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ASTR5900

HW2 - Root Finding

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Question 1

Consider the following equation:

$$x^3 - 7x^2 + 14x - 5 = 0$$

Find a solution to the equation by hand to a precision smaller than 0.01 using

(a) the bisection method (with initial guess of x_i = 0 and 1)

(b) using the Newton Raphson method (with x_i = 0).

(c) How many iterations did it take for each method?

---

a

A description of the bisection method can be found here (https://en.wikipedia.org/wiki/Bisection_method). We first employ it by hand to solve the roots for the equation above.

iteration #1:

We start with the given bounds of a=0 and b=1.

f(a) = -5 and f(b) = 3. The midpoint is (0+1)/2 = 0.5 and f(0.5) = 0.375.

Now because f(a)* f(midpoint) < 0, then we set b equal to the midpoint and repeat.

iteration #2:

a = 0, b = 0.5.

f(a) = -5, f(b) = 0.375

So the midpoint is 0.25 and f(midpoint) = -1.922.

We can now test for precision using the test:

$$\text{precision} = |\frac{\text{midpoint}_{\text{new}} - \text{midpoint}_{\text{old}}}{\text{midpoint}_{\text{old}}}| < 0.01$$

In this case, precision = 1, so we continue.

Because f(a) * f(midpoint) > 0, a = midpoint.

iteration #3:

a = 0.25, b = 0.5.

f(a) = -1.922, f(b) = 0.375

midpoint = 0.375 and f(midpoint) = -0.682

precision = 0.5, so we continue.

Because f(a) * f(midpoint) > 0, a = midpoint.

iteration #4:

a = 0.375, b = 0.5.

f(a) = -0.682, f(b) = 0.375

midpoint = 0.4375 and f(midpoint) = -0.1311

precision = 0.166, so we continue.

Because f(a) * f(midpoint) > 0, a = midpoint.

iteration #5:

a = 0.4375, b = 0.5.

f(a) = -0.1311, f(b) = 0.375

midpoint = 0.46875 and f(midpoint) = 0.1274

precision = 0.071, so we continue.

Because f(a) * f(midpoint) < 0, b = midpoint.

iteration #6:

a = 0.4375, b = 0.46875.

f(a) = -0.1311, f(b) = 0.1274

midpoint = 0.453125 and f(midpoint) = -0.000469

precision = 0.034, so we continue.

Because f(a) * f(midpoint) > 0, a = midpoint.

iteration #7:

a = 0.453125, b = 0.46875.

f(a) = -0.000469, f(b) = 0.1274

midpoint = 0.4609375 and f(midpoint) = 0.0638136

precision = 0.017, so we continue.

Because f(a) * f(midpoint) < 0, b = midpoint.

iteration #8:

a = 0.453125, b = 0.4609375.

f(a) = -0.000469, f(b) = 0.0638136

midpoint = 0.45703125 and f(midpoint) = 0.031758

precision = 0.00854, so we have met our tolerance requirement and we stop.

b

We will again try to find the roots to the equation above. This time, we will use the Newton Raphson method, described here (https://en.wikipedia.org/wiki/Newton%27s_method).

iteration #1

We start with the given intial guess of 0.

We calculate f(0)=-5 and f'(0)=14.

Now we use the Newton iterative function:

$$x_i = x_0 - \frac{f(x_0)}{f(x_0)}$$

We get that x_1 = 0.35714.

We test the precision in the same way as above:

$\text{precision} = |\frac{\mathrm{x_i - x_0}}{\mathrm{x_0}}| < 0.01$

precision = 1. So we continue.

iteration #2

Now we have x_1 = 0.35714.

We calculate f(x_1)=-0.84733 and f'(x_1)=9.38268.

Using the, Newton iterative function, we get that x_2 = 0.4474.

The precision = 0.252. So we continue.

iteration #3

Now we have x_2 = 0.4474.

We calculate f(x_2)=-0.048 and f'(x_2)=8.3369.

Using the, Newton iterative function, we get that x_3 = 0.453157.

The precision = 0.0128. So we continue.

iteration #4

Now we have x_2 = 0.453157.

We calculate f(x_2)=-0.000204 and f'(x_2)=8.27185.

Using the, Newton iterative function, we get that x_4 = 0.45318166.

The precision = 0.0.00005. So we stop here.

Our root is $\boxed{x = 0.45318166}$

c

The bisection method took 8 iterations while the Newton-Raphson method took 4 iterations. The convergence is clearly much faster in the Newton method. The precision decreases at a much faster rate in the Newton method than in the bisection.

Question 2

Write a code that computes roots using each of these methods (bisection and Newton-Raphson). Your code should allow inputs for the initial guess(es), and the maximum error tolerance for the solution (ϵ). It should return the solution and count the number of steps taken.

(a) Use your code to find a solution to the equation in problem 1, to a precision of $\epsilon < 10^{-8}$. Show the outputs of each step. How many iterations did each of the two methods take?

(b) How sensitive is the number of iterations to the location of your initial guess? To answer this, simply try 3 or 4 different initial guesses (further or closer to the known answer), and comment on what you find.

