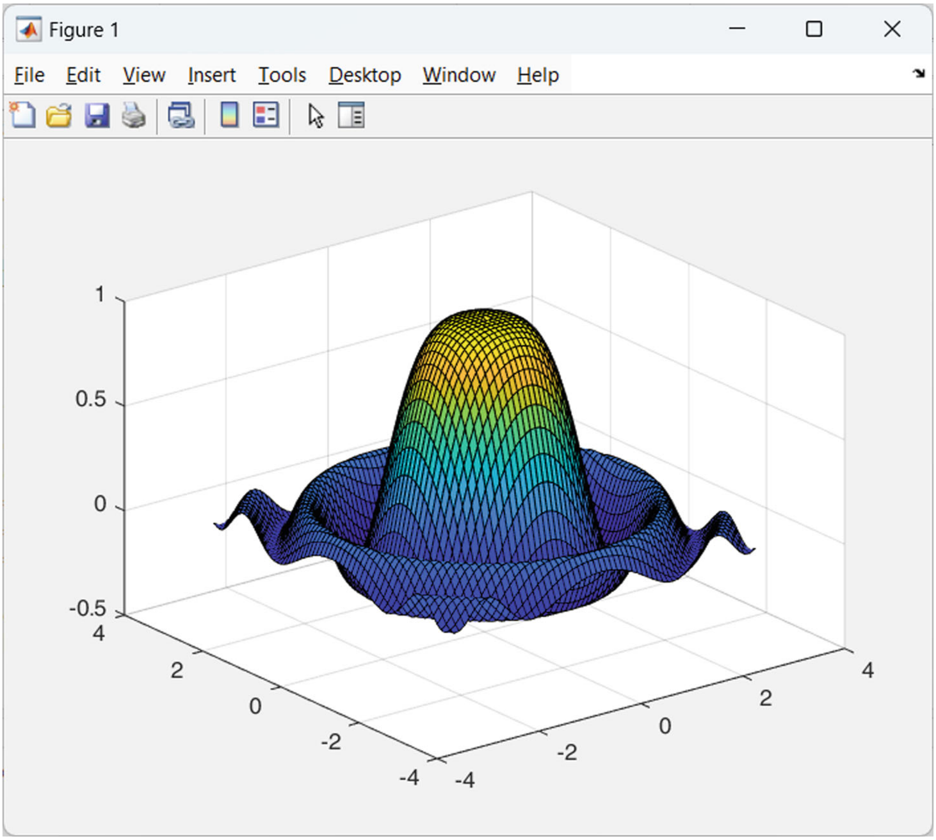


Fig. 1.108 Output of the code shown in Fig. 1.107

References for Further Study

1. Advanced Engineering Mathematics (10^{th} edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8^{th} edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

2 Fundamentals of Trigonometry

2.1 Introduction

Trigonometry is a branch of mathematics that deals with the study of triangles, particularly right-angled triangles. It explores the relationships between the sides and angles of these triangles.

The first part of this chapter covers important trigonometric formulas. The second part demonstrates MATLAB's application to the problems discussed.

2.2 Degree, Radian and Grad

Degree, radian and grad are units of measurement for angles:

Degree: A full circle is divided into 360 degrees.

Radian: A radian is the angle subtended at the center of a circle by an arc equal in length to the radius of the circle. There are 2π radians in a full circle.

Grad: A full circle is divided into 400 grads.

Most of the scientific calculations are done with radian.

Example 2.1 Convert 75° to radians.

1 radian equals to $\frac{180°}{\pi} \cong 57.30°$. Therefore, $75° = \frac{75}{57.30}\,\text{rad} = 1.309\,\text{rad}$.

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_2

Example 2.2 Convert 2.5 radians to degree.

$$2.5 \times 57.30^\circ = 143.25^\circ.$$

Example 2.3 A shaft is rotating at a speed of 1490 revolutions per minute. Convert this rotational speed to radians per second.

$$\begin{aligned} 1490\ \text{rpm} &= 1490 \times 2\pi\ \frac{\text{rad}}{\text{min}} = 1490 \times 2\pi\ \frac{\text{rad}}{60\,\text{s}} \\ &= \frac{1490 \times 2\pi}{60}\ \frac{\text{rad}}{\text{s}} = 156.0324\ \frac{\text{rad}}{\text{s}} \end{aligned}$$

2.3 Trigonometric Functions

Trigonometric functions for the right triangle ABC (Fig. 2.1) are defined as shown in Table 2.1.

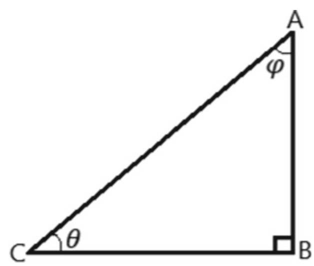


Fig. 2.1 A right triangle

It is recommended to memorize the sine and cosine values for the angles shown in Table 2.2.

Table 2.1 Definition of different trigonometric functions for the right triangle ABC (Fig. 2.1)

For angle $\boldsymbol{\theta}$	For angle $\boldsymbol{\varphi}$
$\sin(\theta) = \frac{AB}{AC}$	$\sin(\varphi) = \frac{BC}{AC}$
$\cos(\theta) = \frac{BC}{AC}$	$\cos(\varphi) = \frac{AB}{AC}$
$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{AB}{BC}$	$\tan(\varphi) = \frac{\sin(\varphi)}{\cos(\varphi)} = \frac{BC}{AB}$
$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} = \frac{BC}{AB}$	$\cot(\varphi) = \frac{\cos(\varphi)}{\sin(\varphi)} = \frac{AB}{BC}$

Table 2.2 Sine and cosine values for common angles

Angle	Sine	Cosine
0°	0	1
15°	0.26	0.97
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
90°	1	0
180°	0	-1

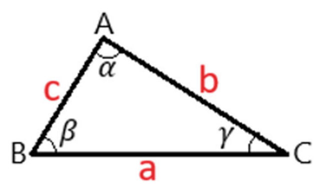
Note that $\sin(x)$ is an odd function since: $\sin(-x) = -\sin(x)$. $\cos(x)$ is an even function since $\cos(-x) = \cos(x)$

2.4 The Law of Sines and Cosines

For a triangle with sides a, b, and c, and their corresponding opposite angles α, β, and γ (Fig. 2.2), the law of sines states:

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

Fig. 2.2 A general triangle



Consider the triangle shown in Fig. 2.2. The Law of Cosines states that:

$$a^2 = b^2 + c^2 - 2 \times b \times c \times \cos(\alpha)$$
$$b^2 = a^2 + c^2 - 2 \times a \times c \times \cos(\beta)$$
$$c^2 = a^2 + b^2 - 2 \times a \times b \times \cos(\gamma)$$

2.5 Isosceles Triangle

In the isosceles triangle ABC depicted in Fig. 2.3, we have: $BC = 2 \times l \times \sin\left(\frac{\theta}{2}\right)$.

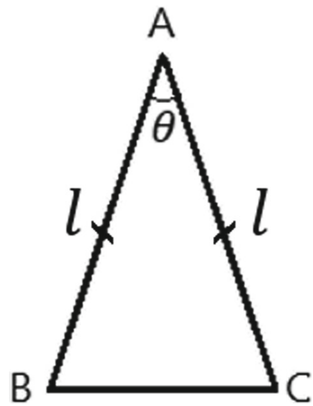


Fig. 2.3 Isosceles triangle ($AB = AC = l$)

Area of the isosceles triangle depicted in Fig. 2.3 is: $S = \frac{1}{2}l^2\sin(\theta)$.

2.6 Area of Triangle

Heron's Formula is used to calculate the area of a triangle when all three side lengths are known. Given a triangle with side lengths a, b, and c (Fig. 2.4), the area (S) can be calculated using the formula $S = \sqrt{p(p-a)(p-b)(p-c)}$. Note that p is the semi-perimeter of the triangle, i.e., $p = \frac{a+b+c}{2}$.

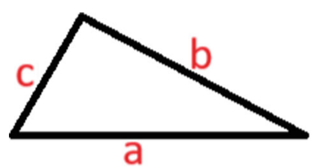


Fig. 2.4 A general triangle with a, b and c sides

Figure 2.5 provides formulas for calculating the areas of various geometric shapes.

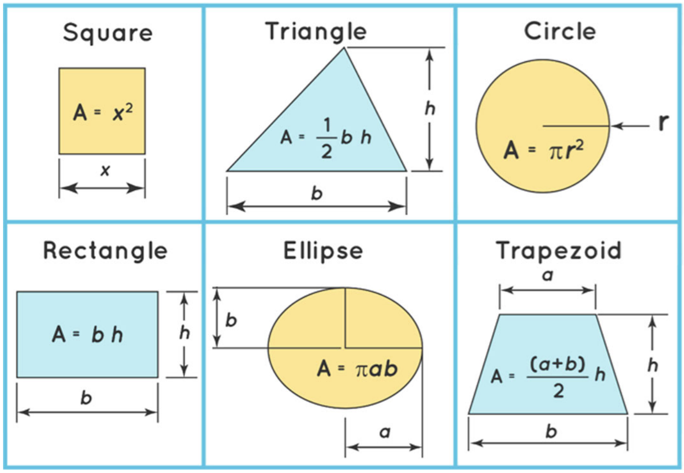


Fig. 2.5 Formulas for calculation of area for different shapes

2.7 Important Trigonometric Relationships

The following are some of the important trigonometric relationships (note that $sign(x) = \begin{cases} +1, \; x > 1 \\ 0, \quad x = 0 \\ -1, \; x < 0 \end{cases}$):

1. $(\sin(x))^2 + (\cos(x))^2 = 1 \Rightarrow \begin{cases} \sin(x) = \sqrt{1 - (\cos(x))^2} \\ \cos(x) = \sqrt{1 - (\sin(x))^2} \end{cases}$
2. $a \times \sin(\omega t) + b \times \cos(\omega t) = sign(a) \cdot \sqrt{a^2 + b^2} \sin\left(\omega t + \tan^{-1}\left(\frac{b}{a}\right)\right)$
3. $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$
4. $\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$
5. $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
6. $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
7. $\sin(\theta \pm \varphi) = \sin(\theta)\cos(\varphi) \pm \cos(\theta)\sin(\varphi)$
8. $\cos(\theta \pm \varphi) = \cos(\theta)\cos(\varphi) \mp \sin(\theta)\sin(\varphi)$
9. $\tan(\theta \pm \varphi) = \frac{\tan(\theta) \pm \tan(\varphi)}{1 \mp \tan(\theta)\tan(\varphi)}$
10. $x + y = \frac{\pi}{2} \Rightarrow \sin(x) = \cos(y)$
11. $x + y = \frac{\pi}{2} \Rightarrow \tan(x) = \text{cotan}(y)$
12. $\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$

13. $\cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$
14. $\tan\left(x + \frac{\pi}{2}\right) = -\text{cotan}(x)$
15. $\sin(x \pm \pi) = -\sin(x)$
16. $\cos(x \pm \pi) = -\cos(x)$
17. $\tan(x \pm \pi) = \tan(x)$
18. $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$
19. $\tan^{-1}(x) + \text{cotan}^{-1}(x) = \frac{\pi}{2}$

Note that $\sin^{-1}(x), \cos^{-1}(x), \tan^{-1}(x)$ and $\text{cotan}^{-1}(x)$ denote the inverse trigonometric functions arcsine, arccosine, arctangent, and arc cotangent, respectively.

Example 2.4 Express the function $5\sin(\theta) + 7\cos(\theta)$ as a single sine function.

$$a \times \sin(\omega t) + b \times \cos(\omega t) = sign(a) \cdot \sqrt{a^2 + b^2}\sin\left(\omega t + \tan^{-1}\left(\frac{b}{a}\right)\right)$$

$$5\sin(\theta) + 7\cos(\theta) = sign(5) \cdot \sqrt{5^2 + 7^2}\sin\left(\theta + \tan^{-1}\left(\frac{7}{5}\right)\right) = +8.60\sin\left(\theta + 54.46^\circ\right)$$

Example 2.5 Express the function $-5\sin(\theta) + 7\cos(\theta)$ as a single sine function.

$$a \times \sin(\omega t) + b \times \cos(\omega t) = sign(a).\sqrt{a^2 + b^2}\sin\left(\omega t + \tan^{-1}\left(\frac{b}{a}\right)\right)$$

$$\begin{aligned} -5\sin(\theta) + 7\cos(\theta) &= sign(-5)\sqrt{5^2 + 7^2}\sin\left(\theta + \tan^{-1}\left(\frac{7}{-5}\right)\right) \\ &= -8.60\sin\left(\theta - 54.46^\circ\right) = 8.60\sin\left(\theta - 54.46^\circ + 180^\circ\right) \\ &= 8.60\sin\left(\theta + 125.54^\circ\right) \end{aligned}$$

Example 2.6 Express the function $5\sin(\theta) - 7\cos(\theta)$ as a single sine function.

$$a \times \sin(\omega t) + b \times \cos(\omega t) = sign(a).\sqrt{a^2 + b^2}\sin\left(\omega t + \tan^{-1}\left(\frac{b}{a}\right)\right)$$

$$\begin{aligned} 5\sin(\theta) - 7\cos(\theta) &= sign(5)\sqrt{5^2 + 7^2}\sin\left(\theta + \tan^{-1}\left(\frac{-7}{5}\right)\right) \\ &= +8.60\sin\left(\theta - 54.46^\circ\right) \end{aligned}$$

Example 2.7 Express the function $-5\sin(\theta) - 7\cos(\theta)$ as a single sine function.

$$a \times \sin(\omega t) + b \times \cos(\omega t) = sign(a) \cdot \sqrt{a^2 + b^2}\sin\left(\omega t + \tan^{-1}\left(\frac{b}{a}\right)\right)$$

$$
\begin{aligned}
-5\sin(\theta) - 7\cos(\theta) &= sign(-5)\sqrt{5^2+7^2}\sin\left(\theta + \tan^{-1}\left(\frac{-7}{-5}\right)\right) \\
&= -8.60\sin(\theta + 54.46^\circ) = 8.60\sin(\theta + 54.46^\circ + 180^\circ) \\
&= 8.60\sin(\theta + 234.46^\circ)
\end{aligned}
$$

Example 2.8: Calculate the $\cos(150^\circ)$ and $\sin(120^\circ)$.

$$\cos(150^\circ) = \sin(-60^\circ) = -\sin(60^\circ) = -\frac{\sqrt{3}}{2}$$

$$\sin(120^\circ) = \cos(-30^\circ) = \cos(30^\circ) = \frac{\sqrt{3}}{2}$$

2.8 Period of Trigonometric Functions

$\sin(kx + \varphi_0)$ or $\cos(kx + \varphi_0)$ has period of $T = \frac{2\pi}{k}$. $\tan(kx + \varphi_0)$ or $\cot(kx + \varphi_0)$ has period of $T = \frac{\pi}{k}$.

Example 2.9 Calculate the period of $\sin\left(3x + \frac{\pi}{6}\right)$ and $\tan\left(6x + \frac{\pi}{3}\right)$.

$$
\begin{aligned}
\sin\left(3x + \frac{\pi}{6}\right) &\Rightarrow T = \frac{2\pi}{3} \\
\tan\left(6x + \frac{\pi}{3}\right) &\Rightarrow T = \frac{\pi}{6}
\end{aligned}
$$

Example 2.10 Calculate the period of $\sin\left(2x + \frac{\pi}{3}\right) + \cos\left(6x + \frac{\pi}{4}\right)$.

$$
\begin{aligned}
\sin\left(2x + \frac{\pi}{3}\right) &\Rightarrow T_1 = \frac{2\pi}{2} = \pi = 1\pi \\
\cos\left(6x + \frac{\pi}{4}\right) &\Rightarrow T_2 = \frac{2\pi}{6} = \frac{\pi}{3} = \frac{1}{3}\pi
\end{aligned}
$$

We need to find the least common (LCM) multiple of T_1 and T_2. LCM of T_1 and T_2 is π. Graph of $\sin\left(2x + \frac{\pi}{3}\right) + \cos\left(6x + \frac{\pi}{4}\right)$ is shown in Fig. 2.6. You can use Fig. 2.6 to verify that period is π.

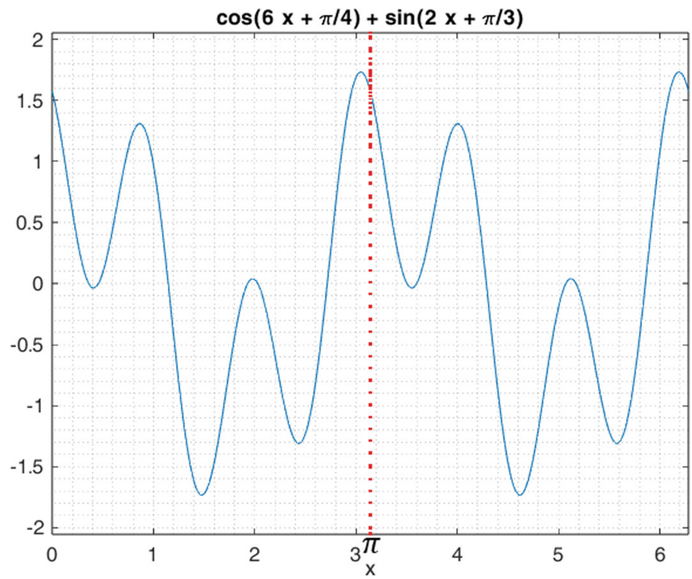


Fig. 2.6 Plot of $\sin\left(2x + \frac{\pi}{3}\right) + \cos\left(6x + \frac{\pi}{4}\right)$ over $[0, 2\pi]$ interval

2.9 Hyperbolic Functions

Hyperbolic functions are defined as:

1. $\sinh(x) = \frac{e^x - e^{-x}}{2}$
2. $\cosh(x) = \frac{e^x + e^{-x}}{2}$
3. $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4. $\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
5. $\mathrm{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$
6. $\mathrm{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$

2.10 Phase Difference

Phase difference is the difference in the phase angle of two sinusoidal waves. It's essentially the difference in the starting point of two waves with the same frequency.

Example 2.11 Determine the phase difference between $y_1(t) = 5\sin\left(314t + \frac{\pi}{3}\right)$ and $y_2(t) = 3\sin\left(314t + \frac{\pi}{2}\right)$.

$$\Delta\varphi = \varphi_{y_1} - \varphi_{y_2} = 314t + \frac{\pi}{3} - \left(314t + \frac{\pi}{2}\right) = \frac{\pi}{3} - \frac{\pi}{2} = -\frac{\pi}{6}$$

$y_1(t)$ lags $y_2(t)$ by $\frac{\pi}{6}$ radians. In other words, $y_2(t)$ leads $y_1(t)$ by $\frac{\pi}{6}$ radians. The graphs of $y_1(t)$ and $y_2(t)$ are shown in Fig. 2.7.

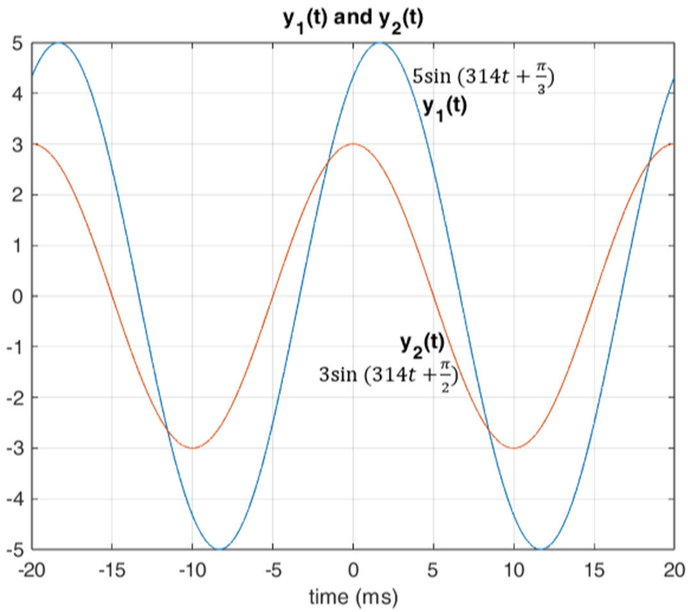


Fig. 2.7 Plot of $y_1(t) = 5\sin\left(314t + \frac{\pi}{3}\right)$ and $y_2(t) = 3\sin\left(314t + \frac{\pi}{2}\right)$

Example 2.12 Determine the phase difference between $y_1(t) = 5\cos\left(314t + \frac{\pi}{6}\right)$ and $y_2(t) = 3\cos(314t + \pi)$.

$$\Delta\varphi = \varphi_{y_1} - \varphi_{y_2} = 314t + \frac{\pi}{6} - (314t + \pi) = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$$

$y_1(t)$ lags $y_2(t)$ by $\frac{5\pi}{6}$ radians. In other words, $y_2(t)$ leads $y_1(t)$ by $\frac{5\pi}{6}$ radians. The graphs of $y_1(t)$ and $y_2(t)$ are shown in Fig. 2.8.

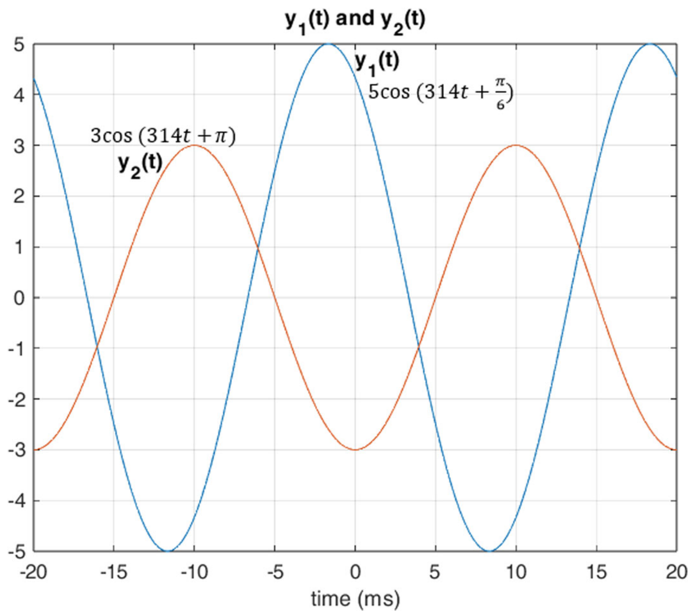


Fig. 2.8 Plot of $y_1(t) = 5\cos\left(314t + \frac{\pi}{6}\right)$ and $y_2(t) = 3\cos(314t + \pi)$

Example 2.13 Determine the phase difference between $y_1(t) = 5\sin(314t)$ and $y_2(t) = 3\cos\left(314t + \frac{\pi}{4}\right)$.

$$y_1(t) = 5\sin(314t)$$

$$y_2(t) = 3\cos\left(314t + \frac{\pi}{4}\right) = 3\sin\left(314t + \frac{\pi}{4} + \frac{\pi}{2}\right) = 3\sin\left(314t + \frac{3\pi}{4}\right)$$

$$\Delta\varphi = \varphi_{y_1} - \varphi_{y_2} = 314t - \left(314t + \frac{3\pi}{4}\right) = -\frac{3\pi}{4}$$

or

$$y_1(t) = 5\sin(314t) = 5\cos\left(314t - \frac{\pi}{2}\right)$$

$$y_2(t) = 3\cos\left(314t + \frac{\pi}{4}\right)$$

$$\Delta\varphi = \varphi_{y_1} - \varphi_{y_2} = \left(314t - \frac{\pi}{2}\right) - \left(314t + \frac{\pi}{4}\right) = -\frac{3\pi}{4}$$

$y_1(t)$ lags $y_2(t)$ by $\frac{3\pi}{4}$ radians. In other words, $y_2(t)$ leads $y_1(t)$ by $\frac{3\pi}{4}$ radians. The graphs of $y_1(t)$ and $y_2(t)$ are shown in Fig. 2.9.

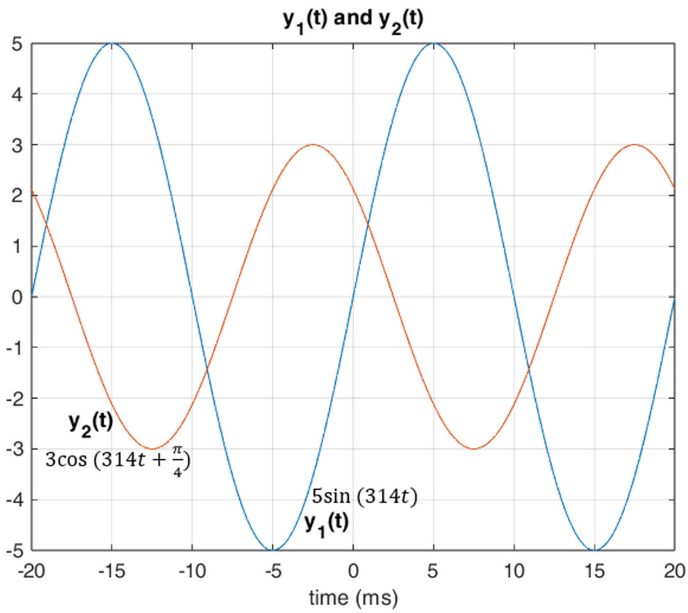


Fig. 2.9 Plot of $y_1(t) = 5\sin(314t)$ and $y_2(t) = 3\cos\left(314t + \frac{\pi}{4}\right)$

2.11 Graphical Phase Difference Measurement

This section demonstrates graphical phase difference measurement. Consider the waveforms shown in Fig. 2.10.

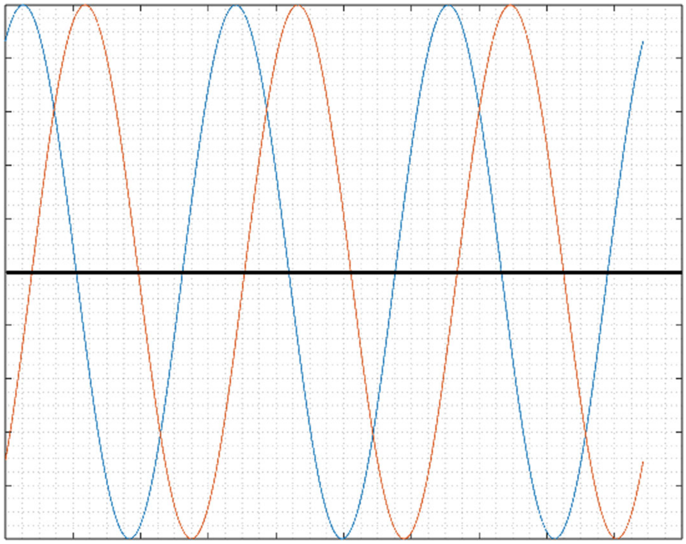


Fig. 2.10 Two sinusoidal signals

We need to measure the period first (Fig. 2.11).

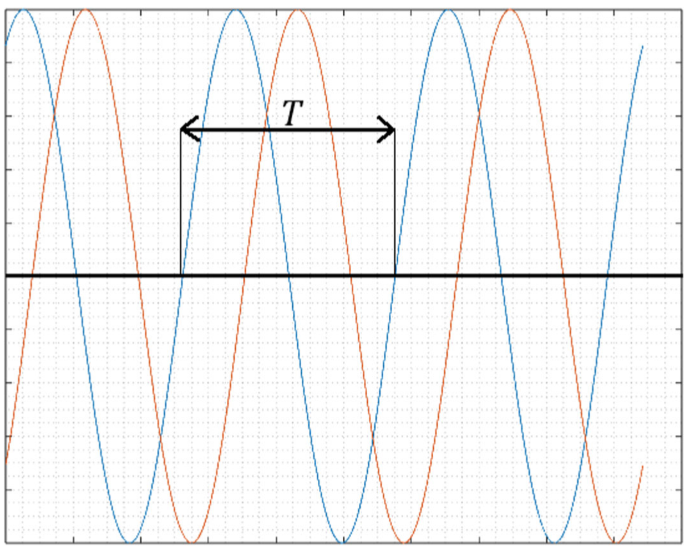


Fig. 2.11 Measurement of period

Next, we determine the time difference between corresponding zero-crossings (or peak points) on the two waveforms (Fig. 2.12). The phase difference can be calculated using $\Delta\varphi = \frac{\Delta t}{T} \times 360°$ formula.

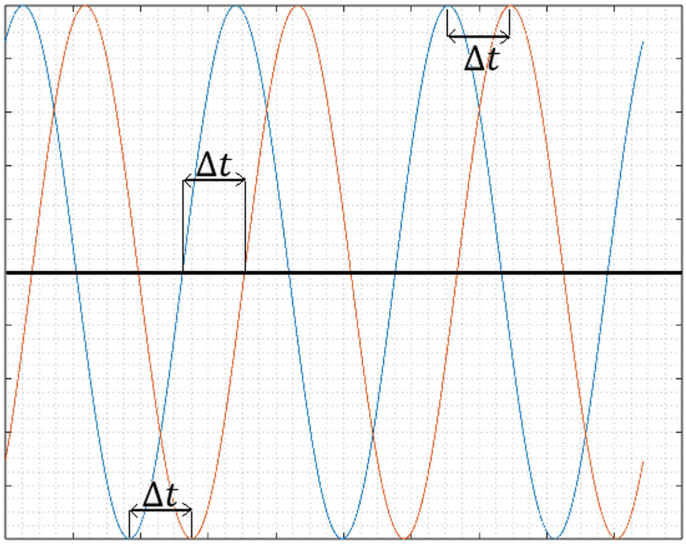


Fig. 2.12 Measurement of time shift

The proposed method does not explicitly identify the leading or lagging waveform. To determine this, we designate one waveform as a reference and identify a zero-crossing point on this waveform where it transitions from negative to positive values.

Next, we examine the value of the second waveform at the selected zero-crossing point. If the value is positive, the second waveform leads the reference. Conversely, if the value is negative, the second waveform lags the reference.

For instance, according to Fig. 2.13, the red waveform lags the blue waveform.

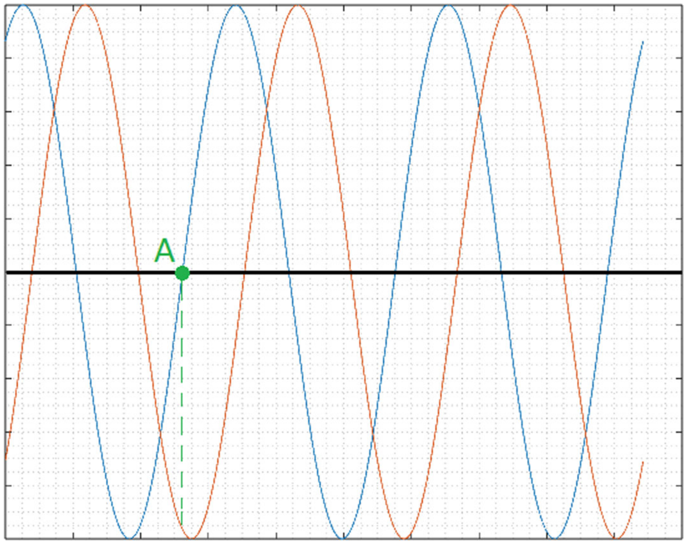


Fig. 2.13 The blue waveform starts to rise at point A

For instance, according to Fig. 2.14, the blue waveform leads the red waveform.

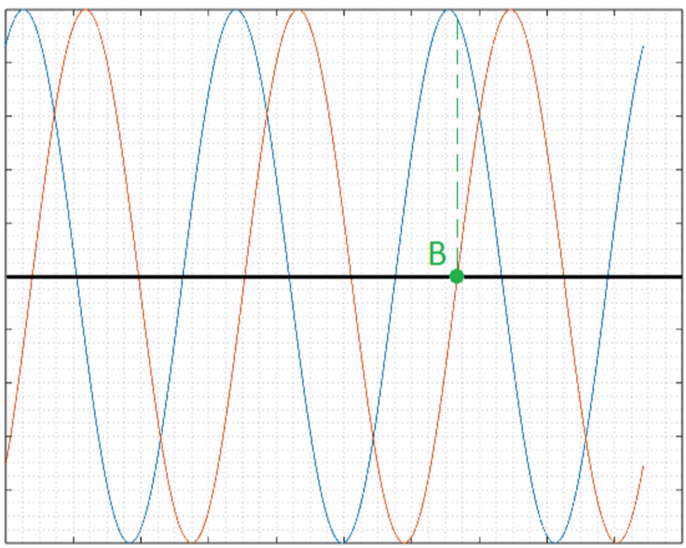


Fig. 2.14 The red waveform starts to rise at point B

2.12 Problem Solving with MATLAB®

This section provides examples of how MATLAB can be employed to solve the types of problems encountered in this chapter.

Example 2.14 The code in Fig. 2.15 converts $\frac{\pi}{6}$ radians to degrees.

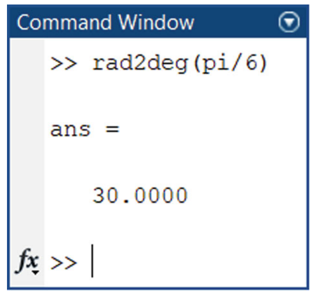


Fig. 2.15 Conversion of $\frac{\pi}{6}$ radians to degrees

Example 2.15 The code in Fig. 2.16 converts 45° to radians.

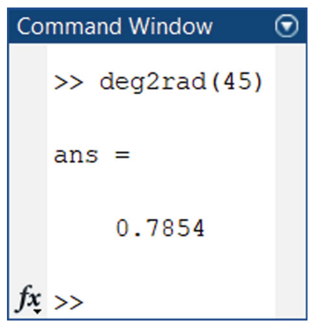


Fig. 2.16 Conversion of 45° to radians

Example 2.16 The code in Fig. 2.17 computes the sine, cosine, tangent, cotangent, secant and cosecant of $\frac{\pi}{6}$ radians.

Fig. 2.17 Calculation of trigonometric ratios for $\frac{\pi}{6}$ radians

Command Window

```
>> x=pi/6;
>> sin(x)

ans =

    0.5000

>> cos(x)

ans =

    0.8660

>> tan(x)

ans =

    0.5774

>> cot(x)

ans =

    1.7321

>> sec(x)

ans =

    1.1547

>> csc(x)

ans =

    2.0000

fx >>
```

Example 2.17 The code in Fig. 2.18 computes the sine, cosine, tangent, cotangent, secant and cosecant of 30°.

Fig. 2.18 Calculation of trigonometric ratios for 30°

```
Command Window
>> x=30;
>> sind(x)

ans =

    0.5000

>> cosd(x)

ans =

    0.8660

>> tand(x)

ans =

    0.5774

>> cotd(x)

ans =

    1.7321

>> secd(x)

ans =

    1.1547

>> cscd(x)

ans =

     2

fx >>
```

Example 2.18 The code in Fig. 2.19 computes the arcsine, arccosine, arctangent, arc cotangent, arc secant and arc cosecant of 0.5. Results are in radians.

Fig. 2.19 Calculation of the inverse trigonometric functions for $x = 0.5$ (Results are in radians)

```
Command Window
>> x=0.5;
>> asin(x)

ans =

    0.5236

>> acos(x)

ans =

    1.0472

>> atan(x)

ans =

    0.4636

>> acot(x)

ans =

    1.1071

fx >>
```

Example 2.19 The code in Fig. 2.20 computes the arcsine, arccosine, arctangent, arc cotangent, arc secant and arc cosecant of 0.5. Results are in degrees.

Fig. 2.20 Calculation of the inverse trigonometric functions for $x = 0.5$ (Results are in Degrees)

```
Command Window
>> x=0.5;
>> asind(x)

ans =

   30.0000

>> acosd(x)

ans =

   60.0000

>> atand(x)

ans =

   26.5651

>> acotd(x)

ans =

   63.4349

fx >> 
```

Example 2.20 The code in Fig. 2.21 computes the hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cotangent, hyperbolic secant and hyperbolic cosecant of 1.2.

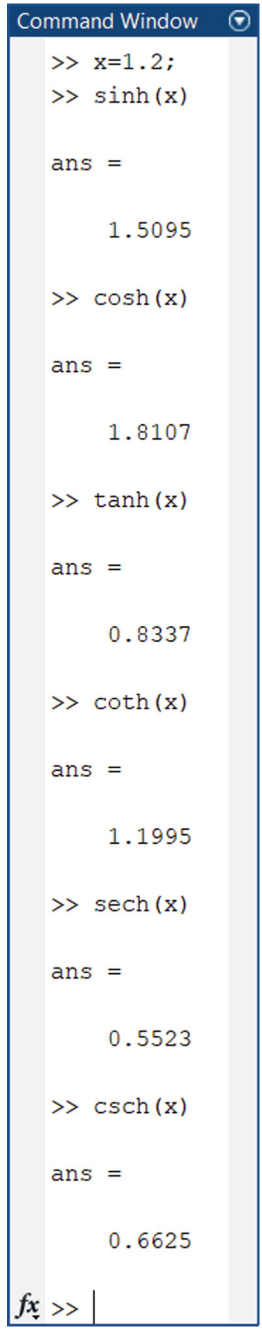


Fig. 2.21 Calculation of hyperbolic ratios for $x = 1.2$

Example 2.21 The code in Fig. 2.22 computes the hyperbolic arcsine, hyperbolic arccosine, hyperbolic arctangent, hyperbolic arc cotangent, hyperbolic arc secant and hyperbolic arc cosecant of 0.5.

```
Command Window
>> x=0.5;
>> asinh(x)

ans =

    0.4812

>> acosh(x)

ans =

   0.0000 + 1.0472i

>> atanh(x)

ans =

    0.5493

>> acoth(x)

ans =

   0.5493 + 1.5708i

fx >>
```

Fig. 2.22 Calculation of inverse hyperbolic functions for $x = 0.5$

Example 2.22 The code in Fig. 2.23 expresses the function $5\sin(\theta) + 7\cos(\theta)$ as a single cosine function.

```
Command Window
>> syms x
>> y=5*sin(x)+7*cos(x);
>> simplify(vpa(y))

ans =

8.6023252670426267717294735350497*cos(x - 0.62024948598282147133653255617593)

fx >>
```

Fig. 2.23 Conversion of $5\sin(\theta) + 7\cos(\theta)$ into a single cosine function

Example 2.23 The code in Fig. 2.24 expresses the function $-5\sin(\theta) + 7\cos(\theta)$ as a single cosine function.

```
Command Window
>> syms x
>> y=-5*sin(x)+7*cos(x);
>> simplify(vpa(y))

ans =

8.6023252670426267717294735350497*cos(x + 0.62024948598282147133653255617593)

fx >>
```

Fig. 2.24 Conversion of $-5\sin(\theta) + 7\cos(\theta)$ into a single cosine function

Example 2.24 The code in Fig. 2.25 expresses the function $5\sin(\theta) - 7\cos(\theta)$ as a single cosine function.

```
Command Window
>> syms x
>> y=5*sin(x)-7*cos(x);
>> simplify(vpa(y))

ans =

-8.6023252670426267717294735350497*cos(x + 0.62024948598282147133653255617593)

fx >>
```

Fig. 2.25 Conversion of $5\sin(\theta) - 7\cos(\theta)$ into a single cosine function

Example 2.25 The code in Fig. 2.26 expresses the function $-5\sin(\theta) - 7\cos(\theta)$ as a single cosine function.

```
Command Window
>> syms x
>> y=-5*sin(x)-7*cos(x);
>> simplify(vpa(y))

ans =

-8.6023252670426267717294735350497*cos(x - 0.62024948598282147133653255617593)

fx >>
```

Fig. 2.26 Conversion of $-5\sin(\theta) - 7\cos(\theta)$ into a single cosine function

Example 2.26 The code in Fig. 2.27 plots the function $f(x) = \sin\left(2x + \frac{\pi}{3}\right) + \cos\left(6x + \frac{\pi}{4}\right)$ on $[0, 2\pi]$ interval. Output of this code is shown in Fig. 2.28.

```
Command Window
>> syms x
>> ezplot(sin(3*x+pi/6)+cos(6*x+pi/4),[0,2*pi])
>> grid minor
fx >>
```

Fig. 2.27 Plotting the $f(x) = \sin\left(2x + \frac{\pi}{3}\right) + \cos\left(6x + \frac{\pi}{4}\right)$ on $[0, 2\pi]$ interval

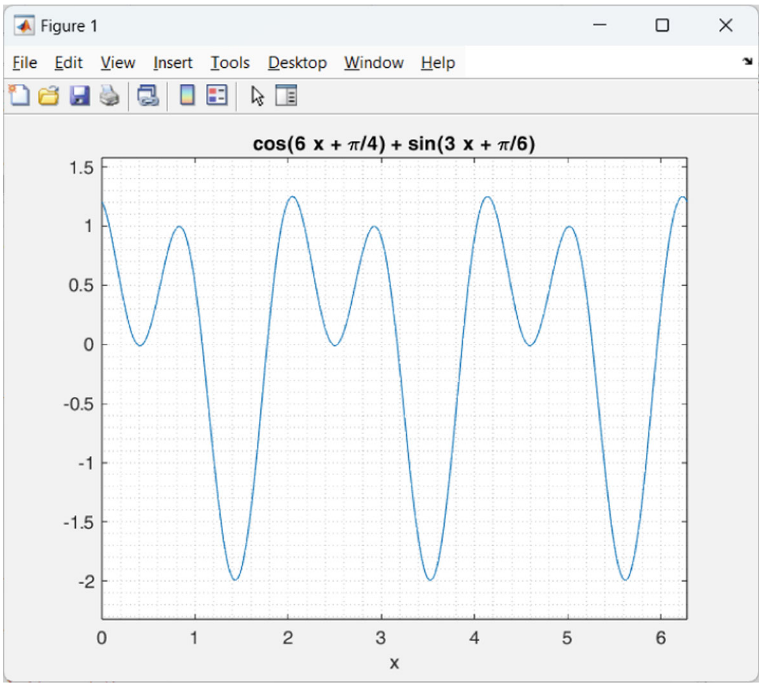


Fig. 2.28 Output of the code shown in Fig. 2.27

According to Fig. 2.29 the period of this function is around $4.13558 - 2.04603 = 2.0896$.

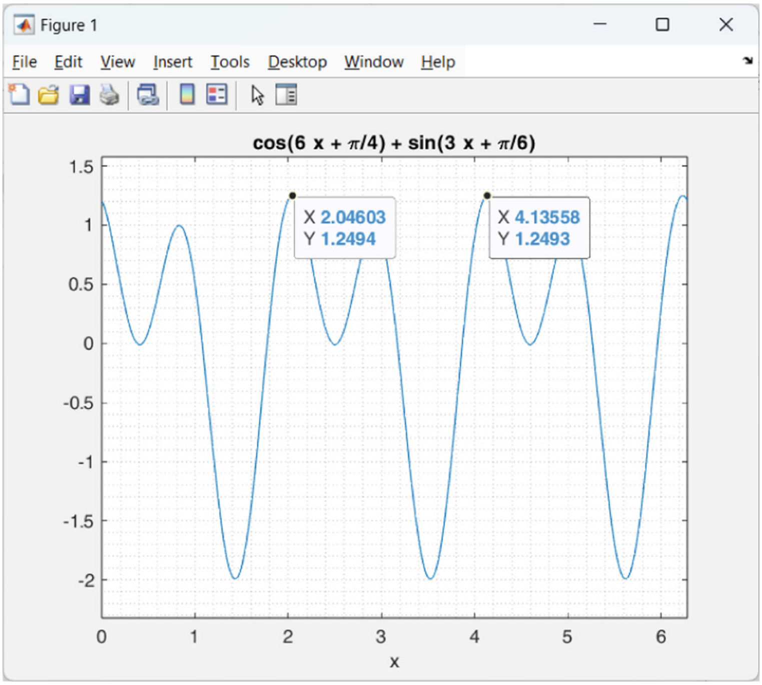


Fig. 2.29 Measuring coordinates of two consecutive maximums

References for Further Study

1. Advanced Engineering Mathematics (10th edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8th edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

3 Equations and Linear Algebra

3.1 Introduction

This chapter begins with a review of equation-solving methods and important linear algebra topics. The second part demonstrates MATLAB's application to the problems discussed.

3.2 First Order Algebraic Equations

Linear equations can be solved as: $ax + b = 0 \Rightarrow x = -\frac{b}{a}$. For instance, $3x + 5 = 0 \Rightarrow x = \frac{-5}{3}$ or $3x + 5 = 6 \Rightarrow x = \frac{6-5}{3} = \frac{1}{3}$.

3.3 Second Order Algebraic Equations

$ax^2 + bx + c = 0$ can be solved as: $ax^2 + bx + c = 0 \Rightarrow x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$ where, $\Delta = b^2 - 4 \times a \times c$. Let's study some numeric examples.

Example 3.1 Solve $x^2 - 3x + 2 = 0$, $x^2 + 6x + 9 = 0$ and $x^2 + 2x + 10 = 0$.

$$x^2 - 3x + 2 = 0 \Rightarrow \Delta = (-3)^2 - 4(1)(2) = 1 \Rightarrow x_{1,2} = \frac{3 \pm \sqrt{1}}{2} \Rightarrow x_1 = 1, x_2 = 2$$

$$x^2 + 6x + 9 = 0 \Rightarrow \Delta = (6)^2 - 4(1)(9) = 0 \Rightarrow x_{1,2} = \frac{-6 \pm \sqrt{0}}{2} \Rightarrow x_{1,2} = -3, x_1 = x_2 = -3$$

$$x^2 + 2x + 10 = 0 \Rightarrow \Delta = (2)^2 - 4(1)(10) = -36$$

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_3

$$\Rightarrow x_{1,2} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm 6j}{2} = -1 \pm 3j$$
$$\Rightarrow x_1 = -1 + 3j, x_2 = -1 - 3j$$

Special case I

When b is an even number, i.e., $b = 2b'$, you can use the following technique:

$$ax^2 + bx + c = 0 \Rightarrow ax^2 + 2b'x + c = 0 \Rightarrow x_{1,2} = \frac{-b' \pm \sqrt{\Delta'}}{a}$$

where $\Delta' = b'^2 - a \times c$.

Working with smaller numbers is the advantage of this technique. Use of this technique decreases the chance of making mistake and is recommended for exams. Let's study a numeric example:

$$x^2 - 10x + 21 = 0 \Rightarrow \frac{5 \pm \sqrt{(-5)^2 - (1)(21)}}{1} = \frac{5 \pm \sqrt{25 - 21}}{1} = \frac{5 \pm 2}{1} \Rightarrow x_1 = 3, x_2 = 7$$

Compare the above solution with the normal solution shown below:

$$x^2 - 10x + 21 = 0 \Rightarrow \frac{10 \pm \sqrt{(-10)^2 - 4(1)(21)}}{2} = \frac{10 \pm \sqrt{100 - 84}}{2} = \frac{10 \pm 4}{2}$$
$$\Rightarrow x_1 = 3, x_2 = 7$$

Special case II

If $b = a + c$ then roots of equation $ax^2 + bx + c = 0$ are $x_1 = -1$ and $x_2 = -\frac{c}{a}$. For instance, $6x^2 + 11x + 5 = 0 \Rightarrow x_1 = -1, x_2 = -\frac{5}{6}$.

Special case III

If $a + b + c = 0$ then roots of equation $ax^2 + bx + c = 0$ are $x_1 = 1$ and $x_2 = \frac{c}{a}$. For instance, $3x^2 - 7x + 4 = 0 \Rightarrow x_1 = 1, x_2 = \frac{4}{3}$.

In general, if the sum of the coefficients of a polynomial is zero, then one of its roots is unity. For instance, consider $x^3 - 9x^2 + 23x - 15 = 0$. Sum of coefficients is zero: $1 - 9 + 23 - 15 = 0$, therefore, one of the roots is 1. If we divide the given polynomial equation by $x - 1$, we obtain:

$$\frac{x^3 - 9x^2 + 23x - 15}{x - 1} = \left(x^2 - 8x + 15\right)$$

The other two roots are 3 and 5 since $\left(x^2 - 8x + 15\right) = 0 \Rightarrow x_{1,2} = \frac{4 \pm \sqrt{4^2 - 1 \times 15}}{1} \Rightarrow x_1 = 3, \Rightarrow x_2 = 5$.

3.4 Constructing a Polynomial with Specified Zeros

A polynomial with $r_1, r_2, r_3, \ldots, r_n$ roots can be formed using the $p(x) = \prod_{i=1}^{n}(x - r_i) = (x - r_1)(x - r_2)(x - r_3)\ldots(x - r_n)$ formula. For instance, a polynomial with roots of 1, 3 and 5 is: $p(x) = (x-1)(x-3)(x-5) = (x-1)\left(x^2 - 8x + 15\right) = x^3 - 9x^2 + 23x - 15$.

When number of given roots is two, you can use the $p(x) = x^2 - Sx + P$ formula. S and P shows the sum and product of given roots. For instance, for $r_1 = 3$ and $r_2 = 5$, $S = 3 + 5 = 8$ and $P = 3 \times 5 = 15$. Therefore, $p(x) = x^2 - 8x + 15$. For $r_1 = -2 + 3j$ and $r_2 = -2 - 3j$, $S = -2 + 3j - 2 - 3j = -4$ and $P = (-2 + 3j) \times (-2 - 3j) = (-2)^2 + 3^2 = 13$. Therefore, $p(x) = x^2 + 4x + 13$.

3.5 Newton–Raphson Method

The Newton–Raphson method is a powerful iterative technique used to find the roots of nonlinear equations, including polynomials. It's based on the idea of linear approximation. In each step the guess is updated with the aid of following formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Le's study a numeric example. We want to find the roots of $f(x) = x^2 - 4$ with the aid of Newton–Raphson method (initial guess is 3, i.e., $x_0 = 3$). We know that roots of $(x) = x^2 - 4$ are ± 2 since, $f(x) = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \sqrt{4} \Rightarrow x = \pm 2$.

Following iterations are converged to $+2.0000$.

$$f'(x) = \frac{df}{dx} = 2x \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 4}{2x_n}$$

$$\begin{aligned}
&x_0 = 3, x_{n+1} = x_n - \frac{x_n^2 - 4}{2x_n} \\
&x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{3^2 - 4}{2 \times 3} = 2.1667 \Rightarrow x_1 = 2.1667 \\
&x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1667 - \frac{2.1667^2 - 4}{2 \times 2.1667} = 2.0064 \Rightarrow x_2 = 2.0064 \\
&x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0064 - \frac{2.0064^2 - 4}{2 \times 2.0064} = 2.0000 \Rightarrow x_3 = 2.0000
\end{aligned}$$

Let's change the initial guess value to $x_0 = -4$. Newton–Raphson iterations with initial guess of -4, i.e., $x_0 = -4$ are shown below. This time the result converged to -2.000.

$$x_0 = -4$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -4 - \frac{(-4)^2 - 4}{2 \times -4} = -2.5000 \Rightarrow x_1 = -2.5000$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -2.5000 - \frac{(-2.5000)^2 - 4}{2 \times -2.5000} = -2.0500 \Rightarrow x_2 = -2.0500$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -2.0500 - \frac{(-2.0500)^2 - 4}{2 \times -2.0500} = 2.0006 \Rightarrow x_3 = -2.0006$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = -2.0006 - \frac{(-2.0006)^2 - 4}{2 \times -2.0006} = 2.0000 \Rightarrow x_4 = -2.0000$$

Exercise: Use Newton–Raphson method to find the two roots of $x + e^{-x} - 2 = 0$. Use − 1 and + 1 as initial guess.

3.6 Absolute Value Equations

This section shows how equations containing absolute values can be solved.

Example 3.2 Solve $|x+4| = 10$.

$$|x+4| = 10 \Rightarrow x + 4 = \pm 10 \Rightarrow \begin{cases} x + 4 = 10 \Rightarrow x = 6 \\ x + 4 = -10 \Rightarrow x = -14 \end{cases}$$

Example 3.3 Solve $|x+4| = |x-1|$.

$$|x+4| = |x-1| \Rightarrow x + 4 = \pm(x-1) \Rightarrow \begin{cases} x + 4 = x - 1 \Rightarrow 4 = -1(!) \\ x + 4 = -(x-1) \Rightarrow 2x = -3 \Rightarrow x = -1.5 \end{cases}$$

Example 3.4 Solve $\left|x^2 + 7x - 3\right| = \left|x^2 + 3x + 1\right|$.

$$\left|x^2 + 7x - 3\right| = \left|x^2 + 3x + 1\right| \Rightarrow x^2 + 7x - 3 = \pm\left(x^2 + 3x + 1\right)$$
$$\Rightarrow \begin{cases} x^2 + 7x - 3 = +\left(x^2 + 3x + 1\right) \Rightarrow x = 1 \\ x^2 + 7x - 3 = -\left(x^2 + 3x + 1\right) \Rightarrow x^2 + 5x - 1 = 0 \Rightarrow x = \frac{-5 \pm \sqrt{29}}{2} \end{cases}$$

3.7 Trigonometric Equations

A trigonometric equation is an equation involving one or more trigonometric ratios of unknown angles. This section shows how this type of equations can be solved.

Trigonometric functions are periodic. A function $f(x)$ is said to be periodic with period T if, for every x in the domain of f, $f(x+T) = f(x)$. $\sin(\omega_0 t + \varphi_0)$ and $\cos(\omega_0 t + \varphi_0)$

are periodic functions with period of $T = \frac{2\pi}{\omega_0}$. $\tan(\omega_0 t + \varphi_0)$ and $\cot(\omega_0 t + \varphi_0)$ are periodic functions with period of $T = \frac{\pi}{\omega_0}$. For instance, period of $\sin(3x)$ is $T = \frac{2\pi}{3}$ and period of $\tan\left(2x + \frac{\pi}{6}\right)$ is $T = \frac{\pi}{2}$.

Inverse trigonometric functions, are the inverse functions of the basic trigonometric functions. They are used to find the angle whose trigonometric function is a given value. The ranges of the important inverse trigonometric functions are as follows:

$$1.\ -\frac{\pi}{2} \le \sin^{-1}(x) \le \frac{\pi}{2}$$
$$2.\ 0 \le \cos^{-1}(x) \le \pi$$
$$3.\ -\frac{\pi}{2} < \tan^{-1}(x) < \frac{\pi}{2}$$
$$4.\ 0 < \cot^{-1}(x) < \pi$$

Example 3.5 Solve $\sin(x) = 0.7$.

Using a calculator $\sin^{-1}(0.7) = 0.7754$ rad (Fig. 3.1).

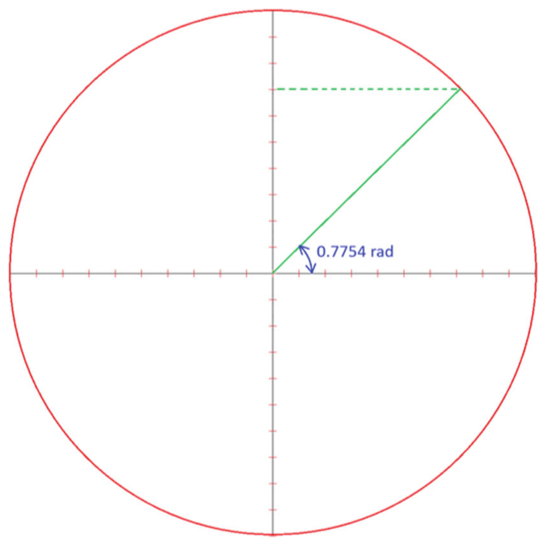


Fig. 3.1 Sine of 0.7754 rad is 0.7

However, there exists another angle whose sine is 0.7. That angle is $\pi - 0.7754 = 2.3662$ rad or $-(\pi + 0.7754) = -3.917$ rad (Figs. 3.2 and 3.3).

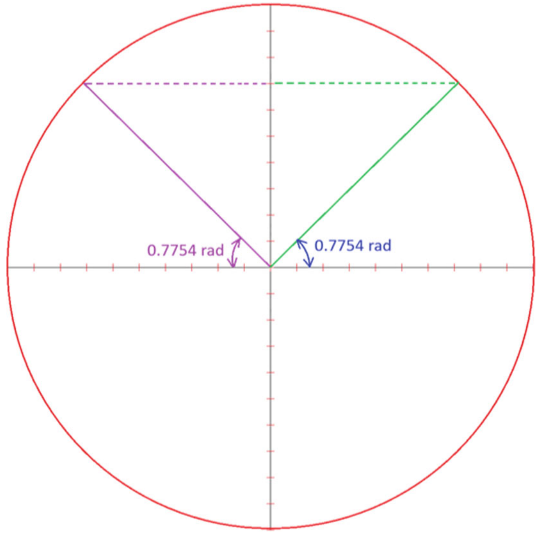


Fig. 3.2 Sine of 2.3662 rad or − 3.917 rad is 0.7

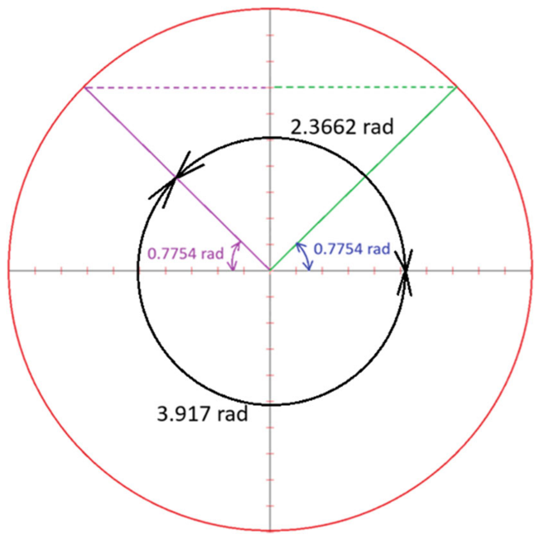


Fig. 3.3 Different ways to measure an angle

$$\sin(x) = 0.7 \Rightarrow \begin{cases} x = 0.7754 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \\ x = \pi - 0.7754 \pm 2k\pi = 2.3662 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \end{cases}$$

or

$$\sin(x) = 0.7 \Rightarrow \begin{cases} x = 0.7754 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \\ x = -(\pi + 0.7754) \pm 2k\pi = -3.917 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \end{cases}$$

Example 3.6 Solve $\sin(x) = -0.7$.

Using a calculator $\sin^{-1}(-0.7) = -0.7754$ rad (Fig. 3.4).

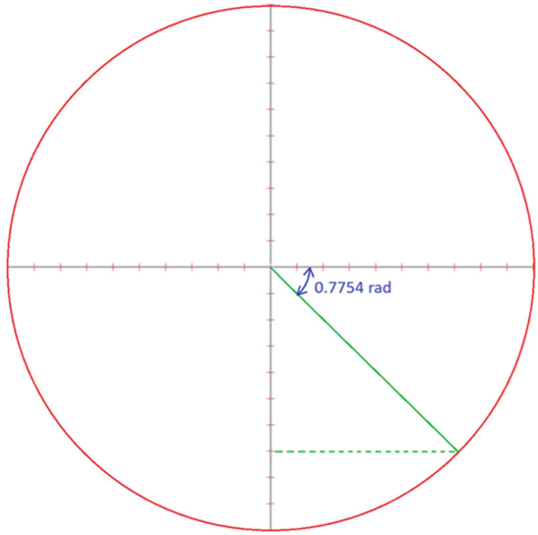


Fig. 3.4 Sine of -0.7754 rad is -0.7

However, there exists another angle whose sine is -0.7. That angle is $-(\pi - 0.7754) = -2.3662$ rad or $(\pi + 0.7754) = +3.917$ rad (Figs. 3.5 and 3.6).

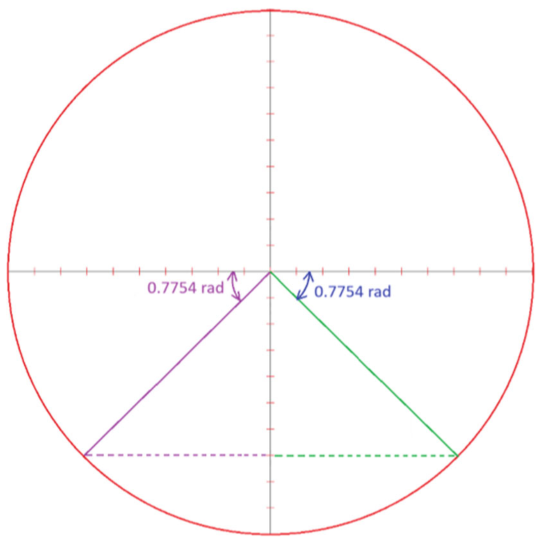


Fig. 3.5 Sine of -2.3662 rad or 3.917 rad is -0.7

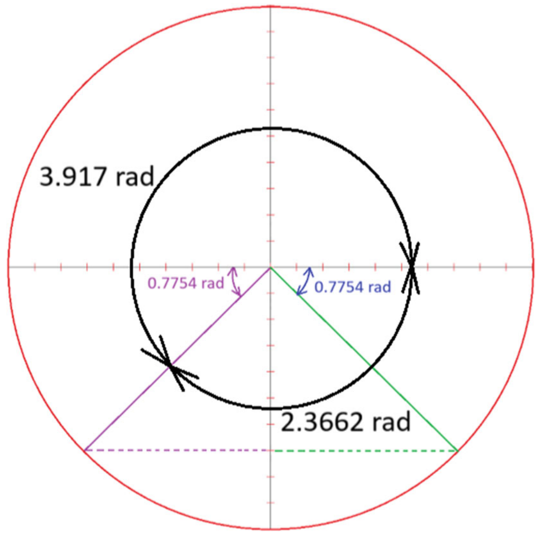


Fig. 3.6 Different ways to measure an angle

$$\sin(x) = -0.7 \Rightarrow \begin{cases} x = -0.7754 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \\ x = \pi - (-0.7754) \pm 2k\pi = 3.9170 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \end{cases}$$

or

$$\sin(x) = -0.7 \Rightarrow \begin{cases} x = -0.7754 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \\ x = -(\pi - 0.7754) \pm 2k\pi = -2.3662 \pm 2k & k = 0, 1, 2, 3, \ldots \end{cases}$$

Example 3.7 Solve $\sin(3x) = 0.7$.

Using a calculator $\sin^{-1}(0.7) = 0.7754\,\text{rad}$.

$$\sin(3x) = 0.7 \Rightarrow \begin{cases} 3x = 0.7754 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \\ 3x = \pi - 0.7754 \pm 2k\pi = 2.3662 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{0.7754 \pm 2k\pi}{3} & k = 0, 1, 2, 3, \ldots \\ x = \frac{\pi - 0.7754 \pm 2k\pi}{3} = \frac{2.3662 \pm 2k\pi}{3} & k = 0, 1, 2, 3, \ldots \end{cases}$$

or

$$\sin(3x) = 0.7 \Rightarrow \begin{cases} 3x = 0.7754 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \\ 3x = -(\pi + 0.7754) \pm 2k\pi = -3.917 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{0.7754 \pm 2k\pi}{3} & k = 0, 1, 2, 3, \ldots \\ x = \frac{-(\pi + 0.7754) \pm 2k\pi}{3} = \frac{-3.917 \pm 2k\pi}{3} & k = 0, 1, 2, 3, \ldots \end{cases}$$

Example 3.8 Solve $y = \sin(x) + \sqrt{2}\cos(x) = 1$.

$a \times \sin(\omega t) + b \times \cos(\omega t) = \text{sign}(a) \cdot \sqrt{a^2 + b^2}\sin\left(\omega t + \tan^{-1}\left(\frac{b}{a}\right)\right)$ where $\text{sign}(x) = \begin{cases} +1, & x > 1 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$.

$$\sin(x) + \sqrt{2}\cos(x) = 1 \Rightarrow \sqrt{3}\sin\left(x + \tan^{-1}\left(\frac{\sqrt{2}}{1}\right)\right) = 1 \Rightarrow \sin(x + 0.9553) = \frac{1}{\sqrt{3}}$$

Using a calculator $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) = 0.6155$ rad. Therefore,

$$\sin(x + 0.9553) = \frac{1}{\sqrt{3}}$$
$$\Rightarrow \begin{cases} x + 0.9553 = 0.6155 \pm 2k\pi \Rightarrow x = -0.3398 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \\ x + 0.9553 = \pi - 0.6155 \pm 2k\pi = 2.5261 \pm 2k\pi \Rightarrow x = +1.5708 \pm 2k\pi & k = 0, 1, 2, 3, \ldots \end{cases}$$

Graph of $y = 1$ and $y = \sin(x) + \sqrt{2}\cos(x)$ for $[-2\pi, 2\pi]$ interval are shown in Fig. 3.7. Intersection points are shown in Fig. 3.8. You can use Fig. 3.8 to verify that obtained solution is correct.

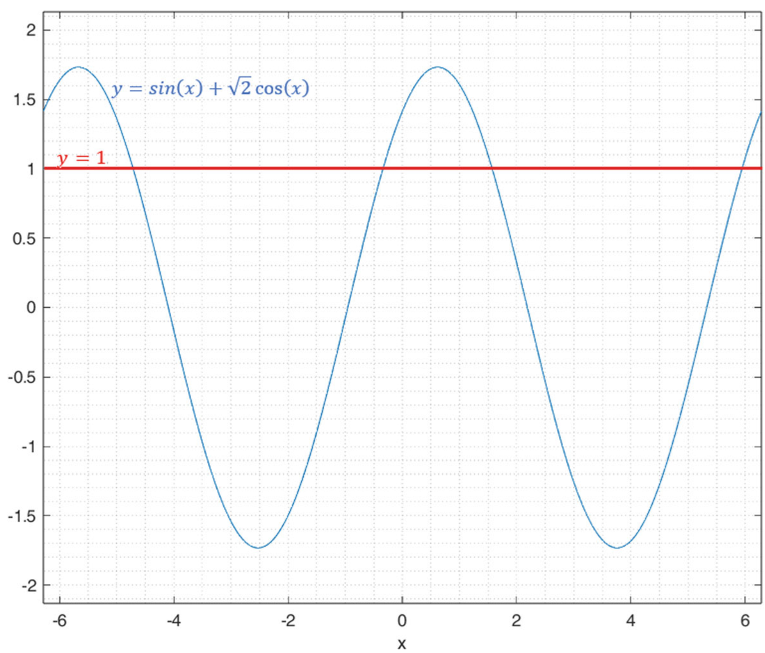


Fig. 3.7 Intersections of $y(x) = \sin(x) + \sqrt{2}\cos(x)$ and $y(x) = 1$

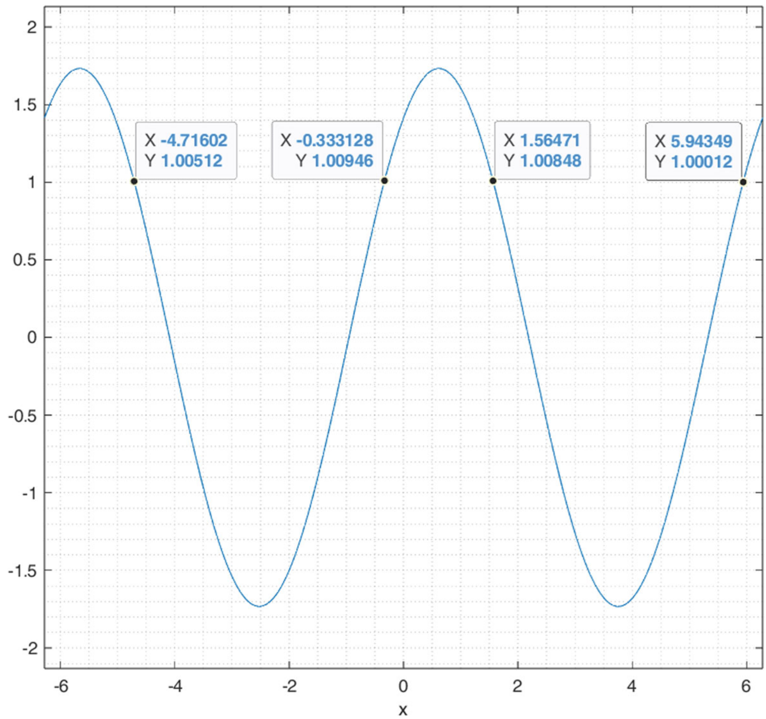


Fig. 3.8 Intersection coordinates

Exercise: Solve $y = \cos(x) - \sqrt{3}\sin(x) = 2$.

Example 3.9 Solve $\cos(x) = 0.7$.

Using a calculator $\cos^{-1}(0.7) = 0.7954\,\text{rad}$ (Fig. 3.9).

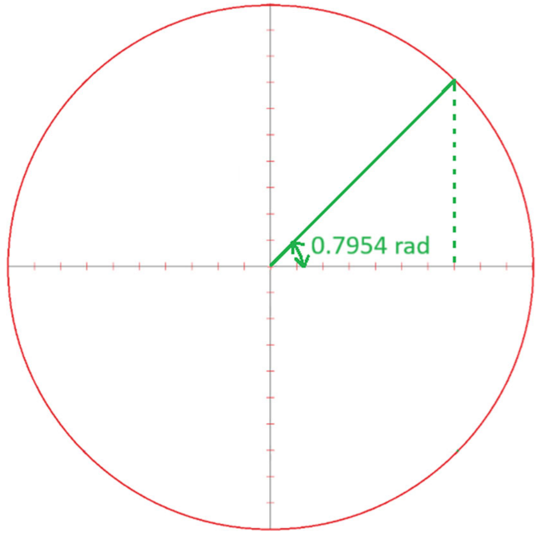


Fig. 3.9 Cosine of 0.7954 rad is 0.7

However, there exists another angle whose cosine is 0.7. That angle is -0.7954 rad or $2\pi - 0.7954 = +5.4878$ rad (Fig. 3.10).

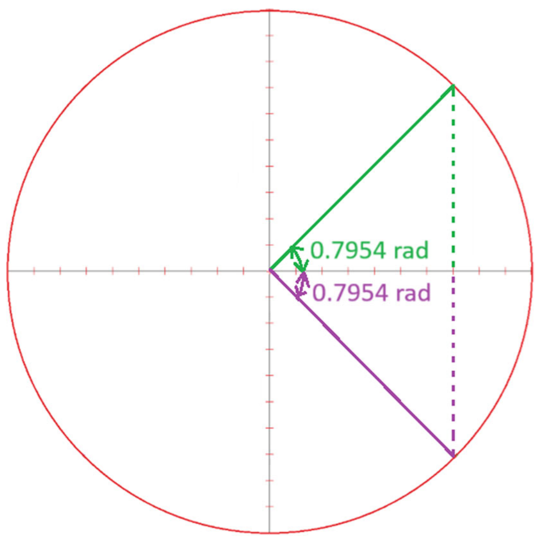


Fig. 3.10 Cosine of 0.7954 rad and -0.7954 rad is 0.7

$$\cos(x) = 0.7 \Rightarrow \begin{cases} x = 0.7954 \pm 2k\pi \\ x = -0.7954 \pm 2k\pi \end{cases} \quad k = 0, 1, 2, 3, \ldots$$

or

$$\cos(x) = 0.7 \Rightarrow \begin{cases} x = 0.7954 \pm 2k\pi \\ x = 5.4878 \pm 2k\pi \end{cases} \quad k = 0, 1, 2, 3, \ldots$$

Example 3.10 Solve $\cos(3x) = 0.7$.

Using a calculator $\cos^{-1}(0.7) = 0.7954$ rad (Fig. 3.10).

$$\begin{aligned} \cos(3x) = 0.7 &\Rightarrow \begin{cases} 3x = 0.7954 \pm 2k\pi \\ 3x = -0.7954 \pm 2k\pi \end{cases} \\ &\Rightarrow \begin{cases} x = \frac{0.7954 \pm 2k\pi}{3} \\ x = \frac{-0.7954 \pm 2k\pi}{3} \end{cases} \\ &\Rightarrow \begin{cases} x = 0.2651 \pm \frac{2k\pi}{3} \\ x = -0.2651 \pm \frac{2k\pi}{3} \end{cases} \quad k = 0, 1, 2, 3, \ldots \end{aligned}$$

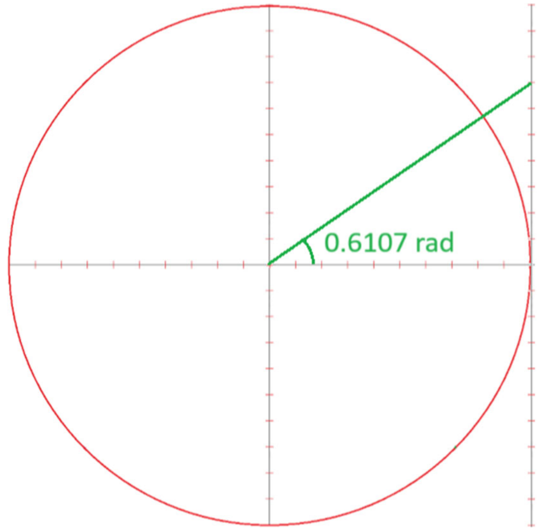
or

$$\begin{aligned} \cos(3x) = 0.7 &\Rightarrow \begin{cases} 3x = 0.7954 \pm 2k\pi \\ 3x = 5.4878 \pm 2k\pi \end{cases} \\ &\Rightarrow \begin{cases} x = \frac{0.7954 \pm 2k\pi}{3} \\ x = \frac{5.4878 \pm 2k\pi}{3} \end{cases} \\ &\Rightarrow \begin{cases} x = 0.2651 \pm \frac{2k\pi}{3} \\ x = 1.8293 \pm \frac{2k\pi}{3} \end{cases} \quad k = 0, 1, 2, 3, \ldots \end{aligned}$$

Example 3.11 Solve $\tan(x) = 0.7$.

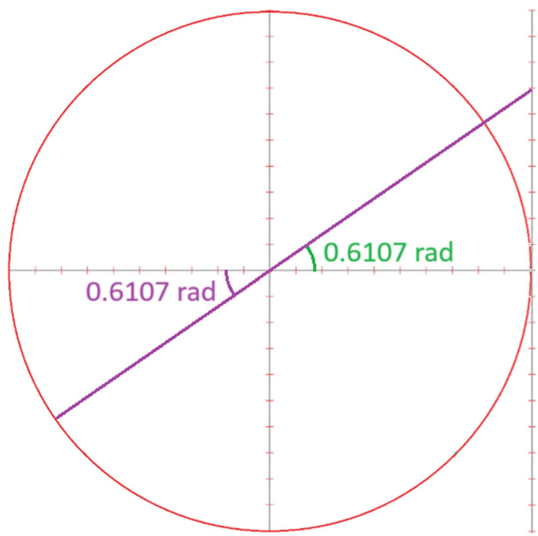
Using a calculator $\tan^{-1}(0.7) = 0.6107$ rad (Fig. 3.11).

Fig. 3.11 Tangent of 0.6107 rad is 0.7



However, there exists another angle whose tangent is 0.7. That angle is $\pi + 0.6107 = 3.7523$ rad or $-(\pi - 0.6107) = -2.5309$ rad (Figs. 3.12 and 3.13).

Fig. 3.12 Tangent of 3.7523 rad or -2.5309 rad is 0.7



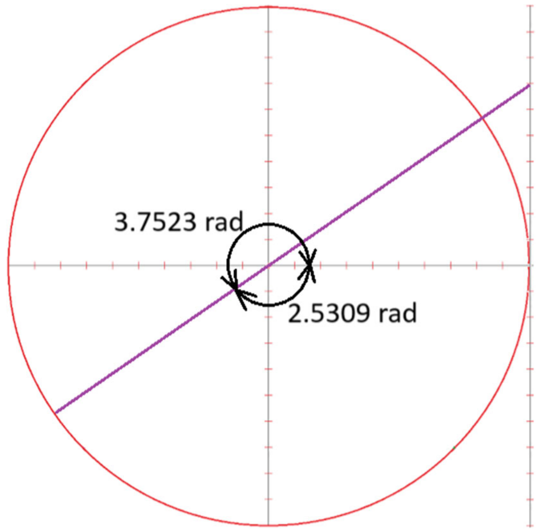


Fig. 3.13 Different ways to measure an angle

Therefore,

$$\tan(x) = 0.7 \Rightarrow \begin{cases} x = 0.6107 \pm k\pi = 0.6107 \pm k\pi & k = 0, 1, 2, 3, \ldots \\ x = \pi + 0.6107 \pm k\pi = 3.7523 \pm k\pi & k = 0, 1, 2, 3, \ldots \end{cases}$$

or

$$\tan(x) = 0.7 \Rightarrow \begin{cases} x = 0.6107 \pm k\pi = 0.6107 \pm k\pi & k = 0, 1, 2, 3, \ldots \\ x = -(\pi - 0.6107) \pm k\pi = -2.5309 \pm k\pi & k = 0, 1, 2, 3, \ldots \end{cases}$$

Example 3.12 Solve $\tan(3x) = 0.7$.

Using a calculator $\tan^{-1}(0.7) = 0.6107$ rad. Therefore,

$$\tan(3x) = 0.7 \Rightarrow \begin{cases} 3x = 0.6107 \pm k\pi & k = 0, 1, 2, 3, \ldots \\ 3x = \pi + 0.6107 \pm k\pi = 3.7523 \pm k\pi & k = 0, 1, 2, 3, \ldots \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{0.6107 \pm k\pi}{3} & k = 0, 1, 2, 3, \ldots \\ x = \frac{3.7523 \pm k\pi}{3} & k = 0, 1, 2, 3, \ldots \end{cases}$$

$$\Rightarrow \begin{cases} x = 0.2036 \pm \frac{k\pi}{3} & k = 0, 1, 2, 3, \ldots \\ x = 1.2508 \pm \frac{k\pi}{3} & k = 0, 1, 2, 3, \ldots \end{cases}$$

or

$$\tan(3x) = 0.7 \Rightarrow \begin{cases} 3x = 0.6107 \pm k\pi & k = 0, 1, 2, 3, \ldots \\ 3x = -(\pi - 0.6107) \pm k\pi = -2.5309 \pm k\pi & k = 0, 1, 2, 3, \ldots \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{0.6107 \pm k\pi}{3} & k = 0, 1, 2, 3, \ldots \\ x = \frac{-2.5309 \pm k\pi}{3} & k = 0, 1, 2, 3, \ldots \end{cases}$$

$$\Rightarrow \begin{cases} x = 0.2036 \pm \frac{k\pi}{3} & k = 0, 1, 2, 3, \ldots \\ x = -0.8436 \pm \frac{k\pi}{3} & k = 0, 1, 2, 3, \ldots \end{cases}$$

Example 3.13 Solve $\tan\left(3x + \frac{\pi}{4}\right) = 0.7$.

Using a calculator $\tan^{-1}(0.7) = 0.6107$. Therefore,

$$\tan\left(3x + \frac{\pi}{4}\right) = 0.7 \Rightarrow \begin{cases} 3x + \frac{\pi}{4} = 0.6107 \pm k\pi \\ 3x + \frac{\pi}{4} = \pi + 0.6107 \pm k\pi \end{cases}$$

$$\Rightarrow \begin{cases} 3x = 0.6107 - \frac{\pi}{4} \pm k\pi \\ 3x = \pi + 0.6107 - \frac{\pi}{4} \pm k\pi \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{0.6107 - \frac{\pi}{4} \pm k\pi}{3} \\ x = \frac{\pi + 0.6107 - \frac{\pi}{4} \pm k\pi}{3} \end{cases}$$

$$\Rightarrow \begin{cases} x = -0.0582 \pm \frac{k\pi}{3} \\ x = 0.989 \pm \frac{k\pi}{3} \end{cases} \quad k = 0, 1, 2, 3$$

or

$$\tan\left(3x + \frac{\pi}{4}\right) = 0.7 \Rightarrow \begin{cases} 3x + \frac{\pi}{4} = 0.6107 \pm k\pi \\ 3x + \frac{\pi}{4} = -(\pi - 0.6107) \pm k\pi \end{cases}$$

$$\Rightarrow \begin{cases} 3x = 0.6107 - \frac{\pi}{4} \pm k\pi \\ 3x = -(\pi - 0.6107) - \frac{\pi}{4} \pm k\pi \end{cases}$$

$$\Rightarrow \begin{cases} x = \frac{0.6107 - \frac{\pi}{4} \pm k\pi}{3} \\ x = \frac{-(\pi - 0.6107) - \frac{\pi}{4} \pm k\pi}{3} \end{cases}$$

$$\Rightarrow \begin{cases} x = -0.0582 \pm \frac{k\pi}{3} \\ x = -1.1054 \pm \frac{k\pi}{3} \end{cases} \quad k = 0, 1, 2, 3, \ldots$$

Exercise: Graph the examples discussed to visually verify the obtained results.

3.8 Matrix Multiplication

The following examples review matrix multiplication.

Example 3.14 Calculate the $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 & 9 \\ 0 & 1 & 2 \\ 3 & 5 & 5 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 & 9 \\ 0 & 1 & 2 \\ 3 & 5 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times 7 + 2 \times 0 + 3 \times 3 & 1 \times 8 + 2 \times 1 + 3 \times 5 & 1 \times 9 + 2 \times 2 + 3 \times 5 \\ 4 \times 7 + 5 \times 0 + 6 \times 3 & 4 \times 8 + 5 \times 1 + 6 \times 5 & 4 \times 9 + 5 \times 2 + 6 \times 5 \end{bmatrix}$$
$$= \begin{bmatrix} 16 & 25 & 28 \\ 46 & 67 & 76 \end{bmatrix}$$

Example 3.15 Calculate the $\begin{bmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \times \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \times \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 7 \times 3 & 0 & 0 \\ 0 & 6 \times 4 & 0 \\ 0 & 0 & 5 \times -2 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & -10 \end{bmatrix}$$

For any positive integer n, the n-th power of a diagonal matrix A, denoted as A^n, is also a diagonal matrix. The diagonal elements of A^n are simply the n-th powers of the corresponding diagonal elements of A. For instance, $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^n = \begin{bmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{bmatrix}$.

3.9 Transpose of a Matrix

If A is an $m \times n$ matrix, then its transpose, denoted as A^T, is an $n \times m$ matrix where: $A^T_{ij} = A_{ji}$.

Example 3.16 Calculate the transpose of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

3.10 Minors of a Matrix

The minor of the element a_{ij} (in the i-th row, j-th column) which is shown by M_{ij} is the determinant of the submatrix obtained by deleting the i-th row and j-th column. For instance, for $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example 3.17 For $A = \begin{bmatrix} 1 & 1 & 2 \\ 5 & 0 & -1 \\ 7 & 2 & 3 \end{bmatrix}, M_{21} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 1 \times 3 - 2 \times 2 = -1.$

3.11 Cofactors of a Matrix

The cofactor of the element a_{ij} (in the i-th row, j-th column) which is shown by C_{ij} equals to $(-1)^{i+j} M_{ij}$. For instance, for $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

$$C_{11} = +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, C_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, C_{13} = +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},$$

$$C_{21} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, C_{22} = +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, C_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix},$$

$$C_{31} = +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, C_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, C_{33} = +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Example 3.18 For $A = \begin{bmatrix} 1 & 1 & 2 \\ 5 & 0 & -1 \\ 7 & 2 & 3 \end{bmatrix}$, $C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 7 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 7 & 2 \end{vmatrix} = -(1 \times 2 - 1 \times 7) = 5$.

3.12 Determinant of a Matrix

The determinant of a matrix is a scalar value that characterizes certain properties of the matrix and the linear transformation it represents. Determinants are defined only for square matrices (matrices with the same number of rows and columns).

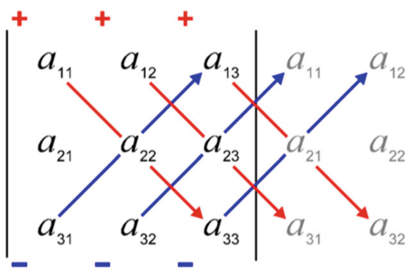
Determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ equals to $a \times d - b \times c$. Determinant of 3×3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ can be calculated using one of the following three methods:

$$\begin{aligned} \det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a \times (e \times i - f \times h) - b \times (d \times i - f \times g) \\ &\quad + c \times (d \times h - e \times g) \end{aligned}$$

$$\begin{aligned} \det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) &= -d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - f\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ &= -d \times (b \times i - c \times h) + e \times (a \times i - c \times g) \\ &\quad - f \times (a \times h - b \times g) \end{aligned}$$

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = g\begin{vmatrix} b & c \\ e & f \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} + i\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= g \times (b \times f - c \times e) - h \times (a \times f - c \times d)$$
$$+ i \times (a \times e - b \times d)$$

The Sarrus method is a quick and efficient way to calculate the determinant of a 3 × 3 matrix. The calculation of the determinant for matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ using the Sarrus method is shown in Fig. 3.14.



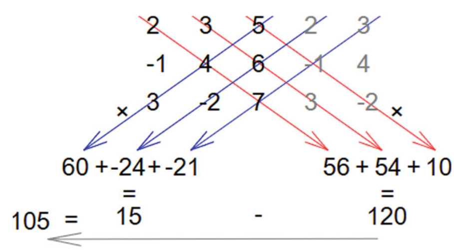
$$det = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Fig. 3.14 Calculation of determinant using Sarrus method

Example 3.19 Use the Sarrus method to calculate the determinant of $\begin{bmatrix} 2 & 3 & 5 \\ -1 & 4 & 6 \\ 3 & -2 & 7 \end{bmatrix}$.

The solution is shown in Fig. 3.15.

Fig. 3.15 Calculation of the determinant of $\begin{bmatrix} 2 & 3 & 5 \\ -1 & 4 & 6 \\ 3 & -2 & 7 \end{bmatrix}$ using Sarrus method



The following are some important properties of determinants.

A. The determinant of a diagonal matrix is simply the product of its diagonal elements. For instance, determinant of $A = \mathrm{diag}(7, 6, -2) = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ equals to $7 \times 6 \times -2 = -84$.
B. The determinant of the product of two square matrices A and B of the same order equals to the product of their individual determinants, i.e., $|A \times B| = |A| \times |B|$.
C. The determinant of a matrix and its transpose are equal. That is, $|A| = \left|A^T\right|$.
D. The determinant of a triangular matrix (upper triangular or lower triangular) is the product of its diagonal elements. For instance, $\begin{vmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & g \end{vmatrix} = a.e.g$ or $\begin{vmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{vmatrix} = a.c.f$.
E. A square matrix A is invertible if and only if $|A| \neq 0$.
F. If a row or column of a matrix is multiplied by a scalar k, the determinant is multiplied by k. For instance,

$$A = \begin{bmatrix} 8 & -2 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 8 & -2 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{vmatrix} = -43$$

$$B = \begin{bmatrix} 8 & -2 & 1 \\ 2 \times 2 & -1 \times 2 & 3 \times 2 \\ 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 8 & -2 & 1 \\ 4 & -2 & 6 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\Rightarrow |B| = \begin{vmatrix} 8 & -2 & 1 \\ 4 & -2 & 6 \\ 1 & 1 & 4 \end{vmatrix} = 2 \times \begin{vmatrix} 8 & -2 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{vmatrix} = 2 \times -43 = -86$$

$$C = \begin{bmatrix} 8 & -2 & 1 \\ 2 \times 2 & -1 \times 2 & 3 \times 2 \\ 1 \times 5 & 1 \times 5 & 4 \times 5 \end{bmatrix} = \begin{bmatrix} 8 & -2 & 1 \\ 4 & -2 & 6 \\ 5 & 5 & 20 \end{bmatrix}$$

$$\Rightarrow |C| = \begin{vmatrix} 8 & -2 & 1 \\ 4 & -2 & 6 \\ 5 & 5 & 20 \end{vmatrix} = 2 \times \begin{vmatrix} 8 & -2 & 1 \\ 2 & -1 & 3 \\ 5 & 5 & 20 \end{vmatrix} = 2 \times 5 \times \begin{vmatrix} 8 & -2 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{vmatrix}$$

$$= 2 \times 5 \times -43 = -430$$

$$D = \begin{bmatrix} 8 \times 4 & -2 \times 4 & 1 \times 4 \\ 2 \times 4 & -1 \times 4 & 3 \times 4 \\ 1 \times 4 & 1 \times 4 & 4 \times 4 \end{bmatrix} = \begin{bmatrix} 32 & -8 & 4 \\ 8 & -4 & 12 \\ 4 & 4 & 16 \end{bmatrix}$$

$$\Rightarrow |D| = \begin{vmatrix} 32 & -8 & 4 \\ 8 & -4 & 12 \\ 4 & 4 & 16 \end{vmatrix} = 4^3 \times \begin{vmatrix} 8 & -2 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{vmatrix} = 64 \times -43 = -2752$$

G. If two rows or columns of a matrix are interchanged, the sign of the determinant changes. For instance,

$$A = \begin{bmatrix} 8 & -2 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 8 & -2 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{vmatrix} = -43$$

$$B = \begin{bmatrix} 8 & -2 & 1 \\ 1 & 1 & 4 \\ 2 & -1 & 3 \end{bmatrix} \Rightarrow |B| = \begin{vmatrix} 8 & -2 & 1 \\ 1 & 1 & 4 \\ 2 & -1 & 3 \end{vmatrix} = 43$$

$$C = \begin{bmatrix} 1 & -2 & 8 \\ 3 & -1 & 2 \\ 4 & 1 & 1 \end{bmatrix} \Rightarrow |C| = \begin{vmatrix} 1 & -2 & 8 \\ 3 & -1 & 2 \\ 4 & 1 & 1 \end{vmatrix} = 43$$

3.13 Matrix Inversion

Inverse of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{a \times d - b \times c} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse of a 3×3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right)} adj\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right)$$

$$= \frac{1}{\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right)} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$C_{11} = +\begin{vmatrix} e & f \\ h & i \end{vmatrix}, C_{12} = -\begin{vmatrix} d & f \\ g & i \end{vmatrix}, C_{13} = +\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$C_{21} = -\begin{vmatrix} b & c \\ h & i \end{vmatrix}, C_{22} = +\begin{vmatrix} a & c \\ g & i \end{vmatrix}, C_{23} = -\begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

$$C_{31} = +\begin{vmatrix} b & c \\ e & f \end{vmatrix}, C_{32} = -\begin{vmatrix} a & c \\ d & f \end{vmatrix}, C_{33} = +\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

Let's study some numeric examples.

Example 3.20 Calculate the inverse of $\begin{bmatrix} 7 & 5 \\ 1 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 7 & 5 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{7 \times 2 - 5 \times 1}\begin{bmatrix} 2 & -5 \\ -1 & 7 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2 & -5 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & \frac{-5}{9} \\ \frac{-1}{9} & \frac{7}{9} \end{bmatrix}$$

Example 3.21 Calculate the inverse of $\begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

$$\det\left(\begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{bmatrix}\right) = 0\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 0 - 0 + 1(2 + 1) = 3$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} +\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} \end{bmatrix}^{T}$$

$$= \frac{1}{3}\begin{bmatrix} -7 & -5 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{bmatrix}^{T} = \begin{bmatrix} \frac{-7}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{-5}{3} & \frac{-1}{3} & \frac{2}{3} \\ 1 & 0 & 0 \end{bmatrix}$$

Example 3.22 Calculate the inverse of $\begin{bmatrix} -4 & -6 & 2 \\ 5 & -1 & 3 \\ -2 & 4 & -3 \end{bmatrix}$.

$$\det\left(\begin{bmatrix} -4 & -6 & 2 \\ 5 & -1 & 3 \\ -2 & 4 & -3 \end{bmatrix}\right) = -4\begin{vmatrix} -1 & 3 \\ 4 & -3 \end{vmatrix} - (-6)\begin{vmatrix} 5 & 3 \\ -2 & -3 \end{vmatrix} + 2\begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix}$$

$$
\begin{aligned}
&= -4(3-12) + 6(-15+6) + 2(20-2) \\
&= 36 - 54 + 36 = 18
\end{aligned}
$$

$$
\begin{bmatrix} -4 & -6 & 2 \\ 5 & -1 & 3 \\ -2 & 4 & -3 \end{bmatrix}^{-1} = \frac{1}{18}\begin{bmatrix} +\begin{vmatrix} -1 & 3 \\ 4 & -3 \end{vmatrix} & -\begin{vmatrix} 5 & 3 \\ -2 & -3 \end{vmatrix} & +\begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix} \\ -\begin{vmatrix} -6 & 2 \\ 4 & -3 \end{vmatrix} & +\begin{vmatrix} -4 & 2 \\ -2 & -3 \end{vmatrix} & -\begin{vmatrix} -4 & -6 \\ -2 & 4 \end{vmatrix} \\ +\begin{vmatrix} -6 & 2 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} -4 & 2 \\ 5 & 3 \end{vmatrix} & +\begin{vmatrix} -4 & -6 \\ 5 & -1 \end{vmatrix} \end{bmatrix}^T
$$

$$
= \frac{1}{18}\begin{bmatrix} -9 & 9 & 18 \\ -10 & -8 & 4 \\ -16 & 22 & 34 \end{bmatrix}^T
$$

$$
= \frac{1}{18}\begin{bmatrix} -9 & -10 & -16 \\ 9 & -8 & 22 \\ 18 & 4 & 34 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{-5}{9} & \frac{-8}{9} \\ \frac{1}{2} & \frac{-4}{9} & \frac{11}{9} \\ 1 & \frac{2}{9} & \frac{17}{9} \end{bmatrix}
$$

3.14 Solving Systems of Equations Using Matrices

This section shows how matrix method can be used to solve system of linear equations. The following examples show how to solve a system of linear equations using the matrix method.

Example 3.23 Solve $\begin{cases} x + 2y = 3 \\ 4x + 5y = 6 \end{cases}$.

$$
\begin{aligned}
\begin{cases} x + 2y = 3 \\ 4x + 5y = 6 \end{cases} &\Rightarrow \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 6 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 6 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \frac{1}{1 \times 5 - 2 \times 4}\begin{bmatrix} 5 & -2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix}
\end{aligned}
$$

$$= \frac{1}{-3}\begin{bmatrix} 5\times 3-2\times 6 \\ -4\times 3+1\times 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$
$$\Rightarrow x=-1,\quad y=2.$$

You can use the following method as well:

$$\begin{cases} x+2y=3 \\ 4x+5y=6 \end{cases} \Rightarrow \begin{cases} -4x-8y=-12 \\ 4x+5y=6 \end{cases}$$
$$\Rightarrow (-4x-8y)+(4x+5y)=-12+6$$
$$\Rightarrow -3y=-6 \Rightarrow y=2.$$

$$x+2y=3 \Rightarrow x+2\times 2=3 \Rightarrow x=3-4=-1 \Rightarrow x=-1$$

The following method is another option.

$$\begin{cases} x+2y=3 \\ 4x+5y=6 \end{cases} \Rightarrow x=3-2y$$

$$4x+5y=6 \Rightarrow 4(3-2y)+5y=6 \Rightarrow 12-8y+5y=6 \Rightarrow -3y=-6 \Rightarrow y=2$$

$$x=3-2y=3-2\times 2=3-4=-1 \Rightarrow x=-1.$$

Example 3.24 Solve $\begin{cases} x+2y+3z=10 \\ 4x+6y+5z=12 \\ 7x+9y+8z=13 \end{cases}$.

$$\begin{cases} x+2y+3z=10 \\ 4x+6y+5z=12 \\ 7x+9y+8z=13 \end{cases} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 13 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix}^{-1}\begin{bmatrix} 10 \\ 12 \\ 13 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0\,0 \\ 0 & 1\,0 \\ 0 & 0\,1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2\,3 \\ 4 & 6\,5 \\ 7 & 9\,8 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 12 \\ 13 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2\,3 \\ 4 & 6\,5 \\ 7 & 9\,8 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 12 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2\,3 \\ 4 & 6\,5 \\ 7 & 9\,8 \end{bmatrix}^{-1} = \frac{1}{\det\left(\begin{bmatrix} 1 & 2\,3 \\ 4 & 6\,5 \\ 7 & 9\,8 \end{bmatrix}\right)} \times adj\left(\begin{bmatrix} 1 & 2\,3 \\ 4 & 6\,5 \\ 7 & 9\,8 \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} 1 & 2\,3 \\ 4 & 6\,5 \\ 7 & 9\,8 \end{bmatrix}\right) = 1 \times \begin{vmatrix} 6\,5 \\ 9\,8 \end{vmatrix} - 2 \times \begin{vmatrix} 4\,5 \\ 7\,8 \end{vmatrix} + 3 \times \begin{vmatrix} 4\,6 \\ 7\,9 \end{vmatrix}$$

$$= 1 \times (6 \times 8 - 5 \times 9) - 2 \times (4 \times 8 - 5 \times 7) + 3 \times (4 \times 9 - 6 \times 7)$$

$$= 3 + 6 - 18 = -9$$

$$adj\left(\begin{bmatrix} 1 & 2\,3 \\ 4 & 6\,5 \\ 7 & 9\,8 \end{bmatrix}\right) = \begin{bmatrix} +\begin{vmatrix} 6\,5 \\ 9\,8 \end{vmatrix} & -\begin{vmatrix} 4\,5 \\ 7\,8 \end{vmatrix} & +\begin{vmatrix} 4\,6 \\ 7\,9 \end{vmatrix} \\ -\begin{vmatrix} 2\,3 \\ 9\,8 \end{vmatrix} & +\begin{vmatrix} 1\,3 \\ 7\,8 \end{vmatrix} & -\begin{vmatrix} 1\,2 \\ 7\,9 \end{vmatrix} \\ +\begin{vmatrix} 2\,3 \\ 6\,5 \end{vmatrix} & -\begin{vmatrix} 1\,3 \\ 4\,5 \end{vmatrix} & +\begin{vmatrix} 1\,2 \\ 4\,6 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & 3 & -6 \\ 11 & -13 & 5 \\ -8 & 7 & -2 \end{bmatrix}^T = \begin{bmatrix} 3 & 11 & -8 \\ 3 & -13 & 7 \\ -6 & 5 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2\,3 \\ 4 & 6\,5 \\ 7 & 9\,8 \end{bmatrix}^{-1} = \frac{1}{\det\left(\begin{bmatrix} 1 & 2\,3 \\ 4 & 6\,5 \\ 7 & 9\,8 \end{bmatrix}\right)} \times adj\left(\begin{bmatrix} 1 & 2\,3 \\ 4 & 6\,5 \\ 7 & 9\,8 \end{bmatrix}\right)$$

$$= \frac{1}{-9}\begin{bmatrix} 3 & 11 & -8 \\ 3 & -13 & 7 \\ -6 & 5 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & -\frac{11}{9} & \frac{8}{9} \\ -\frac{1}{3} & \frac{13}{9} & -\frac{7}{9} \\ \frac{2}{3} & -\frac{5}{9} & \frac{2}{9} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 12 \\ 13 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & -\frac{11}{9} & \frac{8}{9} \\ -\frac{1}{3} & \frac{13}{9} & -\frac{7}{9} \\ \frac{2}{3} & -\frac{5}{9} & \frac{2}{9} \end{bmatrix} \times \begin{bmatrix} 10 \\ 12 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-58}{9} \\ \frac{35}{9} \\ \frac{26}{9} \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-58}{9} \\ \frac{35}{9} \\ \frac{26}{9} \end{bmatrix} = \begin{bmatrix} -6.4444 \\ 3.8889 \\ 2.8889 \end{bmatrix}$$

The following method is another option.

$$\begin{cases} x + 2y + 3z = 10 \\ 4x + 6y + 5z = 12 \\ 7x + 9y + 8z = 13 \end{cases} \Rightarrow x + 2y + 3z = 10 \Rightarrow x = 10 - 2y - 3z$$

$$\begin{cases} 4x + 6y + 5z = 12 \\ 7x + 9y + 8z = 13 \end{cases} \Rightarrow \begin{cases} 4(10 - 2y - 3z) + 6y + 5z = 12 \\ 7(10 - 2y - 3z) + 9y + 8z = 13 \end{cases}$$

$$\Rightarrow \begin{cases} 40 - 8y - 12z + 6y + 5z = 12 \\ 70 - 14y - 21z + 9y + 8z = 13 \end{cases}$$

$$\Rightarrow \begin{cases} -2y - 7z = -28 \\ -5y - 13z = -57 \end{cases} \Rightarrow -2y - 7z = -28$$

$$\Rightarrow y = \frac{-28 + 7z}{-2} = 14 - 3.5z$$

$$-5y - 13z = -57 \Rightarrow -5(14 - 3.5z) - 13z = -57$$

$$\Rightarrow -70 + 17.5z - 13z = -57$$

$$\Rightarrow 4.5z = 13 \Rightarrow z = \frac{13}{4.5} = \frac{26}{9} = 2.8889$$

$$y = 14 - 3.5z = 14 - 3.5 \times 2.8889 = 14 - 10.1111 = 3.8889$$

$$x = 10 - 2y - 3z = 10 - 2 \times 3.8889 - 3 \times 2.8889 = -6.4444$$

3.15 Underdetermined System of Linear Equations

An underdetermined system of equations is a system where there are fewer equations than unknowns. This means that there are more variables than constraints, leading to infinitely many solutions or no solutions at all. An underdetermined system cannot have a unique solution.

