Fundamental Algorithmic Techniques IV

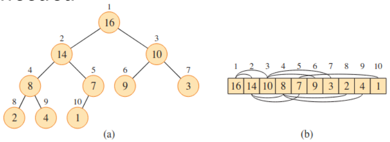
February 2, 2026

Outline

HeapSort

Array $\longleftrightarrow$ Complete Binary Tree

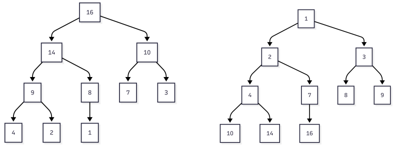
sorts in-place — no extra memory needed



Root: index 1

- Parent(i) $\rightarrow \left\lfloor \frac{i}{2} \right\rfloor$
- Left(i) $\rightarrow 2i$
- Right(i) $\rightarrow 2i + 1$

Goal: Sorting
$[1, 2, 3, 4, 7, 8, 9, 10, 14, 16]$ or
$[16, 14, 10, 9, 8, 7, 4, 3, 2, 1]$



2 operations on Tree:

- `heapify` or max/min heap
- `swap`

quick video link

Core Operations in Heapsort

heapify (max-heapify):

- Restores max-heap property after root removal
- Compares parent with children $\rightarrow$ swaps if needed
- Recurses **downward** toward leaves
- Takes $O(\log n)$ time (at most $2\lfloor \log_2 n \rfloor$ comparisons)

swap:

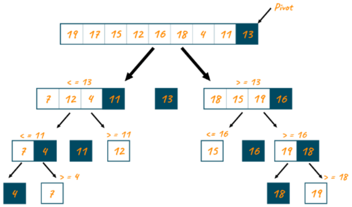
- Exchanges root ($A[0]$) with last element ($A[n-1]$)
- Reduces heap size by 1
- $O(1)$ operation

Example: $[3, 7, 1, 8, 2, 5, 9, 4, 6]$

MergeSort

```
function MergeSort($A, p, r$)
    if $p < r$ then
        $q \leftarrow \left\lfloor \frac{p+r}{2} \right\rfloor$
        MergeSort($A, p, q$)
        MergeSort($A, q+1, r$)
        Merge($A, p, q, r$)
    end if
end function
```

QuickSort



Quicksort Algorithm

```
1: function QUICKSORT($(A, p, r)$)
2:     if $p < r$ then
3:         $q \leftarrow$ PARTITION($(A, p, r)$)
4:         QUICKSORT($(A, p, q-1)$)
5:         QUICKSORT($(A, q+1, r)$)
6:     end if
7: end function
```

Problem Space Reduction

Space of permutations for array v of size n:

$$\approx n!$$

Idea: Reduce permutation space via divide-and-conquer transformations.

MergeSort heuristic: Merging two sorted subarrays of size $n/2$ reduces uncertainty by combining two independent orderings. With $\approx \log_2 n$ levels of recursion and $O(n)$ work per level:

$$\mathcal{O}(n \log n)$$

Analysis of Merge Sort

Simplest analysis for sorting algorithms!

$$T(n) = 2T(n/2) + \mathcal{O}(n)$$

- 2 subproblems of size $n/2$, $c_{\text{crit}} = \log_2 2 = 1$
- Work $f(n) = \mathcal{O}(n)$, $c = 1$

Applying the Master Theorem (balanced case $c_{\text{crit}} = c$):

$$T(n) = \Theta(n^{c_{\text{crit}}} \log n) = \Theta(n \log n)$$

Analysis of Quick Sort

$$T(n) = T(r-1) + T(n-r) + \mathcal{O}(n),$$

where $1 \leq r \leq n$ is the pivot position after partitioning.

Analysis:

- **Balanced:** $T(n) \approx 2T(n/2) + \mathcal{O}(n) \Rightarrow \mathcal{O}(n \log n)$
- **Unbalanced:** $T(n) \approx T(n-1) + \mathcal{O}(n) \Rightarrow \mathcal{O}(n^2)$
- **Average case:** Close to balanced $\Rightarrow \mathcal{O}(n \log n)$

Improved pivots: random selection or median-of-three (first, middle, last)

Analysis of Heap Sort

Master Theorem doesn't apply!

- Subproblems have **different sizes** (not n/b)
- General form: $T(n) = T(n-1) + f(n)$, not $a \cdot T(n/b)$

Instead:

1. Sorting phase: $hs(n) = hs(n-1) + \text{heapify}(n) + \mathcal{O}(1)$
2. Base: $hs(1) = \mathcal{O}(1)$
3. $\text{heapify}(i) = \mathcal{O}(\log i)$ (height of subtree)

$$hs(n) = \mathcal{O}(1) + \sum_{i=2}^{n} \mathcal{O}(\log i) = \mathcal{O}\left(\sum_{i=1}^{n} \log i\right) = \mathcal{O}(\log n!) = \Theta(n \log n)$$

(using $\log n! = \Theta(n \log n)$ via Stirling's approximation)