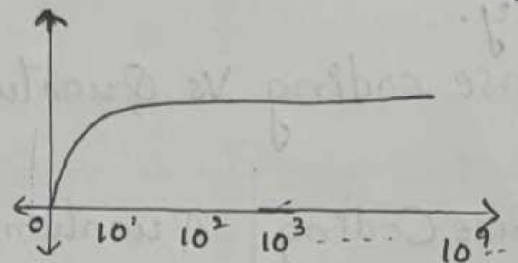
4. Write short notes on Grover's diffusion operator and its significance.

- Grover's Algorithm is also known as Grover's Search Algorithm.
- Grover's Algorithm is a quantum search algorithm that finds a correct item (marked) in an unstructured database of N-items in only about $O(\sqrt{N})$.



* By using Grover's algorithm, we can find target in $O(\sqrt{N})$ steps.

Steps to follow:

1. Create Superposition
2. Oracle (O) operator
3. Diffusion Operator.

$\}$ Iterate step 2 and 3.

Ex: 2-qubit system.

Step 1:

$$|0\rangle - \boxed{H} - \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

$$|0\rangle - \boxed{H} - \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

$$\Rightarrow |00\rangle \xrightarrow{H^{\otimes 2}} \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$$

$$= \frac{|0\rangle + |1\rangle + |2\rangle + |3\rangle}{2}.$$

1) Superposition :

$$|0\rangle^{\otimes n} \;—\; \boxed{H^{\otimes n}} \;—$$

$$S = |\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle .$$

Ex: $N = 4$.

$$\Rightarrow S = |\psi_0\rangle = \frac{1}{\sqrt{4}} \left[|0\rangle + |1\rangle + |2\rangle + |3\rangle \right].$$

Every state has equal probability.
If $|1\rangle$ is target, then probability of getting $|1\rangle$ is $1/4$ and the amplitude of $|1\rangle \cong 1$.
and the amplitude of $|0\rangle$, $|2\rangle$ and $|3\rangle \cong 0$.

$$S = |\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$$

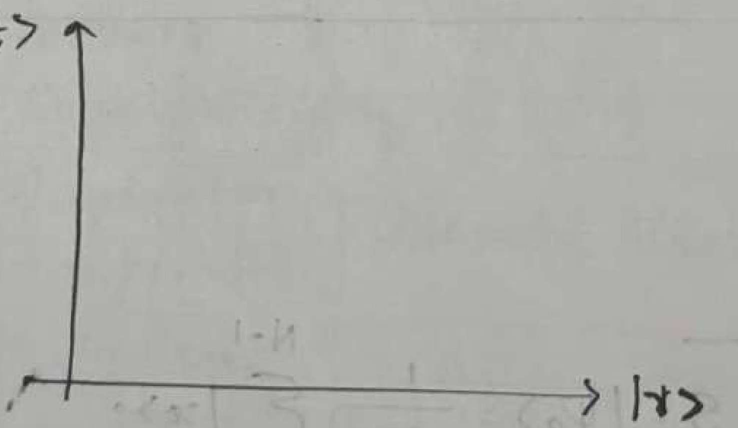
$$S = |\psi_0\rangle = \frac{1}{\sqrt{N}} \left[\sum_{r \neq t} |r\rangle + |t\rangle \right].$$

$$|\psi_0\rangle = \alpha_0 \sum_{r \neq t} |r\rangle + \beta_0 |t\rangle. \quad \text{where, } \alpha_0 = \frac{1}{\sqrt{N}}, \ \beta_0 = \frac{1}{\sqrt{N}}.$$