Row Reduction (Gauss-Jordan elimination) can be used to determine whether an undetermined system has no solution or infinitely many solutions: Write the augmented matrix of the system. Perform row operations to reduce the matrix to row-echelon form or reduced row-echelon form. Analyze the resulting matrix: If you encounter a row of the form $[00...0|c]$ where c is nonzero, the system is inconsistent and has no solution. If you have a row of zeros on the left side and a zero on the right side (a row of the form $[00...0|0]$), the system has infinitely many solutions.

Example 3.25 $2x + 3y = 5$ has two unknowns (x and y) but only one equation. In this case, there are infinitely many solutions. Any pair of numbers that adds up to 5 is a valid solution. For example, $(1, 4)$, $(2, 3)$, $(3, 2)$, and so on, are all solutions to this system.

Example 3.26 $\begin{cases} x + y + z = 1 \\ 2x + 2y + 2z = 2 \end{cases}$ has three unknowns (x, y and z) but only two equations. There are infinitely many solutions to this system. Any combination of x, y, and z that satisfies the first equation will also satisfy the second equation. For example, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are all valid solutions.

Example 3.27 $\begin{cases} x + y + z = 1 \\ 2x + 2y + 2z = 3 \end{cases}$ has three unknowns (x, y and z) but only two equations. As you can see, the second equation is a multiple of the first, but the right-hand side values are inconsistent. This means there's no combination of x, y, and z that can satisfy both equations simultaneously.

Example 3.28 Determine the number of solutions for $\begin{cases} x + 2y + 3z = 1 \\ 11x + 12y + 3z = 2 \end{cases}$.

Let's assume the given underdetermined system has infinitely many solutions.

$$\begin{cases} x + 2y + 3z = 1 \\ 11x + 12y + 3z = 2 \end{cases} \Rightarrow \begin{cases} x + 2y = 1 - 3z \\ 11x + 12y = 2 - 3z \end{cases} \Rightarrow \begin{bmatrix} 1 & 2 \\ 11 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 - 3z \\ 2 - 3z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 11 & 12 \end{bmatrix}^{-1} \begin{bmatrix} 1 - 3z \\ 2 - 3z \end{bmatrix}$$

$$= \begin{bmatrix} -1.2 & 0.2 \\ 1.1 & -0.1 \end{bmatrix} \begin{bmatrix} 1 - 3z \\ 2 - 3z \end{bmatrix}$$

$$= \begin{bmatrix} -1.2 & 0.2 \\ 1.1 & -0.1 \end{bmatrix} \begin{bmatrix} 1 - 3z \\ 2 - 3z \end{bmatrix} = \begin{bmatrix} 3z - 0.8 \\ 0.9 - 3z \end{bmatrix}$$

$$\Rightarrow \begin{cases} x = 3t - 0.8 \\ y = 0.9 - 3t \\ z = t \end{cases}$$

$$\begin{cases} x + 2y + 3z = 1 \\ 11x + 12y + 3z = 2 \end{cases} \Rightarrow \begin{cases} 3t - 0.8 + 2(0.9 - 3t) + 3t = 1 \\ 11(3t - 0.8) + 12(0.9 - 3t) + 3t = 2 \end{cases} \Rightarrow \begin{cases} 1 = 1 \\ 2 = 2 \end{cases}$$

Therefore, the assumption of having infinitely many solutions is correct.

3.16 Overdetermined System of Linear Equations

An overdetermined system of linear equations is a system where there are more equations than unknowns. This means that there are more constraints than degrees of freedom, making it unlikely that an exact solution exists that satisfies all equations simultaneously.

Let's study some numeric examples.

Example 3.29 Consider the following system of linear equations:

$$\begin{cases} x + y = 3 \\ 2x + 3y = 7 \\ x + y = 4 \end{cases}$$

This system has 3 equations and 2 unknowns. Let's solve a subsystem of two equations and two unknowns. For instance, if we select first and second equations:

$$\begin{cases} x + y = 3 \\ 2x + 3y = 7 \end{cases} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \frac{1}{3-2} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Substituting the solution into the third equation yields a contradiction since $x + y = 2 + 1 = 3 \neq 4$.

Let's select another subsystem of two equations and two unknowns: $\begin{cases} x + y = 3 \\ x + y = 4 \end{cases}$. This subsystem has no solution.

The last subsystem of two equations and two unknowns is $\begin{cases} 2x + 3y = 7 \\ x + y = 4 \end{cases}$.

$$\begin{cases} 2x + 3y = 7 \\ x + y = 4 \end{cases} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \frac{1}{2-3} \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

The solution fails to satisfy the remaining equation since $x + y = 5 - 1 \neq 3$.

Therefore, the given system does not have an exact solution that satisfies all equations simultaneously. However, we can find an approximate solution that minimizes the sum of the squared errors. The cost function is defined as:

$$f(x, y) = (x + y - 3)^2 + (2x + 3y - 7)^2 + (x + y - 4)^2$$

Let's find the critical point of cost function:

$$\frac{\partial f}{\partial x} = 2(x + y - 3) + 4(2x + 3y - 7) + 2(x + y - 4) = 12x + 16y - 42$$

$$\frac{\partial f}{\partial y} = 2(x + y - 3) + 6(2x + 3y - 7) + 2(x + y - 4) = 16x + 22y - 56$$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \Rightarrow 12x + 16y - 42 = 0 \Rightarrow 6x + 8y = 21 \\ \frac{\partial f}{\partial y} = 0 \Rightarrow 16x + 22y - 56 = 0 \Rightarrow 8x + 11y = 28 \end{cases} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.5 \\ 0 \end{bmatrix}$$

The critical point is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.5 \\ 0 \end{bmatrix}$. Let's determine the type of obtained critical point. Remember that critical point is:

A) a local minimum if $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$,
B) a local maximum if $D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$, and,
C) a saddle point if $D < 0$.

Note that D shows the discriminant and $D = \frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$. If $D = 0$, the test is inconclusive. For given cost function:

$$\frac{\partial^2 f}{\partial x^2} = 12$$
$$\frac{\partial^2 f}{\partial y^2} = 22$$
$$\frac{\partial^2 f}{\partial x \partial y} = 16$$

$$D = \frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 12 \times 22 - 16^2 = 8$$

Obtained critical point is global minimum since $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$. Value of cost function at critical point is $f(3.5, 0) = 0.5$.

Therefore, $x = 3.5$ and $y = 0$ is the approximate solution to the overdetermined system $\begin{cases} x + y = 3 \\ 2x + 3y = 7 \\ x + y = 4 \end{cases}$.

For the overdetermined system of linear equations $Ax = b$, the approximate solution that minimizes the sum of squared errors is $x = \left(A^T A\right)^{-1} A^T b$. Note that A^T is the transpose of matrix A.

Example 3.30 Approximate solution of $\begin{cases} x + y = 3 \\ 2x + 3y = 7 \\ x + y = 4 \end{cases}$ is:

$$\begin{cases} x + y = 3 \\ 2x + 3y = 7 \\ x + y = 4 \end{cases} \Rightarrow A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = b \text{ where } A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \left(A^T A\right)^{-1} A^T b = \left(\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 \\ 8 & 11 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 8 & 11 \end{bmatrix}^{-1} \begin{bmatrix} 21 \\ 28 \end{bmatrix}$$

$$= \frac{1}{66 - 64} \begin{bmatrix} 11 & -8 \\ -8 & 6 \end{bmatrix} \begin{bmatrix} 21 \\ 28 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 0 \end{bmatrix}$$

Therefore, approximate solution of this overdetermined system is: $x = 3.5$ and $y = 0$.

3.17 Square System of Equations

A linear system of equations in which the number of variables equals the number of equations. The coefficient matrix of a square system is a square matrix.

A square system of linear equations has a unique solution if and only if the determinant of its coefficient matrix is non-zero.

When the determinant of a system of linear equations is zero, the system has either no solutions or infinitely many solutions. If the system has no solutions, it is said to be inconsistent. If the system has infinitely many solutions, it is said to be dependent. This occurs when at least one row of coefficient matrix can be expressed as a linear combination of the other rows.

Row Reduction (Gauss-Jordan elimination) can be used to determine whether a system of linear equations with a zero determinant has no solution or infinitely many solutions: Write the augmented matrix of the system. Perform row operations to reduce the matrix to row-echelon form or reduced row-echelon form. Analyze the resulting matrix: If you encounter a row of the form $[00...0|c]$ where c is nonzero, the system is inconsistent and has no solution. If you have a row of zeros on the left side and a zero on the right side (a row of the form $[00...0|0]$), the system is dependent and has infinitely many solutions.

Example 3.31 $\begin{cases} x+2y+3z=10 \\ 4x+6y+5z=12 \\ 7x+9y+8z=13 \end{cases}$. can be written as: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 13 \end{bmatrix}$.
Determinant of coefficients is non-zero, so a unique solution exists.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{vmatrix} = -9$$

Let's use the Gauss-Jordan elimination method to find the solution. The augmented matrix is:

$$\begin{bmatrix} 1 & 2 & 3 & 10 \\ 4 & 6 & 5 & 12 \\ 7 & 9 & 8 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 10 \\ 0 & -2 & -7 & -28 \\ 7 & 9 & 8 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 10 \\ 0 & -2 & -7 & -28 \\ 0 & -5 & -13 & -57 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & -18 \\ 0 & -2 & -7 & -28 \\ 0 & -5 & -13 & -57 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & -18 \\ 0 & -2 & -7 & -28 \\ 0 & 0 & 4.5 & 13 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -4 & -18 \\ 0 & -2 & 0 & -7.7772 \\ 0 & 0 & 4.5 & 13 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -6.4443 \\ 0 & -2 & 0 & -7.7772 \\ 0 & 0 & 4.5 & 13 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -6.4443 \\ 0 & 1 & 0 & 3.8886 \\ 0 & 0 & 1 & 2.8889 \end{bmatrix}$$

Therefore, solution of the given system is $x = -6.4443$, $y = 3.8886$ and $z = 2.8889$.

Example 3.32 $\begin{cases} 2x+y+z=1 \\ 4x+2y+2z=3 \\ 6x+3y+3z=5 \end{cases}$ can be written as: $\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$. Determinant of coefficients is zero:

$$\begin{vmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} - 1\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0$$

Therefore, the given system may be inconsistent, i.e., have no solution, or it may be dependent, i.e. have infinitely many solutions. The augmented matrix is:

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 4 & 2 & 2 & 3 \\ 6 & 3 & 3 & 5 \end{bmatrix}$$

Basic row operations lead to:

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 4 & 2 & 2 & 3 \\ 6 & 3 & 3 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 6 & 3 & 2 & 5 \end{bmatrix}$$

Since the second has a nonzero entry on the right side, the system is inconsistent and has no solution.

Example 3.33 $\begin{cases} 2x+y+z=1 \\ 4x+2y+2z=2 \\ 6x+3y+3z=3 \end{cases}$ can be written as: $\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Determinant of coefficients is zero:

$$\begin{vmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 6 & 3 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} - 1\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0$$

The given system may be inconsistent, or it may be dependent. The augmented matrix is:

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 4 & 2 & 2 & 2 \\ 6 & 3 & 3 & 3 \end{bmatrix}$$

Basic row operations lead to:

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 4 & 2 & 2 & 2 \\ 6 & 3 & 3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 6 & 3 & 2 & 5 \end{bmatrix}$$

Since the second row is entirely zeros, the system is dependent and has infinitely many solutions. Let's find the solutions:

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 4 & 2 & 2 & 2 \\ 6 & 3 & 3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 6 & 3 & 2 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow 2x + y + z = 1 \Rightarrow \begin{cases} x = t \\ y = u \\ z = 1 - 2t - y \end{cases}$$

For instance, $\begin{bmatrix} 3 \\ 4 \\ -9 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$ satisfy the given system.

Example 3.34 $\begin{cases} x + 2y + 3z = 1 \\ 4x + 8y + 12z = 4 \\ 11x + 12y + 3z = 2 \end{cases}$ can be written as: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \\ 11 & 12 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$.
Determinant of coefficients is zero:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 8 & 12 \\ 11 & 12 & 3 \end{vmatrix} = 1\begin{vmatrix} 8 & 12 \\ 12 & 3 \end{vmatrix} - 2\begin{vmatrix} 4 & 12 \\ 11 & 3 \end{vmatrix} + 3\begin{vmatrix} 4 & 8 \\ 11 & 12 \end{vmatrix}$$

$$= 1(24 - 144) - 2(12 - 132) + 3(48 - 88) = 0$$

The given system may be inconsistent, or it may be dependent. The augmented matrix is:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 8 & 12 & 4 \\ 11 & 12 & 3 & 2 \end{bmatrix}$$

Basic row operations lead to:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 8 & 12 & 4 \\ 11 & 12 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 11 & 12 & 3 & 2 \end{bmatrix}$$

Since the second row is entirely zeros, the system is dependent and has infinitely many solutions. Let's find the solutions:

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 8 & 12 & 4 \\ 11 & 12 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 11 & 12 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -10 & -30 & -9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0.9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -3 & -0.8 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0.9 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x - 3z = -0.8 \\ y + 3z = 0.9 \end{cases} \Rightarrow \begin{cases} x = 3t - 0.8 \\ y = -3t + 0.9 \\ z = t \end{cases}$$

For instance, $\begin{bmatrix} -0.8 \\ 0.9 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2.2 \\ -2.1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -6.8 \\ 6.9 \\ -2 \end{bmatrix}$ satisfy the given system.

3.18 Rank of a Matrix

The rank of a matrix represents the maximum number of linearly independent rows or columns in the matrix. The rank of a matrix is not limited to square matrices. The maximum possible rank of an $m \times n$ matrix is the minimum of m and n.

The rank of a matrix may be determined by employing the following methods:

(A) Row reduction method: Reduce the matrix to row-echelon form or reduced row-echelon form. The number of non-zero rows in the resulting matrix is the rank.
(B) Determinant method: For square matrices, the rank is equal to the size of the largest non-zero minor.

Example 3.35 Calculate the rank of $A = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 2 & -1 \\ 7 & 5 & 5 \end{bmatrix}$.

Given matrix is a square matrix and its determinant is $\begin{vmatrix} 1 & 0 & 5 \\ 2 & 2 & -1 \\ 7 & 5 & 5 \end{vmatrix} = 3$. Therefore, given matrix is full rank (its rank is 3).

Example 3.36 Calculate the rank of $A = \begin{bmatrix} 1 & 1 & 2 \\ 5 & 0 & -1 \\ 7 & 2 & 3 \end{bmatrix}$.

Given matrix is a square matrix and its determinant is $\begin{vmatrix} 1 & 1 & 2 \\ 5 & 0 & -1 \\ 7 & 2 & 3 \end{vmatrix} = 0$. So, the given matrix is not full rank. For instance, $M_{11} = \begin{vmatrix} 0 & -1 \\ 2 & 3 \end{vmatrix} = 0 + 2 = 2 \neq 0$. Therefore, rank of this matrix is 2. We can stop checking other minors since we found a full-rank sub-matrix.

Example 3.37 Calculate the rank of $A = \begin{bmatrix} 1 & 2 & -5 \\ 3 & 6 & -15 \\ 2 & 4 & -10 \end{bmatrix}$.

Given matrix is a square matrix and its determinant is $\begin{vmatrix} 1 & 2 & -5 \\ 3 & 6 & -15 \\ 2 & 4 & -10 \end{vmatrix} = 0$. So, the given matrix is not full rank. Let's see if there exists at least one full-rank 2×2 sub-matrix.2×2 sub-matrices are:

$$\begin{bmatrix} 6 & -15 \\ 4 & -10 \end{bmatrix}, \begin{bmatrix} 3 & -15 \\ 2 & -10 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -5 \\ 4 & -10 \end{bmatrix}, \begin{bmatrix} 1 & -5 \\ 2 & 10 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -5 \\ 6 & -15 \end{bmatrix}, \begin{bmatrix} 1 & -5 \\ 3 & -15 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

However, none of them are full ranked:

$$\begin{vmatrix} 6 & -15 \\ 4 & -10 \end{vmatrix} = 0, \begin{vmatrix} 3 & -15 \\ 2 & -10 \end{vmatrix} = 0, \begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} = 0$$
$$\begin{vmatrix} 2 & -5 \\ 4 & -10 \end{vmatrix} = 0, \begin{vmatrix} 1 & -5 \\ 2 & 10 \end{vmatrix} = 0, \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$$
$$\begin{vmatrix} 2 & -5 \\ 6 & -15 \end{vmatrix} = 0, \begin{vmatrix} 1 & -5 \\ 3 & -15 \end{vmatrix} = 0, \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$$

Therefore, rank of this matrix is 1.

Alternatively, let's apply the row reduction method to the given examples.

Example 3.38 Calculate the rank of $A = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 2 & -1 \\ 7 & 5 & 5 \end{bmatrix}$ with row reduction method.

$$\begin{bmatrix} 1 & 0 & 5 \\ 2 & 2 & -1 \\ 7 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -11 \\ 7 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -11 \\ 0 & 5 & -30 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 2 & -11 \\ 0 & 0 & -2.5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -11 \\ 0 & 0 & -2.5 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2.5 \end{bmatrix}$$

Number of non-zero rows is 3. Therefore, the rank is 3.

Example 3.39 Calculate the rank of $A = \begin{bmatrix} 1 & 1 & 2 \\ 5 & 0 & -1 \\ 7 & 2 & 3 \end{bmatrix}$ with row reduction method.

$$\begin{bmatrix} 1 & 1 & 2 \\ 5 & 0 & -1 \\ 7 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -11 \\ 7 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -11 \\ 0 & -5 & -11 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & -11 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{-1}{5} \\ 0 & -5 & -11 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows is 2. Therefore, the rank is 2.

Example 3.40 Calculate the rank of $A = \begin{bmatrix} 1 & 2 & -5 \\ 3 & 6 & -15 \\ 2 & 4 & -10 \end{bmatrix}$ with row reduction method.

$$\begin{bmatrix} 1 & 2 & -5 \\ 3 & 6 & -15 \\ 2 & 4 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -5 \\ 0 & 0 & 0 \\ 2 & 4 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows is 1. Therefore, the rank is 1.

Example 3.41 Calculate the rank of $A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 7 & 6 & 3 & 2 \\ 1 & 1 & 4 & 9 \end{bmatrix}$ with row reduction method.

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 7 & 6 & 3 & 2 \\ 1 & 1 & 4 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -8 & 17 & -5 \\ 1 & 1 & 4 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -8 & 17 & -5 \\ 0 & -1 & 6 & 8 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -8 & 17 & -5 \\ 0 & 0 & 31/8 & 69/8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & -8 & 17 & -5 \\ 0 & 0 & 1 & 69/31 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 9/4 & -1/4 \\ 0 & -8 & 17 & -5 \\ 0 & 0 & 1 & 69/31 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -652/31 \\ 0 & -8 & 17 & -5 \\ 0 & 0 & 1 & 69/31 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & -652/124 \\ 0 & -8 & 0 & -1328/31 \\ 0 & 0 & 1 & 69/31 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -652/124 \\ 0 & 1 & 0 & 1328/248 \\ 0 & 0 & 1 & 69/31 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -5.2581 \\ 0 & 1 & 0 & 5.3548 \\ 0 & 0 & 1 & 2.2258 \end{bmatrix}$$

Number of non-zero rows is 3. Therefore, the rank is 3.

3.19 Eigenvalues and Eigenvectors of a Matrix

"Eigen" is a German word. It means "own" or "self". In the context of linear algebra, it's used to describe characteristics or properties inherent to a matrix. Eigenvalues and eigenvectors are defined exclusively for square matrices, that is, matrices with an equal number of rows and columns.

An eigenvalue of a square matrix A is a scalar λ such that: $A.v = \lambda v$ where A is a square matrix and v is a non-zero vector, known as the eigenvector corresponding to λ. In other words, an eigenvector of a square matrix A is a non-zero vector v such that, when multiplied by A, it yields a scalar multiple of itself. The scalar multiple is the corresponding eigenvalue.

A vector w that satisfies the $(A - \lambda I)^k w = 0$ for some non-negative integer k is called generalized eigenvector of matrix A. Note that A is a square matrix, λ is an eigenvalue of A, I is the identity matrix, and k is the smallest integer for which the equation holds. Every eigenvector is also a generalized eigenvector. However, not all generalized eigenvectors are eigenvectors.

Following tips help you to speed up in calculations:

A. For a 2 × 2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $|A - \lambda I| = \lambda^2 - (a_{11} + a_{22})\lambda + |A| = 0$. $|A|$ shows the determinant $\left(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}.a_{22} - a_{12}.a_{21}\right)$. For instance, for $A = \begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix}$, $|A - \lambda I| = \lambda^2 - (-1 + 3)\lambda + (-3 - 12) = \lambda^2 - 2\lambda - 15$. The solutions of the equation $\lambda^2 - 2\lambda - 15 = 0$ are the eigenvalues of this matrix.

Exercise: Determine the characteristic equation of $A = \begin{bmatrix} 1 & 3 \\ -1 & 7 \end{bmatrix}$.

B. For a 3 × 3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $|A - \lambda I| = -\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2 - \left(\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right)\lambda + |A| = 0.$ $|A|$ shows the determinant of the matrix A. For instance, for $A = A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$, $|A - \lambda I| = -\lambda^3 + (2 + 3 + 4)\lambda^2 - \left(\begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}\right)\lambda + \begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{vmatrix} = -\lambda^3 + 9\lambda^2 - (12 - 6 + 8 - 3 + 6 - 2)\lambda + 7 = -\lambda^3 + 9\lambda^2 - 15\lambda + 7$. The solutions of the equation $-\lambda^3 + 9\lambda^2 - 15\lambda + 7 = 0$ are the eigenvalues of this matrix.

Exercise: Determine the characteristic equation of $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 7 & 0 \\ 6 & 6 & 9 \end{bmatrix}$.

3.20 Calculation of Eigenvalues and Eigenvectors

Let’s study some numeric examples.

Example 3.42 Calculate the eigenvalues and eigenvectors of matrix $A = \begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix}$.

Let’s start by calculating the eigenvalues (=roots of characteristic equation):

$$A = \begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix} \Rightarrow |A - \lambda I| = 0 \Rightarrow \left|\begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = \begin{vmatrix} -1-\lambda & 6 \\ 2 & 3-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(3-\lambda) - 6 \times 2 = \lambda^2 - 2\lambda - 15 = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = -3$$

Eigen vector associated with $\lambda_1 = 5$ is:

$$(A - \lambda_1 I)V_1 = 0 \Rightarrow \left(\begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \left(\begin{bmatrix} -6 & 6 \\ 2 & -2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -6x_1 + 6x_2 = 0 \Rightarrow x_1 = x_2$$

Therefore, $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $V_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, $V_1 = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$ or any vector with $V_1 = \begin{bmatrix} k \\ k \end{bmatrix}$ form, can be considered as an eigenvector for $\lambda_1 = 5$ case. Let's make a unit eigenvector. The vector $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ can be converted into a unit vector using the $\frac{1}{\sqrt{\alpha^2+\beta^2}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \\ \frac{\beta}{\sqrt{\alpha^2+\beta^2}} \end{bmatrix}$ formula. The vector $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ can be converted into a unit vector using the $\frac{1}{\sqrt{\alpha^2+\beta^2+\gamma^2}} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{\sqrt{\alpha^2+\beta^2+\gamma^2}} \\ \frac{\beta}{\sqrt{\alpha^2+\beta^2+\gamma^2}} \\ \frac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}} \end{bmatrix}$ formula. Unit eigenvector associated with $\lambda_1 = 5$ is:

$$V_1 = \frac{1}{\sqrt{3^2+3^2}} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}.$$

Eigen vector associated with $\lambda_2 = -3$ is:

$$(A - \lambda_2 I)V_2 = 0 \Rightarrow \left(\begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \left(\begin{bmatrix} 2 & 6 \\ 2 & 6 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x_1 + 6x_2 = 0$$
$$\Rightarrow x_1 = -3x_2$$

$V_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$ or $V_2 = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$ can be selected as eigenvector associated with $\lambda_2 = -3$. Let's select one of the mentioned vectors and make it a unit. Unit eigenvector associated with $\lambda_2 = -3$ is:

$$V_2 = \frac{1}{\sqrt{(-3)^2 + 1^2}} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.9487 \\ 0.3162 \end{bmatrix}$$

Example 3.43 Calculate the eigenvalues and eigenvectors of matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

$$\begin{aligned} A &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow |A - \lambda I| = 0 \\ &\Rightarrow \left| \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)(3-\lambda) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3 \end{aligned}$$

Eigen vector associated with $\lambda_1 = 2$ is:

$$\begin{aligned} (A - \lambda_1 I)V_1 = 0 &\Rightarrow \left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0 \end{aligned}$$

For instance, we can select $V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$..

Eigenvector associated with $\lambda_2 = 3$ is:

$$\begin{aligned} (A - \lambda_2 I)V_2 = 0 &\Rightarrow \left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\Rightarrow \left(\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -x_1 = 0 \end{aligned}$$

For instance, we can select $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example 3.44 Calculate the eigenvalues and eigenvectors of matrix $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$.

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \Rightarrow |A - \lambda I| = 0 \Rightarrow \left| \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) - (-8) = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda_1 = 1+2j, \lambda_2 = 1-2j$$

Eigen vectors associated with $\lambda_1 = 1+2j$ and $\lambda_2 = 1-2j$ can be calculated as:

$$\lambda_1 = 1+2j \Rightarrow \left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1+2j)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)V_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 1+2j & 0 \\ 0 & 1+2j \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 3-1-2j & -2 \\ 4 & -1-1-2j \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 2-2j & -2 \\ 4 & -2-2j \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (2-2j)x_1 - 2x_2 = 0 \Rightarrow x_2 = (1-j)x_1$$

$$\Rightarrow V_1 = \begin{bmatrix} 1 \\ 1-j \end{bmatrix}$$

$$\lambda_1 = 1-2j \Rightarrow \left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1-2j)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)V_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 1-2j & 0 \\ 0 & 1-2j \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 3-1+2j & -2 \\ 4 & -1-1+2j \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 2+2j & -2 \\ 4 & -2+2j \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (2+2j)x_1 - 2x_2 = 0 \Rightarrow x_2 = (1+j)x_1 \Rightarrow V_2 = \begin{bmatrix} 1 \\ 1+j \end{bmatrix}$$

Example 3.45 Calculate the eigenvalues and eigenvectors of matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \Rightarrow |A - \lambda I| = 0 \Rightarrow \left|\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right|$$

$$= \begin{vmatrix} 1-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - (-2) = 0$$
$$\Rightarrow \lambda^2 - 2\lambda + 3 = 0 \Rightarrow \lambda_1 = 1 + \sqrt{2}j, \lambda_2 = 1 - \sqrt{2}j$$

$$\lambda_1 = 1 + \sqrt{2}j \Rightarrow \left(\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} - \left(1 + \sqrt{2}j\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) V_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \left(\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1+\sqrt{2}j & 0 \\ 0 & 1+\sqrt{2}j \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \left(\begin{bmatrix} -\sqrt{2}j & -2 \\ 1 & -\sqrt{2}j \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -\sqrt{2}jx_1 - 2x_2 = 0$$
$$\Rightarrow x_2 = -\frac{\sqrt{2}}{2}jx_1 \Rightarrow when x_1 = 1 : V_1 = \begin{bmatrix} 1 \\ \frac{-\sqrt{2}}{2}j \end{bmatrix}$$
$$\lambda_2 = 1 - \sqrt{2}j \Rightarrow \left(\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} - \left(1 - \sqrt{2}j\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) V_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \left(\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1-\sqrt{2}j & 0 \\ 0 & 1-\sqrt{2}j \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \left(\begin{bmatrix} \sqrt{2}j & -2 \\ 1 & \sqrt{2}j \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \sqrt{2}jx_1 - 2x_2 = 0$$
$$\Rightarrow x_2 = \frac{\sqrt{2}}{2}jx_1 \Rightarrow when x_1 = 1 : V_2 = \begin{bmatrix} 1 \\ \frac{\sqrt{2}}{2}j \end{bmatrix}$$

For a 2×2 matrix with a repeated eigenvalue, you will either have two linearly independent eigenvectors or one linearly independent eigenvector and a generalized eigenvector.

Example 3.46 Calculate the eigenvalues and eigenvectors of matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow |A - \lambda I| = 0 \Rightarrow \left| \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$
$$= \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 0 = 0$$
$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 2$$

Eigen vectors associated with $\lambda_1 = \lambda_2 = 2$ can be calculated as:

$$\lambda_1 = \lambda_2 = 2 \Rightarrow \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 0x_1 + 0x_2 = 0 \\ 0x_1 + 0x_2 = 0 \end{cases}$$

For instance, we can select $V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example 3.47 Calculate the eigenvalues and eigenvectors of matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \Rightarrow |A - \lambda I| = 0 \Rightarrow \left|\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right|$$
$$= \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - (0 \times 1) = 0$$
$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 2$$

Eigen vectors associated with $\lambda_1 = \lambda_2 = 2$ can be calculated as:

$$\lambda_1 = \lambda_2 = 2 \Rightarrow \left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) V$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_2 = 0 \\ 0x_1 + 0x_2 = 0 \end{cases}$$

For instance, we can select $V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Using this ordinary eigenvector, we compute the generalized eigenvector w by solving:

$$(A - \lambda I)w = V_1 \Rightarrow \left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} w_2 = 1 \\ 0 = 0 \end{cases}$$

We can choose any vector for w that is linearly independent from V_1. For instance, we can select $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Note that w is a generalized eigenvector since:

$$1.(A - \lambda I)w = \left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = V_1$$

$$2.(A - \lambda I)^2 w = \left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $(A - \lambda I)^2 w = (A - \lambda I)(A - \lambda I)w = (A - \lambda I)V_1 = 0$.

Ordinary eigenvector V_1 and generalized eigenvector w constitute a basis for the vector space since they are linearly independent.

Example 3.48 Calculate the eigenvalues and eigenvectors of matrix $A = \begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix}$.

$$A = \begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix} \Rightarrow |A - \lambda I| = 0$$
$$\Rightarrow \left| \begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$
$$= \begin{vmatrix} -8 - \lambda & -5 \\ 5 & 2 - \lambda \end{vmatrix}$$
$$= (-8 - \lambda)(2 - \lambda) - (-5 \times 5) = 0$$
$$\Rightarrow \lambda^2 + 6\lambda + 9 = 0 \Rightarrow \lambda_1 = -3,\ \lambda_2 = -3$$

Eigen vectors associated with $\lambda_1 = \lambda_2 = -3$ can be calculated as:

$$\lambda_1 = \lambda_2 = -3 \Rightarrow \left(\begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix} - (-3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \left(\begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \left(\begin{bmatrix} -5 & -5 \\ 5 & 5 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -5x_1 - 5x_2 = 0 \\ 5x_1 + 5x_2 = 0 \end{cases} \Rightarrow x_1 = -x_2$$

For instance, we can select $V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Using this ordinary eigenvector, we compute the generalized eigenvector w by solving:

$$
\begin{aligned}
(A - \lambda I)w = V_1 &\Rightarrow \left(\begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} -5 & -5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&\Rightarrow \begin{cases} -5w_1 - 5w_2 = 1 \\ 5w_1 + 5w_2 = -1 \end{cases} \Rightarrow 5w_1 + 5w_2 = -1 \Rightarrow w_1 = -w_2 - \frac{1}{5}
\end{aligned}
$$

For instance, we can select $w = \begin{bmatrix} -\frac{1}{5} \\ 0 \end{bmatrix}$. Note that w is a generalized eigenvector since:

$$
1. (A - \lambda I)w = \left(\begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -\frac{1}{5} \\ 1 \end{bmatrix} = \begin{bmatrix} -5 & -5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = V_1
$$

$$
\begin{aligned}
2. (A - \lambda I)^2 w &= \left(\begin{bmatrix} -8 & -5 \\ 5 & 2 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^2 \begin{bmatrix} -\frac{1}{5} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -5 & -5 \\ 5 & 5 \end{bmatrix}^2 \begin{bmatrix} -\frac{1}{5} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

or $(A - \lambda I)^2 w = (A - \lambda I)(A - \lambda I)w = (A - \lambda I)V_1 = 0$.

Ordinary eigenvector V_1 and generalized eigenvector w constitute a basis for the vector space since they are linearly independent.

Exercise: Calculate the eigenvalues and eigenvectors of matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Example 3.49 Calculate the eigenvalues and eigenvectors of matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

$$
A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \Rightarrow |A - \lambda I| \Rightarrow \left| \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|
$$

$$
= \begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix}
$$

$$= (2-\lambda)\begin{vmatrix} 3-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 3 & 4-\lambda \end{vmatrix} + 1\begin{vmatrix} 2 & 3-\lambda \\ 3 & 3 \end{vmatrix}$$
$$= (2-\lambda)[(3-\lambda)(4-\lambda) - 2\times 3] - 1[2\times(4-\lambda) - 2\times 3] + 1[2\times 3 - 3\times(3-\lambda)]$$
$$= -\lambda^3 + 9\lambda^2 - 15\lambda + 7$$

Typically, finding the roots of a third-degree polynomial requires a calculator or computer software. However, in this specific case, we can find the roots manually. Since the sum of the coefficients is zero ($-1+9-15+7=0$), one root is 1. The remaining roots can be determined using polynomial long division. Note that:

$$\frac{-\lambda^3 + 9\lambda^2 - 15\lambda + 7}{\lambda - 1} = -\lambda^2 - 8\lambda - 7 = -(\lambda-1)(\lambda-7)$$

Therefore, the roots of $-\lambda^3 + 9\lambda^2 - 15\lambda + 7 = (\lambda-1)^2(\lambda-7)$ are $1,\ 1$ and 7. Let's calculate the eigenvectors.

For $\lambda_1 = \lambda_2 = 1$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \Rightarrow (A - \lambda I)V = 0 \Rightarrow (A - I)V$$
$$= \left(\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x + y + z = 0$$

For instance, we can select $V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Selected two vectors are linearly independent. Note that in this example, the repeated eigenvalue 1 yields two eigenvectors:

$$A \times V_1 = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 1 \times V_1$$
$$A \times V_2 = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 1 \times V_2$$

For $\lambda_3 = 7$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \Rightarrow (A - \lambda I)V_3 = 0 \Rightarrow (A - 7I)V_3$$

$$= \left(\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -5x + y + z = 0 \\ 2x - 4y + 2z = 0 \\ 3x + 3y - 3z = 0 \end{cases}$$

$\begin{vmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{vmatrix} = 0$. The augmented matrix of this system is:

$$\begin{bmatrix} -5 & 1 & 1 & 0 \\ 2 & -4 & 2 & 0 \\ 3 & 3 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} -5 & 1 & 1 & 0 \\ 0 & -3.6 & 2.4 & 0 \\ 3 & 3 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} -5 & 1 & 1 & 0 \\ 0 & -3.6 & 2.4 & 0 \\ 0 & 3.6 & -2.4 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} -5 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3.6 & -2.4 & 0 \end{bmatrix} \to \begin{bmatrix} -5 & 0 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3.6 & -2.4 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3.6 & -2.4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2/3 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x - \frac{1}{3}z = 0 \\ y - \frac{2}{3}z = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{3}z \\ y = \frac{2}{3}z \end{cases} \Rightarrow \begin{cases} x = \frac{1}{3}t \\ y = \frac{2}{3}t \\ z = t \end{cases}$$

For instance, for $t = 3$ we obtain $V_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

$$A \times V_3 = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \\ 21 \end{bmatrix} = 7 \times V_3$$

Example 3.50 Show that the vectors V_1, V_2 and V_3 found in the previous example are linearly independent.

If the determinant of a square matrix formed by the vectors as columns (or rows) is non-zero, then the vectors are linearly independent. If the determinant is zero, the vectors are linearly dependent. Vectors found in the previous example are independent since:

$$\begin{aligned}\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & 3 \end{vmatrix} &= 1\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} \\ &= 1(1 \times 3 - 2 \times -1) + 1(0 \times -1 - 1 \times -1) \\ &= 5 + 1 = 6 \neq 0\end{aligned}$$

Exercise: Calculate the eigenvalues and eigenvectors of $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

3.21 Problem Solving with MATLAB®

This section provides examples of how MATLAB can be employed to solve the types of problems encountered in this chapter.

Example 3.51 The code in Fig. 3.16 solves $2x + 7 = 0$.

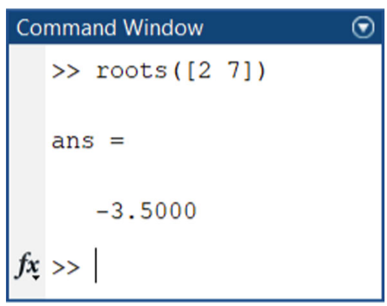
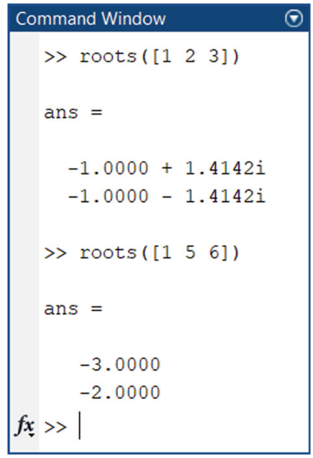


Fig. 3.16 Calculation of $2x + 7$ roots

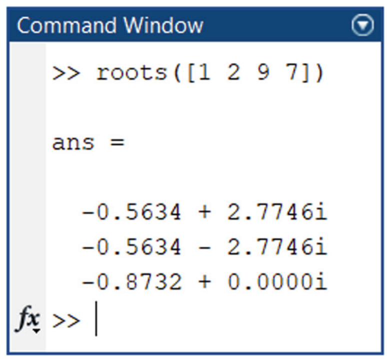
Example 3.52 The code in Fig. 3.17 solves $x^2 + 2x + 3 = 0$.

Fig. 3.17 Calculation of $x^2 + 2x + 3$ and $x^2 + 5x + 6$ roots



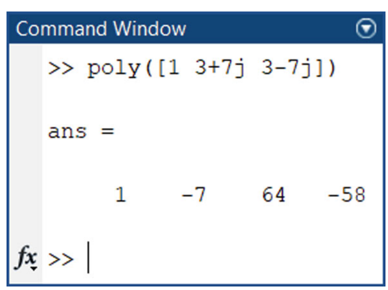
Example 3.53 The code in Fig. 3.18 solves $x^3 + 2x^2 + 9x + 7 = 0$.

Fig. 3.18 Calculation of $x^3 + 2x^2 + 9x + 7$ roots



Example 3.54 The code in Fig. 3.19 calculates the coefficients of the polynomial with roots $1, 3 + 7j$, and $3 - 7j$. According to Fig. 3.19 the resulting polynomial is found to be $x^3 - 7x^2 + 64x - 58 = 0$.

Fig. 3.19 Calculation of coefficients of the polynomial with roots $1, 3 + 7j$, and $3 - 7j$



Example 3.55 The code in Fig. 3.20 calculates the sine, cosine, tangent and cotangent of $\frac{\pi}{3}$ radians.

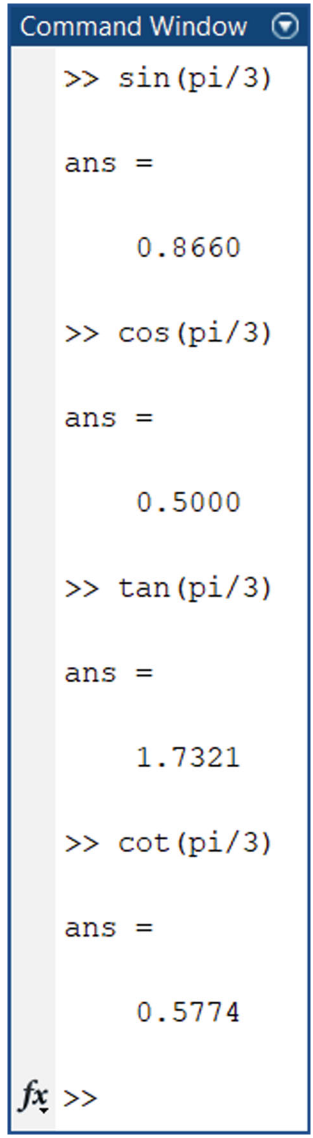


Fig. 3.20 Calculation of trigonometric ratios for $x = \frac{\pi}{3}$

Example 3.56 The code in Fig. 3.21 calculates the sine, cosine, tangent and cotangent of $30°$.

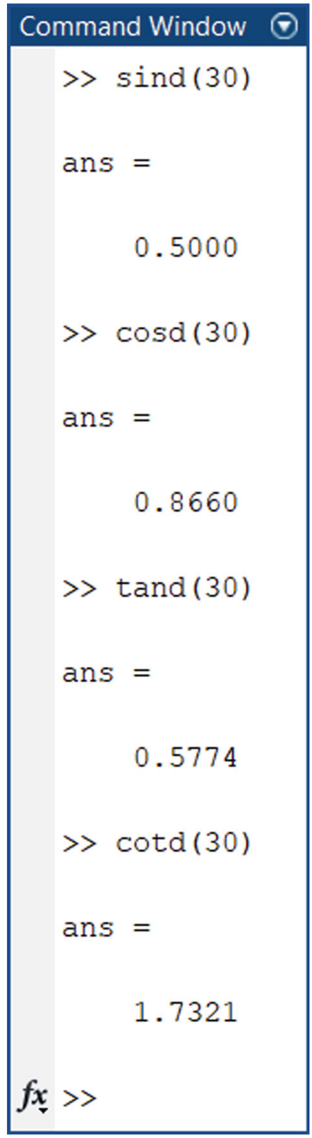


Fig. 3.21 Calculation of trigonometric ratios for $x = 30°$

Example 3.57 The code in Fig. 3.22 calculates the arcsine, arccosine, arctangent and arc cotangent of 0.7, providing results in radians.

Fig. 3.22 Calculation of inverse trigonometric functions for $x = 0.7$ (Results are in radians)

```
Command Window
>> asin(0.7)

ans =

    0.7754

>> acos(0.7)

ans =

    0.7954

>> atan(0.7)

ans =

    0.6107

>> acot(0.7)

ans =

    0.9601

fx >>
```

Example 3.58 The code in Fig. 3.23 calculates the arcsine, arccosine, arctangent and arc cotangent of 0.7, providing results in degrees.

Fig. 3.23 Calculation of inverse trigonometric functions for $x = 0.7$ (Results are in degrees)

```
Command Window
>> asind(0.7)

ans =

   44.4270

>> acosd(0.7)

ans =

   45.5730

>> atand(0.7)

ans =

   34.9920

>> acotd(0.7)

ans =

   55.0080

fx >> 
```

Example 3.59 The code in Fig. 3.24 calculates $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 7 & 5 & 1 & 0 \\ 2 & 2 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 6 & 3 \\ -9 & 1 \end{bmatrix}$.

Fig. 3.24 Matrix multiplication

```
Command Window
>> A=[1 2 0 -1;7 5 1 0;2 2 3 4];
>> B=[1 2;0 -1;6 3;-9 1];
>> A*B

ans =

    10    -1
    13    12
   -16    15

fx >>
```

Example 3.60 The code in Fig. 3.25 demonstrates the process of extracting the first and second columns of matrix $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 7 & 5 & 1 & 0 \\ 2 & 2 & 3 & 4 \end{bmatrix}$ and assigning them to the variables V_1 and V_2, respectively.

Fig. 3.25 Extracting rows and columns of a matrix

```
Command Window
>> A=[1 2 0 -1;7 5 1 0;2 2 3 4];
>> V1=A(:,1)

V1 =

     1
     7
     2

>> V2=A(:,2)

V2 =

     2
     5
     2

fx >>
```

Example 3.61 The code in Fig. 3.26 demonstrates the process of extracting the first and second rows of matrix $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 7 & 5 & 1 & 0 \\ 2 & 2 & 3 & 4 \end{bmatrix}$ and assigning them to the variables W_1 and W_2, respectively.

Fig. 3.26 Extracting rows and columns of a matrix

```
Command Window
>> A=[1 2 0 -1;7 5 1 0;2 2 3 4];
>> W1=A(1,:)

W1 =

     1     2     0    -1

>> W2=A(2,:)

W2 =

     7     5     1     0

>>
```

Example 3.62 Figure 3.27 shows the MATLAB code for generating a 3×2 random matrix.

Fig. 3.27 Generating a random 3×2 matrix

```
Command Window
>> rand(3,2)

ans =

    0.8147    0.9134
    0.9058    0.6324
    0.1270    0.0975

>>
```

Example 3.63 Figure 3.28 shows how to create a 2×3 matrix of ones in MATLAB.

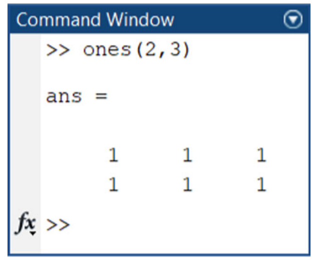


Fig. 3.28 Generating a 2×3 ones matrix

Example 3.64 Figure 3.29 shows how to create a 3×2 matrix of zeros in MATLAB.

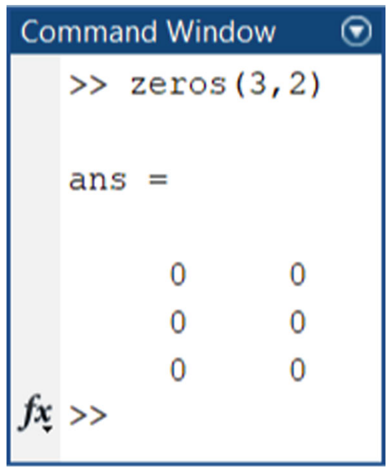


Fig. 3.29 Generating a 3×2 zeros matrix

Example 3.65 Figure 3.30 shows how to create a 3×3 identity matrix in MATLAB.

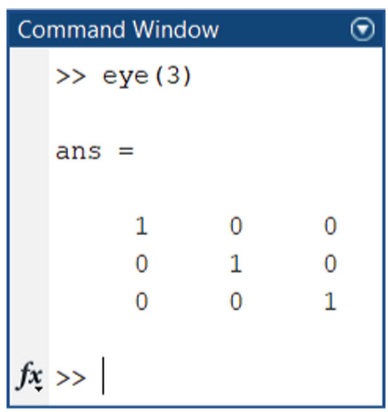


Fig. 3.30 Generating a 3×3 identity matrix

Example 3.66 MATLAB primarily offers two ways to transpose a matrix: Conjugate transpose and non-conjugate transpose. Table 3.1 compares these two operators.

Table 3.1 Transpose operators in MATLAB

Operator	Name	Behavior
’	Conjugate transpose	Transposes and conjugates complex elements
.’	Non-conjugate transpose	Transposes without conjugating complex elements

The code in Fig. 3.31 calculates the conjugate and non-conjugate transpose of matrix $A = \begin{bmatrix} 1 & 1+3i \\ 9-2i & -1+2i \end{bmatrix}$.

Fig. 3.31 Calculating the conjugate and non-conjugate transpose of matrix A

```
Command Window
>> A=[1 1+3j;9-2j -1+2j]

A =

   1.0000 + 0.0000i   1.0000 + 3.0000i
   9.0000 - 2.0000i  -1.0000 + 2.0000i

>> A'

ans =

   1.0000 + 0.0000i   9.0000 + 2.0000i
   1.0000 - 3.0000i  -1.0000 - 2.0000i

>> A.'

ans =

   1.0000 + 0.0000i   9.0000 - 2.0000i
   1.0000 + 3.0000i  -1.0000 + 2.0000i

fx >>
```

Example 3.67 The code in Fig. 3.32 calculates the determinant of $A = \begin{bmatrix} -4 & -6 & 2 \\ 5 & -1 & 3 \\ -2 & 4 & -3 \end{bmatrix}$.

Fig. 3.32 Calculating the determinant of the matrix *A*

```
Command Window
>> A=[-4 -6 2;5 -1 3;-2 4 -3];
>> det(A)

ans =

   18.0000

fx >>
```

Example 3.68 The code in Figs. 3.33 and 3.34 calculates the inverse of $A = \begin{bmatrix} 8 & -2 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$.

Fig. 3.33 Calculating the inverse of the matrix A

```
Command Window
>> A=[8 -2 1;2 -1 3;1 1 4];
>> inv(A)

ans =

    0.1628   -0.2093    0.1163
    0.1163   -0.7209    0.5116
   -0.0698    0.2326    0.0930

fx >>
```

Fig. 3.34 Calculating the inverse of the matrix A

```
Command Window
>> A=[8 -2 1;2 -1 3;1 1 4];
>> A^-1

ans =

    0.1628   -0.2093    0.1163
    0.1163   -0.7209    0.5116
   -0.0698    0.2326    0.0930
fx >>
```

Example 3.69 $\begin{cases} x+2y+3z=10 \\ 4x+6y+5z=12 \\ 7x+9y+8z=13 \end{cases}$ can be written as: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \\ 13 \end{bmatrix}$. Determinant of coefficients is non-zero, so a unique solution exists. The code presented in Figs. 3.35, 3.36, 3.37 and 3.38 demonstrate various approaches to find the solution.

Fig. 3.35 Solving the system of equations given in Example 3.69

```
Command Window
>> A=[1 2 3;4 6 5;7 9 8];
>> b=[10;12;13];
>> A\b

ans =

   -6.4444
    3.8889
    2.8889

fx >> 
```

Fig. 3.36 Solving the system of equations given in Example 3.69

```
Command Window
>> A=[1 2 3;4 6 5;7 9 8];
>> b=[10;12;13];
>> linsolve(A,b)

ans =

   -6.4444
    3.8889
    2.8889

fx >> 
```

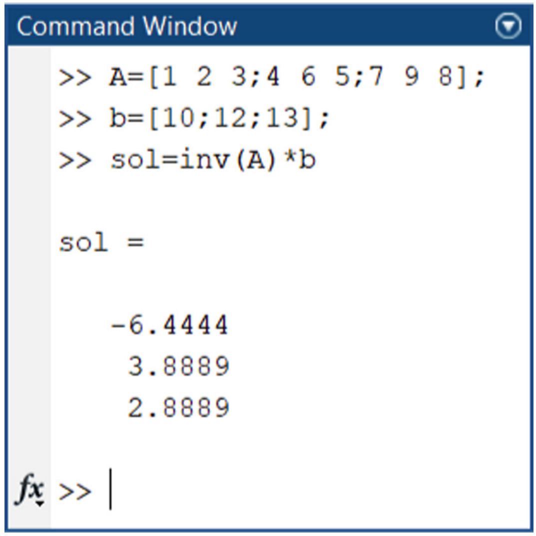


Fig. 3.37 Solving the system of equations given in Example 3.69

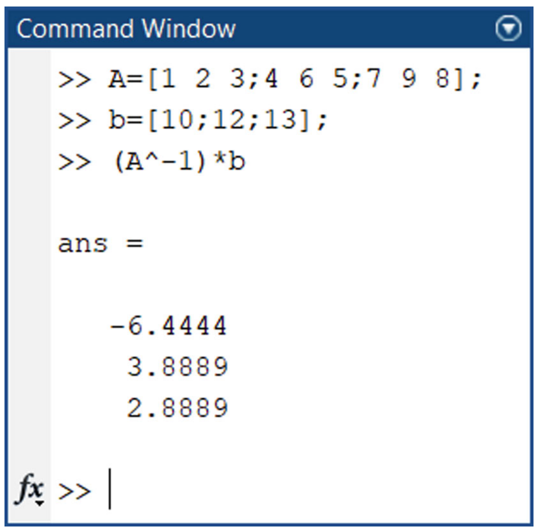
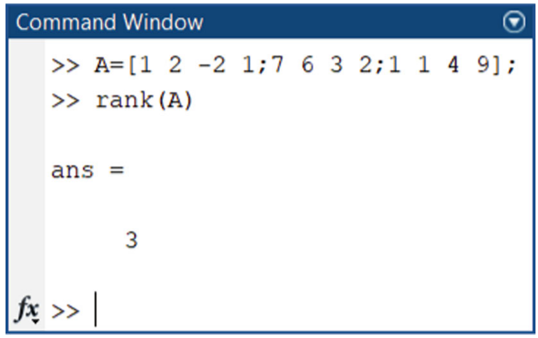


Fig. 3.38 Solving the system of equations given in Example 3.69

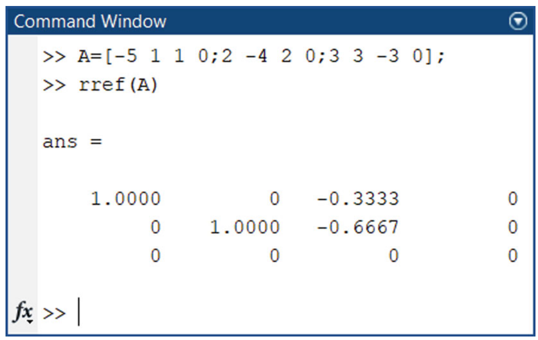
Example 3.70 The code in Fig. 3.39 calculates the rank of $A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 7 & 6 & 3 & 2 \\ 1 & 1 & 4 & 9 \end{bmatrix}$.



```
Command Window
>> A=[1 2 -2 1;7 6 3 2;1 1 4 9];
>> rank(A)

ans =

     3

>>
```

Fig. 3.39 Calculation of rank for matrix A

Example 3.71 The code in Fig. 3.40 calculates the reduced row echelon form of $A = \begin{bmatrix} -5 & 1 & 1 & 0 \\ 2 & -4 & 2 & 0 \\ 3 & 3 & -3 & 0 \end{bmatrix}$.



```
Command Window
>> A=[-5 1 1 0;2 -4 2 0;3 3 -3 0];
>> rref(A)

ans =

    1.0000         0   -0.3333         0
         0    1.0000   -0.6667         0
         0         0         0         0

>>
```

Fig. 3.40 Calculation of reduced row echelon form of matrix A

Example 3.72 The code in Fig. 3.41 calculates the characteristic equation of $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$.

Fig. 3.41 Calculation of characteristic equation for matrix A

```
Command Window
>> syms x
>> A=[2 1 1;2 3 2;3 3 4];
>> det(A-x*eye(3))

ans =

- x^3 + 9*x^2 - 15*x + 7

fx >>
```

The `factor` command can be employed to factorize the characteristic equation. According to Fig. 3.42, $-x^3 + 9x^2 - 15x + 7 = -1 \times (x - 7) \times (x - 1) \times (x - 1) = -(x - 7)(x - 1)^2$.

Fig. 3.42 Factorizing the result

```
Command Window
>> syms x
>> A=[2 1 1;2 3 2;3 3 4];
>> det(A-x*eye(3))

ans =

- x^3 + 9*x^2 - 15*x + 7

>> factor(ans)

ans =

[-1, x - 7, x - 1, x - 1]

fx >>
```

Example 3.73 The code in Fig. 3.43 calculates the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$. According to Fig. 3.43, the eigen values are $\lambda_1 = 1, \lambda_2 = 7$ and $\lambda_3 = 1$.

Eigenvector associated with $\lambda_1 = 1$ is $\begin{bmatrix} -0.8111 \\ 0.3244 \\ 0.4867 \end{bmatrix}$, Eigenvector associated with $\lambda_2 = 1$ is $\begin{bmatrix} 0.2673 \\ 0.5345 \\ 0.8018 \end{bmatrix}$ and Eigenvector associated with $\lambda_3 = 1$ is $\begin{bmatrix} 0.0937 \\ -0.7493 \\ 0.6556 \end{bmatrix}$.

Fig. 3.43 Calculation of eigenvalues and eigenvectors for matrix A

```
Command Window
>> A=[2 1 1;2 3 2;3 3 4];
>> [eigenVector,eigenValue]=eig(A)

eigenVector =

   -0.8111    0.2673    0.0937
    0.3244    0.5345   -0.7493
    0.4867    0.8018    0.6556


eigenValue =

    1.0000         0         0
         0    7.0000         0
         0         0    1.0000

fx >>
```

References for Further Study

1. Advanced Engineering Mathematics (10th edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8th edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

Limits, Derivatives, and Integrals

4

4.1 Introduction

This chapter begins with a review of calculus fundamentals: limits, derivatives, and integrals. The second part demonstrates MATLAB's application to the problems discussed.

4.2 Limit

Limit is a fundamental concept that describes the behavior of a function as its input approaches a certain value. It's a way to understand how a function behaves near a specific point, even if the function is not defined at that point.

Example 4.1 Calculate $\lim\limits_{x \to 3} \left(x^2 + 6\right)$.

$$\lim_{x \to 3} \left(x^2 + 6\right) = 15.$$

Example 4.2 Calculate $\lim\limits_{x \to 0} \left(\frac{x+1}{x+3}\right)$.

$$\lim_{x \to 4} \left(\frac{x + 1)}{x + 3}\right) = \frac{5}{7}.$$

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_4

4.3 L'Hopital's Rule

L'Hopital's rule allows you to evaluate limits of indeterminate forms, such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The rule states: If the limit of the ratio of two functions $f(x)$ and $g(x)$ as x approaches a is an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ where $f'(x)$ and $g'(x)$ are the derivatives of $f(x)$ and $g(x)$, respectively.

For instance, $\lim_{x\to 2}\left(\frac{2x^3-5x^2+3x-2}{2x-4}\right) = \lim_{x\to 2}\left(\frac{6x^2-10x+3}{2}\right) = \frac{7}{2}$.

Example 4.3 Calculate $\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)$.

$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right) = \lim_{x\to 0}\left(\frac{\cos(x)}{1}\right) = 1.$$

Example 4.4 Calculate $\lim_{x\to \infty}\left(\frac{2x+3}{3x+4}\right)$.

$$\lim_{x\to \infty}\left(\frac{2x+3}{3x+4}\right) = \lim_{x\to \infty}\left(\frac{2}{3}\right) = \frac{2}{3}.$$

4.4 Derivative

Derivative measures the rate of change of a function at a specific point. It represents the slope of the tangent line to the graph of the function at that point. The derivative of a function $f(x)$ is often denoted as $f'(x)$ or $\frac{dy}{dx}$.

Here are some of the most important derivative rules (Note that $u(x)$ is a function of x):

1. $\frac{d}{dx}u(x)^n = nu'(x)u(x)^{n-1}$
2. $\frac{d}{dx}\sin(u(x)) = u'(x)\times\cos(u(x))$
3. $\frac{d}{dx}\cos(u(x)) = -u'(x)\times\sin(u(x))$
4. $\frac{d}{dx}\tan(u(x)) = u'(x)\times\left(1+\tan^2(u(x))\right) = u'(x)\times\sec^2(u(x))$
5. $\frac{d}{dx}\arcsin(u(x)) = \frac{u'(x)}{\sqrt{1-u(x)^2}}$
6. $\frac{d}{dx}\arccos(u(x)) = -\frac{u'(x)}{\sqrt{1-u(x)^2}}$
7. $\frac{d}{dx}\arctan(u(x)) = \frac{u'(x)}{1+u(x)^2}$
8. $\frac{d}{dx}e^{u(x)} = u'(x)\times e^{u(x)}$
9. $\frac{d}{dx}a^{u(x)} = u'(x)\times ln(a)\times a^{u(x)}$
10. $\frac{d}{dx}ln(u(x)) = \frac{u'(x)}{u(x)}$

11. $\frac{d}{dx}\log_a u(x) = \frac{u'(x)}{ln(a)\times u(x)}$
12. $\frac{d}{dx}f(x)^{g(x)} = f(x)^{g(x)} \times \left(g'(x)ln(f(x)) + \frac{f'(x)g(x)}{f(x)}\right)$
13. $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
14. $(f(x) \times g(x))' = f'(x)g(x) + g'(x)f(x)$
15. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-g'(x)f(x)}{g(x)^2}$
16. $h(x) = fog(x) \Rightarrow h'(x) = g'(x) \times f'(g(x))$
17. $$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0)^1 + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \ldots \Rightarrow f'(x) = a_1 + 2a_2(x-x_0)^1 + 3a_3(x-x_0)^2 + \ldots = \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}$$

Example 4.5 Key differentiation rules are illustrated in the following samples.

1. $\frac{d}{dx}(3x+7)^4 = 4 \times 3 \times (3x+7)^3$
2. $\frac{d}{dx}\left(\sin\left(x^2+6\right)\right) = 2x \times \cos\left(x^2+6\right)$
3. $\frac{d}{dx}\left(\cos\left(x^4\right)\right) = 4x^3 \times -\sin\left(x^4\right)$
4. $\frac{d}{dx}\left(\tan\left(\sqrt{x}\right)\right) = \frac{1}{2\sqrt{x}}\left(1+\tan^2\left(\sqrt{x}\right)\right)$
5. $\frac{d}{dx}\left(3^{\sin(x)}\right) = \cos(x) \times \ln(3) \times 3^{\sin(x)}$
6. $\frac{d}{dx}(\ln(\sin(6x))) = \frac{6\cos(6x)}{\sin(6x)}$
7. $\frac{d}{dx}\left(\log_5 x^3\right) = \frac{3x^2}{\ln(5)\times x^3}$
8. $\frac{d}{dx}(2x+7)^{x^2} = (2x+7)^{x^2} \times \left(2x \cdot \ln(2x+7) + \frac{2\times x^2}{2x+7}\right)$
9. $\frac{d}{dx}\left(\frac{3x+1}{x+5}\right) = \frac{3(x+5)-1(3x+1)}{(x+5)^2} = \frac{14}{(x+5)^2}$
10. $\frac{d}{dx}\left(\frac{\sin(x)}{e^x}\right) = \frac{\cos(x)e^x - e^x\sin(x)}{(e^x)^2} = \frac{\cos(x)-\sin(x)}{e^x}$
11. $\frac{d}{dx}\left(\sin\left(x^2+6x\right)\right) = (2x+6)\cos\left(x^2+6x\right)$
12. $\frac{d}{dx}\left(\sin(x)^4\right) = \cos(x) \times 4(\sin(x))^3 = 4\cos(x)\sin(x)^3$

Example 4.6 Determine the derivative of function $f(t)$ shown in Fig. 4.1.

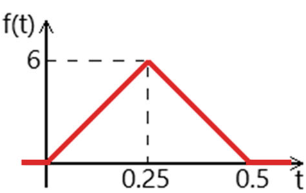


Fig. 4.1 Plot of the $f(t)$

$$f' = \begin{cases} 0 & -\infty < x < 0 \\ \frac{6}{0.25} = 24 & 0 < x < 0.25 \\ \frac{-6}{0.25} = -24 & 0.25 < x < 0.5 \\ 0 & 0.5 < x < \infty \end{cases}$$

Example 4.7 A) Find y' if $x^3 + y^3 - 6xy = 0$. B) Find y'' if $x^4 + y^4 = 16$.

$$3x^2 + 3y'y^2 - 6y - 6xy' = 0 \Rightarrow y'\left(3y^2 - 6x\right)$$
$$= -3x^2 + 6y \Rightarrow y' = \frac{-3x^2 + 6y}{3y^2 - 6x} = \frac{-x^2 + 2y}{y^2 - 2x}$$

$$\frac{d}{dx}(x^4 + y^4) = \frac{d}{dx}(16) \Rightarrow 4x^3 + 4y'y^3 = 0 \Rightarrow y' = -\frac{4x^3}{4y^3} = -\frac{x^3}{y^3}$$

$$\begin{aligned} 4x^3 + 4y'y^3 = 0 \Rightarrow \frac{d}{dx}\left(4x^3 + 4y'y^3\right) &= \frac{d}{dx}(0) \Rightarrow 12x^2 + 4y''y^3 + 4y' \times 3y'y^2 \\ &= 0 \Rightarrow 12x^2 + 4y''y^3 + 12y'^2y^2 = 0 \Rightarrow y'' \\ &= \frac{-12(x^2 + y'^2y^2)}{4y^3} = \frac{-3(x^2 + y'^2y^2)}{y^3} \\ &= \frac{-3\left(x^2 + \left(-\frac{x^3}{y^3}\right)^2 y^2\right)}{y^3} = \frac{-3x^2\left(x^4 + y^4\right)}{y^7} \end{aligned}$$

$x^4 + y^4 = 16$ therefore,

$$y'' = \frac{-3x^2\left(x^4 + y^4\right)}{y^7} = \frac{-3x^2 \times 16}{y^7} = \frac{-48x^2}{y^7}$$

4.5 Partial Derivatives

A partial derivative is a mathematical concept used to find the rate of change of a multivariable function with respect to one of its variables, while keeping the other variables constant.

In order to determine $\frac{\partial f}{\partial x}$ treat y as a constant and differentiate $f(x, y)$ with respect to x. In order to determine $\frac{\partial f}{\partial y}$ treat x as a constant and differentiate $f(x, y)$ with respect to y.

Example 4.8 Determine $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = x^2 + y^2 + 3x^2y^4 - y^3 + \sin(x)$.

$$\frac{\partial f}{\partial x} = 2x + 6xy^4 + \cos(x)$$

$$\frac{\partial f}{\partial y} = 2y + 12x^2y^3 - 3y^2$$

4.6 Integral

An integral is a mathematical concept that represents the area under the curve of a function. There are two main types of integrals: Definite Integral and indefinite integral.

Definite integral calculates the area under the curve of a function between two specific points (limits of integration). The definite integral of a function $f(x)$ between the limits a and b is denoted as $\int_a^b f(x)dx$. The result of a definite integral is a number representing the area.

Indefinite integral finds the antiderivative of a function, which is a function whose derivative is the original function. The indefinite integral of a function $f(x)$ is denoted as $\int f(x)dx$. The result of an indefinite integral is a function plus a constant of integration (C).

Example 4.9 Calculate the $\int_0^1 f(t)dt$ for function $f(t)$ shown in Fig. 4.2.

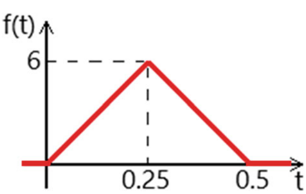


Fig. 4.2 Plot of the $f(t)$

The solution to this problem lies in calculating the area depicted in Fig. 4.3: $\int_0^1 f(t)dt = \frac{1}{2} \times 6 \times 0.5 = 1.5$.

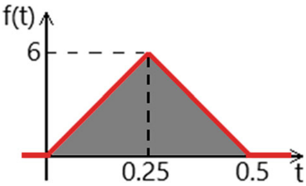


Fig. 4.3 Area under $f(t)$

You can also solve this problem using an analytical method. According to Fig. 4.2, $f(t) = \begin{cases} 24t & 0 \le t \le 0.25 \\ -24t + 12 & 0.25 < t \le 0.5 \\ 0 & otherwise \end{cases}$. Therefore,

$$\int_0^1 f(t)dt = \int_0^{0.25} 24tdt + \int_{0.25}^{0.5} (-24t+12)dt + \int_{0.5}^{1} 0dt = 12t^2\Big|_0^{0.25} -12t^2 + 12t\Big|_{0.25}^{0.5}$$
$$= 12\left(0.25^2 - 0^2\right) + \left(-12 \times 0.5^2 + 12 \times 0.5\right) - \left(-12 \times 0.25^2 + 12 \times 0.25\right)$$
$$= 0.75 + 3 - 2.25 = 1.5$$

Here are some of the most important properties of integrals:

$$a, b \in \mathbb{R} : \int af_1(x) + bf_2(x)dx = a\int f_1(x) + b\int f_2(x)$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$for\ a \le \lambda \le b : \int_a^b f(x)dx = \int_a^{\lambda} f(x)dx + \int_{\lambda}^{b} f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$a \in \mathbb{R} : \frac{d}{dt}\int_a^t f(\tau)d\tau = f(t)$$

If $f(x)$ is an even function, i.e., $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.
If $f(x)$ is an odd function, i.e., $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)dx = 0$.
Here are some of the most important integral formulas:

1. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C \quad n \neq -1$
2. $\int \frac{1}{x}dx = ln|x| + C$
3. $\int \frac{1}{ax+b}dx = \frac{1}{a}ln|ax+b| + C$

4. $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
5. $\int e^{ax}dx = \frac{1}{a}e^{ax} + C$
6. $\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + C$
7. $\int \cos(ax)dx = \frac{1}{a}\sin(ax) + C$

Example 4.10 Calculate $\int_{-1}^{1} x^2 dx$.

x^2 is even therefore, $\int_{-1}^{1} x^2 dx = 2\int_0^1 x^2 dx = 2\frac{x^3}{3}\Big|_0^1 = 2\left(\frac{1^3}{3} - \frac{0^3}{3}\right) = \frac{2}{3}$.

A different approach to solving this problem is:

$$\int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1^3}{3} - \frac{(-1)^3}{3} = \frac{2}{3}.$$

Example 4.11 Calculate $\int_{-1}^{1} x^3 dx$.

x^3 is odd therefore $\int_{-1}^{1} x^3 dx = 0$.

A different approach to solving this problem is:

$$\int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{1} = \left(\frac{1^4}{4} - \frac{(-1)^4}{4}\right) = \left(\frac{1}{4} - \frac{1}{4}\right) = 0.$$

4.7 Tabular Integration

The tabular method, also known as the DI method, is particularly well-suited for integrals that involve products of polynomial functions with exponential, trigonometric, or logarithmic functions. For instance, $\int x^2 e^x dx$, $\int x^3 \sin(x) dx$ or $\int x^4 ln(x) dx$ can be calculated easily with tabular integration technique.

Tabular integration involves creating a table with two columns: one for repeated differentiation of a function and the other for repeated integration of another function. The product of the diagonal entries, with alternating signs, is then summed to obtain the integral. Let's study some examples.

Example 4.12 Calculate $\int x^3 \cos(x)dx$.

The procedure for tabular integration is depicted in Figs. 4.4 and 4.5.

Fig. 4.4 Calculation of $\int x^3 \cos(x)dx$ (First step)

Derivatives	Integrals
x^3	$\cos(x)$
$3x^2$	$\sin(x)$
$6x$	$-\cos(x)$
6	$-\sin(x)$
0	$\cos(x)$

Fig. 4.5 Calculation of $\int x^3 \cos(x)dx$ (Second step)

Derivatives		Integrals
x^3	$+$	$\cos(x)$
$3x^2$	$-$	$\sin(x)$
$6x$	$+$	$-\cos(x)$
6	$-$	$-\sin(x)$
0		$\cos(x)$

According to Fig. 4.5, $\int x^3 \cos(x)dx = x^3 \sin(x) + 3x^2 \cos(x) - 6x \sin(x) - 6 \cos(x) + C$.

Example 4.13 Calculate $\int e^x \cos(x)dx$.

In this example the derivatives of e^x will never reach to zero. However, we can still use the tabular method to set up the integration by parts process. In this case, the key is to recognize that the integral will reappear on the right-hand side of the equation, allowing us to solve for it algebraically.

The procedure for tabular integration is depicted in Figs. 4.6 and 4.7.

Fig. 4.6 Calculation of $\int e^x \cos(x)dx$ (First step)

Derivatives	Integrals
e^x	$cos(x)$
e^x	$sin(x)$
e^x	$-cos(x)$

Fig. 4.7 Calculation of $\int e^x \cos(x)dx$ (Second step)

Derivatives	Integrals
e^x +	$\cos(x)$
e^x −	$\sin(x)$
e^x +	$-\cos(x)$

According to Fig. 4.7,

$$\int e^x \cos(x)dx = e^x \times \sin(x) - e^x \times -\cos(x) + \int -e^x \cos(x)dx + C$$

$$\Rightarrow 2\int e^x \cos(x)dx = e^x \sin(x) + e^x \cos(x) + C$$

$$\Rightarrow \int e^x \cos(x)dx = \frac{e^x \sin(x)}{2} + \frac{e^x \cos(x)}{2} + \frac{C}{2}$$

$$\Rightarrow \int e^x \cos(x)dx = \frac{e^x}{2}(\sin(x) + \cos(x)) + C'$$

Exercise: Calculate the following integrals. Correct answers are provided for verification.

$$\int x^2 e^x dx = e^x\left(x^2 - 2x + 2\right) + C$$

$$\int x^4 \ln(x)dx = \frac{x^5}{5}\left(\ln(x) - \frac{1}{5}\right) + C$$

$$\int x^2 \sin(x)dx = 2x\sin(x) - \left(x^2 - 2\right)\cos(x) + C$$

$$\int e^x \sin(x)dx = -\frac{e^x}{2}(\cos(x) - \sin(x)) + C$$

4.8 Complex Integrals

This section studies the commonly encountered complex integrals and methods to calculate them.

Example 4.14 Calculate $\int_{-\frac{\pi}{6}}^{\pi} e^{j4x}dx$.

$$\int_{-\frac{\pi}{6}}^{\pi} e^{j4x}dx = \frac{1}{j4}e^{j4x}\Big|_{-\frac{\pi}{6}}^{+\pi} = \frac{1}{j4}\left(e^{j4\pi} - e^{-j4\frac{\pi}{6}}\right)$$

$$= \frac{1}{j4}\left(\cos(4\pi) + j\sin(4\pi) - \cos\left(-4\frac{\pi}{6}\right) - j\sin\left(-4\frac{\pi}{6}\right)\right)$$
$$= \frac{1}{j4}\left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \frac{1}{j4}\left(\frac{3}{2} + j\frac{\sqrt{3}}{2}\right) = \frac{3}{8j} + \frac{\sqrt{3}}{8} = \frac{\sqrt{3}}{8} - j\frac{3}{8}$$

You can use the following method as well:

$$\int_{-\frac{\pi}{6}}^{\pi} e^{j4x}dx = \int_{-\frac{\pi}{6}}^{\pi} \cos(4x) + j\sin(4x)dx = \int_{-\frac{\pi}{6}}^{\pi} \cos(4x)dx + j\int_{-\frac{\pi}{6}}^{\pi} \sin(4x)dx$$

$$\int_{-\frac{\pi}{6}}^{\pi} \cos(4x)dx = \frac{\sin(4x)}{4}\bigg|_{-\frac{\pi}{6}}^{\pi} = \frac{\sin(4\pi)}{4} - \frac{\sin\left(-4\frac{\pi}{6}\right)}{4} = \frac{\sqrt{3}}{8}$$

$$j\int_{-\frac{\pi}{6}}^{\pi} \sin(4x)dx = j\frac{-\cos(4x)}{4}\bigg|_{-\frac{\pi}{6}}^{\pi} = -\frac{j}{4}\left(\cos(4\pi) - \cos\left(-4\frac{\pi}{6}\right)\right)$$
$$= -\frac{j}{4}(1 - (-0.5)) = \frac{-3}{8}j$$

$$\int_{-\frac{\pi}{6}}^{\pi} e^{j4x}dx = \int_{-\frac{\pi}{6}}^{\pi} \cos(4x)dx + j\int_{-\frac{\pi}{6}}^{\pi} \sin(4x)dx = \frac{\sqrt{3}}{8} - j\frac{3}{8}$$

Example 4.15 Calculate $\int_{-\pi}^{\pi} x^2 e^{j4x}dx$.

$$\int_{-\pi}^{\pi} x^2 e^{j4x}dx = \int_{-\pi}^{\pi} x^2(\cos(4x) + j\sin(4x))dx = \int_{-\pi}^{\pi} x^2\cos(4x)dx + j\int_{-\pi}^{\pi} x^2\sin(4x)dx$$
$$= \int_{-\pi}^{\pi} x^2\cos(4x)dx = 2\int_{0}^{\pi} x^2\cos(4x)dx$$
$$= 2\left(\frac{x^2}{4}\sin(4x) + \frac{2x}{16}\cos(4x) - \frac{2}{64}\sin(4x)\right)\bigg|_{0}^{\pi} = \frac{\pi}{4}$$

Calculation of $\int_0^\pi x^2\cos(4x)dx$ is shown in Fig. 4.8.

$$\begin{array}{ccc} x^2 & & \cos(4x) \\ & \searrow^{+} & \\ 2x & & \frac{1}{4}\sin(4x) \\ & \searrow^{-} & \\ 2 & & -\frac{1}{16}\cos(4x) \\ & \searrow^{+} & \\ 0 & & -\frac{1}{64}\sin(4x) \end{array}$$

Fig. 4.8 Calculation of $\int x^2\cos(4x)dx$

$$j\int_{-\pi}^{\pi} x^2\sin(4x)dx = 0$$

$$\int_{-\pi}^{\pi} x^2e^{j4x}dx = \int_{-\pi}^{\pi} x^2\cos(4x)dx + j\int_{-\pi}^{\pi} x^2\sin(4x)dx = \frac{\pi}{4} + 0 = \frac{\pi}{4}$$

You can use the tabular integration method to calculate $\int_{-\pi}^{\pi} x^2e^{j4x}dx$, as well (Fig. 4.9).

$$\begin{aligned}\int_{-\pi}^{\pi} x^2e^{j4x}dx &= \left(\frac{x^2}{4j} + \frac{x}{8} - \frac{1}{32j}\right)e^{j4x}\Big|_{-\pi}^{\pi} \\ &= \left(\frac{\pi^2}{4j} + \frac{\pi}{8} - \frac{1}{32j}\right)e^{j4\pi} - \left(\frac{\pi^2}{4j} - \frac{\pi}{8} - \frac{1}{32j}\right)e^{-j4\pi} \\ &= \left(\frac{\pi^2}{4j} + \frac{\pi}{8} - \frac{1}{32j}\right) - \left(\frac{\pi^2}{4j} - \frac{\pi}{8} - \frac{1}{32}j\right) = \frac{\pi}{4}\end{aligned}$$

$$\begin{array}{ccc} x^2 & & e^{j4x} \\ & \searrow^{+} & \\ 2x & & \frac{e^{j4x}}{4j} \\ & \searrow^{-} & \\ 2 & & -\frac{e^{j4x}}{16} \\ & \searrow^{+} & \\ 0 & & -\frac{e^{j4x}}{64j} \end{array}$$

Fig. 4.9 Calculation of $\int x^2e^{j4x}dx$

4.9 Fractional Integrals

This section demonstrates the calculation of fractional integrals. Let's explore some numerical examples.

Example 4.16 Calculate $\int \frac{x^3+x}{x-1}dx$.

According to Fig. 4.10, $x^3 + x = \left(x^2 + x + 2\right)(x - 1) + 2$.

Fig. 4.10 Dividing $x^3 + x$ by $x - 1$

$$
\begin{array}{r|l}
x^3 + x & x - 1 \\ \hline
- \quad x^3 - x^2 & x^2 + x + 2 \\ \cline{1-1}
x^2 + x & \\
- \quad x^2 - x & \\ \cline{1-1}
2x & \\
- \quad 2x - 2 & \\ \cline{1-1}
2 &
\end{array}
$$

$$
\begin{aligned}
\int \frac{x^3 + x}{x - 1}dx &= \int x^2 + x + 2 + \frac{2}{x - 1}dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln(|x - 1|) + C \\
&= \frac{x^3}{3} + \frac{x^2}{2} + 2x + \ln\left(|x - 1|^2\right) + C = \frac{x^3}{3} + \frac{x^2}{2} + 2x + \ln\left(x^2 - 2x + 1\right) + C
\end{aligned}
$$

Example 4.17 Calculate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$.

$$2x^3 + 3x^2 - 2x = x(2x - 1)(x + 2)$$

$$\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} = \frac{A}{x} + \frac{B}{2x - 1} + \frac{C}{x + 2} \overset{\textit{use partial fraction expansion}}{\Rightarrow} A = \frac{1}{2}, B = \frac{1}{5}, C = -\frac{1}{10}$$

$$\int \frac{\frac{1}{2}}{x} + \frac{\frac{1}{5}}{2x - 1} - \frac{\frac{1}{10}}{x + 2}dx = \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x - 1| - \frac{1}{10}\ln|x + 2| + C$$

Example 4.18 Calculate $\int \frac{4}{(2x-7)^2}dx$.

$$\int \frac{4}{(2x - 7)^2}dx = \int 4(2x - 7)^{-2}dx = 2\int 2(2x - 7)^{-2}dx = 2\frac{(2x - 7)^{-2+1}}{-2 + 1} = \frac{-2}{2x - 7}$$

Exercise: Show that for $a \neq 0$ $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} ln\left|\frac{x-a}{x+a}\right| + C$.

Example 4.19 Calculate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx$.

Using polynomial division and partial fraction decomposition, we get:

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1} = x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{-1}{x+1}$$

$$\begin{aligned}\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} dx &= \int \left(x+1+\frac{1}{x-1}+\frac{2}{(x-1)^2}+\frac{-1}{x+1}\right) dx \\ &= \frac{x^2}{2}+x+\ln(|x-1|)-\ln(|x+1|)+2\frac{(x-1)^{-2+1}}{-2+1}+C \\ &= \frac{x^2}{2}+x+\ln\left(\frac{|x-1|}{|x+1|}\right)-\frac{2}{x-1}+C = \frac{x^2}{2}+x-\frac{2}{x-1} \\ &+\ln\left(\left|\frac{x-1}{x+1}\right|\right)+C\end{aligned}$$

Example 4.20 Calculate $\int \frac{2x^2-x+4}{x^3+4x} dx$.

$$\frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x}+\frac{Bx+C}{x^2+4}$$

$$A = \lim_{x \to 0} \frac{2x^2-x+4}{x^2+4} = 1$$

$$\begin{aligned}\frac{Bx+C}{x^2+4} = \frac{2x^2-x+4}{x(x^2+4)} - \frac{1}{x} \Rightarrow Bx+C &= \frac{2x^2-x+4}{x} - \frac{x^2+4}{x} \Rightarrow Bx^2+Cx \\ &= 2x^2-x+4-x^2-4 \Rightarrow Bx^2+Cx \\ &= x^2-x \Rightarrow B=1, C=-1\end{aligned}$$

$$\frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{x(x^2+4)} = \frac{1}{x}+\frac{x-1}{x^2+4}$$

$$\int \frac{2x^2-x+4}{x^3+4x} dx = \int \frac{1}{x}+\frac{x-1}{x^2+4} dx = \int \frac{1}{x} dx + \frac{1}{2}\int \frac{2x}{x^2+4} dx - \int \frac{1}{x^2+4} dx$$

$$\int \frac{1}{x} dx = ln|x|$$

$$\frac{1}{2}\int \frac{2x}{x^2+4} dx = \frac{1}{2} ln\left|x^2+4\right| = \frac{1}{2} ln(x^2+4)$$

$$-\int \frac{1}{x^2+4}dx = -\frac{1}{2}\tan^{-1}\frac{x}{2}$$

$$\int \frac{2x^2-x+4}{x^3+4x}dx = ln|x| + \frac{1}{2}ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

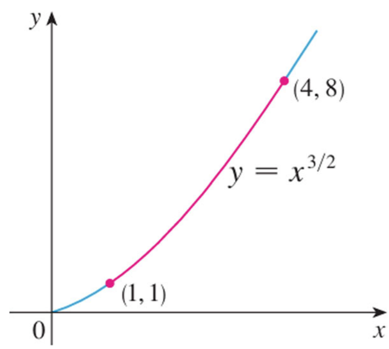
4.10 Calculation of Arc Length

If $f(x)$ is continuous on $[a, b]$, then the length of the curve $y = f(x), a \leq x \leq b$, is $L = \int_a^b \sqrt{1+(f'(x))^2}dx$.

Let's study a numeric example.

Example 4.21 Find the length of the curve $y = x^{\frac{3}{2}}$ from $x = 1$ to $x = 4$ (Fig. 4.11).

Fig. 4.11 Plot of $y = x^{\frac{3}{2}}$



$$L = \int_1^4 \sqrt{1+\left(\frac{3}{2}\sqrt{x}\right)^2}dx = \int_1^4 \sqrt{1+\frac{9}{4}x}dx$$

$$1+\frac{9}{4}x = u \Rightarrow \frac{9}{4}dx = du \Rightarrow dx = \frac{4}{9}du$$

$$x = 1 \Rightarrow u = 1+\frac{9}{4} = \frac{13}{4}$$

$$x = 4 \Rightarrow u = 1+\frac{9}{4}\times 4 = 10$$

$$\int_{1}^{4} \sqrt{1 + \frac{9}{4}x}dx = \int_{\frac{13}{4}}^{10} \sqrt{u} \times \frac{4}{9}du = \frac{4}{9}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} = \frac{8}{27}u^{\frac{3}{2}}\Big|_{\frac{13}{4}}^{10} = \frac{8}{27}\left(10^{\frac{3}{2}} - \left(\frac{13}{4}\right)^{\frac{3}{2}}\right) = 7.634$$

4.11 Area of Surface of Revolution

A surface of revolution is a three-dimensional shape formed by rotating a curve around an axis. To calculate the surface area of such a shape, we use a specific integral formula.

In the case where f is positive and has a continuous derivative, we define the surface area of the surface obtained by rotating the curve $y = f(x), a \leq x \leq b$, about the x-axis as:

$$S = \int_{a}^{b} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

If the curve is described as $x = g(y)$, $c \leq y \leq d$, then the formula for surface area becomes:

$$S = \int_{c}^{d} 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

The above two formula can be summarized symbolically as

$$S = \int 2\pi y ds$$

where $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ or $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$. For rotation about the y-axis, $S = \int 2\pi x ds$.

Example 4.22 The arc of the parabola $y = x^2$ from $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is rotated about the y-axis (Fig. 4.12). Find the area of resulting surface.

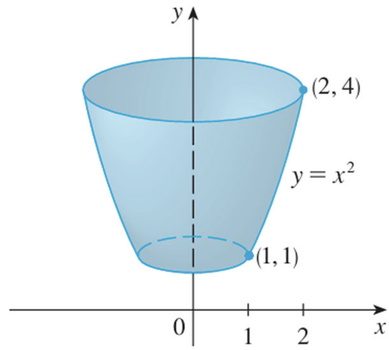


Fig. 4.12 Rotation of $y = x^2$ from $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ about the y-axis

$$y = x^2 \Rightarrow \frac{dy}{dx} = 2x$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + 4x^2} dx$$

$$S = \int 2\pi x ds \Rightarrow S = \int_1^2 2\pi x \sqrt{1 + 4x^2} dx = 2\pi \int_1^2 x\left(1 + 4x^2\right)^{\frac{1}{2}} dx$$

$$S = 2\pi \int_1^2 x\left(1 + 4x^2\right)^{\frac{1}{2}} dx = \frac{2\pi}{8} \int_1^2 8x\left(1 + 4x^2\right)^{\frac{1}{2}} dx$$

$$u = 1 + 4x^2 \Rightarrow du = 8x dx$$

$$x = 1 \Rightarrow u = 1 + 4(1)^2 = 5$$

$$x = 2 \Rightarrow u = 1 + 4(2)^2 = 17$$

$$\frac{2\pi}{8} \int_1^2 8x\left(1 + 4x^2\right)^{\frac{1}{2}} dx = \frac{2\pi}{8} \int_5^{17} u^{\frac{1}{2}} du = \frac{2\pi}{8} \times \left. \frac{u^{\frac{1}{2}+1}}{\frac{1}{2} + 1} \right|_5^{17}$$

$$= \frac{\pi}{6} \times \left. u^{\frac{3}{2}} \right|_5^{17} = \frac{\pi}{6}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right)$$

$$= 30.847$$

You can also solve this problem as follows:

$$x = \sqrt{y}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$$

$$S = \int 2\pi x ds \Rightarrow S = \int_1^4 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}dy = 2\pi \int_1^4 \sqrt{y}\sqrt{1 + \frac{1}{4y}}dy = \pi \int_1^4 \sqrt{4y+1}dy$$

$$u = 4y + 1 \Rightarrow du = 4dy$$

$$y = 1 \Rightarrow u = 5$$

$$y = 4 \Rightarrow u = 17$$

$$S = \pi \int_1^4 \sqrt{4y+1}dy = \frac{\pi}{4} \int_1^4 4\sqrt{4y+1}dy$$

$$= \frac{\pi}{4} \int_5^{17} \sqrt{u}du = \frac{\pi}{4} \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\Bigg|_5^{17}$$

$$= \frac{\pi}{6}u^{\frac{3}{2}}\Bigg|_5^{17} = \frac{\pi}{6}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right) = 30.847$$

Exercise: Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$, about the x-axis (Ans. = 22.943).

4.12 Root Mean Square (RMS) and Average Value

A periodic function is a function that repeats its values at regular intervals. This means that the function's graph has a pattern that repeats itself over and over again. More formally, a function $f(t)$ is periodic with period T if $f(t + T) = f(t)$. For instance, $f(t) = \sin(2t)$ is periodic since: $f(t + \pi) = f(t)$.

Root mean square (RMS) and average value of a periodic signal $f(t)$ are defined as $\sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} f(t)^2 dt}$ and $\frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt = \frac{\textit{Area under one period}}{\textit{period}}$, respectively. The integrals can be evaluated on any interval with length T.

Example 4.23 Calculate the RMS and average value of $f(t) = A\sin(\omega_0 t + \varphi_0)$.

Period of this function is $T = \frac{2\pi}{\omega_0}$.

$$\sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} f(t)^2 dt} = \sqrt{\frac{1}{\frac{2\pi}{\omega_0}}\int_0^{\frac{2\pi}{\omega_0}} (A\sin(\omega_0 t + \varphi_0))^2 dt} = \sqrt{\frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} A^2\sin^2(\omega_0 t + \varphi_0)dt}$$

$$\begin{aligned}
\int_0^{\frac{2\pi}{\omega_0}} A^2\sin^2(\omega_0 t + \varphi_0)dt &= A^2\int_0^{\frac{2\pi}{\omega_0}} \frac{1 - \cos\,(2\omega_0 t + 2\varphi_0)}{2}dt \\
&= A^2\left(\frac{t}{2} - \frac{\sin\,(2\omega_0 t + 2\varphi_0)}{4}\right)\Bigg|_0^{\frac{2\pi}{\omega_0}} \\
&= A^2\left(\frac{\frac{2\pi}{\omega_0}}{2} - \frac{\sin\left(2\omega_0\frac{2\pi}{\omega_0} + 2\varphi_0\right)}{4}\right) - A^2\left(-\frac{\sin(2\varphi_0)}{4}\right) \\
&= A^2\left(\frac{\pi}{\omega_0} - \frac{\sin(4\pi + 2\varphi_0)}{4}\right) - A^2\left(-\frac{\sin(2\varphi_0)}{4}\right) \\
&= A^2\left(\frac{\pi}{\omega_0} - \frac{\sin(2\varphi_0)}{4}\right) - A^2\left(-\frac{\sin(2\varphi_0)}{4}\right) \\
&= A^2\frac{\pi}{\omega_0}
\end{aligned}$$

$$\begin{aligned}
RMS &= \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} f(t)^2 dt} = \sqrt{\frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} A^2\sin^2(\omega_0 t + \varphi_0)dt} \\
&= \sqrt{\frac{\omega_0}{2\pi} \times A^2\frac{\pi}{\omega_0}} = \sqrt{\frac{A^2}{2}} = \frac{A}{\sqrt{2}} = 0.707 \times A
\end{aligned}$$

The average value of given function can be evaluated as follows:

$$\begin{aligned}
\frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt &= \frac{1}{\frac{2\pi}{\omega_0}}\int_0^{\frac{2\pi}{\omega_0}} A\sin(\omega_0 t + \varphi_0)dt \\
&= -\frac{1}{2\pi}A\cos(\omega_0 t + \varphi_0)\Big|_0^{\frac{2\pi}{\omega_0}} = 0
\end{aligned}$$

Example 4.24 RMS value of $f(t) = 10\cos\left(3t + \frac{\pi}{12}\right)$ equals to $\frac{10}{\sqrt{2}} = 10 \times 0.707 = 7.07$. Average value of this function is 0.

4.13 RMS and Average Value of Composite Sinusoidal Signals

It can be shown that RMS of $f(t) = a_0 + \sum_{n=1}^{N} a_n \cos(n\omega_0 t) + \sum_{n=1}^{N} b_n \sin(n\omega_0 t)$ equals to $\sqrt{a_0^2 + \frac{1}{2}\left(\sum_{n=1}^{N} a_n^2 + \sum_{n=1}^{N} b_n^2\right)}$. Average value of given function equals to a_0.

Example 4.25 RMS value of $f(t) = 5 + 10\cos\left(3t + \frac{\pi}{12}\right) + 6\sin(6t) + 4\cos\left(9t + \frac{\pi}{3}\right)$ equals to $\sqrt{5^2 + \frac{1}{2}\left(10^2 + 6^2 + 4^2\right)} = \sqrt{25 + \frac{1}{2}(100 + 36 + 16)} = 10.0499$. Average value of this function is 5.

Example 4.26 Calculate the RMS and average value of periodic function shown in Fig. 4.13. Note that T shows the period and D is a number between 0 and 1 $(0 < D < 1)$.

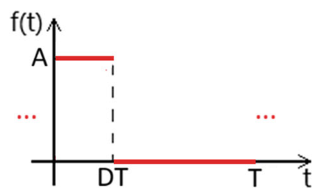


Fig. 4.13 Waveform for Example 4.26

RMS value of this signal is:

$$\sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} f(t)^2 dt} = \sqrt{\frac{1}{T}\int_{0}^{DT} A^2 dt} = \sqrt{\frac{1}{T} \times A^2 \times DT} = A\sqrt{D}$$

Average value of this signal is:

$$\frac{1}{T}\int_{t_0}^{t_0+T} f(t) dt = \frac{1}{T}\int_{0}^{DT} A dt = \frac{1}{T} \times A \times DT = A \times D$$

For instance, for $A = 1$, $T = 0.001$ and $D = 0.5$ RMS and average values are 0.7071 and 0.5, respectively.

Exercise: Calculate the RMS and average value of the periodic function shown in Fig. 4.14.

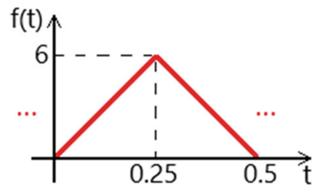


Fig. 4.14 Waveform for the exercise

4.14 Convolution Integral

Convolution is a mathematical operation that combines two functions to produce a third function. It's a fundamental tool in signal processing, image processing, and probability theory. The convolution integral involves multiplying one function by a time-shifted version of the other and integrating over all time shifts. This operation effectively smears or blurs one function with the other.

The convolution operation between two functions $f(t)$ and $g(t)$ is often denoted as $(f * g)(t)$ and defined as:

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau$$

The convolution operation exhibits the following properties:

$$(f * g)(t) = (g * f)(t)$$
$$(f * (g * h))(t) = ((f * g) * h)(t)$$

Example 4.27 Calculate $(f * g)(t)$ for $f(t) = e^{-2t}H(t)$ and $g(t) = e^{-3t}H(t)$. $H(t)$ shows the Heaviside step function, i.e., $H(t) = \begin{cases} 1 t > 0 \\ 0 t < 0 \end{cases}$.

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{+\infty} e^{-2\tau}H(\tau)e^{-3(t-\tau)}H(t-\tau)d\tau$$
$$= \int_{0}^{t} e^{-2\tau}e^{-3(t-\tau)}d\tau = \int_{0}^{t} e^{-3t}e^{-\tau}d\tau = e^{-3t}\int_{0}^{t} e^{-\tau}d\tau$$
$$= e^{-3t}e^{-\tau}\Big|_{0}^{t} = e^{-3t}\left[1 - e^{-t}\right]$$

Exercise: Calculate the $(f * g)(t)$ for functions shown in Figs. 4.15 and 4.16.

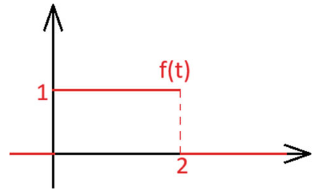


Fig. 4.15 Plot of $f(t)$

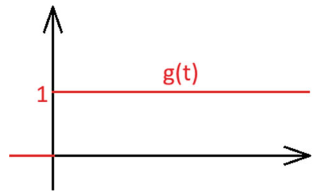


Fig. 4.16 Plot of $g(t)$

4.15 Line Integrals

A line integral is a mathematical concept that involves integrating a function along a curve. This curve can be a simple path in two or three-dimensional space. Line integrals are used to calculate quantities like work done by a force field, mass of a wire, and the circulation of a fluid.

