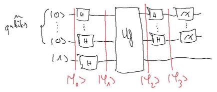
IT41 - Project 2
Deutsch - Jozsa algorithm

Introduction :

* n : number of qubits
* $N = 2^n$: number of differents values with n qubits
* $|0\rangle = |\underbrace{0\cdots0}_{n}\rangle$; $|1\rangle = |0\cdots01\rangle$; $|2\rangle = |0\cdots010\rangle$; ... ; $|N-1\rangle = |1\cdots1\rangle$

Quantum Circuit



Proof of validity

$$|\Psi_0\rangle = |0\rangle|0\rangle\cdots|0\rangle|1\rangle$$

$$\begin{aligned}
|\Psi_1\rangle = H^{\otimes n+1}|\Psi_0\rangle &= H|0\rangle H|0\rangle \cdots H|0\rangle H|1\rangle \\
&= |+\rangle|+\rangle\cdots|+\rangle|-\rangle \\
&= \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\otimes\cdots\otimes\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|-\rangle \\
&= \frac{1}{\sqrt{2^n}}\left(|0\cdots0\rangle+|0\cdots01\rangle+\cdots+|1\cdots1\rangle\right)|-\rangle \\
&= \frac{1}{\sqrt{N}}\left(|0\rangle+|1\rangle+\cdots+|N-1\rangle\right)|-\rangle \\
&= \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle|-\rangle
\end{aligned}$$

$$|\Psi_2\rangle = U_f|\Psi_1\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}U_f(|x\rangle|-\rangle)$$

What's $U_f(|x\rangle|-\rangle)$?

$$U_f(|x\rangle|y\rangle) = |x\rangle|y\oplus f(x)\rangle$$

$$\begin{aligned}
U_f(|x\rangle|-\rangle) &= U_f\left(|x\rangle\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{\sqrt{2}}\left(U_f(|x\rangle|0\rangle)-U_f(|x\rangle|1\rangle)\right) \\
&= \frac{1}{\sqrt{2}}\left(|x\rangle|0\oplus f(x)\rangle-|x\rangle|1\oplus f(x)\rangle\right)
\end{aligned}$$

But $f(x)$ is either 0 or 1

* $f(x)=0$

$$U_f(|x\rangle|-\rangle) = \frac{1}{\sqrt{2}}\left(|x\rangle|0\rangle-|x\rangle|1\rangle\right) = |x\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = |x\rangle|-\rangle$$

* $f(x)=1$

$$U_f(|x\rangle|-\rangle) = \frac{1}{\sqrt{2}}\left(|x\rangle|1\rangle-|x\rangle|0\rangle\right) = -|x\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = -|x\rangle|-\rangle$$

So $U_f(|x\rangle|-\rangle) = (-1)^{f(x)}|x\rangle|-\rangle$

$$\Rightarrow |\Psi_2\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}(-1)^{f(x)}|x\rangle|-\rangle$$

$$\begin{aligned}
|\Psi_3\rangle &= H^{\otimes n}\otimes I|\Psi_2\rangle \\
&= \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}(-1)^{f(x)}\left(H^{\otimes n}|x\rangle\right)|-\rangle
\end{aligned}$$

What is $H^{\otimes n}|x\rangle$?

$x = x_{n-1}2^{n-1}+x_{n-2}2^{n-2}+\cdots+x_0$ with $x_i\in\{0;1\}$ $\forall i$ st. $0\leq i\leq n-1$

$$H^{\otimes n}|x\rangle = H|x_{n-1}\rangle\otimes H|x_{n-2}\rangle\otimes\cdots\otimes H|x_0\rangle$$

* If $x_i=0$ $H|x_i\rangle = \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)$
* If $x_i=1$ $H|x_i\rangle = \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$

$\mapsto$ So $H|x_i\rangle = \left(\frac{|0\rangle+(-1)^{x_i}|1\rangle}{\sqrt{2}}\right)$

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}(-1)^{?}|y\rangle$$

When do we have a "-1" ?

If $x_i=0\Rightarrow$ No "-1"
If $x_i=0\Rightarrow$ No "-1"
If $x_i=1$ and $y_i=0\Rightarrow$ No "-1"
If $x_i=1$ and $y_i=1\Rightarrow$ "-1"

So each time we have a 1 in common a factor -1 appears
→ If we have it an even number of times $\Rightarrow (-1)^{2k}=1$
→ If we have it an odd number of times $\Rightarrow (-1)^{2k+1}=-1$

So ? is "the number of times $x_i=y_i=1$" mod 2

$? = \left(\sum_{i=0}^{N-1}x_iy_i\right) \bmod 2$ and this correspond to $\begin{pmatrix}x_0 & \cdots & x_{n-1}\end{pmatrix}\begin{pmatrix}y_0\\ \vdots\\ y_{n-1}\end{pmatrix} \bmod 2$

and we can write it $x\cdot y$

$$\text{So } H^{\otimes n}|x\rangle = \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}(-1)^{x\cdot y}|y\rangle$$

$$\begin{aligned}
\text{So } |\Psi_3\rangle &= \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}(-1)^{f(x)}\left(\frac{1}{\sqrt{N}}\sum_{y=0}^{N-1}(-1)^{x\cdot y}|y\rangle\right)|-\rangle \\
&= \frac{1}{\sqrt{N}}\times\frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}(-1)^{f(x)}\times(-1)^{x\cdot y}|y\rangle|-\rangle \\
&= \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}(-1)^{f(x)+x\cdot y}|y\rangle|-\rangle
\end{aligned}$$

What is the probability α_y to measure $|y\rangle$?

$$p(|y\rangle) = |\alpha_y|^2 \text{ with } \alpha_y = \frac{1}{N}\sum_{x=0}^{N-1}(-1)^{f(x)+x\cdot y}$$

What is the probability α_0 to measure $|y\rangle = |0\cdots0\rangle$?

$$x\cdot y_0 = \begin{pmatrix}x_0 & \cdots & x_{n-1}\end{pmatrix}\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix} = 0$$

$$\text{So } \alpha_0 = \frac{1}{N}\sum_{x=0}^{N-1}(-1)^{f(x)}$$

And we know that $f(x)$ is either balanced or constant so

If $f(x)$ is balanced : $\alpha_0 = \frac{1}{N}\left(\underbrace{\sum_{x=0}^{N-1}1}_{\text{where } f(x)=0}+\underbrace{\sum_{x=0}^{N-1}-1}_{\text{where } f(x)=1}\right)$

$$= \frac{1}{N}\left(\frac{N}{2}-\frac{N}{2}\right) = 0$$

so $p(|y\rangle = |0\cdots0\rangle) = |\alpha_0|^2 = 0$

If $f(x)$ is constant :

* $f(x)=0 \Rightarrow \alpha_0 = \frac{1}{N}\left(\sum_{x=0}^{N-1}1\right) = \frac{1}{N}\times N = 1$
  and $p(|y\rangle = |0\cdots0\rangle) = |\alpha_0|^2 = 1$
* $f(x)=1 \Rightarrow \alpha_0 = \frac{1}{N}\left(\sum_{x=0}^{N-1}-1\right) = \frac{1}{N}\times(-N) = -1$
  and $p(|y\rangle = |0\cdots0\rangle) = |\alpha_0|^2 = (-1)^2 = 1$

$\boxed{\text{So if we measure } |0\cdots0\rangle,\ f(x) \text{ is constant with the probability } 100\% \text{ otherwise } f(x) \text{ is balanced}}$