Linear Algebra

What is Linear Algebra?

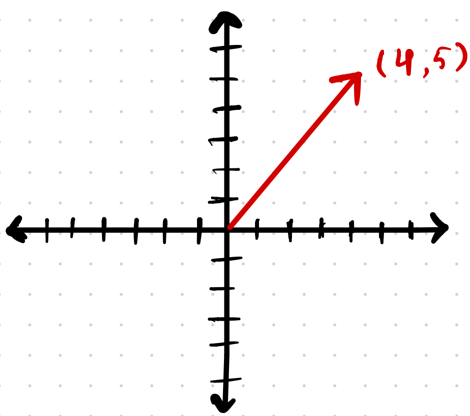
→ Field of mathematics that is concerned with study of lines in high-dimensional space.

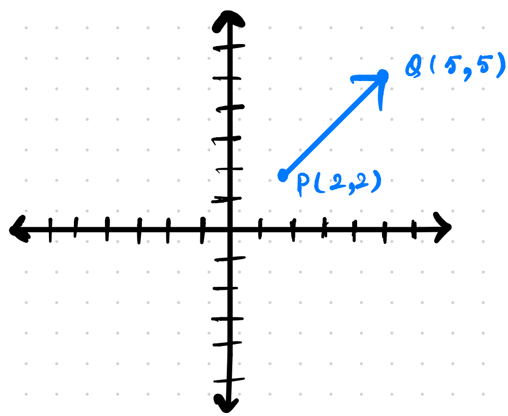
→ It helps in solving unknowns within system of lin. equations.

Examples:- → The sum of two no.'s is 25. One of them exceeds the other by 9. Find the no.'s.

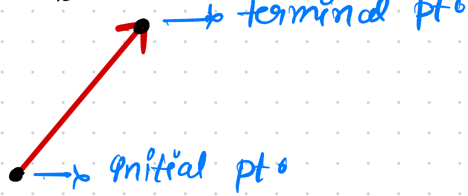
"unknowns"

Vectors:-

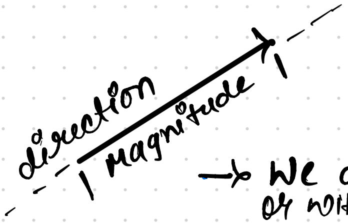




→ A vector is a specific quantity as line seg. with an arrowhead at one end.



→ It is defined by its Magnitude or the length of a line and its directions



A vector is directed line segment.

→ We denote vectors as lower case, boldfaced with or without an arrow like $\vec{a}, \vec{b}$, **c**, **d**

→ Given initial pt. P and terminal pt. Q, a vector can be expressed as $\vec{PQ}$.

→ Given initial pt. (0,0), a vector can be represented as $\begin{bmatrix} a \\ b \end{bmatrix}$.

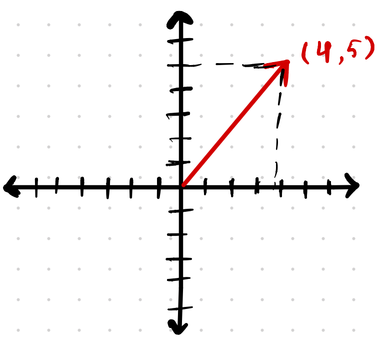
$\begin{bmatrix} a \\ b \end{bmatrix}$ is vector's standard position. The pos. vector has initial pt. as (0,0) and terminal pt. as (a,b).

Most of the time we will only deal with position vector.

Eg:- Given:- P(2,3) and Q(6,4); what's Pos. vector?

$$r = \langle 6-2, 4-3 \rangle$$
$$= \langle 4, 1 \rangle$$
$$= \begin{bmatrix} 4 \\ 1 \end{bmatrix} \begin{matrix} \rightarrow x\text{- component} \\ \rightarrow y\text{- component} \end{matrix}$$

Magnitude of a vector:-



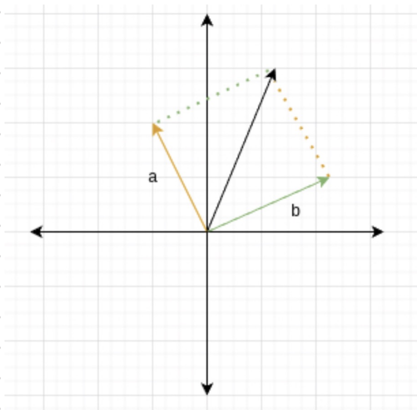
$$v = \begin{bmatrix} 4 \\ 5 \end{bmatrix}; \quad |v| = \sqrt{x^2+y^2}$$
$$= \sqrt{4^2+5^2}$$
$$= \sqrt{41}$$

So, |v| can be found using pythagoras theorem.

Addition of vectors :-

$\vec{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\vec{a} + \vec{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$



- Geometric view of this operation

* Scalar - vector Product :-

Scalar → it's just an integer.

$$c \cdot \vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$$

$$2 \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

it scales your vector by some c.