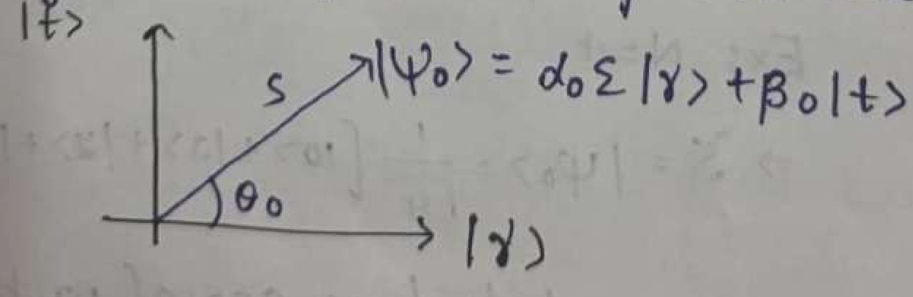
Aim: We need to decrease α_0 and to increase β_0

$$\Rightarrow \underbrace{\left(\frac{|0\rangle + |2\rangle + |3\rangle}{2} \right)}_{\text{remaining state.}} + \underbrace{\left(\frac{|1\rangle}{2} \right)}_{\text{target state}}.$$

$\Rightarrow$ Remaining state and target state are orthogonal to each other.

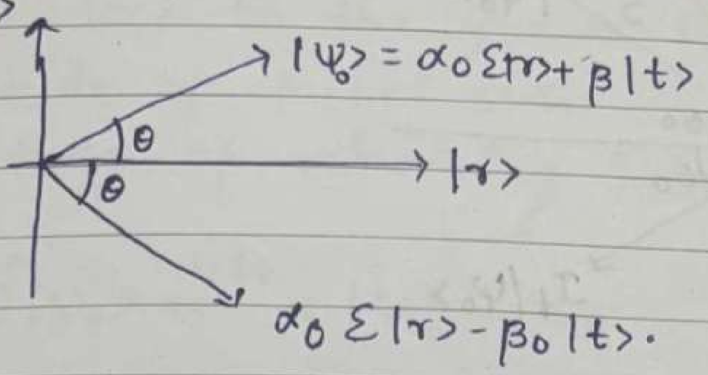


Initially the amplitudes of $|\psi_0\rangle$ are same.



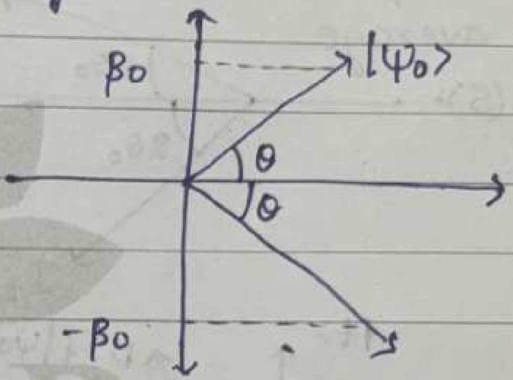
2. Inversion of $|\psi_0\rangle$ about x-axis. (I_t) / Oracle (O).

Flips the phase of the target.



* Inversion of amplitude of target state.

$\Rightarrow \boxed{I_t = I - 2|t\rangle\langle t|}$



$I_t|\psi_0\rangle = \alpha_0 \sum_{r \neq t} |r\rangle - \beta_0|t\rangle$.

where, $|\psi_0\rangle = \dfrac{|0\rangle + |1\rangle + |2\rangle + |3\rangle}{2}$

Inversion Operator I_t.

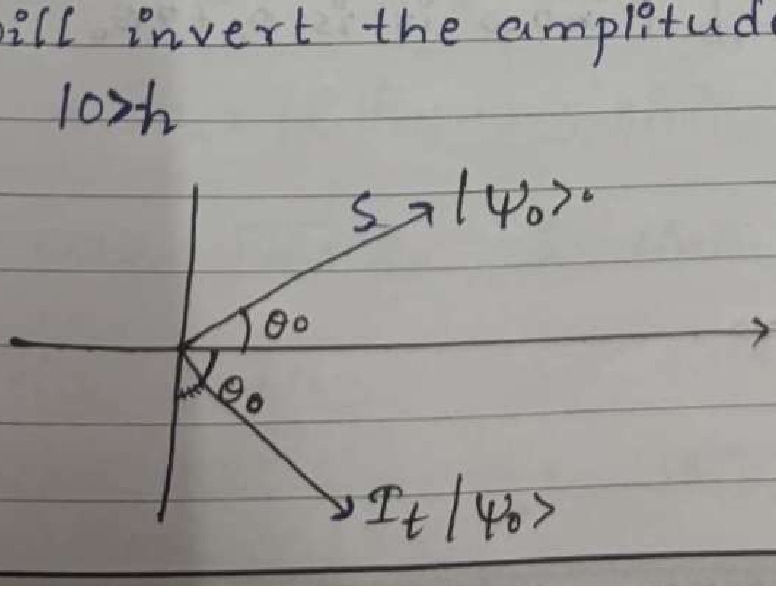
$I_t = \dfrac{|0\rangle - |1\rangle + |2\rangle + |3\rangle}{2}$.

