Consolidation Activities

1. Exercise 1

A big magnetic roll tape needs tape. An experiment is being conducted in which each time 1 meter of the tape is examined randomly. The procedure is repeated 5 times and the number of defects is recorded to be 2,2,6,0 and 3 respectively. The researcher assumes a Poisson distribution for the parameter λ. From previous experience, the beliefs of the researcher about λ can be expressed by a Gamma distribution with mean and variance equal to 3. Derive the posterior distribution that will be obtained. What would be the expected mean and variance of the number of defects per tape meter after the experiment?

▼ Solution

From the section Bayesian Inference Examples we get that for likelihood

$$f(y|\lambda) = \prod_{i=1}^{n} \frac{\exp(-\lambda)\lambda^{y_i}}{y_i!} \propto \exp(-n\lambda)\lambda^{\sum x_i}$$

and a prior $\text{Gamma}(\alpha, \beta)$

$$\pi(\lambda) \propto \lambda^{\alpha-1} \exp(-\beta\lambda)$$

we get a posterior $\text{Gamma}(\alpha + \sum y_i, n + \beta)$.

We know that the prior mean and variance are equal to 3, meaning $\frac{\alpha}{\beta} = 3$, $\frac{\alpha}{\beta^2} = 3$, which implies that $\alpha = 3, \beta = 1$. Also $\sum y_i = 2 + 2 + 6 + 0 + 3 = 13$. Hence the posterior distribution is a $\text{Gamma}(16, 6)$ distribution, implying that the posterior mean is 8/3. This is slightly lower than the prior mean of 3 but higher than what the data suggest as the MLE $\bar{y} = 2.6$. The posterior variance 4/9 which is lower than the prior variance of 3 reflecting the fact that our uncertainty decreased in light of the observed data y.