Rethinking Statistics

EES 4891-06/5891-01

Bayesian Statistical Methods

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Golems and Rethinking

Statistical Tools as Golems

- Statistical tools will do what you tell them to do …
  - but if you're not careful, what you tell them to do may not be what you want them to do
- The goal of this book is to help you:
  - Learn to use statistical golems wisely
  - Learn to choose the right golem for the job
  - Learn to engineer your own golems if the ready-to-use golems aren't right for your job.

The Golem of Prague (Photo: Prague Post)

Hypothesis Testing

- Karl Popper (1902–1994)
  - Science can never prove that a hypothesis is true
  - But it can prove that an incorrect hypothesis is false
  - The more false hypotheses we rule out, we narrow down the list of potentially true hypotheses.

When you have eliminated the impossible,
whatever remains, however improbable,
must be the truth
— Arthur Conan Doyle/Sherlock Holmes

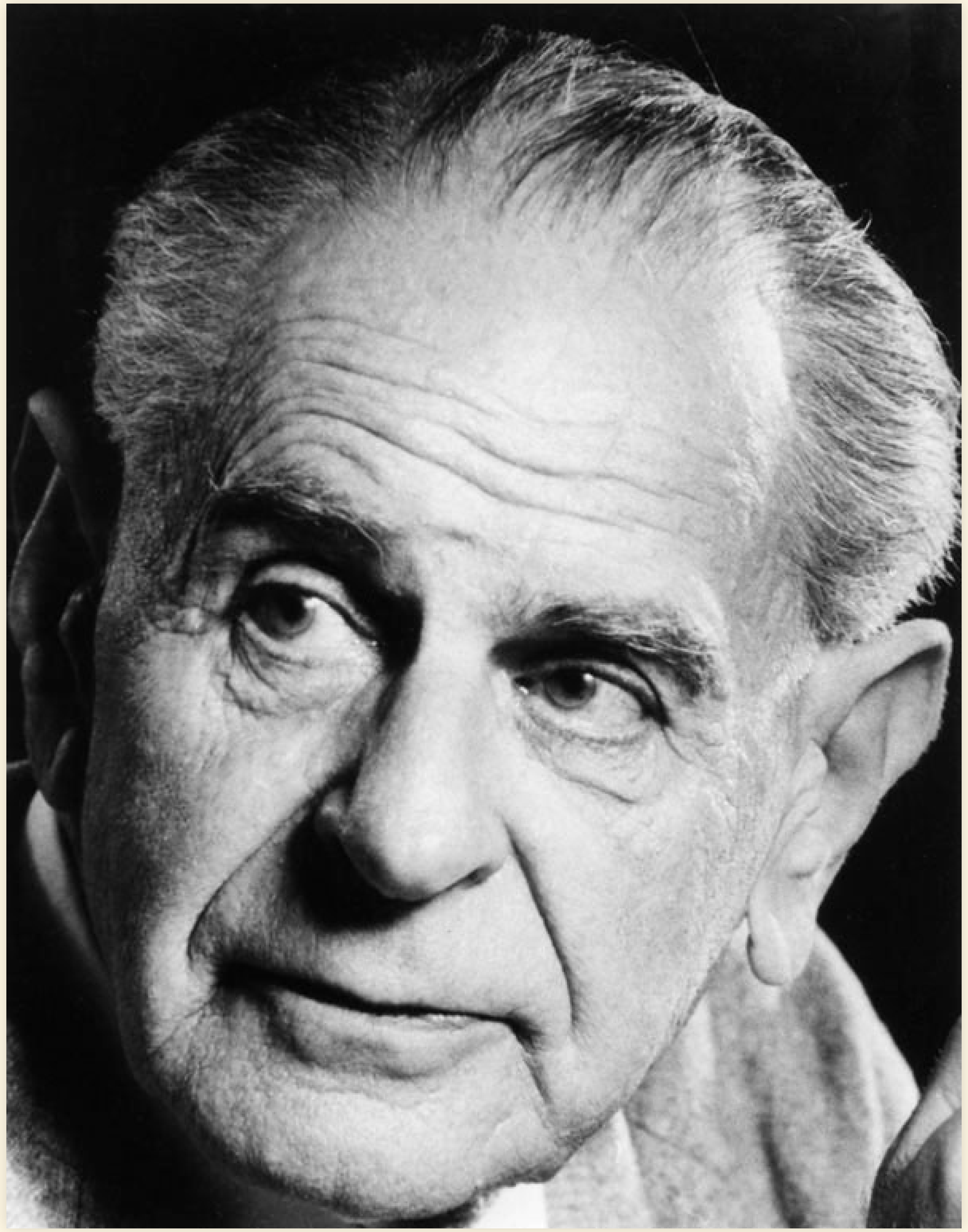
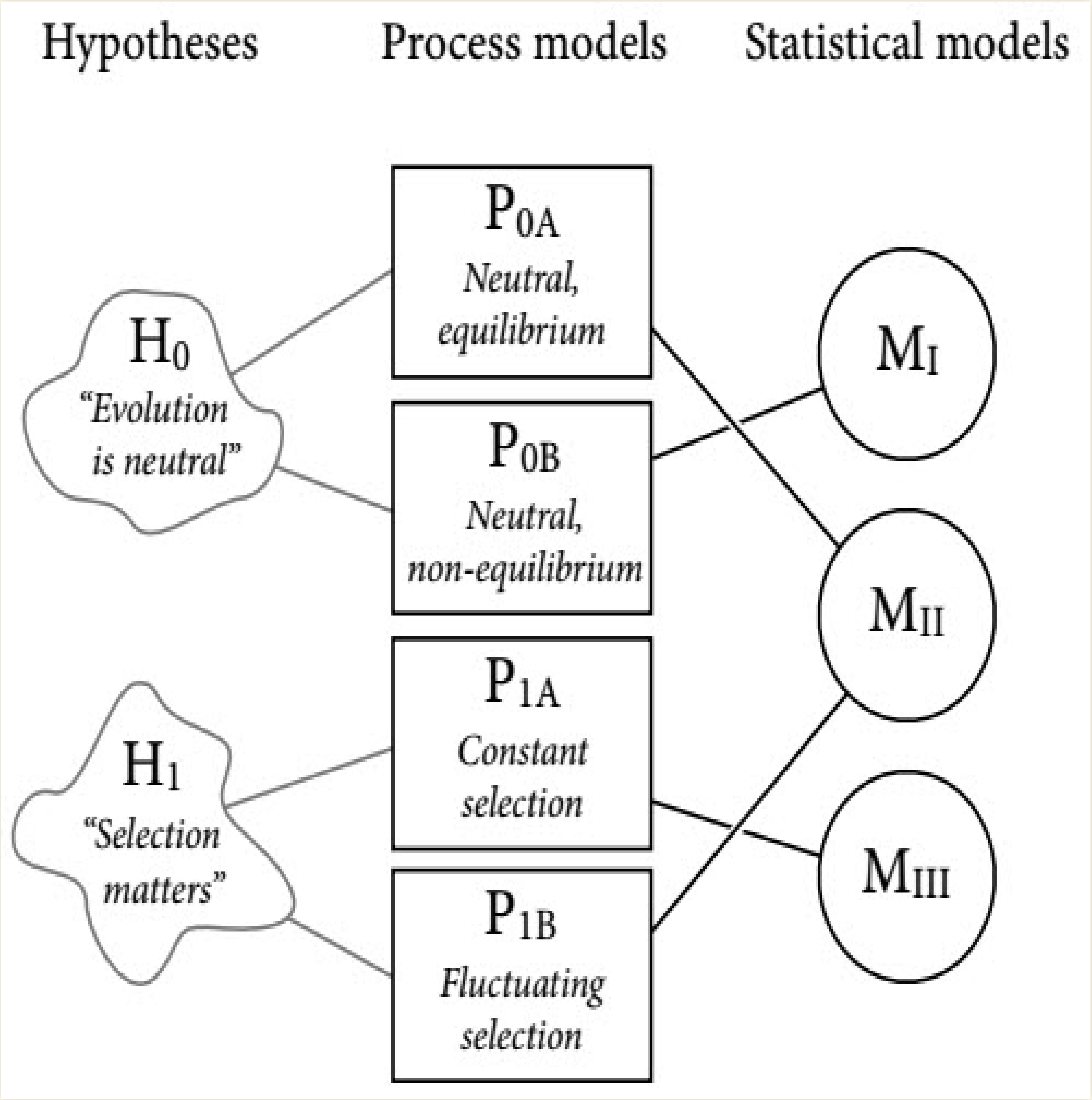