③ Diffusion Operator: Inversion operator about mean (S).

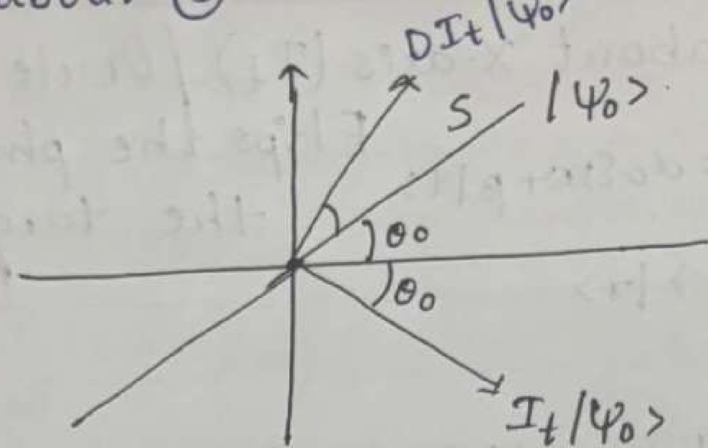
$\boxed{D = -H_n I_0 H_n}$

where $I_0 = I - 2|0\rangle\langle 0|$.

→ It will invert the amplitude of state other than $|0\rangle_n$

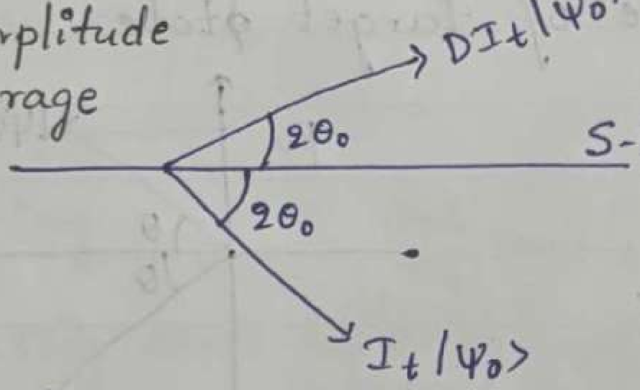


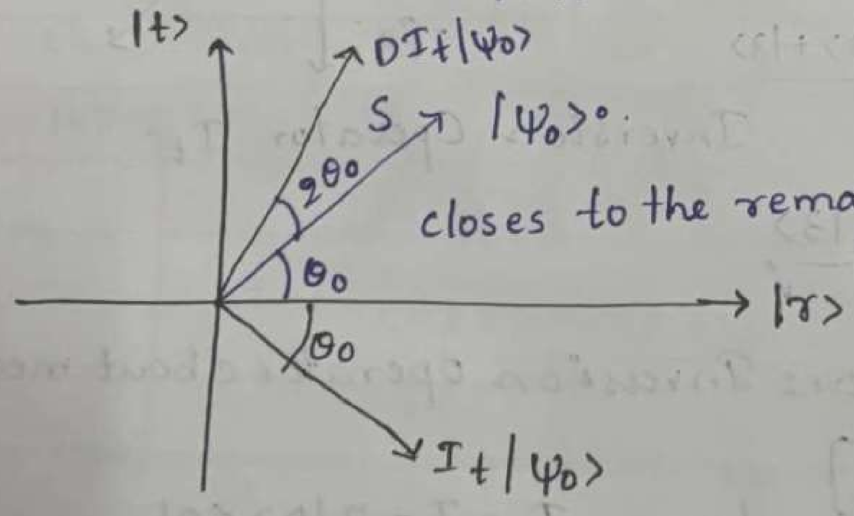
Inverts about (S)



Now inverts about (S).