(c) Using the Newton-Raphson, experiment with intial guesses and see if you can find any that would lead your code astray. Describe and comment on what you find.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

a

we first write out our function and its derivative into python functions:

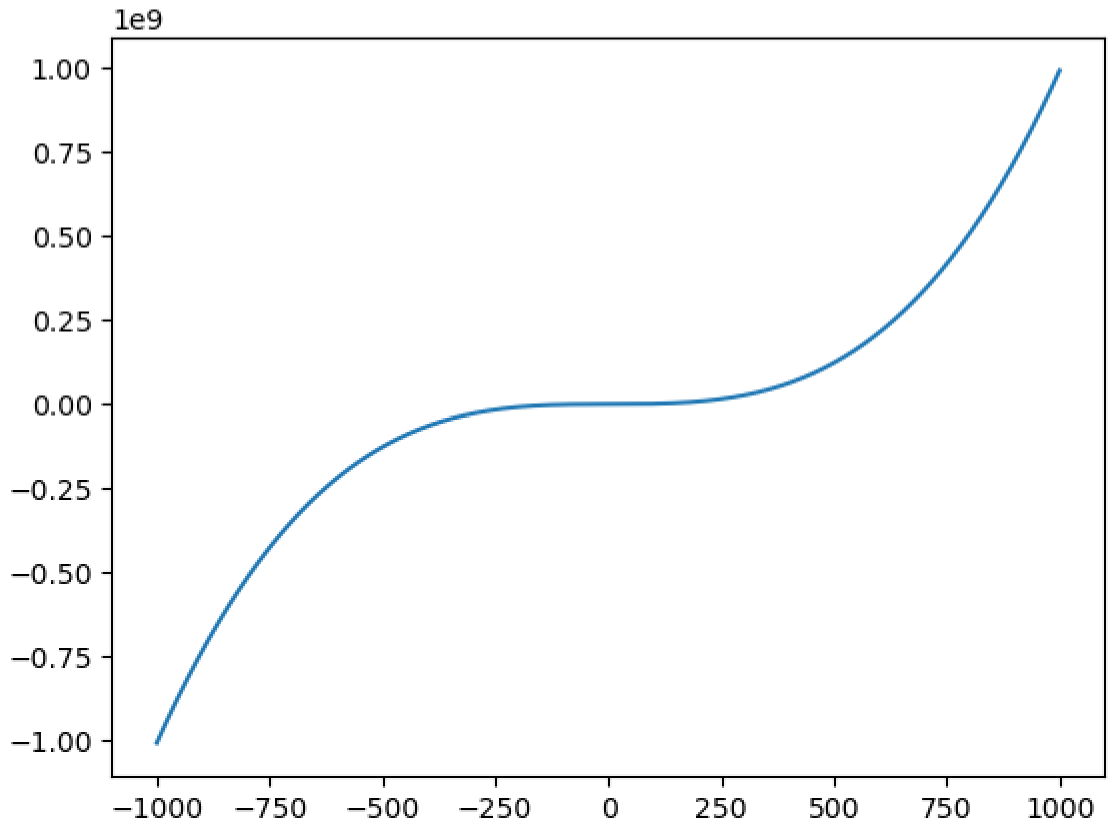
In [2]:
```python
def func1(x):
    return x**3 - 7*x**2 + 14*x - 5

def func1_prime(x):
    return 3*x**2 - 14*x + 14
```

We plot the function to see its behavior:

In [3]:
```python
x_lst = np.linspace(-1_000,1_000,2_000)
f_y_lst = func1(x_lst)

plt.plot(x_lst,f_y_lst)
plt.show()
```



We now code both of the root finding methods:

In [4]:
```python
#bisection method

def bisection(function,a,b,tolerance,max_iterations,print_steps=Fals
    '''
    inputs:
        - function: the function we wish to find a root for.
                the function should be a python function itself.
        - (a,b): the range we expect to find the root to be within
        - tolerance: how small we want the difference between
                the solution and 0 to be
        - max_iterations: if no root exists, we set a maximum iterat
                to prevent infinite loops
        -print_steps: if user wants each iteration's root to be prin
                they can set print_steps=True
    outputs:
        - root:the root of the function if it exists
            or an error if no convergance is reached
        - counter: the number of iterations taken to reach the root
    '''

    #we compute the first iteration of f(a) and f(b)
    f_a,f_b = function(a),function(b)
    #we compute the midpoint of a and b and compute f(midpoint)
    midpoint = (a + b)/2
    f_midpoint = function(midpoint)


    #as long as we haven't reached our max iterations, we continue f
    counter = 0
    while counter <= max_iterations:
        #determine if the midpoint is our new lower or upper bound
        sign_test = f_midpoint * f_a
        if sign_test < 0:
            b = midpoint
        else:
            a = midpoint
        #find the next iteration of f(a), f(b) and midpoint values:
        f_a1,f_b1 = function(a),function(b)
        midpoint1 = (a + b)/2.0
        f_midpoint1 = function(midpoint1)
        #avoid divide by zero errors by not calculating rel diff if 
        if midpoint == 0:
            pass
        else:
            #check if difference in x values is within our tolerance
            relative_x_diff = (midpoint1 - midpoint) / midpoint
            #if it is, our root has been found and we break
            if abs(relative_x_diff) < tolerance:
                root = midpoint1
                break
        #increase the counter
        counter += 1
        #if the user wants, each step will be printed:
        if print_steps == True:
            print('iteration step: {}\t root: {}'.format(counter,mid
        #otherwise, we update the old f(a),f(b), midpoint and f(midp
        f_a,f_b = f_a1,f_b1
        midpoint = midpoint1
        f_midpoint = f_midpoint1
        #raise error if maximum iteration steps is reached
        if counter > max_iterations:
            raise ValueError('maximum iterations steps reached')
    #print('convergence reached in: ',counter,' steps')
    return root,counter
```