Photo: London School of Economics

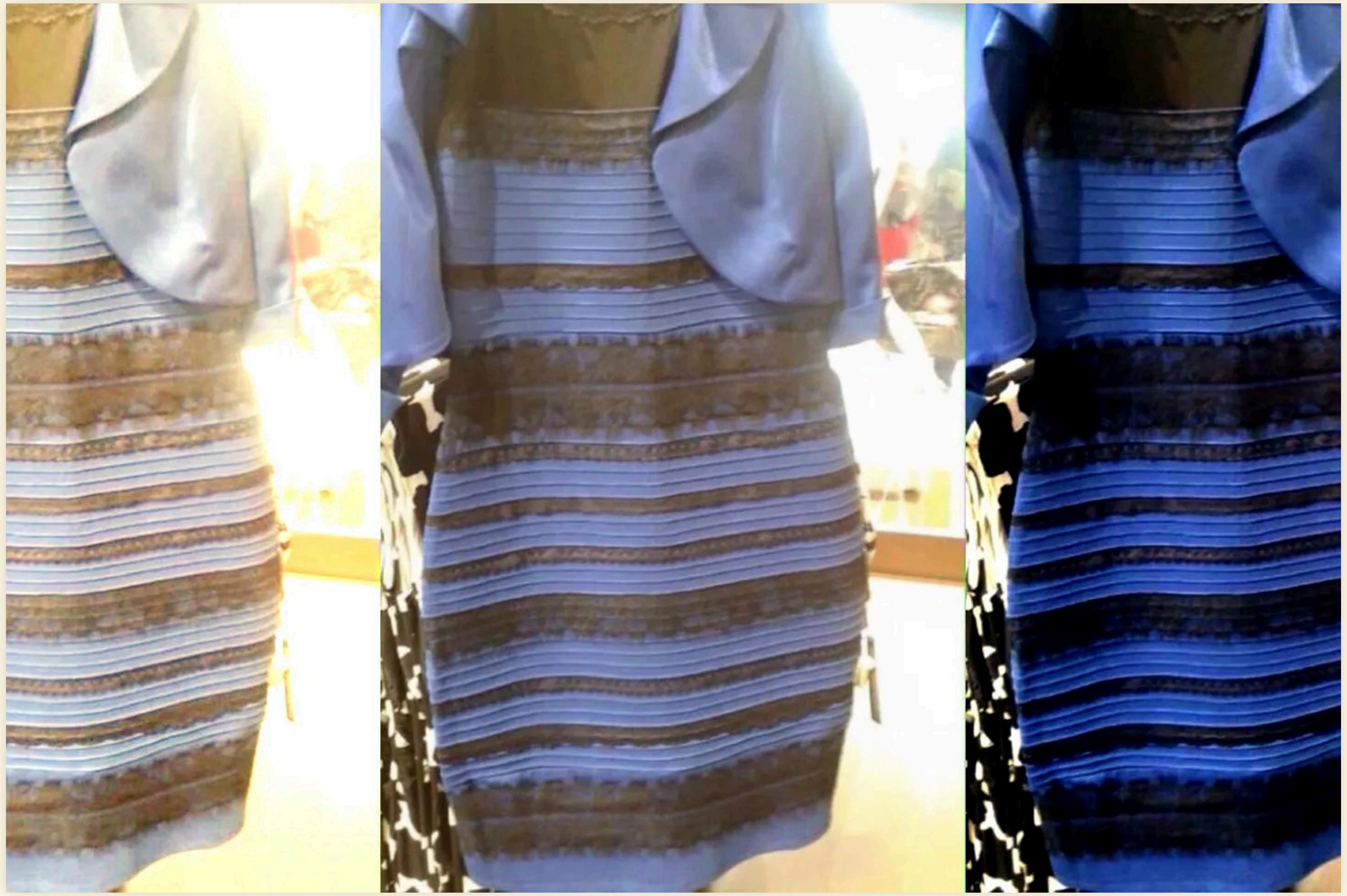
Problems with falsification

- The predictions of a hypothesis may not be as clear as many people assume.
- Depending on what other assumptions you make, two different hypotheses may predict the same kind of data.
  - If your data looks like M_{I}, it rules out (falsifies) H_1
  - But if your data looks like M_{II}, it doesn't rule out either hypothesis.
- A given hypothesis may predict may different possible kinds of data, depending on what other assumptions you make.
  - If your data doesn't look like M_{I}, that doesn't imply that it's less likely H_0 is true.



Experimenter’s Regress

- Can you be sure whether an observation falsifies a hypothesis?
  - Experimental error or uncertainty.
  - Experiments often rest on unstated hypotheses or models.
  - How certain can you be that an experiment was conducted correctly?



In February 2015, the middle photo was posted on tumblr, and people could not agree whether the dress was white & gold (left) or blue & black (right).

Over the next week, more than ten million tweets mentioned the dress.

The actual dress

- Eventually, someone tracked down the dress, and this is a photo from the maker, Roman Originals

Null-Hypothesis Significance Testing

- Most statistical tests aim to rule out a *null hypothesis*, not to falsify the actual research hypothesis.
- Often, there's not one unique alternative hypothesis to the null hypothesis, so even if we reject the null hypothesis, there are many other possibilities.
- Bayesian methods give us better, more powerful golems to answer the questions we're really interested in.