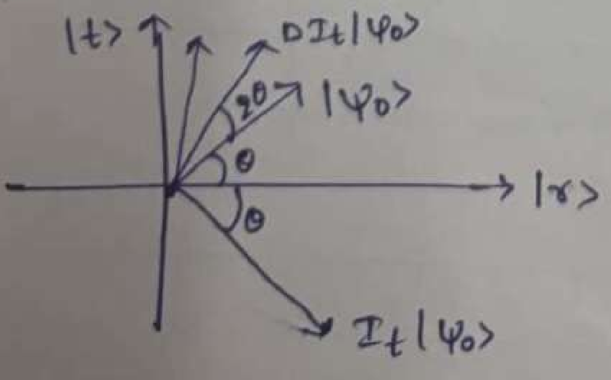
Reflect amplitude about average mean (S).





Initially angle is θ_0, but now it is $\theta_0 + 2\theta_0$

After 'w' iterations..



* So, as the number of iterations increases, our state will be close to the y-axis i.e, target state.
* Amplitude of target state is increasing
* Amplitude of remaining state is decreasing.

$$(DI_t)^n |\psi_0\rangle \xrightarrow{DI_t} |\psi_n\rangle$$

$$|\psi_0\rangle \xrightarrow{DI_t} |\psi_1\rangle$$

$$|\psi_F\rangle \xrightarrow{(DI_t)^\omega} |\psi_F\rangle.$$

We started with,

$$|\psi_0\rangle = \alpha_0 \sum_{r \neq t} |r\rangle + \beta_0 |t\rangle$$

$$(DI_t)^n |\psi_0\rangle = |\psi_n\rangle = \alpha_n \sum_{r \neq t}^{N-1} |r\rangle + \beta_n |t\rangle.$$

Probability of target state $= |\beta_n|^2$

Probability of remaining state $= |\alpha_n|^2 (N-1)$.

$\Rightarrow |\alpha_n|^2 (N-1) + |\beta_n|^2 = 1.$

$\Rightarrow$ We know that, $\cos^2\theta + \sin^2\theta = 1.$

$\Rightarrow$ Now, $\cos^2\theta_n = |\alpha_n|^2 (N-1)$; $\sin^2\theta_n = |\beta_n|^2$

$\Rightarrow \cos\theta_n = \sqrt{N-1}\,\alpha_n$; $\sin\theta_n = \beta_n.$

$\Rightarrow \cos\theta_0 = \alpha_0 \sqrt{N-1}$; $\sin\theta_n = \frac{1}{\sqrt{N}}$

$\Rightarrow \cos\theta_0 = \sqrt{N-1}\,\frac{1}{\sqrt{N}}$; $\sin\theta_0 = \frac{1}{\sqrt{N}}$

After some 'r' iterations.
$|\psi_F\rangle$ should coincide to y-axis (target).

$$2r\theta_0 + \theta_0 = \frac{\pi}{2}. \Rightarrow (2r+1)\theta_0 = \frac{\pi}{2}$$

$$\Rightarrow \boxed{r = \frac{\pi}{4\theta_0} - \frac{1}{2}}$$

$$\Rightarrow \cos\theta_0 = \sqrt{\frac{N-1}{N}} \; ; \; \sin\theta_0 = \frac{1}{\sqrt{N}}.$$

where 'N' is very large then $(N \to \infty)$.
and $\sin\theta_0 \simeq \theta_0$

$$\Rightarrow r = \frac{\pi}{4(\theta)} = \frac{1}{2} \Rightarrow \frac{\pi}{4\left(\sqrt{\frac{1}{N}}\right)} - \frac{1}{2} = \frac{\pi(\sqrt{N})}{4} - \frac{1}{2}.$$

$$\Rightarrow \boxed{r = O(\sqrt{N}).}$$

So, diffusion operator signifies the $O(\sqrt{N})$ by the target state.