In [5]:

```
#we test it using the parameters given in the problem, we also print
lower_bound, upper_bound = 0.,1.
tolerance = 10**(-8)
max_iters = 1_000
root1_bisection = bisection(func1,lower_bound,upper_bound,tolerance,
print()
print('the root using bisection method is {}'.format(root1_bisection
```

```
iteration step: 1        root: 0.5
iteration step: 2        root: 0.25
iteration step: 3        root: 0.375
iteration step: 4        root: 0.4375
iteration step: 5        root: 0.46875
iteration step: 6        root: 0.453125
iteration step: 7        root: 0.4609375
iteration step: 8        root: 0.45703125
iteration step: 9        root: 0.455078125
iteration step: 10       root: 0.4541015625
iteration step: 11       root: 0.45361328125
iteration step: 12       root: 0.453369140625
iteration step: 13       root: 0.4532470703125
iteration step: 14       root: 0.45318603515625
iteration step: 15       root: 0.453155517578125
iteration step: 16       root: 0.4531707763671875
iteration step: 17       root: 0.45317840576171875
iteration step: 18       root: 0.4531822204589844
iteration step: 19       root: 0.45318031311035156
iteration step: 20       root: 0.45318126678466797
iteration step: 21       root: 0.45318174362182617
iteration step: 22       root: 0.45318150520324707
iteration step: 23       root: 0.4531816244125366
iteration step: 24       root: 0.4531816840171814
iteration step: 25       root: 0.4531817138195038
iteration step: 26       root: 0.453181728720665

the root using bisection method is 0.4531817249953747
```

In [6]:
```python
#newton raphson

def newton_raphson(function,function_prime,initial_guess,tolerance,ma
    '''
    inputs:
        - the function we want to find a root for (function should be
        - the derivative of the given function (also passed as a pyth
        - initial_guess: an educated guess of where the root lives
        - tolerance: how small we want the difference between our so
        - max_iterations: if no root exists, we set a maximum iterat
                to prevent infinite loops
        -print_steps: if user wants each iteration's root to be prin
                they can set print_steps=True

    outputs:
        - root: the root of the function if it exists
            or an error if no convergance is reached
        - counter: the number of iterations taken to reach the root
    '''

    #initialize our starting x value
    x0 = initial_guess
    #we evaluate the function and its derivative at the initial gues
    f_x = function(x0)
    fprime_x =function_prime(x0)

    #we initialize our counter
    counter = 0
    #we apply the newton raphson method until we reach our root
    while counter <= max_iterations:
        #compute the newton-raphson iteration
        xi = x0 - (f_x / fprime_x)
        #if this new value is within the range for our root tolerance
        if x0 == 0:
            pass
        else:
            relative_x_diff = (xi - x0)/x0
            #print(abs(relative_x_diff))
            if abs(relative_x_diff) < tolerance:
                root = xi
                break
        x0 = xi
        f_x = function(x0)
        fprime_x = function_prime(x0)
        counter += 1
        #if user wants, each step will be printed:
        if print_steps == True:
            print('iteration step: {}\t root: {}'.format(counter,xi)
        #otherwise, update x0,f(x),f'(x) and repeat
        #raise error if maximum iterations steps is reached
        if counter > max_iterations:
            raise ValueError('maximum iterations steps reached')

    #print('convergence reached in: ',counter,' steps')
    return root,counter
```

In [7]:
```
#we test it using the parameters given in the problem, we also print
initial_guess = 0.
tolerance = 10**(-8)
max_iters = 1_000
root1_newton = newton_raphson(func1,func1_prime,initial_guess,tolerar
print()
print('the root using Newton-Raphson is {}'.format(root1_newton[0]))
```

```
iteration step: 1        root: 0.35714285714285715
iteration step: 2        root: 0.44744814728501514
iteration step: 3        root: 0.453159435118196
iteration step: 4        root: 0.4531817227771845

the root using Newton-Raphson is 0.453181723115918
```

b

Let's now try different sets of initial guesses for bisection method. We will keep the tolerance and maximum iterations allowed the same as above.