Following formula can be used to evaluate line integrals:

$$\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Example 4.28 Evaluate $\int_C (2 + x^2y)ds$ where C is the upper half of the unit circle $x^2 + y^2 = 1$ (Fig. 4.17).

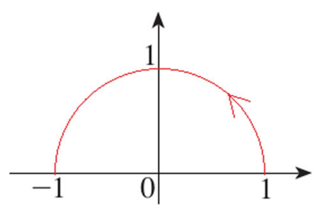


Fig. 4.17 Path of integration

Given path C can be parametrized as: $x = \cos(t)$, $y = \sin(t)$.

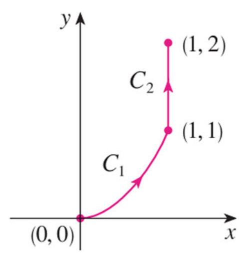
$$\int_C (2 + x^2 y)ds = \int_0^{\pi} (2 + cos^2(t)\sin(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\int_0^{\pi} \left(2 + \cos^2(t)\sin(t)\right)\sqrt{\sin^2(t) + \cos^2(t)}dt = \int_0^{\pi} \left(2 + \cos^2(t)\sin(t)\right)dt$$

$$= 2t - \frac{\cos^3(t)}{3}\bigg|_0^{\pi} = 2\pi + \frac{2}{3}$$

Example 4.29 Evaluate $\int_C 2xds$, where C consists of the arc C_1 of the parabola $y = x^2$ from (0, 0) to (1, 1) followed by the vertical line segment C_2 from (1, 1) to (1, 2) (Fig. 4.18).

Fig. 4.18 Path of integration



For path C_1:

$$C_1 : \begin{cases} x = t \\ y = t^2 \end{cases} \Rightarrow \int_{C_1} 2xds = \int_0^1 2t\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

$$= \int_0^1 2t\sqrt{1^2 + (2t)^2}dt = \int_0^1 2t\sqrt{1 + 4t^2}dt$$

$$\int_0^1 2t\sqrt{1 + 4t^2}dt = \frac{1}{4}\int_0^1 8t\left(1 + 4t^2\right)^{\frac{1}{2}}dt = \frac{1}{4}\frac{\left(1 + 4t^2\right)^{\frac{1}{2}+1}}{\frac{1}{2} + 1}\Bigg|_0^1 = \frac{5\sqrt{5} - 1}{6}$$

For path C_2:

$$C_2 : \begin{cases} x = 1 \\ y = t, \quad 1 \le t \le 2 \end{cases} \Rightarrow \int\limits_{C_2} 2xds = \int\limits_1^2 2 \times 1 \times \sqrt{(0)^2 + (1)^2} dt = \int\limits_1^2 2dt = 2t|_1^2 = 2$$

Therefore,

$$\int\limits_C 2xds = \int\limits_{C_1} 2xds + \int\limits_{C_2} 2xds = \frac{5\sqrt{5} - 1}{6} + 2$$

Exercise: Calculate the line integral $\int_C x^2 ds$, where C is the line segment from $(0, 0)$ to $(2, 4)$ (Ans. $\frac{16}{3}$).

4.16 Gamma Function

The gamma function has many applications in various fields of mathematics and physics, including probability theory, statistics, and complex analysis. The gamma function, denoted by $\Gamma(z)$, is a generalization of the factorial function to complex numbers. The gamma function is defined as:

$$\Gamma(z) = \int\limits_0^\infty t^{z-1} e^{-t} dt$$

For positive integers $n \in \mathbb{N}$, the gamma function is related to the factorial function as follows:

$$\Gamma(n) = (n - 1)!$$

For instance, $\Gamma(5) = 4! = 1 \times 2 \times 3 \times 4 = 24$.

4.17 Problem Solving with MATLAB®

This section provides examples of how MATLAB can be employed to solve the types of problems encountered in this chapter.

Example 4.30 The code in Fig. 4.19 calculates $\lim\limits_{x \to 2} \frac{2x^3 - 5x^2 + 3x - 2}{2x - 4}$.

Fig. 4.19 Calculation of $\lim_{x\to 2} \frac{2x^3-5x^2+3x-2}{2x-4}$

```
Command Window
>> syms x
>> limit((2*x^3-5*x^2+3*x-2)/(2*x-4),x,2)

ans =

7/2

fx >>
```

Example 4.31 The code in Fig. 4.20 calculates $\lim_{x\to 0} \frac{\sin(x)}{x}$.

Fig. 4.20 Calculation of $\lim_{x\to 0} \frac{\sin(x)}{x}$

```
Command Window
>> syms x
>> limit(sin(x)/x,x,0)

ans =

1

fx >>
```

Example 4.32 The code in Fig. 4.21 calculates $\frac{d}{dx}(\ln(\sin(x)))$.

Fig. 4.21 Calculation of $\frac{d}{dx}(\ln(\sin(x)))$

```
Command Window
>> syms x
>> diff(log(sin(x)))

ans =

cos(x)/sin(x)

fx >>
```

Example 4.33 The code in Fig. 4.22 calculates $\frac{d}{dx}\left(\frac{3x+1}{x+5}\right)$.

Fig. 4.22 Calculation of $\frac{d}{dx}\left(\frac{3x+1}{x+5}\right)$

```
Command Window
>> syms x
>> diff((3*x+1)/(x+5))

ans =

3/(x + 5) - (3*x + 1)/(x + 5)^2

>> simplify(ans)

ans =

14/(x + 5)^2

fx >>
```

Example 4.34 The code in Fig. 4.23 calculates $y' = \frac{dy}{dx}$, where y is implicitly defined by the equation $x^3 + y^3 - 6xy = 0$.

Fig. 4.23 Calculation of $y' = \frac{dy}{dx}$ for $x^3 + y^3 - 6xy = 0$

```
Command Window
>> syms x y
>> eqn=x^3+y^3-6*x*y;
>> dydx=simplify(-diff(eqn,x)/diff(eqn,y))

dydx =

-(- 3*x^2 + 6*y)/(- 3*y^2 + 6*x)

>> pretty(dydx)
       2
  - 3 x  + 6 y
- ------------
       2
  - 3 y  + 6 x

fx >>
```

Example 4.35 The code in Fig. 4.24 calculates $y' = \frac{dy}{dx}$, where y is implicitly defined by the equation $x^4 + y^4 = 16$.

```
Command Window
>> syms x y
>> eqn=x^4+y^4-16;
>> dydx=simplify(-diff(eqn,x)/diff(eqn,y))

dydx =

-x^3/y^3

fx >>
```

Fig. 4.24 Calculation of $y' = \frac{dy}{dx}$ for $x^4 + y^4 = 16$

Example 4.36 The code in Fig. 4.25 calculates $\frac{\partial f}{\partial x}$ where, $f(x, y) = x^2 y + \sin(xy)$.

```
Command Window
>> syms x y
>> f = x^2*y+sin(x*y);
>> dfdx = diff(f, x) % Partial derivative with respect to x

dfdx =

2*x*y + y*cos(x*y)
fx
```

Fig. 4.25 Calculation of $\frac{\partial f}{\partial x}$ for $f(x, y) = x^2 y + \sin(xy)$

Example 4.37 The code in Fig. 4.26 calculates $\frac{\partial f}{\partial y}$ where, $f(x, y) = x^2 y + \sin(xy)$.

```
Command Window
>> syms x y
>> f = x^2*y+sin(x*y);
>> dfdx = diff(f, y) % Partial derivative with respect to y

dfdx =

x*cos(x*y) + x^2

fx
```

Fig. 4.26 Calculation of $\frac{\partial f}{\partial y}$ for $f(x, y) = x^2 y + \sin(xy)$

Example 4.38 The code in Fig. 4.27 calculates $\frac{\partial^2 f}{\partial x^2}$ where, $f(x, y) = x^2 y + \sin(xy)$.

```
Command Window
>> syms x y
>> f = x^2*y+sin(x*y);
>> d2fdx2 = diff(f, x, 2) % Second-order partial derivative with respect to x

d2fdx2 =

2*y - y^2*sin(x*y)
fx
```

Fig. 4.27 Calculation of $\frac{\partial^2 f}{\partial x^2}$ for $f(x, y) = x^2 y + \sin(xy)$

Example 4.39 The code in Fig. 4.28 calculates $\frac{\partial^2 f}{\partial y \partial x}$ where, $f(x, y) = x^2 y + \sin(xy)$.

```
Command Window
>> syms x y
>> f = x^2*y+sin(x*y);
>> dfdx = diff(f, x); % First, differentiate with respect to x
>> d2fdxdy = diff(dfdx, y); % Then, differentiate the result with respect to y
>> d2fdxdy = simplify(d2fdxdy) % Simplify the result

d2fdxdy =

2*x + cos(x*y) - x*y*sin(x*y)

fx >> |
```

Fig. 4.28 Calculation of $\frac{\partial^2 f}{\partial y \partial x}$ for $f(x, y) = x^2 y + \sin(xy)$

Example 4.40 The code in Fig. 4.29 calculates $\int \frac{1}{3x+4} dx$.

Fig. 4.29 Calculation of $\int \frac{1}{3x+4} dx$

```
Command Window
>> syms x
>> int(1/(3*x+4),x)

ans =

log(x + 4/3)/3

fx >>
```

Example 4.41 The code in Fig. 4.30 calculates $\int x^3 \cos(x) dx$.

Fig. 4.30 Calculation of $\int x^3 \cos(x) dx$

```
Command Window
>> syms x
>> int(x^3*cos(x),x)

ans =

cos(x)*(3*x^2 - 6) - sin(x)*(- x^3 + 6*x)

fx >> |
```

Example 4.42 The code in Fig. 4.31 calculates $\int_0^1 \frac{1}{3x+4} dx$.

Fig. 4.31 Calculation of $\int_0^1 \frac{1}{3x+4} dx$

```
Command Window
>> int(1/(3*x+4),x,0,1)

ans =

log(14^(1/3)/2)

>> eval(ans)

ans =

    0.1865

fx >>
```

Example 4.43 The code in Fig. 4.32 calculates $\int_0^\infty x^3 e^{-5x} dx$.

Fig. 4.32 Calculation of $\int_0^\infty x^3 e^{-5x} dx$

```
Command Window
>> syms x
>> int(x^3*exp(-5*x),x,0,inf)

ans =

6/625

>> eval(ans)

ans =

    0.0096

fx >>
```

Example 4.44 The code in Fig. 4.33 calculates $\int_{-\frac{\pi}{6}}^{\pi} e^{j4x}dx$.

Fig. 4.33 Calculation of $\int_{-\frac{\pi}{6}}^{\pi} e^{j4x}dx$

```
Command Window
>> syms x
>> int(exp(j*4*x),x,-pi/6,pi)

ans =

3^(1/2)/8 - 3i/8

>> eval(ans)

ans =

   0.2165 - 0.3750i

fx >>
```

Example 4.45 The code in Fig. 4.34 calculates $\int_{-\pi}^{\pi} x^2e^{j4x}dx$.

Fig. 4.34 Calculation of $\int_{-\pi}^{\pi} x^2e^{j4x}dx$

```
Command Window
>> int(x^2*exp(j*4*x),x,-pi,pi)

ans =

pi/4

fx >>
```

Example 4.46 `integral2(fun,xmin,xmax,ymin,ymax)` approximates the integral of the function $fun(x, y)$ over the planar region xmin ≤ x ≤ xmax and ymin(x) ≤ y ≤ ymax(x). The `xmin` and `xmax` arguments in the `integral2` function must be scalar values, while the `ymin` and `ymax` arguments can be either scalar values or function handles.

The code in Fig. 4.35 calculates $\int_0^2 \int_1^2 (x - 3y^2)dydx$.

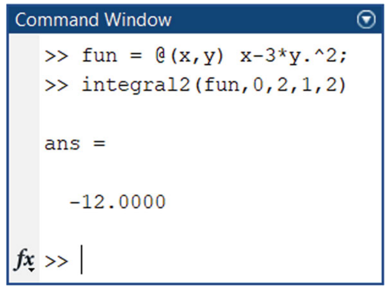


Fig. 4.35 Calculation of $\int_0^2 \int_1^2 \left(x - 3y^2\right)dydx$

Example 4.47 The code in Fig. 4.36 calculates $\int_0^3 \int_1^2 (x^2 y)dydx$.

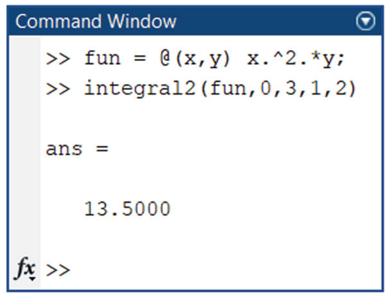


Fig. 4.36 Calculation of $\int_0^3 \int_1^2 \left(x^2 y\right)dydx$

Example 4.48 The code in Fig. 4.37 calculates $\int_{0.5}^1 \int_0^4 (xy)dydx$.

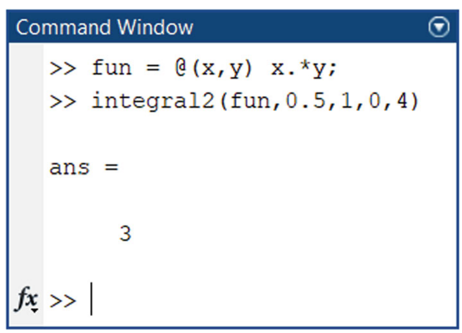


Fig. 4.37 Calculation of $\int_{0.5}^1 \int_0^4 (xy)dydx$

Example 4.49 The code in Fig. 4.38 calculates $\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)dydx$.

Fig. 4.38 Calculation of $\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)dydx$

```
Command Window
>> fun = @(x,y) x+2*y;
>> ymin=@(x) 2*x.^2;
>> ymax=@(x) 1+x.^2;
>> integral2(fun,-1,1,ymin,ymax)

ans =

    2.1333

fx >>
```

Example 4.50 The code in Fig. 4.39 calculates $\int_{0}^{8}\int_{\sqrt[3]{x}}^{2}\frac{1}{1+y^4}dydx$.

Fig. 4.39 Calculation of $\int_{0}^{8}\int_{\sqrt[3]{x}}^{2}\frac{1}{1+y^4}dydx$

```
Command Window
>> fun = @(x,y) 1./(1+y.^4);
>> ymin = @(x) x.^(1/3);
>> integral2(fun,0,8,ymin,2)

ans =

    0.7083
```

Example 4.51 The `integral2` function is designed to evaluate double integrals of the form $\int\int f(x,y)dydx$. This means that the inner integral is with respect to y, and the outer integral is with respect to x. You cannot directly use `integral2` to calculate integrals of the form $\int\int f(x,y)dxdy$. However, you can often manipulate the order of integration or use other numerical techniques to compute such integrals. Let's study an example.

For instance, consider $\int_{0}^{4}\int_{\frac{y}{2}}^{\sqrt{y}}\left(x^2+y\right)dxdy$. The integration region is shown in Fig. 4.40.

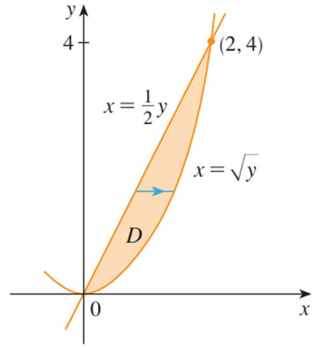


Fig. 4.40 D as a Type II region

From Fig. 4.41, we have: $\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} \left(x^2 + y\right)dxdy = \int_0^2 \int_{x^2}^{2x} \left(x^2 + y\right)dydx$.

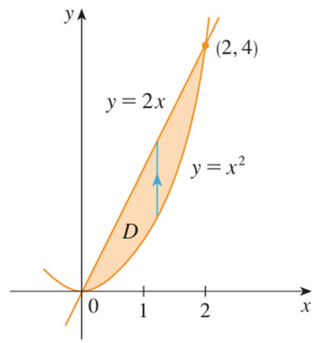


Fig. 4.41 D as a Type I region

The code shown in Fig. 4.42 calculates $\int_0^2 \int_{x^2}^{2x} \left(x^2 + y\right)dydx$.

Fig. 4.42 Calculation of $\int_0^2 \int_{x^2}^{2x} \left(x^2 + y\right)dydx$

Command Window

```
>> fun = @(x,y) x.^2+y;
>> ymin = @(x) x.^2;
>> ymax = @(x) 2.*x;
>> integral2(fun,0,2,ymin,ymax)

ans =

    3.7333

fx >>
```

Example 4.52 `integral3(fun,xmin,xmax,ymin,ymax,zmin,zmax)` approximates the integral of the function *fun*(x, y, z) over the region xmin ≤ x ≤ xmax, ymin(x) ≤ y ≤ ymax(x) and zmin(x,y) ≤ z ≤ zmax(x,y).

`integral3` is designed to calculate triple integrals of the form $\int\int\int f(x, y, z)dzdydx$. This means that the innermost integral is with respect to z, the middle integral is with respect to y, and the outermost integral is with respect to x.

The code in Fig. 4.43 calculates $\int_0^{\pi}\int_0^1\int_{-1}^1 (y\sin(x) + z\cos(x))dzdydx$.

```
Command Window
>> fun=@(x,y,z) y.*sin(x)+z.*cos(x);
>> integral3(fun,0,pi,0,1,-1,1)

ans =

    2.0000

fx >>
```

Fig. 4.43 Calculation of $\int_0^{\pi}\int_0^1\int_{-1}^1 (y\sin(x) + z\cos(x))dzdydx$

Example 4.53 The code in Fig. 4.44 calculates $\int_0^1\int_{-1}^2\int_0^3 (y\sin(x) + z\cos(x))dzdydx$.

```
Command Window
>> fun=@(x,y,z) x.*y.*z.^2;
>> integral3(fun,0,1,-1,2,0,3)

ans =

    6.7500

fx >>
```

Fig. 4.44 Calculation of $\int_0^1\int_{-1}^2\int_0^3 (y\sin(x) + z\cos(x))dzdydx$

Example 4.54 The code in Fig. 4.45 calculates $\int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}}\sqrt{x^2+y^2}dzdydx$.

```
Command Window
>> fun=@(x,y,z)sqrt(x.^2+z.^2);
>> ymin=@(x) x.^2;
>> zmin=@(x,y) -sqrt(y-x.^2);
>> zmax=@(x,y)  sqrt(y-x.^2);
>> integral3(fun,-2,2,ymin,4,zmin,zmax)

ans =

   26.8083

fx >>
```

Fig. 4.45 Calculation of $\int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}}\sqrt{x^2+y^2}dzdydx$

Example 4.55 The code in Fig. 4.46 divides x^3+x by $x-1$. According to the result shown in Fig. 4.46, $x^3+x=\left(x^2+x+2\right)(x-1)+2$.

```
Command Window
>> dividend=[1 0 1 0];
>> divisor=[1 -1];
>> [quotient, remainder] = deconv(dividend, divisor)

quotient =

     1     1     2


remainder =

     0     0     0     2

fx >>
```

Fig. 4.46 Dividing x^3+x by $x-1$

Example 4.56 The code in Fig. 4.47 multiplies $x^2 + 2x + 3$ by $x^5 + x^4 + 2x^2 + 7x + 1$. According to the result shown in Fig. 4.47, $\left(x^2 + 2x + 3\right) \times \left(x^5 + x^4 + 2x^2 + 7x + 1\right) = x^7 + 3x^6 + 5x^5 + 5x^4 + 11x^3 + 21x^2 + 23x + 3$.

```
Command Window
>> conv([1 2 3],[1 1 0 2 7 1])

ans =

     1     3     5     5    11    21    23     3

fx >>
```

Fig. 4.47 Multiplying $x^2 + 2x + 3$ and $x^5 + x^4 + 2x^2 + 7x + 1$

Example 4.57 The code in Fig. 4.48 multiplies $x^2 + 2x + 3$ by $x^5 + x^4 + 2x^2 + 7x + 1$ and then substitute $x = 3$ to obtain the numerical result.

```
Command Window
>> conv([1 2 3],[1 1 0 2 7 1])

ans =

     1     3     5     5    11    21    23     3

>> polyval(ans,3)

ans =

        6552

fx >>
```

Fig. 4.48 Calculation of $\left(x^2 + 2x + 3\right)\left(x^5 + x^4 + 2x^2 + 7x + 1\right)$ at $x = 3$

Example 4.58 The code in Fig. 4.49 calculates the partial fraction decomposition of $\frac{2x^4+x^3-5x^2+3x+3}{x^3-x^2-x+1}$. According to the result shown in Fig. 4.49, $\frac{2x^4+x^3-5x^2+3x+3}{x^3-x^2-x+1} = 2x + 3 + \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-1}{x+1}$.

```
Command Window
>> [r,p,k]=residue([2 1 -5 3 3],[1 -1 -1 1])

r =

    1.0000
    2.0000
   -1.0000


p =

    1.0000
    1.0000
   -1.0000


k =

     2     3

fx >>
```

Fig. 4.49 Partial fraction decomposition of $\frac{2x^4+x^3-5x^2+3x+3}{x^3-x^2-x+1}$

Example 4.59 The code in Fig. 4.50 calculates the partial fraction decomposition of $\frac{2x^2-x+4}{x^3+4x}$. According to the result shown in Fig. 4.50, $\frac{2x^2-x+4}{x^3+4x} = \frac{0.5+0.25j}{x-2j} + \frac{0.5-0.25j}{x-(-2j)} + \frac{1}{x}$.

```
Command Window
>> [r,p,k]=residue([2 -1 4],[1 0 4 0])

r =

   0.5000 + 0.2500i
   0.5000 - 0.2500i
   1.0000 + 0.0000i


p =

   0.0000 + 2.0000i
   0.0000 - 2.0000i
   0.0000 + 0.0000i


k =

     []

fx >>
```

Fig. 4.50 Partial fraction decomposition of $\frac{2x^2-x+4}{x^3+4x}$

Remember that $z + \overline{z} = 2Re\{z\}$. Therefore, $\frac{0.5+0.25j}{x-2j} + \frac{0.5-0.25j}{x-(-2j)} = 2Re\left\{\frac{0.5+0.25j}{x-2j}\right\} = 2Re\left\{\frac{(0.5+0.25j)(x+2j)}{x^2+4}\right\} = 2\frac{0.5x-0.25\times2}{x^2+4} = \frac{x-1}{x^2+4}$. You can use MATLAB to check this result (Fig. 4.51).

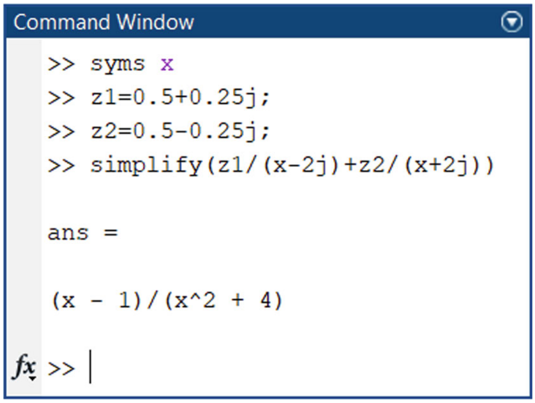


Fig. 4.51 Simplifying $\frac{0.5+0.25j}{x-2j} + \frac{0.5-0.25j}{x-(-2j)}$

So, the partial fraction decomposition of given expression is $\frac{2x^2-x+4}{x^3+4x} = \frac{1}{x} + \frac{x-1}{x^2+4}$.

Example 4.60 The code in Fig. 4.52 calculates $\Gamma(6)$ and $\Gamma(4.3)$.

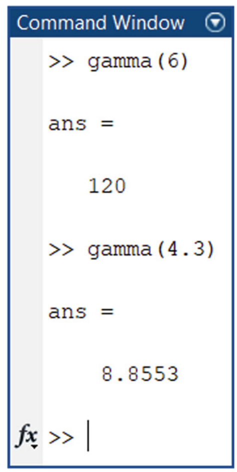


Fig. 4.52 Calculating $\Gamma(6)$ and $\Gamma(4.3)$

Example 4.61 The code in Fig. 4.53 draws the $\Gamma(x)$ on the interval $[0, 6]$. Output of this code is shown in Fig. 4.54.

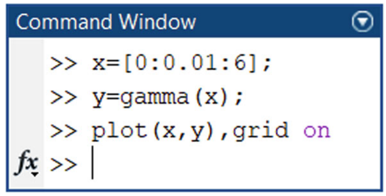


Fig. 4.53 Plotting the $\Gamma(x)$ function on [0, 6]

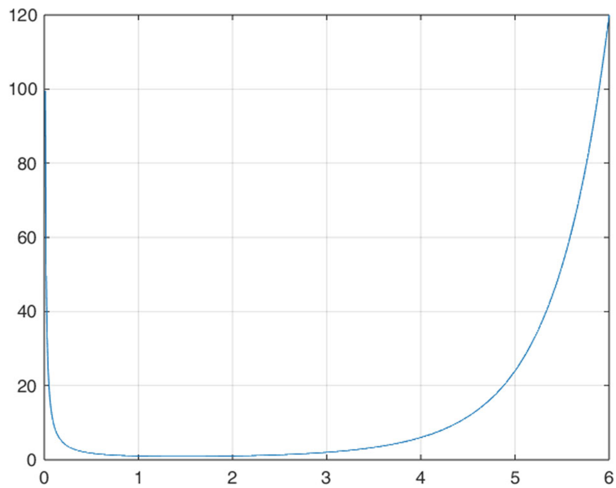


Fig. 4.54 Output of the code shown in Fig. 4.53

References for Further Study

1. Advanced Engineering Mathematics (10th edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8th edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

Power Series Expansions

5

5.1 Introduction

A power series (in one variable) is an infinite series of the form $\sum_{n=0}^{\infty} a_n (x-c)^n$, where a_n are coefficients, x is a variable, and c is a constant called the center of the series. These series provide a powerful tool for approximating functions, solving differential equations, and understanding the behavior of complex functions. By representing functions as power series, we can often simplify calculations, analyze their properties more easily, and even extend their domains of definition.

This chapter begins with a review of Taylor, McLaurin, and Laurent series. The second part demonstrates MATLAB's application to the problems discussed.

5.2 Taylor and Maclaurin Series

The Taylor series (or Taylor expansion) of a real or complex-valued function $f(x)$, that is infinitely differentiable at a real or complex number a, is the power series

$$\begin{aligned} f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Here, $n!$ denotes the factorial of n. The function $f^{(n)}(a)$ denotes the n-th derivative of f evaluated at the point a. The derivative of order zero of f is defined to be f itself and $(x-a)^0$ and 0! are both defined to be 1.

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_5

A Maclaurin series is a special case of a Taylor series where the point of expansion is $a = 0$. Therefore, a Maclaurin series is a Taylor series centered at 0. General form of Maclaurin series is:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Important Maclaurin series are (First 5 formulas worth memorizing):

1. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad for|x| < 1$
2. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad forallx$
3. $\ln(x+1) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad for|x| < 1$
4. $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad forallx$
5. $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad forallx$
6. $\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad for|x| < \frac{\pi}{2}$
7. $\sec(x) = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots \quad for|x| < \frac{\pi}{2}$
8. $\arcsin(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \quad for|x| \le 1$
9. $\arccos(x) = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \cdots \quad for|x| \le 1$
10. $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad for|x| \le 1$
11. $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \quad forallx$
12. $\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \quad forallx$
13. $\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \quad for\,|x| < \frac{\pi}{2}$
14. $\text{arcsinh}(x) = x - \frac{x^3}{6} + \frac{3x^5}{40} - \cdots \quad for\,|x| \le 1$
15. $\text{arctanh}(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \quad for\,|x| < 1$

Example 5.1 Find the Taylor series expansion of $f(x) = ln(x)$ about $x = 1$.

$$f(x) = \ln(x) \Rightarrow f(1) = \ln(1) = 0$$

$$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$$

$$f''^{(x)} = -\frac{1}{x^2} \Rightarrow f''^{(1)} = -1$$

$$f^{(3)}(x) = \frac{2}{x^3} \Rightarrow f^{(3)}(1) = 2$$

$$f^{(4)}(x) = -\frac{2 \times 3}{x^4} \Rightarrow f^{(4)}(1) = -2 \times 3$$

$$\begin{aligned} f(x) &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \end{aligned}$$

$$f(x) = \ln(x) = 0 + 1(x-1) - \frac{1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{(2)(3)(x-1)^4}{4!} + \cdots$$

$$f(x) = \ln(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)!(x-1)^n}{n!}$$

$$f(x) = \ln(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}$$

Exercise: Determine the first five terms of the Maclaurin series for $\sqrt{x}$ and $\cos^2(x)$.

5.3 Taylor Series for Functions of Two Variables

The Taylor series for a function of two variables, $f(x, y)$, expanded around the point (a, b) is given by:

$$\begin{aligned} T(x,y) &= f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) \\ &\quad + \frac{1}{2!}\big((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\big) + \cdots \end{aligned}$$

Example 5.2 Calculate the Taylor series expansion of the function $f(x, y) = e^x \cdot \ln(1 + y)$ up to the 2nd order around the point (0, 0).

$$f_x = e^x \cdot \ln(1+y)$$

$$f_y = \frac{e^x}{1+y}$$

$$f_{xx} = e^x \cdot \ln(1+y)$$

$$f_{yy} = -\frac{e^x}{(1+y)^2}$$

$$f_{xy} = f_{yx} = \frac{e^x}{1+y}$$

Evaluating these derivatives in the origin gives the Taylor coefficients:

$$f_x(0,0) = 0$$

$$f_y(0,0) = 1$$

$$f_{xx}(0,0) = 0$$

$$f_{yy}(0,0) = -1$$

$$f_{xy}(0,0) = f_{yx}(0,0) = 1$$

Substituting these variables in the general formula gives:

$$T(x,y) = y + xy - \frac{1}{2}y^2$$

5.4 Laurent Series

A Laurent series is a representation of an analytic function as an infinite series involving both positive and negative integer powers of the variable $(z - z_0)$, where z_0 is a complex number. It is a generalization of the Taylor series, which only involves non-negative powers of $(z - a)$.

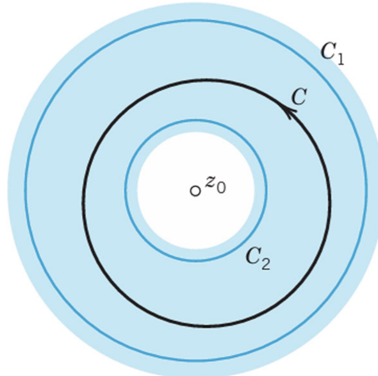
Let complex function $f(z)$ be analytic in a domain containing two concentric circles C_1 and C_2 with center z_0 and the annulus between them (blue in Fig. 5.1). Then can be represented by the Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^* \quad (n = 0, \pm 1, \pm 2, \ldots).$$

Fig. 5.1 Laurent's theorem



In most cases, the Laurent series can be found without using the $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*$ formula. Let's study some numeric examples.

Example 5.3 Find the Laurent series of $z^{-5}\sin(z)$ with center 0.

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \Rightarrow z^{-5}\sin(z)$$
$$= z^{-4} - \frac{z^{-2}}{3!} + \frac{1}{5!} - \frac{z^2}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4}$$

Here the "annulus" of convergence is the whole complex plane without the origin ($|z| > 0$).

Example 5.4 Find the Laurent series of $z^2 e^{\frac{1}{z}}$ with center 0.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} \Rightarrow z^2 e^{\frac{1}{z}}$$
$$= \sum_{n=0}^{\infty} \frac{1}{n! z^{n-2}} = z^2 + z + \frac{1}{2} + \frac{1}{3! z} + \frac{1}{4! z^2} + \cdots$$

Here the "annulus" of convergence is the whole complex plane without the origin ($|z| > 0$)

Example 5.5 Develop $\frac{1}{1-z}$ **(a)** in nonnegative powers of z **(b)** in negative powers of z.

(a) when $|z| < 1$: $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

(b) when $|z| > 1$:
$$\frac{1}{1-z} = \frac{\frac{1}{z}}{\frac{1-z}{z}} = \frac{\frac{1}{z}}{\frac{1}{z}-1} = -\frac{\frac{1}{z}}{1-\frac{1}{z}} = -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)$$
$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = -\frac{1}{z} - \frac{1}{z^2} - \cdots$$

Example 5.6 Find all Laurent series of $\frac{1}{z^3-z^4}$ with center 0.

$$\frac{1}{z^3-z^4} = \frac{1}{z^3(1-z)} = \frac{1}{z^3} \times \frac{1}{1-z}$$

(a) when $0 < |z| < 1$:
$$\frac{1}{z^3} \times \frac{1}{1-z} = \frac{1}{z^3}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-3}$$
$$= \frac{1}{z^{-3}} + \frac{1}{z^{-2}} + \frac{1}{z^{-1}} + 1 + z^1 + z^2 + z^3 + \cdots$$

(b) when $|z| > 0$:
$$\frac{1}{z^3} \times \frac{1}{1-z} = \frac{1}{z^3} \times \frac{\frac{1}{z}}{\frac{1-z}{z}} = -\frac{1}{z^4} \times \frac{1}{1-\frac{1}{z}} = -\frac{1}{z^4}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$
$$= -\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+4} = -\frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \cdots$$

Example 5.7 Find all Taylor and Laurent series of $f(z) = \frac{-2z+3}{z^2-3z+2}$ with center 0.

$$f(z) = \frac{-2z+3}{z^2-3z+2} = \frac{-2z+3}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{-1}{z-2}$$

(a) when $|z| < 1$: $\frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n$

(b) when $|z| > 1$:
$$-\frac{1}{1-z} = -\frac{\frac{1}{z}}{\frac{1-z}{z}} = -\frac{\frac{1}{z}}{\frac{1}{z}-1} = \frac{\frac{1}{z}}{1-\frac{1}{z}} = \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)$$
$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = \frac{1}{z} + \frac{1}{z^2} + \cdots$$

(c) when $|z| < 2$:
$$\frac{-1}{z-2} = \frac{\frac{-1}{2}}{\frac{z-2}{2}} = \frac{\frac{1}{2}}{1-\frac{z}{2}} = \frac{1}{2}\frac{1}{1-\frac{z}{2}}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \frac{z^3}{16} + \cdots$$

(d) when $|z| > 2$:
$$\frac{-1}{z-2} = \frac{\frac{-1}{z}}{\frac{z-2}{z}} = \frac{-\frac{1}{z}}{1-\frac{2}{z}} = -\frac{1}{z}\frac{1}{1-\frac{2}{z}}$$
$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} = -\frac{1}{z} - \frac{2}{z^2} - \frac{4}{z^3} - \frac{8}{z^4} - \cdots$$

We will study three regions (Fig. 5.2):

(a) Region I ($0 < |z| < 1$): $f(z) = \frac{-2z+3}{z^2-3z+2} = \frac{-1}{z-1} + \frac{-1}{z-2} = -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$

(b) Region II ($1 < |z| < 2$): $f(z) = \frac{-2z+3}{z^2-3z+2} = \frac{-1}{z-1} + \frac{-1}{z-2} = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$

(c) Region III ($|z| > 2$): $f(z) = \frac{-2z+3}{z^2-3z+2} = \frac{-1}{z-1} + \frac{-1}{z-2} = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$

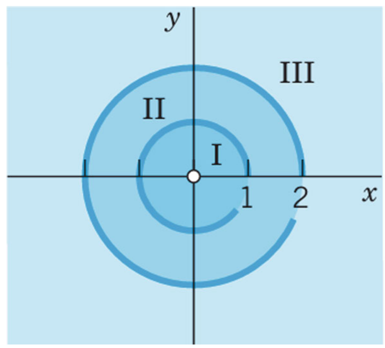


Fig. 5.2 Different complex regions for Example 5.7

5.5 Problem Solving with MATLAB®

This section provides examples of how MATLAB can be employed to solve the types of problems encountered in this chapter.

Example 5.8 The code in Fig. 5.3 calculates the Maclaurin series of $\frac{\sin(x)}{x}$ up to the 10th degree term.

```
Command Window
>> syms x
>> taylor(sin(x)/x,x,0,'order',10)

ans =

x^8/362880 - x^6/5040 + x^4/120 - x^2/6 + 1

>> vpa(ans,3)

ans =

2.76e-6*x^8 - 1.98e-4*x^6 + 0.00833*x^4 - 0.167*x^2 + 1.0

fx >>
```

Fig. 5.3 Taylor series of $\frac{\sin(x)}{x}$

Example 5.9 The code in Fig. 5.4 calculates the Taylor series of $\sin(x)$ centered at $\frac{\pi}{2}$, up to the 6th degree term.

```
Command Window
>> syms x
>> taylor(sin(x),x,pi/2,'order',6)

ans =

(x - pi/2)^4/24 - (x - pi/2)^2/2 + 1

>> vpa(ans,4)

ans =

0.04167*(x - 1.571)^4 - 0.5*(x - 1.571)^2 + 1.0

fx >>
```

Fig. 5.4 Taylor series of $\sin(x)$

Example 5.10 The following code computes the second-order Taylor polynomial of $f(x, y) = 1 + xe^y$ centered at $(1, 0)$.

```
clc
clear all

% Define the function and point of expansion
syms x y;
f = 1+x*exp(y);
a = 1;
b = 0;

% Calculate partial derivatives
dfdx = diff(f, x);
dfdy = diff(f, y);
d2fdx2 = diff(dfdx, x);
d2fdxy = diff(dfdx, y);
d2fdy2 = diff(dfdy, y);

% Evaluate derivatives at the point (a, b)
f0 = subs(f, {x, y}, {a, b});
dfdx0 = subs(dfdx, {x, y}, {a, b});
dfdy0 = subs(dfdy, {x, y}, {a, b});
d2fdx2_0 = subs(d2fdx2, {x, y}, {a, b});
d2fdxy_0 = subs(d2fdxy, {x, y}, {a, b});
d2fdy2_0 = subs(d2fdy2, {x, y}, {a, b});

% Construct the Taylor series up to second order
taylor_series = simplify(f0 + dfdx0*(x-a) + dfdy0*(y-b) + ...
                (1/2)*(d2fdx2_0*(x-a)^2 + 2*d2fdxy_0*(x-a)*(y-b) + ...
                d2fdy2_0*(y-b)^2))
```

Use the `edit` command (Fig. 5.5) to open the Editor (Fig. 5.6).

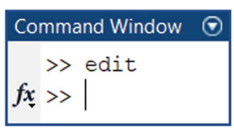


Fig. 5.5 `edit` command

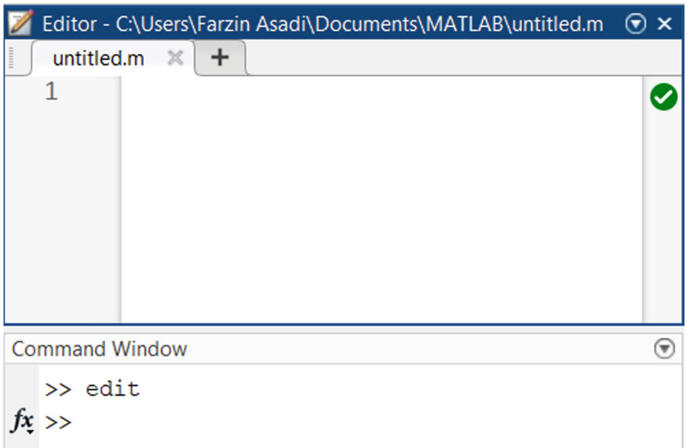


Fig. 5.6 MATLAB editor

Type the given code in the Editor (Fig. 5.7) and press the Ctrl + S to save it.

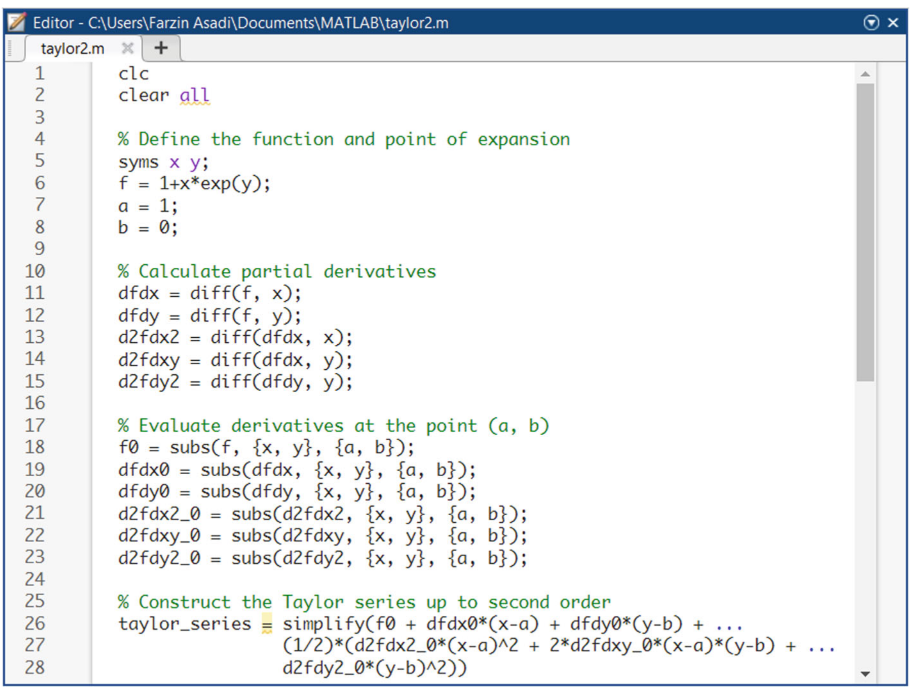


Fig. 5.7 The code typed in the MATLAB editor

Press the F5 key of your keyboard or click the Run button (Fig. 5.8) to execute the code.

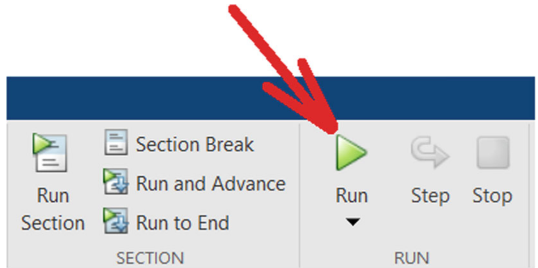


Fig. 5.8 The run button

Output of the code is shown in Fig. 5.9. Therefore, second order Taylor polynomial of $f(x, y) = 1 + xe^y$ centered at $(1, 0)$ is $\frac{y^2}{2} + xy + x + 1$.

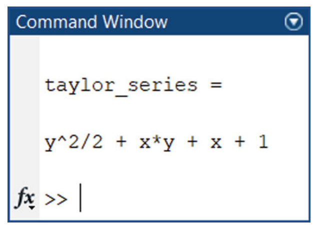


Fig. 5.9 Output of the code shown in Fig. 5.7

The code can be adapted to calculate the second-order Taylor polynomial of other two-variable functions. For instance, in order to calculate the second order Taylor polynomial of $f(x, y) = e^x ln(1 + y)$ centered at $(0, 0)$ change the 6th, 7th and 8th line of the code to:

```
f = exp(x)*log(1 + y);
a = 0;
b = 0;
```

Second order Taylor polynomial of $f(x, y) = e^x \ln(1 + y)$ centered at $(0, 0)$ is shown in Fig. 5.10.

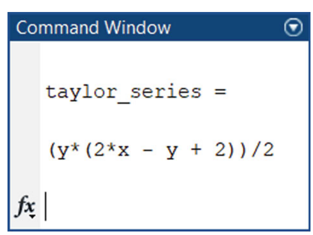


Fig. 5.10 Output for $f(x, y) = e^x \ln(1 + y)$

References for Further Study

1. Advanced Engineering Mathematics (10th edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8th edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

6 Interpolation, Extrapolation, and Curve Fitting

6.1 Introduction

Interpolation, extrapolation, and curve fitting are techniques used to estimate values of a function at points where data is not explicitly given.

Interpolation involves estimating values within the range of known data points, while extrapolation estimates values outside this range. Curve fitting, on the other hand, aims to find a mathematical function that best approximates the given data points, which can be used for both interpolation and extrapolation. These techniques are widely used in various fields, including engineering, science, and statistics, to analyze data, make predictions, and gain insights into underlying trends.

This chapter starts with a review of interpolation, extrapolation, and curve fitting. The second part demonstrates MATLAB's application to the problems discussed.

6.2 Linear Interpolation

Interpolation is a method of estimating unknown values that lie between known data points. It involves constructing a function that passes through the given data points and then using that function to estimate values at intermediate points.

Different types of interpolation exist. Linear interpolation is a simple yet effective method for estimating values between two known data points. It assumes a linear relationship between the data points and constructs a straight line to approximate the function.

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_6

Let's see how linear interpolation works. Two data points (x_1, y_1) and (x_2, y_2) are given (Fig. 6.1). We want to interpolate value y for an arbitrary x_3 between x_1 and x_2 (Fig. 6.2).

Fig. 6.1 Data points A and B

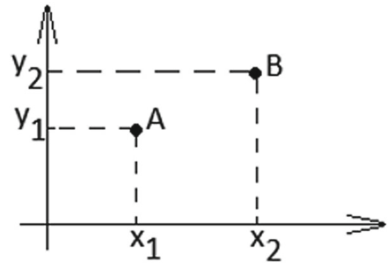


Fig. 6.2 x_3 falls between x_1 and x_2

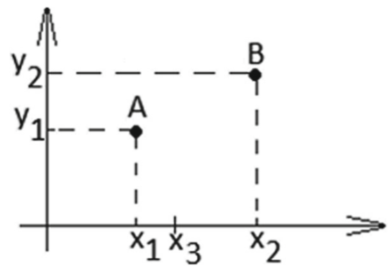
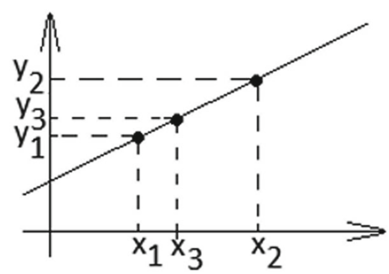


Figure 6.3 illustrates the line passing through points A and B. According to Fig. 6.3, we have:

$$\frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_3}{x_2 - x_3} \Rightarrow y_3x_2 - y_3x_3 - y_1x_2 + y_1x_3$$
$$= y_2x_3 - y_2x_1 - y_3x_3 + y_3x_1 \Rightarrow y_3 = \frac{y_2(x_3 - x_1) - y_1(x_3 - x_2)}{x_2 - x_1}$$

Fig. 6.3 Line connects $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ to $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$



Let's study a numeric example. For instance, value of y_3 for $A = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$ and $x_3 = 1.5$ equals to:

$$y_3 = \frac{y_2(x_3 - x_1) - y_1(x_3 - x_2)}{x_2 - x_1} \Rightarrow y_3 = \frac{11 \times (1.5 - 1) - 7 \times (1.5 - 3)}{3 - 1} = \frac{5.5 + 10.5}{2} = 8$$

6.3 Linear Extrapolation

Linear extrapolation is a method used to estimate values outside the range of known data by extending a straight line beyond the existing data points.

Let's see how linear extrapolation works. Two data points (x_1, y_1) and (x_2, y_2) are given (Fig. 6.4). $x_2 > x_1$. We want to extrapolate value y for an arbitrary $x_3 > x_2$ (Fig. 6.5).

Fig. 6.4 Data points A and B

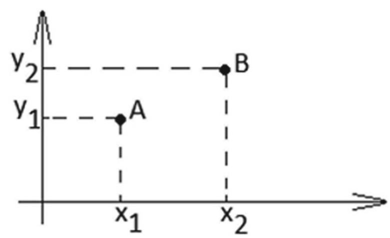


Fig. 6.5 x_3 falls outside of $[x_1, x_2]$ interval $(x_3 > x_2)$

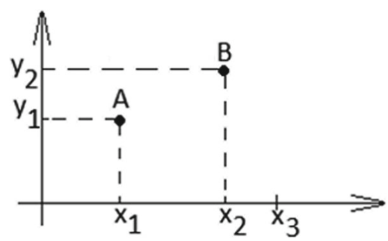
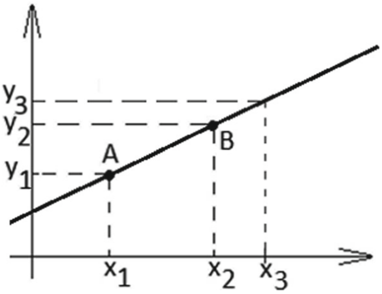


Figure 6.6 illustrates the line passing through points A and B. According to Fig. 6.6, we have:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} \Rightarrow y_3 = \frac{y_2 - y_1}{x_2 - x_1}(x_3 - x_2) + y_2$$

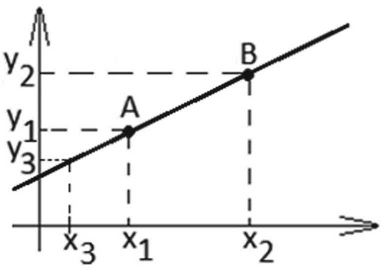
Fig. 6.6 Line connects $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ to $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$



Let's study the $x_3 < x_1$ case (Fig. 6.7), as well. In this case y_3 equals to:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_1}{x_3 - x_1} \Rightarrow y_3 = \frac{y_2 - y_1}{x_2 - x_1}(x_3 - x_1) + y_1$$

Fig. 6.7 x_3 falls outside of $[x_1, x_2]$ interval $(x_3 < x_1)$



Let's study some numeric examples.

Example 6.1 Value of y_3 for $A = \begin{bmatrix} 1 \\ 7 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$ and $x_3 = 3.2$ equals to:

$$y_3 = \frac{y_2 - y_1}{x_2 - x_1}(x_3 - x_2) + y_2 = \frac{11 - 7}{3 - 1}(3.2 - 3) + 11 = 11.4$$

Example 6.2 value of y_3 for $A = \begin{bmatrix} 1 \\ 7 \end{bmatrix}, B = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$ and $x_3 = 0.75$ equals to:

$$y_3 = \frac{y_2 - y_1}{x_2 - x_1}(x_3 - x_1) + y_1 \Rightarrow y_3 = \frac{11 - 7}{3 - 1}(0.75 - 1) + 7 = 6.5$$

6.4 Fitting a Polynomial to a Data

In this section, we will explore the process of fitting a polynomial function to a given dataset. Let's study a numeric example.

Example 6.3 Fit the data points $(1, 16.5), (1.7, 34.57), (3.2, 81.12), (4.9, 153.33)$ and $(6.1, 214.33)$ to the quadratic equation $y = ax^2 + bx + c$.

The given data points are substituted in the given quadratic equation:

$$\begin{cases} a + b + c = 16.5 \\ 2.89a + 1.7b + c = 34.57 \\ 10.24a + 3.2b + c = 81.12 \\ 24.01a + 4.9b + c = 153.33 \\ 37.21a + 6.1b + c = 214.33 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2.89 & 1.7 & 1 \\ 10.24 & 3.2 & 1 \\ 24.01 & 4.9 & 1 \\ 37.21 & 6.1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} 16.5 \\ 34.57 \\ 81.12 \\ 153.33 \\ 214.33 \end{bmatrix} \Rightarrow A. \begin{bmatrix} a \\ b \\ c \end{bmatrix} = d, A = \begin{bmatrix} 1 & 1 & 1 \\ 2.89 & 1.7 & 1 \\ 10.24 & 3.2 & 1 \\ 24.01 & 4.9 & 1 \\ 37.21 & 6.1 & 1 \end{bmatrix}, d = \begin{bmatrix} 16.5 \\ 34.57 \\ 81.12 \\ 153.33 \\ 214.33 \end{bmatrix}$$

The resulting system of equations is overdetermined. Overdetermined systems are systems of equations where there are more equations than unknowns. Typically, such systems don't have an exact solution that satisfies all equations simultaneously.

The optimization approach is employed to find the best possible solution, often in the sense of minimizing the error between the predicted and actual values. The minimization of the sum of squared errors is a widely employed technique for obtaining approximate solutions to overdetermined systems of equations. This method aims to find the solution that minimizes the sum of the squares of the differences between the observed values and the predicted values. So, we want to minimize the following cost function:

$$\begin{aligned} f(a, b, c) = {} & (a + b + c - 16.5)^2 + (2.89a + 1.7b + c - 34.57)^2 \\ & + (10.24a + 3.2b + c - 81.12)^2 + (24.01a + 4.9b + c - 153.33)^2 \\ & + (37.21a + 6.1b + c - 214.33)^2 \end{aligned}$$

The approximate solution (in the least squares sense) can be calculated using the following formula:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \left(A^T \cdot A\right)^{-1} \cdot A^T \cdot d$$

According to the results presented in Fig. 6.8, the quadratic equation $y = 3.1965x^2 + 16.0700x - 2.5187$ provides the best fit to the given data set.

```
Command Window
>> A=[1 1 1;2.89 1.7 1;10.24 3.2 1;24.01 4.9 1;37.21 6.1 1];
>> d=[16.5;34.57;81.12;153.33;214.33];
>> inv(A'*A)*A'*d

ans =

    3.1965
   16.0700
   -2.5187

fx >>
```

Fig. 6.8 Calculation of best quadratic polynomial for given data set

Exercise: Find the optimum point of the cost function $f(a, b, c)$ by setting its partial derivatives to zero. Compare this result with the solution obtained using the formula $\left(A^T \cdot A\right)^{-1} \cdot A^T \cdot d$.

6.5 Problem Solving with MATLAB®

This section provides examples of how MATLAB can be employed to solve the types of problems encountered in this chapter.

Example 6.4 Figure 6.9 shows the code for interpolating between the points $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$ at $x = 1.5$.

Fig. 6.9 Interpolating between the points $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$ at $x = 1.5$

```
Command Window
>> x=[1 3];
>> y=[7 11];
>> yi=interp1(x,y,1.5)

yi =

     8

fx >>
```

Example 6.5 Figure 6.10 shows the code for extrapolating at $x = 3.2$ using the points $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$.

Fig. 6.10 Extrapolating at $x = 3.2$ using the points $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$

```
Command Window
>> x=[1 3];
>> y=[7 11];
>> yi=interp1(x,y,3.2,'linear','extrap')

yi =

   11.4000

fx >>
```

Example 6.6 Figure 6.11 shows the code for extrapolating at $x = 0.75$ using the points $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$.

Command Window
```
>> x=[1 3];
>> y=[7 11];
>> yi=interp1(x,y,0.75,'linear','extrap')

yi =

    6.5000

>>
```

Fig. 6.11 Extrapolating at $x = 0.75$ using the points $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 11 \end{bmatrix}$

Example 6.7 In this example, we will fit a 4th order polynomial to the data provided in Table 6.1.

Table 6.1 Data for Example 6.7

x	y
1	6.1
2	− 8.2
3	− 13.1
4	52.6
5	290.9
6	844.6
7	1897.3
8	3673.40
9	6438.09
10	10,497.4

Input the data into MATLAB (Fig. 6.12).

Command Window
```
>> x=[1 2 3 4 5 6 7 8 9 10];
>> y=[6.10 -8.20 -13.10 52.60 290.9 844.60 1897.30 3673.40 6438.09 10497.4];
>>
```

Fig. 6.12 Given data points

Use the `cftool` command (Fig. 6.13) to run the Curve Fitting Toolbox® (Fig. 6.14).

```
Command Window
>> x=[1 2 3 4 5 6 7 8 9 10];
>> y=[6.10 -8.20 -13.10 52.60 290.9 844.60 1897.30 3673.40 6438.09 10497.4];
>> cftool
fx >>
```

Fig. 6.13 `cftool` command

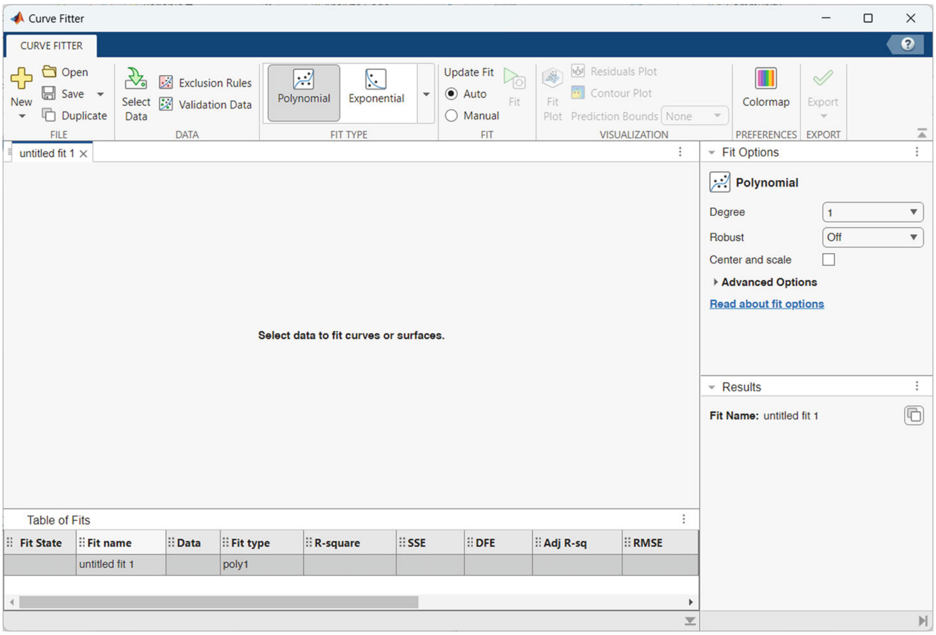
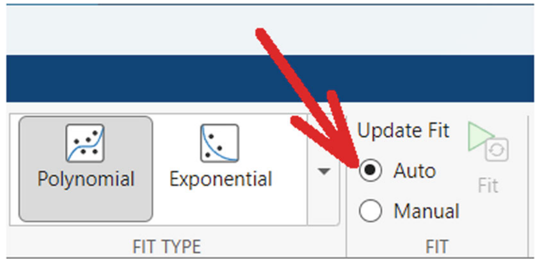


Fig. 6.14 Curve fitter window

Ensure that Auto is selected (Fig. 6.15).

Fig. 6.15 Auto radio button



Select the polynomial regression (Fig. 6.16).

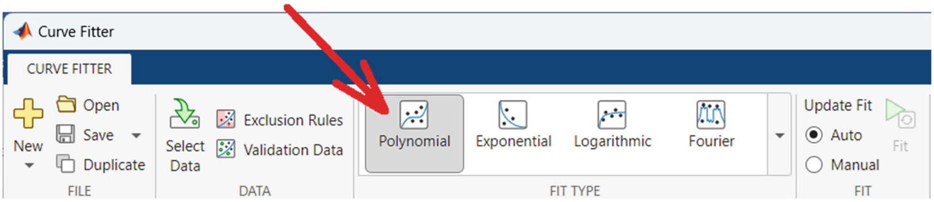


Fig. 6.16 Polynomial regression

To view a comprehensive list of regression models supported by the toolbox, click on the "Show More" button (Fig. 6.17). Supported models are shown in Fig. 6.18.

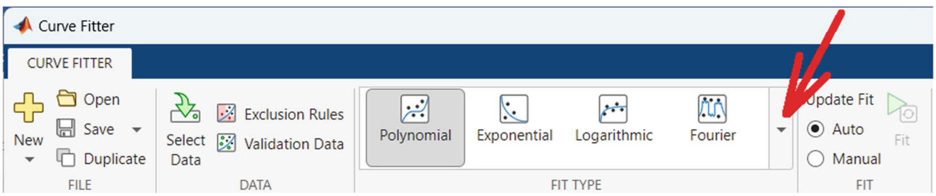


Fig. 6.17 Show more button

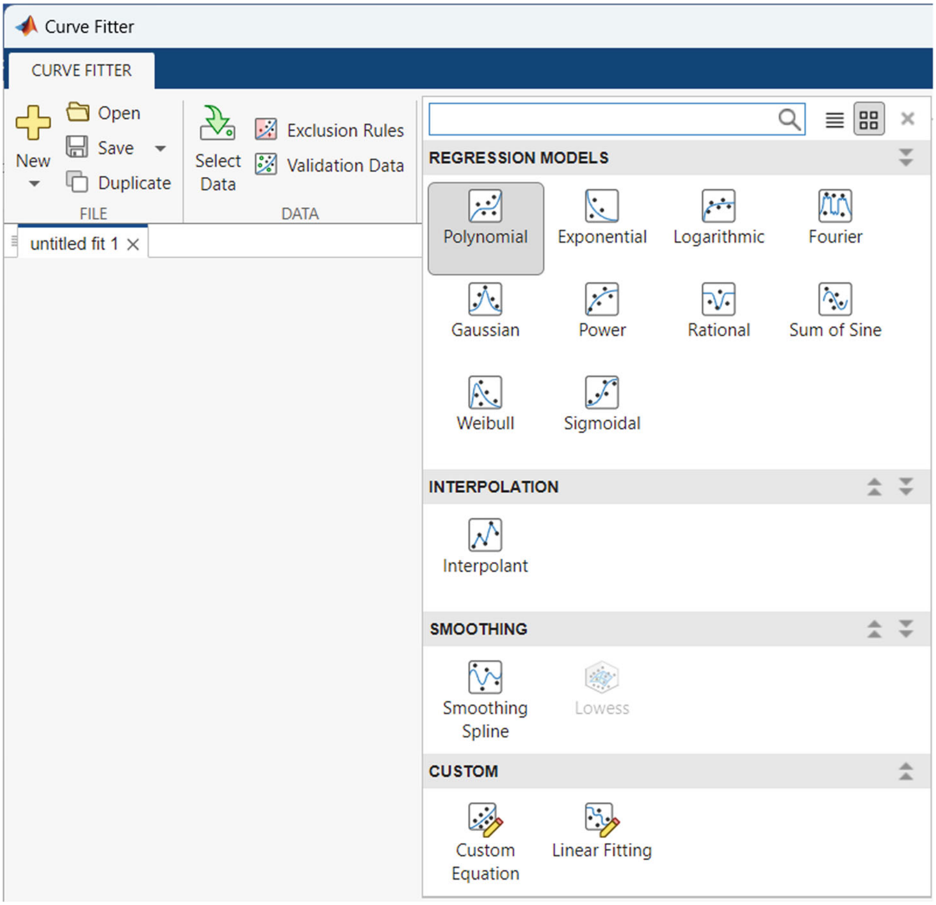
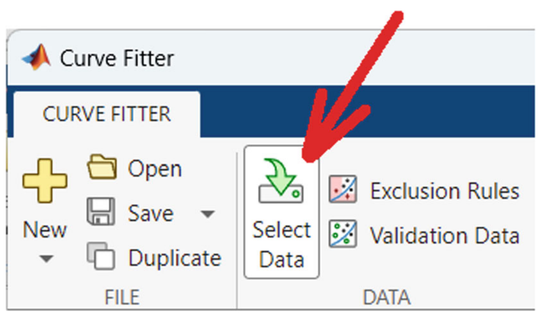


Fig. 6.18 Supported regression models

Proceed by clicking the "Select Data" button (Fig. 6.19).

Fig. 6.19 Select data button



Select "x" for X Data and "y" for Y Data (Fig. 6.20).

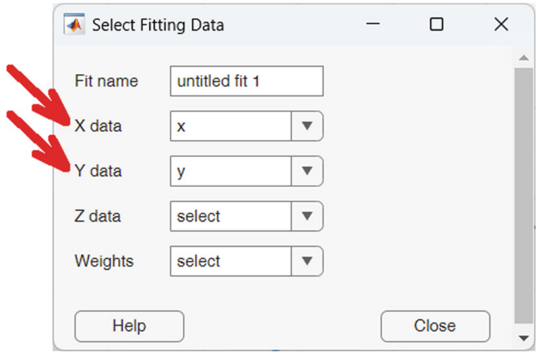


Fig. 6.20 Select fitting data window

Select "4" for Degree (Fig. 6.21).

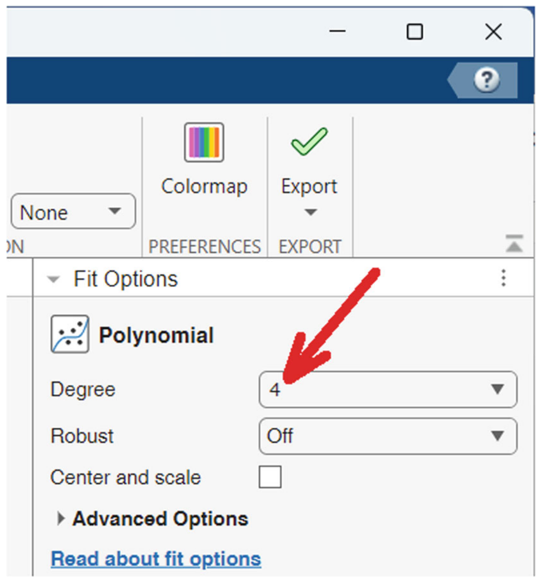


Fig. 6.21 Selection of desired polynomial degree

The toolbox now displays the polynomial fit that minimizes the sum of squared errors between the fitted curve and the original data points (Figs. 6.22 and 6.23). According to Fig. 6.23 the solution is $f(x) = 1.7000x^4 - 6.8007x^3 + 3.0046x^2 - 1.2116x + 9.4083$. The "R-squared" value, as depicted in Fig. 6.23, quantifies the goodness-of-fit of the model to the data. Higher values closer to 1 signify a better fit, while lower values closer to 0 indicate a poorer fit.

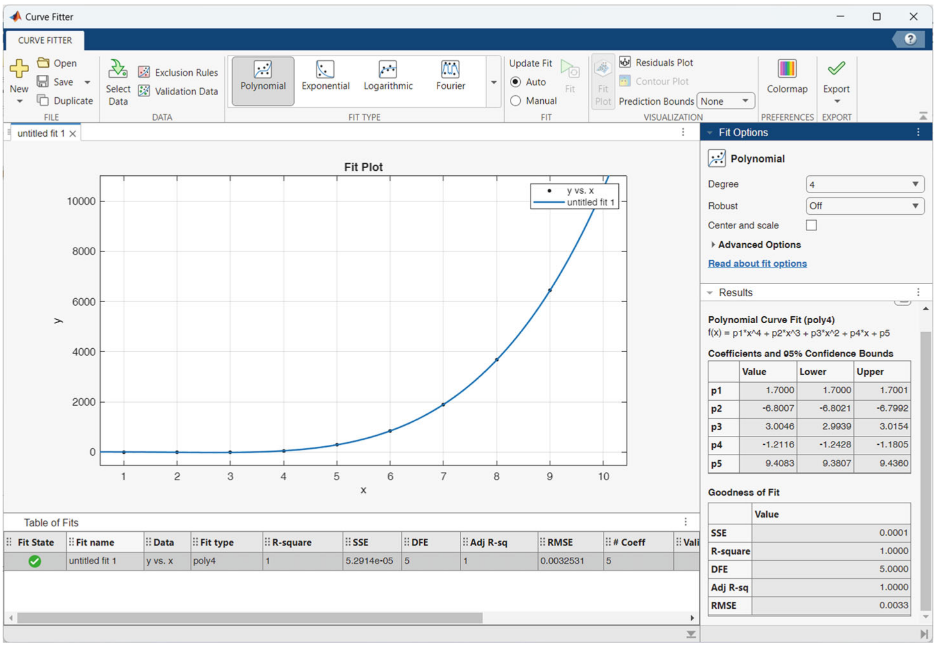
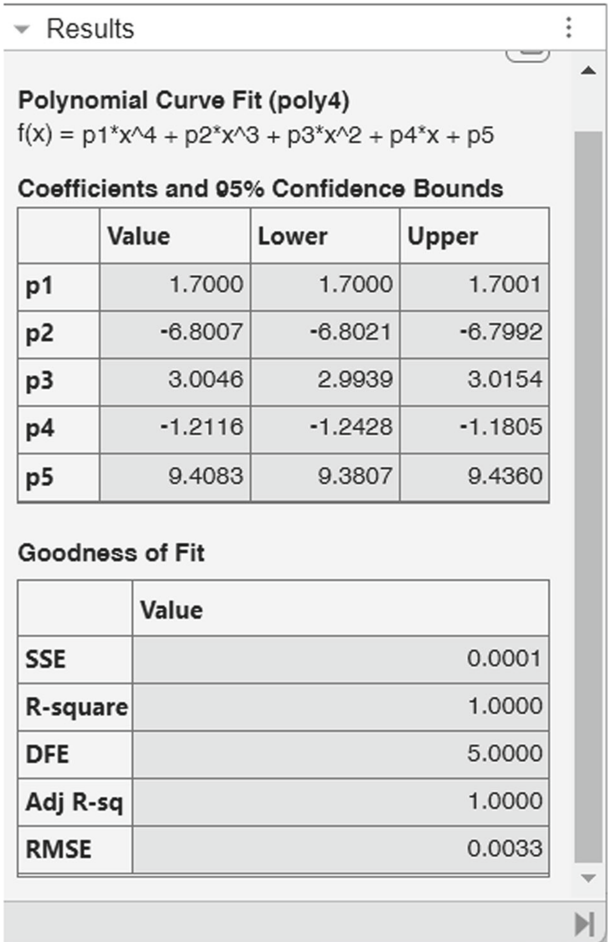


Fig. 6.22 A 4th order polynomial is fitted to given data



Results

Polynomial Curve Fit (poly4)

f(x) = p1*x^4 + p2*x^3 + p3*x^2 + p4*x + p5

Coefficients and 95% Confidence Bounds

	Value	Lower	Upper
p1	1.7000	1.7000	1.7001
p2	-6.8007	-6.8021	-6.7992
p3	3.0046	2.9939	3.0154
p4	-1.2116	-1.2428	-1.1805
p5	9.4083	9.3807	9.4360

Goodness of Fit

	Value
SSE	0.0001
R-square	1.0000
DFE	5.0000
Adj R-sq	1.0000
RMSE	0.0033

Fig. 6.23 Calculated coefficients

Employ the “Export to Workspace” (Fig. 6.24) to export the calculated model into the MATLAB workspace. Upon clicking the “Export to Workspace” button, the “Save Fit to MATLAB Workspace” window (Fig. 6.25) will be displayed.

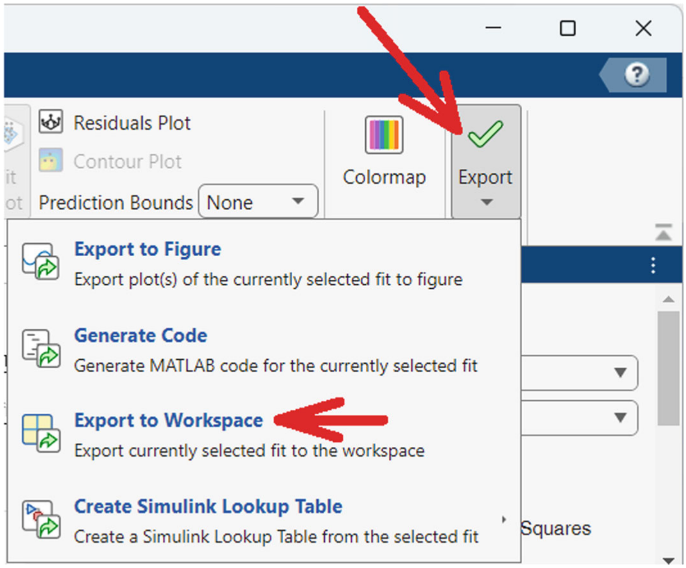


Fig. 6.24 The export button

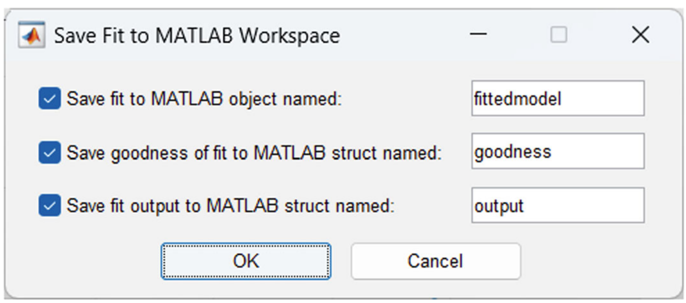


Fig. 6.25 The save fit to MATLAB workspace window

Upon clicking the “OK” button in Fig. 6.25, three new variables will be added to the MATLAB Workspace (Fig. 6.26).

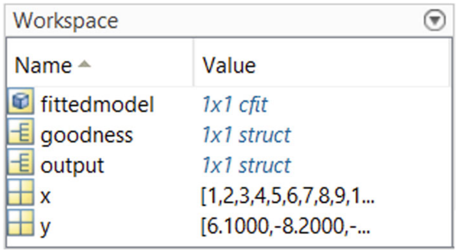


Fig. 6.26 New variables are added to the workspace

The fitted coefficients are stored in the `fittedmodel` variable (Fig. 6.27). Employ the dot operator to retrieve specific coefficients from the `fittedmodel` structure (Fig. 6.28).

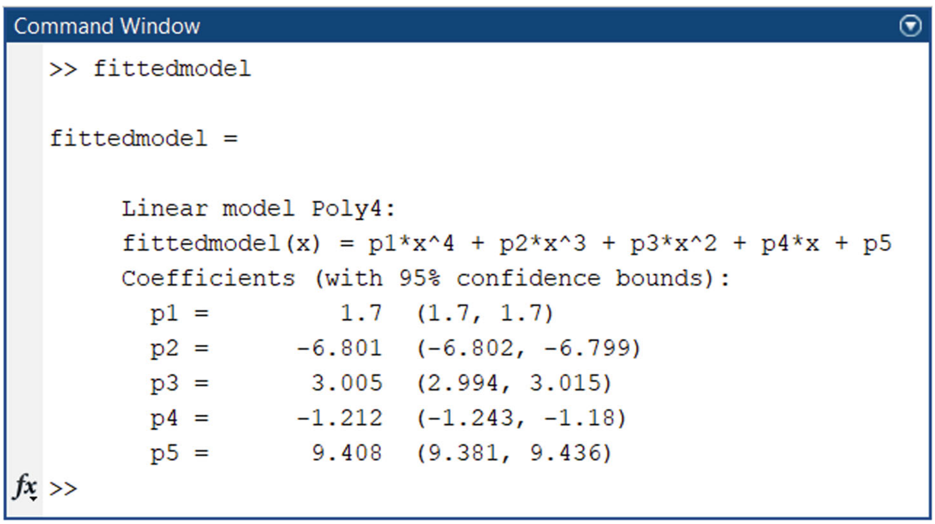


Fig. 6.27 `fittedmodel` structure

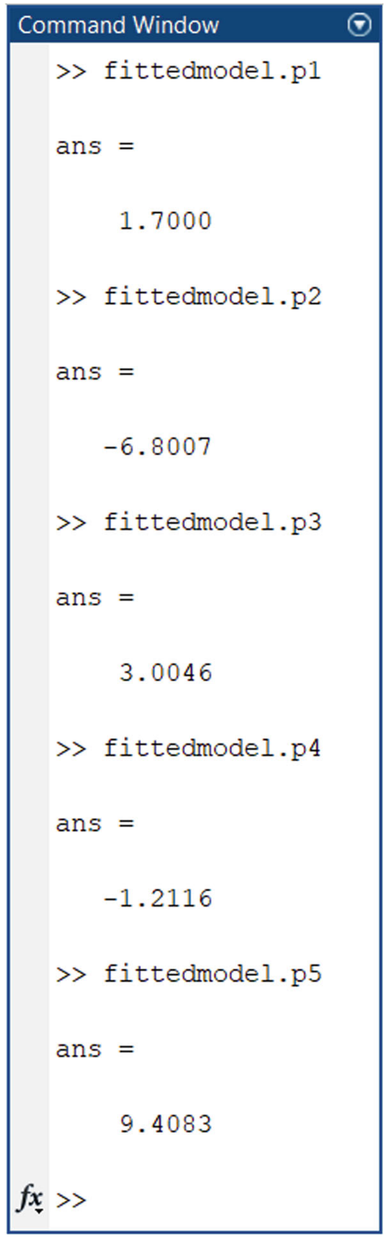


Fig. 6.28 Calculated coefficients p1, p2, p3, p4 and p5

The `fittedmodel` variable can be employed to evaluate the fitted model at any desired input value. For instance, Fig. 6.29 demonstrates the evaluation of the model at $x = 8.7$.

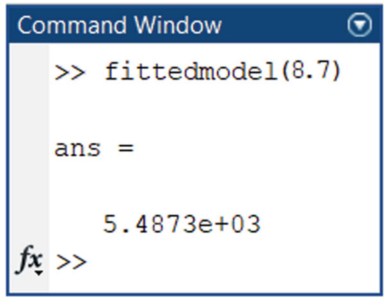


Fig. 6.29 Value of calculated polynomial at $x = 8.7$

Example 6.8 In this example we will fit the data presented in Table 6.2 to $I = I_s\left(e^{\frac{39.06V}{\eta}} - 1\right)$. Model parameters are I_s and η. Given model can be written as $I = I_s\left(e^{\frac{39.06V}{\eta}} - 1\right) = a\left(e^{bV} - 1\right)$ where $a = I_s$ and $b = \frac{39.06}{\eta}$.

Table 6.2 Measured voltage and current for a diode

Diode voltage (V)	Diode current (A)
0.50	0.50×10^{-3}
0.53	0.97×10^{-3}
0.55	1.44×10^{-3}
0.57	1.93×10^{-3}
0.58	2.42×10^{-3}
0.60	3.40×10^{-3}
0.61	4.38×10^{-3}
0.62	5.37×10^{-3}
0.63	6.36×10^{-3}
0.64	7.34×10^{-3}
0.68	17.00×10^{-3}
0.70	24.00×10^{-3}
0.72	34.00×10^{-3}
0.73	39.00×10^{-3}

Input the data into MATLAB. Then use the `cftool` command to run the toolbox (Fig. 6.30).

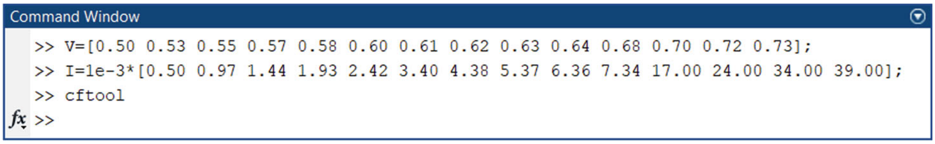


Fig. 6.30 Given data points

Proceed by clicking the "Select Data" button (Fig. 6.31).

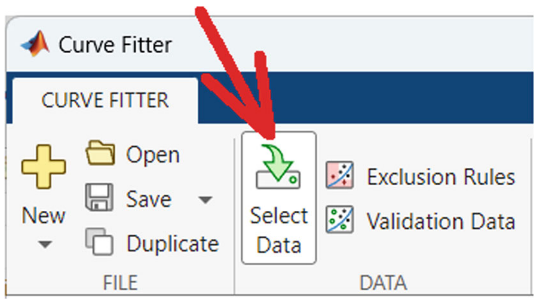


Fig. 6.31 Select data button

Select "V" for X Data and "I" for Y Data (Fig. 6.32).

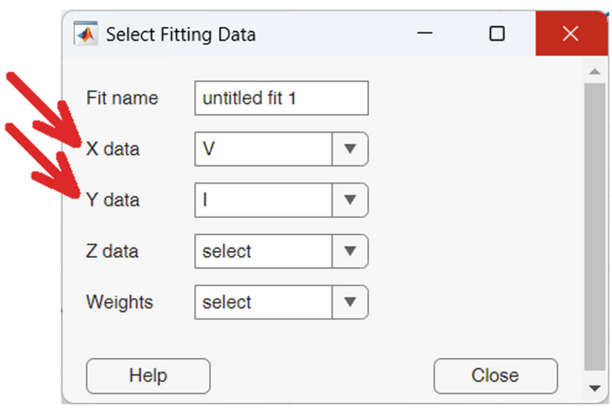


Fig. 6.32 Select fitting data window

Click on the "Show More" button (Fig. 6.33). Then select the "Custom Equation" (Fig. 6.34).

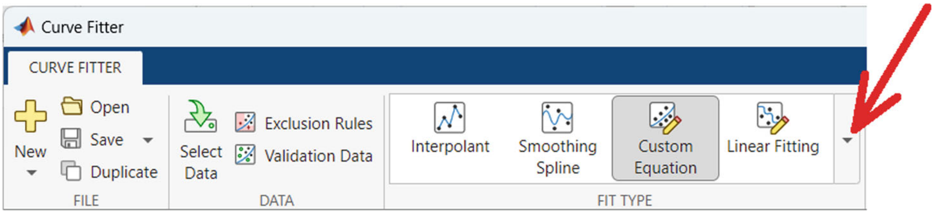


Fig. 6.33 Show more button

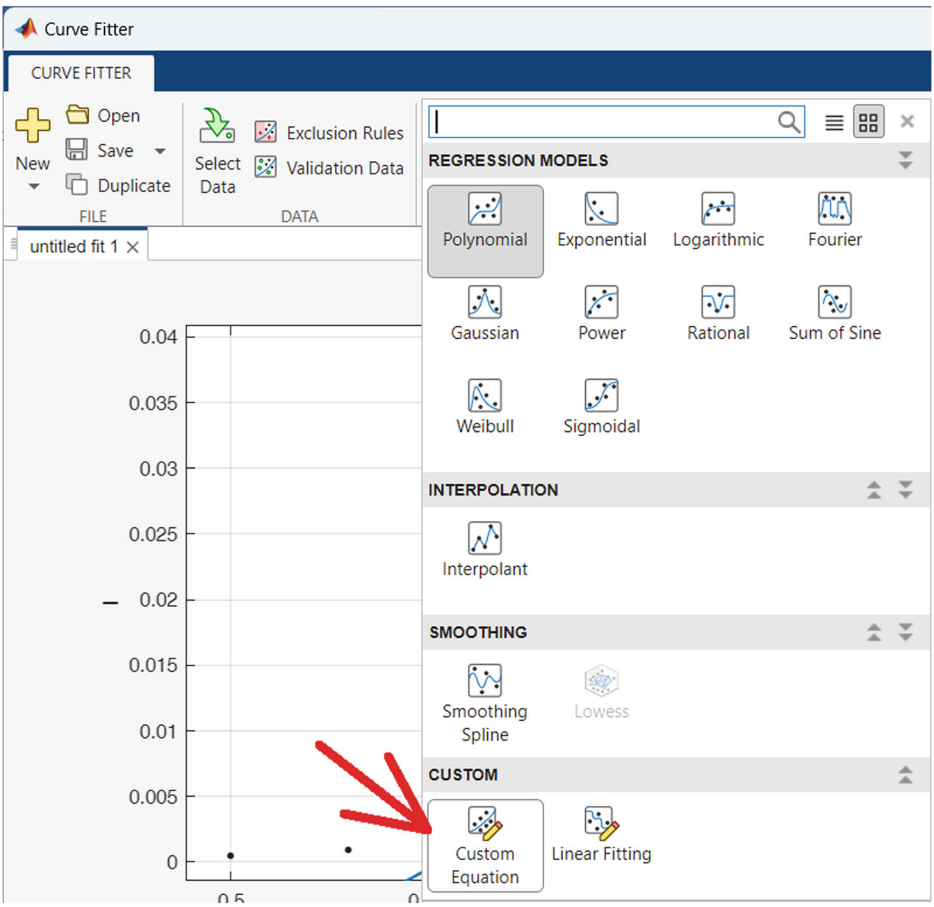


Fig. 6.34 Custom equation button

Type the equation (Fig. 6.35).

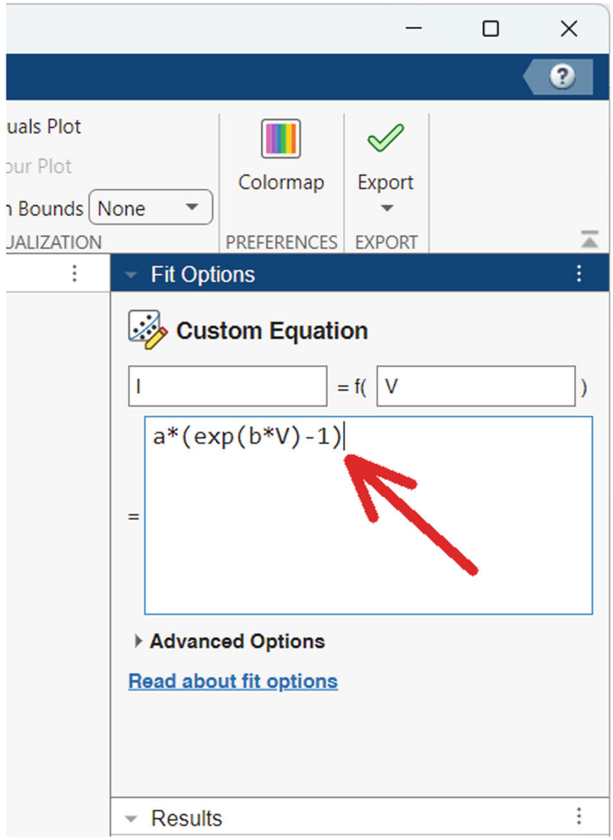


Fig. 6.35 Inputting the equation

The estimated values of the parameters a and b are displayed in Fig. 6.36. The value of a is very small and difficult to read in Fig. 6.36.

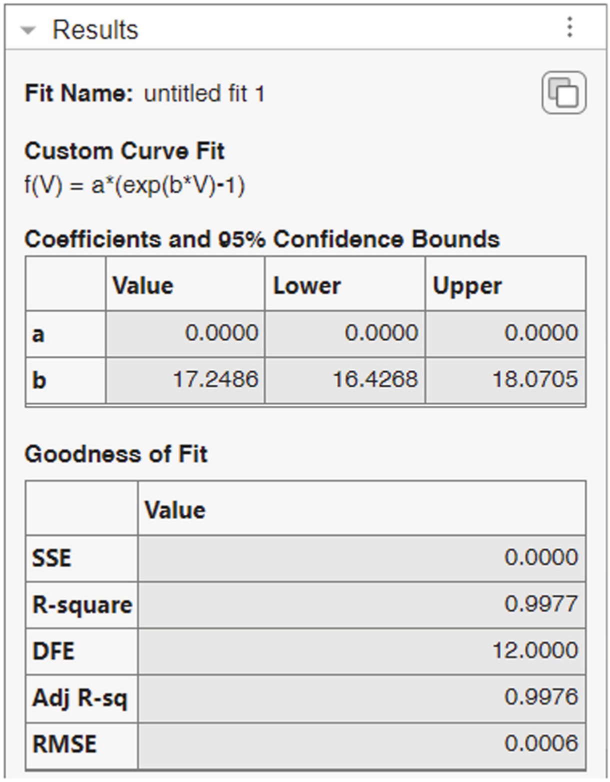


Fig. 6.36 Calculated values for *a* and *bb*

Export the calculated model to MATLAB Workspace (Fig. 6.37).

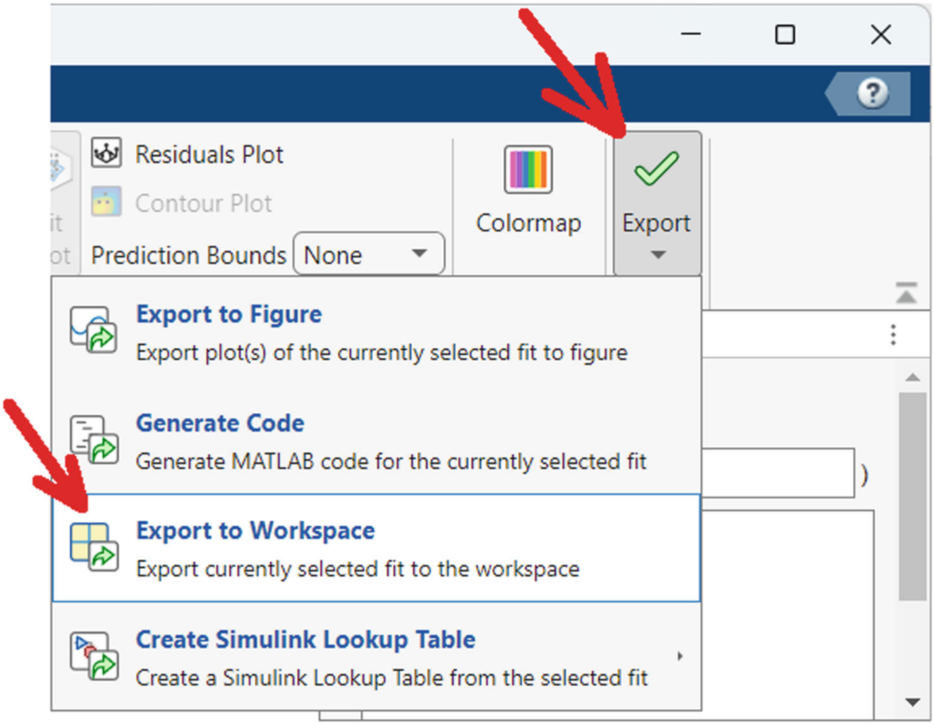


Fig. 6.37 The export button

The values of a and b are shown in Fig. 6.38. According to Fig. 6.38, $I = a\left(e^{bV} - 1\right) = 1.339 \times 10^{-7}\left(e^{17.2486V} - 1\right) = 1.339 \times 10^{-7}\left(e^{\frac{39.06}{2.2645}V} - 1\right) \Rightarrow I_s = 1.339 \times 10^{-7}$, $\eta = 2.2645$.

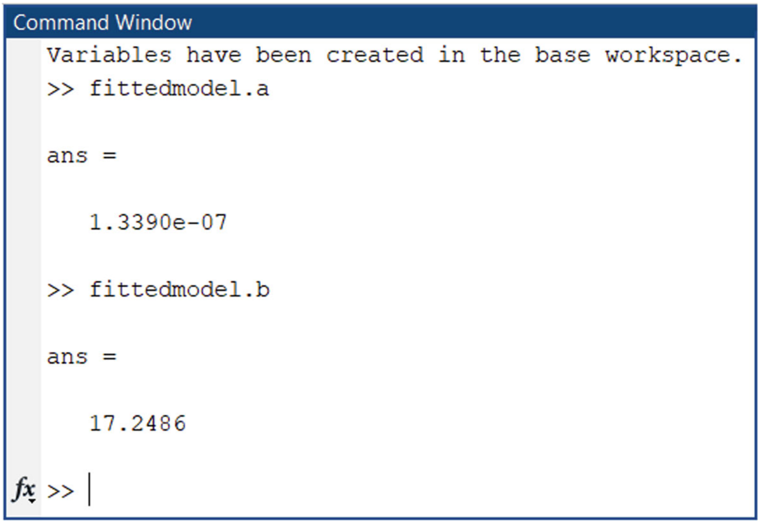


Fig. 6.38 Calculated values for *a* and *b*

References for Further Study

1. Advanced Engineering Mathematics (10^{th} edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8^{th} edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

Partial Fraction Decomposition 7

7.1 Introduction

Partial fraction decomposition is a technique used in algebra and calculus to break down a complex rational expression into simpler fractions. This process involves expressing a rational function, which is a fraction where both the numerator and denominator are polynomials, as a sum of simpler fractions with simpler denominators.

By breaking down a complex rational expression into these simpler components, it becomes easier to integrate, differentiate, or analyze the function. This chapter starts with a review of partial fraction decomposition. The second part demonstrates MATLAB's application to the problems discussed.

7.2 Addition of Rational Expressions

This section reviews the addition of rational expressions with a few examples.

Example 7.1 Add the expression $\frac{s+1}{s^2+5s+6}$ to $\frac{s+4}{s+5}$.

$$
\begin{aligned}
\frac{s+1}{s^2+5s+6}+\frac{s+4}{s+5} &= \frac{(s+1)(s+5)+(s+4)\left(s^2+5s+6\right)}{\left(s^2+5s+6\right)(s+5)} \\
&= \frac{(s+1)(s+5)+(s+4)\left(s^2+5s+6\right)}{\left(s^2+5s+6\right)(s+5)} \\
&= \frac{s^3+10s^2+32s+29}{s^3+10s^2+31s+30}
\end{aligned}
$$

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_7

Example 7.2 Add the expression $\frac{s+1}{(s+2)(s+3)}$ to $\frac{s+4}{(s+3)(s+5)}$.

$$\frac{s+1}{(s+2)(s+3)}+\frac{s+4}{(s+3)(s+5)}=\frac{(s+1)(s+5)+(s+4)(s+2)}{(s+2)(s+3)(s+5)}=\frac{2s^2+12s+13}{s^3+10s^2+31s+30}$$

Example 7.3 Add the expression $\frac{5}{s+2}$ to $\frac{6}{s+3}$.

$$\frac{5}{s+2}+\frac{6}{s+3}=\frac{5s+15+6s+12}{(s+2)(s+3)}=\frac{11s+27}{s^2+5s+6}$$

7.3 Partial Fraction Decomposition

Partial fraction decomposition is a technique used in algebra to break down a complex rational expression into simpler fractions. Let's study some numeric examples.

Example 7.4 Decompose the rational function $\frac{7x-23}{x^2-7x+10}$ into partial fractions.

$$x^2-7x+10=0 \Rightarrow x_{1,2}=\frac{+7\pm\sqrt{7^2-4(1)(10)}}{2}=\frac{7\pm 3}{2}=2\&5$$

$$\frac{7x-23}{x^2-7x+10}=\frac{7x-23}{(x-2)(x-5)}=\frac{A}{(x-2)}+\frac{B}{(x-5)}$$

Cover the $(x-2)$ term with your finger and substitute $x=2$ (the root of covered section) into the remaining expression (Fig. 7.1). The value of A is $\left.\frac{7x-23}{(x-5)}\right|_{x=2}=\frac{7\times 2-23}{(2-5)}=3$.

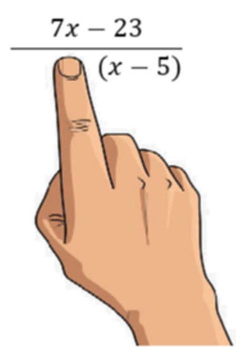


Fig. 7.1 $(x-2)$ term is covered

Now cover the $(x-5)$ term with your finger and substitute $x = 5$ (the root of covered section) into the remaining expression (Fig. 7.2). The value of B is $\left.\frac{7x-23}{(x-2)}\right|_{x=5} = \frac{7\times 5-23}{(5-2)} = 4$.

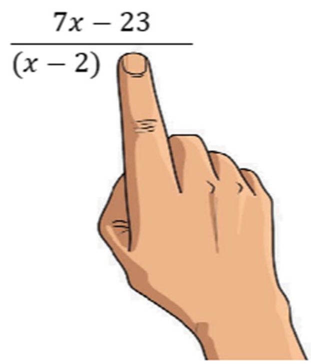


Fig. 7.2 $(x-5)$ term is covered

Therefore, $\frac{7x-23}{x^2-7x+10} = \frac{3}{x-2} + \frac{4}{x-5}$.

Example 7.5 Decompose the rational function $\frac{4x^2-21x+17}{x^2-7x+10}$ into partial fractions.

When the degree of the numerator is greater than or equal to the degree of the denominator, we have an improper rational function. To apply partial fraction decomposition, we first need to perform polynomial division to divide the numerator by the denominator.

$$\frac{4x^2-21x+17}{x^2-7x+10} = \frac{4(x^2-7x+10)+7x-23}{x^2-7x+10} = 4 + \frac{7x-23}{x^2-7x+10} = 4 + \frac{3}{x-2} + \frac{4}{x-5}$$

Example 7.6 Decompose the rational function $\frac{x^3+2x^2+x+1}{x^2+1}$ into partial fractions.

$$\frac{x^3+2x^2+x+1}{x^2+1} = \frac{x(x^2+1)+2x^2+1}{x^2+1} = \frac{x(x^2+1)+2\left(x^2+1\right)-1}{x^2+1} = x+2-\frac{-1}{x^2+1}$$

Example 7.7 Decompose the rational function $\frac{1}{x^2(x+1)}$ into partial fractions.

$$\frac{1}{x^2(x+1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+1}$$

Cover the x^2 term with your finger and substitute $x = 0$ (the root of covered section) into the remaining expression (Fig. 7.3). The value of A is $\left.\frac{1}{x+1}\right|_{x=0} = \frac{1}{0+1} = 1$.

Fig. 7.3 x^2 term is covered

We'll need to use derivatives to find the value of B. $B = \left.\frac{d}{dx}\left(\frac{1}{x+1}\right)\right|_{x=0} = -\left.\frac{1}{(x+1)^2}\right|_{x=0} = -1$.

Now cover the $x+1$ term with your finger and substitute $x = -1$ (the root of covered section) into the remaining expression (Fig. 7.4). The value of C is $\left.\frac{1}{x^2}\right|_{x=-1} = \frac{1}{1} = 1$.

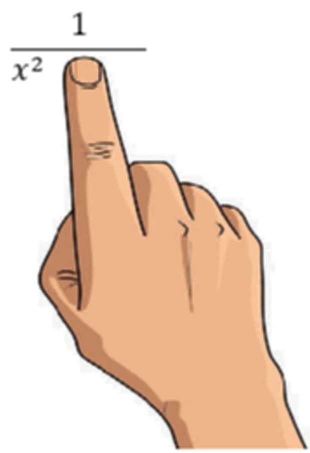


Fig. 7.4 $(x+1)$ term is covered

Therefore, $\frac{1}{x^2(x+1)} = \frac{1}{x^2} + \frac{-1}{x} + \frac{1}{x+1}$.

Example 7.8 Decompose the rational function $\frac{2x+1}{(x+1)(x+2)^2}$ into partial fractions.

$$\frac{2x+1}{(x+1)(x+2)^2} = \frac{A}{(x+2)^2} + \frac{B}{x+2} + \frac{C}{x+1}$$

Cover the $(x+2)^2$ term with your finger and substitute $x = -2$ (the root of covered section) into the remaining expression (Fig. 7.5). The value of A is $\left.\frac{2x+1}{(x+1)}\right|_{x=-2} = \frac{2\times -2+1}{-2+1} = 3$.

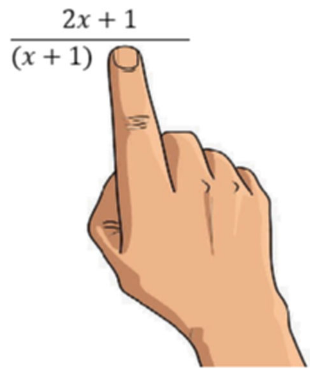


Fig. 7.5 $(x+2)^2$ term is covered

We'll need to use derivatives to find the value of B:

$$B = \frac{d}{dx}\left(\frac{2x+1}{x+1}\right)\Bigg|_{x=-2} = \frac{2(x+1) - 1(2x+1)}{(x+1)^2}\Bigg|_{x=-2} = \frac{2(-2+1) - 1(-4+1)}{(-2+1)^2} = 1$$

Now cover the $(x+1)$ term with your finger and substitute $x = -1$ (the root of covered section) into the remaining expression (Fig. 7.6). The value of C is $\left.\frac{2x+1}{(x+2)^2}\right|_{x=-1} = \frac{-2+1}{(-1+2)^2} = -1$.