  - But they're still golems and we have to be thoughtful and careful about how to use them.

Discussion

Tools for Engineering Golems

Tools for Engineering Golems

1. Bayesian data analysis
2. Graphical causal models
   - Directed Acyclic Graphs (DAGs)
3. Model comparison
4. Multilevel models

Bayes’s Theorem

Bayes's Theorem

- The core of this part of the course is Bayes's theorem.
- Notation:
  - Conditional probability: $P(a|b)$ means *the probability of a, given b*.
- Bayes's theorem:

$$P(H|D) = \frac{P(D|H) \times P(H)}{P(D)},$$

  where
  - $P(H|D)$ is the *posterior*: The probability that H is true, given that you observed D.
  - $P(D|H)$ is the *likelihood*: The probability that you would observe D, if H is true.
  - $P(H)$ is the *prior* probability of H, based on what you knew before observing D
  - $P(D)$ is the *evidence*: The probability that you would observe D, regardless whether H is true.
    - If H is binary (true or false), then

$$P(D) = P(D|H) \times P(H) + P(D|\textbf{not } H) \times (1 - P(H))$$

Bayes's Theorem (cont)

- We can apply Bayes's theorem to a numbers too. For a variable x that we want to predict:

$$P(x|D) = \frac{P(D|x) \times P(x)}{P(D)}$$

- In this case, $P(\ldots)$ refer to *probability density functions* and

$$P(D) = \int_{-\infty}^{\infty} P(D|x) \times P(x) \mathrm{d}x$$

Sampling

Sampling

- You have a globe and want to figure out what fraction of the earth's surface is water.
- Toss the globe in the air, catch it, and note whether your index finger is on water or land: outcomes are *W* and *L*.
- At every toss, use Bayes's theorem to update your estimate of the fraction that is water.