In [8]:
```
lower_bound, upper_bound = -1.,2.
root,counter = bisection(func1,lower_bound,upper_bound,tolerance,max_
print('convergence reached in: ',counter,' steps')
```

```
convergence reached in:  28  steps
```

In [9]:
```
lower_bound, upper_bound = -10.,11.
root,counter = bisection(func1,lower_bound,upper_bound,tolerance,max_
print('convergence reached in: ',counter,' steps')
```

```
convergence reached in:  31  steps
```

In [10]:
```
lower_bound, upper_bound = -(10**2.),1.1*10**2.
root,counter = bisection(func1,lower_bound,upper_bound,tolerance,max_
print('convergence reached in: ',counter,' steps')
```

```
convergence reached in:  34  steps
```

In [11]:
```
lower_bound, upper_bound = -(10**4),2.
root,counter = bisection(func1,lower_bound,upper_bound,tolerance,max_
print('convergence reached in: ',counter,' steps')
```

```
convergence reached in:  40  steps
```

In [12]:
```
lower_bound, upper_bound = -1.,10**(10)
root,counter = bisection(func1,lower_bound,upper_bound,tolerance,max_
print('convergence reached in: ',counter,' steps')
```

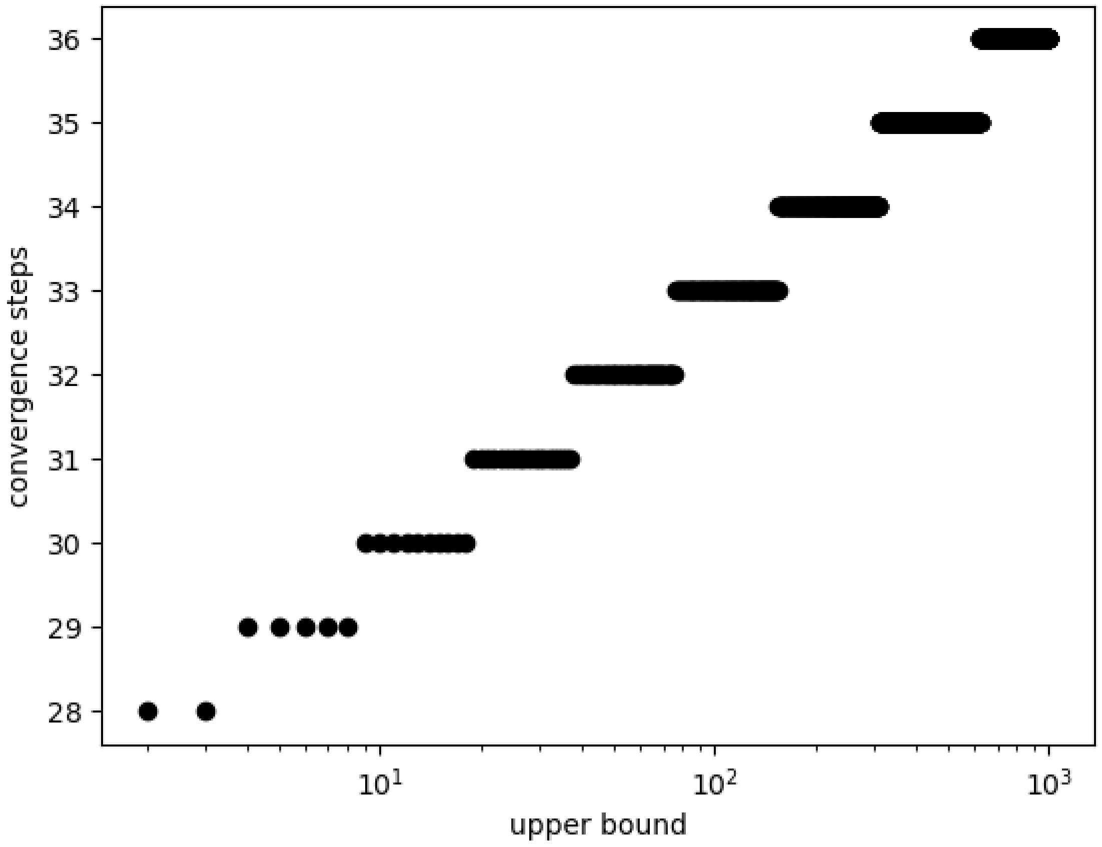
```
convergence reached in:  59  steps
```

We can see that the convergence counter seems to increase steadily as we increase our initial range. We can visualize how the convergence of the bisection looks like. We will keep the lower bound constant at -1 and increase the upper bound from 2 to 1,000:

In [13]:
```
upper_bound_arr = np.linspace(2,10**3,1_000)
```

In [14]:
```
converge_lst = []
for up_bound in upper_bound_arr:
    root,convergence_num = bisection(func1,lower_bound,up_bound,tole
    converge_lst += [convergence_num]
```

In [15]:
```
plt.scatter(upper_bound_arr,converge_lst,color='k')
plt.ylabel('convergence steps')
plt.xlabel('upper bound')
plt.xscale('log')
plt.show()
```



We see that even for a small upper bound, the bisection method takes 28 steps to converge to a tolerance of 10^{-8}. As we increase the upper bound, the convergence takes a functional form similar to log(x). So as we increase the upper bound, the convergence step doesn't increase a whole lot. The plot above has the x-axis on a log scale, which in turn gives a linear functional form to the convergence.