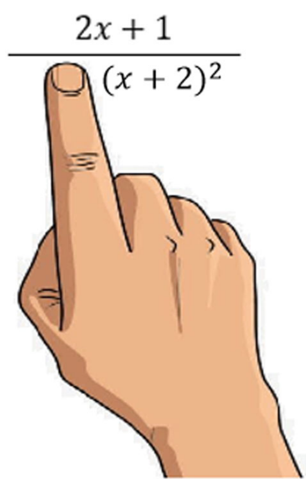


Fig. 7.6 $(x+1)$ term is covered

Therefore, $\frac{2x+1}{(x+1)(x+2)^2} = \frac{3}{(x+2)^2} + \frac{1}{x+2} + \frac{-1}{x+1}$.

Example 7.9 Decompose the rational function $\frac{-2x^2-5x-1}{(x+2)^3}$ into partial fractions.

$$\frac{-2x^2 - 5x - 1}{(x+2)^3} = \frac{A}{(x+2)^3} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)}$$

Cover the $(x+2)^3$ term with your finger and substitute $x = -2$ (the root of covered section) into the remaining expression (Fig. 7.7). The value of A is $-2x^2 - 5x - 1\big|_{x=-2} \Rightarrow -2(-2)^2 + 10 - 1 = 1$.

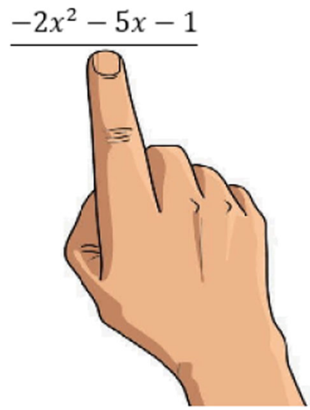


Fig. 7.7 $(x+2)^3$ term is covered

We'll need to use derivatives to find the value of B and C:

$$B = \frac{d}{dx}\left(-2x^2 - 5x - 1\right)\Big|_{x=-2} = -4x - 5|_{x=-2} = -4(-2) - 5 = 3$$

$$C = \frac{1}{2}\frac{d^2}{dx^2}\left(-2x^2 - 5x - 1\right)\Big|_{x=-2} = \frac{1}{2} \times -4|_{x=-2} = -2$$

Example 7.10 Decompose the rational function $\frac{x+1}{(x+2)^3(x+3)^2(x+4)}$ into partial fractions.

$$\frac{x+1}{(x+2)^3(x+3)^2(x+4)} = \frac{A}{(x+2)^3} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)} + \frac{D}{(x+3)^2} + \frac{E}{(x+3)} + \frac{F}{(x+4)}$$

Cover the $(x+2)^3$ term with your finger and substitute $x = -2$ (the root of covered section) into the remaining expression (Fig. 7.8). The value of A is $\frac{x+1}{(x+3)^2(x+4)}\Big|_{x=-2} = \frac{-1}{2}$.

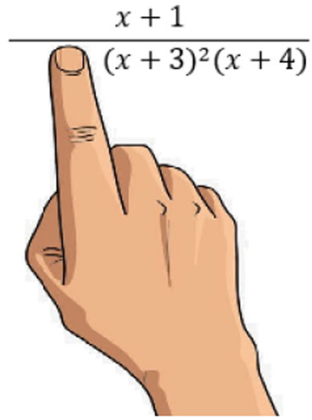


Fig. 7.8 $(x + 2)^3$ term is covered

We'll need to use derivatives to find the value of B and C:

$$B = \frac{d}{dx}\left(\frac{x+1}{(x+3)^2(x+4)}\right)\bigg|_{x=-2} = \frac{7}{4}$$
$$C = \frac{1}{2}\frac{d^2}{dx^2}\left(\frac{x+1}{(x+3)^2(x+4)}\right)\bigg|_{x=-2} = -\frac{27}{8}$$

Cover the $(x + 3)^2$ term with your finger and substitute $x = -3$ (the root of covered section) into the remaining expression (Fig. 7.9). The value of D is $\frac{x+1}{(x+2)^3(x+4)}\Big|_{x=-3} = \frac{-2}{-1^3 \times 1} = 2$.

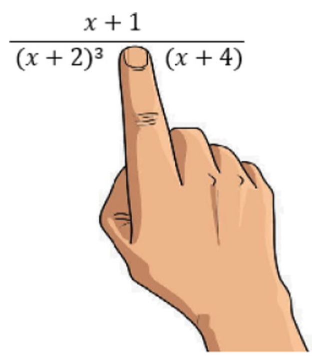


Fig. 7.9 $(x + 3)^2$ term is covered

We'll need to use derivatives to find the value of E:

$$E = \frac{d}{dx}\left(\frac{x+1}{(x+2)^3(x+4)}\right)\bigg|_{x=-3} = 3$$

Cover the $(x+4)$ term with your finger and substitute $x=-4$ (the root of covered section) into the remaining expression (Fig. 7.10). The value of F is $\left.\frac{x+1}{(x+2)^3(x+3)^2}\right|_{x=-4} = \frac{-3}{-8\times 1} = \frac{3}{8}$.

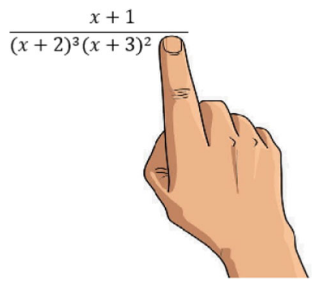


Fig. 7.10 $(x+4)$ term is covered

Therefore, $\frac{x+1}{(x+2)^3(x+3)^2(x+4)} = \frac{\frac{-1}{2}}{(x+2)^3} + \frac{\frac{7}{4}}{(x+2)^2} + \frac{\frac{-27}{8}}{(x+2)} + \frac{2}{(x+3)^2} + \frac{3}{(x+3)} + \frac{\frac{3}{8}}{(x+4)}$.

Example 7.11 Decompose the rational function $\frac{5s^2+s+4}{(s+4)(s^2+4)}$ into partial fractions.

$$\frac{5s^2+s+4}{(s+4)\left(s^2+4\right)} = \frac{5s^2+s+4}{(s+4)(s+2j)(s-2j)} = \frac{A}{s+4} + \frac{B}{s+2j} + \frac{C}{s-2j}$$

Cover the $(s+4)$ term with your finger and substitute $s=-4$ (the root of covered section) into the remaining expression (Fig. 7.11). The value of A is $\frac{5(-4)^2-4+4}{(-4+2j)(-4-2j)} = 4$.

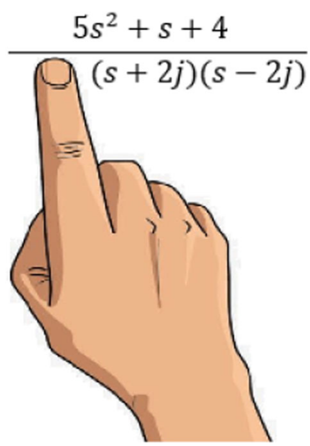
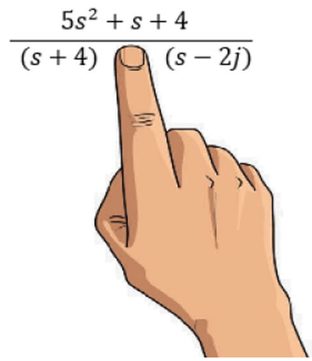


Fig. 7.11 $(s+4)$ term is covered

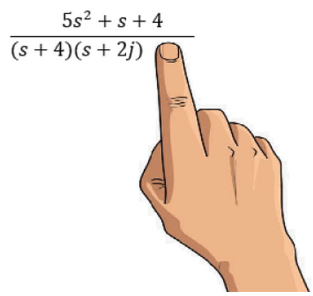
Cover the $(s+2j)$ term with your finger and substitute $s=-2j$ (the root of covered section) into the remaining expression (Fig. 7.12). The value of B is $\frac{5(-2j)^2-2j+4}{(-2j+4)(-2j-2j)}=\frac{-16-2j}{-8-16j}=0.5-0.75j$.

Fig. 7.12 $(s+2j)$ term is covered



Cover the $(s-2j)$ term with your finger and substitute $s=+2j$ (the root of covered section) into the remaining expression (Fig. 7.13). The value of C is $\frac{5(2j)^2+2j+4}{(2j+4)(2j+2j)}=\frac{-16+2j}{-8+16j}=0.5+0.75j$.

Fig. 7.13 $(s-2j)$ term is covered



Therefore, $\frac{5s^2+s+4}{(s+4)(s^2+4)}=\frac{4}{s+4}+\frac{0.5-0.75j}{s+2j}+\frac{0.5+0.75j}{s-2j}=\frac{4}{s+4}+\frac{(0.5-0.75j)(s-2j)+(0.5+0.75j)(s+2j)}{(s-2j)(s+2j)}=\frac{4}{s+4}+\frac{(0.5-0.75j)(s-2j)+(0.5+0.75j)(s+2j)}{s^2+4}=\frac{4}{s+4}+\frac{s-3}{s^2+4}$.

Alternatively, you can solve this problem as follows:

$$\frac{5s^2+s+4}{(s+4)\left(s^2+4\right)}=\frac{A}{s+4}+\frac{Bs+C}{s^2+4}$$

Cover the $(s+4)$ term with your finger and substitute $s=-4$ (the root of covered section) into the remaining expression (Fig. 7.14). The value of A is $\frac{5(-4)^2-4+4}{(-4)^2+4}=4$.

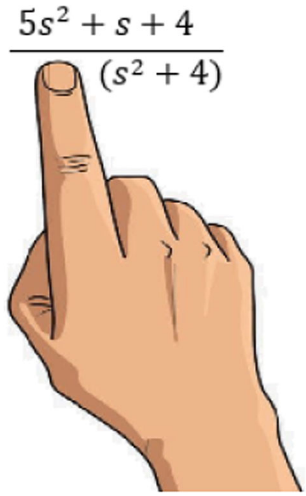


Fig. 7.14 $(s + 4)$ term is covered

Value of B and C can be found as follows:

$$\frac{5s^2 + s + 4}{(s + 4)\left(s^2 + 4\right)} = \frac{4}{s + 4} + \frac{Bs + C}{\left(s^2 + 4\right)} \Rightarrow \frac{5s^2 + s + 4}{(s + 4)\left(s^2 + 4\right)} - \frac{4}{s + 4}$$
$$= \frac{Bs + C}{\left(s^2 + 4\right)} \Rightarrow \frac{s^2 + s - 12}{(s + 4)\left(s^2 + 4\right)} = \frac{Bs + C}{\left(s^2 + 4\right)} \Rightarrow \frac{s^2 + s - 12}{(s + 4)}$$
$$= Bs + C \Rightarrow \frac{(s + 4)(s - 3)}{(s + 4)} = Bs + C \Rightarrow s - 3$$
$$= Bs + C \Rightarrow B = 1, C = -3$$

Therefore, $\frac{5s^2+s+4}{(s+4)\left(s^2+4\right)} = \frac{4}{s+4} + \frac{s-3}{\left(s^2+4\right)}$.

Example 7.12 Decompose the rational function $\frac{3}{(s-1)(s-5)}$ into partial fractions.

$$\frac{k}{(s + a)(s + b)} = \frac{A}{(s + a)} + \frac{B}{(s + b)} = \frac{\frac{k}{b-a}}{(s + a)} + \frac{\frac{k}{a-b}}{(s + b)}$$

$$\frac{3}{(s - 1)(s - 5)} = \frac{\frac{-3}{4}}{(s - 1)} + \frac{\frac{3}{4}}{(s - 5)}$$

Exercise: Show that $\frac{4x}{x^3-x^2-x+1} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-1}{x+1}$.

7.4 Polynomial Division

This section provides a review of polynomial division with a few examples.

Example 7.13 Perform polynomial division on $\frac{3x^2-10}{x^2-4x+4}$.

$$\frac{3x^2-10}{x^2-4x+4} = \frac{3x^2-12x+12+12x-12-10}{x^2-4x+4} = 3 + \frac{12x+22}{x^2-4x+4}$$

You can use the method shown in Fig. 7.15 as well.

Fig. 7.15 Long division of $3x^2-10$ by x^2-4x+4

$$\begin{array}{rr|l} & 3x^2-10 & x^2-4x+4 \\ \cline{3-3} - & 3x^2-12x+12 & 3 \\ \cline{2-2} & 12x+22 & \end{array}$$

Example 7.14 Perform polynomial division on $\frac{x^3+x}{x-1}$.

According to Fig. 7.16, $\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}$.

Fig. 7.16 Long division of x^3+x by $x-1$

$$\begin{array}{rl|l} & x^3+x & x-1 \\ \cline{3-3} - & x^3-x^2 & x^2+x+2 \\ \cline{2-2} & x^2+x & \\ - & x^2-x & \\ \cline{2-2} & 2x & \\ - & 2x-2 & \\ \cline{2-2} & 2 & \end{array}$$

Exercise: Perform polynomial division on $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}$ (Ans. $\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$).

7.5 Problem Solving with MATLAB®

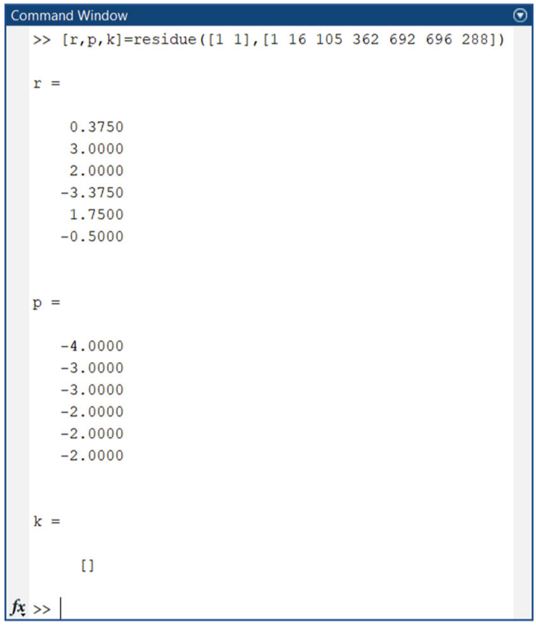
This section provides examples of how MATLAB can be employed to solve the types of problems encountered in this chapter.

Example 7.15 The code in Fig. 7.17 performs a partial fraction decomposition of the rational function $\frac{7x-23}{x^2-7x+10}$. According to the result shown in Fig. 7.17, $\frac{7x-23}{x^2-7x+10} = \frac{3}{x-2} + \frac{4}{x-5}$.

Fig. 7.17 Partial fraction decomposition of $\frac{7x-23}{x^2-7x+10}$

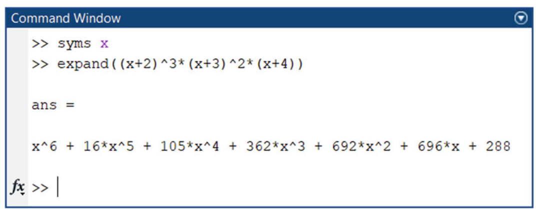
```
Command Window
>> [r,p,k]=residue([7 -23],[1 -7 10])

r =

     4
     3


p =

     5
     2


k =

     []

fx >>
```

Example 7.16 The code in Fig. 7.18 performs a partial fraction decomposition of the rational function $\frac{x+1}{(x+2)^3(x+3)^2(x+4)}$. Note that $(x+2)^3(x+3)^2(x+4) = x^6 + 16x^5 + 105x^4 + 362x^3 + 692x^2 + 696x + 288$ (Fig. 7.19). According to the result shown in Fig. 7.18, $\frac{x+1}{(x+2)^3(x+3)^2(x+4)} = \frac{0.375}{(x+4)} + \frac{3}{(x+3)} + \frac{2}{(x+3)^2} + \frac{-3.375}{(x+2)} + \frac{1.75}{(x+2)^2} + \frac{-0.5}{(x+2)^3}$.



```
Command Window
>> [r,p,k]=residue([1 1],[1 16 105 362 692 696 288])

r =

    0.3750
    3.0000
    2.0000
   -3.3750
    1.7500
   -0.5000


p =

   -4.0000
   -3.0000
   -3.0000
   -2.0000
   -2.0000
   -2.0000


k =

     []

fx >> 
```

Fig. 7.18 Partial fraction decomposition of $\frac{x+1}{(x+2)^3(x+3)^2(x+4)}$



```
Command Window
>> syms x
>> expand((x+2)^3*(x+3)^2*(x+4))

ans =

x^6 + 16*x^5 + 105*x^4 + 362*x^3 + 692*x^2 + 696*x + 288

fx >> 
```

Fig. 7.19 Expansion of $(x+2)^3(x+3)^2(x+4)$

Example 7.17 The code in Fig. 7.20 performs a partial fraction decomposition of the rational function $\frac{x+7}{x^3+4x^2+13x}$. According to the result shown in Fig. 7.20, $\frac{x+7}{x^3+4x^2+13x} = \frac{0.5385}{x} + \frac{-0.2692+0.0128j}{x-(-2+3j)} + \frac{-0.2692-0.0128j}{x-(-2-3j)}$. Note that $\frac{-0.2692+0.0128j}{x-(-2+3j)} + \frac{-0.2692-0.0128j}{x-(-2-3j)} = -\frac{\frac{673}{1250}s+\frac{1442}{1250}}{x^2+4x+13} = -\frac{0.5384s+1.1536}{x^2+4x+13}$ (Fig. 7.21). Therefore, $\frac{x+7}{x^3+4x^2+13x} = \frac{0.5385}{x} - \frac{0.5384s+1.1536}{x^2+4x+13}$.

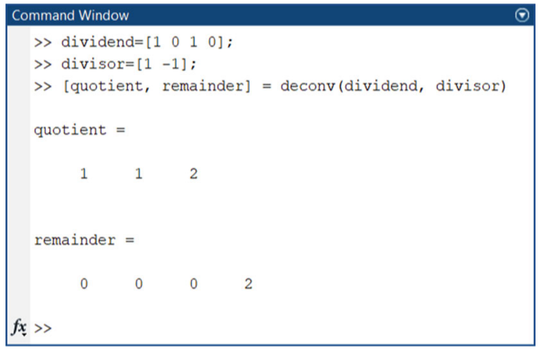
```
Command Window
>> [r,p,k]=residue([1 7],[1 4 13 0])

r =

  -0.2692 + 0.0128i
  -0.2692 - 0.0128i
   0.5385 + 0.0000i


p =

  -2.0000 + 3.0000i
  -2.0000 - 3.0000i
   0.0000 + 0.0000i


k =

     []

fx >>
```

Fig. 7.20 Partial fraction decomposition of $\frac{x+7}{x^3+4x^2+13x}$

```
Command Window
>> syms x
>> simplify((-.2692+0.0128i)/(x+2-3i)+(-.2692-0.0128i)/(x+2+3i))

ans =

-(673*x + 1442)/(1250*(x^2 + 4*x + 13))

>> pretty(ans)
      673 x + 1442
- ---------------------
     2
  (x   + 4 x + 13) 1250

fx >>
```

Fig. 7.21 Simplification of $\frac{-0.2692+0.0128j}{x-(-2+3j)}+\frac{-0.2692-0.0128j}{x-(-2-3j)}$

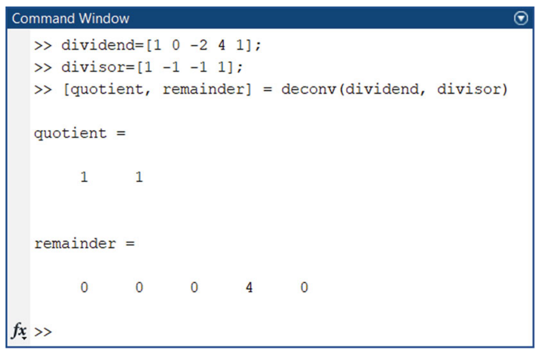
Example 7.18 The code in Fig. 7.22, divides $x^3 + x$ by $x - 1$. According to the result shown in Fig. 7.22, $x^3 + x = (x^2 + x + 2)(x - 1) + 2$.



```
Command Window
>> dividend=[1 0 1 0];
>> divisor=[1 -1];
>> [quotient, remainder] = deconv(dividend, divisor)

quotient =

     1     1     2


remainder =

     0     0     0     2

fx >>
```

Fig. 7.22 Dividing $x^3 + x$ by $x - 1$

Example 7.19 The code in Fig. 7.23, divides $x^4 - 2x^2 + 4x + 1$ by $x^3 - x^2 - x + 1$. According to the result shown in Fig. 7.23, $x^4 - 2x^2 + 4x + 1 = (x + 1)\left(x^3 - x^2 - x + 1\right) + 4x$.



```
Command Window
>> dividend=[1 0 -2 4 1];
>> divisor=[1 -1 -1 1];
>> [quotient, remainder] = deconv(dividend, divisor)

quotient =

     1     1


remainder =

     0     0     0     4     0

fx >>
```

Fig. 7.23 Dividing $x^4 - 2x^2 + 4x + 1$ by $x^3 - x^2 - x + 1$

Example 7.20 The code in Fig. 7.24 evaluates the function $y = \frac{x+1}{(x+3)^2(x+4)}$ at $x = 17$.

Fig. 7.24 Evaluation of $y = \frac{x+1}{(x+3)^2(x+4)}$ at $x = 17$

```
Command Window
>> syms x
>> y=(x+1)/(x+3)^2/(x+4);
>> subs(y,17)

ans =

3/1400

>> eval(ans)

ans =

    0.0021

fx >>
```

Example 7.21 The code in Fig. 7.25 computes $\frac{d}{dx}\left(\frac{x+1}{(x+3)^2(x+4)}\right)$.

```
Command Window
>> syms x
>> dydx=diff((x+1)/(x+3)^2/(x+4),x)

dydx =

1/((x + 3)^2*(x + 4)) - (x + 1)/((x + 3)^2*(x + 4)^2) - (2*(x + 1))/((x + 3)^3*(x + 4))

>> simplify(dydx)

ans =

-(2*x^2 + 7*x - 1)/((x + 3)^3*(x + 4)^2)

fx >>
```

Fig. 7.25 Calculation of $\frac{d}{dx}\left(\frac{x+1}{(x+3)^2(x+4)}\right)$

Example 7.22 The code in Fig. 7.26 computes $\frac{d}{dx}\left(\frac{x+1}{(x+3)^2(x+4)}\right)$ at $x = -2$.

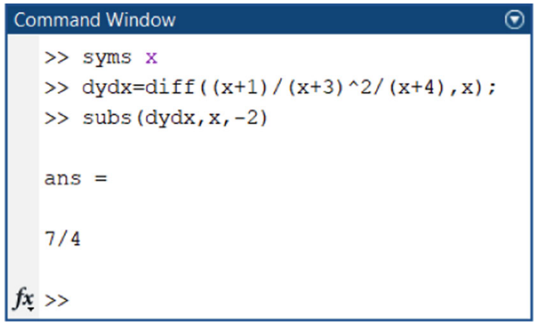


Fig. 7.26 Calculation of $\frac{d}{dx}\left(\frac{x+1}{(x+3)^2(x+4)}\right)$ at $x = -2$

References for Further Study

1. Advanced Engineering Mathematics (10th edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8^{th} edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

Complex Analysis

8

8.1 Introduction

Complex analysis is a branch of mathematics that studies functions of complex numbers. It delves into concepts like complex differentiation, integration, and power series.

This chapter begins with a review of key complex analysis topics. The second part demonstrates MATLAB's application to the problems discussed.

8.2 Complex Numbers

In mathematics, a complex number is an element of a number system that extends the real numbers with a specific element denoted i, called the imaginary unit and satisfying the equation $i^2 = -1$. Every complex numbers can be expressed in the form $a + bi$, where a and b are real numbers.

For the complex number $a + bi$, a is called the real part, and b is called the imaginary part. The set of complex numbers is denoted by the symbol $\mathbb{C}$.

Two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ are equal when $a = c$ and $b = d$.

8.3 Complex Numbers as Roots of Second Order Equations

Roots of $ax^2 + bx + c = 0$ equation are $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$. When $\Delta < 0$ roots are complex. For instance, for $x^2 + 2x + 10 = 0$ the $\Delta = 2^2 - 4(1)(10) = -36$ and the roots are:

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4(1)(10)}}{2} = \frac{-2 \pm \sqrt{-36}}{2} = \frac{-2 \pm \sqrt{36}\sqrt{-1}}{2} = \frac{-2 \pm 6j}{2} = -1 \pm 3j$$

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_8

Note that $\sqrt{-1} = j$. Mathematicians generally use i for $\sqrt{-1}$. Engineers prefer the j notation to avoid conflict with the notation for electrical current. $a + jb$ and $a + bj$ are the same.

8.4 The Geometric View of Complex Numbers

Figure 8.1 shows how points $A = 1 + 2j$ and $B = -1 - 3j$ can be shown on the complex plane.

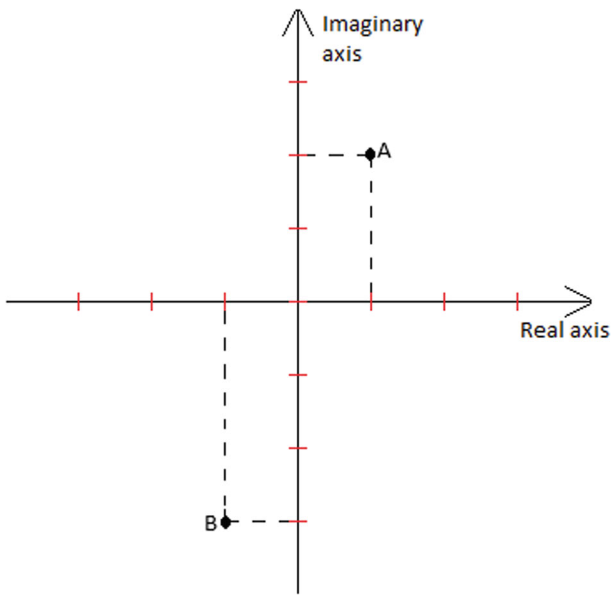


Fig. 8.1 Points $A = 1 + 2j$ and $B = -1 - 3j$

8.5 Euler's Formula

Euler's formula states that, for any real number x, one has:

$$e^{jx} = \cos(x) + j\sin(x)$$

For instance,

(1) $e^{j\frac{\pi}{6}} = \cos\left(\frac{\pi}{6}\right) + j\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + j\frac{1}{2}$
(2) $e^{-j\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) + j\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

(3) $\sqrt{5}e^{j7.3903} = \sqrt{5}\cos(7.3903) + \sin(7.3903) = 1 + j2$
(4) $e^{2+3j} = e^2 e^{3j} = e^2(\cos(3) + j\sin(3)) = e^2(-0.9900 + j0.1411) = -7.3151 + j1.0427$

Sometimes the complex number $re^{jx} = r(\cos(x) + j\sin(x)) = r\cos(x) + jr\sin(x)$ is shown as $r\sphericalangle x$. For instance, $\sqrt{5}e^{j7.3903}$ can be shown as $\sqrt{5}\sphericalangle 7.3903$.

Exercise: Show that $\cos(x) = \frac{e^{jx}+e^{-jx}}{2}$ and $\sin(x) = \frac{e^{jx}-e^{-jx}}{2j}$.

8.6 Magnitude of a Complex Number

Magnitude of a complex number $a + jb$ is shown as $|a + jb|$ and equals to $\sqrt{a^2 + b^2}$. For instance, $|1 + j2| = \sqrt{1^2 + 2^2} = \sqrt{5} = 2.2361$. Magnitude of $z = re^{j\theta}$ equals to r. For instance, $\left|7e^{j\frac{\pi}{4}}\right| = 7$. Magnitude of division of two complex numbers can be calculated with the aid of $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$. For instance, $\left|\frac{1+2j}{3+4j}\right| = \frac{|1+2j|}{|3+4j|} = \frac{\sqrt{1^2+2^2}}{\sqrt{3^2+4^2}} = \frac{\sqrt{5}}{\sqrt{25}} = 0.447$.

8.7 The Principle Argument of a Complex Number

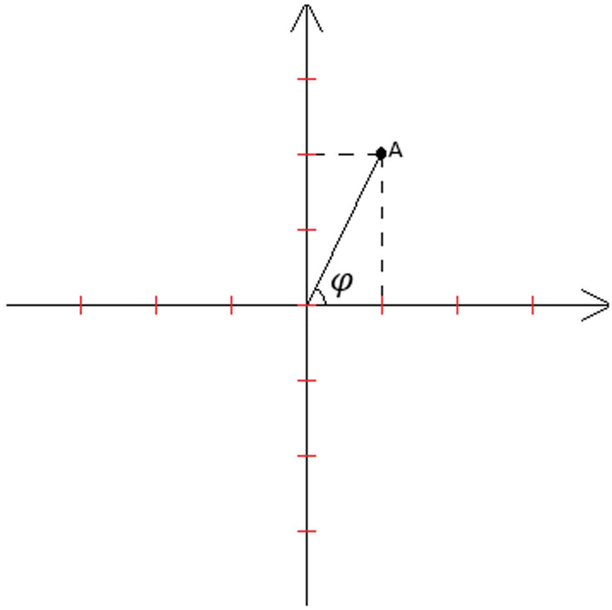
The principal argument of a complex number z is the angle φ formed between the positive real axis and the line representing the complex number in the complex plane. It is typically measured in radians and is defined within the range $(-\pi, \pi]$. The principal argument is often denoted as $Arg(z)$.

Argument of a complex number z is shown with $arg(z)$ and $arg(z) = \{Arg(z) + 2\pi n | n \in \mathbb{Z}\}$.

For a complex number $z = x + yi$, the principal argument can be calculated using the arctangent function. However, special care must be taken depending on the quadrant in which the complex number lies.

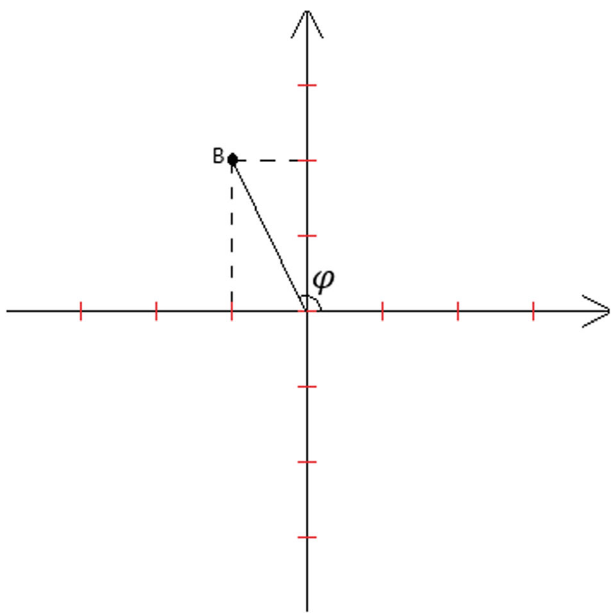
Let's study some numeric examples. The principle argument of $A = 1 + 2j$ is $\varphi = \tan^{-1}\left(\frac{2}{1}\right) = 1.1071rad$ (Fig. 8.2). Argument of A is $1.1071 + 2\pi n, n \in \mathbb{Z}$.

Fig. 8.2 Point $A = 1 + 2j$



The principle argument of $B = -1 + 2j$ is $\varphi = \frac{\pi}{2} + \tan^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2} + 0.463 = 2.034rad$ (Fig. 8.3). Argument of B is $2.034 + 2\pi n, n \in \mathbb{Z}$.

Fig. 8.3 Point $B = -1 + 2j$



The principle argument of $C = 1 - 2j$ is $\varphi = -\tan^{-1}\left(\frac{2}{1}\right) = -1.1071 rad$ (Fig. 8.4). Note that the principle argument of C is NOT $5.1761 rad$ (Fig. 8.5) since $5.1761 \notin (-\pi, \pi]$. Argument of C is $-1.1071 + 2\pi n, n \in \mathbb{Z}$.

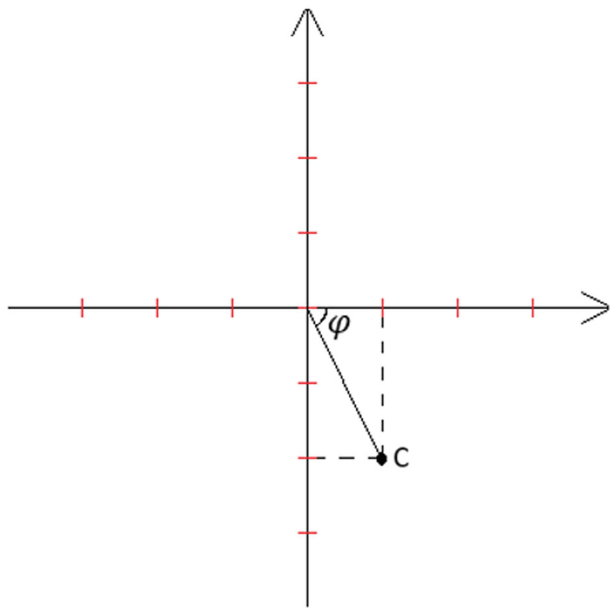


Fig. 8.4 Point $C = 1 - 2j$

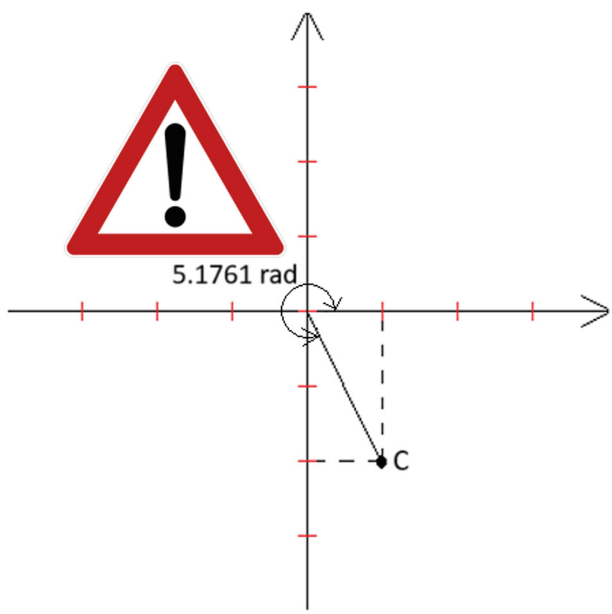


Fig. 8.5 Wrong measurement of principle argument

The principle argument of $D = -1 - 2j$ is $\varphi = -\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{2} - 0.463 = -2.034 rad$ (Fig. 8.6). Note that the principle argument of D is NOT $4.2492 rad$ (Fig. 8.7) since $4.2492 \notin (-\pi, \pi]$. Argument of D is $-2.034 + 2\pi n, n \in \mathbb{Z}$.

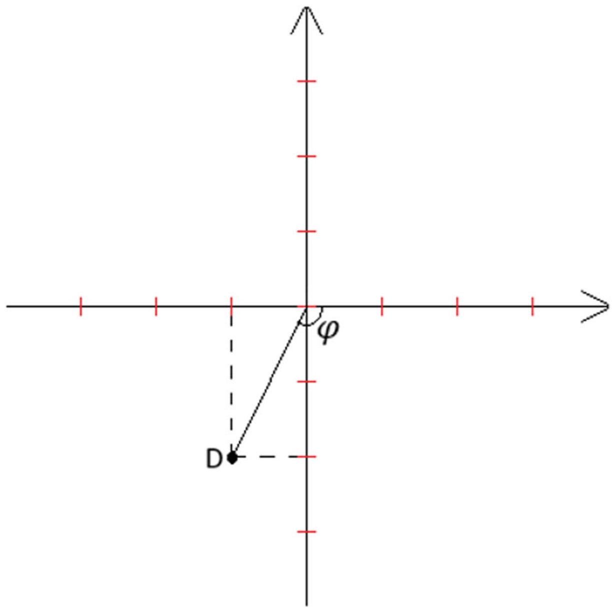


Fig. 8.6 Point $D = -1 - 2j$

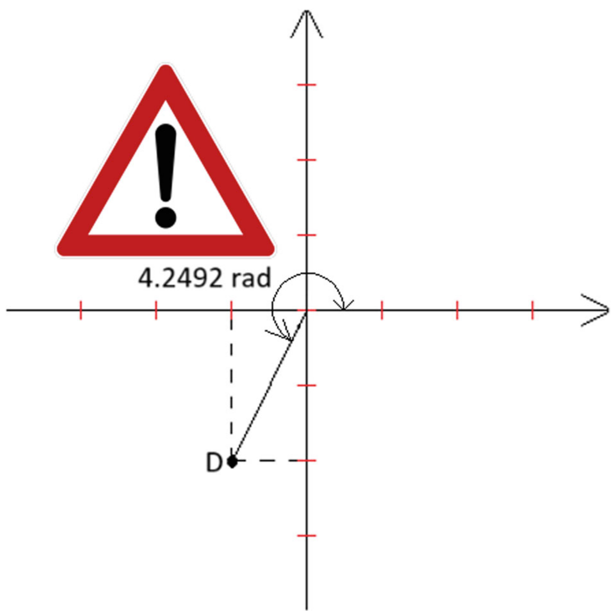


Fig. 8.7 Wrong measurement of principle argument

Additionally, if z lies on the positive real axis, the principle argument is 0. If z lies on the negative real axis, the principle argument is π. The principle argument is not defined for $z = 0$. Note that:

$$Arg\left(\frac{z_1}{z_2}\right) = Arg(z_1) - Arg(z_2)$$

$$Arg(z_1.z_2) = Arg(z_1) + Arg(z_2)$$

If the resulting sum $Arg(z_1) + Arg(z_2)$ or subtraction $Arg(z_1) - Arg(z_2)$ exceeds the $(-\pi, \pi]$ range, you may need to adjust the result by adding or subtracting 2π to bring it back into the principal range. For instance,

$$z_1 = -1 + 2j \Rightarrow Arg(z_1) = 2.0344rad$$

$$z_2 = -2 + j \Rightarrow Arg(z_2) = 2.6779rad$$

$$Arg(z_1.z_2) = 2.0344 + 2.6779 = 4.7123 \notin (-\pi, \pi]$$

$$4.7123 - 2\pi = -1.5709 \in (-\pi, \pi]$$

$$Arg(z_1.z_2) = Arg((-1 + 2j)(-2 + j)) = Arg(-5j) = -\frac{\pi}{2} \approx -1.5709$$

8.8 Polar Form of Complex Numbers

A complex number $z = a + bj$ can be shown in the form of $z = re^{j(\varphi+2k\pi)}k \in \mathbb{Z}$. r and φ show the modulus (magnitude) and the principle argument, respectively. Note that the polar form of a complex number is not unique since $re^{j(\varphi+2k\pi)} = re^{j\varphi}e^{j2k\pi} = re^{j\varphi}(\cos(2k\pi) + j\sin(2k\pi)) = re^{j\varphi}(1 + j0) = re^{j\varphi}, k \in \mathbb{Z}$. For instance, $\sqrt{5}e^{j1.1071}$, $\sqrt{5}e^{j(1.1071+2\pi)} = \sqrt{5}e^{j7.3903}$, $\sqrt{5}e^{j(1.1071-2\pi)} = \sqrt{5}e^{-j5.1761}$ and $\sqrt{5}e^{j(1.1071+20\pi)} = \sqrt{5}e^{j63.9390}$ all represent $1 + 2j$.

Let's study a numeric example. Polar form of $z = -1 - 2j$ is $\sqrt{(-1)^2 + (-2)^2}e^{j(-2.034+2k\pi)} = \sqrt{5}e^{j(-2.034+2k\pi)}, k \in \mathbb{Z}$ (Fig. 8.6).

Example 8.1 Convert $1 + 2j$ to polar form.

$$|1 + 2j| = \sqrt{1+4} = \sqrt{5}$$

$$Arg(1 + 2j) = 1.1071$$

$$1 + 2j = \sqrt{5}e^{j1.1071 \pm 2k\pi} k = 0, 1, 2, 3, \ldots$$

Example 8.2 Convert $-1 - 2j$ to polar form.

$$|-1 - 2j| = \sqrt{1 + 4} = \sqrt{5}$$

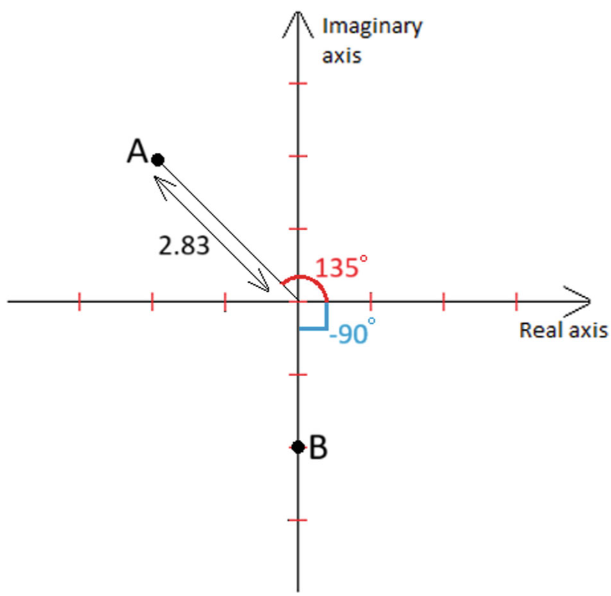
$$Arg(-1 - 2j) = -2.034$$

$$-1 - 2j = \sqrt{5}e^{-j2.034 \pm 2k\pi} k = 0, 1, 2, 3, \ldots$$

8.9 Representation of Polar Form on the Complex Plane

Figure 8.8 shows how points $A = 2.83e^{j\frac{3\pi}{4}}$ and $B = 2e^{-j\frac{\pi}{2}}$ can be shown on the complex plane.

Fig. 8.8 Points $A = 2.83e^{j\frac{3\pi}{4}}$ and $B = 2e^{-j\frac{\pi}{2}}$



8.10 Basic Operations on Complex Numbers

Following examples reviews the basic operations on complex numbers.

(1) $(1+2j)(3+4j) = 1\times 3 + 1\times 4j + 2j\times 3 + 2j\times 4j = 3 + 4j + 6j - 8 = -5 + 10j$
(2) $\frac{(1+2j)}{(3+4j)} = \frac{(1+2j)(3-4j)}{(3+4j)(3-4j)} = \frac{3-4j+6j+8}{3^2+4^2} = \frac{11+2j}{25}$
(3) $\frac{1}{6j} = \frac{1\times j}{6j\times j} = \frac{1\times j}{-6} = -\frac{1}{6}j$
(4) $(1+2j)+(3+4j) = (1+3)+(2+4)j = 4+6j$
(5) $(1+2j)-(3+4j) = (1-3)+(2-4)j = -2-2j$
(6) $\frac{7e^{j\frac{\pi}{4}}}{12e^{j\frac{\pi}{3}}} = \frac{7}{12}e^{j(\frac{\pi}{4}-\frac{\pi}{3})} = \frac{7}{12}e^{j(\frac{-\pi}{12})} = 0.5635 - 0.1510j$
(7) $7e^{j\frac{\pi}{4}} \times 12e^{j\frac{\pi}{3}} = 7\times 12e^{j(\frac{\pi}{4}+\frac{\pi}{3})} = 84e^{j\frac{7\pi}{12}} = -21.7408 + 81.1378j$
(8) $7e^{j\frac{\pi}{4}} + 12e^{j\frac{\pi}{3}} = 7\left(\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right) + 12\left(\cos\left(\frac{\pi}{3}\right) + j\sin\left(\frac{\pi}{3}\right)\right) = 4.9497 + 4.9497i + 6.0000 + 10.3923i = 10.9497 + 15.3420i$

8.11 Complex Conjugate of a Complex Number

A complex conjugate of a complex number is another complex number that has the same real part as the original complex number and the imaginary part has the same magnitude but opposite sign. The product of a complex number and its complex conjugate is a real number. Complex conjugate of $a + jb$ is $a - jb$. The notation for the complex conjugate of z is either $\bar{z}$ or z^*. In this book we use the $\bar{z}$. Following properties are important:

$\overline{(a+jb)} = a - jb$. For instance, $\overline{(1+2j)} = 1 - 2j$.

$\overline{(a+jb)}\ (a+jb) = (a+jb)(a-jb) = a^2 + b^2$. For instance, $(1+2j)(1-2j) = 1^2 + 2^2 = 5$.

$z = re^{j\theta} \Rightarrow \bar{z} = re^{-j\theta}$. For instance, $z = 2e^{j\frac{\pi}{4}} \Rightarrow \bar{z} = 2e^{-j\frac{\pi}{4}}$.

$|z| = |\bar{z}|$. For instance, $|1 - j2| = |1 + j2| = \sqrt{1^2 + 2^2} = \sqrt{5}$.

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$. For instance, $\overline{\left(\frac{1+2j}{3+4j}\right)} = \frac{1-2j}{3-4j}$.

8.12 Geometric Interpretation of Complex Conjugation

Complex conjugation represents a reflection about the real axis. For instance, in Fig. 8.9 point $A = 1 + 2j$ and its complex conjugate $\overline{A} = 1 - 2j$ are shown.

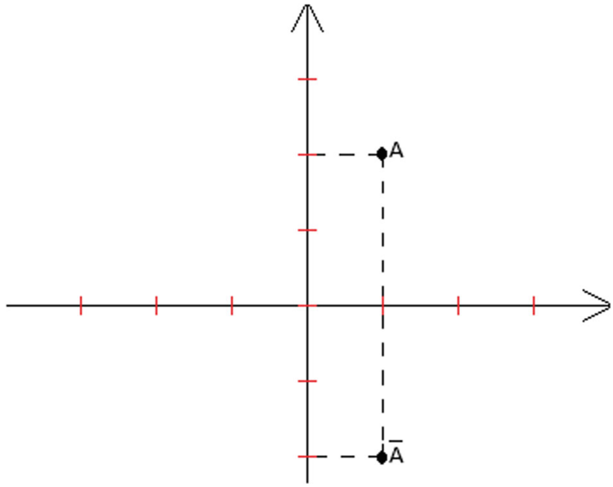


Fig. 8.9 Point A and its complex conjugate

8.13 The Complex Conjugate Root Theorem

In mathematics, the complex conjugate root theorem states that if $P(x)$ is a polynomial in one variable with real coefficients, and $a+jb$ is a root of $P(x)$, then its complex conjugate $a-jb$ is also a root of $P(x)$.

For instance, $P(x) = x^3 + 3x^2 + 7x + 5 = 0$ is polynomial in one variable with real coefficients. Its roots are, -1, $-1+j2$ and $-1-j2$.

8.14 Complex Exponential Function

Value of complex exponential functions can be calculated with the aid of following formula:

$$e^{a+bj} = e^a \times e^{bj} = e^a \times (\cos(b) + j\sin(b))$$

Here are two examples:

(1) $e^{1+2j} = e^1(\cos(2) + j\sin(2)) = e^1(-0.4161 + j0.9093) = -1.1312 + 2.4717j$
(2) $e^{j\frac{\pi}{6}} = \cos\left(\frac{\pi}{6}\right) + j\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + j\frac{1}{2}$

If $a < 0$ then $\lim_{x\to\infty} e^{(a+jb)x} = 0$ (note that j shows the imaginary unit). For instance, $\lim_{x\to\infty} e^{(-3+j4)x}$, $\lim_{x\to\infty} e^{(-3-j4)x}$ or $\lim_{x\to\infty} e^{-6x}$ equals to zero.

If $a > 0$ then $\lim_{x\to-\infty} e^{(a+jb)x} = 0$ (note that j shows the imaginary unit). For instance, $\lim_{x\to-\infty} e^{(7+j2)x}$, $\lim_{x\to-\infty} e^{(7-j2)x}$ or $\lim_{x\to-\infty} e^{12x}$ equals to zero.

8.15 Polar Form of Complex Numbers

A complex number $z = a + bj$ can be shown in the form of $z = re^{j\theta}$. Note that the polar form of a complex number is not unique since:

$$re^{j(\theta+2k\pi)} = re^{j\theta}e^{j2k\pi} = re^{j\theta}(\cos(2k\pi) + j\sin(2k\pi)) = re^{j\theta}(1 + j0) = re^{j\theta}, k \in \mathbb{Z}$$

For instance, $\sqrt{5}e^{j1.1071}$, $\sqrt{5}e^{j(1.1071+2\pi)} = \sqrt{5}e^{j7.3903}$, $\sqrt{5}e^{j(1.1071-2\pi)} = \sqrt{5}e^{-j5.1761}$ and $\sqrt{5}e^{j(1.1071+20\pi)} = \sqrt{5}e^{j63.9390}$ all represent $1 + 2j$.

8.16 Some of Important Equations in Complex Plane

$|z - z_0| = R$ shows a circle with center of z_0 and radius of R. For instance, equation of a circle with center of $A = 1 + 2j$ and radius of 1 (Fig. 8.10) is $|z - (1 + 2j)| = 1$.

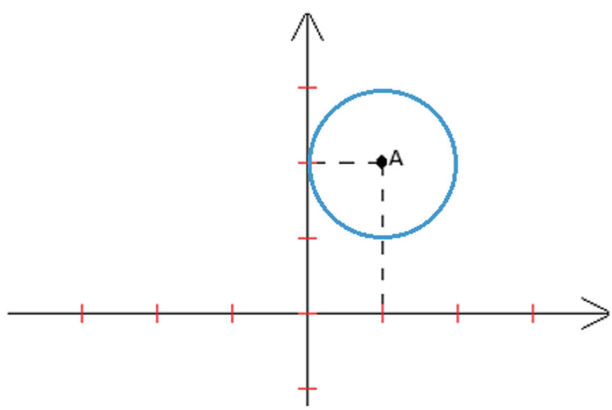
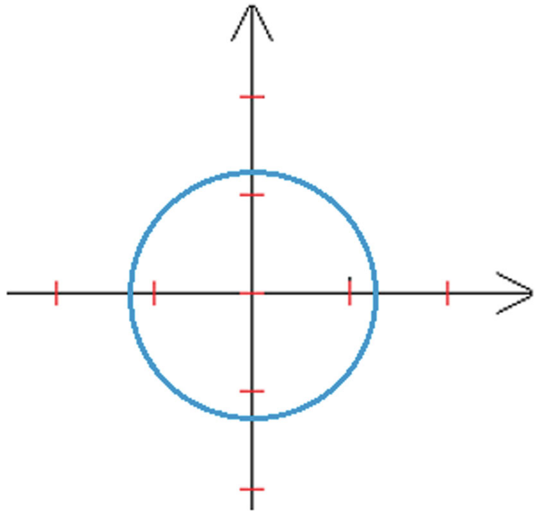


Fig. 8.10 Plot of $|z - (1 + 2j)| = 1$

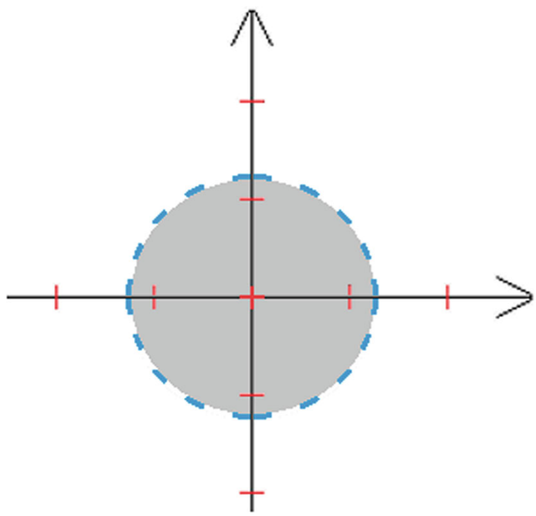
Equation of a circle with center of origin and radius of 1.2 (Fig. 8.11) is $|z - (0 + 0j)| = 1.2$ or $|z| = 1.2$.

Fig. 8.11 Plot of $|z| = 1.2$



$|z - z_0| < R$ shows points fall inside a circle with center of z_0 and radius of R. $|z - z_0| \leq R$ shows points fall inside and on a circle with center of z_0 and radius of R. For instance, $|z| < 1.2$ includes points inside a circle with center of origin and radius of 1.2 (Fig. 8.12). $|z| \leq 1.2$ includes points inside and on a circle with center of origin and radius of 1.2 (Fig. 8.13).

Fig. 8.12 Plot of $|z| < 1.2$



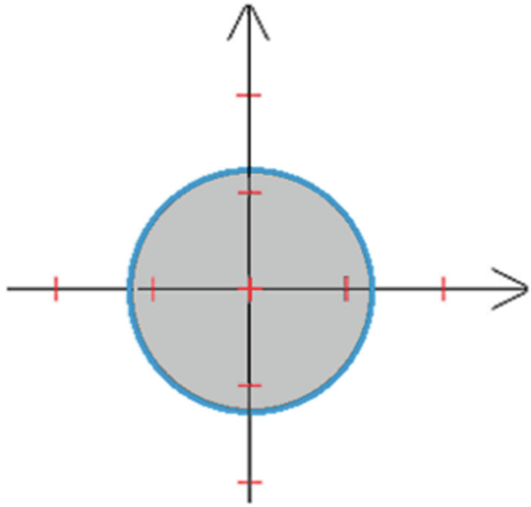


Fig. 8.13 Plot of $|z| \leq 1.2$

$|z - z_0| > R$ shows points fall outside a circle with center of z_0 and radius of R. $|z - z_0| \geq R$ shows points fall outside and on a circle with center of z_0 and radius of R. For instance, $|z - (1 + 2j)| \geq 1$ includes points outside and on a circle with center of $1 + 2j$ and radius of 1 (Fig. 8.14).

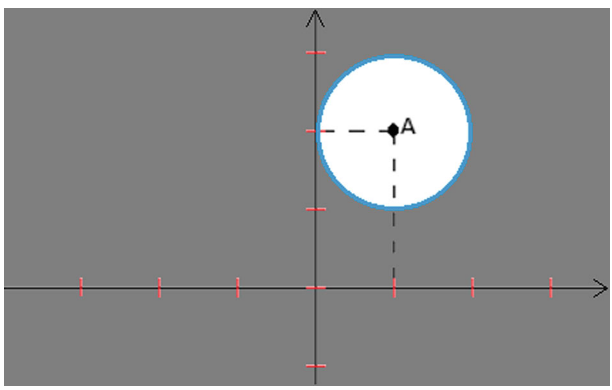


Fig. 8.14 Plot of $|z - (1 + 2j)| \geq 1$

In mathematics, an annulus (pl.: annuli or annuluses) is the region between two concentric circles. $r_1 < |z - z_0| < r_2$ shows the region between two concentric circles with center of z_0 and radiuses r_1 and r_2. For instance, points satisfy $1 \leq |z - 1 - 2j| \leq 1.5$ are shown in Fig. 8.15.

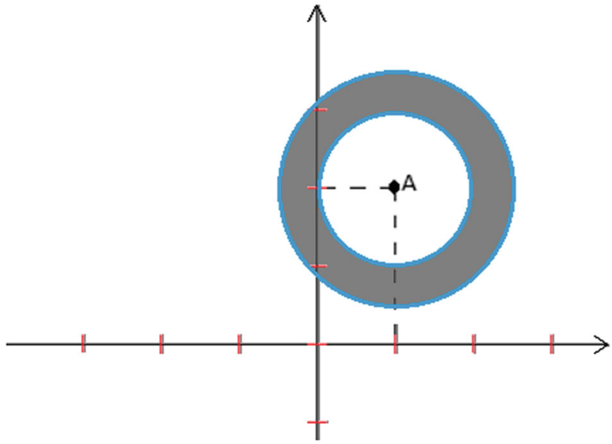


Fig. 8.15 Plot of $1 \leq |z - 1 - 2j| \leq 1.5$

8.17 Complex Trigonometric Functions

The sine and cosine of a complex variable z are defined as follows:

$$\sin(z) = \frac{e^{jz} - e^{-jz}}{2j}$$

$$\cos(z) = \frac{e^{jz} + e^{-jz}}{2}$$

Following formulas help to calculate the value of complex trigonometric functions easily. In the following formulas sinh, cosh, tanh, coth, sech and csch denotes the hyperbolic sine, cosine, tangent, cotangent, secant and cosecant, respectively.

(1) $\sin(a + bj) = \sin(a)\cosh(b) + j\cos(a)\sinh(b)$
(2) $\cos(a + bj) = \cos(a)\cosh(b) - j\sin(a)\sinh(b)$
(3) $tan(a + bj) = \frac{\sin(2a)}{\cos(2a)+\cosh(2b)} + j\frac{\sinh(2b)}{\cos(2a)+\cosh(2b)}$
(4) $cot(a + bj) = \frac{cot(a).coth^2(b)-cot(a)}{cot^2(a)+coth^2(b)} + j\frac{-cot^2(a).coth(b)-coth(b)}{cot^2(a)+coth^2(b)}$
(5) $sec(a + bj) = \frac{1}{\cos(a+bj)} = \frac{\cos(a)\cosh(b)+j\sin(a)\sinh(b)}{\cos^2(a)\cosh^2(b)+\sin^2(a)\sinh^2(b)}$
(6) $\csc(a + bj) = \frac{1}{\sin(a+bj)} = \frac{\sin(a)\cosh(b)-j\cos(a)\sinh(b)}{\sin^2(a)\cosh^2(b)+\cos^2(a)\sinh^2(b)}$

For purely imaginary arguments, i.e., arguments with zero real part,

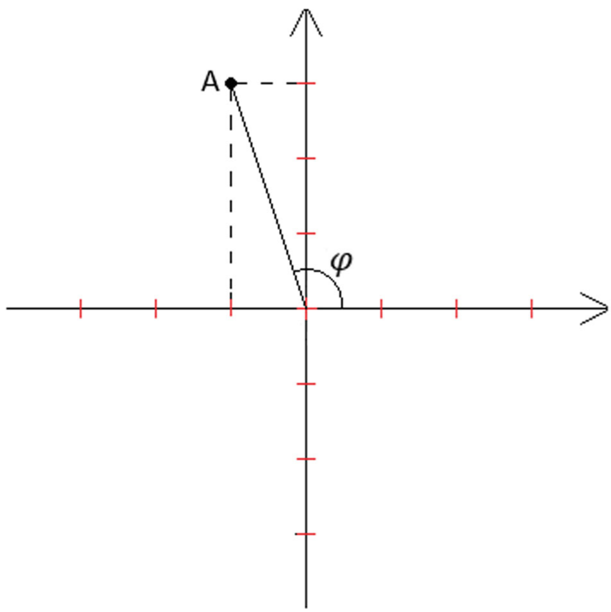
(1) $\sin(jb) = j\sinh(b)$
(2) $\cos(jb) = \cosh(b)$
(3) $\tan(jb) = jtanh(b)$
(4) $\cot(jb) = -jcoth(b)$
(5) $\sec(jb) = sech(b)$
(6) $\csc(jb) = -jcsch(b)$

8.18 Complex Root Function

For any non-zero complex number z and any positive integer n, the equation $w^n = z$ has exactly n distinct complex solutions $w_0, w_1, \ldots, w_{n-1}$.

Procedure for calculation of complex root is shown with an example. Assume that we want to calculate the third root of $-1-3j$, i.e., $\sqrt[3]{(-1-3j)}$. Given Cartesian form needs to be converted into the polar form. $(-1-3j)$ lies in second quadrant (Fig. 8.16).

Fig. 8.16 Point $A = -1 + 3j$



$$\varphi = Arg(-1+3j) = 90 + \tan^{-1}\left(\frac{1}{3}\right) = 90 + 18.4349 = 108.4349^\circ = 1.8925Rad$$

$$-1+3j = \sqrt{(-1)^2 + (-3)^2} = 3.1623e^{j(1.8925+2k\pi)}k = 0, \pm1, \pm2, \pm3, \ldots$$

$$\sqrt[3]{-1+3j} = \sqrt[3]{3.1623}e^{j(\frac{1.8925+2k\pi}{3})} = 1.4678e^{j(\frac{1.8925+2k\pi}{3})}$$

$$k = 0 \Rightarrow z_1 = 1.4678e^{j\left(\frac{1.8925+0}{3}\right)} = 1.4678e^{j0.6308} = 1.1853 + 0.8657j$$

$$k = 1 \Rightarrow z_2 = 1.4678e^{j\left(\frac{1.8925+2\pi}{3}\right)} = 1.4678e^{j2.7252} = -1.3424 + 0.5937j$$

$$k = 2 \Rightarrow z_3 = 1.4678e^{j\left(\frac{1.8925+4\pi}{3}\right)} = 1.4678e^{j4.8196} = 0.1571 - 1.4594j$$

Therefore, $\sqrt[3]{-1+3j} = \begin{cases} 1.1853 + 0.8657j \\ -1.3424 + 0.5937j \\ 0.1571 - 1.4594j \end{cases}.$

8.19 Complex Logarithmic Function

For $z = |z|e^{j(Arg(z)+2k\pi)}k \in \mathbb{Z}$,

$$Ln(z) = \ln(|z|) + jArg(z)$$

$$\ln(z) = Ln(z) + j2k\pi k \in \mathbb{Z}$$

$\ln(z)$ is a multi-valued function with infinitely many possible values. $Ln(z)$ is the principle value of $\ln(z)$. $Ln(z)$ is a single-valued function with a unique value for each complex number within its domain. The domain of the principal complex logarithm is the set of all complex numbers except for the non-positive real numbers. In mathematical notation, this is represented as: $\mathbb{C}\backslash(-\infty, 0]$. $(-\infty, 0]$ denotes the set of all non-positive real numbers, including 0.

For instance, let's find the domain of $Ln(1+z^2)$.

$$1+z^2=0 \Rightarrow z^2=-1 \Rightarrow z=\pm j$$

$$1+z^2=1+(x+jy)^2=1+x^2-y^2+j2xy \Rightarrow \begin{cases} 1+x^2-y^2<0 \\ 2xy=0 \end{cases}$$

$$2xy=0 \Rightarrow x=0 \; or \; y=0$$

when $x=0 \Rightarrow 1+x^2-y^2=1-y^2<0 \Rightarrow 1<y^2 \Rightarrow \begin{cases} y>+1 \\ y<-1 \end{cases}$

when $y=0 \Rightarrow 1+x^2-y^2=1+x^2<0 \Rightarrow impossible\, since\, x \in \mathbb{R}$.

Therefore, at $(0, y \geq +1)$ and $(0, y \leq -1)$, the $Ln(1+z^2)$ is not defined (Fig. 8.17).

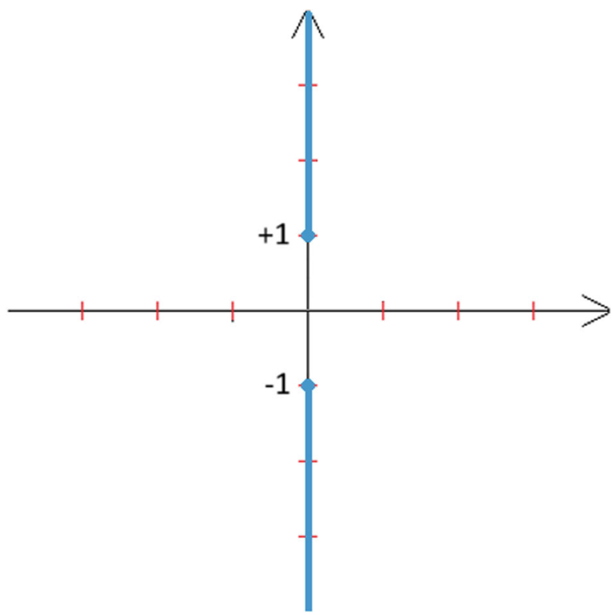
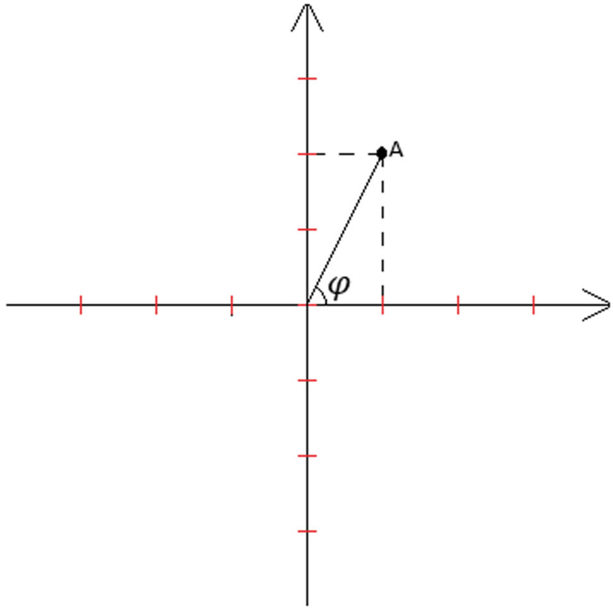


Fig. 8.17 $Ln(1+z^2)$ is defined everywhere except the blue line

Let's study a numeric example to see how complex logarithm can be calculated. For example, let's calculate $\ln(1+2j)$ and $Ln(1+2j)$. $z=1+2j$ lies in the first quadrant (Fig. 8.18).

Fig. 8.18 Point $A = 1 + 2j$



Let's obtain the polar form of $1 + 2j$.

$$\varphi = \tan^{-1}\left(\frac{2}{1}\right) = 1.1071 Rad$$

$$1 + 2j = \sqrt{1+4}e^{j(\varphi+2k\pi)} = \sqrt{5}e^{j(1.1071+2k\pi)}k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Now the ln function can be calculated easily:

$$\ln(1+2j) = \ln\left(\sqrt{5}e^{j(1.1071+2k\pi)}\right) = \ln\left(\sqrt{5}\right) + j(1.1071 + 2k\pi) = 0.8047 + j(1.1071 + 2k\pi), k \in \mathbb{Z}$$

As seen, $\ln(z)$ is a multi-valued function with infinitely many possible values. $k = 0$ gives $Ln(1 + 2j)$.

$$\ln(1+2j) = 0.8047 + j(1.1071 + 2k\pi) \Rightarrow Ln(1+2j) = 0.8047 + j1.1071$$

Note that for $z \in \mathbb{C} \backslash (-\infty, 0]$, $\frac{d\ln(z)}{dz} = \frac{1}{z}$.

8.20 Complex Power Function

${z_1}^{z_2}$ can be calculated with the aid of ${z_1}^{z_2} = e^{z_2 Ln(z_1)}$. Let's study two numeric examples.

(1) $(1+2j)^{(3+4j)} = e^{(3+4j)Ln(1+2j)} = e^{(3+4j)(0.8047+j1.1071)} = e^{-2.0143+j6.5401} = e^{-2.0143}(\cos(6.5401)+j\sin(6.5401)) = 0.1290+j0.0339$

(2) $j^j = e^{jLn(j)} = e^{jLn\left(e^{j\frac{\pi}{2}}\right)} = e^{j.j\frac{\pi}{2}} = e^{-\frac{\pi}{2}} = 0.2079$

8.21 Analyticity

A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D. The function $f(z)$ is said to be analytic at a point $z = z_0$ in D if $f(z)$ is analytic in neighborhood of z_0. Also, by an analytic function we mean a function that is analytic in some domain.

For instance, z^n, $\sum_{i=0}^{n} c_i z^i$, e^z, $\sin(z)$ and $\cos(z)$ are analytic everywhere. Rational functions are not analytic at roots of denominator. For instance, $\frac{z+1}{z-2}$ is analytic everywhere except of $z = 2$, i.e., $\mathbb{C}\backslash\{2\}$. $Log_a(z)$ is analytic everywhere except for the non-positive real numbers, i.e., $\mathbb{C}\backslash(-\infty, 0]$.

8.22 Complex Integrals

This section studies the important complex integrals and methods to calculate them.

8.23 Simply Connected Domain

In complex analysis, a simply connected domain is a region in the complex plane that is both connected and has no "holes." This means that any closed curve within the domain can be continuously deformed to a point without leaving the domain.

For instance, the open unit disk ($|z| < 1$) or the upper half plane ($Im(z) > 0$) are simply connected domains. The annulus ($1 < |z| < 2$) or the complex plane with the origin removed ($\mathbb{C}\backslash\{0\}$) are examples of domains that are not simply connected.

8.24 Cauchy's Integral Theorem

If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D, $\oint_C f(z)dz = 0$ (Fig. 8.19).

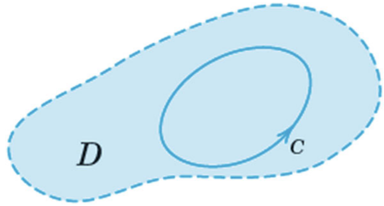


Fig. 8.19 Cauchy's integral theorem

For instance, let C be any closed path. Then $\oint_C e^z dz$, $\oint_C \cos(z) dz$ and $\oint_C z^n dz$ where $n = 0, 1, 2, 3, \ldots$ are all 0 for any closed path C since these functions are all analytic for all z.

8.25 Cauchy's Integral Formula

Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D, and any simple closed path C in D that encloses z_0, $\oint_C \frac{f(z)}{z-z_0} dz = 2\pi i \times f(z_0)$ (Fig. 8.20).

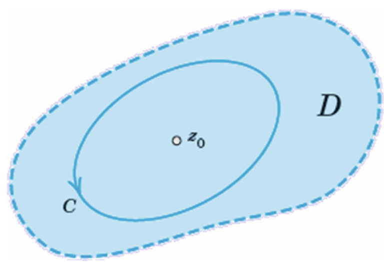


Fig. 8.20 Cauchy's integral formula

Let's study some examples.

Example 8.3 Let C be a contour enclosing $z_0 = 2$. Calculate $\oint_C \frac{e^z}{z-2} dz$.

By the Cauchy integral formula

$$\oint_C \frac{e^z}{z-2} dz = 2\pi i \times e^2$$

Example 8.4 Calculate $\oint_C \frac{z^3-6}{2z-i} dz$ where C is a circuit around the origin with radius of 1.

$z_0 = \frac{1}{2} i$ falls inside the C. Therefore,

$$\oint_C \frac{z^3-6}{2z-i} dz = \oint_C \frac{\frac{1}{2} z^3 - 3}{z - \frac{i}{2}} dz = 2\pi i \times \left(\frac{1}{2} \left(\frac{i}{2} \right)^3 - 3 \right) = \frac{\pi}{8} - 6\pi i$$

Example 8.5 Integrate $f(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z+1)(z-1)}$ around each of the paths shown in Figs. 8.21, 8.22, 8.23 and 8.24.

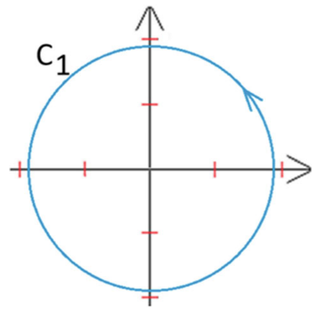


Fig. 8.21 Path C_1

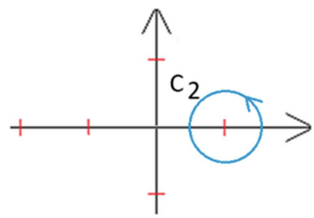


Fig. 8.22 Path C_2

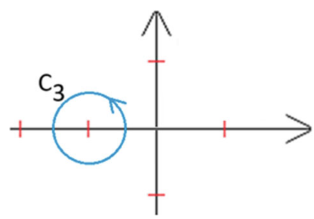


Fig. 8.23 Path C_3

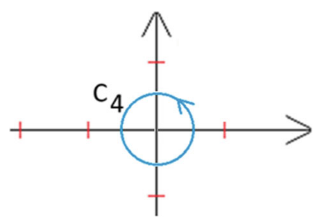


Fig. 8.24 Path C_4

(a) $f(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z+1)(z-1)}$ is not analytic at $z = +1$ and $z = -1$. C_1 encloses both $z = +1$ and $z = -1$. Therefore,

$$f(z) = \frac{z^2+1}{z^2-1} = \frac{z^2-1+2}{z^2-1} = 1 + \frac{2}{z^2-1} = 1 + \frac{2}{(z+1)(z-1)} = 1 + \frac{-1}{(z+1)} + \frac{1}{(z-1)}$$

$$\oint_{C_1}(1+\frac{-1}{(z+1)}+\frac{1}{(z-1)})dz=\oint_{C_1}1dz+\oint_{C_1}\frac{-1}{z+1}dz+\oint_{C_1}\frac{1}{z-1}dz$$

$$\oint_{C_1}1dz=0$$

$$\oint_{C_1}\frac{-1}{z+1}dz=2\pi i\times -1=-2\pi i$$

$$\oint_{C_1}\frac{1}{z-1}dz=2\pi i\times +1=+2\pi i$$

$$\oint_{C_1}1dz+\oint_{C_1}\frac{-1}{z+1}dz+\oint_{C_1}\frac{1}{z-1}dz=0+(-2\pi i)+2\pi i=0$$

(b) $\oint_{C_2}(1+\frac{-1}{(z+1)}+\frac{1}{(z-1)})dz=\oint_{C_2}1dz+\oint_{C_2}\frac{-1}{z+1}dz+\oint_{C_2}\frac{1}{z-1}dz$

$$\oint_{C_2}1dz=0$$

$$\oint_{C_2}\frac{-1}{z+1}dz=0$$

$$\oint_{C_2}\frac{1}{z-1}dz=2\pi i\times +1=+2\pi i$$

$$\oint_{C_2}1dz+\oint_{C_2}\frac{-1}{z+1}dz+\oint_{C_2}\frac{1}{z-1}dz=0+0+2\pi i=2\pi i$$

(c) $\oint_{C_3}(1+\frac{-1}{(z+1)}+\frac{1}{(z-1)})dz=\oint_{C_3}1dz+\oint_{C_3}\frac{-1}{z+1}dz+\oint_{C_3}\frac{1}{z-1}dz$

$$\oint_{C_3}1dz=0$$

$$\oint_{C_3}\frac{-1}{z+1}dz=-2\pi i$$

$$\oint_{C_3}\frac{1}{z-1}dz=2\pi i\times +1=0$$

$$\oint_{C_3} 1dz + \oint_{C_3} \frac{-1}{z+1}dz + \oint_{C_3} \frac{1}{z-1}dz = 0 + (-2\pi i) + 0 = -2\pi i$$

(d) $\oint_{C_4}(1 + \frac{-1}{(z+1)} + \frac{1}{(z-1)})dz = 0 + 0 + 0 = 0$

8.26 Derivatives of an Analytic Function

If $f(z)$ is analytic in a domain D, then it has derivatives of all order in D, which are then also analytic functions in D. The values of these derivatives at a point z_0 in D can be calculated with following formulas:

$$\oint_C \frac{f(z)}{(z-z_0)^2} = 2\pi i \times f\prime(z_0)$$

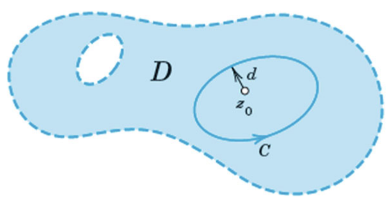
$$\oint_C \frac{f(z)}{(z-z_0)^3} = 2\pi i \times \frac{f''(z_0)}{2}$$

and in general

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} = 2\pi i \times \frac{f^{(n)}(z_0)}{n!} n = 1, 2, 3, \ldots$$

here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D; and we integrate counterclockwise around C (Fig. 8.25).

Fig. 8.25 Path C encloses z_0



Let's study some examples.

Example 8.6 Let C be a contour enclosing $z_0 = \pi i$, oriented counterclockwise. Calculate $\oint_C \frac{\cos(z)}{(z-\pi i)^2}dz$.

$$\oint_C \frac{\cos(z)}{(z-\pi i)^2}dz = -2\pi i \times \sin(\pi i) = 2\pi \times \sinh(\pi)$$

Example 8.7 Calculate $\oint_C \frac{z^4-3z^2+6}{(z+i)^3}dz$. C is any counterclockwise contour enclosing the point $-i$.

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3} dz = \pi i \times \left(z^4 - 3z^2 + 6\right)''\big|_{z=-i} = \pi i \times \left(12z^2 - 6\right)\big|_{z=-i} = -18\pi i$$

Example 8.8 Calculate $\oint_C \frac{e^z}{(z-1)^2(z^2+4)} dz$. C is any counterclockwise contour which 1 lies inside and $\pm 2i$ lies outside.

$$\oint_C \frac{e^z}{(z-1)^2(z^2+4)} dz = \oint_C \frac{\frac{e^z}{(z^2+4)}}{(z-1)^2} dz = 2\pi i \times \left(\frac{e^z}{z^2+4}\right)'\bigg|_{z=1}$$
$$= 2\pi i \left(\frac{e^z(z^2+4) - e^z \times 2z}{\left(z^2+4\right)^2}\right)\Bigg|_{z=1} = \frac{6\pi e}{25} i = 2.05i$$

8.27 Problem Solving with MATLAB®

This section provides examples of how MATLAB can be employed to solve the types of problems encountered in this chapter.

Example 8.9 Define the complex number $z_1 = 1 + 2i$ in MATLAB.

You can use either of the methods shown in Figs. 8.26 or 8.27.

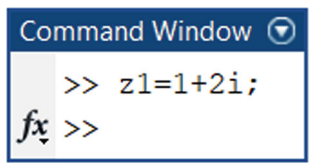


Fig. 8.26 Defining `z1` as $1 + 2i$

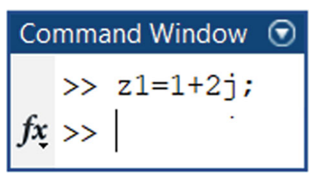


Fig. 8.27 Defining `z1` as $1 + 2i$

Example 8.10 Define the complex number $z_2 = 3e^{j\frac{\pi}{3}}$ in MATLAB.

You can use the command shown in Fig. 8.28. According to Fig. 8.28, $z_2 = 3e^{j\frac{\pi}{3}} = 1.5 + 2.5981j$ as well.

Fig. 8.28 Defining z2 as $3e^{j\frac{\pi}{3}}$

```
Command Window
>> z2=3*exp(j*pi/3)
z2 =
   1.5000 + 2.5981i
fx >>
```

Example 8.11 Visualize the complex numbers $z_1 = 1 + 2i$, $z_2 = -1 - 3i$ and $z_3 = 3e^{j\frac{\pi}{6}}$ on the complex plane using MATLAB.

The MATLAB code in Fig. 8.29 visualizes the complex numbers z_1, z_2 and z_3 as small red circles on the complex plane. Output of this code is shown in Fig. 8.30.

```
Command Window
>> z1=1+j;
>> z2=-1-3j;
>> z3=3*exp(j*pi/6);
>> plot(real([z1 z2 z3]),imag([z1 z2 z3]),'ro');
>> xlim([-2 3]);
>> ylim([-3.5 2]);
>> grid
fx >>
```

Fig. 8.29 Visualization of $z_1 = 1 + 2i$, $z_2 = -1 - 3i$ and $z_3 = 3e^{j\frac{\pi}{6}}$ on the complex plane

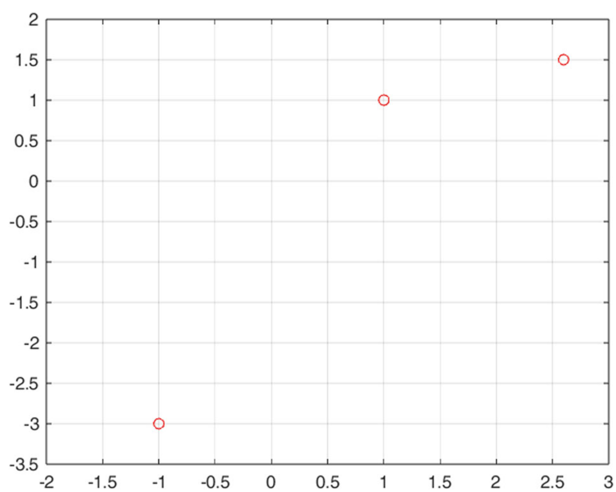


Fig. 8.30 Output of the code shown in Fig. 8.29

Example 8.12 Visualize the complex numbers $z_1 = e^{j\frac{\pi}{3}}$, $z_2 = 2e^{j\frac{\pi}{4}}$ and $z_3 = 3.4e^{j\frac{\pi}{2}}$ on the complex plane using MATLAB.

The `polarplot` command can be used to visualize complex numbers given in polar format. The MATLAB code in Fig. 8.31 visualizes the complex numbers z_1, z_2 and z_3 as small x markers on the complex plane. The output of this code is shown in Fig. 8.32.

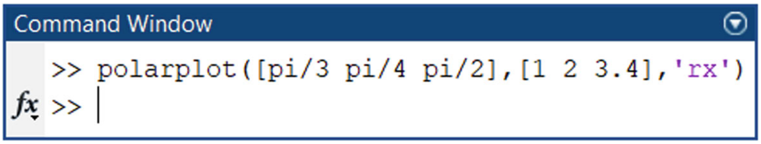
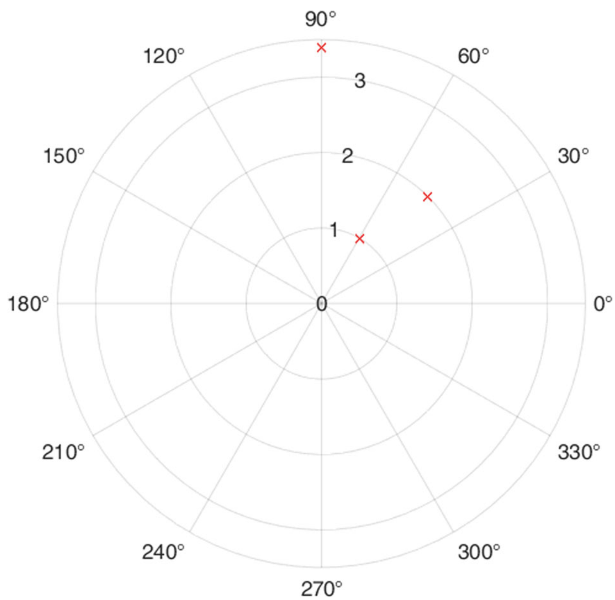


Fig. 8.31 Visualization of $z_1 = e^{j\frac{\pi}{3}}$, $z_2 = 2e^{j\frac{\pi}{4}}$ and $z_3 = 3.4e^{j\frac{\pi}{2}}$ on the complex plane

Fig. 8.32 Output of the code shown in Fig. 8.31



Example 8.13 Use MATLAB to calculate the real part and imaginary part of $z_2 = 3e^{j\frac{\pi}{3}}$.

The commands shown in Fig. 8.33 calculates the real part and imaginary part of $z_2 = 3e^{j\frac{\pi}{3}}$.

Fig. 8.33 Real and imaginary part of $z_2 = 3e^{j\frac{\pi}{3}}$

```
Command Window
>> z2=3*exp(j*pi/3)
z2 =
   1.5000 + 2.5981i
>> real(z2)
ans =
    1.5000
>> imag(z2)
ans =
    2.5981
fx >> 
```

Example 8.14 Given $z_1 = 1+2i$ and $z_2 = 3+4i$, use MATLAB to calculate $z_1 \cdot z_2$, $\frac{z_1}{z_2}$, z_1+z_2 and $z_1 - z_2$.

Figure 8.34 shows the MATLAB code for calculating $z_1 \cdot z_2$, $\frac{z_1}{z_2}$, $z_1 + z_2$ and $z_1 - z_2$.

Fig. 8.34 Calculation of $z_1 \cdot z_2$, $\frac{z_1}{z_2}$, $z_1 + z_2$ and $z_1 - z_2$ for $z_1 = 1 + 2i$ and $z_2 = 3 + 4i$

```
Command Window
>> z1=1+2j;
>> z2=3+4j;
>> z1*z2
ans =
  -5.0000 +10.0000i
>> z1/z2
ans =
   0.4400 + 0.0800i
>> z1+z2
ans =
   4.0000 + 6.0000i
>> z1-z2
ans =
  -2.0000 - 2.0000i
fx >> 
```

Example 8.15 Determine the modulus and principal argument of the complex number $z_1 = 1 - 2i$ using MATLAB.

Figures 8.35 and 8.36 present two approaches to determine the modulus and principal argument of $z_1 = 1 - 2i$. According to Figs. 8.35 and 8.36, $|z_1| = 2.2361$ and $Arg(z_1) = -1.1071$.

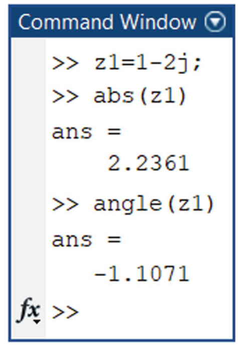


Fig. 8.35 Modulus and principle argument of $z_1 = 1 - 2i$

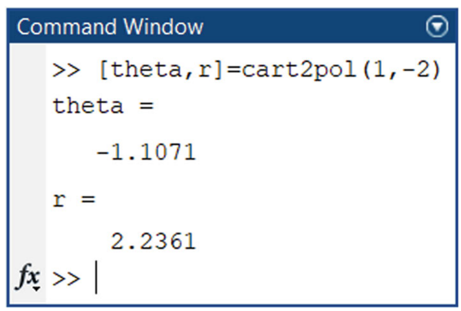


Fig. 8.36 Modulus and principle argument of $z_1 = 1 - 2i$

Example 8.16 Use MATLAB to determine the Cartesian form of $z = 2.2361e^{-j1.1071}$.

You can use either of the methods shown in Figs. 8.37 or 8.38. According to Figs. 8.37 and 8.38, $z = 2.2361e^{-j1.1071} = 1.0001 - 2.0000j$.

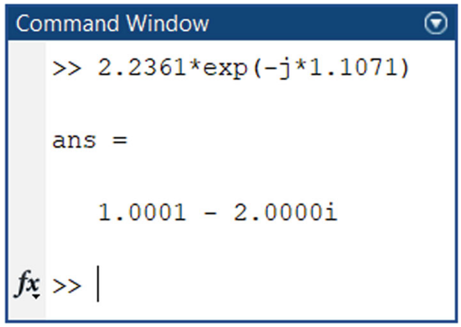


Fig. 8.37 Conversion of $z = 2.2361e^{-j1.1071}$ to Cartesian coordinates

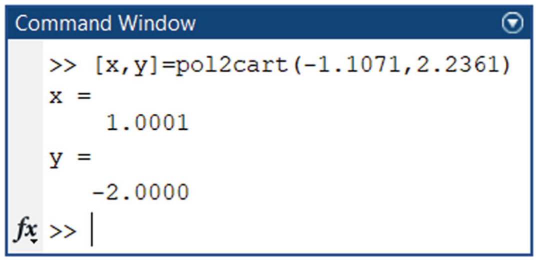


Fig. 8.38 Conversion of $z = 2.2361e^{-j1.1071}$ to Cartesian coordinates

Example 8.17 Use MATLAB to calculate the complex conjugate of $z_1 = 1 + 2j$ and $z_2 = 3e^{j\frac{\pi}{3}}$.

Figure 8.39 shows the MATLAB code for calculating the complex conjugates of $z_1 = 1 + 2j$ and $z_2 = 3e^{j\frac{\pi}{3}}$.

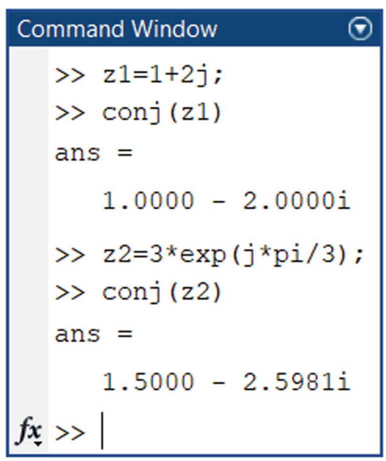


Fig. 8.39 Calculation of complex conjugate of $z_1 = 1 + 2j$ and $z_2 = 3e^{j\frac{\pi}{3}}$

Example 8.18 Given $z_1 = 1 + 2i$, use MATLAB to calculate $\sin(z_1)$, $\cos(z_1)$, $\tan(z_1)$, $\cot(z_1)$, $\sec(z_1)$ and $\text{cosec}(z_1)$.

Figure 8.40 shows the MATLAB code for calculating $\sin(z_1)$, $\cos(z_1)$, $\tan(z_1)$, $\cot(z_1)$, $\sec(z_1)$ and $\text{cosec}(z_1)$.

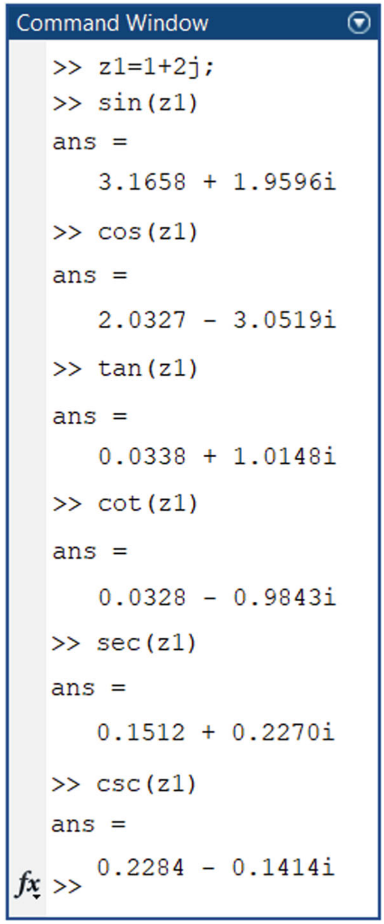


Fig. 8.40 Calculation of trigonometric ratios for $z_1 = 1 + 2i$

Example 8.19 Given $z_1 = 1 + 2i$ and $z_2 = 3 + 4i$, use MATLAB to calculate ${z_1}^{z_2}$, $Ln(z_1)$ and e^{z_1}.

Figure 8.41 shows the MATLAB code for calculating ${z_1}^{z_2}$, $Ln(z_1)$ and e^{z_1}.

Fig. 8.41 Calculation of ${z_1}^{z_2}$, $Ln(z_1)$ and e^{z_1} for $z_1 = 1 + 2i$ and $z_2 = 3 + 4i$

```
Command Window
>> z1=1+2j;
>> z2=3+4i;
>> z1^z2
ans =
   0.1290 + 0.0339i
>> log(z1)
ans =

   0.8047 + 1.1071i
>> exp(z1)
ans =
  -1.1312 + 2.4717i
fx >>
```

Example 8.20 Given $z = -1 + 3i$, use MATLAB to calculate $\sqrt[3]{z}$.

As shown in Sect. 8.18, $\sqrt[3]{-1+3j} = \begin{cases} 1.1853 + 0.8657j \\ -1.3424 + 0.5937j \\ 0.1571 - 1.4594j \end{cases}$. The MATLAB command shown in Fig. 8.42 calculates one of the roots only. The remaining two roots can be obtained by multiplying the initial root by $e^{j\frac{2\pi}{3}}$ and $e^{j\frac{4\pi}{3}}$ (Fig. 8.43).

Fig. 8.42 Calculating one of the three cube roots of $-1 + 3j$

```
Command Window
>> r1=(-1+3j)^(1/3)
r1 =
   1.1853 + 0.8658i
fx >>
```

```
Command Window
>> r1=(-1+3j)^(1/3)
r1 =
   1.1853 + 0.8658i
>> r2=r1*exp(j*2*pi/3)
r2 =
  -1.3424 + 0.5936i
>> r3=r1*exp(j*4*pi/3)
r3 =
   0.1571 - 1.4594i
fx >>
```

Fig. 8.43 Determining the three complex third roots of $z = -1 + 3j$

Example 8.21 Given $z = -1 + 3i$, use MATLAB to calculate $\sqrt[5]{z}$.

Figure 8.44 shows the MATLAB code for calculating the 5 roots of $\sqrt[5]{z}$.

```
Command Window
>> r1=(-1+3j)^(1/5)
r1 =
   1.1698 + 0.4652i
>> r2=r1*exp(j*2*pi/5)
r2 =
  -0.0810 + 1.2563i
>> r3=r1*exp(j*2*2*pi/5)
r3 =
  -1.2198 + 0.3112i
>> r4=r1*exp(j*3*2*pi/5)
r4 =
  -0.6730 - 1.0640i
>> r5=r1*exp(j*4*2*pi/5)
r5 =
   0.8039 - 0.9688i
fx >>
```

Fig. 8.44 Determining the five complex fifth roots of $z = 1 + 3j$

References for Further Study

1. Advanced Engineering Mathematics (10^{th} edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8^{th} edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

9 Differential Equations

9.1 Introduction

A differential equation is an equation that relates an unknown function to its derivatives. In essence, it describes the relationship between a function and its rate of change. Differential equations are widely used in various fields, including physics, engineering, biology, and economics, to model real-world phenomena. They help us understand and analyze how systems change over time.

This chapter begins with a review of ordinary differential equation solving techniques. The second part demonstrates MATLAB's application to the problems discussed.

9.2 Dot Notation

Dot notation is a common notation for derivatives, particularly in physics and engineering, where the variable of differentiation is often time. A single dot above a variable indicates its first derivative with respect to time. $\dot{y}(t) = \frac{dy(t)}{dt}$ means "the rate of change of y with respect to time". Two dots above a variable indicate its second derivative with respect to time. $\ddot{y}(t) = \frac{d^2y(t)}{dt^2}$ means "the rate of change of the rate of change of y with respect to time". For instance, the differential equation $\frac{dy(t)}{dt} + ay(t) = 0$ can be written as $\dot{y}(t) + ay(t) = 0$. For conciseness, we can represent the differential equation $\dot{y}(t) + ay(t) = 0$ as $\dot{y} + ay = 0$.

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_9

9.3 First Order Differential Equations with Constant Stimulation

$\dot{y} + ay = b$ Where $a, b \in \mathbb{R}$ can be solved as follows:

$$\lambda + a = 0 \Rightarrow \lambda = -a \Rightarrow y(t) = \frac{b}{a} + C_1 e^{-at}$$

$\lambda + a$ is the characteristic equation associated with the given differential equation.

Example 9.1 Find the general solution of $\dot{y} + 3y = 0$.

$$\lambda + 3 = 0 \Rightarrow \lambda = -3 \Rightarrow y(t) = C_1 e^{-3t}$$

Example 9.2 Find the solution of $\dot{y} - 5y = 0, y(0) = 7$.

$$\lambda - 5 = 0 \Rightarrow \lambda = 5 \Rightarrow y(t) = C_1 e^{5t}$$

$$y(0) = 7 \Rightarrow y(0) = C_1 e^{5\times 0} = C_1 = 7 \Rightarrow y(t) = 7e^{5t}$$

Example 9.3 Find the general solution of $\dot{y} + 4y = 8, y(0) = 5$.

$$\lambda + 4 = 0 \Rightarrow \lambda = -4 \Rightarrow y(t) = C_1 e^{-4t} + \frac{8}{4} = C_1 e^{-3t} + 2$$

$$y(0) = 5 \Rightarrow y(0) = C_1 e^{-3\times 0} + 2 = 5 \Rightarrow C_1 = 5 - 2 = 3 \Rightarrow y(t) = 3e^{-3t} + 2$$

9.4 First Order Differential Equations with Polynomial Stimulation

In this section we want to find the general solution of $\dot{y} + ay = p(t)$ where $p(t) = k_n t^n + k_{n-1} t^{n-1} + k_{n-2} t^{n-2} + \cdots + k_0$ and $a \in \mathbb{R}$. General solution of this equation is $y(t) = C_1 e^{-at} + A_n t^n + A_{n-1} t^{n-1} + A_{n-2} t^{n-2} + \cdots + A_0$. Value of $A_n, A_{n-1}, A_{n-2}, \ldots, A_0$ must be determined. Given initial value determines the value of C_1. Let's study a numeric example.

Example 9.4 Find the solution of $\dot{y} + 3y = t^2 + 5, y(0) = 5$.

$$\lambda + 3 = 0 \Rightarrow \lambda = -3 \Rightarrow y(t) = C_1 e^{-3t} + y_p = C_1 e^{-3t} + A_2 t^2 + A_1 t + A_0$$

$$\begin{aligned}\dot{y}_p + 3y = t^2 + 5 &\Rightarrow 2A_2t + A_1 + 3A_2t^2 + 3A_1t + 3A_0 \\ &= t^2 + 5 \Rightarrow 3A_2t^2 + (3A_1 + 2A_2)t + (3A_0 + A_1) \\ &= t^2 + 5\end{aligned}$$

$$\begin{cases} 3A_2 = 1 \\ 3A_1 + 2A_2 = 0 \\ 3A_0 + A_1 = 5 \end{cases} \Rightarrow \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 3 \\ 0 & 3 & 2 \\ 3 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \Rightarrow \begin{cases} A_0 = \frac{47}{27} \\ A_1 = \frac{-2}{9} \\ A_2 = \frac{1}{3} \end{cases}$$

$$y(t) = C_1e^{-3t} + \frac{1}{3}t^2 + \frac{-2}{9}t + \frac{47}{27}$$

$$y(0) = 5 \Rightarrow C_1 + \frac{47}{27} = 5 \Rightarrow C_1 = 5 - \frac{47}{27} = \frac{88}{27}$$

$$y(t) = \frac{88}{27}e^{-3t} + \frac{1}{3}t^2 + \frac{-2}{9}t + \frac{47}{27}$$

9.5 First Order Differential Equations with Exponential Stimulation

In this section we want to solve the $\dot{y} + ay = ke^{\alpha t}$ where $a, \alpha \in \mathbb{R}$. We study two cases: When $\alpha \neq -a$ and when $\alpha = -a$.

Case I: When $\boldsymbol{\alpha \neq -a}$

$$\dot{y} + ay = ke^{-at}$$

$$\lambda + a = 0 \Rightarrow \lambda = -a$$

$$y(t) = C_1e^{-at} + Ae^{\alpha t} = C_1e^{-at} + y_p(t)$$

$$y_p(t) = Ae^{\alpha t} \Rightarrow \dot{y}_p(t) = A\alpha e^{\alpha t}$$

$$\begin{aligned}\dot{y}_p + ay_p = ke^{\alpha t} &\Rightarrow A\alpha e^{\alpha t} + aAe^{\alpha t} = ke^{\alpha t} \Rightarrow A\alpha + aA \\ &= k \Rightarrow A = \frac{k}{a + \alpha} \Rightarrow y_p(t) = \frac{k}{a + \alpha}e^{\alpha t}\end{aligned}$$

$$y(t) = C_1 e^{-at} + \frac{k}{a+\alpha} e^{\alpha t}$$

Let's study a numeric example.