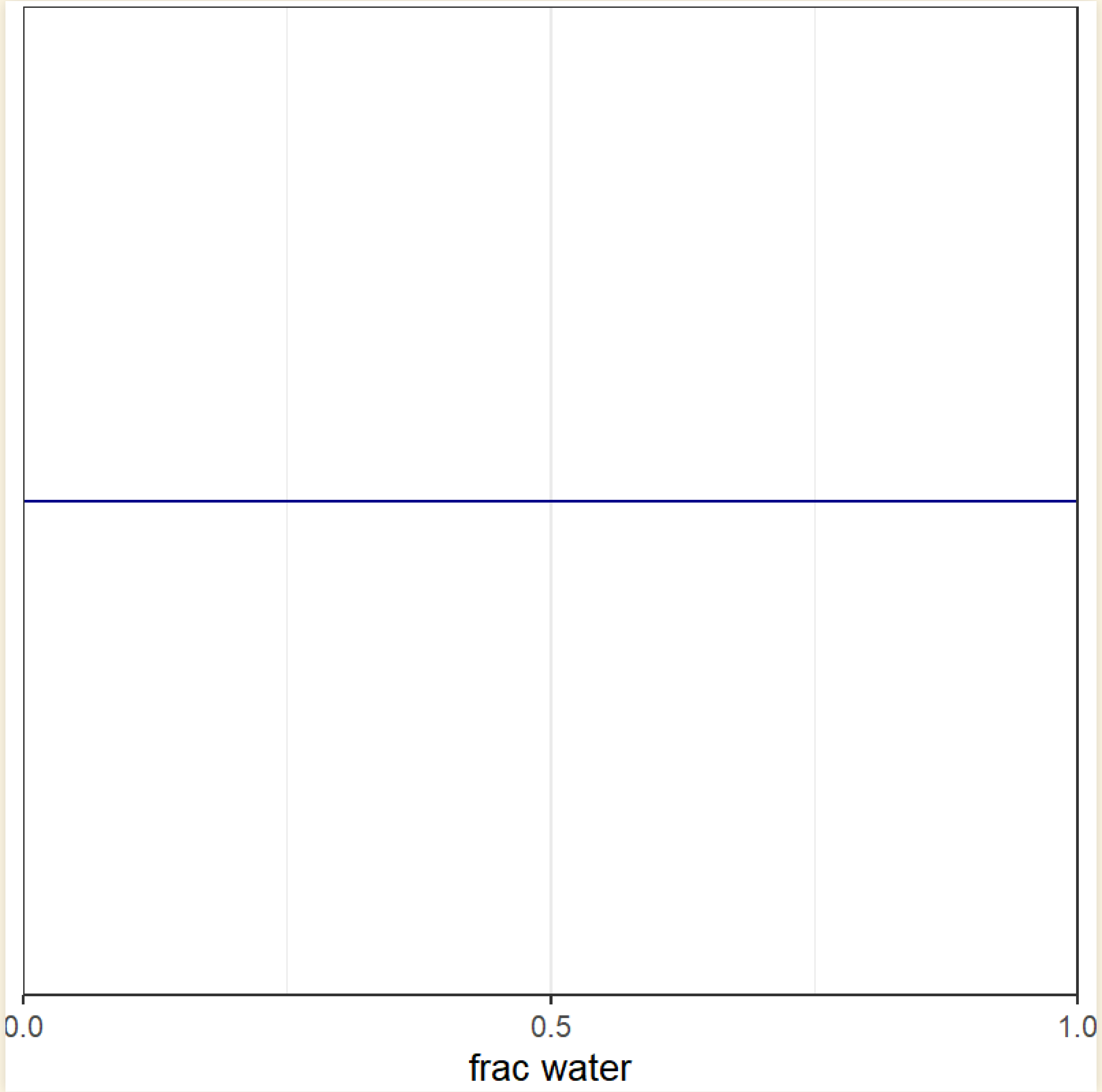
First toss

- Before you toss the globe, pick a prior probability distribution for the fraction that's water.
- Suppose we don't know anything.
  - Pick $p \sim \text{Uniform}(0, 1)$, a uniform prior:
- Toss the globe and your finger lands on water.
- Update the probability:

$$P(p|W) = \frac{P(W|p)P(p)}{p(W)},$$

where p is the probability of water, and W is measuring water.