Vector - Vector Product

$$\vec{a} * \vec{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} * \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

Transpose of vector :-

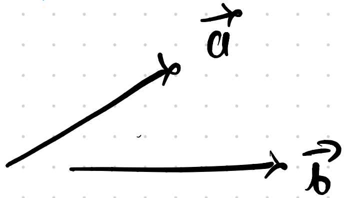
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}^T = [x_1, x_2, x_3 \cdots x_m]$$

make rows from column

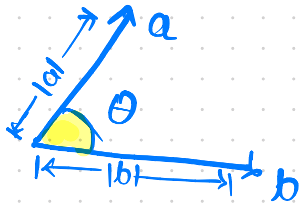
$$\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}^T = [2, 3, 4]$$

Strategies for Multiplying two vectors :-

* Dot product :-



we can multiply vectors using dot product.



length of vector b

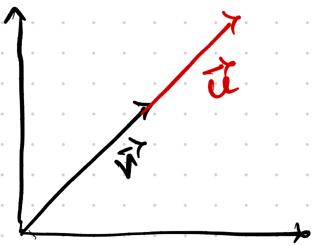
$a \cdot b = |a| \times |b| \times \cos\theta$

angle b/w a & b.

it's the magnitude of a.

→ It is about combining two vectors into a single no. It tells us about much two vectors point in the same direction

case :- when $\theta = 0$



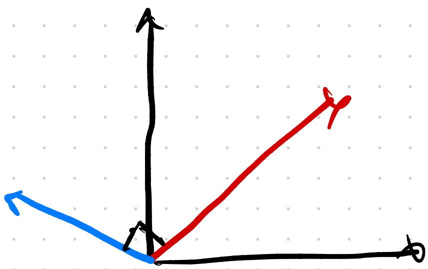
(here dot product is largest as it points in same direction)

$\vec{v} \cdot \vec{u} = |v| \times |u| \times \cos(0)$

$\cos(0) = 1$

The more two vector points in same direction, the larger dot product will be!

case :- when $\theta = 90^0$



$\cos(90) = 0$; $a \cdot b = |a| \times |b| \times 0$
$= 0$

* These type of vector are called orthogonal vectors.

A better way to calculate dot product :-

$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$\vec{a} \cdot \vec{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$

$= \sum_{i=0}^{n} a_i \cdot b_i$

Cross Product :-

$\vec{a} \times \vec{b} = \vec{c}$ → resulting vector

$|\vec{c}| = |\vec{a}||\vec{b}| \sin(\theta)$

Properties :-

→ $\vec{c}$ is $\perp$ to both $\vec{a}$ & $\vec{b}$.

→ $\vec{c}$ is a measure of how far apart $\vec{a}$ & $\vec{b}$ are pointing augmented by their magnitudes.

Case : when $\theta = 90°$

$\sin(90°) = 1$

$|\vec{c}| = |\vec{a}||\vec{b}| \cdot 1$

→ in this case cross product is the largest.

Linear Combination :-

* Given any no. of vectors, linear combination when we multiply each vector by a scalar and sum it all up!

$u = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$ & $v = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, when we multiply with some scalars.

$(1)\begin{bmatrix} -5 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$; $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ is the lr. combination of $\vec{u}$ & $\vec{v}$.

→ A vector r is said to be a lr. combination of a, b, c ... etc. if there exists scalar $x, y, z, \ldots$ etc. s.t $r = xa + yb + cz + \ldots$ etc.

$\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \underbrace{\left(-\frac{1}{5}\right)}_{a} u + \underbrace{2}_{b} v$ → $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is a lr. combination of u & v.

Linear dependence & Independence :-

* A set of vectors is said to be lr. independent if no vector can be represented as lr. combination of the remaining vectors.

* if a vector is said to be lr. dependent if a vector can be represented as lr. combination of the vector.

Span :-

* The set of all the possible lr. combination of given group of vectors is called the span of those vectors.

The span of vectors v_1, v_2, v_3 is written as span of $\{v_1, v_2, v_3\}$.

Eg :- $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ is the span of $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Norms :-

* L2 Norm :- Euclidean Norm, Calculates the dist. from the origin;

$$||x||_2 = \sqrt{\sum_i x_i^2}$$

* L1 Norm :- Manhattan distance, calculates the sum of absolute values.

$$|x|_1 = \sum_{r=1}^{n} |x_r|$$

* Max Norm :- distance by taking out the max element.

$$||x||_\infty = \max_i |x_i| \qquad \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix} \quad \max \rightarrow \boxed{9}$$

Orthogonal vectors :-

* Two vectors (assume a & b) are orthogonal, if they are perpendicular to each other.

$$a \cdot b = 0$$

↳ dot product

Matrices :-

- It is a rectangular array or table of numbers, symbols arranged in rows and columns.
- data which have n, no. of rows and m, no. of cols.

$$\begin{array}{c|cccc} & 1 & 2 & \dots & n \\ \hline 1 & a_{11} & a_{12} & \dots & a_{1n} \\ 2 & a_{21} & a_{22} & \dots & a_{2n} \\ 3 & a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m & a_{m1} & a_{m2} & \dots & a_{mn} \end{array}$$