Example 9.5 Find the solution of $\dot{y} + 3y = 5e^{2t}$, $y(0) = 6$.

$$\lambda + 3 = 0 \Rightarrow \lambda = -3$$

$$y(t) = C_1 e^{-3t} + Ae^{2t} = C_1 e^{-3t} + y_p(t)$$

$$\dot{y}_p + 3y_p = 5e^{2t} \Rightarrow 2Ae^{2t} + 3Ae^{2t} = 5e^{2t} \Rightarrow 5A = 5 \Rightarrow A = 1 \Rightarrow y_p(t) = e^{2t}$$

$$y(t) = C_1 e^{-3t} + e^{2t}$$

$$y(0) = 6 \Rightarrow y(0) = C_1 + 1 = 6 \Rightarrow C_1 = 5$$

$$y(t) = 5e^{-3t} + e^{2t}$$

Case II: When $\alpha = -a$

$$\dot{y} + ay = ke^{-at}$$

$$\lambda + a = 0 \Rightarrow \lambda = -a$$

$$y(t) = C_1 e^{-at} + Ate^{-at} = C_1 e^{-at} + y_p(t)$$

$$y_p(t) = Ate^{-at} \Rightarrow \dot{y}_p(t) = Ae^{-at} - Aate^{-at}$$

$$\dot{y}_p + ay_p = ke^{-at} \Rightarrow Ae^{-at} - Aate^{-at} + aAte^{-at} = ke^{-at} \Rightarrow A = k \Rightarrow y_p(t) = kte^{-at}$$

$$y(t) = C_1 e^{-at} + kte^{-at}$$

Let's study a numeric example for this case.

Example 9.6 Find the solution of $\dot{y} + 3y = 5e^{-3t}$, $y(0) = 9$.

$$\lambda + 3 = 0 \Rightarrow \lambda = -3$$

$$y(t) = C_1 e^{-3t} + Ate^{-3t} = C_1 e^{-3t} + y_p(t)$$

$$y_p(t) = Ate^{-3t} \Rightarrow \dot{y}_p(t) = Ae^{-3t} - 3Ate^{-3t}$$

$$\dot{y}_p + 3y_p = 5e^{-3t} \Rightarrow Ae^{-3t} - 3Ate^{-3t} + 3Ate^{-3t} = 5e^{-3t} \Rightarrow A = 5 \Rightarrow y_p(t) = 5te^{-3t}$$

$$y(t) = C_1 e^{-3t} + 5te^{-3t}$$

$$y(0) = 9 \Rightarrow C_1 = 9 \Rightarrow y(t) = 9e^{-3t} + 5te^{-3t} = (9 + 5t)e^{-3t}$$

9.6 First Order Differential Equations with Sinusoidal Stimulation

In this section we want to solve the $\dot{y} + ay = k\sin(\omega_0 t + \varphi_0).a, k, \omega_0, \varphi_0 \in \mathbb{R}$.

$$\dot{y} + ay = k\sin(\omega_0 t + \varphi_0)$$

$$\lambda + a = 0 \Rightarrow \lambda = -a$$

$$y(t) = C_1 e^{-at} + \left|\frac{1}{j\omega_0 + a}\right| \times k\sin\left(\omega_0 t + \varphi_0 + Arg\left(\frac{1}{j\omega_0 + a}\right)\right)$$

$$y(t) = C_1 e^{-at} + \frac{1}{\sqrt{\omega_0^2 + a^2}} \times k\sin\left(\omega_0 t + \varphi_0 + Arg\left(\frac{1}{j\omega_0 + a}\right)\right)$$

When stimulation is a cosine function, i.e., $\dot{y} + ay = k\cos(\omega_0 t + \varphi_0)$, $a, k, \omega_0, \varphi_0 \in \mathbb{R}$:

$$\dot{y} + ay = k\cos(\omega_0 t + \varphi_0)$$

$$\lambda + a = 0 \Rightarrow \lambda = -a$$

$$y(t) = C_1 e^{-at} + \left| \frac{1}{j\omega_0 + a} \right| \times k \cos\left(\omega_0 t + \varphi_0 + Arg\left(\frac{1}{j\omega_0 + a} \right) \right)$$

$$y(t) = C_1 e^{-at} + \frac{1}{\sqrt{\omega_0^2 + a^2}} \times k \cos\left(\omega_0 t + \varphi_0 + Arg\left(\frac{1}{j\omega_0 + a} \right) \right)$$

When stimulation has two terms, i.e., $\dot{y} + ay = k_0 \cos(\omega_0 t + \varphi_0) + k_1 \sin(\omega_1 t + \varphi_1), a, k_0, k_1, \omega_0, \omega_1, \varphi_0, \varphi_1 \in \mathbb{R}$:

$$\dot{y} + ay = k_0 \cos(\omega_0 t + \varphi_0) + k_1 \sin(\omega_1 t + \varphi_1)$$

$$y(t) = C_1 e^{-at} + \frac{k_0}{\sqrt{\omega_0^2 + a^2}} \cos\left(\omega_0 t + \varphi_0 + Arg\left(\frac{1}{j\omega_0 + a} \right) \right) + \frac{k_1}{\sqrt{\omega_0^2 + a^2}} \sin\left(\omega_1 t + \varphi_1 + Arg\left(\frac{1}{j\omega_1 + a} \right) \right)$$

Let's study some numeric examples.

Example 9.7 Find the general solution of $\dot{y} + 6y = 20 \sin\left(7t + \frac{\pi}{4}\right)$.

$$\lambda + 6 = 0 \Rightarrow \lambda = -6$$

$$y(t) = C_1 e^{-6t} + \frac{1}{\sqrt{7^2 + 6^2}} 20 \sin\left(7t + \frac{\pi}{4} + Arg\left(\frac{1}{j7 + 6} \right) \right)$$

$$Arg\left(\frac{1}{j7 + 6} \right) = Arg(1) - Arg(j7 + 6) = 0 - \tan^{-1}\left(\frac{7}{6} \right) = -0.8622 Rad$$

$$y(t) = C_1 e^{-6t} + 2.1693 \sin\left(7t + \frac{\pi}{4} - 0.8622 \right) = C_1 e^{-6t} + 2.1693 \sin(7t - 0.0768)$$

Example 9.8 Find the general solution of $\dot{y} + 6y = 30 \sin\left(4t + \frac{\pi}{3}\right)$.

$$\lambda + 6 = 0 \Rightarrow \lambda = -6$$

$$y(t) = C_1 e^{-6t} + \frac{1}{\sqrt{4^2 + 6^2}} 30 \sin\left(4t + \frac{\pi}{3} + Arg\left(\frac{1}{j4 + 6} \right) \right)$$

$$Arg\left(\frac{1}{j4 + 6} \right) = Arg(1) - Arg(j4 + 6) = 0 - \tan^{-1}\left(\frac{4}{6} \right) = -0.5880 Rad$$

$$
\begin{aligned}
y(t) &= C_1 e^{-6t} + 4.1603 \sin\left(4t + \frac{\pi}{3} - 0.5880\right) \\
&= C_1 e^{-6t} + 4.1603 \sin(4t + 0.4592)
\end{aligned}
$$

Example 9.9 Find the general solution of $\dot{y} + 6y = 20\sin\left(7t + \frac{\pi}{4}\right) + 30\sin\left(4t + \frac{\pi}{3}\right)$.

$$\lambda + 6 = 0 \Rightarrow \lambda = -6$$

$$y(t) = C_1 e^{-6t} + y_p(t)$$

$$y_p(t) = y_{p1}(t) + y_{p2}(t)$$

$y_{p1}(t)$ and $y_{p2}(t)$ are particular solutions of $\dot{y} + 6y = 20\sin\left(7t + \frac{\pi}{4}\right)$ and $\dot{y} + 6y = 30\sin\left(4t + \frac{\pi}{3}\right)$, respectively. According to previous examples $y_{p1}(t) = 2.1693\sin(7t - 0.0768)$ and $y_{p2}(t) = 4.1603\sin(4t + 0.4592)$. Therefore,

$$y(t) = C_1 e^{-6t} + 2.1693\sin(7t - 0.0768) + 4.1603\sin(4t + 0.4592)$$

Example 9.10 Find the general solution of $\dot{y} + 4y = 12\cos\left(3t + \frac{\pi}{3}\right)$.

$$\lambda + 4 = 0 \Rightarrow \lambda = -4$$

$$y(t) = C_1 e^{-4t} + \frac{1}{\sqrt{3^2 + 4^2}} \times 12\cos\left(3t + \frac{\pi}{3} + Arg\left(\frac{1}{j3 + 4}\right)\right)$$

$$Arg\left(\frac{1}{j3 + 4}\right) = Arg(1) - Arg(j3 + 4) = 0 - \tan^{-1}\left(\frac{3}{4}\right) = -0.6435 Rad$$

$$y(t) = C_1 e^{-4t} + 2.4\cos\left(3t + \frac{\pi}{3} - 0.6435\right) = C_1 e^{-4t} + 2.4\cos(3t + 0.4037)$$

Example 9.11 Find the general solution of $\dot{y} + 4y = 24\cos\left(3t + \frac{\pi}{3}\right)$.

$$\lambda + 4 = 0 \Rightarrow \lambda = -4$$

$$y(t) = C_1 e^{-4t} + y_p(t)$$

Particular solution of this example is two times bigger than previous example: $y_p(t) = 2 \times 2.4\cos(3t + 0.4037) = 4.8\cos(3t + 0.4037)$. Therefore,

$$y(t) = C_1 e^{-4t} + 4.8\cos(3t + 0.4037)$$

9.7 Second Order Differential Equations with Constant Stimulation

$\ddot{y} + a\dot{y} + by = c$. Where $a \neq 0, b \neq 0, c \in \mathbb{R}$ can be solved as follows:

$$\lambda^2 + a\lambda + b = 0 \Rightarrow \lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Three different cases must be studied.

Case 1: $\boldsymbol{a}^2 - 4\boldsymbol{b} > 0$

In this case characteristic equation has two separate real roots λ_1 and λ_2. General solution of given differential equation is:

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \frac{c}{b}$$

Case 2: $\boldsymbol{a}^2 - 4\boldsymbol{b} = 0$

In this case characteristic equation has repeated real roots $\lambda_1 = \lambda_2 = -\frac{a}{2}$. General solution of given differential equation is:

$$y(t) = C_1 e^{-\frac{a}{2}t} + C_2 t e^{-\frac{a}{2}t} + \frac{c}{b}$$

Case 3: $\boldsymbol{a}^2 - 4\boldsymbol{b} < 0$

In this case characteristic equation has two complex roots $\lambda_1 = \alpha + \beta j$ and $\lambda_2 = \alpha - \beta j$. General solution of given differential equation is:

$$y(t) = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t)) + \frac{c}{b}$$

Let's study some numeric examples.

Example 9.12 Find the solution of $\ddot{y} - 3\dot{y} + 2y = 0, y(0) = 3, \dot{y}(0) = 5$.

$$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda_{1=}1, \lambda_{2=}2 \Rightarrow y(t) = C_1 e^t + C_2 e^{2t}$$

$$y(0) = 3 \Rightarrow C_1 + C_2 = 3$$

$$\dot{y} = C_1 e^t + 2C_2 e^{2t} \Rightarrow \dot{y}(0) = C_1 + 2C_2 = 5$$

$$\begin{cases} C_1 + C_2 = 3 \\ C_1 + 2C_2 = 5 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{1 \times 2 - 1 \times 1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \times 3 - 1 \times 5 \\ -1 \times 3 + 1 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow C_1 = 1, C_2 = 2 \Rightarrow y(t) = e^t + 2e^{2t}$$

Example 9.13 Find the general solution of $\ddot{y} - 6\dot{y} + 9y = 0$.

$$\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 3$$

$$y(t) = C_1 e^{3t} + C_2 t e^{3t}$$

Example 9.14 Find the general solution of $\ddot{y} + 2\dot{y} + 5y = 0$.

$$\lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda_1 = -1 + 2j, \lambda_2 = -1 - 2j$$

$$y(t) = e^{-t}(\mathrm{C}_1 \cos(2t) + \mathrm{C}_2 \sin(2t))$$

Special case I

When $a = 0$ differential equation $\ddot{y} + a\dot{y} + by = c$ simplifies to $\ddot{y} + by = c$. Differential equation $\ddot{y} + by = c$ with $b > 0$ can be solved as:

$$\lambda^2 + b = 0 \Rightarrow \lambda_1 = +j\sqrt{b}, \lambda_2 = -j\sqrt{b}$$

$$y(t) = C_1 \sin\left(\sqrt{b}t\right) + C_2 \cos\left(\sqrt{b}t\right) + \frac{c}{b}$$

Solution of $\ddot{y} + by = c$ with $b < 0$ is:

$$\lambda^2 + b = 0 \Rightarrow \lambda_1 = +\sqrt{-b}, \lambda_2 = -\sqrt{-b}$$

$$y(t) = C_1 e^{\sqrt{-b}t} + C_2 e^{-\sqrt{-b}t} + \frac{c}{b}$$

Let's study some numeric examples.

Example 9.15 Find the general solution of $\ddot{y} + \frac{4}{25}y = 0$.

$$\lambda^2 + \frac{4}{25} = 0 \Rightarrow \lambda_1 = +\frac{2}{5}j, \lambda_2 = -\frac{2}{5}j$$

$$y(t) = C_1\cos\left(\frac{2}{5}t\right) + C_2\sin\left(\frac{2}{5}t\right)$$

Example 9.16 Find the general solution of $\ddot{y} + \frac{4}{25}y = 8$.

$$\lambda^2 + \frac{4}{25} = 0 \Rightarrow \lambda_1 = +\frac{2}{5}j, \lambda_2 = -\frac{2}{5}j$$

$$y(t) = C_1\cos\left(\frac{2}{5}t\right) + C_2\sin\left(\frac{2}{5}t\right) + \frac{8}{\frac{4}{25}} = C_1\cos\left(\frac{2}{5}t\right) + C_2\sin\left(\frac{2}{5}t\right) + 50$$

Example 9.17 Find the general solution of $\ddot{y} - \frac{4}{25}y = 0$.

$$\lambda^2 - \frac{4}{25} = 0 \Rightarrow \lambda_1 = +\frac{2}{5}, \lambda_2 = -\frac{2}{5}$$

$$y(t) = C_1e^{\frac{2}{5}t} + C_2e^{-\frac{2}{5}t}$$

Example 9.18 Find the general solution of $\ddot{y} - \frac{4}{25}y = 8$.

$$\lambda^2 - \frac{4}{25} = 0 \Rightarrow \lambda_1 = +\frac{2}{5}, \lambda_2 = -\frac{2}{5}$$

$$y(t) = C_1e^{\frac{2}{5}t} + C_2e^{-\frac{2}{5}t} + \frac{8}{-\frac{4}{25}} = C_1e^{\frac{2}{5}t} + C_2e^{-\frac{2}{5}t} - 50$$

Special case II

When $b = 0$ differential equation $\ddot{y} + a\dot{y} + by = c$ simplifies to $\ddot{y} + a\dot{y} = c$. Differential equation $\ddot{y} + a\dot{y} = c$ can be solved as:

$$\lambda^2 + a\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -a$$

$$y(t) = C_1 + C_2e^{-at} + \frac{c}{a}t$$

Let's study some numeric examples.

Example 9.19 Find the general solution of $\ddot{y} + \frac{4}{25}\dot{y} = 0$.

$$\lambda^2 + \frac{4}{25}\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -\frac{4}{25}$$

$$y(t) = C_1 + C_2e^{-\frac{4}{25}t}$$

Example 9.20 Find the solution of $\ddot{y} + \frac{4}{25}\dot{y} = 8, y(0) = 1, \dot{y}(0) = 2$.

$$\lambda^2 + \frac{4}{25}\lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -\frac{4}{25}$$

$$y(t) = \mathrm{C}_1 + \mathrm{C}_2 e^{-\frac{4}{25}t} + \frac{8}{\frac{4}{25}}t = \mathrm{C}_1 + \mathrm{C}_2 e^{-\frac{4}{25}t} + 50t$$

It is time to apply the given initial condition:

$$y(0) = 1 \Rightarrow C_1 + C_2 = 1$$

$$\begin{aligned} y(t) = \mathrm{C}_1 + \mathrm{C}_2 e^{-\frac{4}{25}t} + 50t \Rightarrow \dot{y}(t) &= -\frac{4}{25}\mathrm{C}_2 e^{-\frac{4}{25}t} + 50 \\ \Rightarrow \dot{y}(0) = -\frac{4}{25}\mathrm{C}_2 + 50 = 2 \Rightarrow &-\frac{4}{25}\mathrm{C}_2 = 2 - 50 \\ \Rightarrow C_2 = 300 \end{aligned}$$

$$C_1 + C_2 = 1 \Rightarrow C_1 + 300 = 1 \Rightarrow C_1 = -299$$

$$y(t) = -299 + 300e^{-\frac{4}{25}t} + 50t$$

9.8 Second Order Differential Equations with Polynomial Stimulation

General solution of differential equation $\ddot{y} + a\dot{y} + by = p(t)$ with $b \neq 0, a \in \mathbb{R}$ and $p(t) = k_n t^n + k_{n-1}t^{n-1} + k_{n-2}t^{n-2} + \cdots + k_0$ is:

$$\begin{aligned} y(t) = y_h(t) + y_p(t) = y_h(t) + A_n t^n + A_{n-1}t^{n-1} + A_{n-2}t^{n-2} \\ + \cdots + A_0 \end{aligned}$$

Value of $A_n, A_{n-1}, A_{n-2}, \ldots, A_0$ must be determined. $y_h(t)$ shows the homogenous solution and can take three forms based on the roots of characteristic equations:

$$y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

$$y_h(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$$

$$y_h(t) = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$$

Therefore, general solution of differential equation $\ddot{y}+a\dot{y}+by = p(t)$ with $b \neq 0$, $a \in \mathbb{R}$ and $p(t) = k_n t^n + k_{n-1}t^{n-1} + k_{n-2}t^{n-2} + \cdots + k_0$ is one of the following three forms:

$$y(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} + A_n t^n + A_{n-1}t^{n-1} + A_{n-2}t^{n-2} + \cdots + A_0$$

$$y(t) = C_1e^{\lambda_1 t} + C_2te^{\lambda_1 t} + A_n t^n + A_{n-1}t^{n-1} + A_{n-2}t^{n-2} + \cdots + A_0$$

$$\begin{aligned} y(t) = e^{\alpha t}(C_1\cos(\beta t) + C_2\sin(\beta t)) + A_n t^n + A_{n-1}t^{n-1} \\ + A_{n-2}t^{n-2} + \cdots + A_0 \end{aligned}$$

Let's study a numeric example.

Example 9.21 Find the solution of $\ddot{y} + 2\dot{y} + y = 6t + 7$, $y(0) = 5$, $\dot{y}(0) = 3$.

$$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda + 1)^2 = 0 \Rightarrow \lambda_{1,2} = -1$$

$$y(t) = C_1e^{-t} + C_2te^{-t} + A_1t + A_0 = C_1e^{-t} + C_2te^{-t} + y_p(t)$$

$$y_p(t) = A_1t + A_0 \Rightarrow \dot{y}_p(t) = A_1, \ddot{y}_p(t) = 0$$

$$\ddot{y}_p(t) + 2\dot{y}_p(t) + y_p(t) = 6t + 7 \Rightarrow 0 + 2A_1 + A_1t + A_0 = 6t + 7$$

$$\begin{cases} A_0 + 2A_1 = 7 \\ A_1 = 6 \end{cases} \Rightarrow \begin{cases} A_0 = -5 \\ A_1 = 6 \end{cases} \Rightarrow y_p(t) = 6t - 5$$

$$y(t) = C_1e^{-t} + C_2te^{-t} + 6t - 5$$

$$\dot{y}(t) = -C_1e^{-t} + C_2e^{-t} - C_2te^{-t} + 6$$

$$y(0) = 5 \Rightarrow C_1 - 5 = 5 \Rightarrow C_1 = 10$$

$$\dot{y}(0) = 3 \Rightarrow -C_1 + C_2 - 0 + 6 = 3 \Rightarrow -10 + C_2 = -3 \Rightarrow C_2 = 7$$

$$y(t) = 10e^{-t} + 7te^{-t} + 6t - 5$$

Special case I

Differential equation $\ddot{y} + a\dot{y} + by = p(t)$ with $b = 0, a \in \mathbb{R}$ and $p(t) = k_n t^n + k_{n-1} t^{n-1} + k_{n-2} t^{n-2} + \cdots + k_0$ simplifies to $\ddot{y} + a\dot{y} = p(t)$. General solution of differential equation $\ddot{y} + a\dot{y} = p(t)$ with $a \in \mathbb{R}$ and $p(t) = k_n t^n + k_{n-1} t^{n-1} + k_{n-2} t^{n-2} + \cdots + k_0$ is:

$$\lambda^2 + a\lambda = 0 \Rightarrow \lambda(\lambda + a) = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = -a$$

$$\begin{aligned} y(t) = y_h(t) + y_p(t) &= C_1 + C_2 e^{-at} \\ &+ t \times \left(A_n t^n + A_{n-1} t^{n-1} + A_{n-2} t^{n-2} + \cdots + A_0\right) \end{aligned}$$

Let's study a numeric example.

Example 9.22 Find the solution of $\ddot{y} - \dot{y} = 2t - 1, y(0) = 1, \dot{y}(0) = 2$.

$$\lambda^2 - \lambda = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 1$$

$$y(t) = C_1 + C_2 e^t + t(A_1 t + A_0) = C_1 + C_2 e^t + A_1 t^2 + A_0 t = C_1 + C_2 e^t + y_p(t)$$

$$y_p(t) = A_1 t^2 + A_0 t \Rightarrow \dot{y}_p(t) = 2A_1 t + A_0 \Rightarrow \ddot{y}_p(t) = 2A_1$$

$$\begin{aligned} &2A_1 - 2A_1 t - A_0 = 2t - 1 \Rightarrow \begin{cases} 2A_1 - A_0 = -1 \\ -2A_1 = 2 \end{cases} \Rightarrow \begin{cases} A_0 = -1 \\ A_1 = -1 \end{cases} \\ &\Rightarrow y_p(t) = -t^2 - t \end{aligned}$$

$$y(t) = C_1 + C_2 e^t - t^2 - t$$

$$\dot{y}(t) = C_2 e^t - 2t - 1$$

$$\begin{cases} y(0) = 1 \Rightarrow y(0) = C_1 + C_2 = 1 \\ \dot{y}(0) = 2 \Rightarrow C_2 - 1 = 2 \end{cases} \Rightarrow \begin{cases} C_1 = -2 \\ C_2 = 3 \end{cases}$$

$$y(t) = -2 + 3e^t - t^2 - t$$

Special case II

Differential equation $\ddot{y} + a\dot{y} + by = p(t)$ with $b = 0, a = 0$ and $p(t) = k_n t^n + k_{n-1} t^{n-1} + k_{n-2} t^{n-2} + \cdots + k_0$ simplifies to $\ddot{y} = p(t)$. General solution of differential equation $\ddot{y} = p(t)$ is $y(t) = \int\int p(t)dt$.

Let's study a numeric example.

Example 9.23 Find the general solution of $\ddot{y} = 2t - 1$.

$$\ddot{y} = 2t - 1 \Rightarrow \dot{y} = \int (2t - 1)dt = t^2 - t + C_1 \Rightarrow y = \int (t^2 - t + C_1)dt$$
$$= \frac{t^3}{3} - \frac{t^2}{2} + C_1 t + C_2$$

9.9 Second Order Differential Equations with Sinusoidal Excitation

$\ddot{y} + a\dot{y} + by = k_1 \sin(\omega t) + k_2 \cos(\omega t)$ can be solved as follows (either of k_1 or k_2 can be zero):

Case 1: $a \neq 0$

In this case the particular solution has the following form:

$$y_p(t) = A \sin(\omega t) + B \cos(\omega t)$$

The values of coefficients A and B must be determined.

Case 2: $a = 0$ and $b \neq \sqrt{\omega}$

This case is similar to the first case. In this case the particular solution has the following form:

$$y_p(t) = A \sin(\omega t) + B \cos(\omega t)$$

The values of coefficients A and B must be determined.

Case 3: $a = 0$ and $b = \sqrt{\omega}$

In this case the particular solution has the following form:

$$y_p(t) = t \times (A \sin(\omega t) + B \cos(\omega t))$$

The values of coefficients A and B must be determined.
Let's study some numeric examples.

Example 9.24 Find the general solution of $\ddot{y} + 4\dot{y} + 3y = 2\cos(2t)$.

$$\lambda^2 + 4\lambda + 3 = 0 \Rightarrow \lambda_{1,2} = \frac{-2 \pm \sqrt{2^2 - 1 \times 3}}{1} = -1, -3$$

$$y_h(t) = C_1 e^{-t} + C_2 e^{-3t}$$

$$\ddot{y}_p + 4\dot{y}_p + 3y_p = 2\cos(2t)$$

$$y_p(t) = A\sin(2t) + B\cos(2t)$$

$$\dot{y}_p(t) = 2A\cos(2t) - 2B\sin(2t)$$

$$\ddot{y}_p(t) = -4A\sin(2t) - 4B\cos(2t)$$

Substitution in the given differential equation:

$$\begin{aligned} &-4A\sin(2t) - 4B\cos(2t) + 8A\cos(2t) - 8B\sin(2t) \\ &+ 3A\sin(2t) + 3B\cos(2t) = 2\cos(2t) \end{aligned}$$

$$(-A - 8B)\sin(2t) + (-B + 8A)\cos(2t) = 2\cos(2t)$$

$$\Rightarrow \begin{cases} -A - 8B = 0 \\ 8A - B = 2 \end{cases} \Rightarrow \begin{bmatrix} -1 & -8 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 8 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{1}{1-(-64)} \begin{bmatrix} -1 & 8 \\ -8 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{65} \begin{bmatrix} 16 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{16}{65} \\ \frac{-2}{65} \end{bmatrix} \Rightarrow A = \frac{16}{65},\ B = \frac{-2}{65}$$

$$y_p(t) = \frac{16}{65}\sin(2t) + \frac{-2}{65}\cos(2t)$$

$$\begin{aligned} y(t) = y_h(t) + y_p(t) = C_1 e^{-t} + C_2 e^{-3t} + \frac{16}{65}\sin(2t) \\ + \frac{-2}{65}\cos(2t) \end{aligned}$$

Example 9.25 Find the general solution of $\ddot{y} + 9y = \sin(4t) - \cos(4t)$.

$$\lambda^2 + 9 = 0 \Rightarrow \lambda^2 = -9 \Rightarrow \lambda = \pm 3j$$

$$y_h(t) = C_1\cos(3t) + C_2\sin(3t)$$

$$\ddot{y}_p + 9y_p = \sin(4t) - \cos(4t)$$

$$y_p(t) = A\sin(4t) + B\cos(4t)$$

$$\dot{y}_p(t) = 4A\cos(4t) - 4B\sin(4t)$$

$$\ddot{y}_p(t) = -16A\sin(4t) - 16B\cos(4t)$$

Substitution in the given differential equation:

$$\begin{aligned} &-16A\sin(4t) - 16B\cos(4t) + 9A\sin(4t) + 9B\cos(4t) \\ &= \sin(4t) - \cos(4t) \Rightarrow \begin{cases} -16A + 9A = 1 \\ -16B + 9B = -1 \end{cases} \\ &A = \frac{-1}{7}, B = \frac{1}{7} \Rightarrow y_p(t) = \frac{1}{7}(\cos(4t) - \sin(4t)) \end{aligned}$$

$$y(t) = C_1\cos(3t) + C_2\sin(3t) + \frac{1}{7}(\cos(4t) - \sin(4t))$$

Example 9.26 Find the general solution of $\ddot{y} + 9y = \sin(3t) - \cos(3t)$.

$$\lambda^2 + 9 = 0 \Rightarrow \lambda^2 = -9 \Rightarrow \lambda = \pm 3j$$

$$y_h(t) = C_1\cos(3t) + C_2\sin(3t)$$

Given differential equation falls in third case, i.e., $a = 0$ and $b = \sqrt{\omega}$.

$$\ddot{y}_p + 9y_p = \sin(3t) - \cos(3t)$$

$$y_p(t) = t(A\sin(3t) + B\cos(3t))$$

$$\dot{y}_p(t) = A\sin(3t) + B\cos(3t) + t(3A\cos(3t) - 3B\sin(3t))$$

$$\begin{aligned} \ddot{y}_p(t) = &\ 3A\cos(3t) - 3B\sin(3t) + 3A\cos(3t) - 3B\sin(3t) \\ &+ t(-9B\cos(3t) - 9A\sin(3t)) \end{aligned}$$

$$\dot{y}_p(t) = (A - 3Bt)\sin(3t) + (B + 3At)\cos(3t)$$

$$\ddot{y}_p(t) = (6A - 9Bt)\cos(3t) - (9At)\sin(3t)$$

Substitution in the given differential equation:

$$(6A - 9Bt)\cos(3t) - (6B + 9At)\sin(3t) + 9At\sin(3t) + 9Bt\cos(3t) = \sin(3t) - \cos(3t)$$

$$(6A)\cos(3t) - (6B)\sin(3t) = \sin(3t) - \cos(3t)$$

$$\Rightarrow \begin{cases} A = \frac{-1}{6} \\ B = \frac{-1}{6} \end{cases} \Rightarrow y_p(t) = -\frac{t}{6}(\sin(3t) + \cos(3t))$$

$$y(t) = y_h(t) + y_p(t) \Rightarrow y(t) = C_1\cos(3t) + C_2\sin(3t) - \frac{t}{6}(\sin(3t) + \cos(3t))$$

or

$$y(t) = \left(C_1 - \frac{t}{6}\right)\cos(3t) + \left(C_2 - \frac{t}{6}\right)\sin(3t)$$

9.10 Second Order Differential Equations with Exponential Input

General solution of $\ddot{y} + a\dot{y} + by = k_1 e^{\alpha t}$ where $a, b, k_1, \alpha \in \mathbb{R}$ can be found using the following technique.

$$\lambda^2 + a\lambda + b = 0 \Rightarrow \lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$\Rightarrow \begin{cases} \lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \\ \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2} \end{cases}$$

Case 1: λ_1 and λ_2 are real numbers and $\alpha \neq \lambda_1$ and $\alpha \neq \lambda_2$

In this case $y_p(t) = Ae^{\alpha t}$. A is unknown and its value must be found. Therefore,

$$\dot{y}_p(t) = A\alpha e^{\alpha t}$$

$$\ddot{y}_p(t) = A\alpha^2 e^{\alpha t}$$

$$\ddot{y}_p + a\dot{y}_p + by_p = k_1 e^{\alpha t} \Rightarrow A\alpha^2 e^{\alpha t} + aA\alpha e^{\alpha t} + bAe^{\alpha t} = k_1 e^{\alpha t}$$

$$\left(\alpha^2 + \alpha a + b\right)A = k_1 \Rightarrow A = \frac{k_1}{\alpha^2 + \alpha a + b} \Rightarrow y_p(t)$$

$$= \frac{k_1}{\alpha^2 + \alpha a + b} e^{\alpha t}$$

Case 2: $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ are real numbers and $\boldsymbol{\alpha = \lambda_1}$ but $\boldsymbol{\alpha \neq \lambda_2}$

In this case $y_p(t) = Ate^{\alpha t}$. A is unknown and its value must be found. Therefore,

$$\dot{y}_p(t) = Ae^{\alpha t} + A\alpha te^{\alpha t}$$

$$\ddot{y}_p(t) = A\alpha e^{\alpha t} + A\alpha e^{\alpha t} + A\alpha^2 te^{\alpha t}$$

$$\ddot{y}_p + a\dot{y}_p + by_p = k_1 e^{\alpha t} \Rightarrow A\alpha e^{\alpha t} + A\alpha e^{\alpha t} + A\alpha^2 te^{\alpha t} + Aae^{\alpha t} + A\alpha ate^{\alpha t} + Abte^{\alpha t} = k_1 e^{\alpha t}$$

$$A(2\alpha + a)e^{\alpha t} + A(\alpha^2 + \alpha a + b)te^{\alpha t} = k_1 e^{\alpha t}$$

Note that $\lambda_1^2 + a\lambda_1 + b = 0$. $\alpha = \lambda_1$ therefore, $\alpha^2 + a\alpha + b = 0$, too. So,

$$\alpha^2 + \alpha a + b = 0 \Rightarrow A(2\alpha + a)e^{\alpha t} = k_1 e^{\alpha t} \Rightarrow A(2\alpha + a) = k_1 \Rightarrow A = \frac{k_1}{2\alpha + a}$$

$$y_p(t) = \frac{k_1}{2\alpha + a} te^{\alpha t}$$

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \frac{k_1}{2\alpha + a} te^{\alpha t}$$

Case 3: $\boldsymbol{a = -2\alpha}$ and $\boldsymbol{b = \alpha^2}$

In this case $\alpha = \lambda_1 = \lambda_2$ and particular solution is $y_p(t) = At^2 e^{\alpha t}$. Therefore,

$$\dot{y}_p(t) = 2Ate^{\alpha t} + A\alpha t^2 e^{\alpha t}$$

$$\ddot{y}_p(t) = 2Ae^{\alpha t} + 2A\alpha te^{\alpha t} + 2A\alpha te^{\alpha t} + A\alpha^2 t^2 e^{\alpha t}$$

$$\ddot{y}_p + a\dot{y}_p + by_p = k_1 e^{\alpha t} \Rightarrow 2Ae^{\alpha t} + 2A\alpha te^{\alpha t} + 2A\alpha te^{\alpha t} + A\alpha^2 t^2 e^{\alpha t} + 2Aate^{\alpha t} + A\alpha at^2 e^{\alpha t} + bAt^2 e^{\alpha t} = k_1 e^{\alpha t}$$

$$2Ae^{\alpha t} + 2A(2\alpha + a)te^{\alpha t} + A\left(\alpha^2 + \alpha a + b\right)t^2 e^{\alpha t} = k_1 e^{\alpha t}$$

$$\begin{cases} 2\alpha + a = 2\alpha - 2\alpha = 0 \\ \alpha^2 + \alpha a + b = \alpha^2 - 2\alpha^2 + \alpha^2 = 0 \end{cases}$$

$$2Ae^{\alpha t} + 2A(2\alpha + a)te^{\alpha t} + A\left(\alpha^2 + \alpha a + b\right)t^2 e^{\alpha t}$$
$$= k_1 e^{\alpha t} \Rightarrow 2Ae^{\alpha t} = k_1 e^{\alpha t} \Rightarrow A = \frac{k_1}{2}$$

$$y_p(t) = \frac{k_1}{2} t^2 e^{\alpha t}$$

$$y(t) = C_1 e^{\alpha t} + C_2 t e^{\alpha t} + \frac{k_1}{2} t^2 e^{\alpha t}$$

Let's study some numeric examples.

Example 9.27 Find the general solution of $\ddot{y} + 5\dot{y} + 6y = 3e^{-4t}$.

$$\lambda^2 + 5\lambda + 6 = 0 \Rightarrow (\lambda + 2)(\lambda + 3) = 0 \Rightarrow \lambda_1 = -2,$$
$$\lambda_2 = -3 \Rightarrow y(t) = C_1 e^{-2t} + C_2 e^{-3t} + y_p(t)$$

$$\ddot{y}_p + 5\dot{y}_p + 6y_p = 3e^{-4t}$$

In this case $\alpha \neq \lambda_1$ and $\alpha \neq \lambda_2$ therefore, $y_p(t) = Ae^{-4t}$. Therefore,

$$\dot{y}_p(t) = -4Ae^{-4t}$$

$$\ddot{y}_p(t) = 16Ae^{-4t}$$

$$\ddot{y}_p + 5\dot{y}_p + 6y_p = 3e^{-4t} \Rightarrow 16Ae^{-4t} - 20Ae^{-4t} + 6Ae^{-4t}$$
$$= 3e^{-4t} \Rightarrow 2A = 3 \Rightarrow A = \frac{3}{2}$$

$$y_p(t) = \frac{3}{2} e^{-4t}$$

$$y(t) = C_1 e^{-2t} + C_2 e^{-3t} + \frac{3}{2} e^{-4t}$$

Example 9.28 Find the general solution of $\ddot{y} + 5\dot{y} + 6y = 7e^{-2t}$.

$$\lambda^2 + 5\lambda + 6 = 0 \Rightarrow (\lambda + 2)(\lambda + 3) = 0 \Rightarrow \lambda_1 = -2,$$
$$\lambda_2 = -3 \Rightarrow y(t) = C_1 e^{-2t} + C_2 e^{-3t} + y_p(t)$$

$$\ddot{y}_p + 5\dot{y}_p + 6y_p = 7e^{-2t}$$

In this case $\alpha = \lambda_1$ but $\alpha \neq \lambda_2$ therefore, $y_p(t) = Ate^{-2t}$.

$$\dot{y}_p(t) = Ae^{-2t} - 2Ate^{-2t}$$

$$\ddot{y}_p(t) = -2Ae^{-2t} - 2Ae^{-2t} + 4Ate^{-2t} = -4Ae^{-2t} + 4Ate^{-2t}$$

$$\begin{aligned} &\ddot{y}_p + 5\dot{y}_p + 6y_p = 7e^{-2t} \Rightarrow -4Ae^{-2t} + 4Ate^{-2t} + 5Ae^{-2t} \\ &- 10Ate^{-2t} + 6Ate^{-2t} = 7e^{-2t} \end{aligned}$$

$$A(-4+5)e^{-2t} + A(4-10+6)te^{-2t} = 7e^{-2t} \Rightarrow A = \frac{7}{-4+5} = 7 \Rightarrow y_p(t) = 7te^{-2t}$$

$$y(t) = C_1e^{-2t} + C_2e^{-3t} + 7te^{-2t}$$

Example 9.29 Find the general solution of $\ddot{y} + 5\dot{y} + 6y = 7e^{-2t} + 3e^{-4t}$.

$$\begin{aligned} &\lambda^2 + 5\lambda + 6 = 0 \Rightarrow (\lambda+2)(\lambda+3) = 0 \Rightarrow \lambda_1 = -2, \\ &\lambda_2 = -3 \Rightarrow y_h(t) = C_1e^{-2t} + C_2e^{-3t} \end{aligned}$$

$$y(t) = y_h(t) + y_{p1}(t) + y_{p2}(t) = C_1e^{-2t} + C_2e^{-3t} + y_{p1}(t) + y_{p2}(t)$$

$$\ddot{y}_{p1} + 5\dot{y}_{p1} + 6y_{p1} = 7e^{-2t} \Rightarrow y_{p1} = 7te^{-2t}$$

$$\ddot{y}_{p2} + 5\dot{y}_{p2} + 6y_{p2} = 3e^{-4t} \Rightarrow y_{p2} = \frac{3}{2}e^{-4t}$$

$$y(t) = C_1e^{-2t} + C_2e^{-3t} + 7te^{-2t} + \frac{3}{2}e^{-4t}$$

Example 9.30 Find the general solution of $\ddot{y} + \dot{y} + \frac{1}{4}y = \frac{1}{4}e^{-\frac{1}{2}t}$.

$$\begin{aligned} &\lambda^2 + \lambda + \frac{1}{4} = 0 \Rightarrow \lambda_1 = -\frac{1}{2}, \lambda_2 = -\frac{1}{2} \Rightarrow y(t) \\ &= C_1e^{-\frac{1}{2}t} + C_2te^{-\frac{1}{2}t} + y_p(t) \end{aligned}$$

$$\ddot{y}_p + \dot{y}_p + \frac{1}{4}y_p = \frac{1}{4}e^{-\frac{1}{2}t}$$

In this case $\alpha = \lambda_1 = \lambda_2$ therefore, $y_p(t) = At^2e^{-\frac{1}{2}t}$.

$$\dot{y}_p(t) = 2Ate^{-\frac{1}{2}t} - \frac{A}{2}t^2e^{-\frac{1}{2}t}$$

$$\begin{aligned}\ddot{y}_p(t) &= 2Ae^{-\frac{1}{2}t} - Ate^{-\frac{1}{2}t} - Ate^{-\frac{1}{2}t} + \frac{A}{4}t^2e^{-\frac{1}{2}t} \\ &= 2Ae^{-\frac{1}{2}t} - 2Ate^{-\frac{1}{2}t} + \frac{A}{4}t^2e^{-\frac{1}{2}t}\end{aligned}$$

$$\begin{aligned}&\ddot{y}_p + \dot{y}_p + \frac{1}{4}y_p = \frac{1}{4}e^{-\frac{1}{2}t} \Rightarrow 2Ae^{-\frac{1}{2}t} - 2Ate^{-\frac{1}{2}t} + \frac{A}{4}t^2e^{-\frac{1}{2}t} \\ &+ 2Ate^{-\frac{1}{2}t} - \frac{A}{2}t^2e^{-\frac{1}{2}t} + \frac{A}{4}t^2e^{-\frac{1}{2}t} = \frac{1}{4}e^{-\frac{1}{2}t}\end{aligned}$$

$$\begin{aligned}&2Ae^{-\frac{1}{2}t} + 2A(-1+1)te^{-\frac{1}{2}t} + A\left(\frac{1}{4} - \frac{1}{2} + \frac{1}{4}\right)t^2e^{-\frac{1}{2}t} \\ &= \frac{1}{4}e^{-\frac{1}{2}t} \Rightarrow 2A = \frac{1}{4} \Rightarrow A = \frac{1}{8}\end{aligned}$$

$$y_p(t) = \frac{1}{8}t^2e^{-\frac{1}{2}t}$$

$$y(t) = C_1e^{-\frac{1}{2}t} + C_2te^{-\frac{1}{2}t} + \frac{1}{8}t^2e^{-\frac{1}{2}t}$$

9.11 Transfer Functions

In the Laplace domain, the transfer function, $H(s)$, is defined as the ratio of the Laplace transform of the output signal, $Y(s)$, to the Laplace transform of the input signal, $U(s)$, assuming zero initial conditions: $H(s) = \frac{Y(s)}{U(s)}$.

For instance, for $\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = u(t) \Rightarrow \frac{Y(s)}{U(s)} = \frac{1}{s^2+4s+3}$. Figure 9.1 shows the block diagram representation of given differential equation.

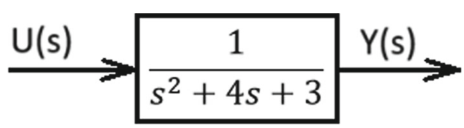


Fig. 9.1 Block diagram of given differential equation

The transfer function can help us to calculate the system response to sinusoidal inputs easily. For instance, transfer function associated with $\ddot{y}(t)+a\dot{y}(t)+by(t)=u(t)$ is $H(s)=\frac{Y(s)}{U(s)}=\frac{1}{s^2+as+b}$. Assume that input $u(t)$ is sinusoidal: $\ddot{y}(t)+a\dot{y}(t)+by(t)=Asin(\omega_0 t+\varphi_0)$. The particular solution of this equation can be written as $y_p(t)=|H(j\omega_0)|\times A\times sin(\omega_0 t+\varphi_0+Arg(H(j\omega_0)))$. Note that $H(j\omega_0)=\left.\frac{1}{s^2+as+b}\right|_{s=j\omega_0}=\frac{1}{(j\omega_0)^2+a(j\omega_0)+b}=\frac{1}{b-{\omega_0}^2+ja\omega_0}$.

Let's study a numeric example.

Example 9.31 Find the general solution of $\ddot{y}+4\dot{y}+3y=2\cos(2t)$.

Homogenous solution can be found easily:

$$\lambda^2+4\lambda+3=0\Rightarrow\lambda_1=-1,\lambda_2=-3\Rightarrow y_h(t)=C_1e^{-t}+C_2te^{-3t}$$

Now we find the particular solution with the aid of transfer function:

$$\ddot{y}(t)+4\dot{y}(t)+3y(t)=u(t)\Rightarrow\left(s^2+4s+3\right)Y(s)$$
$$=U(s)\Rightarrow\frac{Y(s)}{U(s)}=\frac{1}{s^2+4s+3}$$

$$\left.\frac{1}{s^2+4s+3}\right|_{s=2j}=\frac{1}{(j2)^2+4\times(2j)+3}=\frac{1}{-4+8j+3}=\frac{1}{-1+8j}$$

$$\left|\frac{1}{-1+8j}\right|=\frac{1}{\sqrt{65}}$$

Principle argument $\frac{1}{-1+8j}$ can be calculated with the aid of Fig. 9.2:

$$Arg\left(\frac{1}{-1+8j}\right)=Arg\left(\frac{1\times(-1-8j)}{(-1+8j)\times(-1-8j)}\right)$$
$$=Arg\left(\frac{-1-8j}{65}\right)=Arg\left(\frac{-1}{65}+\frac{-8j}{65}\right)=-\left(\frac{\pi}{2}+\tan^{-1}\left(\frac{\frac{1}{65}}{\frac{8}{65}}\right)\right)$$
$$=-1.6952Rad\in(-\pi,\pi]$$

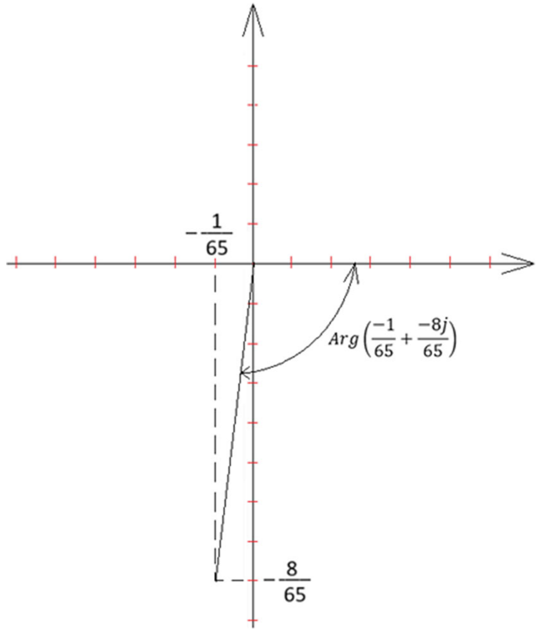


Fig. 9.2 Principle argument of $\frac{1}{-1+8j} = -\frac{1}{65} - \frac{8}{65}j$

You can use the following method and Fig. 9.3 to calculate the principle argument as well.

$$Arg\left(\frac{1}{-1+8j}\right) = Arg(1) - Arg(-1+8j) = 0 - \left(\frac{\pi}{2} + \tan^{-1}\left(\frac{1}{8}\right)\right)$$
$$= -1.6952Rad \in (-\pi, \pi]$$

$$y_p(t) = \frac{1}{\sqrt{65}} \times 2\cos(2t - 1.6952)$$

$$y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2te^{-3t} + \frac{1}{\sqrt{65}} \times 2\cos(2t - 1.6952)$$

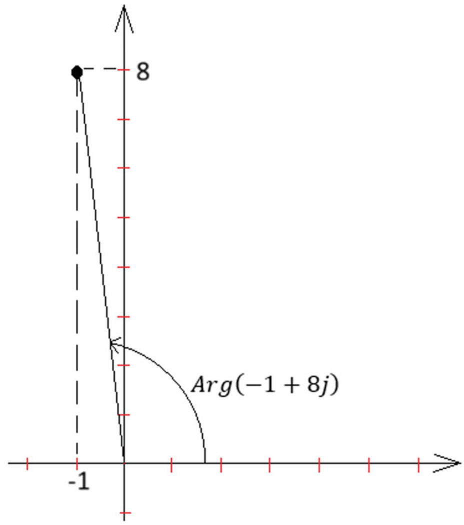


Fig. 9.3 Principle argument of $-1+8j$

Example 9.32 Find the general solution of $\dot{y}+6y = 30\sin(4t+\frac{\pi}{3})$.

$$\lambda + 6 = 0 \Rightarrow \lambda = -6 \Rightarrow y_h(t) = C_1 e^{-6t}$$

$$\dot{y}(t) + 6y(t) = u(t) \Rightarrow (s+6)Y(s) = U(s) \Rightarrow \frac{Y(s)}{U(s)} = \frac{1}{s+6}$$

$$\left.\frac{1}{s+6}\right|_{s=4j} = \frac{1}{4j+6}$$

$$\left|\frac{1}{6+4j}\right| = \frac{1}{\sqrt{6^2+4^2}} = \frac{1}{\sqrt{52}}$$

The principle argument of $\frac{1}{6+4j}$ can be calculated with the aid of Fig. 9.4:

$$Arg\left(\frac{1}{6+4j}\right) = Arg(1) - Arg(6+4j) = 0 - \left(\tan^{-1}\left(\frac{4}{6}\right)\right) = -0.5880Rad$$

$$y_p(t) = \frac{1}{\sqrt{52}} \times 30\sin\left(4t + \frac{\pi}{3} - 0.5880\right) = 4.1603\sin(4t+0.4592)$$

$$y(t) = y_h(t) + y_p(t) = C_1 e^{-6t} + 4.1603\sin(4t+0.4592)$$

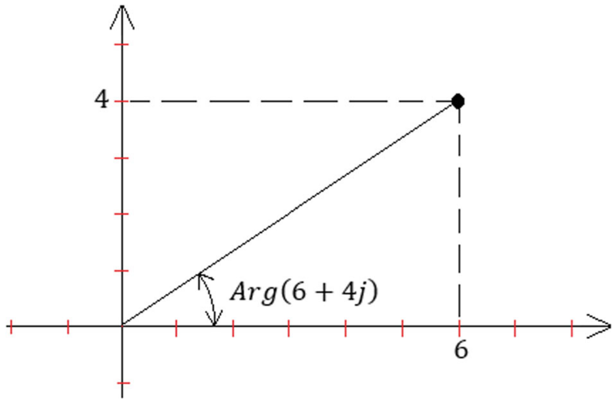


Fig. 9.4 Principle argument of $6+4j$

Example 9.33 Find the general solution of $\ddot{y}+4y=6\cos(2t)$.

$$s^2+4=0 \Rightarrow s=\sqrt{-4}=\pm 2j \Rightarrow y_h=C_1\cos(2t)+C_2\sin(2t)$$

$$\ddot{y}(t)+4y(t)=u(t) \Rightarrow \left(s^2+4\right)Y(s)=U(s) \Rightarrow \frac{Y(s)}{U(s)}=\frac{1}{s^2+4}$$

$$\left.\frac{1}{s^2+4}\right|_{s=2j}=\frac{1}{(j2)^2+4}=\frac{1}{-4+4}=\frac{1}{0}(!)$$

Therefore, the transfer function method can’t be used here. Let’s assume the following form for particular solution:

$$y_p=t\times(A\cos(2t)+B\sin(2t))$$

$$\dot{y}_p=A\cos(2t)+B\sin(2t)+t(-2A\sin(2t)+2B\cos(2t))$$

$$\begin{aligned}\ddot{y}_p&=-2A\sin(2t)+2B\cos(2t)-2A\sin(2t)+2B\cos(2t)+t(-4A\cos(2t)-4B\sin(2t))\\&=-4A\sin(2t)+4B\cos(2t)-4t(A\cos(2t)+B\sin(2t))\end{aligned}$$

Substitution in the given differential equation:

$$\begin{aligned}&\ddot{y}_p+4y_p=6\cos(2t) \Rightarrow -4A\sin(2t)+4B\cos(2t)\\&\quad-4t(A\cos(2t)+B\sin(2t))+4t(A\cos(2t)+B\sin(2t))\\&=-4A\sin(2t)+4B\cos(2t)\end{aligned}$$

$$-4A\sin(2t)+4B\cos(2t)=6\cos(2t) \Rightarrow A=0, B$$

$$= \frac{6}{4} = 1.5 \Rightarrow y_p = \frac{3}{2}t\sin(2t)$$

$$y = y_h + y_p = C_1\cos(2t) + C_2\sin(2t) + \frac{3}{2}t\sin(2t)$$

9.12 Matrix Exponential

Matrix exponentiation is a powerful tool for solving systems of linear differential equations. It provides a systematic and efficient method to find solutions, especially for systems with constant coefficients.

The matrix exponential of a square matrix A is defined as the infinite series:

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

Let's study some numeric examples.

Example 9.34 $A = \begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix}$. Calculate e^{At}.

$$A = \begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix} \Rightarrow |A - \lambda I| = 0 \Rightarrow \left\| \begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\|$$
$$= \begin{vmatrix} -1-\lambda & 6 \\ 2 & 3-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda) - 12 = 0 \Rightarrow \lambda^2 - 2\lambda$$
$$- 15 = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = -3$$

Eigen vectors associated with $\lambda_1 = 5$ and $\lambda_2 = -3$ are $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\lambda_1 = 5$ and $V_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ for $\lambda_2 = -3$. Define the matrix P as the concatenation of the eigenvectors V_1 and V_2:

$$P = \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$$

Note that the use of unit eigenvectors is not required in forming the matrix P. Calculate the invers of P:

$$P^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Now use the following formula to calculate e^{At}:

$$e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} P^{-1}$$

$$e^{At} = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{5t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4}e^{5t} & \frac{3}{4}e^{5t} \\ -\frac{1}{4}e^{-3t} & \frac{1}{4}e^{-3t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}e^{5t} + \frac{3}{4}e^{-3t} & \frac{3}{4}e^{5t} - \frac{3}{4}e^{-3t} \\ \frac{1}{4}e^{5t} - \frac{1}{4}e^{-3t} & \frac{3}{4}e^{5t} + \frac{1}{4}e^{-3t} \end{bmatrix}$$

Note that order of matrix multiplication is not important, i.e., $(A \times B) \times C = A \times (B \times C)$, since:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}, C = \begin{bmatrix} g & h \\ i & j \end{bmatrix}$$

$$\Rightarrow (A \times B) \times C = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \right) \times \begin{bmatrix} g & h \\ i & j \end{bmatrix} = \begin{bmatrix} ae & bf \\ ce & df \end{bmatrix} \times \begin{bmatrix} g & h \\ i & j \end{bmatrix}$$

$$= \begin{bmatrix} aeg + bfi & aeh + bfj \\ ceg + dfi & ceh + dfj \end{bmatrix}$$

$$A \times (B \times C) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \left(\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \times \begin{bmatrix} g & h \\ i & j \end{bmatrix} \right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} eg & eh \\ fi & fj \end{bmatrix}$$

$$= \begin{bmatrix} aeg + bfi & aeh + bfj \\ ceg + dfi & ceh + dfj \end{bmatrix}$$

In general, for $n \times n$ square matrices A, B and C, $(A \times B) \times C = A \times (B \times C)$.

Example 9.35 $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Calculate e^{At}.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow |A - \lambda I| = 0 \Rightarrow \left| \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)(3-\lambda) = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3$$

Eigen vector associated with $\lambda_1 = 2$ is:

$$(A-\lambda_1 I)V_1 = 0 \Rightarrow \left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_2 = 0$$

For instance, we can select $V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Eigenvector associated with $\lambda_2 = 3$ is:

$$(A-\lambda_2 I)V_2 = 0 \Rightarrow \left(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \left(\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -x_1 = 0$$

For instance, we can select $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Similar to the previous example,

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$e^{At} = P\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} e^{1t} & 0 \\ 0 & e^{3t} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{1t} & 0 \\ 0 & e^{3t} \end{bmatrix}$$

Example 9.36 $A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$. Calculate e^{At}.

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \Rightarrow |A-\lambda I| = 0 \Rightarrow \left|\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right|$$
$$= \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = (3-\lambda)(-1-\lambda) - (-8) = 0$$
$$\Rightarrow \lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda_1 = 1+2j, \lambda_2 = 1-2j$$

Eigen vectors associated with $\lambda_1 = 1+2j$ and $\lambda_2 = 1-2j$ can be calculated as:

$$\lambda_1 = 1 + 2j \Rightarrow \left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1+2j) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) V_1$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 1+2j & 0 \\ 0 & 1+2j \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left(\begin{bmatrix} 3-1-2j & -2 \\ 4 & -1-1-2j \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 = 1 - 2j \Rightarrow \left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - (1-2j) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) V_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 1-2j & 0 \\ 0 & 1-2j \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left(\begin{bmatrix} 3-1+2j & -2 \\ 4 & -1-1+2j \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 2+2j & -2 \\ 4 & -2+2j \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow (2+2j)x_1 - 2x_2 = 0$$

$$\Rightarrow x_2 = (1+j)x_1 \Rightarrow V_2 = \begin{bmatrix} 1 \\ 1+j \end{bmatrix}$$

Similar to the previous example,

$$P = \begin{bmatrix} 1 & 1 \\ 1-j & 1+j \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 0.5-0.5j & 0.5j \\ 0.5+0.5j & -0.5j \end{bmatrix} = 0.5 \begin{bmatrix} 1-j & j \\ 1+j & -j \end{bmatrix}$$

$$e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 1 \\ 1-j & 1+j \end{bmatrix} \times \begin{bmatrix} e^{(1+2j)t} & 0 \\ 0 & e^{(1-2j)t} \end{bmatrix} \times 0.5 \begin{bmatrix} 1-j & j \\ 1+j & -j \end{bmatrix}$$

$$= 0.5 \begin{bmatrix} 1 & 1 \\ 1-j & 1+j \end{bmatrix} \begin{bmatrix} (1-j)e^{(1+2j)t} & je^{(1+2j)t} \\ (1+j)e^{(1-2j)t} & -je^{(1-2j)t} \end{bmatrix}$$

$$= 0.5 \begin{bmatrix} (1-j)e^{(1+2j)t} + (1+j)e^{(1-2j)t} & je^{(1+2j)t} - je^{(1-2j)t} \\ (1-j)^2 e^{(1+2j)t} + (1+j)^2 e^{(1-2j)t} & j(1-j)e^{(1+2j)t} - j(1+j)e^{(1-2j)t} \end{bmatrix}$$

Remember that $z + \overline{z} = 2Re\{z\}$. Therefore,

$$e^{At} = 0.5 \begin{bmatrix} (1-j)e^{(1+2j)t} + (1+j)e^{(1-2j)t} & je^{(1+2j)t} - je^{(1-2j)t} \\ (1-j)^2 e^{(1+2j)t} + (1+j)^2 e^{(1-2j)t} & j(1-j)e^{(1+2j)t} - j(1+j)e^{(1-2j)t} \end{bmatrix}$$

$$= 0.5\begin{bmatrix} 2Re\{(1-j)e^{(1+2j)t}\} & 2Re\{je^{(1+2j)t}\} \\ 2Re\{(1-j)^2 e^{(1+2j)t}\} & 2Re\{j(1-j)e^{(1+2j)t}\} \end{bmatrix}$$
$$= \begin{bmatrix} Re\{(1-j)e^{(1+2j)t}\} & Re\{je^{(1+2j)t}\} \\ Re\{(1-j)^2 e^{(1+2j)t}\} & Re\{(1+j)e^{(1+2j)t}\} \end{bmatrix}$$

Let's calculate the real part of each element:

$$Re\left\{(1-j)e^{(1+2j)t}\right\} = e^t Re\left\{(1-j)e^{2jt}\right\} = e^t Re\{(1-j)(\cos(2t) + j\sin(2t))\}$$
$$= e^t(\cos(2t) + \sin(2t))$$

$$Re\left\{je^{(1+2j)t}\right\} = e^t Re\left\{je^{j2t}\right\} = e^t Re\{j(\cos(2t) + j\sin(2t))\} = -e^t\sin(2t)$$

$$Re\{(1-j)^2 e^{(1+2j)t}\} = Re\{-2je^{(1+2j)t}\} = -e^t Re\{2j(\cos(2t) + j\sin(2t))\} = 2e^t\sin(2t)$$

$$Re\left\{(1+j)e^{(1+2j)t}\right\} = e^t Re\left\{(1+j)e^{j2t}\right\} = e^t Re\{(1+j)(\cos(2t) + jsin(2t))\} = e^t(\cos(2t)$$
$$- \sin(2t))$$

Therefore,

$$e^{At} = \begin{bmatrix} Re\{(1-j)e^{(1+2j)t}\} & Re\{je^{(1+2j)t}\} \\ Re\{(1-j)^2 e^{(1+2j)t}\} & Re\{j(1-j)e^{(1+2j)t}\} \end{bmatrix}$$
$$= e^t\begin{bmatrix} \cos(2t) + \sin(2t) & -\sin(2t) \\ 2\sin(2t) & \cos(2t) - \sin(2t) \end{bmatrix}$$

Example 9.37 $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$. Calculate e^{At}.

$$A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix} \Rightarrow |A - \lambda I| = 0 \Rightarrow \left|\begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right|$$
$$= \begin{vmatrix} 7-\lambda & -1 \\ 4 & 3-\lambda \end{vmatrix} = (7-\lambda)(3-\lambda) + 4 = 0 \Rightarrow \lambda^2 - 10\lambda$$
$$+ 25 = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = 5$$

Eigen vectors associated with $\lambda_1 = \lambda_2 = 5$:

$$(A - \lambda I)V = 0 \Rightarrow \left(\begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 2x - y = 0 \Rightarrow y = 2x$$

For instance, we can select $V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The generalized eigenvector is:

$$(A - \lambda I)w = V_1 \Rightarrow \left(\begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow 2w_1 - w_2 = 1$$
$$\Rightarrow w_2 = 2w_1 - 1$$

For instance, we can select $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore, given matrix has an ordinary eigenvector and a generalized eigenvector.

Define the matrix P as the concatenation of the eigenvectors V_1 and w_1:

$$P = \begin{bmatrix} V_1 & w_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

The invers of P is:

$$P^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

Given matrix has one eigenvalue and one eigenvector. Therefore, we need to use the following formula to calculate the e^{At}:

$$e^{At} = e^{\lambda t} (P \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} P^{-1})$$

$$e^{At} = e^{5t} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = e^{5t} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2t-1 & 1-t \\ 2 & -1 \end{bmatrix} = e^{5t} \begin{bmatrix} 2t+1 & -t \\ 4t & 1-2t \end{bmatrix}$$

Matrix exponentials provide a powerful tool for solving systems of linear differential equations. Let's study a numeric example.

Example 9.38 Solve $\begin{cases} \dot{x}(t) = 3x(t) - 2y(t) \\ \dot{y}(t) = 4x(t) - y(t) \end{cases}$ with $\begin{bmatrix} x(0) = 1 \\ y(0) = 7 \end{bmatrix}$.

$$\begin{cases} \dot{x}(t) = 3x(t) - 2y(t) \\ \dot{y}(t) = 4x(t) - y(t) \end{cases} \Rightarrow \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{At} . \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = e^t \begin{bmatrix} \cos(2t) + \sin(2t) & -\sin(2t) \\ 2\sin(2t) & \cos(2t) - \sin(2t) \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$= e^t \begin{bmatrix} \cos(2t) + \sin(2t) - 7\sin(2t) \\ 2\sin(2t) + 7\cos(2t) - 7\sin(2t) \end{bmatrix} = e^t \begin{bmatrix} \cos(2t) - 6\sin(2t) \\ 7\cos(2t) - 5\sin(2t) \end{bmatrix} \Rightarrow \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$= \begin{bmatrix} e^t(\cos(2t) - 6\sin(2t)) \\ e^t(7\cos(2t) - 5\sin(2t)) \end{bmatrix} \Rightarrow \begin{cases} x(t) = e^t(\cos(2t) - 6\sin(2t)) \\ y(t) = e^t(7\cos(2t) - 5\sin(2t)) \end{cases}$$

(Note that e^{At} is calculated in Example 9.36.)

An alternative approach would be to employ the Laplace transform. For the sake of brevity, we will represent the Laplace transform of $x(t)$ and $y(t)$ by X and Y, respectively.

$$\begin{cases} \dot{x}(t) = 3x(t) - 2y(t) \\ \dot{y}(t) = 4x(t) - y(t) \end{cases} \Rightarrow \begin{cases} L(\dot{x}(t)) = L(3x(t)) - L(2y(t)) \\ \mathcal{L}(\dot{y}(t)) = L(4x(t)) - L(y(t)) \end{cases}$$

$$\Rightarrow \begin{cases} sX - x(0) = 3X - 2Y \\ sY - y(0) = 4X - Y \end{cases} \Rightarrow \begin{cases} sX - 1 = 3X - 2Y \\ sY - 7 = 4X - Y \end{cases}$$

$$\Rightarrow \begin{cases} (s-3)X + 2Y = 1 \\ -4X + (s+1)Y = 7 \end{cases} \Rightarrow \begin{bmatrix} s-3 & 2 \\ -4 & s+1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$= \begin{bmatrix} s-3 & 2 \\ -4 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \frac{1}{(s-3)(s+1) - (2)(-4)} \begin{bmatrix} s+1 & -2 \\ 4 & s-3 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$= \frac{1}{s^2 - 2s + 5} \begin{bmatrix} s + 1 - 14 \\ 4 + 7(s-3) \end{bmatrix} = \frac{1}{s^2 - 2s + 5} \begin{bmatrix} s - 13 \\ 7s - 17 \end{bmatrix} = \begin{bmatrix} \frac{s-13}{s^2-2s+5} \\ \frac{7s-17}{s^2-2s+5} \end{bmatrix}$$

$$\Rightarrow \begin{cases} X = \frac{s-13}{s^2-2s+5} \\ Y = \frac{7s-17}{s^2-2s+5} \end{cases}$$

Therefore,

$$\begin{cases} x(t) = \mathcal{L}^{-1}(X) = \mathcal{L}^{-1}\left(\frac{s-13}{s^2-2s+5}\right) = \mathcal{L}^{-1}\left(\frac{s-1}{(s-1)^2+2^2} - 6\frac{2}{(s-1)^2+2^2}\right) \\ \qquad = e^t\cos(2t) - 6e^t\sin(2t) \\ y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{7s-17}{s^2-2s+5}\right) = \mathcal{L}^{-1}\left(\frac{7(s-1)}{(s-1)^2+2^2} - 5\frac{2}{(s-1)^2+2^2}\right) \\ \qquad = 7e^t\cos(2t) - 5e^t\sin(2t) \end{cases}$$

The result obtained is identical to the preceding one.

Example 9.39 Solve $\begin{cases} \dot{x}(t) = 7x(t) - y(t) \\ \dot{y}(t) = 4x(t) + 3y(t) \end{cases}$ with $\begin{bmatrix} x(0) = 1 \\ y(0) = 2 \end{bmatrix}$.

$$\begin{cases} \dot{x}(t) = 7x(t) - y(t) \\ \dot{y}(t) = 4x(t) + 3y(t) \end{cases} \Rightarrow \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \cdot \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{At} . \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = e^{5t} \begin{bmatrix} 2t+1 & -t \\ 4t & 1-2t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = e^{t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} x(t) = e^{t} \\ y(t) = 2e^{t} \end{cases}$$

(Note that is e^{At} calculated in Example 9.37.)

An alternative approach would be to employ the Laplace transform. For the sake of brevity, we will represent the Laplace transform of $x(t)$ and $y(t)$ by X and Y, respectively.

$$\begin{cases} \dot{x}(t) = 7x(t) - y(t) \\ \dot{y}(t) = 4x(t) + 3y(t) \end{cases} \Rightarrow \begin{cases} L(\dot{x}(t)) = L(7x(t)) - L(y(t)) \\ L(\dot{y}(t)) = L(4x(t)) + L(3y(t)) \end{cases}$$

$$\Rightarrow \begin{cases} sX - x(0) = 7X - Y \\ sY - y(0) = 4X + 3Y \end{cases} \Rightarrow \begin{cases} sX - 1 = 7X - Y \\ sY - 2 = 4X + 3Y \end{cases}$$

$$\Rightarrow \begin{cases} (s-7)X + Y = 1 \\ -4X + (s-3)Y = 2 \end{cases} \Rightarrow \begin{bmatrix} s-7 & 1 \\ -4 & s-3 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} s-7 & 1 \\ -4 & s-3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{(s-7)(s-3)-(1)(-4)} \begin{bmatrix} s-3 & -1 \\ 4 & s-7 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{s^2 - 10s + 25} \begin{bmatrix} s-5 \\ 2s-10 \end{bmatrix} = \frac{1}{(s-5)^2} \begin{bmatrix} s-5 \\ 2(s-5) \end{bmatrix} = \begin{bmatrix} \frac{1}{s-5} \\ \frac{2}{s-5} \end{bmatrix} \Rightarrow \begin{cases} X = \frac{1}{s-5} \\ Y = \frac{2}{s-5} \end{cases}$$

Therefore,

$$\begin{cases} x(t) = \mathcal{L}^{-1}(X) = \mathcal{L}^{-1}\left(\frac{1}{s-5}\right) = e^{5t} \\ y(t) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{2}{s-5}\right) = 2e^{t} \end{cases}$$

The result obtained is identical to the preceding one.

9.13 Converting a Higher-Order Differential Equation to a First-Order System

The following example illustrates the technique for converting a higher-order differential equation to a first-order system.

Example 9.40 Convert $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = e^{-5t}, y(0) = 1, \dot{y}(0) = 0$ to a system of first-order differential equations.

We define $x_1(t)$ and $x_2(t)$ as $\begin{cases} x_1(t) = y(t) \\ x_2(t) = \dot{y}(t) \end{cases}$. Therefore, $\dot{x}_1(t) = \dot{y}(t) = x_2(t)$ and $\dot{x}_2(t) = \ddot{y}(t)$. Given differential equation can be written as: $\ddot{y}(t) = -6y(t) - 5\dot{y}(t) + e^{-5t} \Rightarrow \dot{x}_2(t) = -6x_1(t) - 5x_2(t) + e^{-5t}$. Therefore, $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = e^{-5t}, y(0) = 1, \dot{y}(0) = 0$ can be written as:

$$\begin{cases} \dot{x}_1(t) = x_2(t), x_1(0) = 1 \\ x_2(t) = -6x_1(t) - 5x_2(t) + e^{-5t}, x_2(0) = 0 \end{cases}$$

Example 9.41 Convert $\ddot{y}(t) + 2\dot{y}(t) + 3y(t) = 4u + 5\dot{u}$ to a system of first-order differential equations. $y(t)$ and $u(t)$ represent the output and input, respectively.

We define $x_1(t)$ and $x_2(t)$ as $\begin{cases} x_1(t) = y(t) \\ x_2(t) = \dot{y}(t) - 5u(t) \end{cases}$. Therefore, $\dot{x}_1(t) = \dot{y}(t) = x_2(t) + 5u(t)$ and $\dot{x}_2(t) = \ddot{y}(t) - 5\dot{u}(t) = -2\dot{y}(t) - 3y(t) + 4u(t) = -2(x_2(t) + 5u(t)) - 3x_1(t) + 4u(t)$. Therefore, $\dot{x}_2(t) = -3x_1(t) - 2x_2(t) - 6u(t)$.

$$\ddot{y}(t) + 2\dot{y}(t) + 3y(t) = 4u + 5\dot{u}$$
$$\Rightarrow \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 5 \\ -6 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

For instance, when input is $u(t) = e^{-5t}$:

$$\ddot{y}(t) + 2\dot{y}(t) + 3y(t) = 4e^{-5t} + 5 \times -5e^{-5t} = -21e^{-5t}$$
$$\Rightarrow \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 5 \\ -6 \end{bmatrix} e^{-5t} \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

Example 9.42 Find the transfer function associated with $\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 5 \\ -6 \end{bmatrix}u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}.$

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 5 \\ -6 \end{bmatrix}u(t) \right. \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_1(t) = x_2(t) + 5u(t) \\ \dot{x}_2(t) = -3x_1(t) - 2x_2(t) - 6u(t) \\ y(t) = x_1(t) \end{cases}$$

$$\stackrel{\mathcal{L}}{\Rightarrow} \begin{cases} sX_1(s) = X_2(s) + 5U(s) \\ sX_2(s) = -3X_1(s) - 2X_2(s) - 6U(s) \\ Y(s) = X_1(s) \end{cases}$$

$$\Rightarrow \begin{cases} sX_1(s) - X_2(s) = 5U(s) \\ 3X_1(s) + (s+2)X_2(s) = -6U(s) \end{cases}$$

$$\Rightarrow \begin{bmatrix} s & -1 \\ 3 & s+2 \end{bmatrix}\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 5U(s) \\ -6U(s) \end{bmatrix} \Rightarrow \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} s & -1 \\ 3 & s+2 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ -6 \end{bmatrix}U(s)$$

$$= \begin{bmatrix} \frac{5s+4}{s^2+2s+3} \\ \frac{-6s-15}{s^2+2s+3} \end{bmatrix}U(s)$$

$$\begin{aligned} Y(s) = X_1(s) = \frac{5s+4}{s^2+2s+3}U(s) \Rightarrow T(s) = \frac{Y(s)}{U(s)} \\ = \frac{5s+4}{s^2+2s+3} \end{aligned}$$

In general, the transfer function of a system described by the state-space equations:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

is given by (I shows the identity matrix):

$$T(s) = C(sI - A)^{-1}B + D$$

In this example, $A = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $D = 0$. Therefore,

$$T(s) = C(sI - A)^{-1}B + D = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 5 \\ -6 \end{bmatrix} + 0$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 3 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2+2s+3} \begin{bmatrix} s+2 & 1 \\ -3 & s \end{bmatrix} \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

$$= \frac{1}{s^2+2s+3} \times \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ -3 & s \end{bmatrix} \begin{bmatrix} 5 \\ -6 \end{bmatrix} = \frac{1}{s^2+2s+3} \times \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 5s+10-6 \\ -15-6s \end{bmatrix}$$

$$= \frac{1}{s^2+2s+3} \times \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 5s+4 \\ -15-6s \end{bmatrix} = \frac{5s+4}{s^2+2s+3}$$

The result obtained matches the result from the previous calculation.

Example 9.43 Use the Laplace transform to solve the

$$\begin{cases} \dot{x}_1(t) = x_2(t), x_1(0) = 1 \\ x_2(t) = -6x_1(t) - 5x_2(t) + e^{-5t}, x_2(0) = 0 \end{cases}$$

For the sake of brevity, we will represent the Laplace transform of $x_1(t)$ and $x_2(t)$ by X_1 and X_2, respectively.

$$\begin{cases} \dot{x}_1(t) = x_2(t), x_1(0) = 1 \\ x_2(t) = -6x_1(t) - 5x_2(t) + e^{-5t}, x_2(0) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} sX_1 - x_1(0) = X_2 \\ sX_2 - x_2(0) = -6X_1 - 5X_2 + \frac{1}{s+5} \end{cases}$$

$$\Rightarrow \begin{cases} sX_1 - 1 = X_2 \\ sX_2 - 0 = -6X_1 - 5X_2 + \frac{1}{s+5} \end{cases} \Rightarrow \begin{cases} sX_1 - X_2 = 1 \\ 6X_1 + (s+5)X_2 = \frac{1}{s+5} \end{cases}$$

$$\Rightarrow \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{s+5} \end{bmatrix} \Rightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \frac{1}{s+5} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{s^2+5s+6} \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{s+5} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$= \frac{1}{s^2+5s+6} \begin{bmatrix} s+5+\frac{1}{s+5} \\ -6+\frac{s}{s+5} \end{bmatrix} = \begin{bmatrix} \frac{s+5}{s^2+5s+6} + \frac{1}{(s+5)(s^2+5s+6)} \\ \frac{-6}{s^2+5s+6} + \frac{s}{(s+5)(s^2+5s+6)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+5}{(s+2)(s+3)} + \frac{1}{(s+5)(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} + \frac{s}{(s+5)(s+2)(s+3)} \end{bmatrix}$$

Using partial fraction decomposition, we have:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \frac{s+5}{(s+2)(s+3)} + \frac{1}{(s+5)(s+2)(s+3)} \\ \frac{-6}{(s+2)(s+3)} + \frac{s}{(s+5)(s+2)(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{s+2} + \frac{-2}{s+3} + \frac{\frac{1}{3}}{s+2} + \frac{-\frac{1}{2}}{s+3} + \frac{\frac{1}{6}}{s+5} \\ \frac{-6}{s+2} + \frac{6}{s+3} + \frac{\frac{-2}{3}}{s+2} + \frac{\frac{3}{2}}{s+3} + \frac{\frac{-5}{6}}{s+5} \end{bmatrix} = \begin{bmatrix} \frac{\frac{10}{3}}{s+2} + \frac{\frac{-5}{2}}{s+3} + \frac{\frac{1}{6}}{s+5} \\ \frac{-\frac{20}{3}}{s+2} + \frac{\frac{15}{2}}{s+3} + \frac{\frac{-5}{6}}{s+5} \end{bmatrix}$$

Therefore,

$$= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \mathcal{L}^{-1}\left(\begin{bmatrix} \frac{\frac{10}{3}}{s+2} + \frac{\frac{-5}{2}}{s+3} + \frac{\frac{1}{6}}{s+5} \\ \frac{-\frac{20}{3}}{s+2} + \frac{\frac{15}{2}}{s+3} + \frac{\frac{-5}{6}}{s+5} \end{bmatrix}\right) = \begin{bmatrix} \frac{10}{3}e^{-2t} + \frac{-5}{2}e^{-3t} + \frac{1}{6}e^{-5t} \\ -\frac{20}{3}e^{-2t} + \frac{15}{2}e^{-3t} - \frac{5}{6}e^{-5t} \end{bmatrix}$$

9.14 4th Order Runge–Kutta Method

The most commonly used Runge–Kutta method to find the solution of a differential equation is the 4th order Runge–Kutta method. Only the first-order ODEs can be solved using the 4th order Runge–Kutta method.

$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ is the given differential equation. The formula for 4th order Runge–Kutta method is given as,

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

whereas

$$k_1 = h \times f(x_n, y_n)$$

$$k_2 = h \times f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h \times f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h \times f\left(x_n + \frac{h}{2}, y_n + k_3\right)$$

Example 9.44 Consider an ordinary differential equation $\frac{dy}{dx} = (1 + 4x)\sqrt{y}, y(0) = 1$. Find $y(1.5)$ using the 4th order Runge–Kutta method with $h = 0.5$.

First iteration (n = 0)

$h = 0.5, x_0 = 0, y_0 = 1$ and $f(x, y) = (1 + 4x)\sqrt{y}$.

$$k_1 = h \times f(x_0, y_0) \Rightarrow k_1 = 0.5 \times f(0, 1)$$

$$= 0.5 \times (1 + 4 \times 0)\sqrt{1} = 0.5$$

$$k_2 = h \times f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \Rightarrow k_2 = 0.5 \times f\left(0 + \frac{0.5}{2}, 1 + \frac{0.5}{2}\right)$$
$$= 0.5 \times f(0.25, 1.25) = 0.5 \times (1 + 4 \times 0.25)\sqrt{1.25} = 1.1180$$

$$k_3 = h \times f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \Rightarrow k_3 = 0.5 \times f\left(0 + \frac{0.5}{2}, 1 + \frac{1.1180}{2}\right)$$
$$= 0.5 \times f(0.25, 1.5590) = 0.5 \times (1 + 4 \times 0.25)\sqrt{1.5590} = 1.2485$$

$$k_4 = h \times f(x_0 + h, y_0 + k_3) \Rightarrow k_4 = 0.5 \times f(0 + 0.5, 1 + 1.2485)$$
$$= 0.5 \times f(0.5, 2.2485) = 0.5 \times (1 + 4 \times 0.5)\sqrt{2.2485} = 2.2492$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \Rightarrow y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 1 + \frac{1}{6}(0.5 + 2 \times 1.1180 + 2 \times 1.2485 + 2.2492) = 2.2470$$

Second iteration (n = 1)
$h = 0.5, x_1 = 0.5, y_1 = 2.2470$ and $f(x, y) = (1 + 4x)\sqrt{y}$.

$$k_1 = h \times f(x_1, y_1) \Rightarrow k_1 = 0.5 \times f(0.5, 2.2470) = 2.2484$$

$$k_2 = h \times f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) \Rightarrow k_2 = 0.5$$
$$\times f\left(0.5 + \frac{0.5}{2}, 2.2470 + \frac{2.2484}{2}\right) = 0.5 \times f(0.75, 3.3712)$$
$$= 3.6721$$

$$k_3 = h \times f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) \Rightarrow k_3 = 0.5$$
$$\times f\left(0.5 + \frac{0.5}{2}, 2.2470 + \frac{3.6721}{2}\right) = 0.5 \times f(0.75, 4.083)$$
$$= 4.0413$$

$$k_4 = h \times f(x_1 + h, y_1 + k_3) \Rightarrow k_4 = 0.5 \times f(0.5 + 0.5, 2.2470 + 4.0413)$$
$$= 0.5 \times f(0.75, 6.2883) = 6.2691$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \Rightarrow y_2$$
$$= y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2.2470$$
$$+ \frac{1}{6}(2.2484 + 2 \times 3.6721 + 2 \times 4.0413 + 6.2691) = 2.2470$$
$$+ 3.9907 = 6.2377$$

Third iteration (n = 2)
$h = 0.5, x_2 = 1, y_2 = 6.2377$ and $f(x, y) = (1 + 4x)\sqrt{y}$.

$$k_1 = h \times f(x_2, y_2) \Rightarrow k_1 = 0.5 \times f(1, 6.2377) = 6.2438$$

$$k_2 = h \times f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) \Rightarrow k_2 = 0.5 \times f\left(1 + \frac{0.5}{2}, 6.2377 + \frac{6.2438}{2}\right)$$
$$= 0.5 \times f(1.25, 9.3596) = 9.1780$$

$$k_3 = h \times f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) \Rightarrow k_3 = 0.5 \times f\left(1 + \frac{0.5}{2}, 6.2377 + \frac{9.1780}{2}\right)$$
$$= 0.5 \times f(1.25, 10.8267) = 9.8711$$

$$k_4 = h \times f(x_2 + h, y_2 + k_3) \Rightarrow k_4 = 0.5 \times f(1 + 0.5, 6.2377 + 9.8711)$$
$$= 0.5 \times f(1.5, 16.1088) = 14.0475$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \Rightarrow y_3 = y_2$$
$$+ \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 6.2377 + \frac{1}{6}(6.2438$$
$$+ 2 \times 9.1780 + 2 \times 9.8711 + 14.0475) = 15.9692$$

Therefore, $y(0) = 1, y(0.5) = 2.2470, y(1) = 6.2377$ and $y(1.5) = 15.9692$.

Example 9.45 Write a MATLAB code to do the calculations of previous example.

Enter edit and press the Enter key on your keyboard (Fig. 9.5).

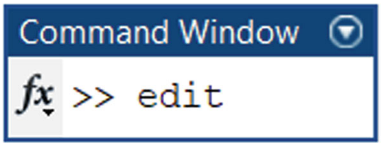


Fig. 9.5 `edit` command

Write the following code in the editor (Fig. 9.6).

```
function yout = ode4(F,x0,h,xfinal,y0)
   y = y0;
   yout = y;
    for x = x0 : h : xfinal-h
      k1 = h*F(x,y);
      k2 = h*F(x+h/2, y+k1/2);
      k3 = h*F(x+h/2, y+k2/2);
      k4 = h*F(x+h, y+k3);
      y = y + (k1 + 2*k2 + 2*k3 + k4)/6;
      yout = [yout; y];
    end
```

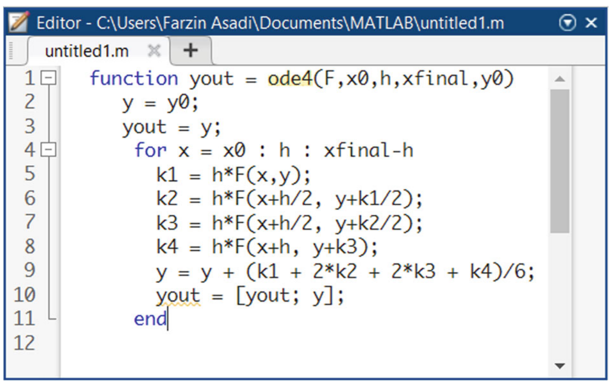


Fig. 9.6 Entering the code into MATLAB Editor

Press Ctrl + S to save the file as “ode4” (Fig. 9.7).

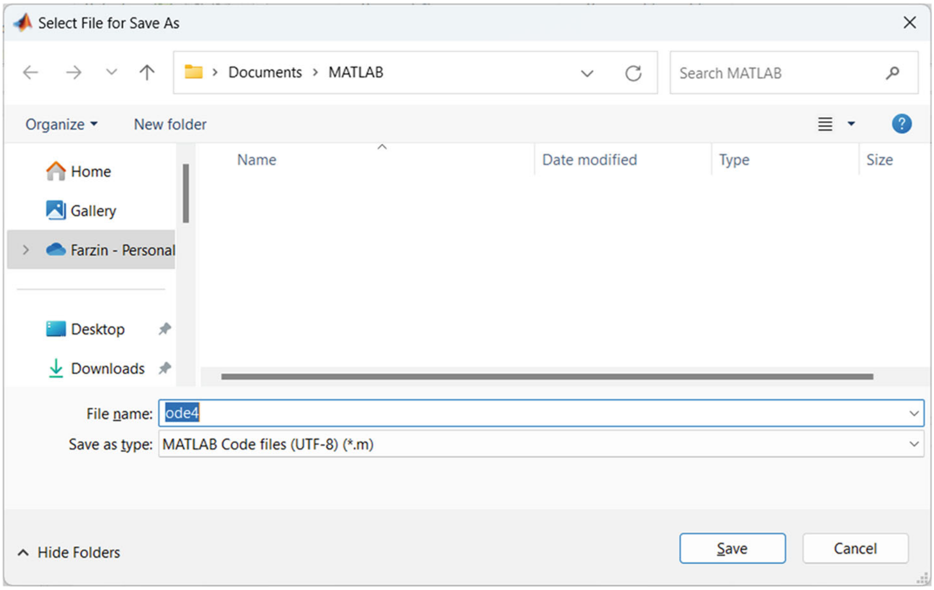
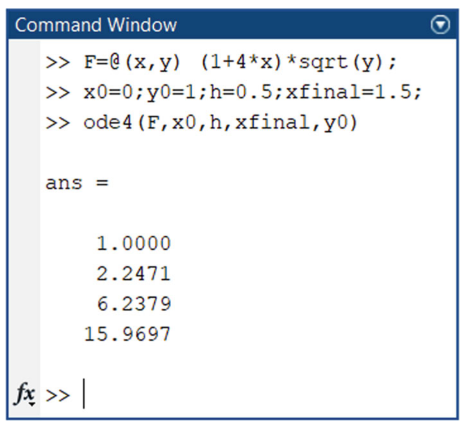


Fig. 9.7 Select File for Save As window

Enter the commands shown in Fig. 9.8 to solve $\frac{dy}{dx} = (1 + 4x)\sqrt{y}$, $y(0) = 1$ using the 4th order Runge–Kutta method with the step size $h = 0.5$ up to $x = 1.5$. Obtained result is the same as the result of hand analysis.

Fig. 9.8 Solving $\frac{dy}{dx} = (1 + 4x)\sqrt{y}$, $y(0) = 1$ using the 4th order Runge–Kutta method (step size $= 0.5$)



Let's decrease the step size to $h = 0.05$ and solve the equation up to $x = 5$. The code shown in Fig. 9.9 does this job. Output of this code is shown in Fig. 9.10.

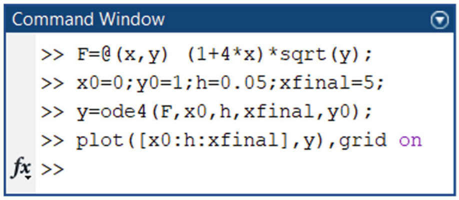


Fig. 9.9 Solving $\frac{dy}{dx} = (1 + 4x)\sqrt{y}$, $y(0) = 1$ using the 4th order Runge–Kutta method (step size $= 0.05$)

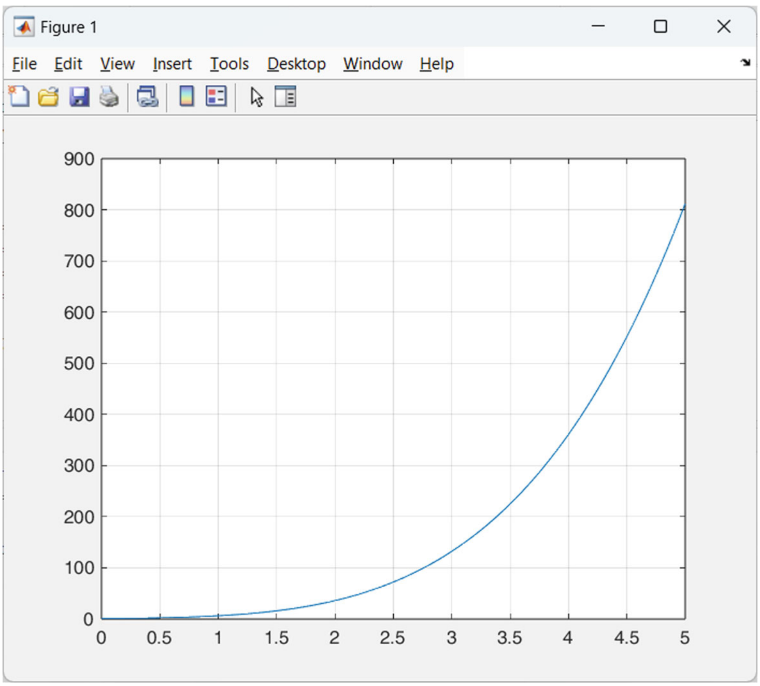


Fig. 9.10 Output of the code shown in Fig. 9.9

Example 9.46 4th order Runge–Kutta method can be used to solve $\ddot{y} + 2\dot{y} + y = \sin(t), y(0) = 1, \dot{y}(0) = 2$ as well. However, we need to convert the given differential equation to a system of linear equations.

$$y_1 = y \Rightarrow \dot{y}_1 = \dot{y} \Rightarrow \dot{y}_1 = y_2$$

$$y_2 = \dot{y} \Rightarrow \dot{y}_2 = \ddot{y}$$

$$\begin{aligned} y + 2\dot{y} + y = \sin(t) \Rightarrow \ddot{y} = -2\dot{y} - y + \sin(t) \Rightarrow \dot{y}_2 \\ = -2y_2 - y_1 + \sin(t) \end{aligned}$$

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -2y_2 - y_1 + \sin(t) \end{cases}, y_0 = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Following code solves the obtained system of first order differential equations using the 4th order Runge–Kutta method. Output of the code is shown in Fig. 9.11.

```
function runge_kutta_example()
    % Define the time span and initial conditions
    t0 = 0;       % Initial time
    tfinal = 10;      % Final time
    h = 0.01;    % Time step
    t = t0:h:tfinal; % Time vector

    % Initial conditions
    y1_0 = 1;     % Initial value of y (y1)
    y2_0 = 2;     % Initial value of y' (y2)
    y = [y1_0; y2_0]; % Initial state vector

    % Prepare to store results
    results = zeros(length(t), 2);
    results(1, :) = y; % Store the initial condition

    % Runge-Kutta method
    for i = 1:length(t)-1
        k1 = h * system_equations(t(i), y);
        k2 = h * system_equations(t(i) + h/2, y + k1/2);
        k3 = h * system_equations(t(i) + h/2, y + k2/2);
        k4 = h * system_equations(t(i) + h, y + k3);

        % Update the state vector
        y = y + (k1 + 2*k2 + 2*k3 + k4) / 6;
        results(i+1,:) = y'; % Store the results
    end

    % Plot results
    plot(t, results(:, 1));
    title('Solution of y'''' + 2y'' + y = sin(t)');
    xlabel('Time t');
    ylabel('y(t)');
    grid on;
end

function dydt = system_equations(t, y)
    % System of equations
    dydt = zeros(2, 1);
    dydt(1) = y(2);                          % y1' = y2
    dydt(2) = sin(t) - 2*y(2) - y(1); % y2' = sin(t) - 2y2 - y1
end
```

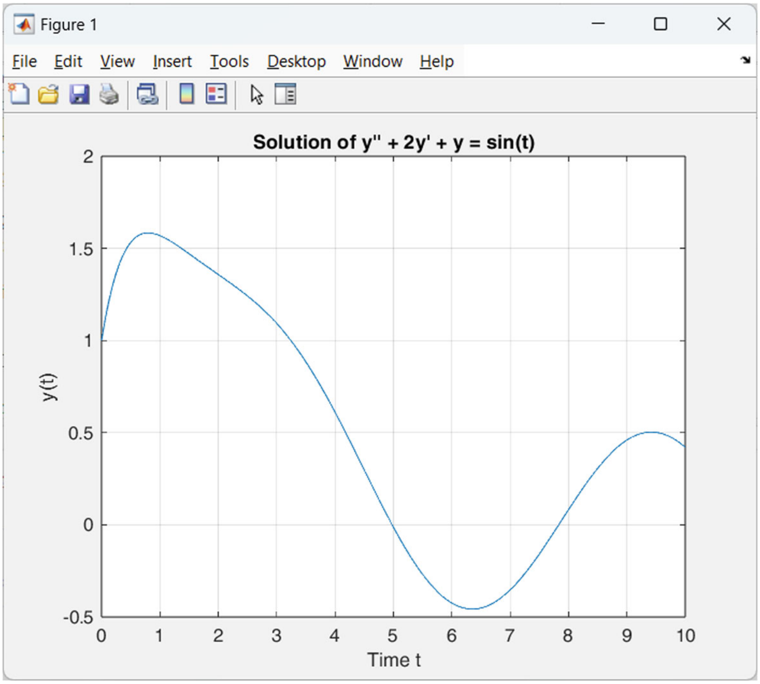


Fig. 9.11 Solution of $\ddot{y}+2\dot{y}+y=\sin(t)$, $y(0)=1$, $\dot{y}(0)=2$ for [0, 10] interval

Analytic solution of $\ddot{y}+2\dot{y}+y=\sin(t)$, $y(0)=1$, $\dot{y}(0)=2$ is $y(t)=\frac{3}{2}e^{-t}+\frac{7}{2}te^{-t}-\frac{1}{2}\cos(t)$ (why?). Graph of this function for [0, 10] interval is drawn with the aid of the code shown in Fig. 9.12. Output is shown in Fig. 9.13. You can compare different points of Fig. 9.11 with this figure in order to ensure that obtained result is correct.

```
Command Window
>> t=[0:0.01:10];
>> y=1.5*exp(-t)+3.5*t.*exp(-t)-0.5*cos(t);
>> plot(t,y),grid on
fx >>
```

Fig. 9.12 Plotting $y(t)=\frac{3}{2}e^{-t}+\frac{7}{2}te^{-t}-\frac{1}{2}\cos(t)$ on [0, 10] interval

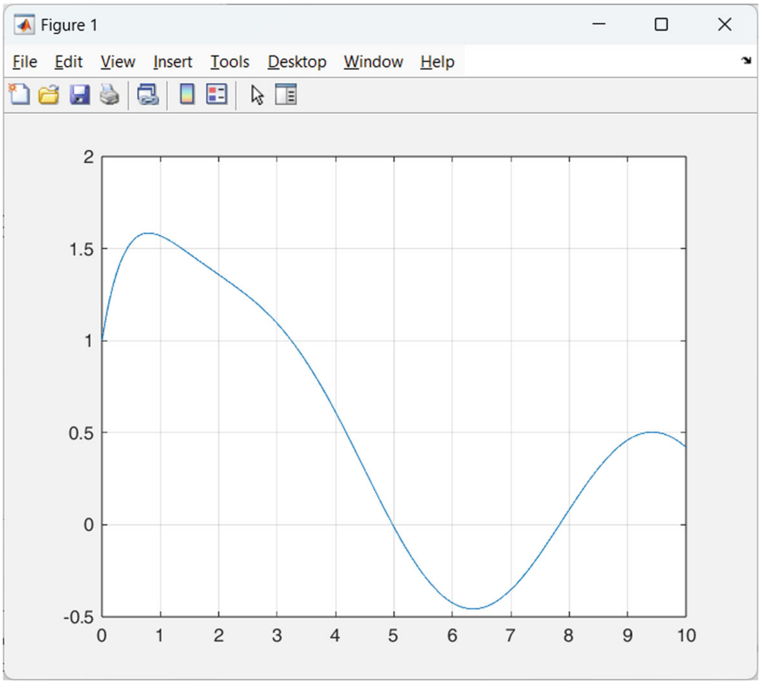


Fig. 9.13 Output of the code shown in Fig. 9.12

Example 9.47 In this example we will use the `ode45` command to solve $\ddot{\theta}(t) + 0.2\dot{\theta}(t) + 19.62\theta(t) = 0$, $\theta(0) = \frac{\pi}{18}$, $\dot{\theta}(0) = 0$. This equation models pendulum motion.

We need to convert the given second order differential equation to two first order equations:

$$\begin{cases} x_1 = \theta(t) \\ x_2 = \dot{\theta}(t) \end{cases} \Rightarrow \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -0.2x_1(t) - 19.62\sin(x_1(t)) \end{cases}, \quad \begin{bmatrix} x_1(0) = \frac{\pi}{18} \\ x_2(0) = 0 \end{bmatrix}$$

Now we can use the `ode45` command to solve this system. Use the `edit` command (Fig. 9.14) to open the Editor (Fig. 9.15).

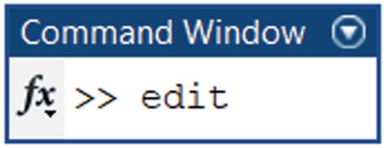


Fig. 9.14 `edit` command

Fig. 9.15 MATLAB Editor

Type the following code into the Editor (Fig. 9.16). After typing the code, press the Ctrl + S and save the file with desired name.

```
function dydt = odefun(t,y)
    dydt = [y(2);-0.2*y(2)-19.62*sin(y(1))];
end

% Define the time span and initial condition
tspan = [0 10];
y0 = [pi/18;0];

% Solve the ODE
[t,y] = ode45(@odefun, tspan, y0);

% Plot the solution
plot(t,y(:,1))
xlabel('Time')
ylabel('\theta(t)')
grid on
```

```
function dydt = odefun(t,y)
    dydt = [y(2);-0.2*y(2)-19.62*sin(y(1))];
end

% Define the time span and initial condition
tspan = [0 10];
y0 = [pi/18;0];

% Solve the ODE
[t,y] = ode45(@odefun, tspan, y0);

% Plot the solution
plot(t,y(:,1))
xlabel('Time')
ylabel('\theta(t)')
grid on
```

Fig. 9.16 Entering the code into the Editor

Press the F5 key of your keyboard or click the run button (Fig. 9.17) to run the code.

