

# Fundamental Algorithmic Techniques III

February 3, 2026



# Outline

Divide & Conquer

Recurrence Relations

Master Theorem

## Multiplying Square Matrices

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B},$$

$n$  operations  $\forall i, j:$   $c_{ij} = \sum_{k=0}^n a_{ik} \cdot b_{kj}.$

Divide and conquer:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}. \quad (4.4)$$

Decomposing with 8 sub-operations:  $T(n) = 8T(n/2) + \Theta(1)$ ,  
so  $\mathbf{T(n)} = (\mathbf{n^3})$  (master theorem  $c = 3 = \log_2(8)$ ).

## Strassen Algorithm

$$T(n) = 7 T(n/2) + \Theta(1), \text{ so } \mathbf{T(n)} = (\mathbf{n^{2.81}})$$

$$M_1 = (A_{11} + A_{22})(B_{11} + B_{22})$$

$$M_2 = (A_{21} + A_{22})B_{11}$$

$$M_3 = A_{11}(B_{12} - B_{22})$$

$$M_4 = A_{22}(B_{21} - B_{11})$$

$$M_5 = (A_{11} + A_{12})B_{22}$$

$$M_6 = (A_{21} - A_{11})(B_{11} + B_{12})$$

$$M_7 = (A_{12} - A_{22})(B_{21} + B_{22})$$

### Result Blocks:

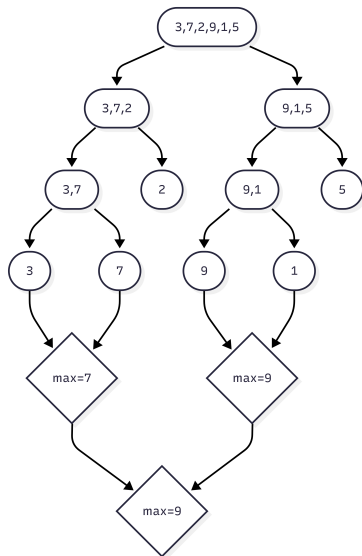
$$C_{11} = M_1 + M_4 - M_5 + M_7$$

$$C_{12} = M_3 + M_5$$

$$C_{21} = M_2 + M_4$$

$$C_{22} = M_1 - M_2 + M_3 + M_6$$

## Find Maximum: Divide and Conquer



**Problem:** Find the maximum element in an array of  $n$  numbers.

**Approach:**

- **Divide:** Split into two halves
- **Conquer:** Recursively find max
- **Combine:**  $\max(\text{left}, \text{right})$

**Complexity:**

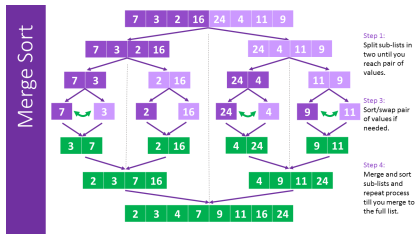
$$T(n) = 2T(n/2) + cn$$
$$\Rightarrow \mathcal{O}(n)$$

# Sorting

**Insertion sort:**  $T(n) = an^2 + bn + c$ ,  $a, b, c \in \mathbb{N}$   
So  $T(N) = \mathcal{O}(n^2)$ .

But can we do better?

Yes, actually  $\mathcal{O}(n \log(n))$  with MergeSort and QuickSort.



Merge Sort

## Recurrence Relation of D&C: Mathematical Description

Let  $T(n)$  be a recurrence relation defined for  $n \geq 1$  by:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where:

- $a \geq 1$  is the number of subproblems in the recursion,
- $b > 1$  is the factor by which the input size is reduced in each subproblem,
- $f(n)$  is the cost of dividing the problem and combining the results.

# Recursive: Not always good!

## Iterative vs Recursive Factorial: Complexity Comparison

	Iterative	Recursive
Time Complexity	$O(n)$	$O(n)$
Space Complexity	$O(1)$	$O(n)$
Stack Overflow?	No	Yes

```
1
2 function factorial_iter(n::Int)
3     result = 1
4     for i in 2:n
5         result *= i
6     end
7     return result
8 end
9
10 function factorial_recu(n::Int)
11     n <= 1 ? 1 : n * factorial(n - 1)
12 end
13
```

Factorial in Julia



# Master Theorem

Asymptotic behavior of  $T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$ :

**critical exponent:**  $c_{\text{crit}} = \log_b a$

**1 Case 1 (Subproblem Dominated):**

If  $f(n) = \mathcal{O}(n^c)$  where  $c < c_{\text{crit}}$ , then:

$$T(n) = \mathcal{O}(n^{\log_b a})$$

**2 Case 2 (Balanced):**

If  $f(n) = \mathcal{O}(n^{c_{\text{crit}}})$ , then:

$$T(n) = \mathcal{O}(n^{c_{\text{crit}}} \log n) = \mathcal{O}(n^{\log_b a} \log n)$$

**3 Case 3 (Work Dominated):**

If  $f(n) = \mathcal{O}(n^c)$  where  $c > c_{\text{crit}}$ , and if the **regularity condition** holds:

$$af\left(\frac{n}{b}\right) \leq kf(n) \quad \text{for constant } k < 1 \text{ and all sufficiently large } n,$$

then:

$$T(n) = \mathcal{O}(f(n))$$

# Master Theorem : Limitations & Examples

## Limitations:

- $T(n)$  not monotone, e.g.  $T(n) = \sin(n)$
- $f(n)$  not polynomial, e.g.  $f(n) = 2^n$
- $a$  not a constant, e.g.  $a = 2n$

## Examples: verify!

- 1  $T(n) = 4T(\frac{n}{2}) + n$ ,  
 $\Rightarrow$  Case 1, Subproblem dominated,  $T(n) = \Theta(n^2)$
- 2  $T(n) = 2T(\frac{n}{2}) + n$ ,  
 $\Rightarrow$  Case 2, Balanced,  $T(n) = \Theta(n \log(n))$
- 3  $T(n) = 3T(\frac{n}{2}) + n^2$ ,  
 $\Rightarrow$  Case 3, Work dominated,  $T(n) = \Theta(n^2)$
- 4  $T(n) = 2T(2n) + n \log(n)$ ,  
 $\Rightarrow$  trivially not applicable!
- 5  $T(n) = T(n-1) + 1$ ,  
 $\Rightarrow$  Not applicable!  $n-1 \neq n/b$ , actually  $T(n) = \Theta(n!)$

## Master Theorem : Proof with Tree Approach

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

- level 0: work  $f(n)$ ,  $a$  subproblems of size  $n/b$

- ...

- level  $i$ : work  $a^i \cdot f(n/b^i)$ ,  $a^i$  subproblems of size  $n/b^i$

stops when  $i = \log_b(n)$ , and using  $a \cdot \log_b(n) = n \cdot \log_b(a)$ :

$$T(n) = \sum_{i=1}^{\log_b n} a^i \cdot f(n/b^i) + \Theta(n^{\log_b a}),$$

with **work** and **leaf decomposition** contributions.

...Study each 3 cases separately for the proof...