Operations :-

* Scalar - Matrix Product :-

$$2 \cdot \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 8 \end{bmatrix}$$

* **Matrix Addition :-**

$$\underset{\uparrow}{A} + \underset{\uparrow}{B} = C ; \quad \underset{\substack{\downarrow \\ A}}{\begin{bmatrix} 2 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}} + \underset{\substack{\downarrow \\ B}}{\begin{bmatrix} 2 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}} = \underset{\substack{\downarrow \\ C}}{\begin{bmatrix} 4 & 6 & 4 \\ 4 & 8 & 6 \end{bmatrix}}$$

Properties :-

* Commutative Property :-

$A + B = B + A$

* Associative Property :- $A + (B + C) = (A + B) + C$

* Dimensions should be same.

Operations on matrices

Matrix-Matrix Multiplication

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} a_{11}b_{11} + \cdots + a_{1n}b_{n1} & a_{11}b_{12} + \cdots + a_{1n}b_{n2} & \cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ a_{21}b_{11} + \cdots + a_{2n}b_{n1} & a_{21}b_{12} + \cdots + a_{2n}b_{n2} & \cdots & a_{21}b_{1p} + \cdots + a_{2n}b_{np} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}b_{11} + \cdots + a_{mn}b_{n1} & a_{m1}b_{12} + \cdots + a_{mn}b_{n2} & \cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{pmatrix}$$

Number of columns in the first matrix must be equal to the number of rows in the second matrix.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

$$A.B = \begin{bmatrix} 2 \cdot 2 + 3 \cdot 5 & 2 \cdot 3 + 3 \cdot 6 & 2 \cdot 4 + 3 \cdot 7 \\ - & - & - \end{bmatrix}$$

Credit: Wikipedia

$A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} ; \quad C = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}$

$$A + (B + C) = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} + \left\{ \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 \\ 9 & 3 \end{bmatrix}_{2 \times 2}$$

$(A + B) + C$

$$\begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 9 & 3 \end{bmatrix}$$

Operations on matrices

Matrix-Vector Product

$$A = \begin{bmatrix} 2 & 6 \\ 3 & 4 \\ 2 & 2 \end{bmatrix} \cdot \; v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} so, \; \vec{s} \; = \begin{bmatrix} 4+18 \\ 6+12 \\ 4+6 \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 10 \end{bmatrix}$$

Note: You will write out the findings from this operation which is given in your problem set.

Operations on matrices

Transpose of a Matrix

Transpose of a matrix is an operator which flips over its diagonal.

- Reflects A over it's main diagonal to obtain A^T.
- Write the rows of A as the cols of AT.
- Write the cols of A as the rows of AT.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Special Types of Matrices

A Matrix is called symmetric matrix if it is equal to its transpose.

$$B = \begin{pmatrix} 0 & 4 & 7 \\ -4 & 0 & -3 \\ -7 & 3 & 0 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 0 & -4 & -7 \\ 4 & 0 & 3 \\ 7 & -3 & 0 \end{pmatrix}$$

Identity matrix is a matrix where all diagonal elements are 1 and remaining elements are 0.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant of a Matrix

The determinant is a scalar value that is a function of the entries of a square matrix. It gives you the area under the parallelogram.

The determinant of a matrix A is denoted det(A), det A, or |A|. In the case of a 2 × 2 matrix the determinant can be defined as:-

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

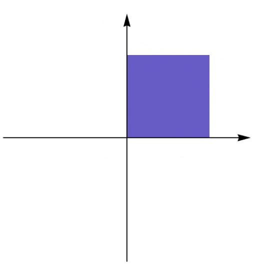
Similarly,

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - bdi - afh.$$

Minor of Matrix A

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{vmatrix} = 1.$$



Co-Factor, Minor of a Matrix

A minor of matrix A is the determinant of some smaller square matrix, cut down from A by removing one or more of its rows and columns.

We obtain minors by removing just one row and one column from a square matrices and this is required for calculating matrix co-factor.

It is useful in calculation of determinant and Inverse!

$$\begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix} \quad \text{Minor of a Matrix:- } M_{2,3} = \det\begin{bmatrix} 1 & 4 \\ -1 & 9 \end{bmatrix} = 9 - (-4) = 13$$

The Co-Factor of (2,3) entry is, $C_{2,3} = (-1^{2+3}) * M_{2,3} = -13$

$$\det(A) = \sum_{i=1}^{n} a_{ij}(-1)^{i+1} M_{ij}$$

Adjugate of a Matrix

The adjugate of a square matrix is just the transpose of the it's Co-Factor Matrix.

Inverse Matrix

Let's start with a simple example, $A^{-1} = \frac{1}{A}$ But when we talk about Inverse of matrix, it's just high-dimensional generalization.

- Calculate the minors of all elements of matrix A.
- Compute the Co-Factors of all elements of matrix A.
- Take out the adj A by taking out transposing of matrix A.
- Multiply the adj A by reciprocal of det(A).

Note:- We will look into usecases as we go in depth of ML.