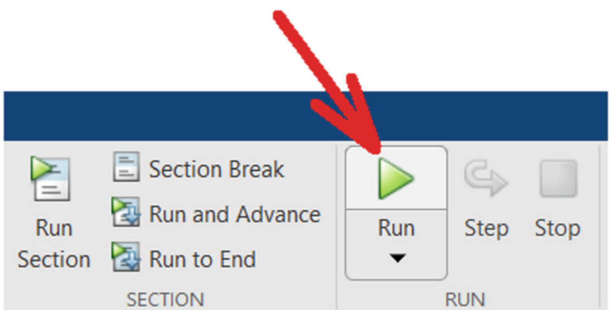


Fig. 9.17 The Run button

Output of the code is shown in Fig. 9.18.

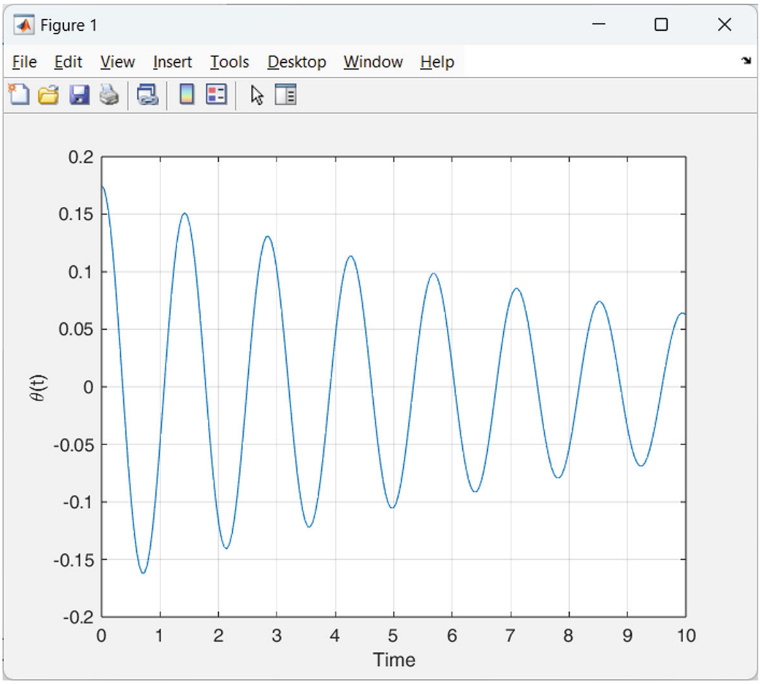


Fig. 9.18 Output of the code shown in Fig. 9.16

9.15 Problem Solving with MATLAB®

This section provides examples of how MATLAB can be employed to solve the types of problems encountered in this chapter.

Example 9.48 Figures 9.19, 9.20 and 9.21 illustrate various approaches to solve $\begin{cases} x + 7y = 5 \\ -x + 3y = 12 \end{cases}$.

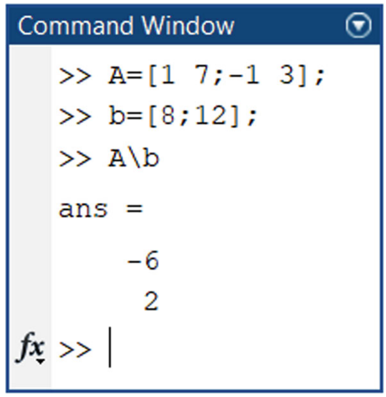


Fig. 9.19 The code for solving $\begin{cases} x + 7y = 5 \\ -x + 3y = 12 \end{cases}$

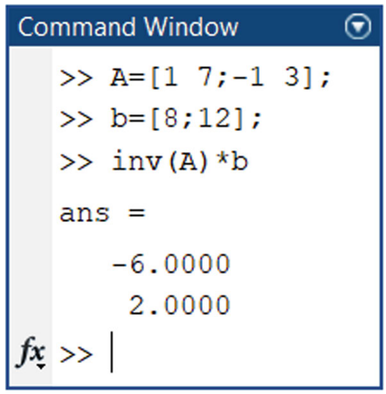


Fig. 9.20 The code for solving $\begin{cases} x + 7y = 5 \\ -x + 3y = 12 \end{cases}$

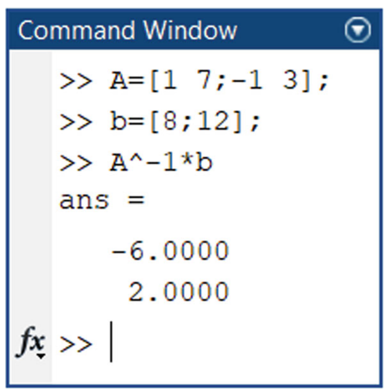


Fig. 9.21 The code for solving $\begin{cases} x + 7y = 5 \\ -x + 3y = 12 \end{cases}$

Example 9.49 The code shown in Fig. 9.22 solves $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 12$.

Fig. 9.22 The code for solving $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 12$

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+5*diff(y,t)+6*y==12;
>> sol=dsolve(ode)

sol =

C1*exp(-3*t) + C2*exp(-2*t) + 2

fx >>
```

Example 9.50 The code shown in Fig. 9.23 solves $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 12, y(0) = 2, \dot{y}(0) = 3$.

Fig. 9.23 The code for solving $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 12, y(0) = 2, \dot{y}(0) = 3$

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+5*diff(y,t)+6*y==12;
>> ic=[y(0)==2, subs(diff(y,t),0)==3];
>> sol = dsolve(ode,ic)

sol =

3*exp(-2*t) - 3*exp(-3*t) + 2

fx >>
```

Example 9.51 The code shown in Fig. 9.24 solves $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 12, y(0) = 2, \dot{y}(0) = 3$ and evaluates the solution at $t = 1$.

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+5*diff(y,t)+6*y==12;
>> ic=[y(0)==2,subs(diff(y,t),0)==3];
>> sol=dsolve(ode,ic)

sol =

3*exp(-2*t) - 3*exp(-3*t) + 2

>> eval(subs(sol,t,1))

ans =

    2.2566

fx >>
```

Fig. 9.24 Value of $y(1)$ for $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 12, y(0) = 2, \dot{y}(0) = 3$

Example 9.52 The code shown in Fig. 9.25 solves $\ddot{y}(t) + 4\dot{y}(t) + 7y(t) = 10, y(0) = 1, \dot{y}(0) = 1$ and plots the solution over the interval $[0, 4]$. Output of this code is shown in Fig. 9.26.

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+4*diff(y,t)+7*y==10;
>> ic=[y(0)==1,subs(diff(y,t),0)==1];
>> sol=dsolve(ode,ic)

sol =

(3^(1/2)*exp(-2*t)*sin(3^(1/2)*t))/21 - (3*exp(-2*t)*cos(3^(1/2)*t))/7 + 10/7

>> ezplot(sol,[0,4]), grid on
fx >>
```

Fig. 9.25 The code for plotting the solution of $\ddot{y}(t) + 4\dot{y}(t) + 7y(t) = 10, y(0) = 1, \dot{y}(0) = 1$ on $[0, 4]$

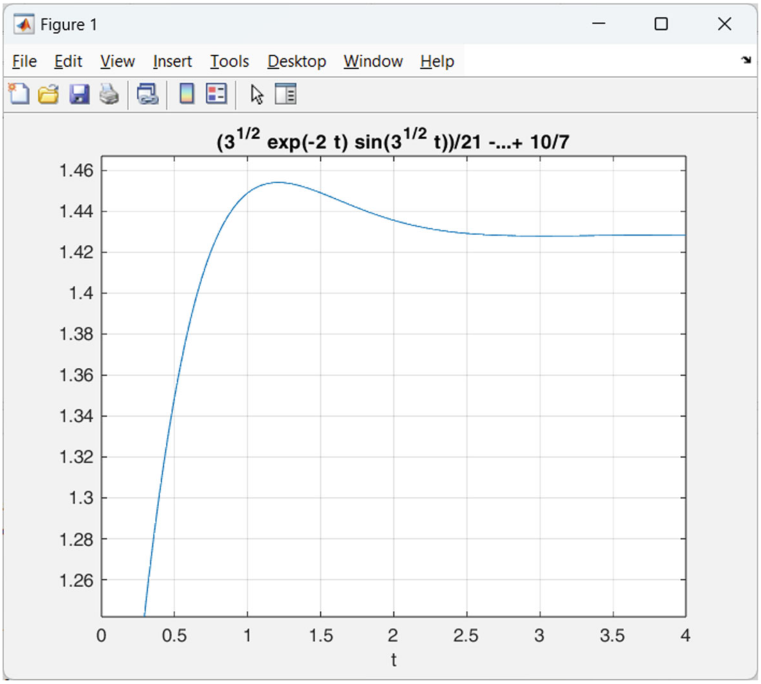


Fig. 9.26 Output of the code shown in Fig. 9.25

Example 9.53 The code shown in Fig. 9.27 solves $\ddot{y}(t) + 2\dot{y}(t) + y(t) = 6t + 7, y(0) = 5, \dot{y}(0) = 3$.

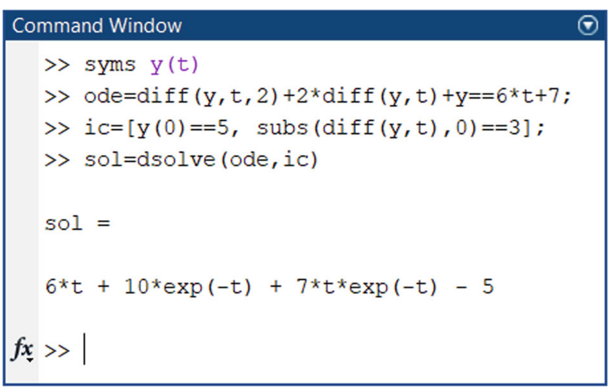


Fig. 9.27 The code for solving $\ddot{y}(t) + 2\dot{y}(t) + y(t) = 6t + 7, y(0) = 5, \dot{y}(0) = 3$

Example 9.54 The code shown in Fig. 9.28 solves $\ddot{y}(t) + 2\dot{y}(t) + y(t) = \delta(t), y(0) = 0, \dot{y}(0) = 0$. $\delta(t)$ shows the unit impulse function (see Sect. 10.3 in Chap. 10).

Fig. 9.28 The code for solving $\ddot{y}(t) + 2\dot{y}(t) + y(t) = \delta(t), y(0) = 0, \dot{y}(0) = 0$

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+2*diff(y,t)+y==dirac(t);
>> ic=[y(0)==0,subs(diff(y,t),0)==0];
>> dsolve(ode,ic)

ans =

(t*exp(-t)*sign(t))/2

fx >>
```

Remember that $\mathrm{sign}(x) = \begin{cases} +1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$. Therefore, for $t > 0$, $\frac{te^{-t}\mathrm{sign}(t)}{2} = \frac{te^{-t}}{2}$.

Example 9.55 The code shown in Fig. 9.29 solves $\ddot{y}(t) + 2\dot{y}(t) + y(t) = 6H(t), y(0) = 1, \dot{y}(0) = 2$. $H(t)$ shows the Heaviside unit step function (see Sect. 10.2 in Chap. 10).

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+2*diff(y,t)+y==6*heaviside(t);
>> ic=[y(0)==1,subs(diff(y,t),0)==2];
>> dsolve(ode,ic)

ans =

-exp(-t)*(3*sign(t) - 3*exp(t) - 3*exp(t)*sign(t) + 3*t*sign(t) + 2)

>> expand(ans)

ans =

3*sign(t) - 2*exp(-t) - 3*exp(-t)*sign(t) - 3*t*exp(-t)*sign(t) + 3

fx >>
```

Fig. 9.29 The code for solving $\ddot{y}(t) + 2\dot{y}(t) + y(t) = 6H(t), y(0) = 1, \dot{y}(0) = 2$

$\mathrm{sign}(x) = \begin{cases} +1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$. Therefore, for $t > 0$, $3\mathrm{sign}(t) - 2e^{-t} - 3e^{-t}\mathrm{sign}(t) - 3te^{-t}\mathrm{sign}(t) + 3 = 3 - 2e^{-t} - 3e^{-t} - 3te^{-t} + 3 = 6 - 5e^{-t} - 3te^{-t}$.

Example 9.56 The code shown in Fig. 9.30 solves $\dot{y}(t)^2 + y(t)^2 = 1$.

Fig. 9.30 The code for solving $\dot{y}(t)^2 + y(t)^2 = 1$

```
Command Window
>> syms y(t)
>> ode=diff(y,t)^2+y^2==1;
>> sol=dsolve(ode)

sol =

                                                     -1
                                                      1
(exp(C2*1i + t*1i)*(exp(- C2*2i - t*2i) + 1))/2
(exp(C1*1i - t*1i)*(exp(- C1*2i + t*2i) + 1))/2

>> simplify(expand(sol))

ans =

         -1
          1
cos(C2 + t)
cos(C1 - t)

fx >>
```

Example 9.57 The code shown in Fig. 9.31 solves $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = \sin(5t)$, $y(0) = 1$, $\dot{y}(0) = 2$.

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+5*diff(y,t)+6*y==sin(5*t);
>> ic=[y(0)==1, subs(diff(y,t),0)==2];
>> sol = dsolve(ode,ic)

sol =

(150*exp(-2*t))/29 - (141*exp(-3*t))/34 - (986^(1/2)*cos(5*t - atan(19/25)))/986

fx >>
```

Fig. 9.31 The code for solving $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = \sin(5t)$, $y(0) = 1$, $\dot{y}(0) = 2$

The vpa function takes the symbolic expression as input and the desired number of decimal places as the second argument. It then evaluates the expression numerically and displays the result with the specified precision. By using vpa, you can ensure that the numerical values in the solution are displayed with the desired level of accuracy.

To obtain a more precise numerical approximation, let's apply the vpa function to the calculated result (Fig. 9.32).

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+5*diff(y,t)+6*y==sin(5*t);
>> ic=[y(0)==1, subs(diff(y,t),0)==2];
>> sol = dsolve(ode,ic)

sol =

(150*exp(-2*t))/29 - (141*exp(-3*t))/34 - (986^(1/2)*cos(5*t - atan(19/25)))/986

>> vpa(sol,2)

ans =

5.2*exp(-2.0*t) - 0.032*cos(5.0*t - 0.65) - 4.1*exp(-3.0*t)

fx >>
```

Fig. 9.32 Utilize the vpa command to numerically evaluate the symbolic expression

Example 9.58 The code shown in Fig. 9.33 solves $\dot{y}(t) + 6y(t) = 20\sin\left(7t + \frac{\pi}{4}\right) + 30\sin\left(4t + \frac{\pi}{3}\right)$.

```
Command Window
>> syms y(t)
>> ode=diff(y,t)+6*y==20*sin(7*t+pi/4)+30*sin(4*t+pi/3);
>> sol=dsolve(ode)

sol =

C1*exp(-6*t) - (2*2^(1/2)*170^(1/2)*cos(7*t + atan(13)))/17 + cos(4*t - atan(((15*3^(1/2))/13 +

fx >>
```

Fig. 9.33 The code for solving $\dot{y}(t) + 6y(t) = 20\sin\left(7t + \frac{\pi}{4}\right) + 30\sin\left(4t + \frac{\pi}{3}\right)$

The result in Fig. 9.33 is too long. We can use the vpa command to obtain a more compact representation (Fig. 9.34).

```
Command Window
>> syms y(t)
>> ode=diff(y,t)+6*y==20*sin(7*t+pi/4)+30*sin(4*t+pi/3);
>> sol=dsolve(ode)

sol =

C1*exp(-6*t) - (2*2^(1/2)*170^(1/2)*cos(7*t + atan(13)))/17 + cos(4*t - atan(((15*3^(1/2))/13 +

>> sol1=vpa(sol,5)

sol1 =

4.1603*cos(4.0*t - 1.1116) - 2.1693*cos(7.0*t + 1.494) + C1*exp(-6.0*t)
fx >>
```

Fig. 9.34 Utilize the `vpa` command to numerically evaluate the symbolic expression

In Example 9.9, we derived the general solution to the differential equation $\dot{y} + 6y = 20\sin\left(7t + \frac{\pi}{4}\right) + 30\sin\left(4t + \frac{\pi}{3}\right)$, which is $y(t) = C_1e^{-6t} + 2.1693\sin(7t - 0.0768) + 4.1603\sin(4t + 0.4592)$. We will now compare our solution to the solution obtained from MATLAB (Fig. 9.34).

Remember that $\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$, $-\sin(x) = \sin(x + \pi)$ and $\sin(x - 2\pi) = \sin(x)$. Therefore,

$$\begin{aligned}
&C_1e^{-6t} + 4.1603\cos(4t - 1.1116) - 2.1693\cos(7t + 1.494)\\
&= C_1e^{-6t} + 4.1603\sin\left(4t - 1.1116 + \frac{\pi}{2}\right) - 2.1693\sin\left(7t + 1.494 + \frac{\pi}{2}\right)\\
&= C_1e^{-6t} + 4.1603\sin(4t - 1.1116 + 1.5708) - 2.1693\sin(7t + 1.494 + 1.5708)\\
&= C_1e^{-6t} + 4.1603\sin(4t + 0.4592) - 2.1693\sin(7t + 3.0648)\\
&= C_1e^{-6t} + 4.1603\sin(4t + 0.4592) + 2.1693\sin(7t + 3.0648 + \pi)\\
&= C_1e^{-6t} + 4.1603\sin(4t + 0.4592) + 2.1693\sin(7t + 6.2064)\\
&= C_1e^{-6t} + 4.1603\sin(4t + 0.4592) + 2.1693\sin(7t + 6.2064 - 2\pi)\\
&= C_1e^{-6t} + 4.1603\sin(4t + 0.4592) + 2.1693\sin(7t - 0.0768)
\end{aligned}$$

Therefore, we have verified that the MATLAB solution matches our solution.

A more visual approach would be to plot the difference between the two functions. If the difference is negligible across the domain of interest, it provides strong evidence that the two solutions are essentially identical. The code shown in Fig. 9.35 plots the absolute difference between $4.1603\cos(4t - 1.1116) - 2.1693\cos(7t + 1.494)$ and $4.1603\sin(4t + 0.4592) + 2.1693\sin(7t - 0.0768)$ over the interval $[0, 10\pi]$.

```
Command Window
>> MatlabSolution=4.1603*cos(4.0*t-1.1116)-2.1693*cos(7.0*t+1.494);
>> bookSolution=2.1693*sin(7*t-0.0768)+4.1603*sin(4*t+0.4592);
>> ezplot(abs(bookSolution-MatlabSolution),[0,10*pi]), grid on
fx >>
```

Fig. 9.35 Visualizing the discrepancy between the book and MATLAB solutions

Figure 9.36 shows the output of the code shown in Fig. 9.35.

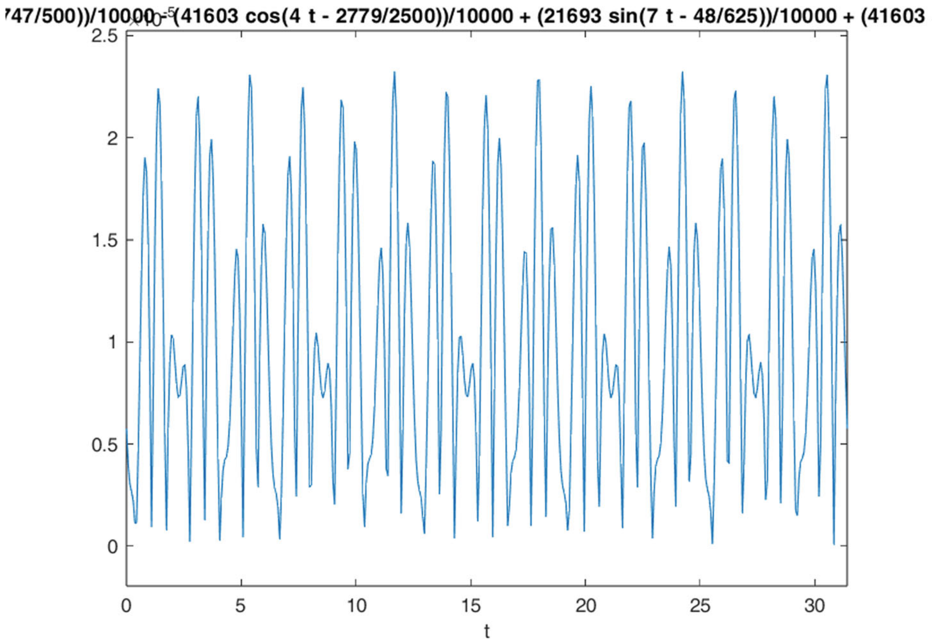


Fig. 9.36 Output of the code shown in Fig. 9.35

Let's try to find the maximum difference between the two functions. According to Fig. 9.37, the maximum difference is around 2.3×10^{-5} which is sufficiently small to conclude that the two solutions are virtually identical.

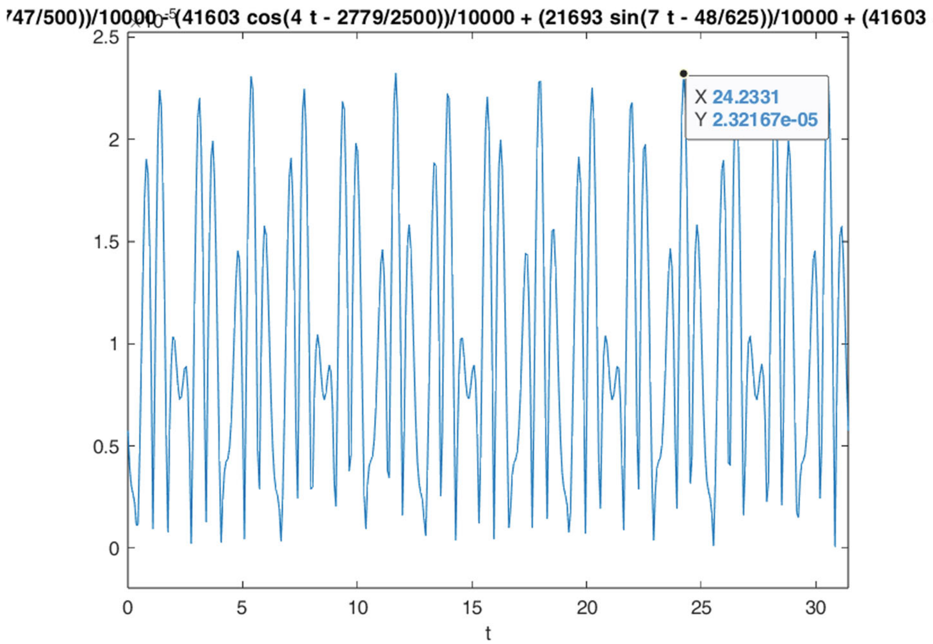


Fig. 9.37 Peak value is around 2.32×10^{-5}

Example 9.59 The code shown in Fig. 9.38 solves $y^{(3)}(t)+10\ddot{y}(t)+31\dot{y}(t)+30y(t) = e^{-5t}$, $y(0) = 1, \dot{y}(0) = 2, \ddot{y}(0) = 4$.

```
Command Window
>> syms y(t)
>> ode=diff(y,t,3)+10*diff(y,t,2)+31*diff(y,t)+30*y==exp(-5*t);
>> ic=[y(0)==1, subs(diff(y,t),0)==2, subs(diff(y,t,2),0)==4];
>> sol = dsolve(ode,ic)

sol =

(exp(-5*t)*(6*t - 513*exp(2*t) + 424*exp(3*t) + 125))/36

fx >>
```

Fig. 9.38 The code for solving $y^{(3)}(t) + 10\ddot{y}(t) + 31\dot{y}(t) + 30y(t) = e^{-5t}$, $y(0) = 1, \dot{y}(0) = 2, \ddot{y}(0) = 4$

Employ the `pretty` command to enhance the readability of the solution (Fig. 9.39).

```
Command Window
>> syms y(t)
>> ode=diff(y,t,3)+10*diff(y,t,2)+31*diff(y,t)+30*y==exp(-5*t);
>> ic=[y(0)==1, subs(diff(y,t),0)==2, subs(diff(y,t,2),0)==4];
>> sol = dsolve(ode,ic)

sol =

(exp(-5*t)*(6*t - 513*exp(2*t) + 424*exp(3*t) + 125))/36

>> pretty(sol)
exp(-5 t) (6 t - exp(2 t) 513 + exp(3 t) 424 + 125)
----------------------------------------------------
                         36

fx >>
```

Fig. 9.39 The `pretty` command enhances the readability of the solution

Example 9.60 The code shown in Fig. 9.40 solves $\dot{y}(t) + ty(t) = (t - 3)e^{-3t}$.

```
Command Window
>> syms y(t)
>> ode=diff(y,t)+t*y==(t-3)*exp(-3*t);
>> sol=dsolve(ode)

sol =

C1*exp(-t^2/2) + exp(-t^2/2)*exp(t^2/2 - 3*t)

fx >>
```

Fig. 9.40 The code for solving $\dot{y}(t) + ty(t) = (t - 3)e^{-3t}$

The solution obtained in Fig. 9.40 is not in its simplest form. To simplify the expression, we can utilize the `simplify` command (Fig. 9.41).

Fig. 9.41 Utilize the `simplify` command to simplify the solution

```
Command Window
>> syms y(t)
>> ode=diff(y,t)+t*y==(t-3)*exp(-3*t);
>> sol=dsolve(ode)

sol =

C1*exp(-t^2/2) + exp(-t^2/2)*exp(t^2/2 - 3*t)

>> simplify(sol)

ans =

exp(-3*t) + C1*exp(-t^2/2)

fx >>
```

Example 9.61 The code in Fig. 9.42 evaluates $T(s) = \frac{1}{s^2+4s+3}$ at $s = 2j$. According to Fig. 9.42, $T(s) = \frac{1}{(2j)^2+4\times 2j+3} = -\frac{1}{65} - j\frac{8}{65} = -0.0154 - 0.1234j$.

Fig. 9.42 Evaluates $T(s) = \frac{1}{s^2+4s+3}$ at $s = 2j$

```
Command Window
>> syms s
>> T=1/(s^2+4*s+3);
>> A=subs(T,2j)

A =

- 1/65 - 8i/65

>> eval(A)

ans =

  -0.0154 - 0.1231i

fx >>
```

The code in Fig. 9.43 calculates the modulus and principle argument of $T(2j)$. According to Fig. 9.43, $T(2j) = 0.1240e^{-j1.6952}$.

Fig. 9.43 Calculation of modulus and principle argument of $T(s) = \frac{1}{s^2+4s+3}$ at $s = 2j$

```
Command Window
>> syms s
>> T=1/(s^2+4*s+3);
>> A=subs(T,2j)

A =

- 1/65 - 8i/65

>> eval(A)

ans =

  -0.0154 - 0.1231i

>> eval(abs(A))

ans =

    0.1240

>> eval(angle(A))

ans =

   -1.6952

fx >> 
```

Example 9.62 The code in Fig. 9.44 calculates $e^{\begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix} t}$.

```
Command Window
>> syms t
>> expm([-1 6;2 3]*t)

ans =

[(3*exp(-3*t))/4 + exp(5*t)/4, (3*exp(5*t))/4 - (3*exp(-3*t))/4]
[     exp(5*t)/4 - exp(-3*t)/4,      exp(-3*t)/4 + (3*exp(5*t))/4]
fx >>
```

Fig. 9.44 Calculation of $e^{\begin{bmatrix}-1 & 6\\ 2 & 3\end{bmatrix}t}$

You can employ the `pretty` command to enhance the readability of the solution (Fig. 9.45).

```
Command Window
>> syms t
>> expm([-1 6;2 3]*t)

ans =

[(3*exp(-3*t))/4 + exp(5*t)/4, (3*exp(5*t))/4 - (3*exp(-3*t))/4]
[     exp(5*t)/4 - exp(-3*t)/4,      exp(-3*t)/4 + (3*exp(5*t))/4]

>> pretty(ans)
/ exp(-3 t) 3   exp(5 t)   exp(5 t) 3   exp(-3 t) 3 \
\
| ----------- + --------, ---------- - ----------- |
|      4            4          4             4      |
|                                                   |
|  exp(5 t)   exp(-3 t)     exp(-3 t)   exp(5 t) 3  |
|  -------- - ---------,    --------- + ----------  |
\      4          4             4            4      /

fx >>
```

Fig. 9.45 The `pretty` command enhances the readability of the solution

Example 9.63 The code in Fig. 9.46 calculates $e^{\begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix} t}$ at $t = 1.2$.

```
Command Window
>> syms t
>> A=[-1 6;2 3];
>> B=expm(A*t)

B =

[(3*exp(-3*t))/4 + exp(5*t)/4, (3*exp(5*t))/4 - (3*exp(-3*t))/4]
[    exp(5*t)/4 - exp(-3*t)/4,     exp(-3*t)/4 + (3*exp(5*t))/4]

>> C=subs(B,t,1.2)

C =

[exp(6)/4 + (3*exp(-18/5))/4, (3*exp(6))/4 - (3*exp(-18/5))/4]
[    exp(6)/4 - exp(-18/5)/4,     (3*exp(6))/4 + exp(-18/5)/4]

fx >> 
```

Fig. 9.46 Evaluation of $e^{\begin{bmatrix} -1 & 6 \\ 2 & 3 \end{bmatrix} t}$ at $t = 1.2$

You can use `eval` or `vpa` functions to compute the numerical value of symbolic matrix C (Fig. 9.47).

```
Command Window
>> syms t
>> A=[-1 6;2 3];
>> B=expm(A*t)

B =

[(3*exp(-3*t))/4 + exp(5*t)/4, (3*exp(5*t))/4 - (3*exp(-3*t))/4]
[    exp(5*t)/4 - exp(-3*t)/4,     exp(-3*t)/4 + (3*exp(5*t))/4]

>> C=subs(B,t,1.2)

C =

[exp(6)/4 + (3*exp(-18/5))/4, (3*exp(6))/4 - (3*exp(-18/5))/4]
[    exp(6)/4 - exp(-18/5)/4,     (3*exp(6))/4 + exp(-18/5)/4]

>> eval(C)

ans =

  100.8777  302.5511
  100.8504  302.5784

>> vpa(C,7)

ans =

[100.8777, 302.5511]
[100.8504, 302.5784]

fx >>
```

Fig. 9.47 Numerical value of $e^{\begin{bmatrix}-1 & 6\\ 2 & 3\end{bmatrix}t}$ at $t = 1.2$

Example 9.64 The code in Fig. 9.48 solves $\begin{cases} \dot{x}(t) = 3x(t) - 2y(t) \\ \dot{y}(t) = 4x(t) - y(t) \end{cases} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$.

```
Command Window
>> syms x(t) y(t)
>> eq1=diff(x,t)==3*x-2*y;
>> eq2=diff(y,t)==4*x-y;
>> ic=[x(0)==1,y(0)==7];
>> [xSol(t),ySol(t)]=dsolve([eq1,eq2],ic)

xSol(t) =

cos(2*t)*exp(t) - 6*sin(2*t)*exp(t)


ySol(t) =

7*cos(2*t)*exp(t) - 5*sin(2*t)*exp(t)

fx >>
```

Fig. 9.48 The code for solving $\begin{cases} \dot{x}(t) = 3x(t) - 2y(t) \\ \dot{y}(t) = 4x(t) - y(t) \end{cases} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

Example 9.65 The code in Fig. 9.49 solves $\begin{cases} \dot{x}(t) = 7x(t) - y(t) \\ \dot{y}(t) = 4x(t) + 3y(t) \end{cases} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

```
Command Window
>> syms x(t) y(t)
>> eq1=diff(x,t)==7*x-y;
>> eq2=diff(y,t)==4*x+3*y;
>> ic=[x(0)==1,y(0)==2];
>> [xSol(t),ySol(t)]=dsolve([eq1,eq2],ic)

xSol(t) =

exp(5*t)


ySol(t) =

2*exp(5*t)

fx >>
```

Fig. 9.49 The code for solving $\begin{cases} \dot{x}(t) = 7x(t) - y(t) \\ \dot{y}(t) = 4x(t) + 3y(t) \end{cases} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Example 9.66 The following code numerically solves

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -6x_1(t) - 5x_2(t) + e^{-5t} \end{cases} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

over the [0, 3] interval using the `ode45` function.

```
% Define the system of differential equations
function dydt = system(t,y)
    dydt = [y(2);-6*y(1)-5*y(2)+exp(-5*t)];
end

% Initial conditions
y0 = [1; 0]; % Initial values for x and y

% Time span
tspan = [0 3];

% Solve the system numerically
[t,y] = ode45(@system, tspan, y0);

% Plot the solutions
plot(t, y(:,1), 'b', t, y(:,2), 'r');
xlabel('Time');
ylabel('State Variables');
legend('x_1(t)', 'x_2(t)');
grid on
```

Use the `edit` command (Fig. 9.50) to open the Editor (Fig. 9.51).

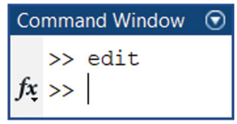


Fig. 9.50 `edit` command

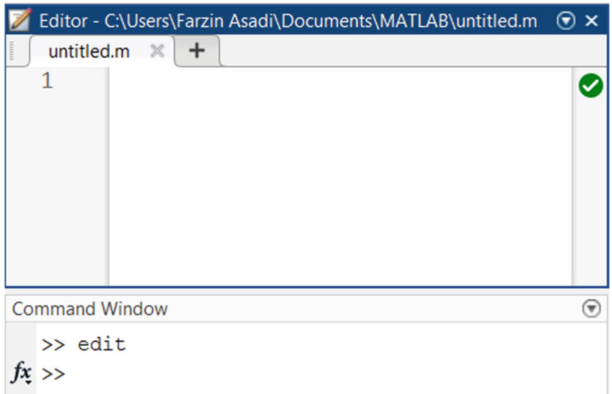


Fig. 9.51 MATLAB Editor

Type the given code into the Editor (Fig. 9.52). Press the Ctrl + S to save it.

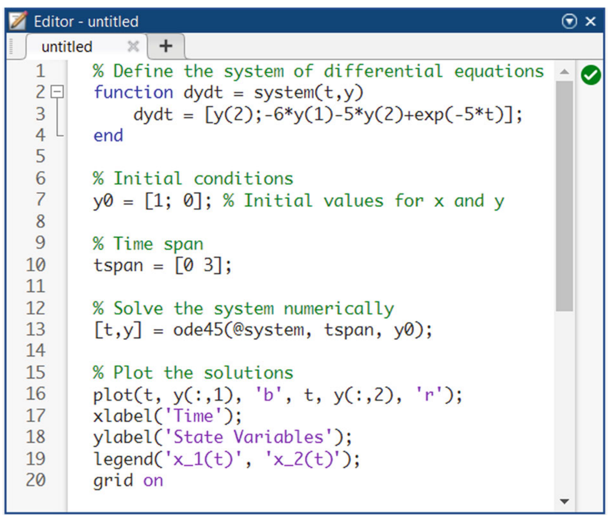


Fig. 9.52 Entering the code into MATLAB Editor

Press the F5 key of your keyboard or click the run button (Fig. 9.53) to run the code.

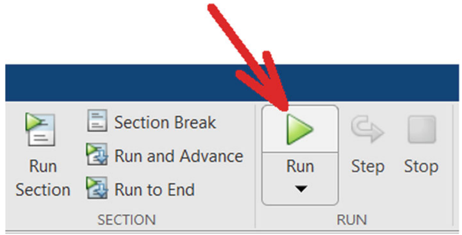


Fig. 9.53 The Run button

Output of the code is shown in Fig. 9.54.

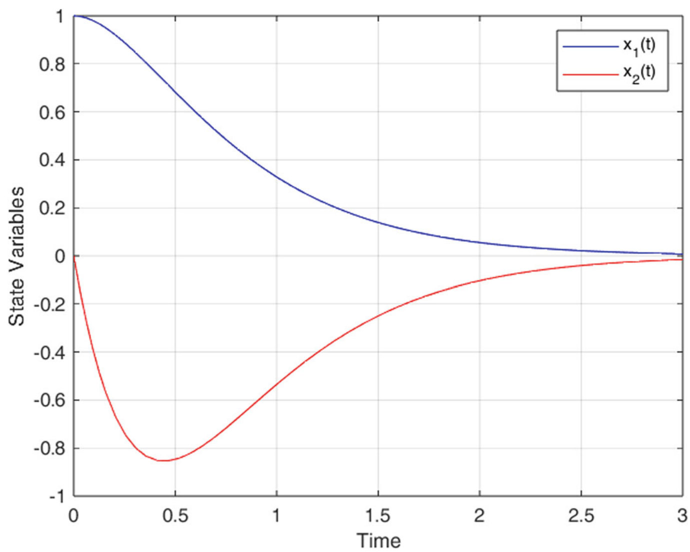


Fig. 9.54 Output of the code shown in Fig. 9.52

Example 9.67 The code in Fig. 9.55 calculates the transfer function, i.e., $T(s) = \frac{Y(s)}{U(s)}$, associated with $\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 5 \\ -6 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]u(t) \end{cases}$. According to Fig. 9.55, $T(s) = \frac{Y(s)}{U(s)} = \frac{5s+4}{s^2+2s+3}$.

Fig. 9.55 Calculation of transfer function associated with the given equation

```
Command Window
>> A=[0 1;-3 -2];
>> B=[5;-6];
>> C=[1 0];
>> D=0;
>> [num,den]=ss2tf(A,B,C,D);
>> T=tf(num,den)

T =

     5 s + 4
  -------------
  s^2 + 2 s + 3

Continuous-time transfer function.
Model Properties
fx >>
```

Example 9.68 The code in Fig. 9.56 simplifies $\frac{s^2+2s+1}{s^3+3s^2+2s}$.

Fig. 9.56 The code for simplifying $\frac{s^2+2s+1}{s^3+3s^2+2s}$

```
Command Window
>> syms s
>> H=(s^2+2*s+1)/(s^3+3*s^2+2*s);
>> simplify(H)

ans =

(s + 1)/(s*(s + 2))

fx >>
```

Let's verify the obtained result: $\frac{s^2+2s+1}{s^3+3s^2+2s} = \frac{(s+1)^2}{s(s^2+3s+2)} = \frac{(s+1)^2}{s(s+1)(s+2)} = \frac{s+1}{s(s+2)}$.

Example 9.69 The code in Fig. 9.57 calculates the partial fraction decomposition of $\frac{s^2+2s+1}{s^3+3s^2+2s}$.

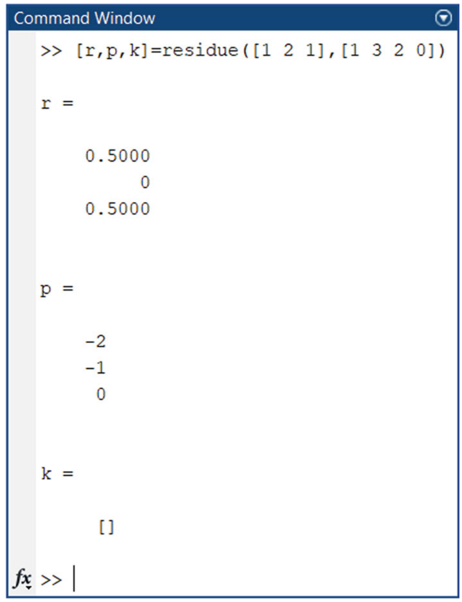


Fig. 9.57 Partial fraction decomposition of $\frac{s^2+2s+1}{s^3+3s^2+2s}$

According to Fig. 9.57, $\frac{s^2+2s+1}{s^3+3s^2+2s} = \frac{0.5}{s+2} + \frac{0}{s+1} + \frac{0.5}{s}$.

Example 9.70 The code in Fig. 9.58 calculates the partial fraction decomposition of $\frac{12s^2+57s+27}{s^3+9s^2+15s+7}$.

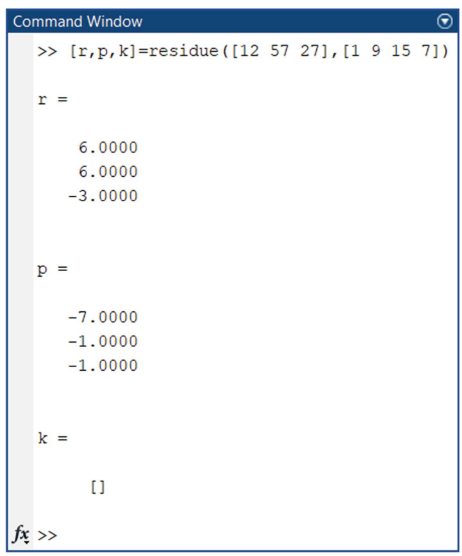


Fig. 9.58 Partial fraction decomposition of $\frac{12s^2+57s+27}{s^3+9s^2+15s+7}$

According to Fig. 9.58, $\frac{12s^2+57s+27}{s^3+9s^2+15s+7} = \frac{6}{s+7} + \frac{6}{s+1} + \frac{-3}{(s+1)^2}$.

Example 9.71 The code in Fig. 9.59 calculates $\frac{6}{s+7} + \frac{6}{s+1} - \frac{3}{(s+1)^2}$. According to Fig. 9.59, $\frac{6}{s+7} + \frac{6}{s+1} - \frac{3}{(s+1)^2} = 3 \times \frac{4s+19s+9}{(s+7)(s+1)^2}$.

```
Command Window
>> syms s
>> H=6/(s+7)+6/(s+1)-3/(s+1)^2;
>> simplify(H)

ans =

(3*(4*s^2 + 19*s + 9))/((s + 1)^2*(s + 7))

>> pretty(ans)
    2
(4 s  + 19 s + 9) 3
-------------------
          2
  (s + 1)  (s + 7)

fx >>
```

Fig. 9.59 Calculation of $\frac{6}{s+7} + \frac{6}{s+1} - \frac{3}{(s+1)^2}$

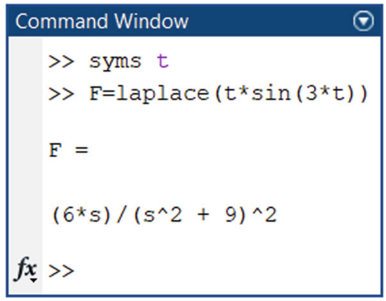
Example 9.72 The code in Fig. 9.60 calculates the Laplace transform of $t.H(t)$, where $H(t)$ denotes the unit step function.

```
Command Window
>> syms t
>> F=laplace(t)

F =

1/s^2

fx >>
```

Fig. 9.60 Laplace transform of $t.H(t)$

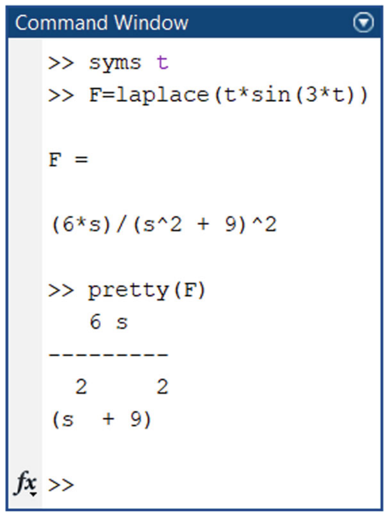
Example 9.73 The code in Fig. 9.61 calculates the Laplace transform of $t \cdot \sin(3t) \cdot H(t)$, where $H(t)$ denotes the unit step function.

Fig. 9.61 Laplace transform of $t \cdot \sin(3t) \cdot H(t)$



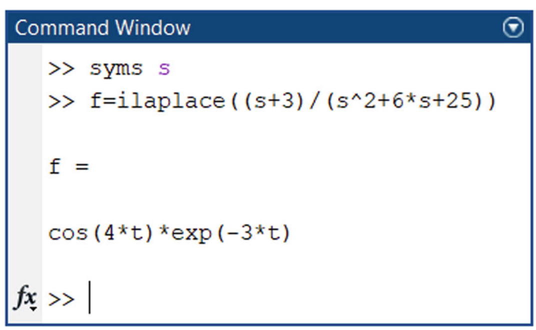
You can employ the `pretty` command to enhance the readability of the solution (Fig. 9.62).

Fig. 9.62 The `pretty` command enhances the readability of the solution



Example 9.74 The code in Fig. 9.63 calculates the inverse Laplace transform of $\frac{s+3}{s^2+6s+25}$.

Fig. 9.63 Inverse Laplace transform of $\frac{s+3}{s^2+6s+25}$



Example 9.75 The code in Fig. 9.64 calculates the inverse Laplace transform of $\frac{s+6}{s^2+s+1}$.

```
Command Window
>> syms s
>> f=ilaplace((s+6)/(s^2+s+1))

f =

exp(-t/2)*(cos((3^(1/2)*t)/2) + (11*3^(1/2)*sin((3^(1/2)*t)/2))/3)

fx >>
```

Fig. 9.64 Inverse Laplace transform of $\frac{s+6}{s^2+s+1}$

Employ the `vpa` function to obtain a numerical approximation of the symbolic function `f` (Fig. 9.65).

```
Command Window
>> syms s
>> f=ilaplace((s+6)/(s^2+s+1))

f =

exp(-t/2)*(cos((3^(1/2)*t)/2) + (11*3^(1/2)*sin((3^(1/2)*t)/2))/3)

>> vpa(f,4)

ans =

exp(-0.5*t)*(cos(0.866*t) + 6.351*sin(0.866*t))

fx >>
```

Fig. 9.65 Numerical value of inverse Laplace transform of $\frac{s+6}{s^2+s+1}$

References for Further Study

1. Advanced Engineering Mathematics (10th edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8th edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

10 The Laplace Transform

10.1 Introduction

The Laplace transform is a mathematical tool that converts a function of a real variable (often representing time) into a function of a complex variable (often denoted as s). This transformation is particularly useful for solving differential equations, as it transforms differential equations into algebraic equations, which are often easier to solve.

This chapter begins with a review of the Laplace transform and its applications. The second part demonstrates MATLAB's application to the problems discussed.

10.2 Unit Step Function

The unit step function (Fig. 10.1), also known as the Heaviside step function, is a mathematical function that is zero for negative arguments and one for positive arguments. It is often denoted by the symbol $u(t)$ or $H(t)$. Here's the formal definition: $H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$.

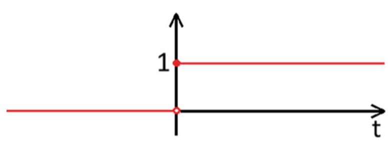


Fig. 10.1 Unit step function

Example 10.1 When $0 < a < b$ the function shown in Fig. 10.2 can be written as $f(t) = k(H(t-a) - H(t-b))$.

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_10

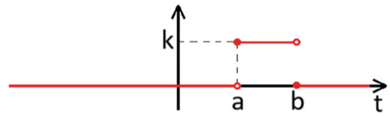


Fig. 10.2 Function $f(t)$

10.3 Unit Impulse Function

The unit impulse function, also known as the Dirac delta function, is an idealized mathematical function that is zero everywhere except at zero, where it is infinitely high, and its integral over the entire real line is equal to one. The Dirac delta function is typically denoted by the symbol $\delta(t)$.

The functions shown in Figs. 10.3 and 10.4 can be considered as approximations of the Dirac delta function. While the true Dirac delta function is an idealized concept with infinite height and zero width, these functions represent practical approximations with finite height and width.

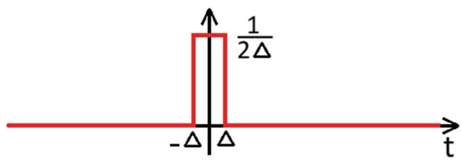


Fig. 10.3 Rectangular pulse approximation of Dirac delta function ($\Delta \to 0$)

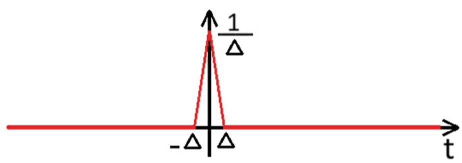


Fig. 10.4 Triangular pulse approximation of Dirac delta function ($\Delta \to 0$)

Sifting property is the most important property of unit impulse function. When the unit impulse function is multiplied by another function and integrated, it effectively "sifts out" the value of the other function at the point where the impulse is located. The sifting property of the Dirac delta function is mathematically expressed as:

$$\int_{-\infty}^{+\infty} f(t)\delta(t-a)dt = f(a)$$

Sifting property is crucial in many applications. While the unit impulse function is an idealized concept, it's a powerful tool for analyzing systems and signals, particularly in the context of linear systems theory.

10.4 Laplace Transform

The Laplace transform is a mathematical tool that converts a function of a real variable (often representing time) into a function of a complex variable (often representing frequency). This transformation can be useful for solving differential equations, particularly those that arise in engineering and physics.

The Laplace transform of a function $f(t)$ is typically denoted as $F(s)$ or $\mathcal{L}(f(t))$. The Laplace transform of a function $f(t)$ is defined as (note that s is a complex variable, i.e., $s = \sigma + j\omega$):

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty f(t)e^{-st}dt$$

The inverse Laplace transform is the reverse operation of the Laplace transform. It takes a function in the s-domain (the complex frequency domain) and transforms it back into a function in the time domain. Fortunately, a Laplace transform table can be employed to determine the inverse Laplace transform of most functions.

Example 10.2 Determine the Laplace transform of the Heaviside step function.

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$H(s) = \mathcal{L}(H(t)) = \int_0^\infty H(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \left.\frac{e^{-(\sigma+j\omega)t}}{-(\sigma+j\omega)}\right|_0^\infty$$

When $\sigma > 0$, $\lim_{t\to\infty} e^{-(\sigma+j\omega)t} = 0 \Rightarrow \lim_{t\to\infty} \frac{e^{-(\sigma+j\omega)t}}{-(\sigma+j\omega)} = 0$. Therefore,

$$\left.\frac{e^{-(\sigma+j\omega)t}}{-(\sigma+j\omega)}\right|_0^\infty = 0 - \frac{e^{-(\sigma+j\omega)0}}{-(\sigma+j\omega)} = \frac{1}{(\sigma+j\omega)} = \frac{1}{s}$$

Example 10.3 Determine the Laplace transform of $f(t) = tH(t)$. Note that $H(t)$ shows the Heaviside step function.

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty tH(t)e^{-st}dt = \int_0^\infty te^{-st}dt$$

Tabular integration (Fig. 10.5) can be used to calculate $\int_0^\infty te^{-st}dt$.

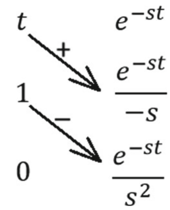


Fig. 10.5 Calculation of $\int te^{-st}dt$

According to Fig. 10.5, $\int_0^{\infty} te^{-st}dt = (\frac{t}{-s} - \frac{1}{s^2})e^{-st}$. Therefore,

$$\int_0^{\infty} te^{-st}dt = \left(\frac{t}{-s} - \frac{1}{s^2}\right)e^{-st}\Big|_0^{\infty} = \lim_{t\to\infty}\left(\frac{t}{-s} - \frac{1}{s^2}\right)e^{-st} - \left(\frac{0}{-s} - \frac{1}{s^2}\right)e^{-s0} = \lim_{t\to\infty}\left(\frac{t}{-s} - \frac{1}{s^2}\right)e^{-st} + \frac{1}{s^2}$$
$$= \lim_{t\to\infty}\left(\frac{t}{-se^{st}} - \frac{1}{s^2e^{st}}\right) + \frac{1}{s^2}$$
$$= \lim_{t\to\infty}\left(\frac{t}{-se^{st}}\right) - \lim_{t\to\infty}\left(\frac{1}{s^2e^{st}}\right) + \frac{1}{s^2}$$

Remember that s is a complex variable, i.e., $s = \sigma + j\omega$. When $Re(s) = \sigma > 0$, $\lim_{t\to\infty}\left(\frac{t}{-se^{st}}\right) - \lim_{t\to\infty}\left(\frac{1}{s^2e^{st}}\right) = 0 - 0 = 0$. Note that $\lim_{t\to\infty}\left(\frac{t}{-se^{st}}\right) = 0$ since exponential function grows much faster than linear functions. $\lim_{t\to\infty}\left(\frac{1}{s^2e^{st}}\right) = 0$ since numerator is a constant divided by infinity. Therefore,

$$F(s) = \mathcal{L}(f(t)) = \int_0^{\infty} te^{-st}dt = \frac{1}{s^2}$$

10.5 Commonly Used Laplace Transforms

Table 10.1 provides a compendium of common Laplace transform pairs, essential for solving differential equations and analyzing linear systems.

10.6 Inverse Laplace Transform

Partial fraction decomposition (Chap. 7) and a table of Laplace transform pairs (Table 10.1) can help us to calculate the inverse Laplace transform of most functions that appear in engineering. Let's study some numeric examples.

Example 10.4 Determine the inverse Laplace transform of $\frac{5s^2+s+4}{(s+4)(s^2+4)}$.

Table 10.1 Laplace transform of commonly used functions

Signal	$f(t)$	$F(s)$
Impulse	$\delta(t)$	1
Step function	$H(t)$	$\frac{1}{s}$
Ramp	$tH(t)$	$\frac{1}{s^2}$
Exponential	$e^{-\alpha t}H(t)$	$\frac{1}{s+\alpha}$
Damped ramp	$te^{-\alpha t}H(t)$	$\frac{1}{(s+\alpha)^2}$
Sine	$\sin(\omega t)H(t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)H(t)$	$\frac{s}{s^2+\omega^2}$
Damped sine	$e^{-\alpha t}\sin(\omega t)H(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
Damped cosine	$e^{-\alpha t}\cos(\omega t)H(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

$$\frac{5s^2+s+4}{(s+4)(s^2+4)} = \frac{4}{s+4} + \frac{s-3}{s^2+4} = \frac{4}{s+4} + \frac{s}{s^2+4} + \frac{-3}{s^2+4}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{5s^2+s+4}{(s+4)(s^2+4)}\right) &= \mathcal{L}^{-1}\left(\frac{4}{s+4}\right) + \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \mathcal{L}^{-1}\left(\frac{-3}{s^2+4}\right) \\
&= 4\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) + \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \frac{-3}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) \\
&= \left(4e^{-4t} + \cos(2t) - \frac{3}{2}\sin(2t)\right) \cdot H(t)
\end{aligned}$$

Example 10.5 Determine the inverse Laplace transform of $\frac{10(s+2)}{s(s^2+2s+2)}$.

$$\frac{10(s+2)}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

$$A = \left.\frac{10(s+2)}{s^2+2s+2}\right|_{s=0} = \frac{20}{2} = 10$$

$$\frac{10(s+2)}{s\left(s^2+2s+2\right)} = \frac{10}{s} + \frac{Bs+C}{s^2+2s+2} \Rightarrow \frac{10(s+2)}{s\left(s^2+2s+2\right)} - \frac{10}{s} = \frac{Bs+C}{s^2+2s+2}$$

$$\frac{10(s+2)}{s\left(s^2+2s+2\right)} - \frac{10\left(s^2+2s+2\right)}{s\left(s^2+2s+2\right)} = \frac{-10s^2-10s}{s(s^2+2s+2)} = \frac{-10s-10}{s^2+2s+2} = \frac{Bs+C}{s^2+2s+2}$$

$$-10s-10 = Bs+C \Rightarrow B=-10,\ C=-10$$

$$\frac{10(s+2)}{s(s^2+2s+2)} = \frac{10}{s} + \frac{-10s-10}{s^2+2s+2} = \frac{10}{s} - \frac{10(s+1)}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}\left(\frac{10(s+2)}{s\left(s^2+2s+2\right)}\right) = 10(\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{(s+1)}{(s+1)^2+1}\right) = 10(1-e^{-t}\cos(t))H(t)$$

Example 10.6 Determine the inverse Laplace transform of $\frac{s+1}{s^2-6s+13}$.

$$\mathcal{L}^{-1}\left(\frac{s+1}{s^2-6s+13}\right) = \mathcal{L}^{-1}\left(\frac{s-3+4}{(s-3)^2+4}\right) = \mathcal{L}^{-1}\left(\frac{s-3}{(s-3)^2+4}\right)$$
$$+\,2\mathcal{L}^{-1}\left(\frac{2}{(s-3)^2+4}\right) = e^{3t}\cos(2t)H(t) + 2e^{3t}\sin\,(2t)\cdot H(t)$$
$$= e^{3t}(\cos(2t) + 2\sin\,(2t)).H(t)$$

Example 10.7 Determine the inverse Laplace transform of $\frac{10(s+2)}{s(s^2+2s+2)}$

$$\mathcal{L}^{-1}\left(\frac{10(s+2)}{s(s^2+2s+2)}\right) = \mathcal{L}^{-1}\left(\frac{10(s+2)}{s((s+1)^2+1)}\right) = \mathcal{L}^{-1}\left(\frac{A}{s}\right) + \mathcal{L}^{-1}\left(\frac{B(s+1)+C}{(s+1)^2+1}\right)$$

$$A = \left.\frac{10(s+2)}{(s+1)^2+1}\right|_{s=0} = 10$$

$$\frac{10(s+2)}{s(s^2+2s+2)} - \frac{10}{s} = \frac{10(s+2)}{s(s^2+2s+2)} - \frac{10(s^2+2s+2)}{s(s^2+2s+2)}$$
$$= \frac{10s+20-10s^2-20s-20}{s(s^2+2s+2)} = \frac{-10s-10s^2}{s(s^2+2s+2)} = \frac{-10-10s}{s^2+2s+2}$$
$$= \frac{-10(s+1)+0}{(s+1)^2+1} \Rightarrow B = -10, C = 0$$

$$\mathcal{L}^{-1}\left(\frac{10(s+2)}{s(s^2+2s+2)}\right) = \mathcal{L}^{-1}\left(\frac{10}{s}\right) - \mathcal{L}^{-1}\left(\frac{10(s+1)}{(s+1)^2+1}\right) = (10-10e^{-t}\cos(t))H(t).$$

10.7 Properties of Laplace Transform

The followings are the important properties of Laplace transform. Note that $f(t) \leftrightarrow F(s), f_1(t) \leftrightarrow F_1(s), f_2(t) \leftrightarrow F_2(s), H(t)$ shows the Heaviside step function, and $k_1, k_2 \in \mathbb{R}$.

1. $k_1 f(t) \leftrightarrow k_1 F(s)$
2. $k_1 f_1(t) + k_2 f_2(t) \leftrightarrow k_1 F_1(s) + k_2 F_2(s)$

3. $for\, a > 0, f(at) \leftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right)$
4. $f(t - t_0)H(t - t_0) \leftrightarrow e^{-t_0 s}F(s)$
5. $e^{-at}f(t) \leftrightarrow F(s + a)$
6. $\dot{f}(t) = \frac{df}{dt} \leftrightarrow sF(s) - f(0)$
7. $\ddot{f}(t) = \frac{d^2f}{dt^2} \leftrightarrow s^2F(s) - sf(0) - \dot{f}(0)$
8. $\dddot{f}(t) = \frac{d^3f}{dt^3} \leftrightarrow s^3F(s) - s^2f(0) - s\dot{f}(0) - \ddot{f}(0)$
9. $\int\limits_0^t f(\tau)d\tau \leftrightarrow \frac{F(s)}{s}$
10. $t.f(t) \leftrightarrow -\frac{d}{ds}(F(s)) = -F'(s)$
11. $\frac{f(t)}{t} \leftrightarrow \int\limits_s^\infty F(\sigma)d\sigma$
12. $\lim\limits_{t\to 0^+} f(t) = \lim\limits_{s\to\infty} s.F(s)$
13. $\lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} s.F(s)$
14. $f_1(t) * f_2(t) = \int\limits_{-\infty}^{+\infty} f_1(\tau)f_2(t - \tau)d\tau \leftrightarrow F_1(s) \cdot F_2(s)$
15. $if\ \forall t \geq 0 : f(t + T) = f(t) \Rightarrow \mathcal{L}(f(t)) = \frac{\int_0^T e^{-st}f(t)dt}{1-e^{-Ts}}$

Example 10.8 Determine the Laplace transform of $f(t) = \begin{cases} +1 & 0 \leq t < 1 \\ -1 & 1 \leq t < 2 \end{cases}, f(t + 2) = f(t)$(Fig. 10.6).

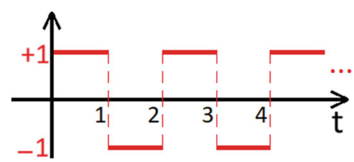


Fig. 10.6 Periodic function $f(t)$

$$if\ \forall t \geq 0 : f(t + T) = f(t) \Rightarrow \mathcal{L}(f(t)) = \frac{\int_0^T e^{-st}f(t)dt}{1 - e^{-Ts}}$$

$$T = 2 \Rightarrow \mathcal{L}(f(t)) = \frac{\int_0^2 e^{-st}f(t)dt}{1 - e^{-2s}} = \frac{\int_0^1 e^{-st}dt + \int_1^2 -e^{-st}dt}{1 - e^{-2s}}$$

$$= \frac{\frac{e^{-st}}{-s}\Big|_0^1 + \frac{e^{-st}}{s}\Big|_1^2}{1 - e^{-2s}} = \frac{\frac{e^{-s}}{-s} + \frac{1}{s} + \frac{e^{-2s}}{s} - \frac{e^{-s}}{s}}{1 - e^{-2s}} = \frac{1 - 2e^{-s} + e^{-2s}}{\left(1 - e^{-2s}\right)s}$$

$$\mathcal{L}(f(t)) = \frac{1 - 2e^{-s} + e^{-2s}}{\left(1 - e^{-2s}\right)s} = \frac{\left(1 - e^{-s}\right)^2}{\left(1 - e^{-s}\right)\left(1 + e^{-s}\right)s} = \frac{\left(1 - e^{-s}\right)}{\left(1 + e^{-s}\right)s}$$

Example 10.9 Determine the Laplace transform of $f(t) = \sin(t)$.

$$if\ \forall t \geq 0 : f(t+T) = f(t) \Rightarrow \mathcal{L}(f(t)) = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-Ts}}$$

$$f(t) = \sin(t) \Rightarrow T = 2\pi \Rightarrow \mathcal{L}(f(t)) = \frac{\int_0^{2\pi} e^{-st} \sin(t)dt}{1 - e^{-2\pi s}}$$

$$= \frac{\frac{1-e^{-2\pi s}}{s^2+1}}{1 - e^{-2\pi s}} = \frac{1}{s^2+1}$$

10.8 Laplace Transform for Solving Differential Equations

The Laplace transform converts differential equations into algebraic equations. Initial conditions are directly included in the transformed equation, making it easier to handle. Memorizing the following two formulas is highly recommended, as they are instrumental in solving a majority of problems.

$$\dot{f}(t) = \frac{df}{dt} \leftrightarrow sF(s) - f(0)$$

$$\ddot{f}(t) = \frac{d^2 f}{dt^2} \leftrightarrow s^2 F(s) - sf(0) - \dot{f}(0)$$

Let's study some numeric examples.

Example 10.10 Calculate the inverse Laplace transform of $X(s) = \frac{s-13}{s^2-2s+5}$ and $Y(s) = \frac{7s-17}{s^2-2s+5}$.

We use the partial fraction decomposition and Table 10.1 to calculate the given inverse Laplace transforms.

$$\begin{cases} x(t) = \mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(\frac{s-13}{s^2-2s+5}\right) = \mathcal{L}^{-1}\left(\frac{s-1}{(s-1)^2+2^2} - 6\frac{2}{(s-1)^2+2^2}\right) = e^t \cos(2t) \\ \quad -6e^t sin(2t) \\ y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{7s-17}{s^2-2s+5}\right) = \mathcal{L}^{-1}\left(\frac{7(s-1)}{(s-1)^2+2^2} - 5\frac{2}{(s-1)^2+2^2}\right) = 7e^t \cos(2t) \\ \quad -5e^t sin(2t) \end{cases}$$

Example 10.11 Solve the $\ddot{y}(t) + 2\dot{y}(t) + y(t) = 6H(t), y(0) = 1, \dot{y}(0) = 2$.

Taking the Laplace transform of both sides:

$$\mathcal{L}(\ddot{y}(t)) + 2\mathcal{L}(\dot{y}(t)) + \mathcal{L}(y(t)) = 6\mathcal{L}(H(t))$$

$$\mathcal{L}(\ddot{y}(t)) = s^2Y(s) - sy(0) - \dot{y}(0) = s^2Y(s) - s - 2$$

$$\mathcal{L}(\dot{y}(t)) = sY(s) - y(0) = sY(s) - 1$$

$$s^2Y(s) - s - 2 + 2(sY(s) - 1) + Y(s) = \frac{6}{s} \Rightarrow Y(s)$$
$$= \frac{s + 4 + \frac{6}{s}}{s^2 + 2s + 1} = \frac{s + 4}{(s + 1)^2} + \frac{6}{s(s + 1)^2}$$

$$\frac{s + 4}{(s + 1)^2} = \frac{A}{s + 1} + \frac{B}{(s + 1)^2}$$

$$B = (s + 4)|_{s=-1} = -1 + 4 = 3$$

$$A = \frac{d}{ds}(s + 4)\bigg|_{s=-1} = 1|_{s=-1} = 1$$

Therefore, $\frac{s+4}{(s+1)^2} = \frac{1}{s+1} + \frac{3}{(s+1)^2}$.

$$\frac{6}{s(s + 1)^2} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{(s + 1)^2}$$

$$A = \frac{6}{(s + 1)^2}\bigg|_{s=0} = \frac{6}{1} = 6$$

$$C = \frac{6}{s}\bigg|_{s=-1} = \frac{6}{-1} = -6$$

$$B = \frac{d}{ds}(\frac{6}{s})\bigg|_{s=-1} = -\frac{6}{s^2}\bigg|_{s=-1} = -6$$

Therefore, $\frac{6}{s(s+1)^2} = \frac{6}{s} - \frac{6}{s+1} - \frac{6}{(s+1)^2}$.

$$Y(s) = \frac{s + 4}{(s + 1)^2} + \frac{6}{s(s + 1)^2} = \frac{s + 1 + 3}{(s + 1)^2} + \frac{6}{s} - \frac{6}{s + 1} - \frac{6}{(s + 1)^2}$$

$$Y(s) = \frac{s + 1 + 3}{(s + 1)^2} + \frac{6}{s} - \frac{6}{s + 1} - \frac{6}{(s + 1)^2} = \frac{1}{s + 1} + \frac{3}{(s + 1)^2} + \frac{6}{s} - \frac{6}{s + 1} - \frac{6}{(s + 1)^2}$$

$$Y(s) = \frac{-5}{s + 1} - \frac{3}{(s + 1)^2} + \frac{6}{s}$$

Using Table 10.1, $y(t) = (-5e^{-t} - 3te^{-t} + 6)H(t)$.

Example 10.12 Solve the $\ddot{y}(t) + 4y(t) = H(t) - H(t-1)$, $y(0) = \dot{y}(0) = 0$.

Taking the Laplace transform of both sides:

$$\mathcal{L}(\ddot{y}(t)) + \mathcal{L}(4y(t)) = \mathcal{L}(H(t)) - \mathcal{L}(H(t-1))$$

$$\mathcal{L}(\ddot{y}(t)) + 4\mathcal{L}(y(t)) = \mathcal{L}(H(t)) - \mathcal{L}(H(t-1))$$

$$s^2Y(s) - sy(0) - \dot{y}(0) + 4Y(s) - 4y(0) = \frac{1}{s} - \frac{1}{s}e^{-s}$$

$$Y(s) = \frac{1}{s(s^2+4)} - \frac{1}{s(s^2+4)}e^{-s}$$

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$A = \left.\frac{1}{s^2+4}\right|_{s=0} = \frac{1}{4}$$

$$\frac{1}{s\left(s^2+4\right)} = \frac{\frac{1}{4}}{s} + \frac{Bs+C}{s^2+4} \Rightarrow \frac{1}{s\left(s^2+4\right)} - \frac{\frac{1}{4}}{s} = \frac{Bs+C}{s^2+4} \Rightarrow \frac{1-\frac{1}{4}\left(s^2+4\right)}{s\left(s^2+4\right)} = \frac{Bs+C}{s^2+4}$$

$$\frac{1-\frac{1}{4}s^2-1}{s\left(s^2+4\right)} = \frac{Bs+C}{s^2+4} \Rightarrow B = \frac{-1}{4}, C = 0 \Rightarrow \frac{1}{s\left(s^2+4\right)} = \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}s}{s^2+4}$$

$$Y(s) = \frac{1}{s\left(s^2+4\right)} - \frac{1}{s\left(s^2+4\right)}e^{-s} = \frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}s}{s^2+4} - \frac{\frac{1}{4}}{s}e^{-s} + \frac{\frac{1}{4}s}{s^2+4}e^{-s}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{\frac{1}{4}}{s}\right) - \mathcal{L}^{-1}\left(\frac{\frac{1}{4}s}{s^2+4}\right) - \mathcal{L}^{-1}\left(\frac{\frac{1}{4}}{s}e^{-s}\right) + \mathcal{L}^{-1}\left(\frac{\frac{1}{4}s}{s^2+4}e^{-s}\right)$$

$$y(t) = \frac{1}{4}H(t) - \frac{1}{4}\cos(2t)H(t) - \frac{1}{4}H(t-1) + \frac{1}{4}\cos(2(t-1))H(t-1)$$

$$y(t) = \frac{1}{4}(1-\cos(2t))H(t) - \frac{1}{4}(1-\cos(2(t-1))H(t-1)$$

$$y(t) = \begin{cases} \frac{1}{4}(1-\cos(2t)) & 0 \le t < 1 \\ \frac{1}{4}\cos(2t-2) - \frac{1}{4}\cos(2t) & t \ge 1 \end{cases}$$

Example 10.13 Solve the $\ddot{y}(t) + \dot{y}(t) + y(t) = \delta(t)$, $y(0) = \dot{y}(0) = 0$.

Taking the Laplace transform of both sides:

$$\mathcal{L}(\ddot{y}(t)) + \mathcal{L}(\dot{y}(t)) + \mathcal{L}(y(t)) = \mathcal{L}(\delta(t))$$

$$\left(s^2+s+1\right)Y(s) = 1 \Rightarrow Y(s) = \frac{1}{\left(s^2+s+1\right)}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+s+1}\right) = \mathcal{L}^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right) = \mathcal{L}^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$
$$= \frac{1}{\frac{\sqrt{3}}{2}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)H(t)$$

Therefore, $y(t) = \frac{2}{\sqrt{3}}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t)H(t)$ is the solution of this problem.

Example 10.14 Find the general solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = sin(3t)$.

Homogenous solution can be found easily:

$$s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2 \Rightarrow y_h(t) = C_1e^{-t} + C_2e^{-2t}$$

Let's find the particular solution with undetermined coefficient method. According to the method of undetermined coefficients, a particular solution to the differential equation $\ddot{y}(t)+3\dot{y}(t)+2y(t) = sin(3t)$ can be assumed to have the form $y_p(t) = Asin(3t)+Bcos(3t)$. Therefore,

$$y_p(t) = Asin(3t) + Bcos(3t)$$

$$\dot{y}_p(t) = 3Acos(3t) - 3Bsin(3t)$$

$$\ddot{y}_p(t) = -9Asin(3t) - 9Bcos(3t)$$

Substitution in the $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = sin(3t)$ leads to:

$$-9Asin(3t) - 9Bcos(3t) + 3(3Acos(3t) - 3Bsin(3t)) + 2(Asin(3t) + Bcos(3t)) = \sin(t)$$

which can be simplified to $(-7A - 9B)sin(3t) + (9A - 7B)cos(3t) = sin(3t)$. Therefore,

$$\begin{cases} -7A - 9B = 1 \\ 9A - 7B = 0 \end{cases} \Rightarrow A = \frac{-7}{130}, B = \frac{-9}{130} \Rightarrow y_p(t) = A\sin(3t)$$
$$+ B\cos(3t) = \frac{-7}{130}\sin(3t) + \frac{-9}{130}\cos(3t)$$

Note that $a \times \sin(\omega t) + b \times \cos(\omega t) = sign(a).\sqrt{a^2 + b^2}\sin(\omega t + \tan^{-1}(\frac{b}{a}))$. Therefore,

$$\begin{aligned} y_p(t) &= \frac{-7}{130}\sin(3t) + \frac{-9}{130}\cos(3t) = -\sqrt{\frac{1}{130}}\sin\left(3t + \tan^{-1}\left(\frac{\frac{-9}{130}}{\frac{-7}{130}}\right)\right) \\ &= -\sqrt{\frac{1}{130}}\sin\left(3t + \tan^{-1}\left(\frac{9}{7}\right)\right) = -0.0877\sin(3t + 0.9098) \\ &= 0.0877\sin(3t + 0.9098 - \pi) = 0.0877\sin(3t - 2.2318) \end{aligned}$$

The general solution of this equation is:

$$y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{-2t} + 0.0877\sin(3t - 2.2318)$$

Steady-state response ($y_{ss}(t)$) is the behavior of a dynamic system after a long enough time, when all transient effects (solutions of homogenous equation) have died out. In simpler terms, it's the long-term behavior of the system. Steady-state response is commonly studied with sinusoidal inputs. In this example, the first two terms go toward zero as t increases. The steady-state response in this example is $y_{ss}(t) = 0.877\sin(3t - 2.2318)$ since $\lim_{t\to\infty} C_1e^{-t} + C_2e^{-2t} = 0$.

Sometimes, the transient behavior of the system (solutions of the homogenous equation) is not important and we just look for the steady-state response of the system. The Laplace transform can be used to find the steady-state response. Let's find the steady-state response of this example using the Laplace transform.

Given differential equation $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = sin(3t)$ can be written as of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t)$ where of $u(t)$ shows the system input. In this example input is sinusoidal, i.e., $u(t) = \sin(3t)$. Transfer function of this system is:

$$\begin{aligned} &\mathcal{L}(\ddot{y}(t)) + \mathcal{L}(3\dot{y}(t)) + \mathcal{L}(2y(t)) = \mathcal{L}(u(t)) \Rightarrow \mathcal{L}(\ddot{y}(t)) \\ &\quad + 3\mathcal{L}(\dot{y}(t)) + 2\mathcal{L}(y(t)) = \mathcal{L}(u(t)) \Rightarrow s^2Y(s) + 3sY(s) \\ &\quad + 2Y(s) = U(s) \Rightarrow (s^2 + 3s + 2)Y(s) = U(s) \Rightarrow T(s) \\ &\quad = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 3s + 2} \end{aligned}$$

Note that the initial conditions are assumed to be zero when calculating the transfer function. The input is $u(t) = sin(3t)$, so we need to calculate $T(3j)$:

$$\begin{aligned} T(s) &= \frac{1}{s^2 + 3s + 2} \Rightarrow T(3j) = \frac{1}{(3j)^2 + 3 \times 3j + 2} = \frac{1}{-7 + 9j} \\ &= 0.0877e^{-j2.2318} \end{aligned}$$

Therefore, steady state solution (particular solution) of given differential equation is:

$$y_{ss}(t) = y_p(t) = 0.0877\sin(3t - 2.2318)$$

which is identical to the result obtained from the previous method.

Example 10.15 Find the general solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = sin\left(3t - \frac{\pi}{4}\right)$.

Homogenous solution can be found easily:

$$s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2 \Rightarrow y_h(t) = C_1 e^{-t} + C_2 e^{-2t}$$

In this example the input is $u_2(t) = sin\left(3t - \frac{\pi}{4}\right) = sin\left(3(t - \frac{\pi}{12})\right)$. As seen in the previous example, for the input $u_1(t) = sin(3t)$, a particular solution is $y_{p1}(t) = 0.0877\sin(3t - 2.2318)$. Therefore, particular solution for $u_2(t) = u_1\left(t - \frac{\pi}{12}\right) = sin\left(3(t - \frac{\pi}{12})\right) = sin\left(3t - \frac{\pi}{4}\right)$ must be $y_{p2}(t) = y_{p1}\left(t - \frac{\pi}{12}\right) = 0.0877\sin\left(3\left(t - \frac{\pi}{12}\right) - 2.2318\right) = 0.0877\sin\left(3t - \frac{\pi}{4} - 2.2318\right) = 0.0877\sin(3t - 3.0172)$. Therefore, the general solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = sin\left(3t - \frac{\pi}{4}\right)$ is:

$$y(t) = y_h(t) + y_{p2}(t) = C_1 e^{-t} + C_2 e^{-2t} + 0.0877\sin(3t - 3.0172)$$

Example 10.16 Find the general solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 6sin\left(5t + \frac{\pi}{6}\right)$.

Homogenous solution can be found easily:

$$s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2 \Rightarrow y_h(t) = C_1 e^{-t} + C_2 e^{-2t}$$

We found the transfer function in the previous example: $T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2}$. Input is $6sin\left(5t + \frac{\pi}{6}\right)$ therefore, we need to calculate $T(5j)$.

$$\begin{aligned} T(s) &= \frac{1}{s^2 + 3s + 2} \Rightarrow T(5j) = \frac{1}{(5j)^2 + 3 \times 5j + 2} = \frac{1}{-23 + 15j} \\ &= 0.0364e^{-j2.5637} \end{aligned}$$

$$y_{ss}(t) = y_p(t) = 0.0364 \times 6 \times \sin\left(5t + \frac{\pi}{6} - 2.5637\right) = 0.2184\sin(5t - 2.0401)$$

General solution of this equation is:

$$y(t) = y_h(t) + y_p(t) = C_1 e^{-t} + C_2 e^{-2t} + 0.2184\sin(5t - 2.0401)$$

Example 10.17 Find the general solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 9cos\left(7t + \frac{\pi}{3}\right)$.

Homogenous solution can be found easily:

$$s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2 \Rightarrow y_h(t) = C_1 e^{-t} + C_2 e^{-2t}$$

The transfer function for $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t)$ is: $T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2}$. Input is $9cos\left(7t + \frac{\pi}{3}\right)$ therefore, we need to calculate $T(7j)$.

$$T(s) = \frac{1}{s^2 + 3s + 2} \Rightarrow T(7j) = \frac{1}{(7j)^2 + 3 \times 7j + 2} = \frac{1}{-47 + 21j}$$
$$= 0.0194e^{-j2.7214}$$

$$y_{ss}(t) = y_p(t) = 0.0194 \times 9 \times \cos\left(7t + \frac{\pi}{3} - 2.7214\right)$$
$$= 0.1746\sin(7t - 1.6742)$$

General solution of this equation is:

$$y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{-2t} + 0.1746\sin(7t - 1.6742)$$

Example 10.18 Find the general solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 9cos\left(7t + \frac{\pi}{3}\right) + 6sin\left(5t + \frac{\pi}{6}\right)$.

Homogenous solution can be found easily:

$$s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2 \Rightarrow y_h(t) = C_1e^{-t} + C_2e^{-2t}$$

We use the superposition principle to find the particular solution. According to previous examples, particular solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 9cos\left(7t + \frac{\pi}{3}\right)$ is $y_{p1}(t) = 0.1746\sin(7t - 1.6742)$. Particular solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 6sin\left(5t + \frac{\pi}{6}\right)$ is $y_{p2}(t) = 0.2184\sin(5t - 2.0401)$. Therefore, particular solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 9cos\left(7t + \frac{\pi}{3}\right) + 6sin\left(5t + \frac{\pi}{6}\right)$ is:

$$y_p(t) = y_{p1}(t) + y_{p2}(t) = 0.1746\sin(7t - 1.6742) + 0.2184\sin(5t - 2.0401)$$

General solution of this equation is:

$$y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{-2t} + 0.1746\sin(7t - 1.6742)$$
$$+ 0.2184\sin(5t - 2.0401)$$

Example 10.19 Find the general solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 7e^{-4t}$.

Homogenous solution can be found easily:

$$s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2 \Rightarrow y_h(t) = C_1e^{-t} + C_2e^{-2t}$$

The transfer function for $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t)$ is: $T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2}$. Input is $7e^{-4t}$. Therefore, we need to calculate $T(-4)$.

$$T(s) = \frac{1}{s^2 + 3s + 2} \Rightarrow T(-4) = \frac{1}{(-4)^2 + 3 \times -4 + 2} = \frac{1}{6}$$

$$y_p(t) = \frac{1}{6} \times 7e^{-4t} = \frac{7}{6}e^{-4t}$$

General solution of this equation is:

$$y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{-2t} + \frac{7}{6}e^{-4t}$$

Example 10.20 Find the general solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 5e^{4t}$.

Homogenous solution can be found easily:

$$s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2 \Rightarrow y_h(t) = C_1e^{-t} + C_2e^{-2t}$$

The transfer function for $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t)$ is: $T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2}$. Input is $5e^{4t}$. Therefore, we need to calculate $T(+4)$.

$$T(s) = \frac{1}{s^2 + 3s + 2} \Rightarrow T(4) = \frac{1}{(4)^2 + 3 \times 4 + 2} = \frac{1}{30}$$

$$y_p(t) = \frac{1}{30} \times 5e^{4t} = \frac{1}{6}e^{4t}$$

General solution of this equation is:

$$y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{-2t} + \frac{1}{6}e^{4t}$$

Example 10.21 Find the general solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 7e^{j4t}$.

Homogenous solution can be found easily:

$$s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2 \Rightarrow y_h(t) = C_1e^{-t} + C_2e^{-2t}$$

The transfer function for $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t)$ is: $T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2}$. Input is $7e^{j4t}$. Therefore, we need to calculate $T(4j)$.

$$T(s) = \frac{1}{s^2 + 3s + 2} \Rightarrow T(4) = \frac{1}{(4j)^2 + 3 \times 4j + 2}$$

$$= \frac{1}{-14 + 12j}$$

$$y_p(t) = \frac{1}{-14 + 12j} \times 7e^{j4t} = \frac{7}{-14 + 12j} \times e^{j4t} = (-0.2882 - j0.2471)e^{j4t}$$
$$= 0.3796e^{-j2.433}e^{j4t} = 0.3796e^{j(4t-2.433)}$$

Therefore, the general solution of this equation is:

$$y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{-2t} + 0.3796e^{j(4t-2.433)}$$

Let's employ the undetermined coefficient method to see whether the obtained result is correct.

$$y_p(t) = Ae^{j4t}$$

$$\dot{y}_p(t) = j4Ae^{j4t}$$

$$\ddot{y}_p(t) = -16Ae^{j4t}$$

Note that $A \in \mathbb{C}$, i.e., $A = a + bj$.

$$\ddot{y}_p(t) + 3\dot{y}_p(t) + 2y_p(t) = 7e^{j4t} \Rightarrow -16Ae^{j4t} + 3 \times j4Ae^{j4t}$$
$$+ 2 \times Ae^{j4t} = 7e^{j4t} \Rightarrow (-14 + 12j)A = 7 \Rightarrow (-14 + 12j)(a + bj)$$
$$= 7 + 0j \Rightarrow -14a - 12b + j(12a - 14b) = 7 + 0j$$

Remember that two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. Therefore,

$$\begin{cases} -14a - 12b = 7 \\ 12a - 14b = 0 \end{cases} \Rightarrow a = -0.2882, b = -0.2471$$

Particular solution is:

$$y_p(t) = (-0.2882 - j0.2471)e^{j4t} = 0.3796e^{-j2.433}e^{j4t} = 0.3796e^{j(4t-2.433)}$$

Therefore, the general solution is:

$$y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{-2t} + 0.3796e^{j(4t-2.433)}$$

The result obtained is identical to the one from the previous method.

Example 10.22 Find the general solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 3e^{j4t}$.

Homogenous solution can be found easily:
$s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2 \Rightarrow y_h(t) = C_1e^{-t} + C_2e^{-2t}$.
The transfer function for $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t)$ is: $T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2}$. Input is $3e^{j4t}$.herefore, we need to calculate $T(j4)$.

$$T(s) = \frac{1}{s^2 + 3s + 2} \Rightarrow T(j4) = \frac{1}{(j4)^2 + 3 \times j4 + 2} = \frac{1}{-14 + 12j}$$

$$y_p(t) = \frac{1}{-14 + 12j} \times 3e^{j4t} = 0.0542e^{-j2.433} \times 3e^{j4t} = 0.1626e^{j(4t-2.433)}$$

Therefore, the general solution of this equation is:

$$y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{-2t} + 0.1626e^{j(4t-2.433)}$$

Example 10.23 Find the general solution of **(A)** $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 3\cos(4t)$ **(B)**$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 3\cos(4t)$.

Homogenous solution can be found easily:

$$s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2 \Rightarrow y_h(t) = C_1e^{-t} + C_2e^{-2t}$$

As demonstrated in the preceding example, a particular solution to the input $u = 3e^{j4t}$ is given by $y_p(t) = 0.1626e^{j(4t-2.433)}$. The input in part (A) is $u_A(t) = 3\cos(4t) = Re\{3e^{j4t}\}$. Therefore, particular solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 3\cos(4t)$ is $y_p(t) = Re\left\{0.1626e^{j(4t-2.433)}\right\} = 0.1626\cos(4t - 2.433)$. The general solution of part (A) is: $y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{-2t} + 0.1626\cos(4t - 2.433)$.

The input in part (B) is $u_B(t) = 3\sin(4t) = Im\{3e^{j4t}\}$. Therefore, particular solution of $\ddot{y}(t)+3\dot{y}(t)+2y(t) = 3\sin(4t)$ is $y_p(t) = Im\left\{0.1626e^{j(4t-2.433)}\right\} = 0.1626\sin(4t - 2.433)$. The general solution of part (B) is: $y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{-2t} + 0.1626\sin(4t - 2.433)$.

Example 10.24 Find the general solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 6$.

Homogenous solution can be found easily:

$$s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2 \Rightarrow y_h(t) = C_1e^{-t} + C_2e^{-2t}$$

The transfer function for $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t)$ is: $T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2}$. Input is $6e^{j0t}$. Therefore, we need to calculate $T(0)$.

$$T(s) = \frac{1}{s^2 + 3s + 2} \Rightarrow T(0) = \frac{1}{(0)^2 + 3 \times 0 + 2} = \frac{1}{2}$$

$$y_p(t) = \frac{1}{2} \times 6e^{j0t} = 3$$

Therefore, the general solution of this equation is:

$$y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{-2t} + 3$$

Example 10.25 Find the general solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 4 + 7\cos(3t + \frac{\pi}{12})$.

Homogenous solution can be found easily:

$$s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, s_2 = -2 \Rightarrow y_h(t) = C_1e^{-t} + C_2e^{-2t}$$

The transfer function for $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = u(t)$ is: $T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+3s+2}$. Input is $4 + 7\cos(3t + \frac{\pi}{12})$. Let's use superposition to find the particular solution. For $u_1(t) = 4$, output is:

$$y_{p1}(t) = T(0) \times u_1(t) = \frac{1}{(0)^2 + 3 \times 0 + 2} \times 4 = \frac{1}{2} \times 4 = 2$$

For $u_2(t) = 7\cos(3t + \frac{\pi}{12})$, output is:

$$T(j3) = \frac{1}{(j3)^2 + 3 \times j3 + 2} = \frac{1}{-7 + j9} = 0.0877e^{-j2.2318}$$

$$y_{p2}(t) = 0.0877 \times 7\cos\left(3t + \frac{\pi}{12} - 2.2318\right) = 0.6139\cos(3t - 1.9700)$$

Therefore, the particular solution is

$$y_p(t) = y_{p1}(t) + y_{p2}(t) = 2 + 0.6139\cos(3t - 1.9700)$$

Therefore, the general solution of this equation is:

$$y(t) = y_h(t) + y_p(t) = C_1e^{-t} + C_2e^{-2t} + 2 + 0.6139\cos(3t - 1.9700)$$

Exercise: Use the superposition principle to fine the particular solution of $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 3e^{j4t} = 3\cos(4t) + j3\sin(4t)$.

Example 10.26 Solve the differential equation $\begin{cases} \frac{dx(t)}{dt} = 2x(t) + 3y(t) \\ \frac{dy(t)}{dt} = x(t) + 4y(t) \end{cases}$ with initial conditions $x(0) = 0$ and $y(0) = 1$.

Applying the Laplace transform to the given equations:

$$\begin{cases} sX(s) = 2X(s) + 3Y(s) \\ sY(s) - 1 = X(s) + 4Y(s) \end{cases}$$

To simplify notation, let $X(s)$ be denoted by X and $Y(s)$ by Y:

$$\begin{cases} sX = 2X + 3Y \\ sY - 1 = X + 4Y \end{cases} \Rightarrow \begin{cases} (2-s)X + 3Y = 0 \\ X + (4-s) = -1 \end{cases} \Rightarrow \begin{bmatrix} 2-s & 3 \\ 1 & 4-s \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 2-s & 3 \\ 1 & 4-s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \frac{1}{(2-s)(4-s)-3} \begin{bmatrix} 4-s & -3 \\ -1 & 2-s \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$= \frac{1}{s^2 - 6s + 5} \begin{bmatrix} 3 \\ s-2 \end{bmatrix} = \frac{1}{(s-1)(s-5)} \begin{bmatrix} 3 \\ s-2 \end{bmatrix} \Rightarrow \begin{cases} X(s) = \frac{3}{(s-1)(s-5)} \\ Y(s) = \frac{s-2}{(s-1)(s-5)} \end{cases}$$

$$x(t) = \mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(\frac{3}{(s-1)(s-5)}\right) = \mathcal{L}^{-1}\left(\frac{-\frac{3}{4}}{s-1} + \frac{\frac{3}{4}}{s-5}\right) = \frac{3}{4}\left(e^{5t} - e^{t}\right)$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{s-2}{(s-1)(s-5)}\right) = \mathcal{L}^{-1}\left(\frac{\frac{1}{4}}{s-1} + \frac{\frac{3}{4}}{s-5}\right) = \frac{1}{4}\left(3e^{5t} - e^{t}\right)$$

Exercise: Solve $\begin{cases} \frac{dx(t)}{dt} = 3x(t) - 2y(t) \\ \frac{dy(t)}{dt} = 4x(t) - y(t) \end{cases}$ with the initial condition of $x(0) = 1$ and $y(0) = 1$.

10.9 Problem Solving with MATLAB®

This section provides examples of how MATLAB can be employed to solve the types of problems encountered in this chapter.

Example 10.27 The code in Fig. 10.7 calculates $\int te^{-st} dt$.

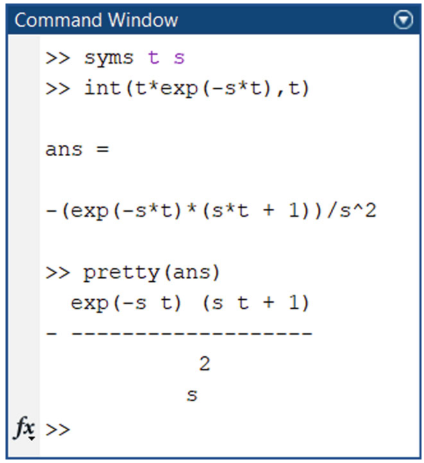


Fig. 10.7 Calculation of $\int te^{-st}dt$

Example 10.28 The code in Fig. 10.8 calculates $\int e^{-j\omega_0 t}e^{-st}dt$.

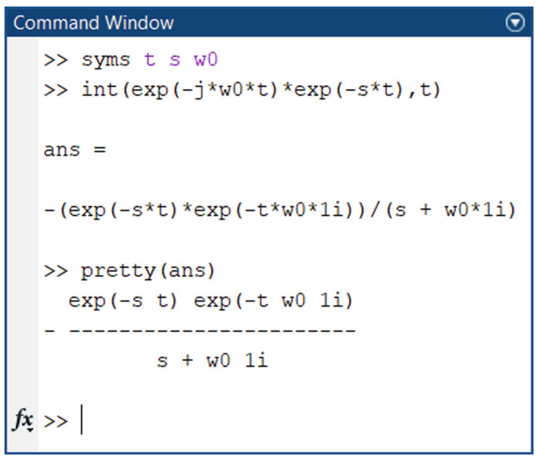


Fig. 10.8 Calculation of $\int e^{-j\omega_0 t}e^{-st}dt$

Example 10.29 The code in Fig. 10.9 calculates Laplace transform of the unit step function.

Fig. 10.9 Laplace transform of unit step function

```
>> syms t
>> laplace(heaviside(t))

ans =

1/s

>>
```

Example 10.30 The code in Fig. 10.10 calculates Laplace transform of the unit impulse function.

Fig. 10.10 Laplace transform of unit impulse function

```
>> syms t
>> laplace(dirac(t))

ans =

1

>>
```

Example 10.31 The code in Fig. 10.11 calculates Laplace transform of $f(t) = e^{-2t}\sin(7t)H(t)$. Note that $H(t)$ shows the unit step function.

Fig. 10.11 Laplace transform of $f(t) = e^{-2t}\sin(7t)H(t)$

```
>> syms t
>> laplace(exp(-2*t)*sin(7*t))

ans =

7/((s + 2)^2 + 49)

>> pretty(ans)
      7
-------------
       2
(s + 2)  + 49

>>
```

Example 10.32 The code in Fig. 10.12 calculates Laplace transform of $f(t) = \sin(5t + \frac{\pi}{4})H(t)$.

Fig. 10.12 Laplace transform of $f(t) = \sin(5t + \frac{\pi}{4})H(t)$

```
Command Window
>> syms t
>> laplace(sin(5*t+pi/4))

ans =

((2^(1/2)*s)/2 + (5*2^(1/2))/2)/(s^2 + 25)

>> pretty(ans)
sqrt(2) s    5 sqrt(2)
--------- + ---------
    2           2
---------------------
         2
        s  + 25

>> pretty(vpa(ans,4))
0.7071 s + 3.536
----------------
      2
     s  + 25.0

fx >>
```

Example 10.33 The code in Fig. 10.13 calculates Laplace transform of $f(t) = te^{-6t}H(t)$.

Fig. 10.13 Laplace transform of $f(t) = te^{-6t}H(t)$

```
Command Window
>> syms t
>> laplace(t*exp(-6*t))

ans =

1/(s + 6)^2

fx >>
```

Example 10.34 The code in Fig. 10.14 calculates inverse Laplace transform of $\frac{5s^2+s+4}{(s+4)(s^2+4)}$. According to Fig. 10.14, $\mathcal{L}^{-1}\{\frac{5s^2+s+4}{(s+4)(s^2+4)}\} = \big(4e^{-4t} + \cos(2t) - \frac{3}{2}\sin(2t)\big)H(t)$.

Fig. 10.14 Inverse Laplace transform of $\frac{5s^2+s+4}{(s+4)(s^2+4)}$

```
Command Window
>> syms s
>> ilaplace((5*s^2+s+4)/((s+4)*(s^2+4)))

ans =

cos(2*t) + 4*exp(-4*t) - (3*sin(2*t))/2

>> pretty(ans)
                               sin(2 t) 3
cos(2 t) + exp(-4 t) 4 - ----------
                                    2
fx >>
```

Example 10.35 The code in Fig. 10.15 calculates inverse Laplace transform of $\frac{s+1}{s^2-6s+13}$. According to Fig. 10.15, $\mathcal{L}^{-1}\{\frac{s+1}{s^2-6s+13}\} = e^{3t}(\cos(2t) + 2\sin\,(2t))H(t)$.

Fig. 10.15 Inverse Laplace transform of $\frac{s+1}{s^2-6s+13}$

```
Command Window
>> syms s
>> ilaplace((s+1)/(s^2-6*s+13))

ans =

exp(3*t)*(cos(2*t) + 2*sin(2*t))

fx >>
```

Example 10.36 The code in Fig. 10.16 calculates inverse Laplace transform of $\frac{10(s+2)}{s(s^2+2s+2)}$. According to Fig. 10.16, $\mathcal{L}^{-1}\{\frac{10(s+2)}{s(s^2+2s+2)}\} = (10 - 10e^{-t}\cos(t))H(t)$.

Fig. 10.16 Inverse Laplace transform of $\frac{10(s+2)}{s(s^2+2s+2)}$

```
Command Window
>> syms s
>> ilaplace(10*(s+2)/s/(s^2+2*s+2))

ans =

10 - 10*exp(-t)*cos(t)

fx >>
```

Example 10.37 The code in Fig. 10.17 calculates the partial fraction decomposition of $\frac{5s^2+s+4}{(s+4)(s^2+4)}$. According to Fig. 10.17, $\frac{5s^2+s+4}{(s+4)(s^2+4)} = \frac{4}{s+4} + \frac{0.5+0.75j}{s-2j} + \frac{0.5-0.75j}{s+2j}$.