Let's now take the same approach for the Newton-Raphson method. We start by printing the convergence step at various initial guesses:

In [16]:
```
initial_guess = 0.
newton,counter = newton_raphson(func1,func1_prime,initial_guess,tole
print('convergence reached in: ',counter,' steps')
```

```
convergence reached in:  4  steps
```

In [17]:
```
initial_guess = -5.
newton,counter = newton_raphson(func1,func1_prime,initial_guess,tole
print('convergence reached in: ',counter,' steps')
```

```
convergence reached in:  8  steps
```

In [18]:
```
initial_guess = 20.
newton,counter = newton_raphson(func1,func1_prime,initial_guess,tole
print('convergence reached in: ',counter,' steps')
```

```
convergence reached in:  22  steps
```

In [19]:
```
initial_guess = 10**2.
newton,counter = newton_raphson(func1,func1_prime,initial_guess,tole
print('convergence reached in: ',counter,' steps')
```

```
convergence reached in:  89  steps
```

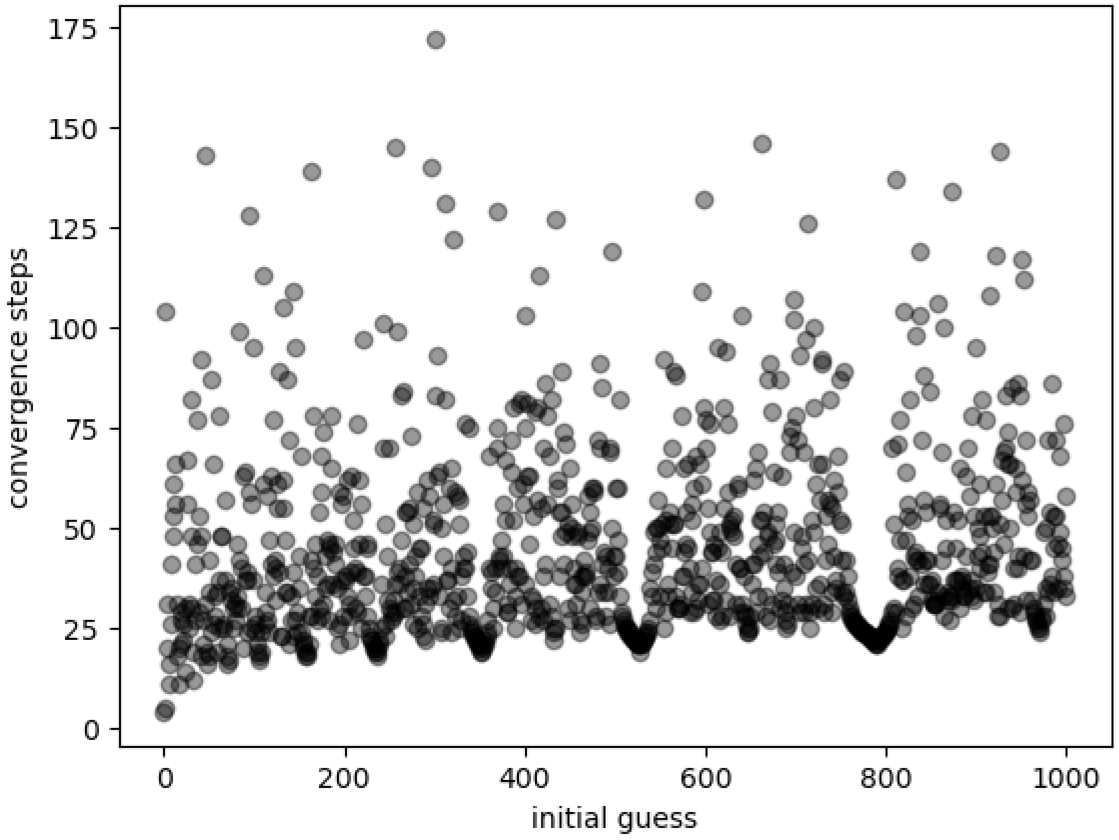
In [20]:
```
initial_guess = 10**3.
newton,counter = newton_raphson(func1,func1_prime,initial_guess,tole
print('convergence reached in: ',counter,' steps')
```

