Consolidation Activities

2. Exercise 2

Let $x = (x_1, \ldots, x_n)$ be a random sample from a $\mathrm{N}(\theta, \sigma^2)$ distribution with σ^2 known.

a. Show that the likelihood is proportional to

$$f(x|\theta) \propto \exp\left(-\frac{n(\bar{x}-\theta)^2+(n-1)S^2}{2\sigma^2}\right).$$

where $\bar{x}$ is the sample mean and S^2 is the sample variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2.$$

Hence the likelihood simplifies to

$$f(x|\theta) \propto \exp\left(-\frac{(\theta-\bar{x})^2}{2\frac{\sigma^2}{n}}\right)$$

▼ Solution

The joint density of the sample x is

$$\begin{aligned} f(x|\theta,\sigma^2) = f(x_1,\ldots,x_n|\theta,\sigma^2) &= \prod_{i=1}^{n}(2\pi\sigma^2)^{-1/2}\exp\left(-\frac{(x_i-\theta)^2}{2\sigma^2}\right) \\ &= (2\pi\sigma^2)^{-n/2}\exp\left(-\frac{\sum_{i=1}^{n}(x_i-\theta)^2}{2\sigma^2}\right) \end{aligned}$$

Hence, it suffices to show that

$$\sum_{i=1}^{n}(x_i-\theta)^2 = n(\bar{x}-\theta)^2+\sum_{i=1}^{n}(x_i-\bar{x})^2.$$

Note that

$$\sum_{i=1}^{n}(x_i-\theta)^2 = \sum_{i=1}^{n}(x_i^2-2\theta x_i+\theta^2) = \sum_{i=1}^{n}x_i^2-2\theta n\bar{x}+n\theta^2$$

$$\sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=1}^{n}(x_i^2-2\bar{x}x_i+\bar{x}^2) = \sum_{i=1}^{n}x_i^2-2n\bar{x}^2+n\bar{x}^2 = \sum_{i=1}^{n}x_i^2-n\bar{x}^2$$

Subtracting the second of these equations from the first yields

$$\sum_{i=1}^{n}(x_i-\theta)^2-\sum_{i=1}^{n}(x_i-\bar{x})^2 = n\bar{x}^2-2\theta n\bar{x}+n\theta^2 = n(\bar{x}-\theta)^2$$

Since σ^2 is known, we are interested in $f(x|\theta)$ which is proportional to

$$\begin{aligned} f(x|\theta) &= (2\pi\sigma^2)^{-n/2}\exp\left(-\frac{n(\bar{x}-\theta)^2+(n-1)S^2}{2\sigma^2}\right) \\ &\propto \exp\left(-\frac{n(\bar{x}-\theta)^2}{2\sigma^2}\right)\exp\left(-\frac{(n-1)S^2}{2\sigma^2}\right) \\ &\propto \exp\left(-\frac{(\theta-\bar{x})^2}{2\frac{\sigma^2}{n}}\right) \end{aligned}$$

b. Set the prior for θ to be $\mathrm{N}(\mu, \tau^2)$ and derive its posterior distribution. (You can use the above result)

▼ Solution

The prior for θ is set to be $\mathrm{N}(\mu, \tau^2)$. Hence we can write:

$$\pi(\theta) \propto \exp\left(-\frac{(\theta-\mu)^2}{2\tau^2}\right).$$

Using the result of part (a), the posterior is then proportional to

$$\begin{aligned}
\pi(\theta|x) \propto f(x|\theta)\pi(\theta) \propto \exp\left(-\frac{(\theta-\bar{x})^2}{2\frac{\sigma^2}{n}}\right)\exp\left(-\frac{(\theta-\mu)^2}{2\tau^2}\right) \\
&= \exp\left(-\frac{\theta^2+2\theta\bar{x}}{2\frac{\sigma^2}{n}}-\frac{\theta^2-2\theta\mu}{2\tau^2}\right)\exp\left(-\frac{\bar{x}^2}{2\frac{\sigma^2}{n}}\right)\exp\left(-\frac{\mu^2}{2\tau^2}\right) \\
&\propto \exp\left(-\frac{\theta^2+2\theta\bar{x}}{2\frac{\sigma^2}{n}}-\frac{\theta^2-2\theta\mu}{2\tau^2}\right) \\
&= \exp\left(-\frac{\tau^2\theta^2-2\theta\bar{x}\tau^2+\frac{\sigma^2}{n}\theta^2-2\theta\mu\frac{\sigma^2}{n}}{2\frac{\sigma^2}{n}\tau^2}\right) \\
&= \exp\left(-\frac{(\frac{\sigma^2}{n}+\tau^2)\theta^2-2\theta(\bar{x}\tau^2+\mu\frac{\sigma^2}{n})}{2\frac{\sigma^2}{n}\tau^2}\right) \\
&= \exp\left(-\frac{\theta^2-2\theta\frac{\bar{x}\tau^2+\mu\frac{\sigma^2}{n}}{(\frac{\sigma^2}{n}+\tau^2)}}{2\frac{\frac{\sigma^2}{n}\tau^2}{(\frac{\sigma^2}{n}+\tau^2)}}\right) \stackrel{\mathcal{D}}{=} \mathrm{N}\left(\frac{\frac{\sigma^2}{n}\mu+\tau^2\bar{x}}{\tau^2+\frac{\sigma^2}{n}}, \frac{\tau^2\frac{\sigma^2}{n}}{\tau^2+\frac{\sigma^2}{n}}\right)
\end{aligned}$$