Fig. 10.17 Partial fraction decomposition of $\frac{5s^2+s+4}{(s+4)(s^2+4)}$

```
Command Window
>> [r,p,k]=residue([5 1 4],conv([1 4],[1 0 4]))

r =

   4.0000 + 0.0000i
   0.5000 + 0.7500i
   0.5000 - 0.7500i


p =

  -4.0000 + 0.0000i
   0.0000 + 2.0000i
   0.0000 - 2.0000i


k =

     []

fx >>
```

The code shown in Fig. 10.18 simplifies $\frac{0.5+0.75j}{s-2j} + \frac{0.5-0.75j}{s+2j}$. According to Fig. 10.18, $\frac{0.5+0.75j}{s-2j} + \frac{0.5-0.75j}{s+2j} = \frac{s-3}{s^2+4}$. Therefore, $\frac{5s^2+s+4}{(s+4)(s^2+4)} = \frac{4}{s+4} + \frac{0.5+0.75j}{s-2j} + \frac{0.5-0.75j}{s+2j} = \frac{4}{s+4} + \frac{s-3}{s^2+4}$.

```
Command Window
>> simplify((.5+.75*j)/(s-2*j)+(.5-.75*j)/(s+2*j))

ans =

(s - 3)/(s^2 + 4)

fx >>
```

Fig. 10.18 Simplifying $\frac{0.5+0.75j}{s-2j} + \frac{0.5-0.75j}{s+2j}$

You can use the following technique to simplify the obtained expression as well (Remember that $z + \overline{z} = 2Re\{z\}$):

$$\frac{5s^2+s+4}{(s+4)\left(s^2+4\right)}=\frac{4}{s+4}+\frac{0.5+0.75j}{s-2j}+\frac{0.5-0.75j}{s+2j}=\frac{4}{s+4}$$
$$+2\mathrm{Re}\left\{\frac{0.5+0.75j}{s-2j}\right\}=\frac{4}{s+4}+2\mathrm{Re}\left\{\frac{(0.5+0.75j)(s+2j)}{s^2+4}\right\}$$
$$=\frac{4}{s+4}+2\mathrm{Re}\left\{\frac{0.5s+j+0.75js-1.5}{s^2+4}\right\}=\frac{4}{s+4}+2\times\frac{0.5s-1.5}{s^2+4}$$
$$=\frac{4}{s+4}+\frac{s-3}{s^2+4}$$

Example 10.38 The code in Fig. 10.19 calculates the partial fraction decomposition of $\frac{10(s+2)}{s(s^2+2s+2)}$. According to Fig. 10.19, $\frac{10(s+2)}{s(s^2+2s+2)}=\frac{-5}{s-(-1+j)}+\frac{-5}{s-(-1-j)}+\frac{10}{s}$.

```
Command Window
>> [r,p,k]=residue([10 20],conv([1 0],[1 2 2]))

r =

  -5.0000 + 0.0000i
  -5.0000 - 0.0000i
  10.0000 + 0.0000i


p =

  -1.0000 + 1.0000i
  -1.0000 - 1.0000i
   0.0000 + 0.0000i


k =

     []

fx >>
```

Fig. 10.19 Partial fraction decomposition of $\frac{10(s+2)}{s(s^2+2s+2)}$

The code shown in Fig. 10.20 simplifies $\frac{-5}{s-(-1+j)}+\frac{-5}{s-(-1-j)}$. According to Fig. 10.20, $\frac{-5}{s-(-1+j)}+\frac{-5}{s-(-1-j)}=\frac{-(10s+10)}{s^2+2s+2}$. Therefore, $\frac{10(s+2)}{s(s^2+2s+2)}=\frac{10}{s}+\frac{-5}{s-(-1+j)}+\frac{-5}{s-(-1-j)}=\frac{10}{s}-\frac{(10s+10)}{s^2+2s+2}$.

```
Command Window
>> simplify(-5/(s+1-j)-5/(s+1+j))

ans =

-(10*s + 10)/(s^2 + 2*s + 2)

fx >>
```

Fig. 10.20 Simplifying $\frac{-5}{s-(-1+j)}+\frac{-5}{s-(-1-j)}$

Example 10.39 The code in Fig. 10.21 solves $\ddot{y}(t)+5\dot{y}(t)+6y(t)=4\sin(t+\frac{\pi}{3})$.

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+5*diff(y,t)+6*y==4*sin(t+pi/3);
>> dsolve(ode)

ans =

C1*exp(-3*t) - (2*2^(1/2)*cos(t + (7*pi)/12))/5 + C2*exp(-2*t)

fx >>
```

Fig. 10.21 The code for solving $\ddot{y}(t)+5\dot{y}(t)+6y(t)=4\sin(t+\frac{\pi}{3})$

The `vpa` function takes the symbolic expression as input and the desired number of decimal places as the second argument. It then evaluates the expression numerically and displays the result with the specified precision. By using vpa, you can ensure that the numerical values in the solution are displayed with the desired level of accuracy.

To obtain a more precise numerical approximation, let's apply the vpa function to the calculated result (Fig. 10.22).

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+5*diff(y,t)+6*y==4*sin(t+pi/3);
>> dsolve(ode)

ans =

C1*exp(-3*t) - (2*2^(1/2)*cos(t + (7*pi)/12))/5 + C2*exp(-2*t)

>> vpa(ans,5)

ans =

C2*exp(-2.0*t) - 0.56569*cos(t + 1.8326) + C1*exp(-3.0*t)

fx >>
```

Fig. 10.22 Utilize the `vpa` command to numerically evaluate the symbolic expression

Example 10.40 The code in Fig. 10.23 solves $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 4\sin\left(t + \frac{\pi}{3}\right)$, $y(0) = 0$, $\dot{y}(0) = 1$.

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+5*diff(y,t)+6*y==4*sin(t+pi/3);
>> ic=[y(0)==0,subs(diff(y,t),0)==1];
>> dsolve(ode,ic)

ans =

exp(-2*t)*((6*2^(1/2)*(2^(1/2)/4 - 6^(1/2)/4))/5 - (2*2^(1/2)*(2^(1/2)/4 +

>> vpa(ans,5)

ans =

0.014359*exp(-2.0*t) - 0.16077*exp(-3.0*t) - 0.56569*cos(t + 1.8326)

fx >>
```

Fig. 10.23 The code for solving $\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 4\sin\left(t + \frac{\pi}{3}\right)$, $y(0) = 0$, $\dot{y}(0) = 1$

Example 10.41 The code in Fig. 10.24 solves $\ddot{y}(t) + 2\dot{y}(t) + y(t) = \delta(t)$, $y(0) = \dot{y}(0) = 0$. Note that $\delta(t)$ shows the unit impulse function.

Fig. 10.24 The code for solving $\ddot{y}(t) + 2\dot{y}(t) + y(t) = \delta(t), y(0) = \dot{y}(0) = 0$

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+2*diff(y,t)+y==dirac(t);
>> ic=[y(0)==0,subs(diff(y,t),0)==0];
>> dsolve(ode,ic)

ans =

(t*exp(-t)*sign(t))/2

fx >>
```

According to Fig. 10.24 the solution is $y(t) = \frac{te^{-t}}{2} sign(t)$. Remember that $sign(t) = \begin{cases} +1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$. Therefore, the solution of $\ddot{y}(t) + 2\dot{y}(t) + y(t) = \delta(t), y(0) = \dot{y}(0) = 0$ for $t > 0$ is $y(t) = \frac{te^{-t}}{2}$.

Example 10.42 The code in Fig. 10.25 solves $\ddot{y}(t)+2\dot{y}(t)+y(t) = 6H(t), y(0) = \dot{y}(0) = 0$. Note that $H(t)$ shows the unit step function.

```
Command Window
>> syms y(t)
>> ode=diff(y,t,2)+2*diff(y,t)+y==6*heaviside(t);
>> ic=[y(0)==1,subs(diff(y,t),0)==2];
>> dsolve(ode,ic)

ans =

-exp(-t)*(3*sign(t) - 3*exp(t) - 3*exp(t)*sign(t) + 3*t*sign(t) + 2)

>> expand(ans)

ans =

3*sign(t) - 2*exp(-t) - 3*exp(-t)*sign(t) - 3*t*exp(-t)*sign(t) + 3

fx >>
```

Fig. 10.25 The code for solving $\ddot{y}(t) + 2\dot{y}(t) + y(t) = 6H(t), y(0) = \dot{y}(0) = 0$

According to Fig. 10.25, the solution for $t > 0$ is $y(t) = 6 - 5e^{-t} - 3te^{-t}$.

Example 10.43 The code in Fig. 10.26 solves $\begin{cases} \frac{dx(t)}{dt} = 3x(t) + 4y(t) \\ \frac{dy(t)}{dt} = -4x(t) + 3y(t) \end{cases} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

```
Command Window
>> syms x(t) y(t)
>> eq1=3*x+4*y==diff(x,t);
>> eq2=-4*x+3*y==diff(y,t);
>> ic=[x(0)==0,y(0)==1];
>> [xSol,ySol]=dsolve([eq1,eq2],ic)

xSol =

sin(4*t)*exp(3*t)


ySol =

cos(4*t)*exp(3*t)

fx >>
```

Fig. 10.26 The code for solving $\begin{cases} \frac{dx(t)}{dt} = 3x(t) + 4y(t) \\ \frac{dy(t)}{dt} = -4x(t) + 3y(t) \end{cases} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Example 10.44 The code in Fig. 10.27 plots the impulse response, i.e., the output for the unit impulse input with zero initial conditions, of $\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = 8\dot{u}(t) + u(t)$ on $[0, 6]$ interval. Note that $y(t)$ and $u(t)$ show the output and input, respectively. The transfer function of given differential equation is (the initial conditions are zero):

$$\begin{aligned} &\mathcal{L}\{\ddot{y}(t)\} + \mathcal{L}\{3\dot{y}(t)\} + \mathcal{L}\{4y(t)\} = \mathcal{L}\{8\dot{u}(t)\} \\ &\quad + \mathcal{L}\{u(t)\} \Rightarrow (s^2 + 3s + 4)Y(s) = (8s + 1)U(s) \Rightarrow T(s) \\ &= \frac{Y(s)}{U(s)} = \frac{8s + 1}{s^2 + 3s + 4} \end{aligned}$$

Fig. 10.27 Plotting $\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = 8\dot{u}(t) + u(t)$ on $[0, 6]$ interval

```
Command Window
>> T=tf([8 1],[1 3 4]);
>> impulse(T,[0,6])
>> grid on
fx >>
```

Output of the code shown in Fig. 10.27 is shown in Fig. 10.28.

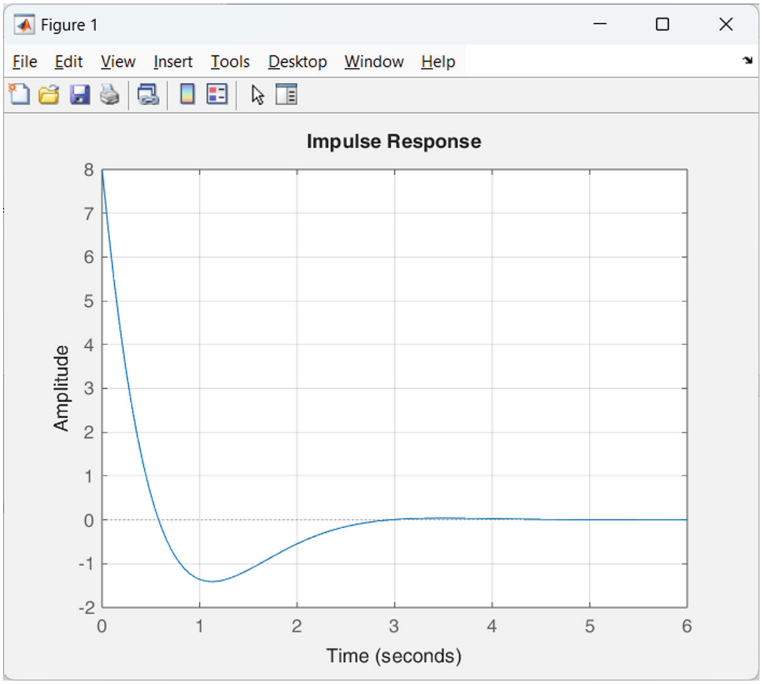
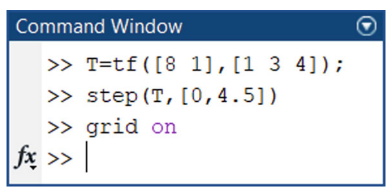


Fig. 10.28 Output of the code shown in Fig. 10.27

Example 10.45 The code in Fig. 10.29 plots the step response, i.e., the output for the unit step input with zero initial conditions, of $\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = 8\dot{u}(t) + u(t)$ on $[0, 4.5]$ interval. Note that $y(t)$ and $u(t)$ show the output and input, respectively. Output of the code shown in Fig. 10.29 is shown in Fig. 10.30.

Fig. 10.29 Plotting step response of $\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = 8\dot{u}(t) + u(t)$ on $[0, 4.5]$ interval



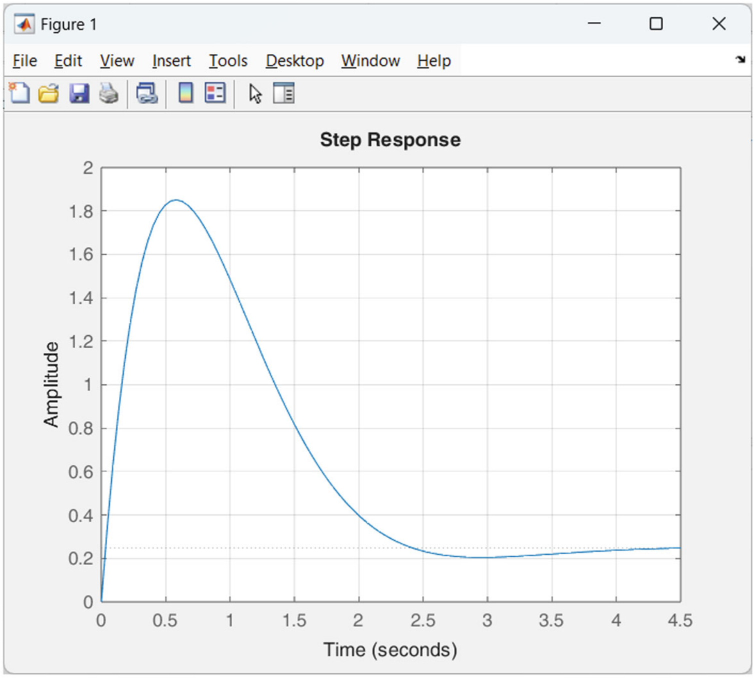


Fig. 10.30 Output of code shown in Fig. 10.29

References for Further Study

1. Advanced Engineering Mathematics (10th edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8th edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

11 The Fourier Series and Transform

11.1 Introduction

Fourier series and transforms are powerful mathematical tools used to analyze functions in terms of their frequency components.

Fourier series represent periodic functions as a sum of sine and cosine waves of different frequencies and amplitudes. This decomposition allows us to understand the underlying frequencies that make up a complex signal. The Fourier transform extends this concept to non-periodic functions, breaking them down into a continuous spectrum of frequencies.

These techniques find applications in various fields, including signal processing, image processing, audio engineering, and physics, enabling us to analyze and manipulate complex signals in the frequency domain.

This chapter begins with a review of Fourier series and transforms. The second part demonstrates MATLAB's application to the problems discussed.

11.2 Periodic Functions and Fourier Series

A function $f(t)$ is said to be periodic with period T if, for every t in the domain of f, $f(t+T)=f(t)$. Fourier Series is a mathematical technique used to represent periodic functions as a sum of sine and cosine functions. It's a powerful tool in signal processing, engineering, and physics.

Periodic function $f(t)=f(t+T)$ can be written as:

$$f(t)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos(n\omega_0 t)+b_n\sin(n\omega_0 t)\right)$$

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_11

where

$$\omega_0 = \frac{2\pi}{T}$$

$$a_0 = \frac{1}{T} \int\limits_{t_0}^{t_0+T} f(t)dt$$

$$a_n = \frac{2}{T} \int\limits_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)dt \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{2}{T} \int\limits_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)dt \quad n = 1, 2, 3, \ldots$$

t_0 is an optional number. Therefore, integrals can be calculated on any interval with length of T. For instance, integrals can be calculated on $\left[\frac{-T}{2}, \frac{+T}{2}\right]$, $[0, T]$ or $[10, 10 + T]$. a_0 is usually called dc component or average value of $f(t)$. $\sqrt{{a_1}^2 + {b_1}^2}$ is called amplitude of the fundamental harmonic. $\sqrt{{a_n}^2 + {b_n}^2}$ is called amplitude of the nth harmonic. For instance, $\sqrt{{a_3}^2 + {b_3}^2}$ is called amplitude of the 3rd harmonic.

Special case I

Fourier series of a periodic even function $f(t)$ is:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)$$

where

$$a_0 = \frac{1}{T} \int\limits_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt = \frac{2}{T} \int\limits_{0}^{\frac{T}{2}} f(t)dt$$

$$a_n = \frac{2}{T} \int\limits_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)dt = \frac{2}{T} \int\limits_{-\frac{T}{2}}^{+\frac{T}{2}} f(t)\cos(n\omega_0 t)dt$$

$$= \frac{4}{T} \int\limits_{0}^{+\frac{T}{2}} f(t)\cos(n\omega_0 t)dt \quad n = 1, 2, 3, \ldots$$

Remember that function $f(t)$ is even when $\forall t : f(-t) = f(t)$.

Special case II

Fourier series of a periodic odd function $f(t)$ is:

$$f(t) = \sum_{n=1}^{\infty} (b_n \sin(n\omega_0 t))$$

where

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)dt = \frac{2}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f(t)\sin(n\omega_0 t)dt$$

$$= \frac{4}{T}\int_{0}^{\frac{T}{2}} f(t)\sin(n\omega_0 t)dt \quad n = 1, 2, 3, \ldots$$

Remember that function $f(t)$ is odd when $\forall t : f(-t) = -f(t)$ and the product of two odd functions $f(t)$ and $\sin(n\omega_0 t)$ is an even function.

11.3 Review of Some Important Relationships

The following formulas help us in calculating Fourier series.

(A) $\forall \text{n} \in \mathbb{Z} : \sin(n\pi) = 0$

(B) $\cos(n\pi) = (-1)^n = \begin{cases} +1 & n\,is\,even \\ -1 & n\,is\,odd \end{cases}$

(C) $e^{j2n\pi} = \cos(2n\pi) + j\sin(2n\pi) = 1 + j0 = 1$

(D) $e^{jn\pi} = \cos(n\pi) + j\sin(n\pi) = (-1)^n + j0 = \begin{cases} +1 & n\,is\,even \\ -1 & n\,is\,odd \end{cases}$

Table 11.1 helps you to see why $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$.

Table 11.1 Values of $\cos(n\pi)$ and $\sin(n\pi)$ for $0 \le n \le 6$

n	$\cos(n\pi)$	$\sin(n\pi)$
0	$\cos(0) = +1$	$\sin(0) = 0$
1	$\cos(\pi) = -1$	$\sin(\pi) = 0$
2	$\cos(2\pi) = +1$	$\sin(2\pi) = 0$
3	$\cos(3\pi) = -1$	$\sin(3\pi) = 0$
4	$\cos(4\pi) = +1$	$\sin(4\pi) = 0$
5	$\cos(5\pi) = -1$	$\sin(5\pi) = 0$
6	$\cos(6\pi) = +1$	$\sin(6\pi) = 0$

11.4 Complex Form of Fourier Series

Trigonometric form of Fourier series is introduced in the previous section. This section introduces the complex form of Fourier series.

Periodic function $f(t) = f(t+T)$ can be written as:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where

$$\omega_0 = \frac{2\pi}{T}$$

$$c_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

The relationship between the complex and trigonometric form coefficients is:

$$c_0 = a_0$$

$$c_n = \frac{a_n - jb_n}{2} \quad n = 1, 2, 3, \ldots$$

$$c_{-n} = \overline{c_n} = \frac{a_n + jb_n}{2} \quad n = 1, 2, 3, \ldots$$

11.5 Studied Examples

Let's study some examples.

Example 11.1 Find the value (a_0) of the waveform shown in Fig. 11.1.

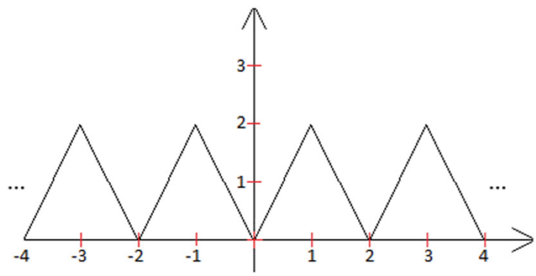


Fig. 11.1 Waveform of Example 11.1

a_0 can be found by calculating the area under one period (Fig. 11.2) and dividing it by the period. According to Fig. 11.1 period T equals to 2 and the area under one period is $\frac{1}{2} \times 2 \times 2 = 2$. Therefore, $a_0 = \frac{2}{2} = 1$.

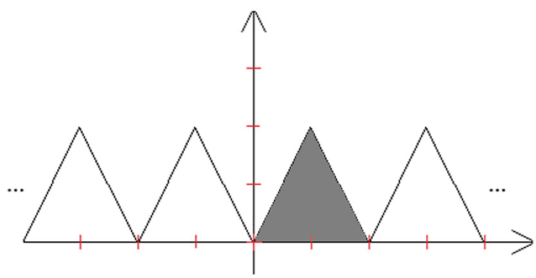


Fig. 11.2 Area under one period

You can use the graphical method to calculate the value of a_0 as well. Visually estimate a horizontal line (Fig. 11.3) that divides the area enclosed by the function curve into equal positive and negative parts (Fig. 11.4). This line represents the average value of the function.

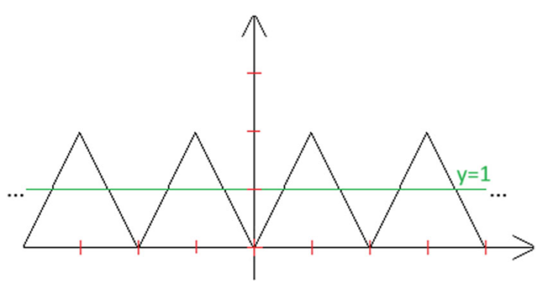


Fig. 11.3 Graphical method to calculate a_0

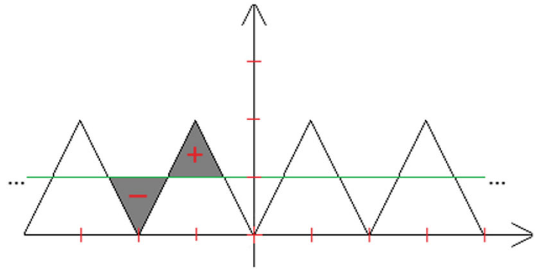


Fig. 11.4 The areas above and below the line are equal

Example 11.2 Find the Fourier series of $f(t) = 1 + 3\cos(t) + 2\cos(3t) + 0.5\sin(5t)$.

Compare the given function with $f(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t))$. For this function $a_0 = 1, a_1 = 3, a_3 = 2$ and $b_5 = 0.5$. Other coefficients are zero.

Example 11.3 Find the Fourier series representation of periodic signal shown in Fig. 11.5.

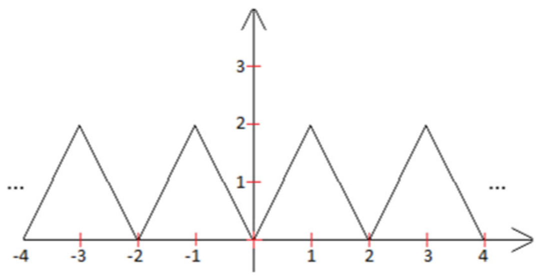


Fig. 11.5 Waveform of Example 11.3

Let's derive the equation for one period of the function shown in Fig. 11.5.

$$f(t) = \begin{cases} +2t & 0 < t \le 1 \\ -2t & -1 \le t \le 0 \end{cases}$$

$$T = 2 \Rightarrow \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

Given function is even. Therefore, there is no need to calculate b_n terms.

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\pi t)$$

Value of a_0 can be found by calculating the area under one period (Fig. 11.6) and dividing it by the period. Period T equals to 2 and the area under one period is $S = \frac{1}{2} \times 2 \times 2 = 2$. Therefore, $a_0 = \frac{S}{T} = \frac{\frac{1}{2}\times 2\times 2}{2} = 1$.

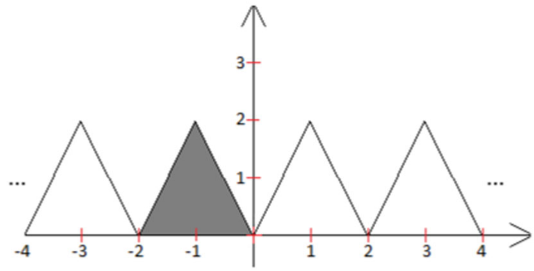


Fig. 11.6 Area under one period

You can use the following methods to calculate a_0 as well:

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt \overset{f(t) is\, even}{\Rightarrow} \frac{2}{T}\int_0^{\frac{T}{2}} f(t)dt = \frac{2}{2}\int_0^1 2tdt = t^2\Big|_0^1 = 1$$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt = \frac{1}{2}\left(\int_{-1}^{0} -2tdt + \int_0^1 2tdt\right) = \frac{1}{2}\left(-t^2\Big|_{-1}^{0} + t^2\Big|_0^1\right)$$
$$= \frac{1}{2}(0-(-1)+1-0) = 1$$

Let's calculate $a_{n=1,2,3,\ldots}$ terms.

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)dt = \frac{2}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f(t)\cos(n\omega_0 t)dt$$
$$= \frac{4}{T}\int_0^{+\frac{T}{2}} f(t)\cos(n\omega_0 t)dt \quad n = 1, 2, 3, \ldots$$

$$a_n = \frac{4}{2}\int_0^1 2t\cos(n\pi t)dt = 2\int_0^1 2t\cos(n\pi t)dt = 4\int_0^1 t\cos(n\pi t)dt$$

By means of tabular integration (Fig. 11.7) we obtain:

$$a_n = 4\int_0^1 t\cos(n\pi t)dt = 4\left(t\frac{\sin(n\pi t)}{n\pi} + \frac{\cos(n\pi t)}{n^2\pi^2}\right)\Big|_0^1$$
$$= 4\left(\frac{\cos(n\pi)}{n^2\pi^2} - \frac{\cos(0)}{n^2\pi^2}\right) = \begin{cases} 0 & for\ even\ n \\ \frac{-8}{n^2\pi^2} & for\ odd\ n \end{cases}$$

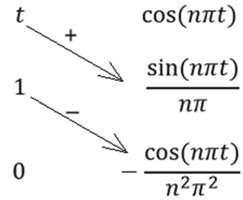


Fig. 11.7 Calculation of $\int t\cos(n\pi t)dt$

Therefore, Fourier series of the function shown in Fig. 11.5 is $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi t)$, where:

$$a_n = \begin{cases} 0 & for\, even\, n \\ \frac{-8}{n^2\pi^2} & for\, odd\; n \\ 1 & n = 0 \end{cases}$$

Example 11.4 Determine the complex Fourier series representation of the function depicted in Fig. 11.8.

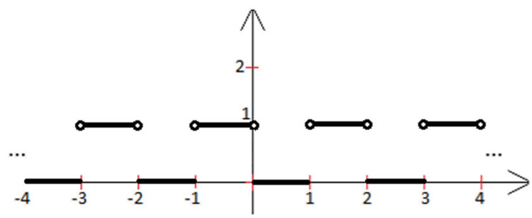


Fig. 11.8 Waveform of Example 11.4

Let's derive the equation for one period of the function shown in Fig. 11.4.

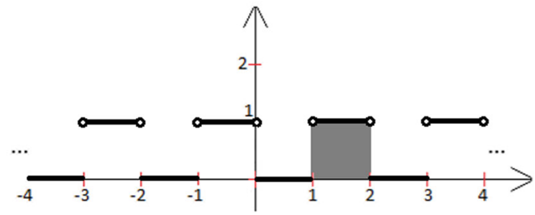
$$f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 1 & 1 < t < 2 \end{cases}$$

$$T = 2 \Rightarrow \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi t}$$

Let's calculate c_0. $c_0 = a_0$. Therefore, we can calculate the area under one period and divide it by the period. According to Fig. 11.9, $c_0 = a_0 = \frac{S}{T} = \frac{1\times 1}{2} = 0.5$.

Fig. 11.9 Area under one period



Let's calculate $c_{n=1,2,3,\ldots}$:

$$c_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-jn\omega_0 t}dtn = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$c_n = \frac{1}{2}\int_0^2 f(t)e^{-jn\pi t}dt = \frac{1}{2}\left(\int_0^1 0e^{-jn\pi t}dt + \int_1^2 e^{-jn\pi t}dt\right) = \frac{1}{2}\left(\int_1^2 e^{-jn\pi t}dt\right)$$

when $n \neq 0$:

$$\frac{1}{2}\left(\int_1^2 e^{-jn\pi t}dt\right) = \frac{1}{2}\left(\frac{e^{-jn\pi t}}{-jn\pi}\Bigg|_1^2\right)n \neq 0$$

Let's simplify $\frac{1}{2}\left(\frac{e^{-jn\pi t}}{-jn\pi}\Big|_1^2\right)$:

$$\frac{1}{-j2\pi n}\left(e^{-jn\pi t}\Big|_1^2\right) = \frac{j}{2\pi n}\left(e^{-jn\pi t}\Big|_1^2\right) = \frac{j}{2\pi n}\left(e^{-j2\pi n} - e^{-j\pi n}\right) = \frac{j}{2\pi n}\left(1 - (-1)^n\right)$$
$$= \begin{cases} 0 & n \text{ is even} \\ \frac{j}{2\pi n} & n \text{ is odd} \end{cases}$$

Therefore, complex Fourier series representation of given function is:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\pi t}$$

where:

$$c_n = \begin{cases} 0 & n \text{ is even} \\ 0.5 & n = 0 \\ \frac{j}{2\pi n} & n \text{ is odd} \end{cases}$$

11.6 Fourier Transform

Fourier transform takes a time-domain signal (function) and converts it into a frequency-domain representation. Fourier transform of function $f(t)$ is defined as:

$$F(j\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$

The inverse Fourier transform converts a frequency-domain signal back into a time-domain signal. Inverse Fourier transform is defined as:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)e^{j\omega t}d\omega$$

Special case I
When $f(t)$ is an even function, the Fourier transform simplifies to:

$$\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} f(t)(\cos(\omega t) - j\sin(\omega t))dt = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)dt$$
$$-j\int_{-\infty}^{+\infty} f(t)\sin(\omega t)dt$$

$$f(-t) = f(t) \Rightarrow \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)dt$$
$$= 2\int_{0}^{+\infty} f(t)\cos(\omega t)dt$$

Special case II
When $f(t)$ is an odd function, the Fourier transform simplifies to:

$$\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} f(t)(\cos(\omega t) - j\sin(\omega t))dt = \int_{-\infty}^{+\infty} f(t)\cos(\omega t)dt$$
$$-j\int_{-\infty}^{+\infty} f(t)\sin(\omega t)dt$$

$$f(-t) = -f(t) \Rightarrow \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt = -j\int_{-\infty}^{+\infty} f(t)\sin(\omega t)dt$$
$$= -2j\int_{0}^{+\infty} f(t)\sin(\omega t)dt$$

11.7 Fourier Transform Properties

The following are the most important Fourier transform properties:

$$af_1(t) + bf_2(t) \leftrightarrow aF_1(j\omega) + bF_2(j\omega)$$

$$f(at) \leftrightarrow \frac{1}{|a|}F\left(j\frac{\omega}{a}\right)$$

$$f(t - t_0) \leftrightarrow e^{-j\omega t_0}F(j\omega)$$

$$tf(t) \leftrightarrow j\frac{d}{d\omega}(F(j\omega))$$

$$f_1(t) * f_2(t) = \int_{-\infty}^{+\infty} f_1(\tau)f_2(t-\tau)d\tau \leftrightarrow F_1(j\omega) \times F_2(j\omega)$$

$$f_1(t) \times f_2(t) \leftrightarrow \frac{1}{2\pi}F_1(j\omega) * F_2(j\omega)$$

$$\frac{df(t)}{dt} \leftrightarrow j\omega \times F(j\omega)$$

$$\frac{d^2f(t)}{dt^2} \leftrightarrow -\omega^2 \times F(j\omega)$$

Table 11.2 presents the Fourier transforms of frequently encountered functions in engineering.

Table 11.2 Brief table of Fourier transform pairs

$\delta(t) \leftrightarrow 1$
$\delta(t-t_0) \leftrightarrow e^{-j\omega t_0}$
$1 \leftrightarrow 2\pi\delta(\omega)$
$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$
$e^{-at}H(t), a>0 \leftrightarrow \frac{1}{a+j\omega}$
$te^{-at}H(t), a>0 \leftrightarrow \frac{1}{(a+j\omega)^2}$
$H(t) \leftrightarrow \frac{1}{j\omega}+\pi\delta(\omega)$
$\sin(\omega_0 t) \leftrightarrow -j\pi(\delta(\omega-\omega_0)-\delta(\omega+\omega_0))$
$\cos(\omega_0 t) \leftrightarrow \pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$
$\frac{\sin(\omega_c t)}{\pi t} \leftrightarrow \begin{cases} 1, -\omega_c < \omega < \omega_c \\ 0, otherwise \end{cases}$
$\begin{cases} 1, -M \le t \le M \\ 0, otherwise \end{cases} \leftrightarrow \frac{\sin(\omega M)}{\omega/2}$

Example 11.5 Determine the Fourier transform of $f(t)=e^{-|t|} = \begin{cases} e^{-t} & t \ge 0 \\ e^{t} & t<0 \end{cases}$.

Graph of $f(t)$ is shown in Fig. 11.10.

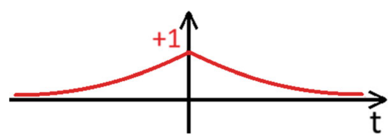


Fig. 11.10 Plot of $f(t)=e^{-|t|}$

$$F(j\omega)=\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{0} e^{t}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-t}e^{-j\omega t}dt$$

Let's calculate the obtained integrals one by one:

$$\int_{-\infty}^{0} e^{t}e^{-j\omega t}dt = \int_{-\infty}^{0} e^{(1-j\omega)t}dt = \frac{e^{(1-j\omega)t}}{1-j\omega}\bigg|_{-\infty}^{0} = \frac{e^{(1-j\omega)\times 0}}{1-j\omega} - \frac{e^{(1-j\omega)\times -\infty}}{1-j\omega}$$

$$= \frac{1}{1-j\omega} - 0 = \frac{1}{1-j\omega}$$

Remember that $\lim_{t\to-\infty} e^{(a+jb)t} = 0$ when $a > 0$.

$$\int_0^\infty e^{-t}e^{-j\omega t}dt = \int_0^\infty e^{(-1-j\omega)t}dt = \left.\frac{e^{(-1-j\omega)t}}{-1-j\omega}\right|_0^\infty = \frac{e^{(-1-j\omega)\times\infty}}{-1-j\omega} - \frac{e^{(-1-j\omega)\times 0}}{-1-j\omega}$$
$$= 0 - \frac{1}{-1-j\omega} = \frac{1}{1+j\omega}$$

Remember that $\lim_{t\to+\infty} e^{(a+jb)t} = 0$ when $a < 0$.
Fourier transform of $f(t)$ is:

$$F(j\omega) = \int_{-\infty}^0 e^t e^{-j\omega t}dt + \int_0^\infty e^{-t}e^{-j\omega t}dt = \frac{1}{1-j\omega} + \frac{1}{1+j\omega} = \frac{1+j\omega+1-j\omega}{(1-j\omega)(1+j\omega)}$$
$$= \frac{2}{1+\omega^2}$$

Example 11.6 Determine the Fourier transform of $f(t) = \begin{cases} 1 & |t| < 1 \\ 0 & otherwise \end{cases}$.

Graph of $f(t)$ is shown in Fig. 11.11.

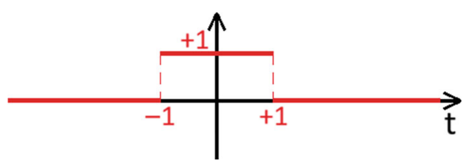


Fig. 11.11 Plot of $f(t) = \begin{cases} 1 & |t| < 1 \\ 0 & otherwise \end{cases}$

$$F(j\omega) = \int_{-1}^1 1\times e^{-j\omega t}dt = \int_{-1}^1 (\cos(\omega t) - j\sin(\omega t))dt = \int_{-1}^1 \cos(\omega t)dt$$
$$-j\int_{-1}^1 \sin(\omega t)dt$$

Let's calculate the obtained integrals one by one:

$$\int_{-1}^1 \cos(\omega t)dt = 2\int_0^1 \cos(\omega t)dt = 2\left.\frac{\sin(\omega t)}{\omega}\right|_0^1 = 2\frac{\sin(\omega)}{\omega}$$

$$j\int_{-1}^{1}\sin(\omega t)dt = j0 = 0$$

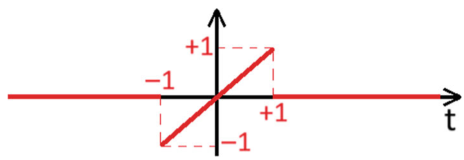
Fourier transform of $f(t)$ is:

$$F(j\omega) = \int_{-1}^{1} 1\times e^{-j\omega t}dt = \int_{-1}^{1}\cos(\omega t)dt - j\int_{-1}^{1}\sin(\omega t)dt = 2\frac{\sin(\omega)}{\omega}$$

Example 11.7 Determine the Fourier transform of $f(t) = \begin{cases} t & |t| < 1 \\ 0 & otherwise \end{cases}$.

Graph of $f(t)$ is shown in Fig. 11.12.

Fig. 11.12 Plot of $f(t) = \begin{cases} t & |t| < 1 \\ 0 & otherwise \end{cases}$



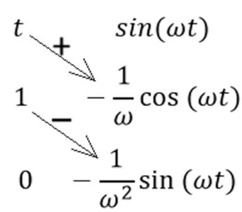
$$F(j\omega) = \int_{-1}^{1} t\times e^{-j\omega t}dt = \int_{-1}^{1} t(\cos(\omega t) - j\sin(\omega t))dt = \int_{-1}^{1} t\cos(\omega t)dt$$
$$-j\int_{-1}^{1} t\sin(\omega t)dt$$

Let’s calculate the obtained integrals one by one:

$\int_{-1}^{1}t\cos(\omega t)dt = 0$ since integrand is an odd function. Note that the product of an even and an odd function is odd.

$t\sin(\omega t)$ is even. Therefore, $\int_{-1}^{1}t\sin(\omega t)dt = 2\int_{0}^{1}t\sin(\omega t)dt$. The tabular integration technique is employed to compute the $\int_{-1}^{1}t\sin(\omega t)dt$ (Fig. 11.13).

Fig. 11.13 Calculation of $\int t\sin(\omega t)dt$



$$-j\int_{-1}^{1} t\sin(\omega t)dt = -2j\int_{0}^{1} t\sin(\omega t)dt = -2j\left(-\frac{t}{\omega}\cos(\omega t) + \frac{1}{\omega^2}\sin(\omega t)\Big|_0^1\right)$$

$$= -2j \times \left(-\frac{1}{\omega}\cos(\omega) + \frac{1}{\omega^2}\sin(\omega)\right)$$

Fourier transform of $f(t)$ is:

$$F(j\omega) = \int_{-1}^{1} te^{-j\omega t}dt = -2j \times \left(-\frac{1}{\omega}\cos(\omega) + \frac{1}{\omega^2}\sin(\omega)\right)$$

11.8 Problem Solving with MATLAB®

This section provides examples of how MATLAB can be employed to solve the types of problems encountered in this chapter.

Example 11.8 The code in Fig. 11.14 calculates $\int t \cdot \cos(n\pi t)dt$.

Fig. 11.14 Calculation of $\int t \cdot \cos(n\pi t)dt$

```
Command Window
>> syms n t
>> int(t*cos(n*pi*t),t)

ans =

(cos(pi*n*t) + n*t*pi*sin(pi*n*t))/(n^2*pi^2)

>> pretty(ans)
cos(pi n t) + n t pi sin(pi n t)
--------------------------------
              2   2
             n  pi
fx >>
```

Example 11.9 The code in Fig. 11.15 calculates $\int_0^1 t \cdot \cos(n\pi t)dt$.

Fig. 11.15 Calculation of $\int_0^1 t \cdot \cos(n\pi t)dt$

```
Command Window
>> syms n t
>> int(t*cos(n*pi*t),t,0,1)

ans =

-(2*sin((pi*n)/2)^2 - n*pi*sin(pi*n))/(n^2*pi^2)

>> pretty(ans)
      / pi n \2
  sin| ---- |   2 - n pi sin(pi n)
      \  2  /
- ------------------------------
                  2  2
                 n  pi
fx >>
```

Example 11.10 The code in Fig. 11.16 calculates $\int_{-1}^{1} t \cdot \sin(\omega t)dt$.

Fig. 11.16 Calculation of $\int_{-1}^{1} t \cdot \sin(\omega t)dt$

```
Command Window
>> syms w t
>> int(t*sin(w*t),t,-1,1)

ans =

(2*(sin(w) - w*cos(w)))/w^2

>> pretty(ans)
2 (sin(w) - w cos(w))
----------------------
            2
           w
fx >>
```

Example 11.11 MATLAB calculates the Fourier transform using $F(\omega) = c \int_{-\infty}^{+\infty} f(t)e^{s\times j\omega t}$. Value of c and s can be determined by the user using the `sympref('FourierParameters',[c s])` command. The code in Fig. 11.17 or 11.18 asks MATLAB to calculate the Fourier transform using the formula $1 \times \int_{-\infty}^{+\infty} f(t)e^{-1\times j\omega t} = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$. In most books the Fourier transform is defined as $\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$.

You only need to run the code in Fig. 11.17 or 11.18 once to set the Fourier parameters. Subsequent Fourier transform calculations will use the new settings. The MATLAB's default settings can be restored with the code shown in Fig. 11.18.

Fig. 11.17 $\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$ is used to calculate the Fourier transform

```
Command Window
>> sympref('FourierParameters',[1 -1]);
fx >> |
```

```
Command Window
fx >> sympref('FourierParameters', 'default');
```

Fig. 11.18 $\int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$ is used to calculate the Fourier transform

All Fourier transform computations in this chapter are performed using the parameter settings specified in Fig. 11.17 or 11.18.

Example 11.12 The code in Fig. 11.19 calculates the Fourier transform of unit impulse function.

Fig. 11.19 Fourier transform of unit impulse function

```
Command Window
>> syms t
>> fourier(dirac(t))

ans =

1

fx >>
```

Example 11.13 The code in Fig. 11.20 calculates the Fourier transform of unit step function.

Fig. 11.20 Fourier transform of unit step function

```
Command Window
>> syms t
>> fourier(heaviside(t))

ans =

pi*dirac(w) - 1i/w

fx >>
```

Example 11.14 The code in Fig. 11.21 calculates the Fourier transform of $f(t) = e^{-3t}H(t)$. Note that $H(t)$ represent the unit step function.

Fig. 11.21 Fourier transform of $f(t) = e^{-3t}H(t)$

```
Command Window
>> syms t
>> fourier(exp(-3*t)*heaviside(t))

ans =

1/(3 + w*1i)

fx >>
```

Example 11.15 The code in Fig. 11.22 calculates the Fourier transform of $f(t) = e^{-|t|}$.

Fig. 11.22 Fourier transform of $f(t) = e^{-|t|}$

```
Command Window
>> syms t
>> fourier(exp(-abs(t)))

ans =

2/(w^2 + 1)

fx >>
```

Example 11.16 The code in Fig. 11.23 calculates the Fourier transform of $f(t) = \frac{\sin(t)}{\pi t}$.

Fig. 11.23 Fourier transform of $f(t) = \frac{\sin(t)}{\pi t}$

```
Command Window
>> syms t
>> fourier(sin(t)/pi/t)

ans =

-(pi*heaviside(- w - 1) - pi*heaviside(1 - w))/pi

>> simplify(ans)

ans =

heaviside(w + 1) - heaviside(w - 1)

fx >>
```

Example 11.17 The code in Fig. 11.24 calculates the Fourier transform of $f(t) = \sin(3t)$.

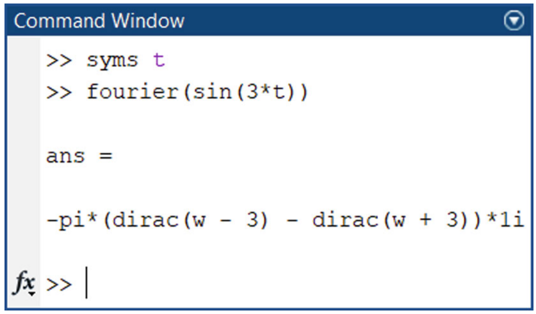


Fig. 11.24 Fourier transform of $f(t) = \sin(3t)$

References for Further Study

1. Advanced Engineering Mathematics (10^{th} edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8^{th} edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

Simulating Dynamic Systems with Simulink® 12

12.1 Introduction

Simulink® is a powerful graphical programming environment designed for modeling, simulating, and analyzing dynamical systems. It's part of the MATLAB suite of tools, offering a user-friendly interface for creating block diagrams that represent system components and their interactions.

This chapter shows how to simulate dynamic systems using Simulink.

12.2 Integrator Block

Integrator blocks are fundamental building blocks in simulation tools for modeling dynamic systems. This section illustrates the application of integrator blocks in modeling dynamic systems.

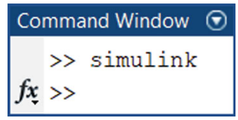
An unstimulated dynamic system is one that evolves over time without any external input. Its behavior is solely governed by its internal dynamics. The following example models an unstimulated system in Simulink.

Example 12.1 Use Simulink to model and solve the $\begin{cases} \frac{dx(t)}{dt} = 2x(t) + 3y(t) \\ \frac{dy(t)}{dt} = x(t) + 4y(t) \end{cases}$ with initial conditions $x(0) = 0$ and $y(0) = 1$.

Use the `simulink` command (Fig. 12.1) to run the Simulink. The Simulink Start Page (Fig. 12.2) will appear shortly.

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_12

Fig. 12.1 `simulink` command



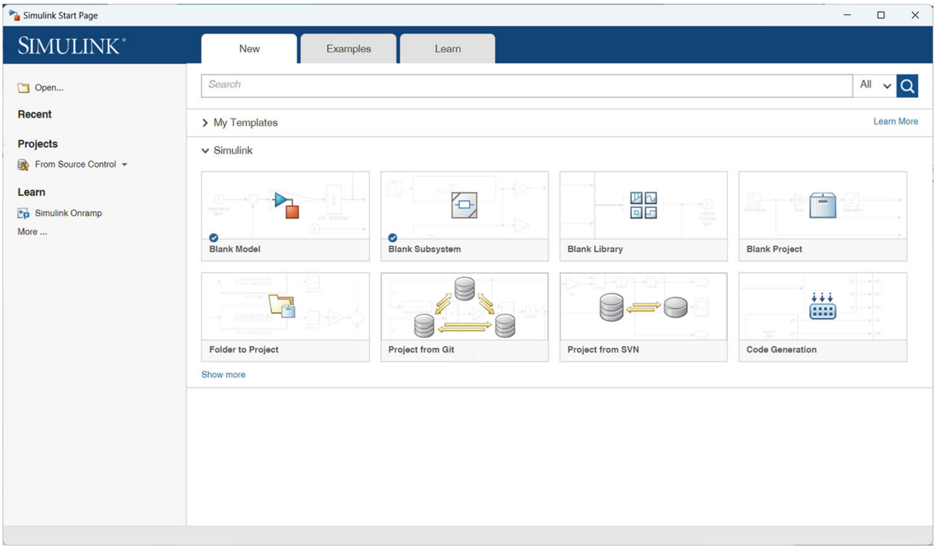
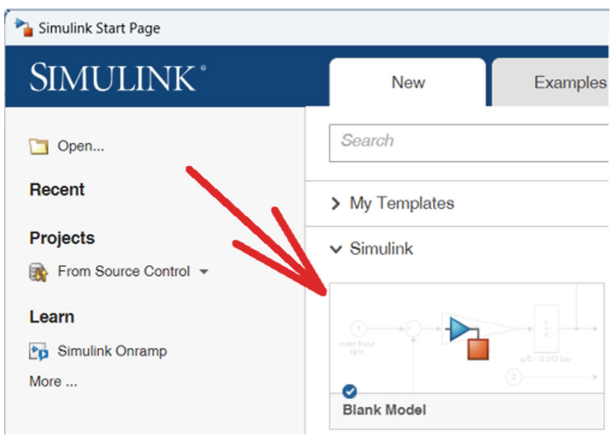


Fig. 12.2 Simulink Start Page window

Choose the Blank Model template (Fig. 12.3) to create a new, empty model (Fig. 12.4).

Fig. 12.3 Blank Model



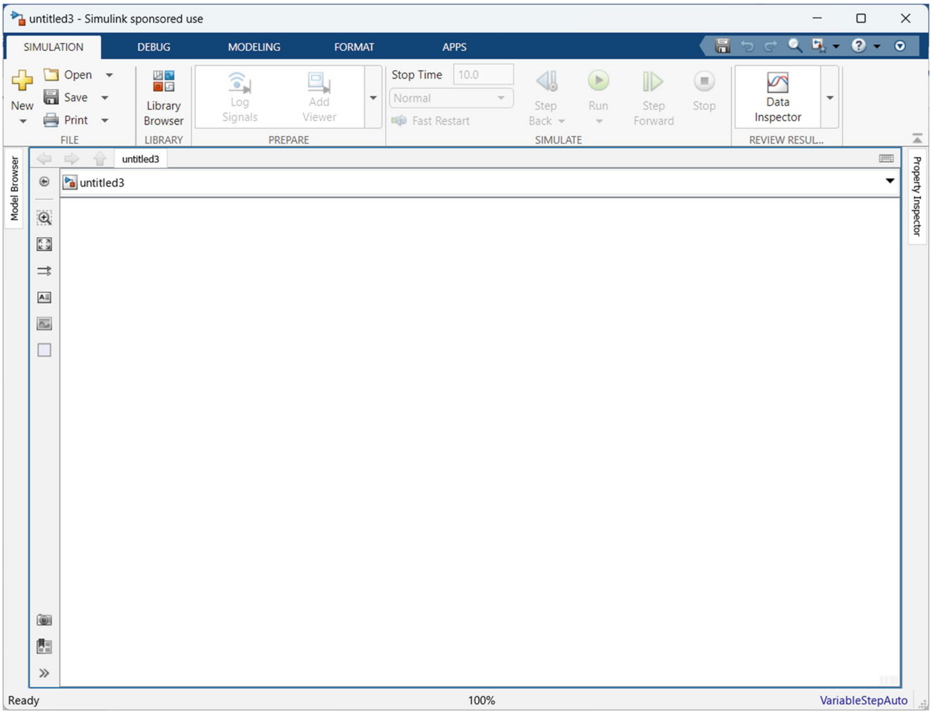
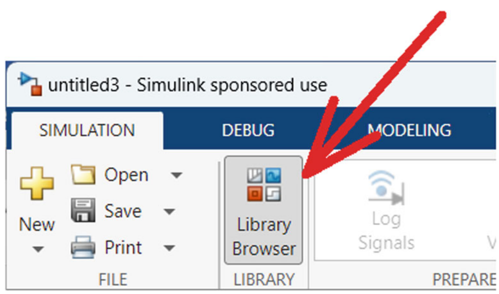


Fig. 12.4 Simulink environment

Clicking on the Library Browser icon (Fig. 12.5) will open the Library Browser window (Fig. 12.6).

Fig. 12.5 The Library Browser button



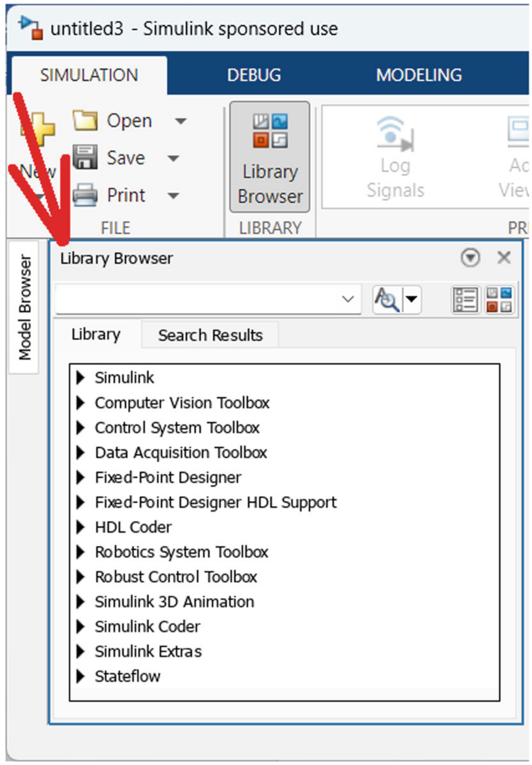


Fig. 12.6 The Library Browser window

Select Simulink (Fig. 12.7).

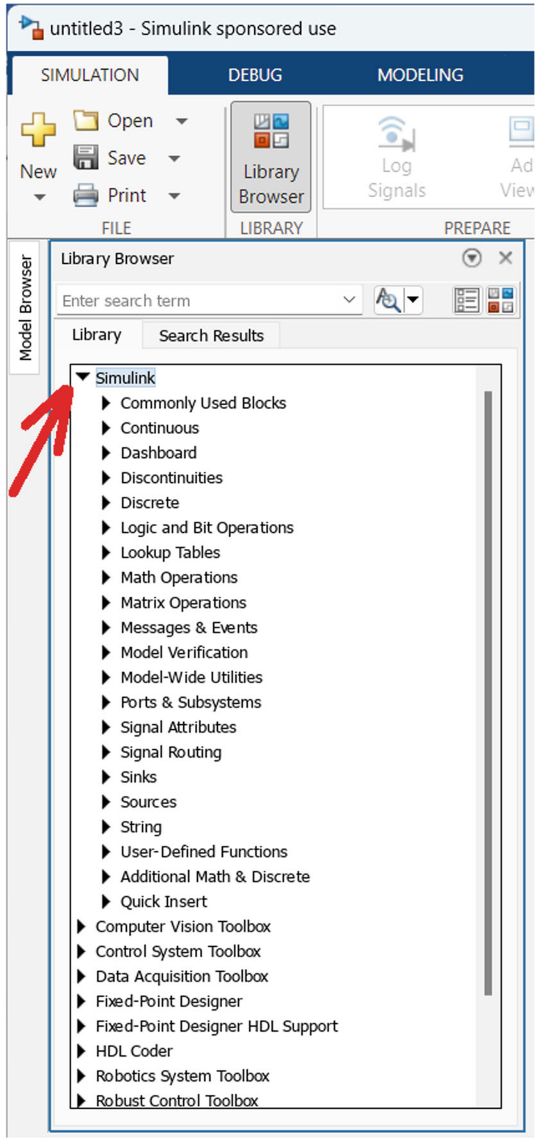
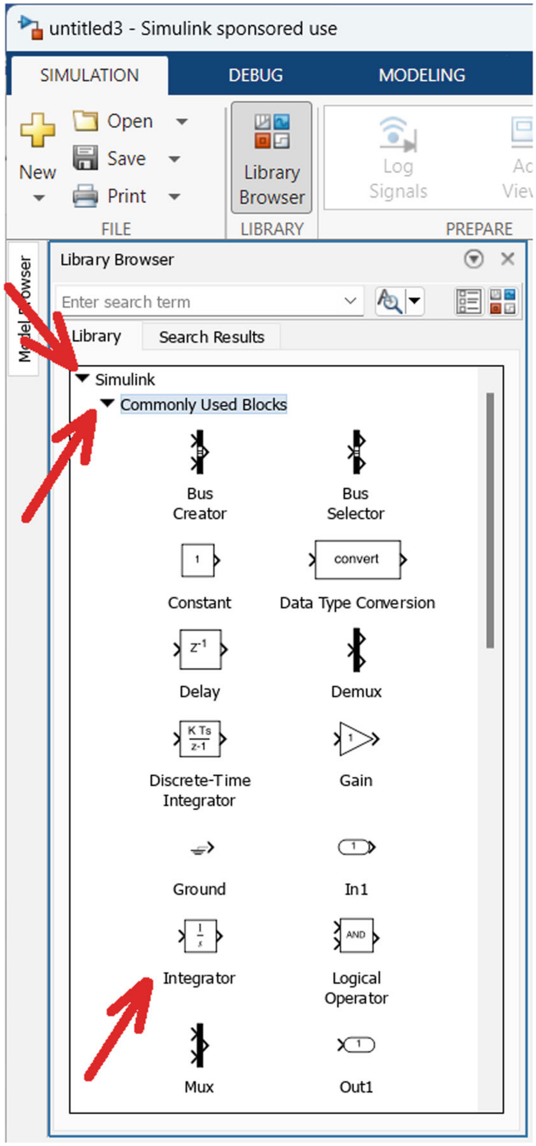


Fig. 12.7 The Simulink library

Drag two Integrator blocks (Fig. 12.8) onto the Simulink canvas (Fig. 12.9).

Fig. 12.8 The Integrator block



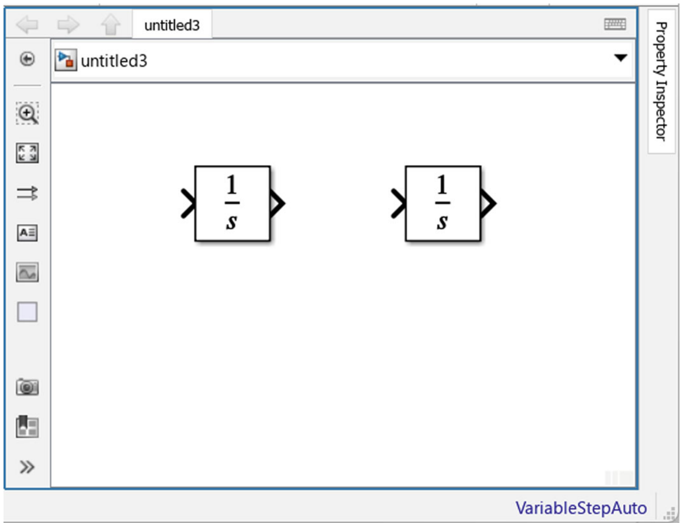


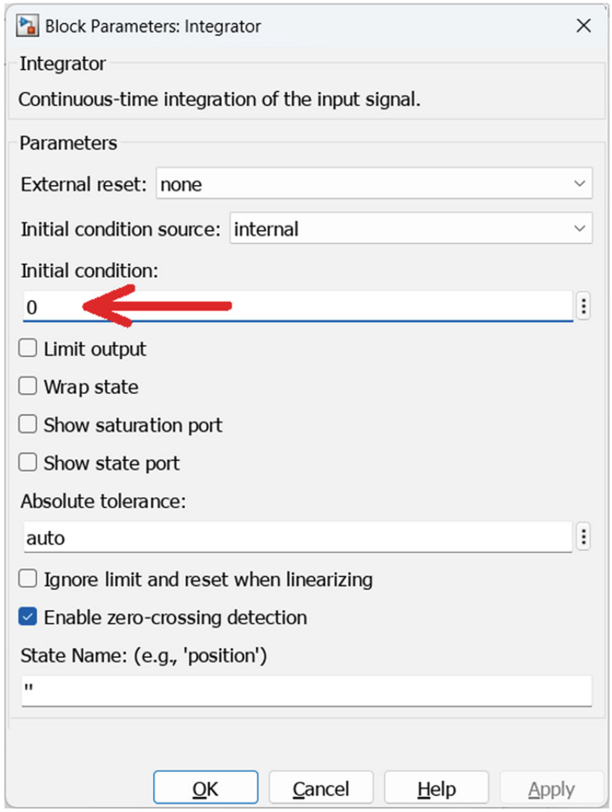
Fig. 12.9 Two integrator blocks are added to the Simulink model

The left integrator generates the $x(t)$, and the right integrator generates the $y(t)$ (Fig. 12.10).

Fig. 12.10 Inputs and outputs of integrators

Double-click the left integrator in Fig. 12.9 (the integrator that generates $x(t)$) and set its initial condition to 0 (Fig. 12.11).

Fig. 12.11 Settings of left integrator



Double-click the right integrator in Fig. 12.9 (the integrator that generates $y(t)$) and set its initial condition to 1 (Fig. 12.12).

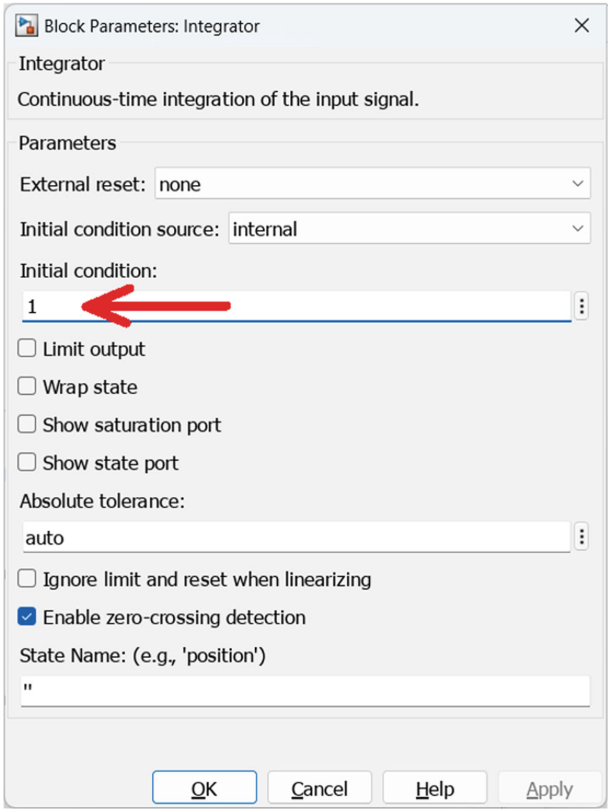


Fig. 12.12 Settings of right integrator

Add three Gain blocks (Fig. 12.13) and two Sum blocks (Fig. 12.14) to the Simulink model (Fig. 12.15). You can rotate blocks by pressing Ctrl+R.

Fig. 12.13 The Gain block

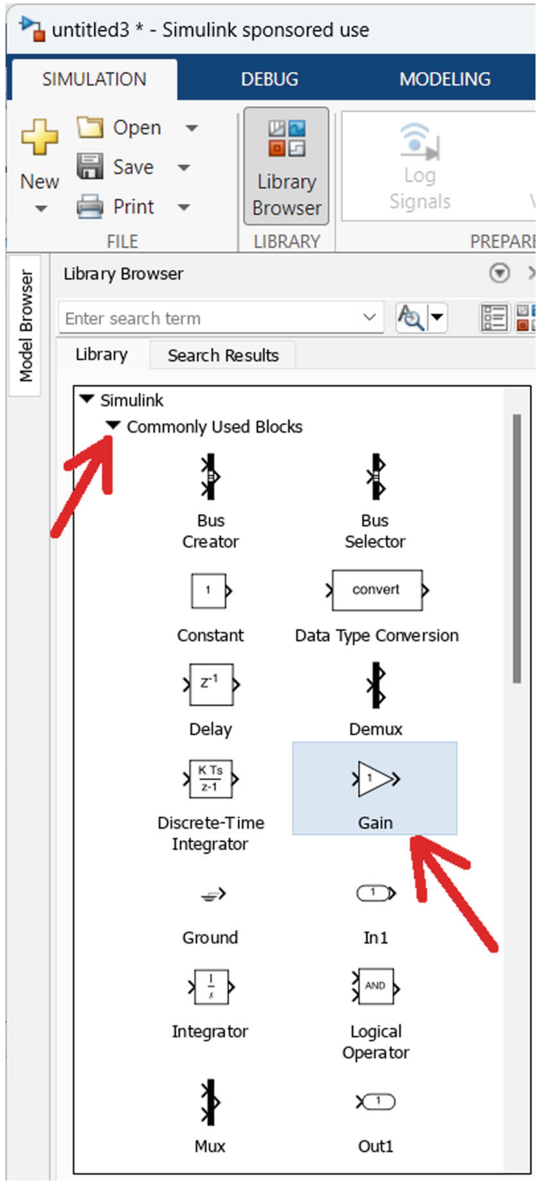
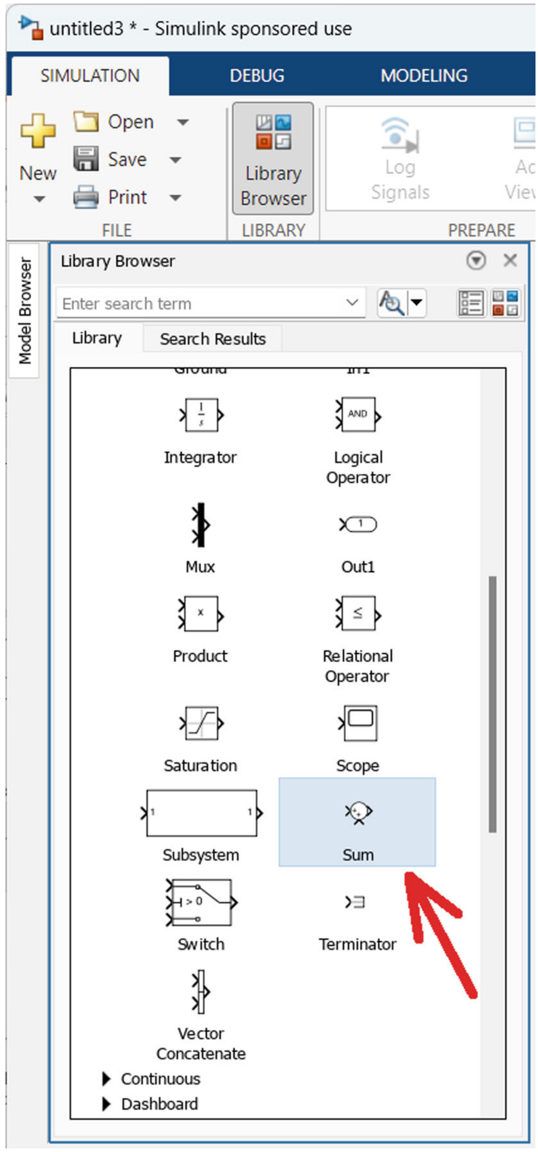


Fig. 12.14 The Sum block



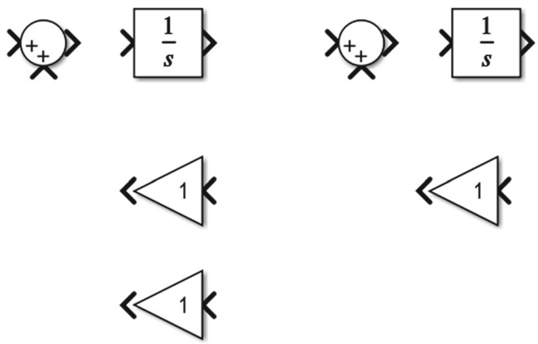


Fig. 12.15 Blocks added to the model

Connect the blocks as shown in Fig. 12.16.

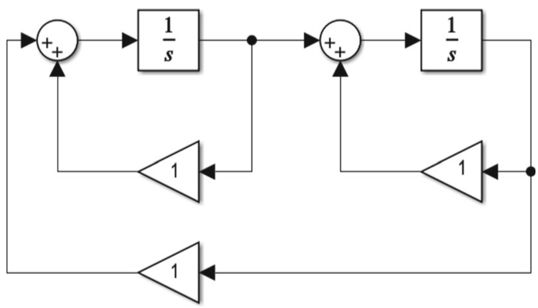


Fig. 12.16 Blocks are connected

Double-click the Gain blocks and set their Gain values (Fig. 12.17) as shown in Fig. 12.18

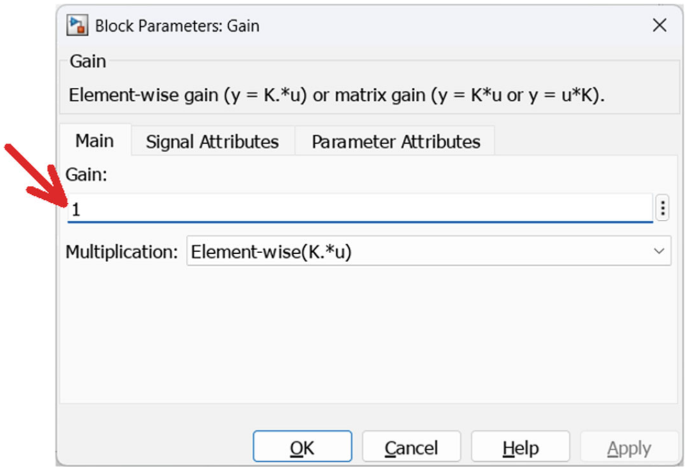


Fig. 12.17 The Gain box

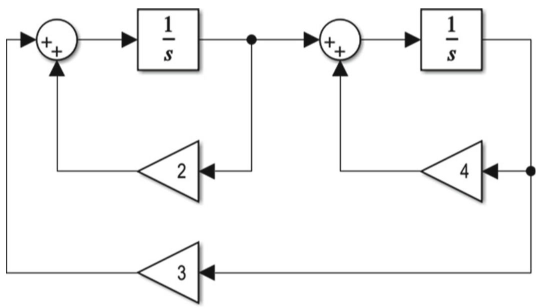


Fig. 12.18 Simulink model with updated gains

Add a Multiplexer (Fig. 12.19) and a Scope (Fig. 12.20) to the Simulink model (Fig. 12.21).

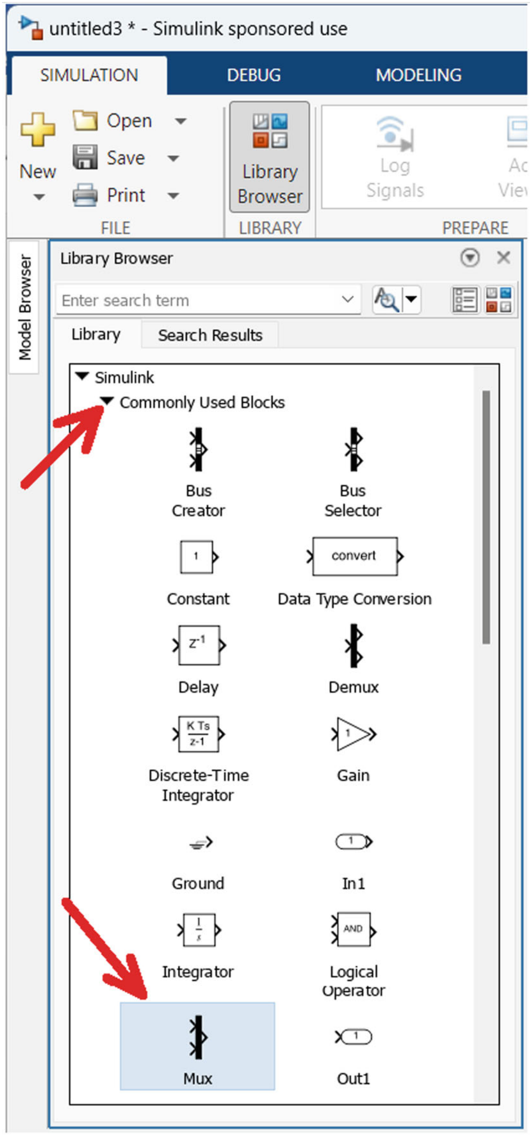


Fig. 12.19 The Multiplexer block

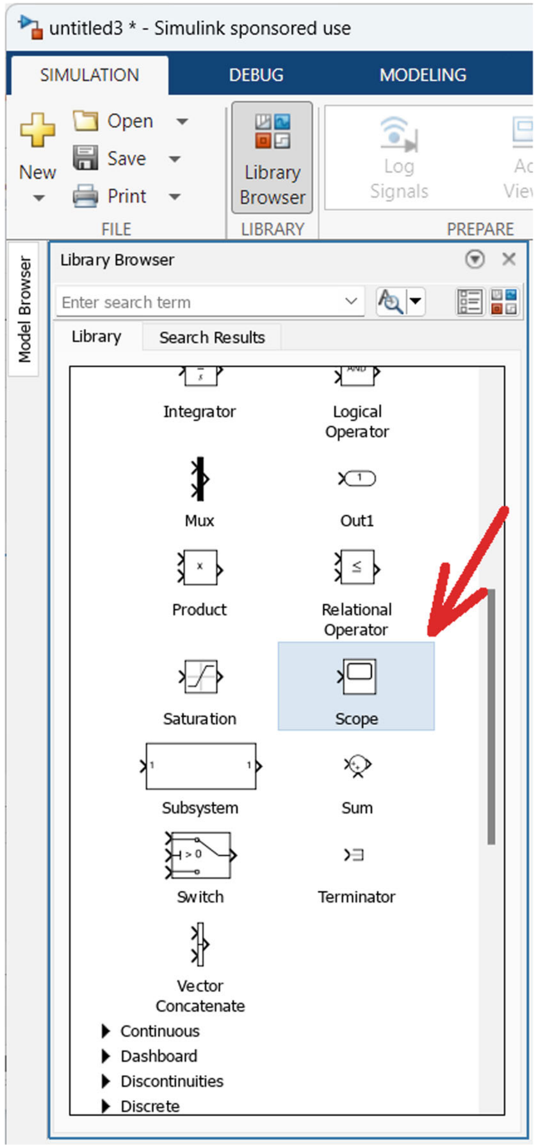


Fig. 12.20 The Scope block

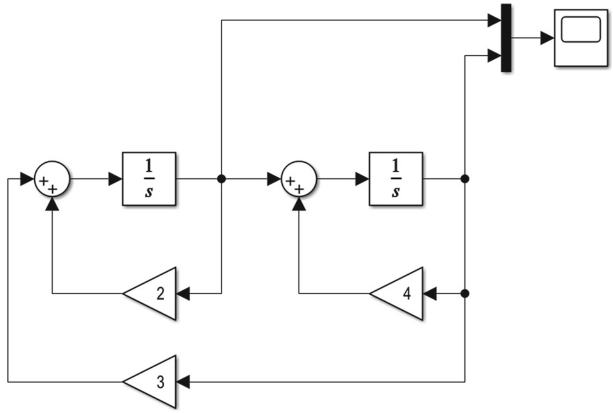


Fig. 12.21 A multiplexer and scope block are added to the model

The multiplexer in Fig. 12.21 merges the $x(t)$ and $y(t)$ signals into a single output, which is then visualized on the Scope. Figure 12.22 shows the default colors of a 2×1 multiplexer block.

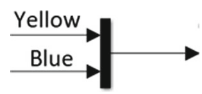


Fig. 12.22 Default color scheme for multiplexer blocks

The scope displays $x(t)$ in yellow and $y(t)$ in blue (Fig. 12.23).

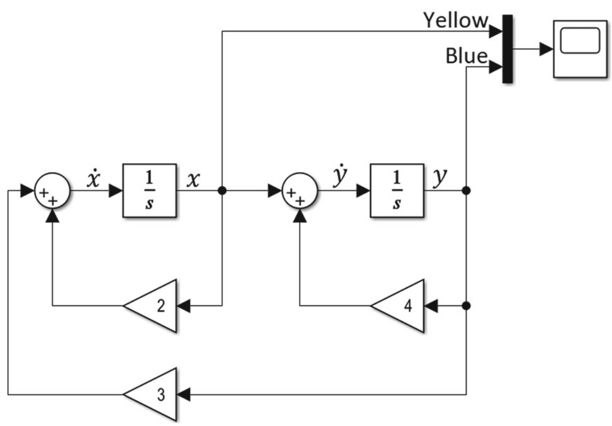


Fig. 12.23 The left and right integrators output $x(t)$ and $y(t)$, respectively

The numerical method employed by Simulink to solve the differential equation is shown on the right side of the screen (Fig. 12.24).

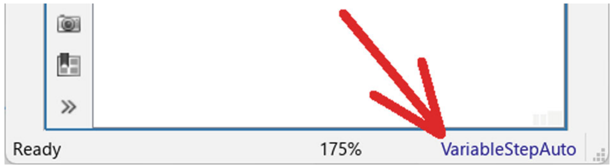


Fig. 12.24 Numerical method used by Simulink to solve the model

You can pick the solver you want by clicking on Model Settings (Fig. 12.25).

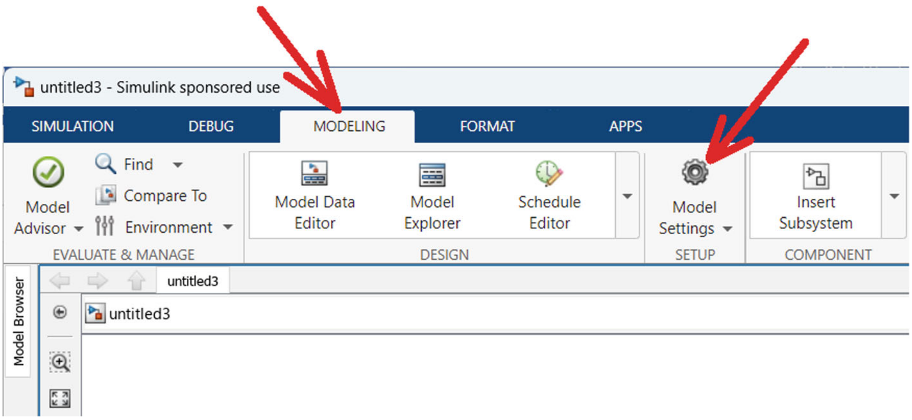


Fig. 12.25 The Model Settings button

Click on Solver (Fig. 12.26) to see what solvers you can use (Fig. 12.27).

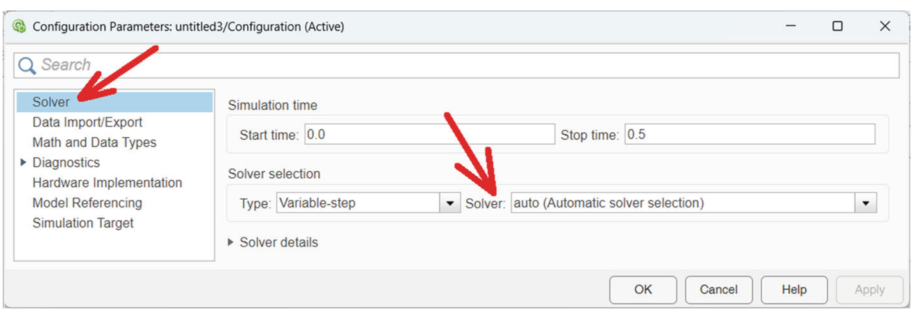


Fig. 12.26 Configuration Parameters window

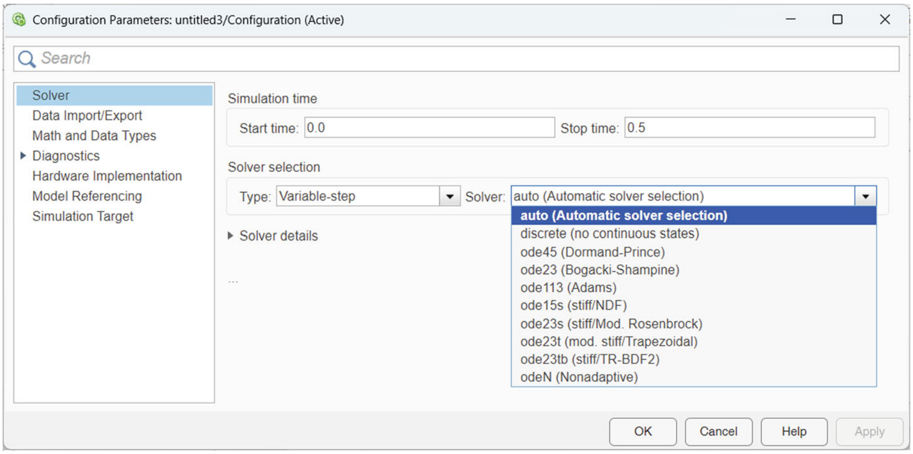


Fig. 12.27 Supported solvers

The default solver is sufficient for this example. Click the Cancel button in Fig. 12.27 and set the Stop time to 0.5 (Fig. 12.28). This configures Simulink to solve the differential equation over the time span from 0 to 0.5 s.

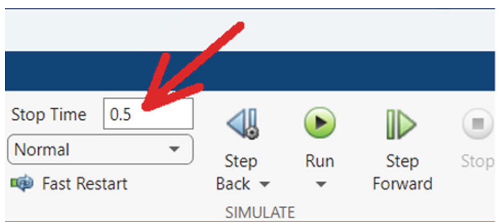
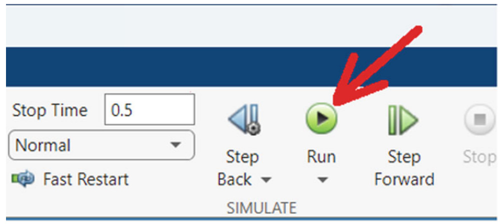


Fig. 12.28 The Stop Time box

Click the Run button (Fig. 12.29) to start the simulation. Double click the scope block to view the waveforms. The output of the simulation is visualized in Fig. 12.30.

Fig. 12.29 The Run button



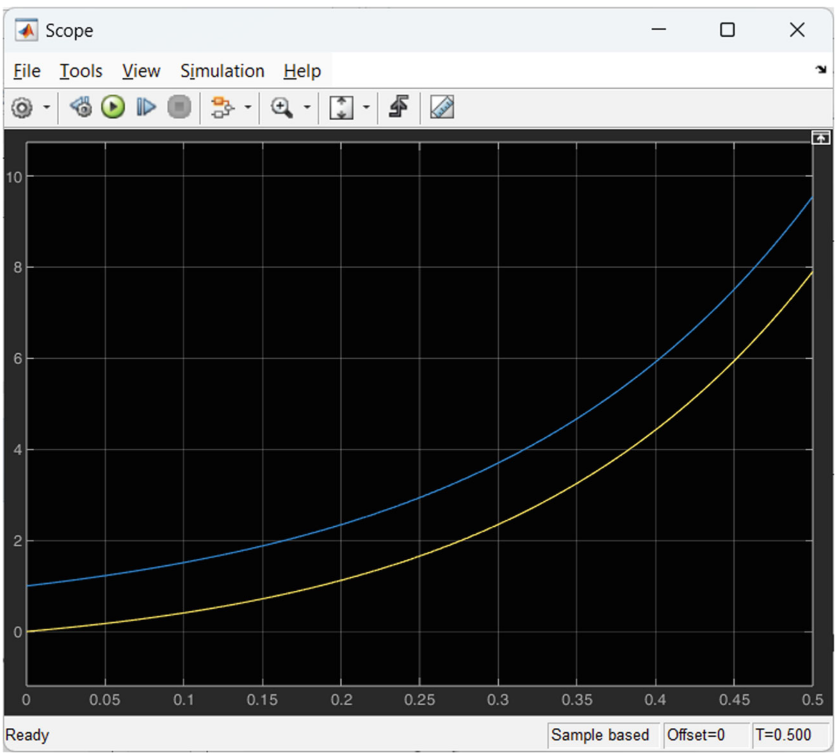


Fig. 12.30 $x(t)$ and $y(t)$ over [0, 0.5] interval

To copy the Scope plot, navigate to File > Copy to Clipboard (Fig. 12.31). You can then paste this image into other software. Pasted image has a white background (Fig. 12.32).

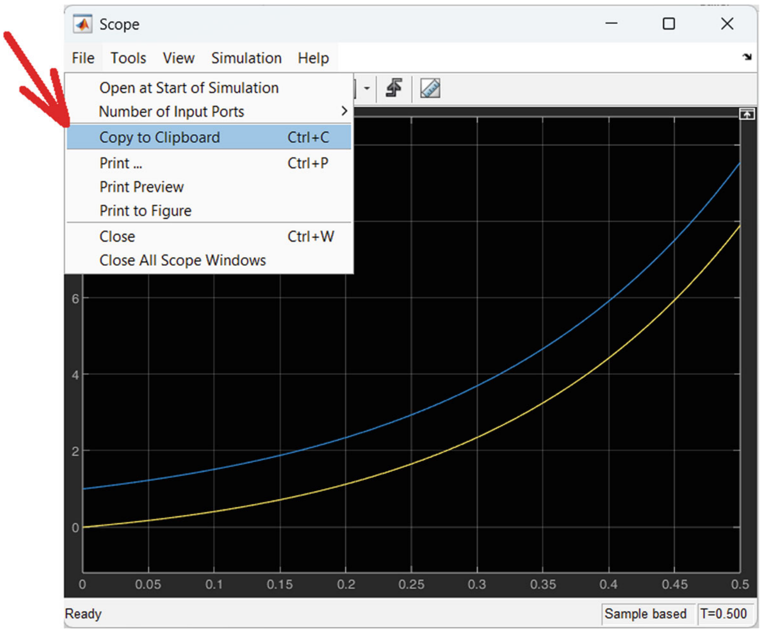


Fig. 12.31 File > Copy to Clipboard

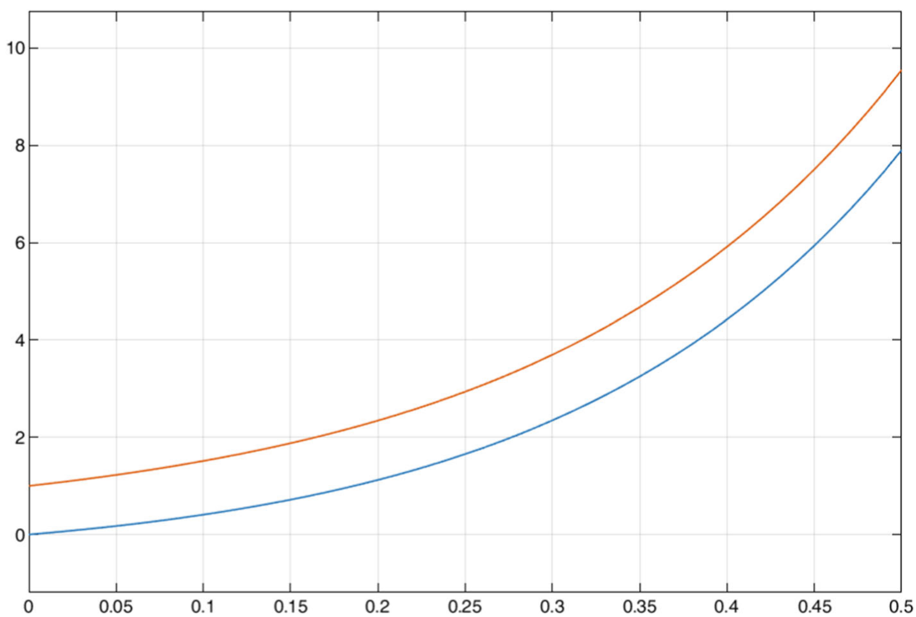


Fig. 12.32 The pasted plot has a plain white background

Example 12.2 In the previous example, we used two integrator blocks. Here, we aim to solve the same problem with only one. We will utilize a single integrator block in Simulink to solve the differential equation $\begin{cases} \frac{dx(t)}{dt} = 2x(t) + 3y(t) \\ \frac{dy(t)}{dt} = x(t) + 4y(t) \end{cases}$ with initial conditions $x(0) = 0$ and $y(0) = 1$.

Implement the model shown in Fig. 12.33 in Simulink.

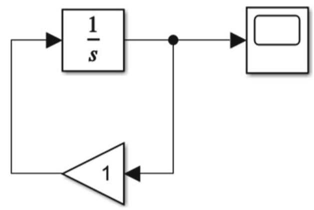


Fig. 12.33 Simulink model of Example 12.2

Double click the integrator block and set the initial condition as shown in Fig. 12.34.

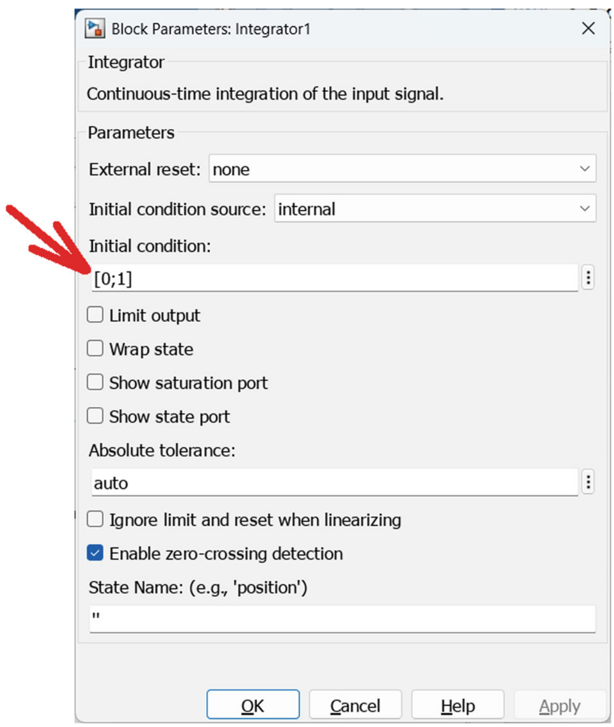


Fig. 12.34 Determining the initial condition

Double click the gain block and configure it as shown in Fig. 12.35.

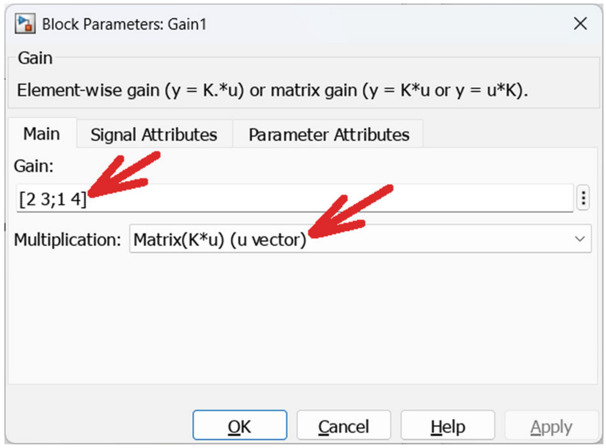


Fig. 12.35 The gain block settings

Set the simulation stop time to 0.5 and run the simulation (Fig. 12.36).

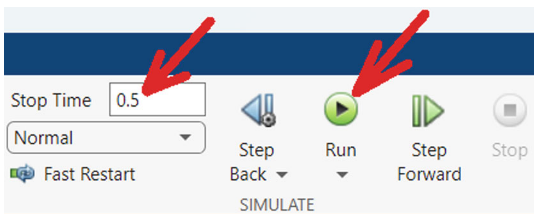


Fig. 12.36 The stop time box and the run button

Double click the scope block to view the simulation results (Fig. 12.37).

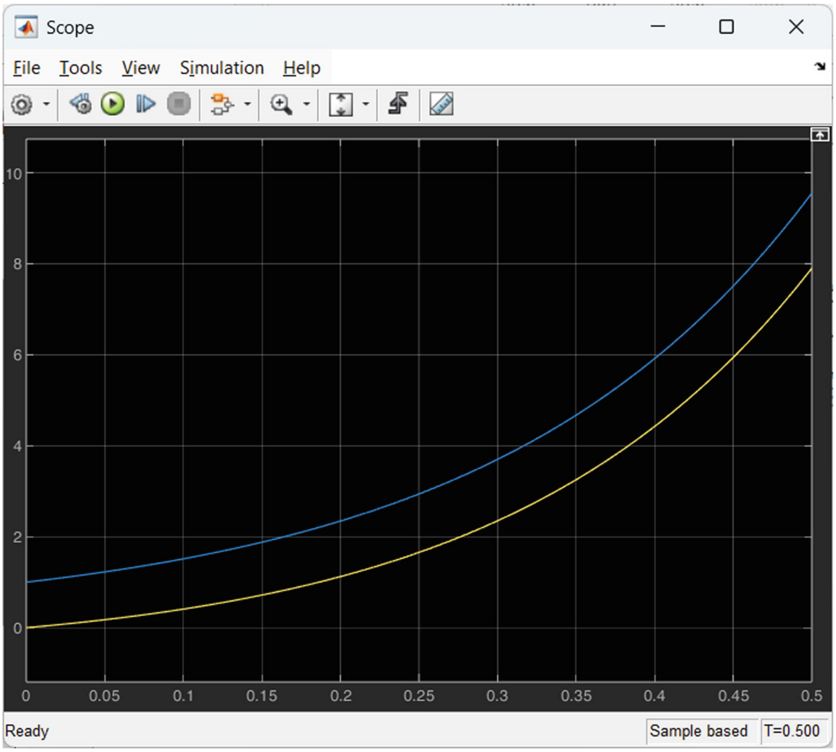


Fig. 12.37 $x(t)$ and $y(t)$ over [0, 0.5] interval

12.3 State Space Block

Simulink's state-space block efficiently simulates systems described by the equations

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases},$$

where

$x(t)$: State vector—Represents the internal state of the system at time t,
$\dot{x}(t)$: Derivative of the state vector with respect to time,
$u(t)$: Input vector—Represents the external inputs to the system,
$y(t)$: Output vector—Represents the system's outputs,
A, B, C, D: System matrices that define the dynamics of the system.

Example 12.3 Solve the differential equation $\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} -1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t) \end{cases}$, $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ and $u(t) = 1$ using Simulink.

Implement the Simulink model shown in Fig. 12.38 using State-Space (Fig. 12.39) and Constant (Fig. 12.40) blocks.

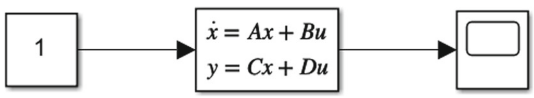


Fig. 12.38 Simulink model of Example 12.3

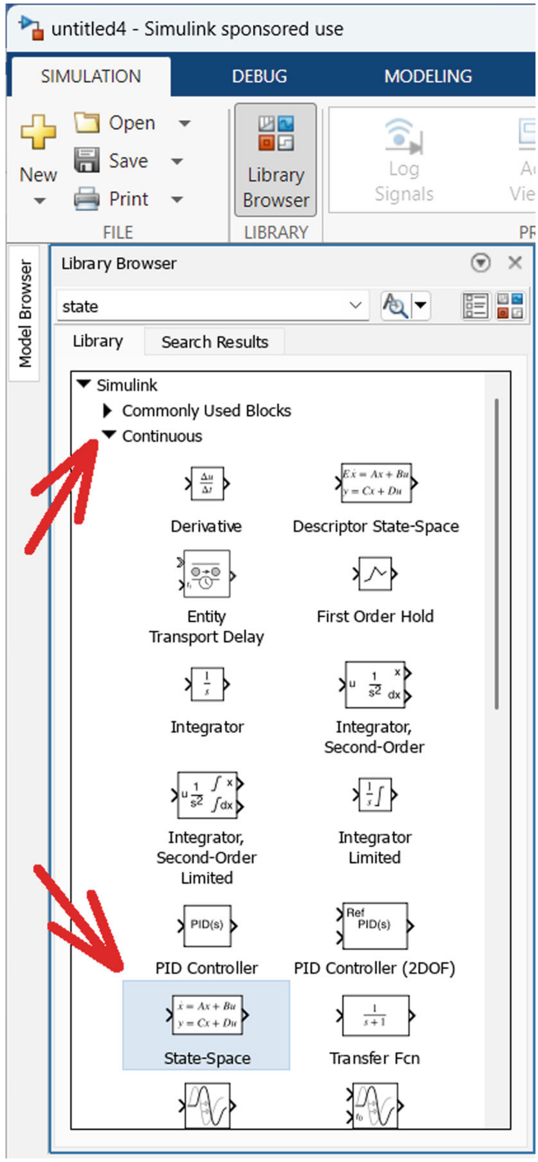


Fig. 12.39 The State-Space block

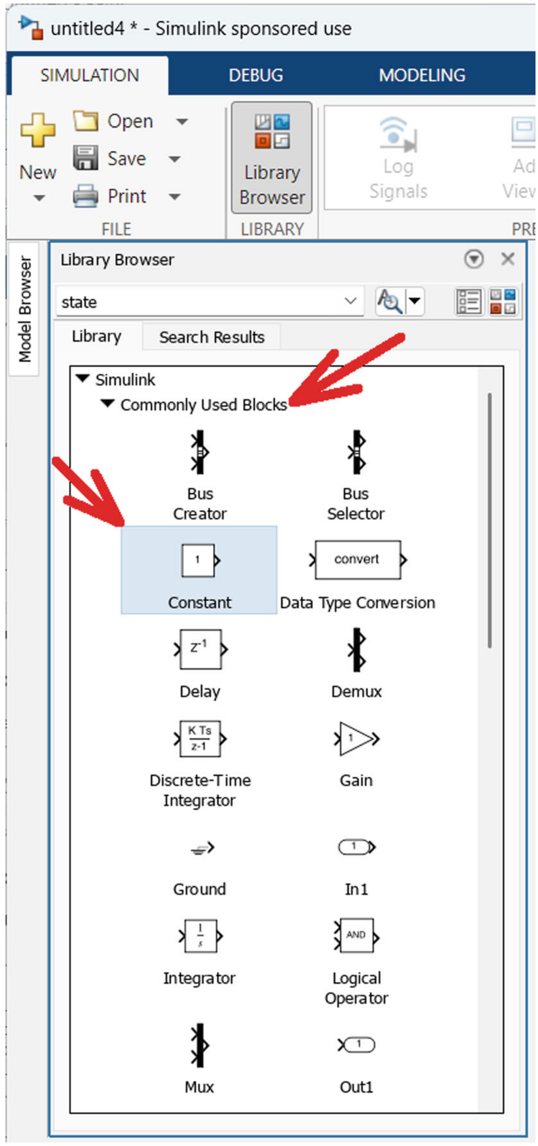


Fig. 12.40 The Constant block

Double click the state-space and gain blocks and input the values specified in Figs. 12.41 and 12.42 to configure them accordingly.

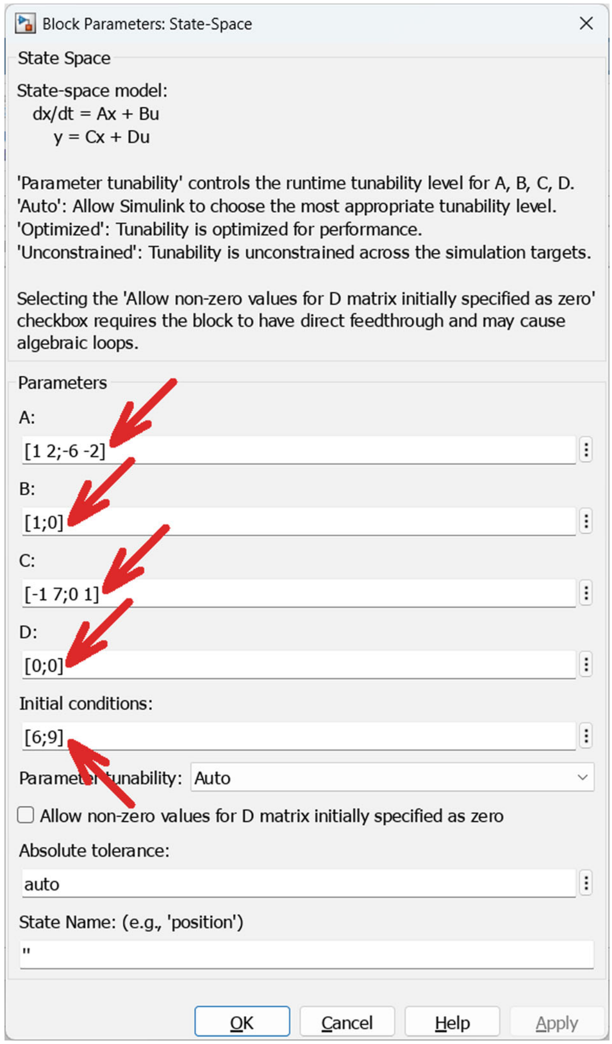


Fig. 12.41 The State-Space block settings

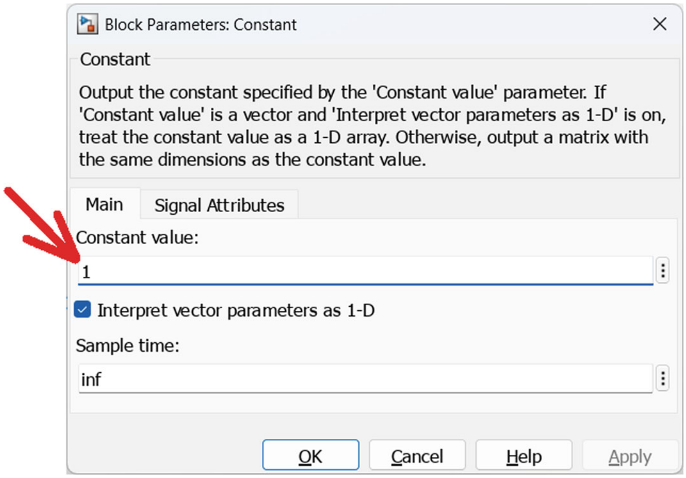


Fig. 12.42 The gain block settings

Set the simulation stop time to 0.3 and run the simulation (Fig. 12.43).