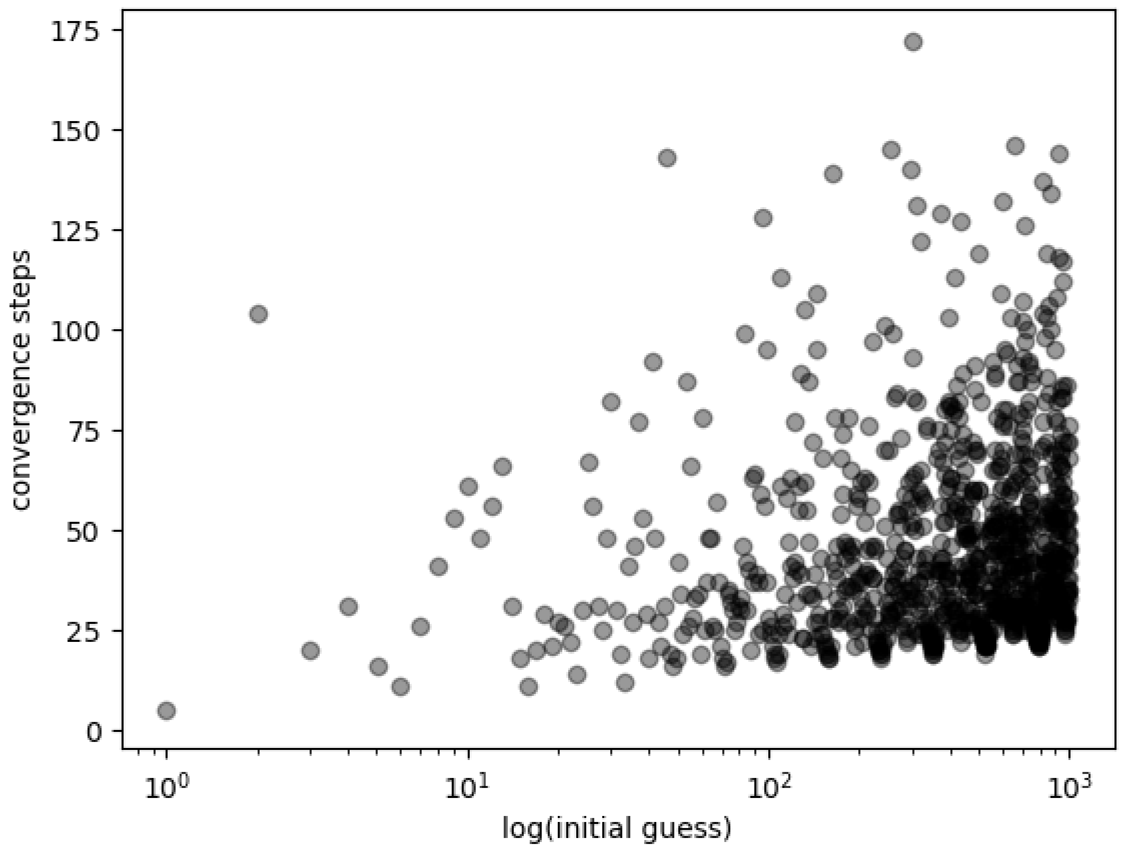
```
convergence reached in:  58  steps
```

In [21]:
```
initial_guess_lst1 = np.linspace(0,10**3,1_000)
newton_counter = []
for guess in initial_guess_lst1:
    root, counter = newton_raphson(func1,func1_prime,guess,tolerance
    newton_counter += [counter]
```

In [22]:
```python
plt.scatter(initial_guess_lst1,newton_counter,color='k',alpha=0.4)
plt.ylabel('convergence steps')
plt.xlabel('initial guess')
#plt.xscale('log')
plt.show()

plt.scatter(initial_guess_lst1,newton_counter,color='k',alpha=0.4)
plt.ylabel('convergence steps')
plt.xlabel('log(initial guess)')
plt.xscale('log')
plt.show()
```





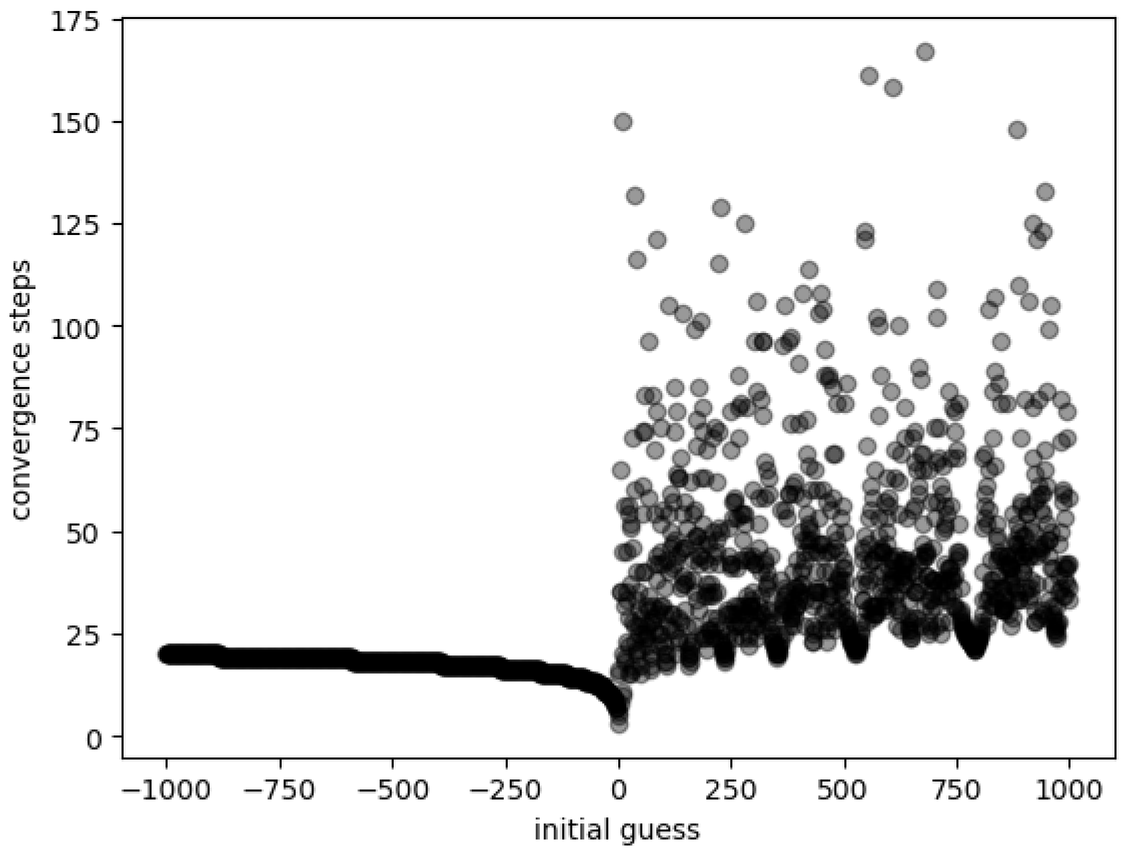
From the plot, the Newton Raphson has a lot more noise than the bisection method. However, the underlying form is a very slow increase in convergence steps as we increase the initial guess. The slope of the underlying line here is smaller than the bisection method and seems to hover around 25 iteration steps. This indicates that the convergence of the Newton method is faster generally than the bisection method.

c

We can make a similar as the one above for this question. This time, we will test inital guesses from -10^3 to 10^3.

In [23]:
```
initial_guess_lst = np.linspace(-(10**3),10**3,2_000)
newton_counter = []
for guess in initial_guess_lst:
    root, counter = newton_raphson(func1,func1_prime,guess,tolerance
    newton_counter += [counter]
```

In [24]:
```python
plt.scatter(initial_guess_lst,newton_counter,color='k',alpha=0.4)
plt.ylabel('convergence steps')
plt.xlabel('initial guess')
#plt.xscale('log')
plt.show()
```



We plot in a linear scale so that we can see the negative values more clearly. It is apparent that regardless of the initial guess (at least for the range above), the Newton Raphson converges to the root. Oddly, there is almost zero noise in the convergence of the negative numbers while the positive initial guesses have a lot of noise in the convergence steps.

Question 3

Find an equation that is of interest to you that does not have an analytical solution but can be solved with one of the above methods. Give and describe the equation and why you find it interesting. Use your own code to find the solution to this equation (perhaps for a few different values of the coefficients/constants in your equation). Show and discuss your findings.

We will be looking at the equations of orbital motion often discussed in Classical Mechanics. The energy of a spherically symmetric potential energy can be written as

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L_z^2}{mr^2} + V(r)$$

where $L_z = mr^2\dot{\phi}$.

We can invert the problem to solve for $\dot{r}^2$:

$$\dot{r}^2 = \frac{2}{m}\left(E - \frac{1}{2}\frac{L_z^2}{mr^2} + V(r)\right) = F(r)$$

Now, we can find turning points of the orbit when $\dot{r} = 0$, which gives $F(r) = 0$. So, finding the turning points of the orbit is a root finding problem in orbital mechanics.

Let's consider a simple case where we have Newtonian gravity:

$$V(r) = \frac{1}{r}$$

Our equation becomes:

$$\frac{2}{m}\left(E - \frac{1}{2}\frac{L_z^2}{mr^2} + \frac{1}{r}\right) = 0$$

For ease, let $L_z = m = 1$ and $E = 0$, where the trajectory is a parabola

In [25]:

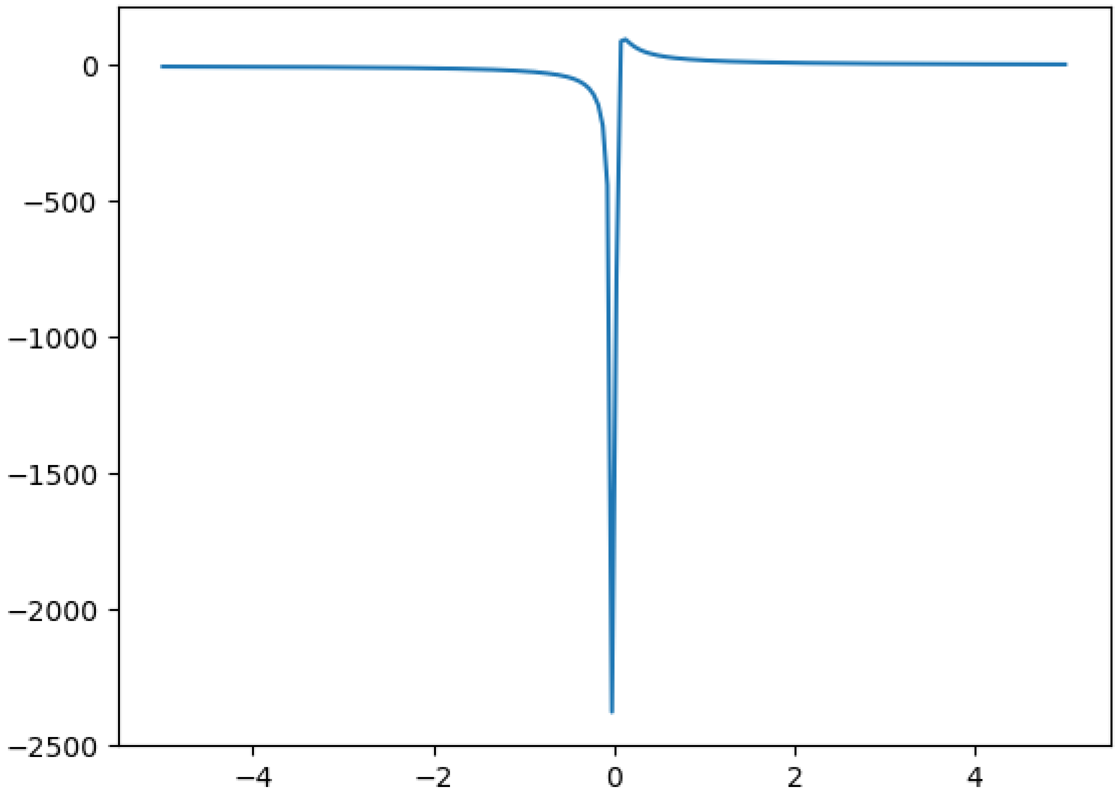
```python
#we first code the equation for energy of the orbit
def orbital_speed(r):
    '''
    inputs:
        r: distance of orbit from center
    outputs:
        velocity of orbit
    '''
    E = 0
    Lz, m = 0.1,0.1

    term1 = (1/2) * (Lz**2/(m*r**2))
    term2 = 1/r
    return (2/m) * (E - term1 + term2)
```