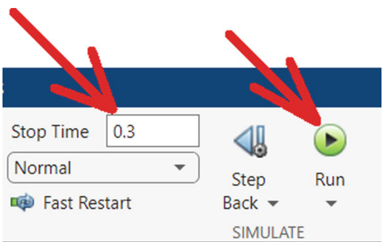


Fig. 12.43 The stop time and the run button

Double click the scope block to view the simulation results (Fig. 12.44).

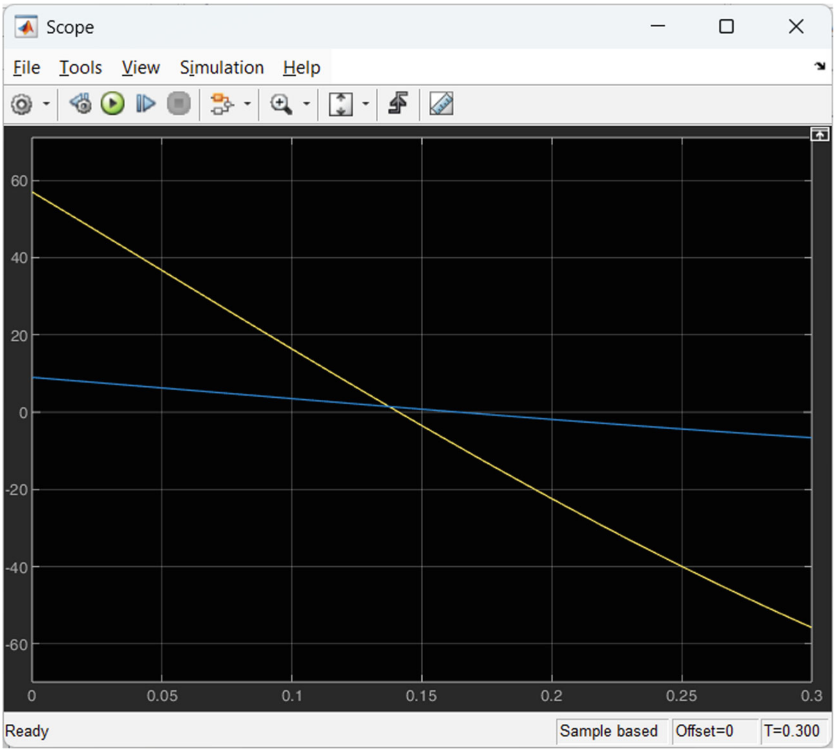


Fig. 12.44 Output $y(t)$ over [0, 0.3] interval

Example 12.4 Solve the differential equation $\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} -1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t) \end{cases}$, $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ and $u(t) = 2 + 7\sin(4t + \frac{\pi}{6})$ using Simulink.

Open the Simulink model that we created earlier (Fig. 12.45).

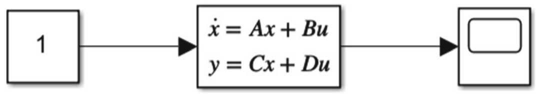


Fig. 12.45 Simulink model of Example 12.3

Click on the gain block to select it (Fig. 12.46). Then, press the Delete key on your keyboard to remove it from the Simulink model (Fig. 12.47).

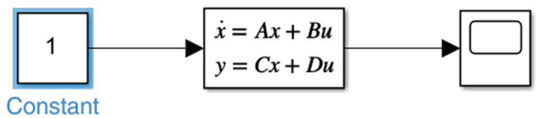


Fig. 12.46 Constant block is selected

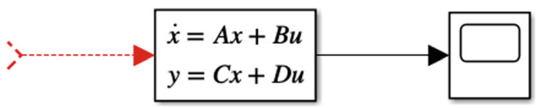


Fig. 12.47 Constant block is removed

Add a sine wave block to the model (Fig. 12.48). The sine wave block is located in the Sources library (Figs. 12.49 and 12.50).

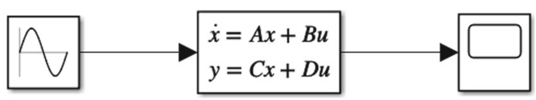
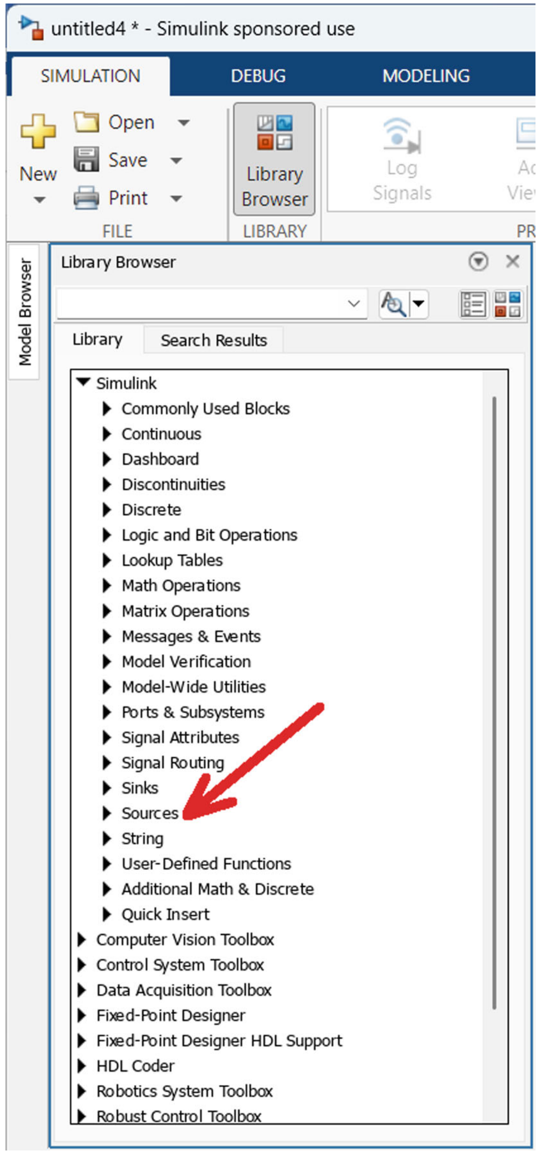


Fig. 12.48 A sine wave block is added to the model

Fig. 12.49 The Sources library



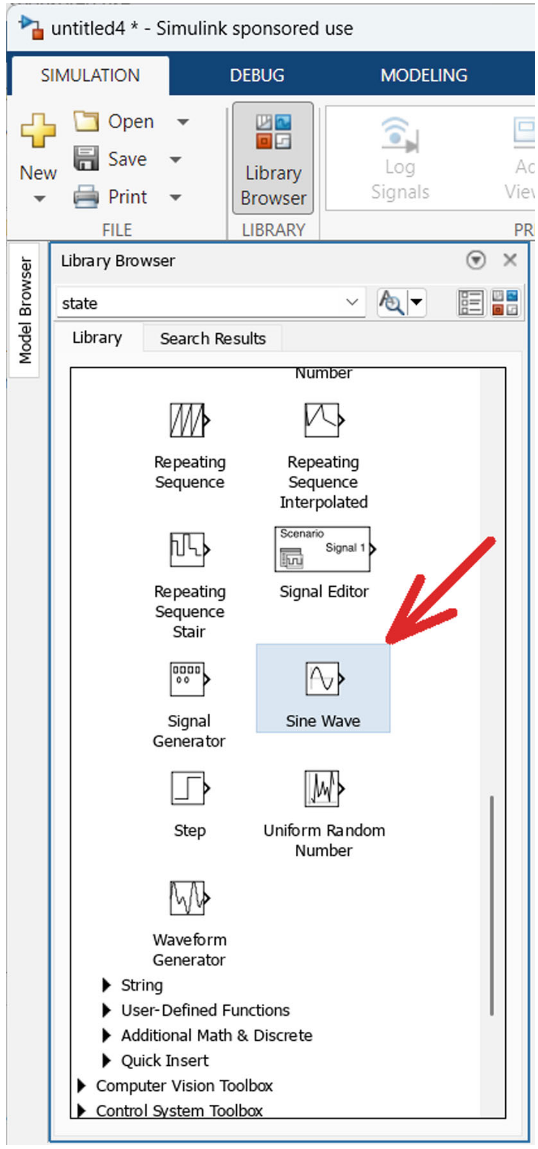


Fig. 12.50 The Sine Wave block

Double click on the sine wave block and configure it as shown in Fig. 12.51.

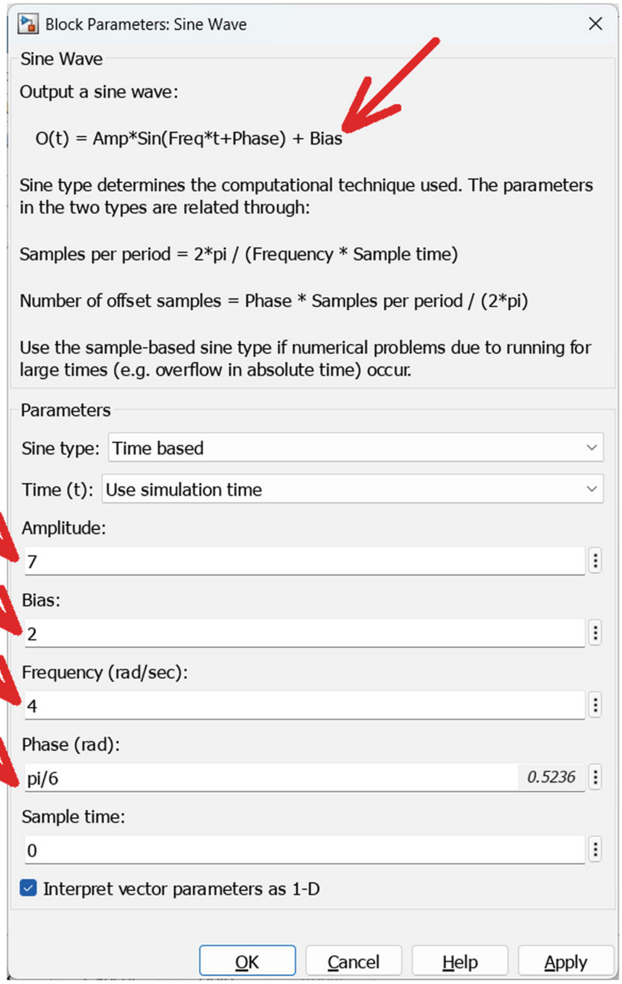


Fig. 12.51 Required settings to generate $u(t) = 2 + 7\sin(4t + \frac{\pi}{6})$

Set the simulation stop time to 0.3 and run the simulation (Fig. 12.52).

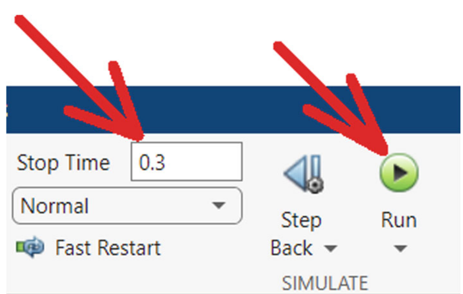


Fig. 12.52 The stop time and the run button

Double click the scope block to view the simulation results (Fig. 12.53).

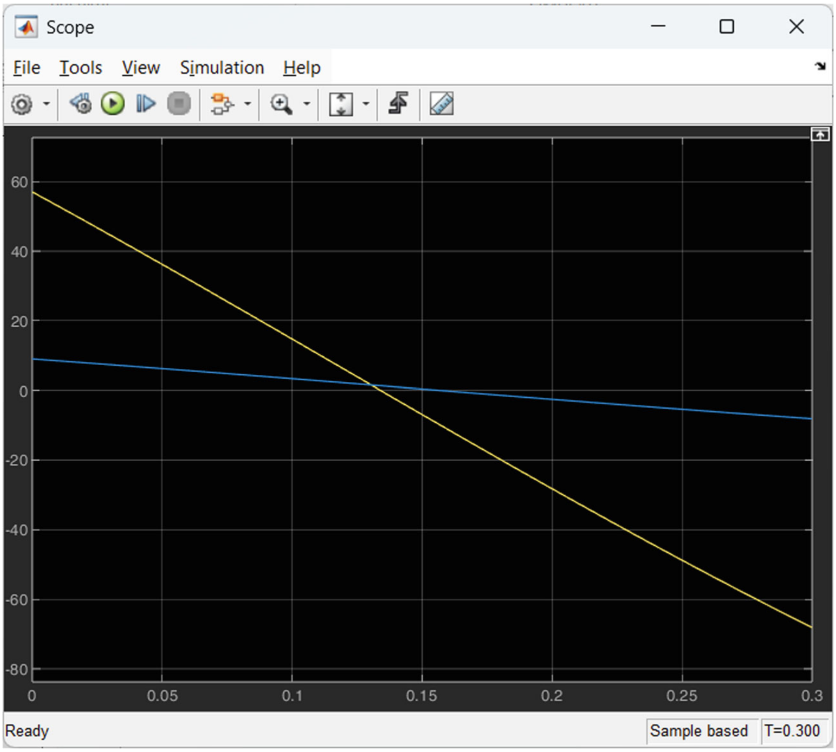


Fig. 12.53 $y_1(t) = -x_1(t) + 7x_2(t)$ and $y_2(t) = x_2(t)$ over $[0, 0.3]$ interval

Example 12.5 In this example we will simulate the response of the system $y^{(4)}(t)+7y^{(3)}(t)+6\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \dot{x}(t) + 2x(t)$ to a unit step input multiplied by three, assuming zero initial conditions. The transfer function for this system is $T(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^4+7s^3+6s^2+3s+2}$.

Implement the model shown in Fig. 12.54. To find the transfer function and step blocks, refer to Figs. 12.55 and 12.56. These figures show where to locate these blocks in the Simulink library.

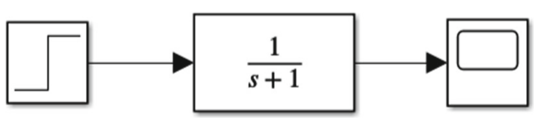


Fig. 12.54 Initial model for Example 12.5

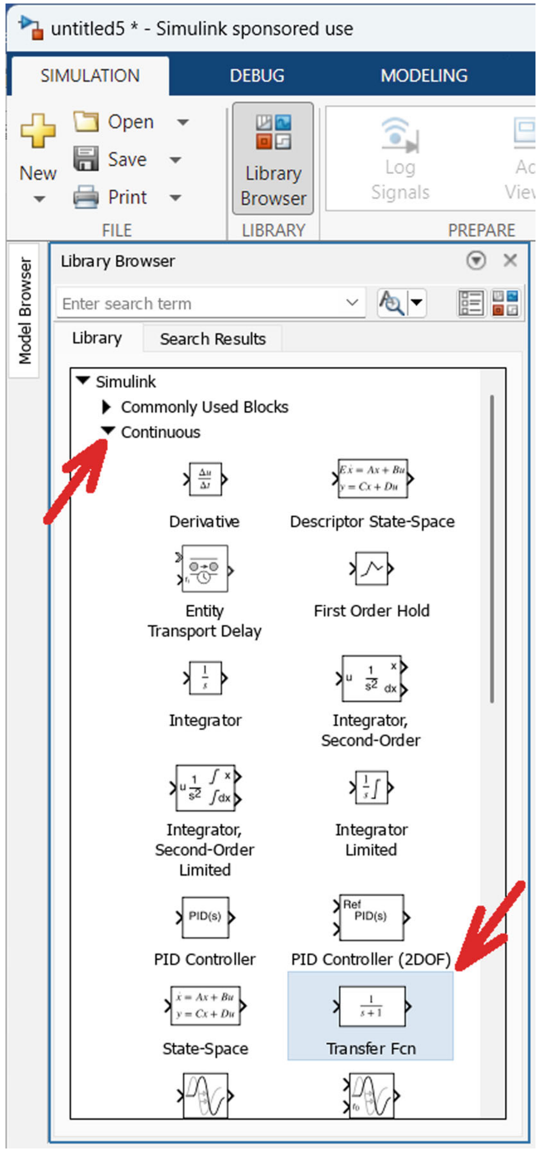


Fig. 12.55 The Transfer Function block

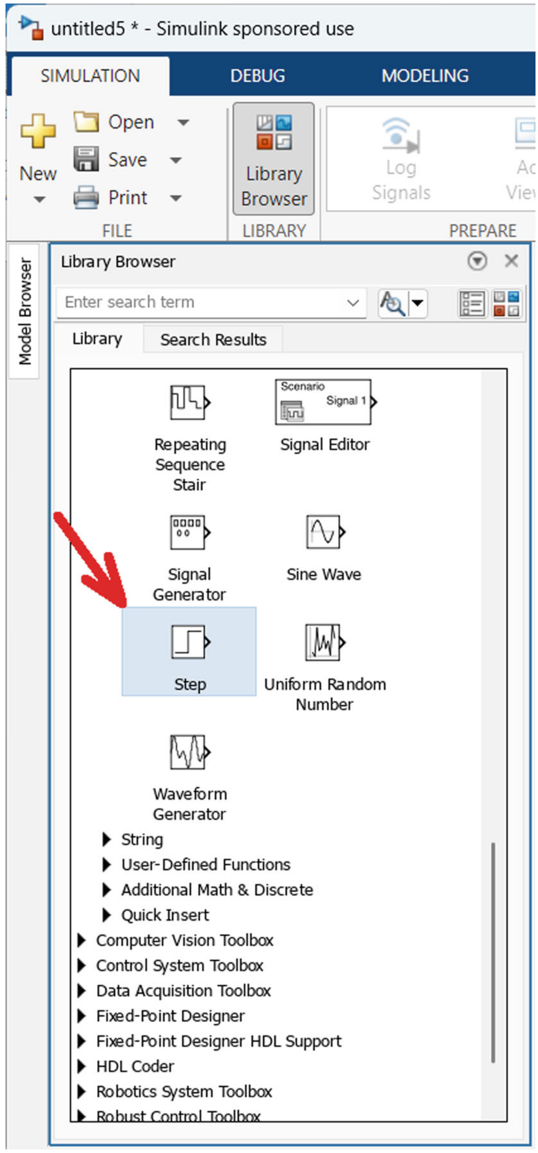


Fig. 12.56 The Step block can be found in the Sources library

Double click the step block and configure it as shown in Fig. 12.57.

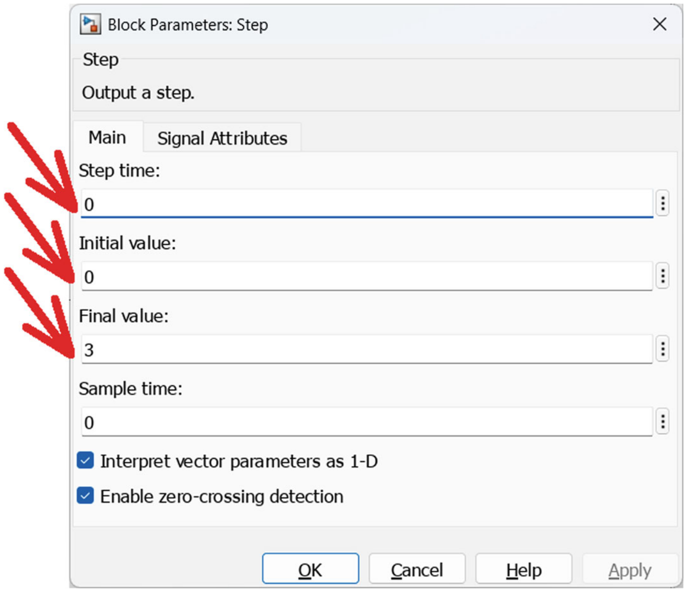


Fig. 12.57 Required settings to generate $x(t) = 3H(t)$

Double-click the transfer function block and input the numerator and denominator coefficients as specified in Fig. 12.58. This will configure the block to represent the transfer function $T(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^4+7s^3+6s^2+3s+2}$.

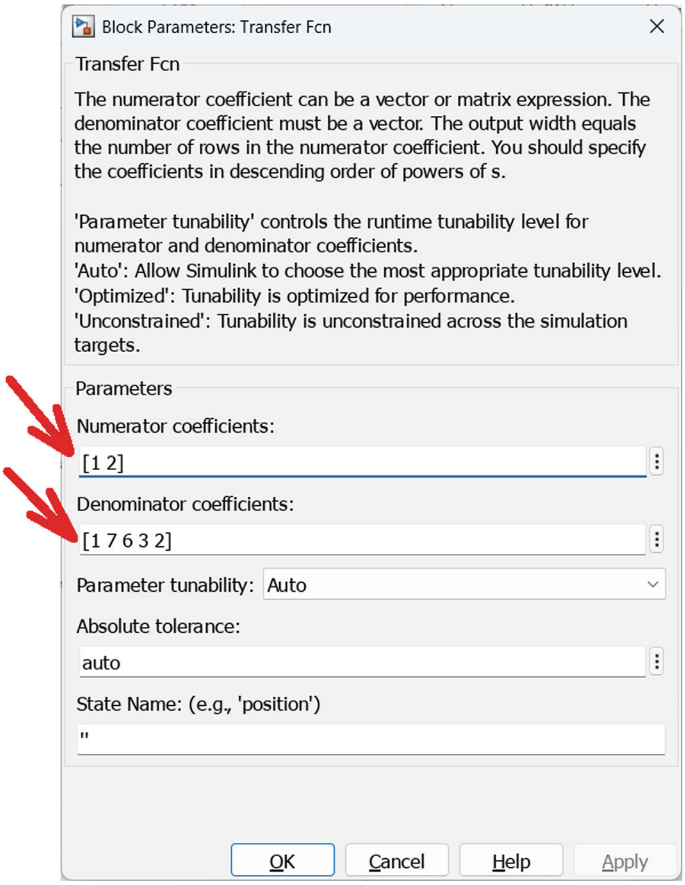


Fig. 12.58 Required settings to define $T(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^4+7s^3+6s^2+3s+2}$

Click the OK button in Fig. 12.58, then resize the transfer function block for clearer visualization of the transfer function equation (Fig. 12.59).

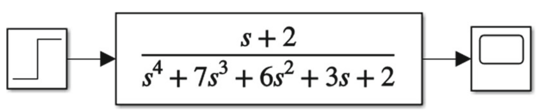


Fig. 12.59 Block diagram of Example 12.5

Set the simulation stop time to 30 and run the simulation (Fig. 12.60).

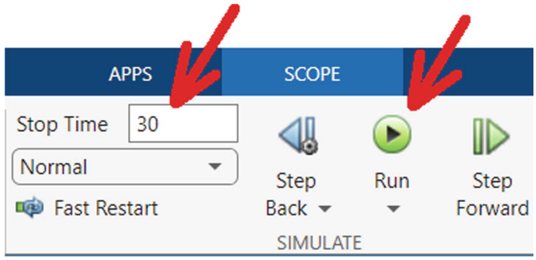


Fig. 12.60 The stop time and the run button

Double click the scope block to view the simulation results (Fig. 12.61).

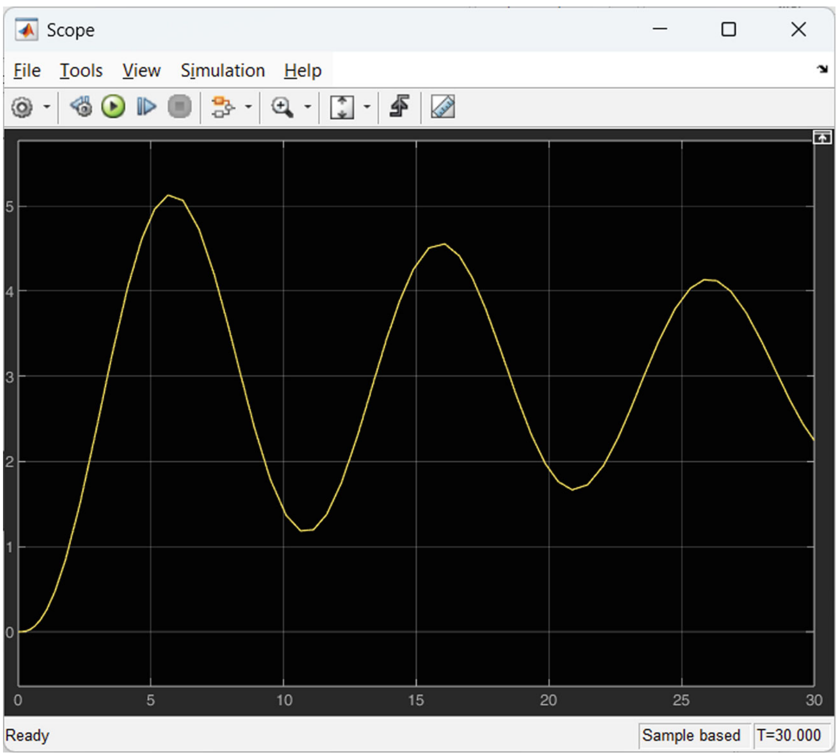


Fig. 12.61 Simulation results

The simulation result in Fig. 12.61 is not smooth enough. Let's improve its smoothness. Click the Model Settings icon (Fig. 12.62) to open the Configuration Parameters window (Fig. 12.63). Then, click the Solver details (Fig. 12.63).

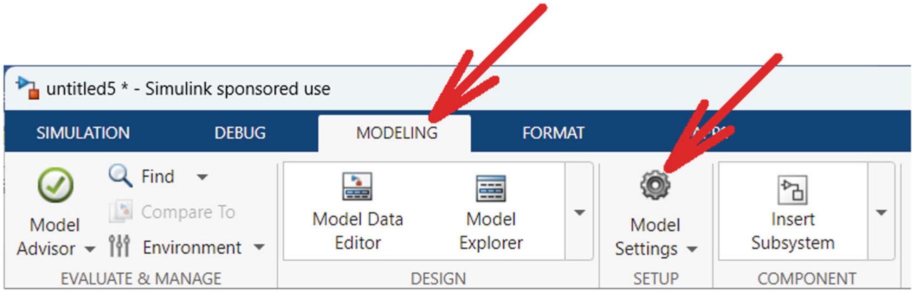


Fig. 12.62 The Model Settings button

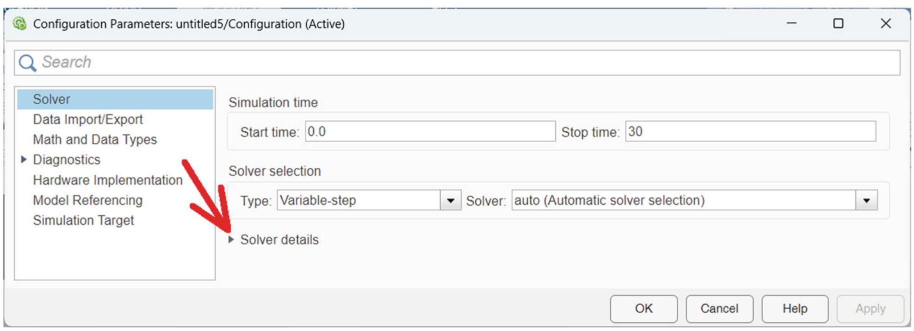


Fig. 12.63 The Solver details

Configure the solver to use a maximum step size of 0.001 (Fig. 12.64).

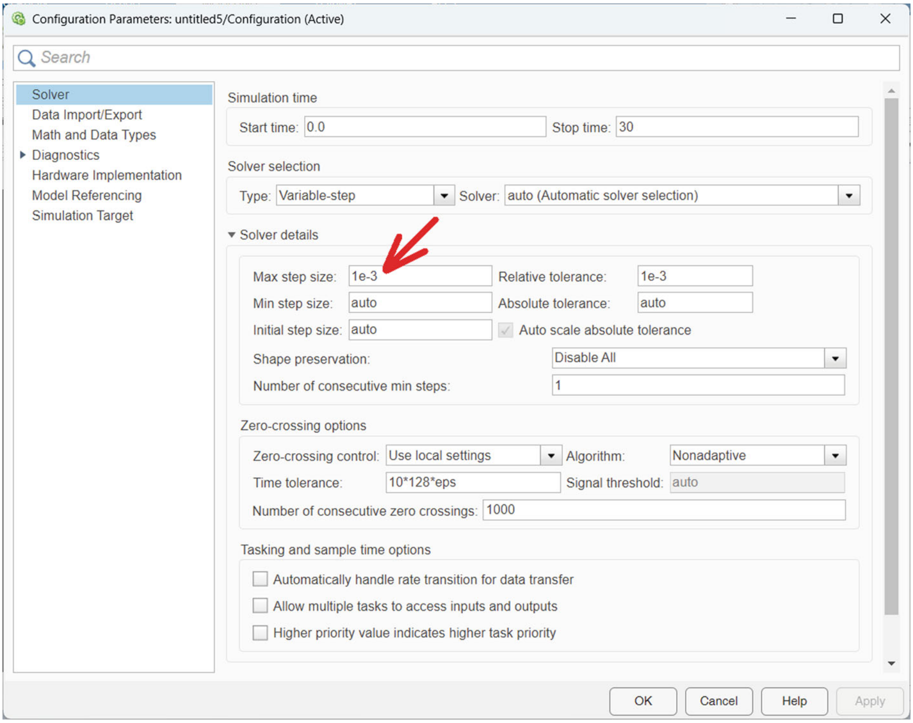


Fig. 12.64 Max step size box

Click the OK button in Fig. 12.64 and rerun the simulation. The simulation output is depicted in Fig. 12.65. Reducing the step size enhances the smoothness of the simulation results, albeit at the cost of increased computational effort.

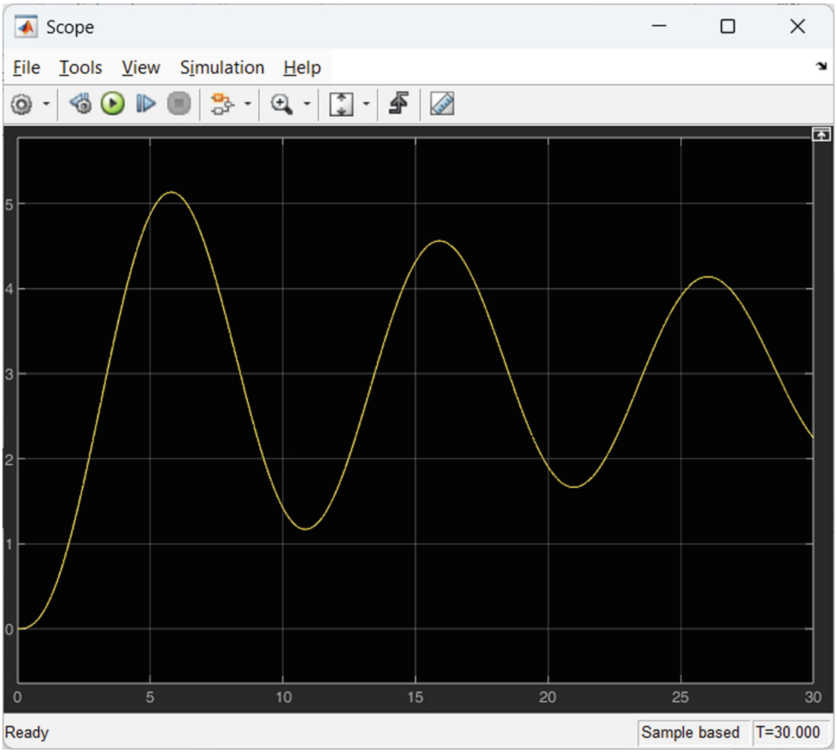


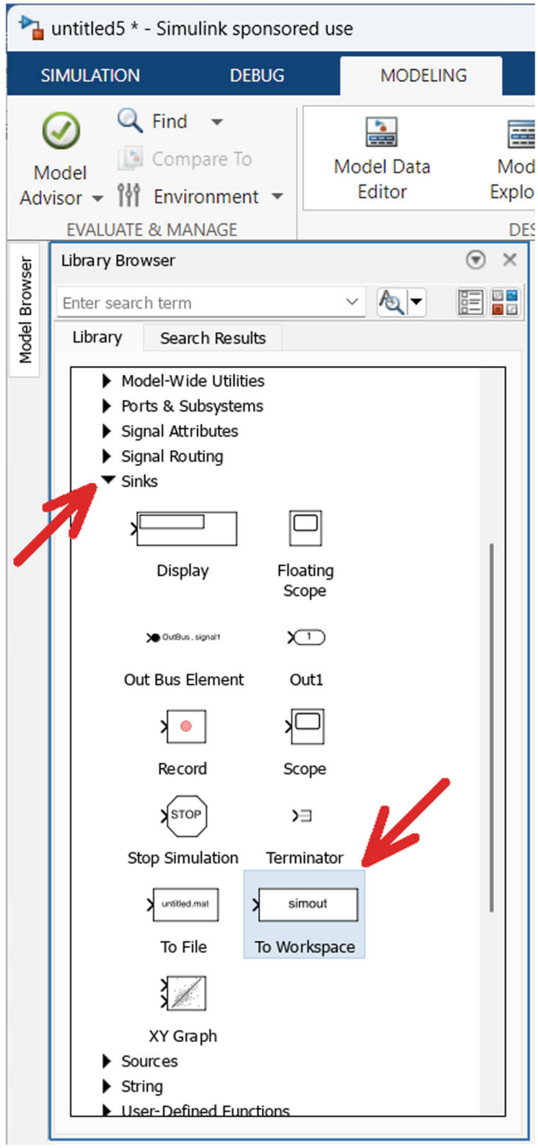
Fig. 12.65 Simulation results

Exercise: Simulate the response of the system $y^{(4)}(t) + 7y^{(3)}(t) + 6\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \dot{x}(t) + 2x(t)$ to a unit step input multiplied by three and delayed by two time units, i.e., $x(t) = 3H(t-2)$, assuming zero initial conditions.

12.4 To Workspace Block

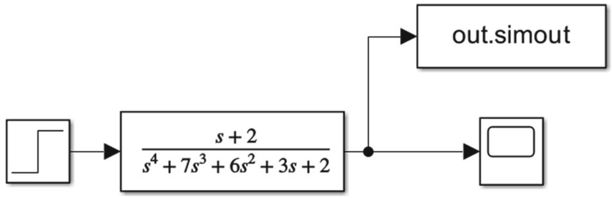
The To Workspace block (Fig. 12.66) is used to export simulation data from the Simulink model to the MATLAB workspace. This allows you to analyze, visualize, and further process the data using MATLAB commands and functions.

Fig. 12.66 The To Workspace block



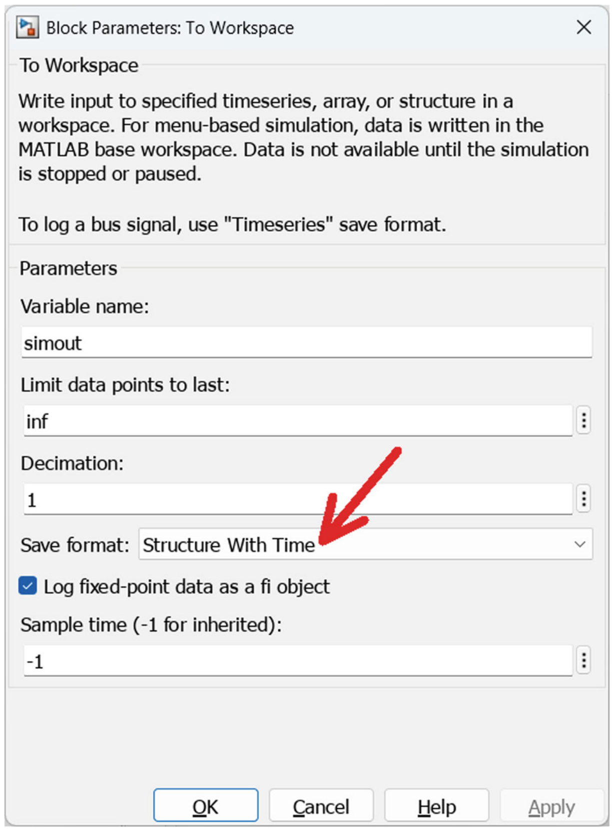
Example 12.6 In this example we will export the output of previous simulation to MATLAB environment. Change the Simulink model of Example 12.3 to what shown in Fig. 12.67.

Fig. 12.67 The Simulink model of Example 12.4



Double click the To Workspace block and configure it as shown in Fig. 12.68.

Fig. 12.68 The To Workspace block settings



Set the simulation stop time to 30 and run the simulation (Fig. 12.69).

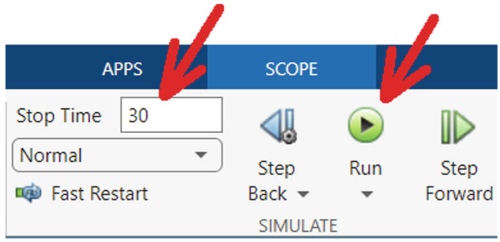


Fig. 12.69 The stop time and the run button

A new variable is added to the MATLAB Workspace upon simulation completion (Fig. 12.70).

Fig. 12.70 A new variable is added to the workspace

Figure 12.71 shows how to extract the time and signal values. Output of this code is shown in Fig. 12.72. This figure is identical to Fig. 12.65.

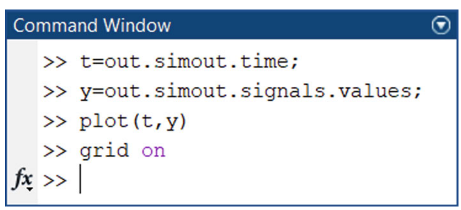


Fig. 12.71 Plotting the imported data

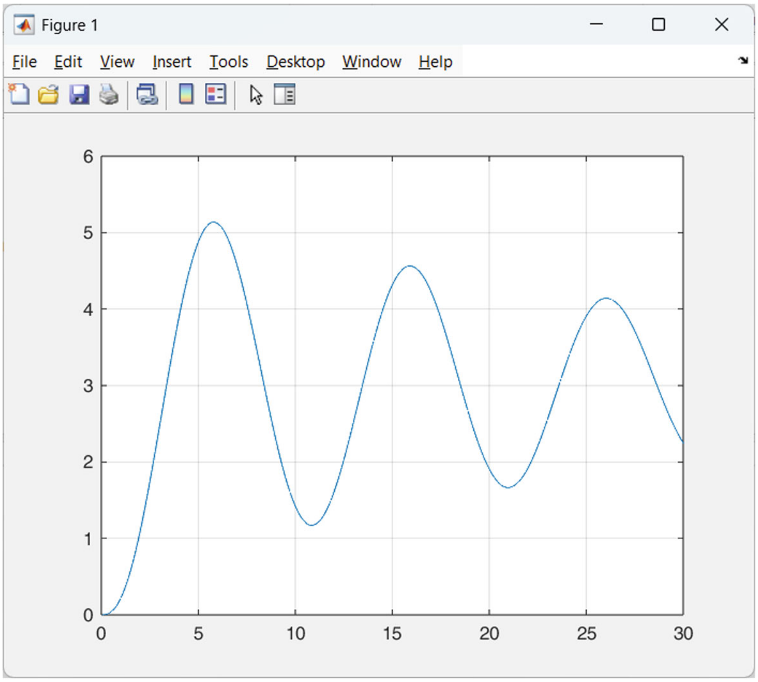


Fig. 12.72 Output of the code shown in Fig. 12.71

12.5 MATLAB® Function Block

MATLAB Function block can be used to simulate nonlinear systems. Let's study an example.

Example 12.7 We want to simulate $\begin{cases} \dot{x_1}(t) = x_2(t) \\ \dot{x}_2(t) = -\sin(x_1(t)) - 0.1x_2(t) + u(t) \end{cases}$, $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ over the time interval [0, 5]. $u(t) = t.H(t)$ is the system input. $H(t)$ represents the unit step function.

Create a Simulink model that matches the diagram in Fig. 12.73. The Ramp block (Fig. 12.74) is used to generate a ramp signal. The MATLAB Function block (Fig. 12.75) is used to implement custom functions.

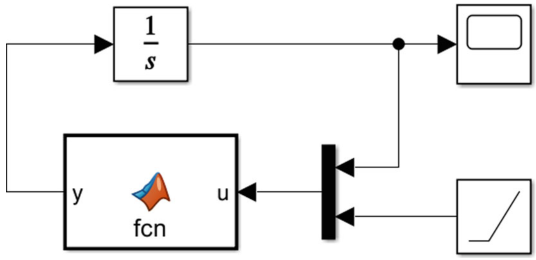


Fig. 12.73 Simulink model of Example 12.7

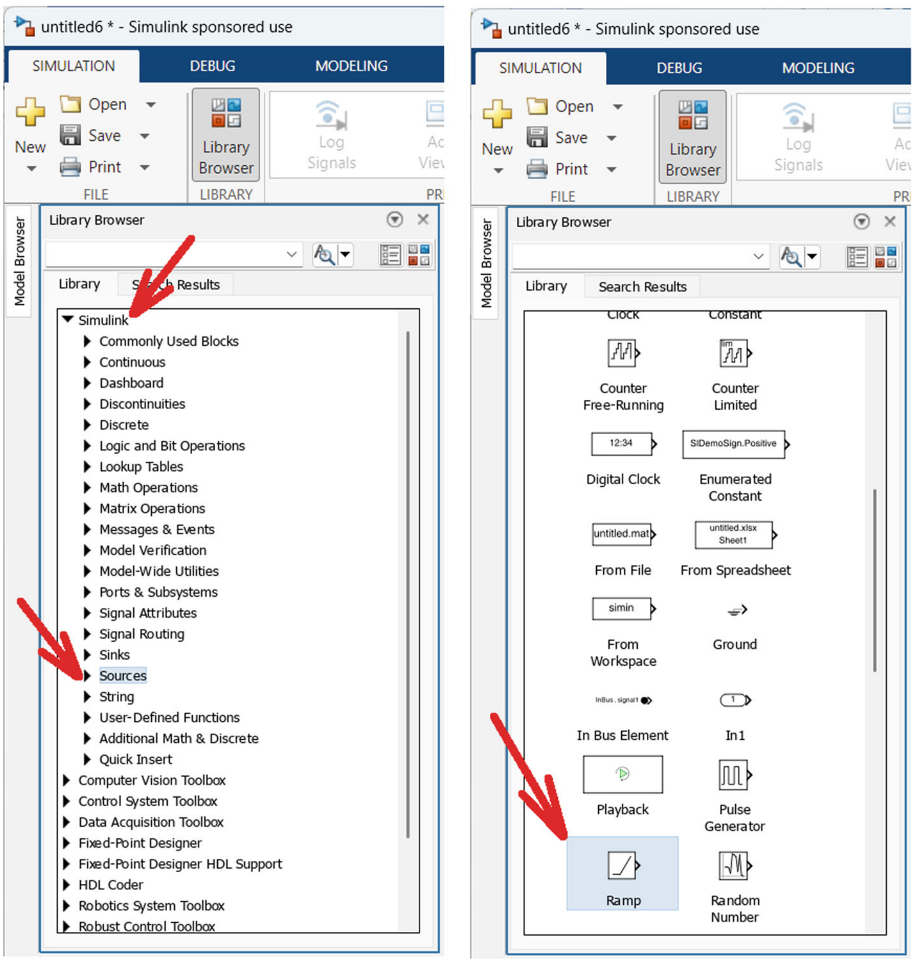


Fig. 12.74 The Ramp block

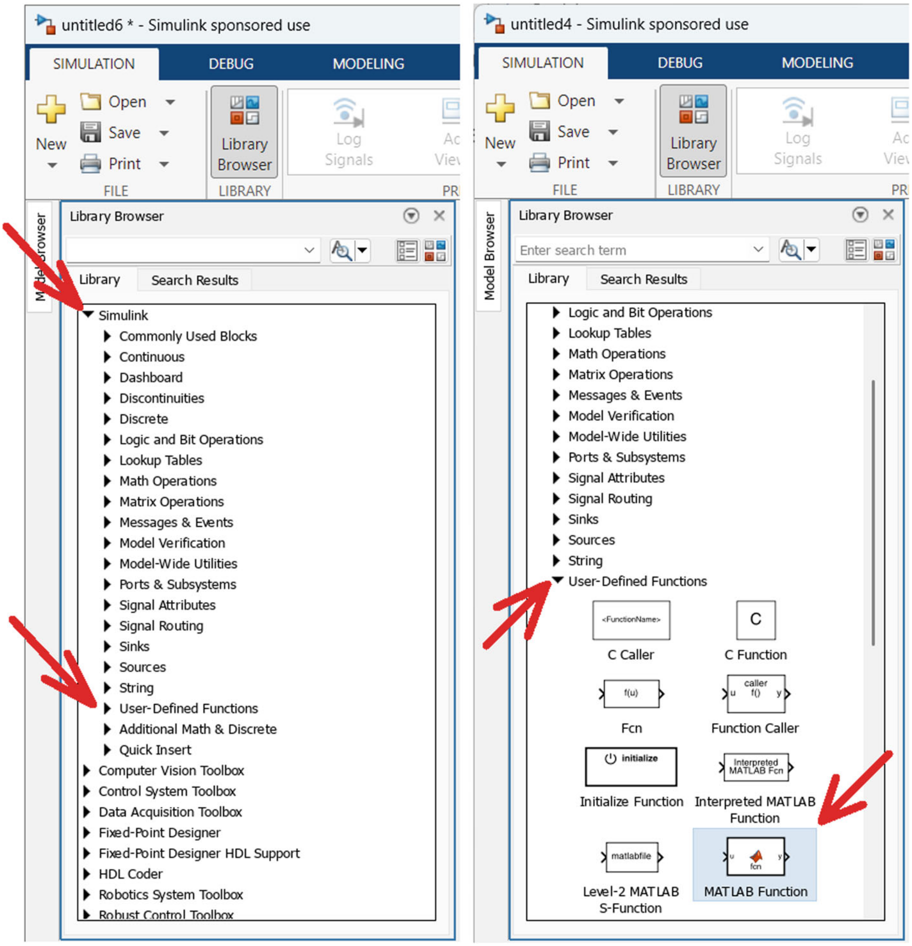


Fig. 12.75 The MATLAB Function block

Double click the integrator block and configure it as shown in Fig. 12.76.

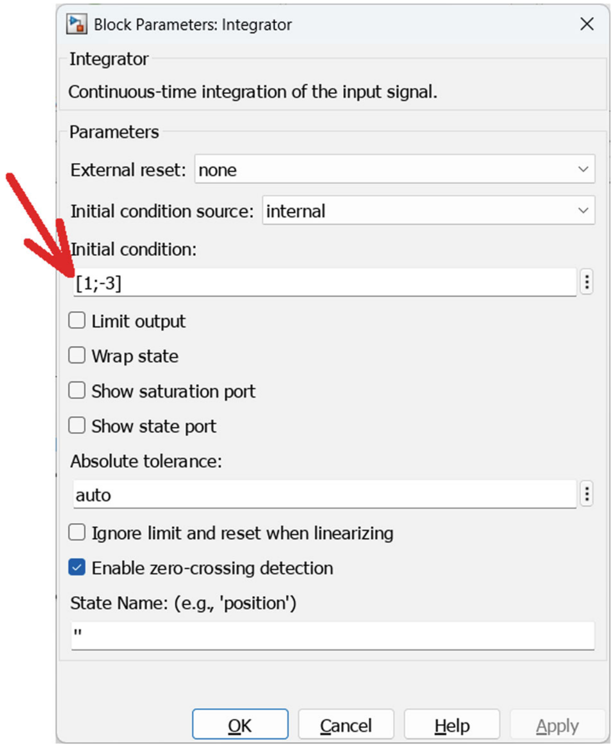


Fig. 12.76 The integrator block settings

Double click the MATLAB Function block to open the function editor. Type the function code as shown in Fig. 12.77. Once you're done, click the arrows to return to the Simulink model.

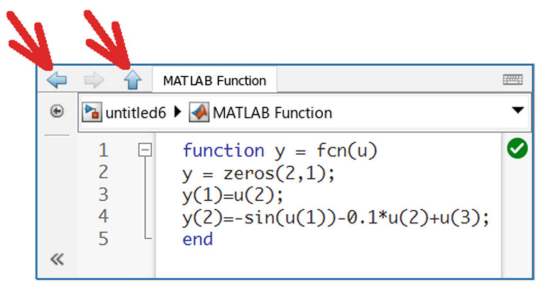
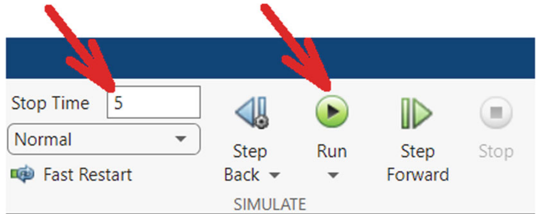


Fig. 12.77 MATLAB Function block code

Set the simulation stop time to 5 and run the simulation (Fig. 12.78).

Fig. 12.78 The stop time and the run button



Double click the scope block to view the simulation results (Fig. 12.79).

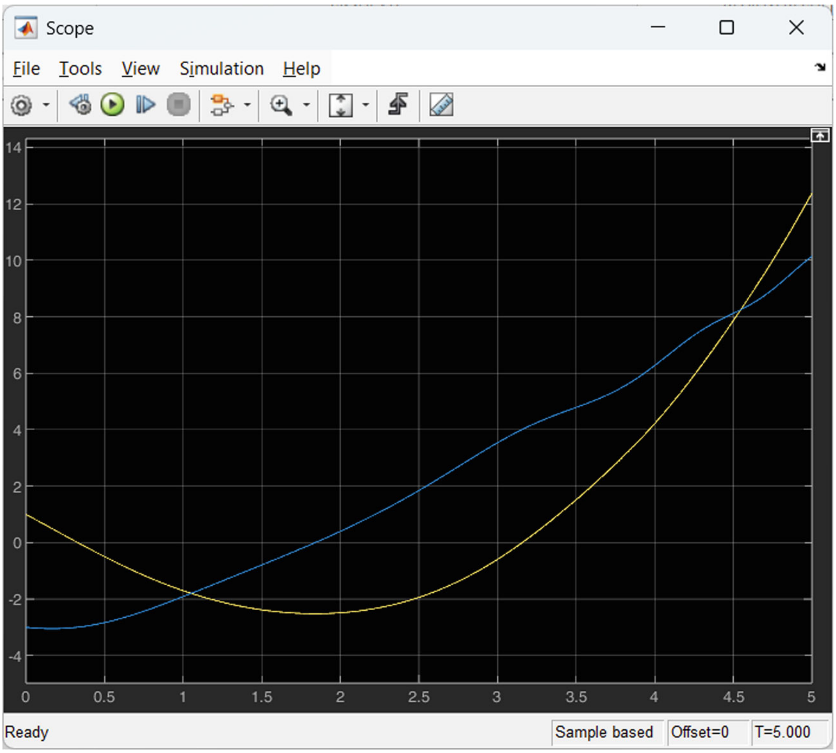


Fig. 12.79 Simulation results

References for Further Study

1. Advanced Engineering Mathematics (10th edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8th edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

Probability and Statistics

13

13.1 Introduction

The Statistics and Machine Learning Toolbox® is a powerful toolset within MATLAB® that provides a comprehensive range of functions and apps for data analysis, statistical modeling, and machine learning. This chapter introduces key MATLAB functions for statistical analysis and probability theory. The Statistics and Machine Learning Toolbox is required for this chapter.

13.2 Probability Distribution

A probability distribution is a mathematical function that describes the likelihood of occurrence of different possible values of a random variable. In simpler terms, it's a way to model the randomness inherent in many real-world phenomena. MATLAB supports a wide range of probability distributions, both continuous and discrete. Here are some of the most common ones:

(A) **Continuous distributions:** Normal (Gaussian), Uniform, Exponential, Gamma, Beta, Weibull, Chi-Square, F, t, Inverse Gaussian, Rayleigh, Rician, Nakagami, etc.
(B) **Discrete distributions:** Binomial, Poisson, Geometric, Negative Binomial, Hypergeometric, Discrete Uniform.

For each of these distributions, MATLAB provides functions for:

(A) **Probability Density Function (PDF):** Calculates the probability density at a specific value.

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_13

(B) **Cumulative Distribution Function (CDF):** Calculates the probability of a value being less than or equal to a specific value.
(C) **Inverse CDF (Quantile Function):** Calculates the value corresponding to a given probability.
(D) **Random Number Generation:** Generates random numbers from the distribution.
(E) **Parameter Estimation:** Estimates the parameters of the distribution from a sample of data.

13.3 Core MATLAB Functions for Probability and Statistics

Example 13.1 The code in Fig. 13.1 calculates $\binom{6}{2} = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!}$.

Fig. 13.1 Calculation of $\binom{6}{2}$

```
Command Window
>> nCr=nchoosek(6,2)

nCr =

    15

fx >> 
```

The code in Fig. 13.2 calculates $P(6, 2) = \frac{6!}{(6-2)!}$.

Fig. 13.2 Calculation of $P(6, 2)$

```
Command Window
>> nPr=factorial(6)/factorial(6-2)

nPr =

    30

fx >> 
```

Example 13.2 The code in Fig. 13.3 calculates the minimum and maximum of $\{170, 180, 175, 170, 162, 160, 180, 185\}$.

Fig. 13.3 Calculation of minimum and maximum

```
Command Window
>> x=[170 180 175 170 162 160 180 185];
>> min(x)

ans =

   160

>> max(x)

ans =

   185

fx >>
```

Example 13.3 The code in Fig. 13.4 calculates the range of {170, 180, 175, 170, 162, 160, 180, 185}. Remember that the range is the difference between the maximum and minimum values.

Fig. 13.4 Calculation of range

```
Command Window
>> x=[170 180 175 170 162 160 180 185];
>> range(x)

ans =

    25
fx >>
```

Example 13.4 The code in Fig. 13.5 calculates the sum of {170, 180, 175, 170, 162, 160, 180, 185}.

Fig. 13.5 Calculation of the sum of the data

```
Command Window
>> x=[170 180 175 170 162 160 180 185];
>> sum(x)

ans =

        1382

fx >>
```

Example 13.5 The code in Fig. 13.6 counts the elements in the set {170, 180, 175, 170, 162, 160, 180, 185}.

Fig. 13.6 Counting the number of data elements

```
Command Window
>> x=[170 180 175 170 162 160 180 185];
>> length(x)

ans =

     8

fx >>
```

Example 13.6 The code presented in Figs. 13.7 and 13.8 computes the average of {170, 180, 175, 170, 162, 160, 180, 185}. Remember that $\overline{x} = \frac{\sum_{i=1}^{N} x_i}{N}$.

Fig. 13.7 Computation of the mean

```
Command Window
>> x=[170 180 175 170 162 160 180 185];
>> mean(x)

ans =

  172.7500

fx >>
```

Fig. 13.8 Computation of the mean

```
Command Window
>> x=[170 180 175 170 162 160 180 185];
>> sum(x)/length(x)

ans =

  172.7500
fx >>
```

Example 13.7 The code presented in Fig. 13.9 computes the variance of the {170, 180, 175, 170, 162, 160, 180, 185}. Remember that $\sigma^2 = \sum_{i=1}^{N} \frac{(x_i - \overline{x})^2}{N-1}$.

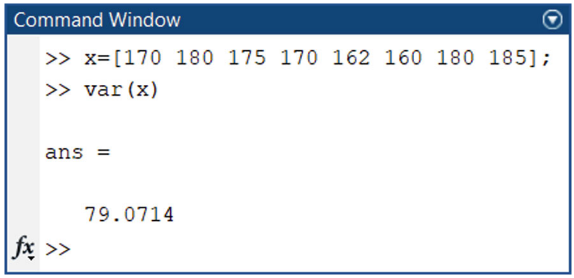


Fig. 13.9 Computation of the variance

Example 13.8 The code presented in Fig. 13.10 computes the standard deviation of the {170, 180, 175, 170, 162, 160, 180, 185}. Remember that $\sigma = \sqrt{\sum_{i=1}^{N} \frac{(x_i - \bar{x})^2}{N-1}}$.

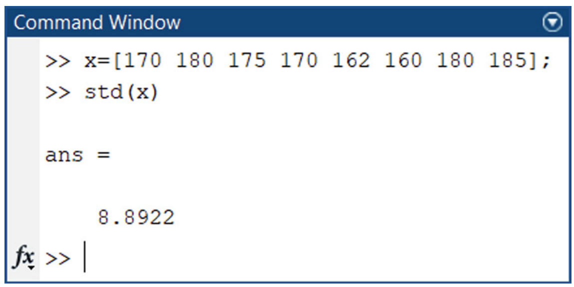


Fig. 13.10 Computation of the standard deviation

Example 13.9 The code presented in Fig. 13.11 computes the median of the {170, 180, 175, 170, 162, 160, 180, 185}.

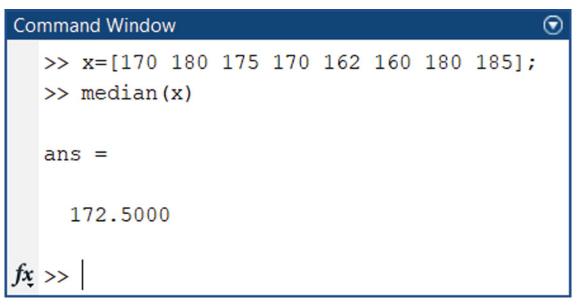


Fig. 13.11 Computation of the median

Let's check the result. Sorted data set is {160, 162, 170, 170, 175, 180, 180, 185}. The data set has 8 members. Therefore, median is $\frac{170+175}{2} = 172.5$.

13.4 Random Number Generation

`rand` and `randn` are two functions in MATLAB used to generate random numbers, but they differ in the type of distribution they follow.

The `rand` function generates random numbers from a uniform distribution between 0 and 1. This function is useful for tasks like simulating random events with equal probabilities.

The `randn` function generates random numbers from a normal (Gaussian) distribution with a mean of 0 and a standard deviation of 1. This function is useful for tasks like simulating noise, modeling natural phenomena, and machine learning applications.

Random numbers generated with `randn` can theoretically range from negative infinity to positive infinity. However, in practice, the majority of the numbers will fall within a few standard deviations of the mean (which is 0 for a standard normal distribution). While there's no strict upper or lower bound, you can expect most of the generated numbers to be between -3 and 3. However, it's important to remember that there's a small probability of getting values outside this range.

Example 13.10 The code presented in Figs. 13.12 and 13.13 produces a 1-by-4 vector containing randomly generated elements.

```
Command Window
>> randn(1,4)

ans =

   -1.1658   -1.1480    0.1049    0.7223

fx >>
```

Fig. 13.12 Random number generation with `randn` function

```
Command Window
>> rand(1,4)

ans =

    0.9340    0.1299    0.5688    0.4694

fx >>
```

Fig. 13.13 Random number generation with `rand` function

The `randi` is a MATLAB function used to generate random integers within a specified range. It follows a uniform distribution, meaning each integer within the range has an equal probability of being generated.

Example 13.11 The code presented in Fig. 13.14 models the process of flipping a fair coin four times.

Fig. 13.14 Simulating four coin flips

```
Command Window
>> num_flips = 4;
>> coin_flips = randi([0 1], num_flips, 1)

coin_flips =

     0
     1
     0
     1

fx >>
```

Example 13.12 The code presented in Fig. 13.15 models the process of rolling a die five times.

Fig. 13.15 Simulating five dice rolls

```
Command Window
>> num_rolls=5;
>> dice_rolls=randi(6,num_rolls,1)

dice_rolls =

     1
     3
     6
     3
     4

fx >>
```

The `unifrnd` function generates random numbers from a continuous uniform distribution.

Example 13.13 The code shown in Fig. 13.16 generates 4 random numbers between − 3 and 12.

Fig. 13.16 Generating four random numbers between − 3 and 12

```
Command Window
>> random_numbers = unifrnd(-3, 12, 4, 1)

random_numbers =

    1.2876
    8.3580
    8.3059
    2.7067

fx >>
```

13.5 Linear Regression

Linear regression is a statistical method used to model the relationship between a dependent variable and one or more independent variables. It aims to find the best-fitting line through a set of data points, allowing us to make predictions and understand the relationship between variables.

Example 13.14 In this example, a linear regression model will be fitted to the dataset presented in Table 13.1.

Table 13.1 Data for Example 13.14

x	y
1	1.2
2	2.5
3	3.1
4	4.3
5	5.7

The code presented in Fig. 13.17 implements a linear regression model to fit a line to the given data points, minimizing the sum of squared errors. As shown in Fig. 13.17, the equation of the fitted line is $y = 1.08x + 0.12$.

Fig. 13.17 The data is fitted to a linear model

```
Command Window
>> x = [1 2 3 4 5];
>> y = [1.2 2.5 3.1 4.3 5.7];
>> polyfit(x,y,1)

ans =

    1.0800    0.1200

fx >>
```

To verify the solution, we aim to minimize the following cost function: $f(m, b) = \sum_{x=1}^{5}(mx + b - y_i) = (m + b - 1.2)^2 + (2m + b - 2.5)^2 + (3m + b - 3.1)^2 + (4m + b - 4.3)^2 + (5m + b - 5.7)^2$. The code in Fig. 13.18 calculates the $\frac{\partial f}{\partial m}$ and $\frac{\partial f}{\partial b}$. According to Fig. 13.18, $\frac{\partial f}{\partial m} = 30b + 110m - \frac{612}{5}$ and $\frac{\partial f}{\partial b} = 10b + 30m - \frac{168}{5}$.

```
Command Window
>> syms m b
>> f=(m+b-1.2)^2+(2*m+b-2.5)^2+(3*m+b-3.1)^2+(4*m+b-4.3)^2+(5*m+b-5.7)^2;
>> simplify(diff(f,m))

ans =

30*b + 110*m - 612/5

>> simplify(diff(f,b))

ans =

10*b + 30*m - 168/5

fx >>
```

Fig. 13.18 Calculation of $\frac{\partial f}{\partial m}$ and $\frac{\partial f}{\partial b}$

Let's calculate the critical point.

$$\begin{cases} \frac{\partial f}{\partial m} = 0 \\ \frac{\partial f}{\partial b} = 0 \end{cases} \Rightarrow \begin{cases} 30b + 110m - \frac{612}{5} = 0 \\ 10b + 30m - \frac{168}{5} = 0 \end{cases} \Rightarrow \begin{cases} 30b + 110m = \frac{612}{5} \\ 10b + 30m = \frac{168}{5} \end{cases}$$

$$\Rightarrow \begin{bmatrix} 30 & 110 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} \frac{612}{5} \\ \frac{168}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 30 & 110 \\ 10 & 30 \end{bmatrix}^{-1} \begin{bmatrix} \frac{612}{5} \\ \frac{168}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 0.1200 \\ 1.0800 \end{bmatrix}$$

Therefore, the linear regression model $y = 1.08x + 0.12$ provides the best fit to the data presented in Table 13.1.

13.6 Polynomial Regression

Polynomial regression is a form of regression analysis in which the relationship between the independent variable x and the dependent variable y is modeled as an nth degree polynomial in x.

Example 13.15 In this example, we will fit a 4th order polynomial to the data provided in Table 13.2.

Table 13.2 Data for Example 13.15

x	y
1	6.1
2	− 8.2
3	− 13.1
4	52.6
5	290.9
6	844.6
7	1897.3
8	3673.40
9	6438.09
10	10,497.4

The code shown in Fig. 13.19 fits a 4th order polynomial to given data points, minimizing the sum of squared errors. According to Fig. 13.19, the found polynomial is: $1.7000x^4 - 6.8007x^3 + 3.0046x^2 - 1.2116x + 914083$.

```
Command Window
>> x=[1 2 3 4 5 6 7 8 9 10];
>> y=[6.10 -8.20 -13.10 52.60 290.9 844.60 1897.30 3673.40 6438.09 10497.4];
>> polyfit(x,y,4)

ans =

    1.7000   -6.8007    3.0046   -1.2116    9.4083

fx >>
```

Fig. 13.19 The data is fit to a 4th order polynomial

Figure 13.20 shows code that evaluates the polynomial $1.7000x^4 - 6.8007x^3 + 3.0046x^2 - 1.2116x + 914083$ at $x = 7.3$.

```
Command Window
>> x=[1 2 3 4 5 6 7 8 9 10];
>> y=[6.10 -8.20 -13.10 52.60 290.9 844.60 1897.30 3673.40 6438.09 10497.4];
>> polyfit(x,y,4)

ans =

    1.7000   -6.8007    3.0046   -1.2116    9.4083

>> polyval(ans,7.3)

ans =

   2.3429e+03

fx >>
```

Fig. 13.20 The value of the fitted polynomial at $x = 7.3$

13.7 Binomial Distribution

`Y = binopdf(x,n,p)` computes the binomial probability mass function (PMF) at each value in `x`, given the number of trials `n` and the probability of success `p` for each trial.

Example 13.16 Given a biased coin with a probability of heads equal to 0.3, the probability of getting exactly 5 heads in 10 tosses is $P(X = 5) = \binom{10}{5}(0.3)^5(0.7)^{10-5} = 252 \times 0.004 = 0.1029$ (Fig. 13.21).

```
Command Window
>> binopdf(5,10,0.3)

ans =

    0.1029

fx >>
```

Fig. 13.21 Calculating the probability of exactly 5 heads

The code in Fig. 13.22 plots the binomial probability mass function at each integer value from 0 to 10.

```
Command Window
>> binopdf(0:10,10,0.3)

ans =

    0.0282    0.1211    0.2335    0.2668    0.2001    0.1029    0.0368    0.0090    0.0014    0.0001    0.0000

fx >> |
```

Fig. 13.22 Calculating probabilities for 0–10 heads (use `format long` for more precision)

Figure 13.23 visualizes the binomial probability mass function for values between 0 and 10. Output of this code is shown in Fig. 13.24.

```
Command Window
>> binopdf(0:10,10,0.3)

ans =

    0.0282    0.1211    0.2335    0.2668    0.2001    0.1029    0.0368    0.0090    0.0014    0.0001    0.0000

>> stem([0:10],ans)
fx >> |
```

Fig. 13.23 Visualizing the binomial probability mass function

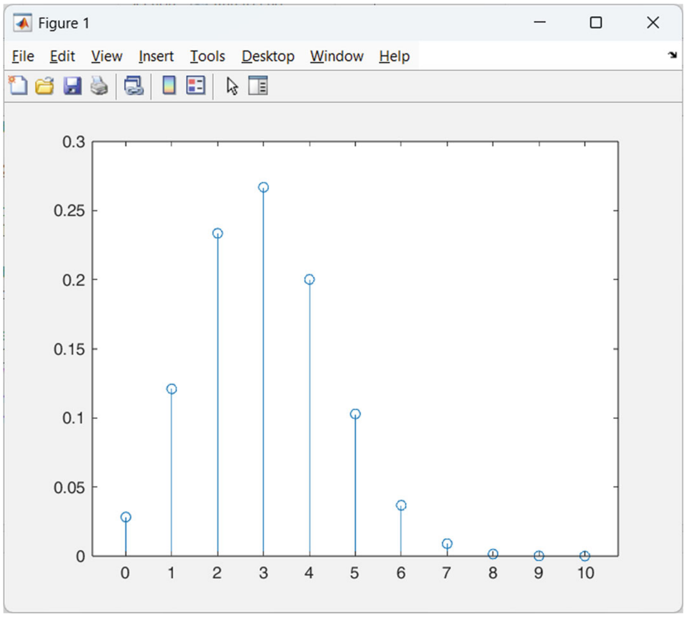


Fig. 13.24 Output of the code shown in Fig. 13.23

Example 13.17 Given a biased coin with a probability of heads equal to 0.3, the probability of getting 4 or fewer heads in 10 tosses is $F_X(4) = P(X \leq 4) = \sum_{k=0}^{4} P(X = k) = 0.0282 + 0.1211 + 0.2335 + 0.2668 + 0.2001 = 0.8497$ (Fig. 13.25).

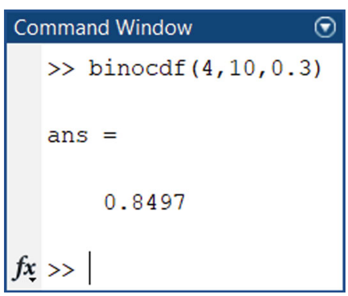


Fig. 13.25 Probability of 4 or fewer heads

13.8 Normal (Gaussian) Distribution

A normal distribution is defined by the probability density function (PDF) $f\left(x;\mu,\sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$. The notation $N(\mu,\sigma^2)$ is used to denote a normal distribution with a mean of μ and a variance of σ^2.

Example 13.18 The code in Fig. 13.26 computes the PDF value at $x = 0.56$ for a $N(0.4, 0.7^2)$ distribution.

Fig. 13.26 Value of $N(0.4, 0.7^2)$ at $x = 0.56$

```
Command Window
>> normpdf(0.56,0.4,.7)

ans =

    0.5552

fx >>
```

Let's verify the result. According to Fig. 13.27 the obtained result is correct.

```
Command Window
>> sigma=0.7;
>> mu=0.4;
>> eval(subs(1/sigma/sqrt(2*pi)*exp(-0.5*((x-mu)/sigma)^2),x,.56))

ans =

    0.5552
fx >>
```

Fig. 13.27 Verifying the Fig. 13.26 result

Example 13.19 Given $X \sim N(2,9)$, find $P(X < 3.6)$.

$X \sim N(2,9) \Rightarrow \begin{cases} \mu = 2 \\ \sigma^2 = 9 \Rightarrow \sigma = 3 \end{cases}$. According to Fig. 13.28 $P(X < 3.6) = 0.7031$.

Fig. 13.28 Calculation of $P(X < 3.6)$

```
Command Window
>> sigma=sqrt(9);
>> mu=2;
>> normcdf(3.6,mu,sigma)

ans =

    0.7031

fx >>
```

Let's verify the result. According to Fig. 13.29 the obtained result is correct.

```
Command Window
>> sigma=sqrt(9);
>> mu=2;
>> syms x
>> eval(int(1/sigma/sqrt(2*pi)*exp(-0.5*((x-mu)/sigma)^2),x,-inf,3.6))

ans =

    0.7031

fx >>
```

Fig. 13.29 Calculation of $\int_{-\infty}^{3.6} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$ with $\mu = 2$ and $\sigma = 3$

Example 13.20 Let's say we have a normally distributed dataset with a mean (μ) of 6 and a standard deviation (σ) of 11. Locate the value of x that satisfy $P(X < x) = 0.65$.

According to the code shown in Fig. 13.30, $P(X < 10.2385) = 0.65$.

Fig. 13.30 Finding x such that $P(X < x) = 0.65$

```
Command Window
>> mu=6;
>> sigma=11;
>> p=0.65;
>> x=norminv(p,mu,sigma)

x =

    10.2385

fx >>
```

Let's verify the result. According to Fig. 13.31 the obtained result is correct.

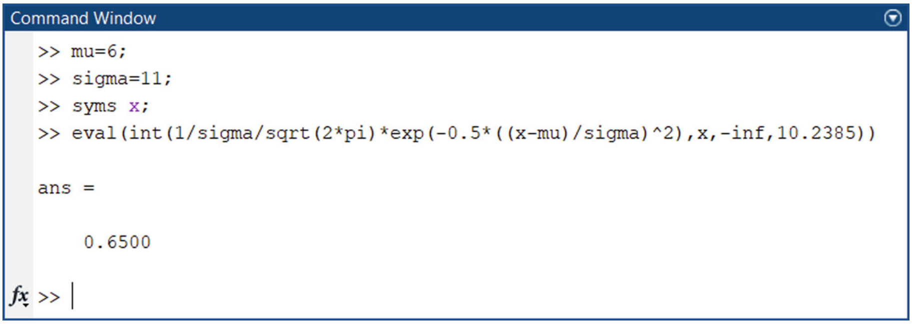


Fig. 13.31 Calculation of $\int_{-\infty}^{10.2385} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$ with $\mu = 6$ and $\sigma = 11$

Example 13.21 In this example we will visualize the probability density function of the standard normal distribution ($\mu = 0$ and $\sigma = 1$) over the range -3 to 3.

Use the `edit` command (Fig. 13.32) to open the Editor (Fig. 13.33).

Fig. 13.32 `edit` command

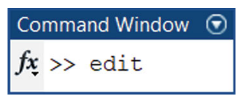
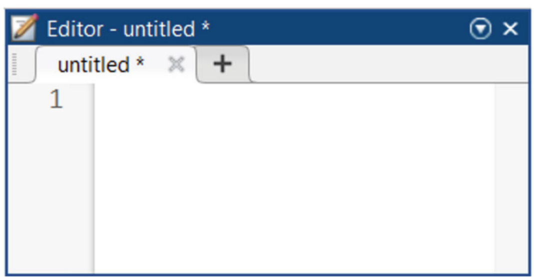


Fig. 13.33 MATLAB Editor



Enter the following code into the Editor (Fig. 13.34) and save it using Ctrl+S.

```
% Define the mean and standard deviation
mu = 0;
sigma = 1;

% Create a vector of x-values
x = -3:0.1:3;

% Calculate the PDF values
y = normpdf(x, mu, sigma);

% Plot the PDF
plot(x, y);
xlabel('x');
ylabel('Probability Density');
title('Normal Distribution PDF');
grid on;
```

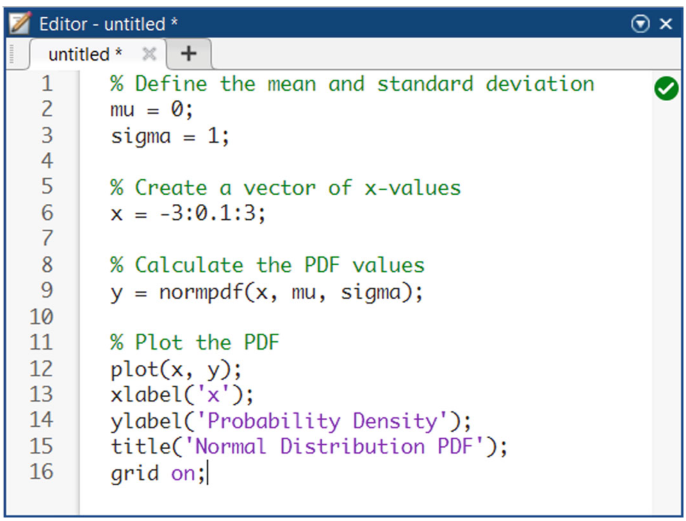


Fig. 13.34 The code is entered into the MATLAB Editor

Press F5 to run the code. Output of the code is shown in Fig. 13.35.

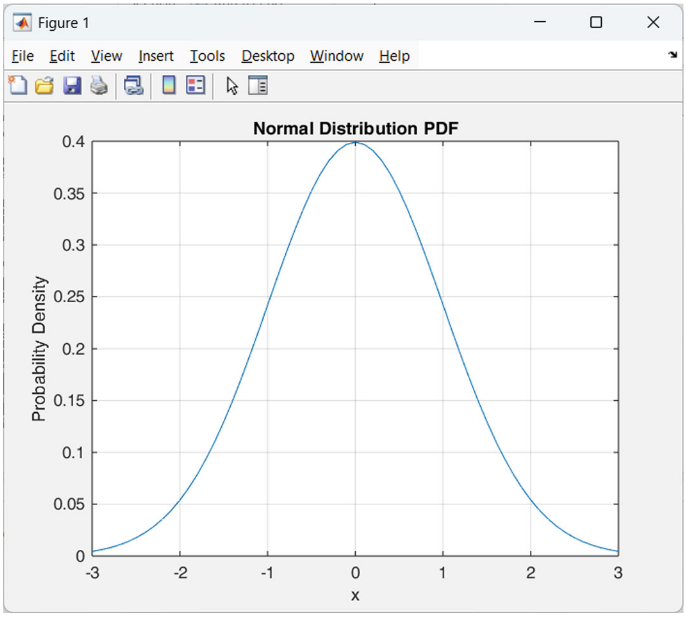


Fig. 13.35 Output of the code shown in Fig. 13.34

Example 13.22 The code in Fig. 13.36 visualizes the probability density function of the standard normal distribution ($\mu = 0$ and $\sigma = 1$) over the range -3 to 3 using the `ezplot` command. Output of this code is shown in Fig. 13.37.

```
Command Window
>> syms x
>> sigma=1;
>> mu=0;
>> ezplot(1/sigma/sqrt(2*pi)*exp(-0.5*((x-mu)/sigma)^2),[-3,3]), grid on
fx >>
```

Fig. 13.36 Visualizing the $N(0, 1)$ on $[-3, 3]$ interval

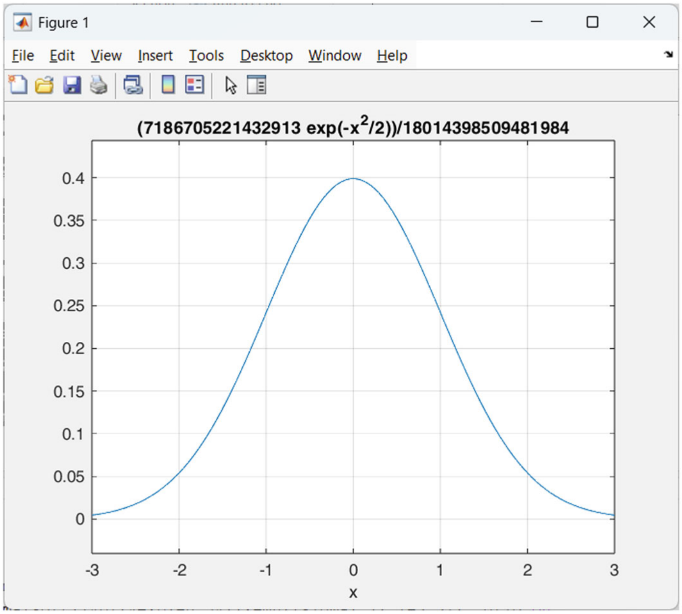


Fig. 13.37 Output of the code shown in Fig. 13.36

13.9 Error Function

The error function, denoted as $erf\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$, is a mathematical function that is closely related to the normal distribution.

Example 13.23 The code shown in Fig. 13.38 calculates the $erf\,(1.3)$.

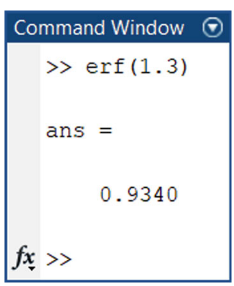


Fig. 13.38 Calculation of $erf\,(1.3)$

Let's verify the result. According to Fig. 13.39 the obtained result is correct.

Fig. 13.39 Calculation of *erf* (1.3)

```
Command Window
>> syms t
>> x=1.3;
>> eval(2/sqrt(pi)*int(exp(-t^2),t,0,x))

ans =

    0.9340

fx >>
```

13.10 Gamma Distribution

The gamma distribution is a versatile two-parameter family of continuous probability distributions. It is often used to model positive-valued random variables, such as waiting times, time to failure or interest rates.

The PDF of a gamma distribution is given by $Gamma(\alpha, \beta) = \frac{x^{\alpha-1} e^{\frac{-x}{\beta}}}{\beta^{\alpha}.\Gamma(\alpha)}, x > 0$. The PDF of the gamma distribution is zero for $x \leq 0$.

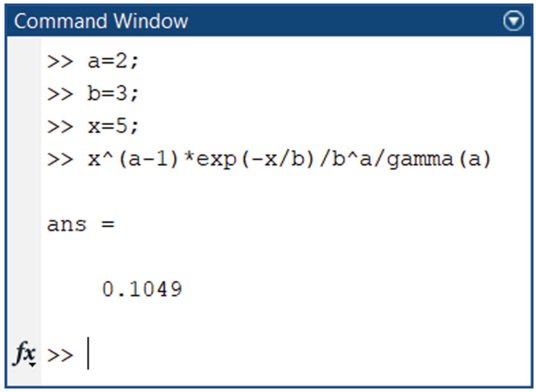
Example 13.24 The code in Figs. 13.40 computes the PDF value at $x = 5$ for a $Gamma(2, 3)$ distribution.

Fig. 13.40 Calculation of $Gamma(2, 3)$ for $x = 5$

```
Command Window
>> a=2;
>> b=3;
>> gampdf(5,a,b)

ans =

    0.1049

fx >>
```

Let's verify the result. According to Fig. 13.41 the obtained result is correct.

Fig. 13.41 Checking the validity of the result in Fig. 13.40



Example 13.25 Given $X \sim Gamma(2, 3)$, find $P(X < 5)$.

According to Fig. 13.42, $P(X < 5) = 0.4963$.

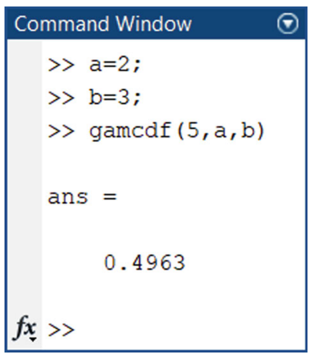


Fig. 13.42 Calculation of $P(X < 5)$

Let's verify the result. According to Fig. 13.43 the obtained result is correct.

```
Command Window
>> a=2;
>> b=3;
>> eval(int(x^(a-1)*exp(-x/b)/b^a/gamma(a),x,0,5))

ans =

    0.4963

fx >>
```

Fig. 13.43 Calculation of $\int_0^5 \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}.\Gamma(\alpha)}dx$ with $\alpha = 2$ and $\beta = 3$

Example 13.26 Let's say we have a gamma-distributed random variable with shape parameter $\alpha = 2$ and scale parameter $\beta = 3$. Locate the value of x that satisfy $P(X < x) = 0.85$.

According to the code shown in Fig. 13.44, $P(X < 10.1173) = 0.85$.

```
Command Window
>> alpha = 2;
>> beta = 3;
>> p = 0.85;
>> x = gaminv(p, alpha, beta)

x =

   10.1173

fx >>
```

Fig. 13.44 Finding x such that $P(X < x) = 0.85$

Let's verify the result. According to Fig. 13.45 the obtained result is correct.

```
Command Window
>> a=2;
>> b=3;
>> syms x
>> eval(int(x^(a-1)*exp(-x/b)/b^a/gamma(a),x,0,10.1173))

ans =

    0.8500

fx >> 
```

Fig. 13.45 Calculation of $\int_0^{10.1173} \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^{\alpha}.\Gamma(\alpha)}dx$ with $\alpha = 2$ and $\beta = 3$

13.11 MATLAB Functions for Other Distributions

Tables 13.3, 13.4, 13.5, 13.6, 13.7, 13.8, 13.9 and 13.10 provide a summary of commonly encountered distributions and their corresponding MATLAB functions.

Table 13.3 MATLAB functions for exponential distribution

`exprnd`	Generates random numbers from an exponential distribution
`expcdf`	Calculates the cumulative distribution function (CDF) of the exponential distribution
`exppdf`	Calculates the probability density function (PDF) of the exponential distribution
`expinv`	Calculates the inverse CDF (quantile function) of the exponential distribution

Table 13.4 MATLAB functions for beta distribution

`betarnd`	Generates random numbers from a beta distribution
`betapdf`	Calculates the PDF of the beta distribution
`betacdf`	Calculates the CDF of the beta
`betainv`	Calculates the inverse CDF (quantile function) of the beta distribution
`betastat`	Calculates the mean and variance of a beta distribution

Table 13.5 MATLAB functions for Weibull distribution

wblrnd	Generates random numbers from a Weibull distribution
wblpdf	Calculates the PDF of the Weibull distribution
wblcdf	Calculates the CDF of the Weibull distribution
wblinv	Calculates the inverse CDF (quantile function) of the Weibull distribution
wblstat	Calculates the mean and variance of a Weibull distribution
wblfit	Estimates the parameters of a Weibull distribution from a given sample of data

Table 13.6 MATLAB functions for χ^2 distribution

chi2rnd	Generates random numbers from a chi-square distribution
chi2pdf	Calculates the PDF of the chi-square distribution
chi2cdf	Calculates the CDF of the chi-square distribution
chi2inv	Calculates the inverse CDF (quantile function) of the chi-square distribution
chi2stat	Calculates the mean and variance of a chi-square distribution

Table 13.7 MATLAB functions for t-distribution

trnd	Generates random numbers from a t-distribution
tpdf	Calculates the PDF of the t-distribution
tcdf	Calculates the CDF of the t-distribution
tinv	Calculates the inverse CDF (quantile function) of the t-distribution

Table 13.8 MATLAB functions for Rayleigh distribution

raylrnd	Generates random numbers from a Rayleigh distribution
raylpdf	Calculates the PDF of the Rayleigh distribution
raylcdf	Calculates the CDF of the Rayleigh distribution
raylinv	Calculates the inverse CDF (quantile function) of the Rayleigh distribution

Table 13.9 MATLAB functions for Poison distribution

poissrnd	Generates random numbers from a Poisson distribution
Poisspdf	Calculates the probability mass function (PMF) of the Poisson distribution
poisscdf	Calculates the CDF of the Poisson distribution

Table 13.10 MATLAB functions for geometric distribution

`geornd`	Generates random numbers from a geometric distribution
`geopdf`	Calculates the probability mass function (PMF) of the geometric distribution
`geocdf`	Calculates the CDF of the geometric distribution
`geostat`	Calculates the mean and variance of a geometric distribution

References for Further Study

1. Probability and Statistics for Engineering and the Sciences (8^{th} edition), Jay L. Devore, Cengage Learning, 2011.
2. Advanced Engineering Mathematics (10^{th} edition), Erwin Kreyszig, Wiley, 2011.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.

Calculator Skills and Formula Writing in Microsoft® Word®

14

14.1 Introduction

Calculators have become an essential tool for engineers, significantly enhancing their efficiency and accuracy in a wide range of tasks. Calculators play an important role in university exams, as well. So, it is important to learn how to work with your calculator correctly. By mastering your calculator, you can significantly improve your problem-solving skills and achieve better results in your engineering studies.

A common mistake in calculator usage is setting the wrong angle mode. If your calculator is in radian mode but you input an angle in degrees, the result will be incorrect. Always double-check your calculator's mode setting to ensure it's in degrees or radians, as needed, before performing calculations.

First part of this chapter offers reference operations to aid in your calculator usage. Endeavor to replicate the provided correct results. Depending on your calculator's model, some operations may not be possible. Second part of this chapter shows how to write a formula in Microsoft® Word®.

14.2 Basic Operations

The following are some basic operations and their values.

1. $12345679 \times 8 = 98765432$
2. $54.63 \times 632.58 = 34557.8454$
3. $12.653 \times 98.74 = 1249.3572$
4. $\frac{542.68}{6.24} = 86.9679$
5. $\frac{364.56}{5.25} = 69.44$

F. Asadi, *Engineering Mathematics with MATLAB® and Simulink®*,
https://doi.org/10.1007/978-3-031-85244-2_14

6. $(1+2j)+(3-8j)=4-6j$
7. $(1+2j)-(3-8j)=-2+10j$
8. $(1+2j)\times(3-8j)=19-2j$
9. $\frac{(1+2j)}{3-8j}=-0.1781+0.1918j$

14.3 Exponents and Radicals

The following are some example calculations for exponents and radicals.

1. $14^{2.5}=733.3648$
2. $3.2^{4.3}=148.6432$
3. $12.96^{2.24}=310.6241$
4. $\sqrt{17.63}=4.1988$
5. $\sqrt{965.22}=31.0680$
6. $\sqrt[3]{752.53}=9.0958$
7. $\sqrt[5]{12036}=6.5478$

14.4 Logarithm

The following are some example calculations for logarithms.

1. $\log_{10}56.3=1.7505$
2. $\log_{10}1200.698=3.0794$
3. $\ln(56.3)=4.0307$
4. $\ln(1200.698)=7.0907$
5. $\ln(0.33)=-1.1087$

14.5 Conversion of Complex Numbers to Polar Form

The following are some examples of converting complex numbers between Cartesian and polar forms.

1. $1+6j=6.0828e^{1.4056j}$
2. $-3+5j=5.8310e^{j2.1112}$
3. $-16-3j=16.2788e^{-2.9562j}$
4. $10-9j=13.4536e^{-j0.7328}$
5. $12.56e^{-j1.8}=-2.8537-12.2315j$
6. $7.23e^{j.8}=5.0372+5.1865j$

14.6 Roots of Polynomial and Non-linear Equations

The following are some examples of finding the roots of polynomials and non-linear equations.

1. $x^2 - 3x + 2 = 0 \Rightarrow x_1 = 1, x_2 = 2$
2. $x^3 - 11.77x^2 + 41.985x - 47.025 = 0 \Rightarrow x_1 = 2.5, x_2 = 3, x_3 = 6.27$
3. $x^2 + 2x + 10 = 0 \Rightarrow x_1 = -1 + 3j, x_2 = -1 - 3j$
4. $x^3 + 2x^2 - 51x - 232 = 0 \Rightarrow x_1 = -5 - 2j, x_2 = -5 + 2j, x_3 = 8$
5. $x^2 - 2\cos(x) = 0 \Rightarrow x_1 = -1.02162, x_2 = 1.02162$
6. $x + e^{-x} - 2 \Rightarrow x_1 = -1.146189, x_2 = +1.841402$

14.7 Derivative and Integrals

The following are some examples of evaluating integrals and derivatives.

1. $\int_0^3 \sin(x)dx = 1.9900$
2. $\int_0^1 x^2 dx = 0.3333$
3. $\int_0^2 x^2 \sin(2x)dx = 0.1371$
4. $\int_0^{0.5} e^{-6x}dx = 0.1584$
5. $\int_0^2 \frac{3}{x+1}dx = 3.2958$
6. $\frac{d}{dx}(\cos(x))\Big|_{\frac{\pi}{3}} = -0.8660$
7. $\frac{d}{dx}(1 + \tan^2(x))\Big|_{0.25} = 0.5440$
8. $\frac{d}{dx}(\frac{1+x}{\sin(x)})\Big|_{0.5} = -3.6413$
9. $\frac{d}{dx}(\frac{x^2+9}{(x^3+6x-4)})\Big|_{1.5} = -1.6868$

14.8 Trigonometric Ratios

Table 14.1 presents trigonometric ratios for several angles.

Table 14.1 Trigonometric ratios for some sample angles

Radian	Degrees
$\sin(0.2563) = 0.2535$ $\sin(0.5236) = 0.5000$ $\sin(0.7238) = 0.6622$ $\sin(1.5) = 0.9975$ $\sin(2) = 0.9093$	$\sin(10°) = 0.1736$ $\sin(30°) = 0.5000$ $\sin(50°) = 0.7660$ $\sin(70°) = 0.9397$ $\sin(123°) = 0.8387$
$\cos(0.2563) = 0.9673$ $\cos(0.5236) = 0.8660$ $\cos(0.7238) = 0.7493$ $\cos(1.5) = 0.0707$ $\cos(2) = -0.4161$	$\cos(10°) = 0.9848$ $\cos(30°) = 0.866$ $\cos(50°) = 0.6428$ $\cos(70°) = 0.3420$ $\cos(123°) = -0.5446$
$\tan(0.2563) = 0.2621$ $\tan(0.5236) = 0.5774$ $\tan(0.7238) = 0.8838$ $\tan(1.5) = 14.1014$ $\tan(2) = -2.1850$	$\tan(10°) = 0.1763$ $\tan(30°) = 0.5774$ $\tan(50°) = 1.1918$ $\tan(70°) = 2.7475$ $\tan(123°) = -1.5399$

14.9 Inverse Trigonometric Ratios

Table 14.2 shows the inverse trigonometric functions for a sample of trigonometric ratios.

Table 14.2 Inverse trigonometric functions for a sample of trigonometric ratios

$Arcsin(0.310) = 0.3152Rad = 18.0596°$ $Arcsin(0.652) = 0.7102Rad = 40.6915°$ $Arcsin(0.711) = 0.7909Rad = 45.3163°$	$Arccos(0.52) = 1.0239Rad = 58.6677°$ $Arccos(0.73) = 0.7525Rad = 43.1136°$ $Arccos(0.95) = 0.3176Rad = 18.1949°$
$Arctan(1.45) = 0.967Rad = 55.4077°$ $Arctan(3.35) = 1.2807Rad = 73.3792°$ $Arctan(8.62) = 1.4553Rad = 83.3827°$	$Arccotan(0.65) = 0.9944Rad = 56.9761°$ $Arccotan(7.35) = 0.1352Rad = 7.7477°$ $Arccotan(10.87) = 0.0917Rad = 5.2562°$

14.10 Microsoft Word Equation Editor

Click the Insert button (Fig. 14.1) to open the Insert ribbon (Fig. 14.2).

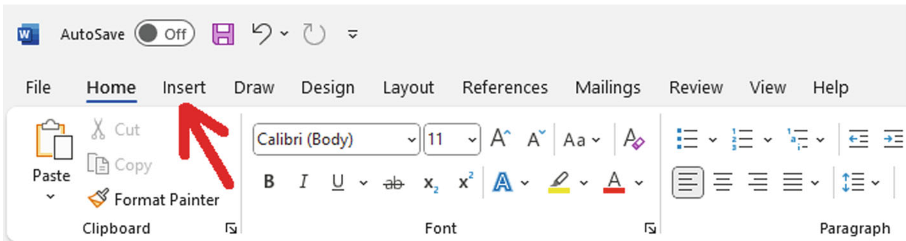


Fig. 14.1 Insert button

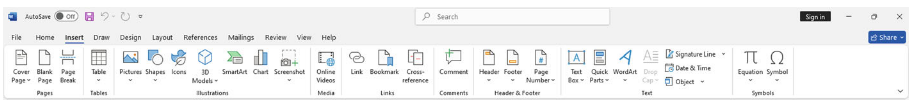


Fig. 14.2 Insert ribbon

Click on the Equations button (Fig. 14.3).

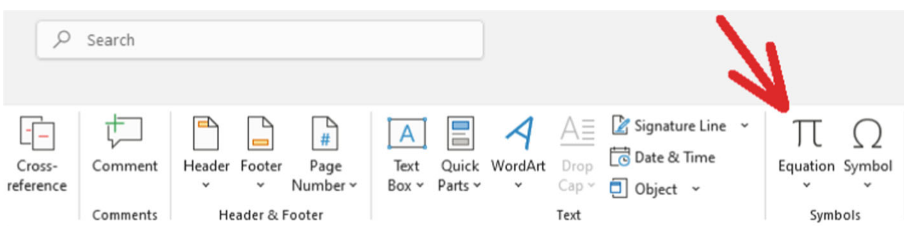


Fig. 14.3 Equations button

Clicking the Equation button will insert an equation box (Fig. 14.4) into your Word document. You can then use the tools provided in the toolbar to input your mathematical expressions (Fig. 14.5).

Fig. 14.4 Equation box

Fig. 14.5 Tools for writing mathematical expressions

Clicking the "Other" button (Fig. 14.6) will open a new window (Fig. 14.7), providing access to additional tools. From there, expand the "Basic Math" drop-down menu (Fig. 14.8) to explore a wider range of mathematical symbols and functions.

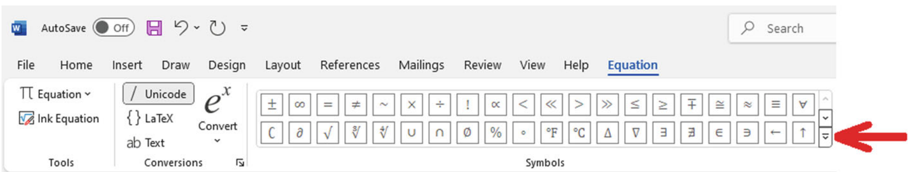


Fig. 14.6 Other button

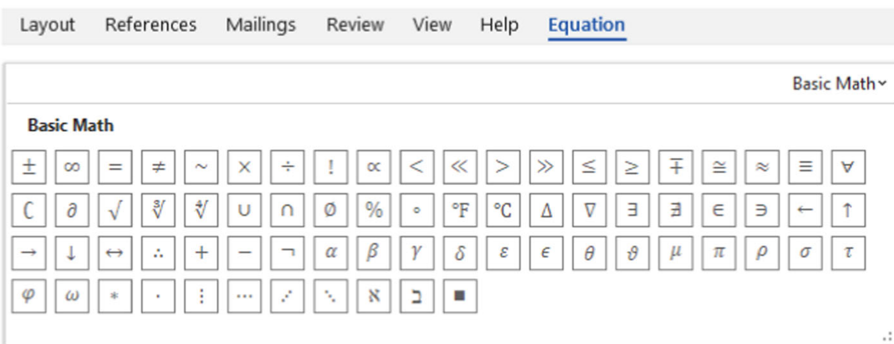


Fig. 14.7 Opened window

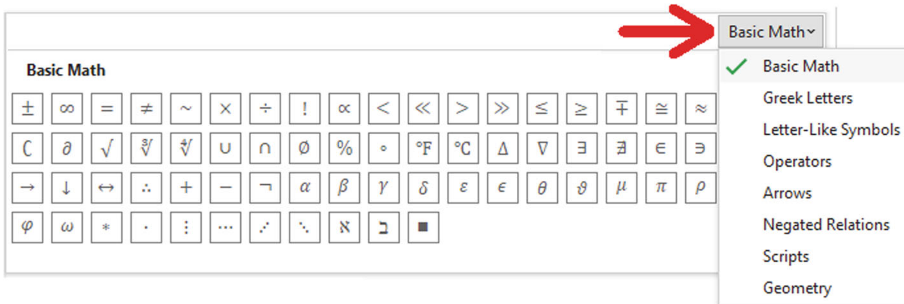


Fig. 14.8 Provided mathematical symbols and functions

References for Further Study

1. Advanced Engineering Mathematics (10th edition), Erwin Kreyszig, Wiley, 2011.
2. Calculus (8th edition), James Stewart, Cengage Learning, 2015.
3. Applied Numerical Analysis with MATLAB®/Simulink®: For Engineers and Scientists, Farzin Asadi, Springer, 2023.