In [26]:
```
r_values = np.linspace(-5,5,200)
Lz, m, E = 0.1,0.1,0

orbital_speeds = orbital_speed(r_values)

plt.plot(r_values,orbital_speeds)
plt.show()
```



In [27]:
```
lower_bound, upper_bound = -1.,2.
tolerance = 10**(-8)
max_iters = 1_000
root,counter = bisection(orbital_speed,lower_bound,upper_bound,toler
print('convergence reached in: ',counter,' steps')
```

```
iteration step: 1         root: 0.5
iteration step: 2         root: -0.25
iteration step: 3         root: 0.125
iteration step: 4         root: -0.0625
iteration step: 5         root: 0.03125
iteration step: 6         root: 0.078125
iteration step: 7         root: 0.0546875
iteration step: 8         root: 0.04296875
iteration step: 9         root: 0.048828125
iteration step: 10        root: 0.0517578125
iteration step: 11        root: 0.05029296875
iteration step: 12        root: 0.049560546875
iteration step: 13        root: 0.0499267578125
iteration step: 14        root: 0.05010986328125
iteration step: 15        root: 0.050018310546875
iteration step: 16        root: 0.0499725341796875
iteration step: 17        root: 0.04999542236328125
iteration step: 18        root: 0.050006866455078125
iteration step: 19        root: 0.05000114440917969
iteration step: 20        root: 0.04999828338623047
iteration step: 21        root: 0.04999971389770508
iteration step: 22        root: 0.05000042915344238
iteration step: 23        root: 0.05000007152557373
iteration step: 24        root: 0.049999892711639404
iteration step: 25        root: 0.04999998211860657
iteration step: 26        root: 0.05000002682209015
iteration step: 27        root: 0.05000000447034836
iteration step: 28        root: 0.04999999329447746
iteration step: 29        root: 0.04999999888241291
iteration step: 30        root: 0.050000001676380634
iteration step: 31        root: 0.05000000027939677
convergence reached in:  31  steps
```

In [28]:
```
print('the root is x={}'.format(root))
print('the corresponding y value is {}'.format(orbital_speed(root)))
```

```
the root is x=0.04999999993015081
the corresponding y value is -5.587935447692871e-07
```

To use the newton raphson method, we need to know the derivative:

In [29]:
```
def orbital_acceleration(r):
    Lz,m = 0.1,0.1
    term1 = 2*Lz**2 / (m**2*r**3)
    term2 = -2 / (m*r**2)
    return term1 + term2
```

In [30]:
```
initial_guess = 10
tolerance = 10**(-8)
max_iters = 1_000
newton,counter = newton_raphson(orbital_speed,orbital_acceleration,i
print('convergence reached in: ',counter,' steps')
```

```
---------------------------------------------------------------------------
OverflowError                             Traceback (most recent call last)
Cell In[30], line 4
      2 tolerance = 10**(-8)
      3 max_iters = 1_000
----> 4 newton,counter = newton_raphson(orbital_speed,orbital_acceleration,initial_guess,tolerance,max_iters,False)
      5 print('convergence reached in: ',counter,' steps')

Cell In[6], line 44, in newton_raphson(function, function_prime, initial_guess, tolerance, max_iterations, print_steps)
     42 x0 = xi
     43 f_x = function(x0)
---> 44 fprime_x = function_prime(x0)
     45 counter += 1
     46 #if user wants, each step will be printed:

Cell In[29], line 3, in orbital_acceleration(r)
      1 def orbital_acceleration(r):
      2     Lz,m = 0.1,0.1
----> 3     term1 = 2*Lz**2 / (m**2*r**3)
      4     term2 = -2 / (m*r**2)
      5     return term1 + term2

OverflowError: (34, 'Result too large')
```

We see that the bisection method correctly picks out the zero, which is at x=0.05. This makese sense physically for a parabolic orbit to speed up really fast near the inflection point and then coast near zero else where. However, the newton raphson had a lot of issues with this function. This is likely due to the large slopes near the root.

In [ ]:
```

```