Prior:



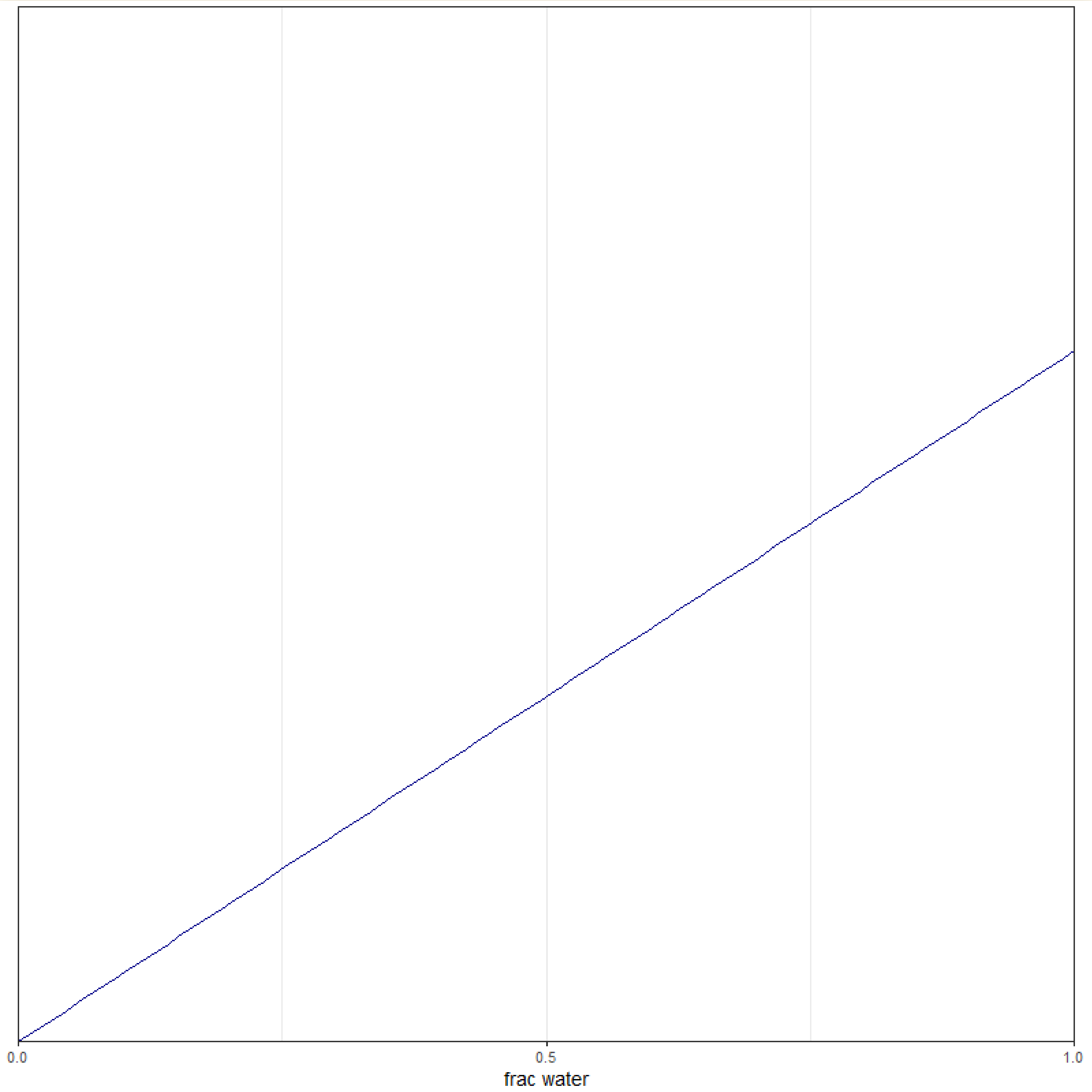
The calculation:

$$P(p|W) = \frac{P(W|p)P(p)}{p(W)},$$

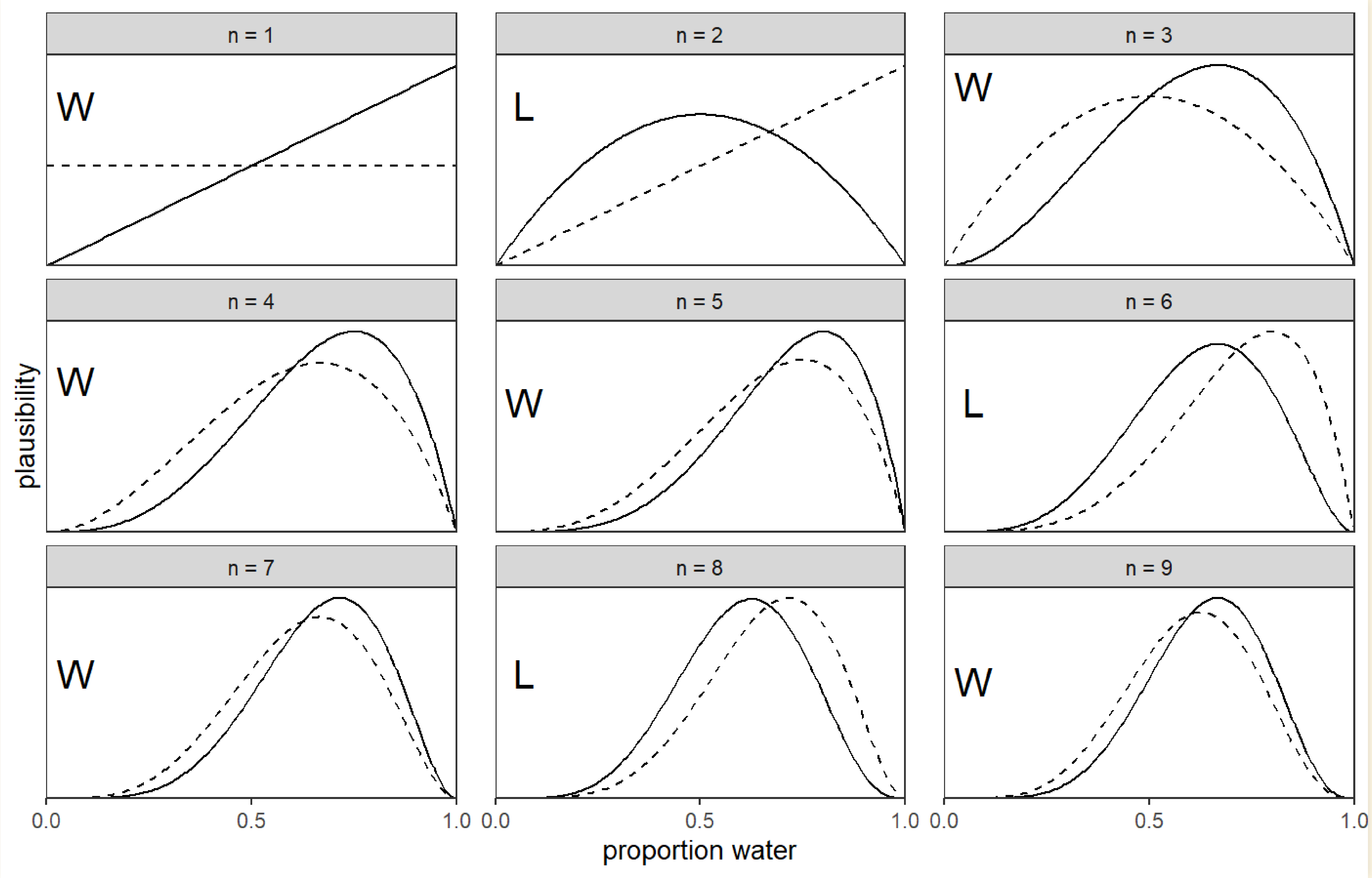
- $P(W|p) = p$
- $P(p) = 1$ (it's a uniform distribution

$$P(W) = \int_0^1 P(W|p)P(p)\mathrm{d}p = \int_0^1 p \times 1\mathrm{d}p$$
$$= (p^2/2)\Big|_0^1 = 1/2$$

- so the posterior $P(p|W) = 2p$
- Use this posterior as the prior for the next toss...



Subsequent tosses



- Dashed line = *prior*
- Letter = observation
  - W = water
  - L = land
- Solid line = *posterior*

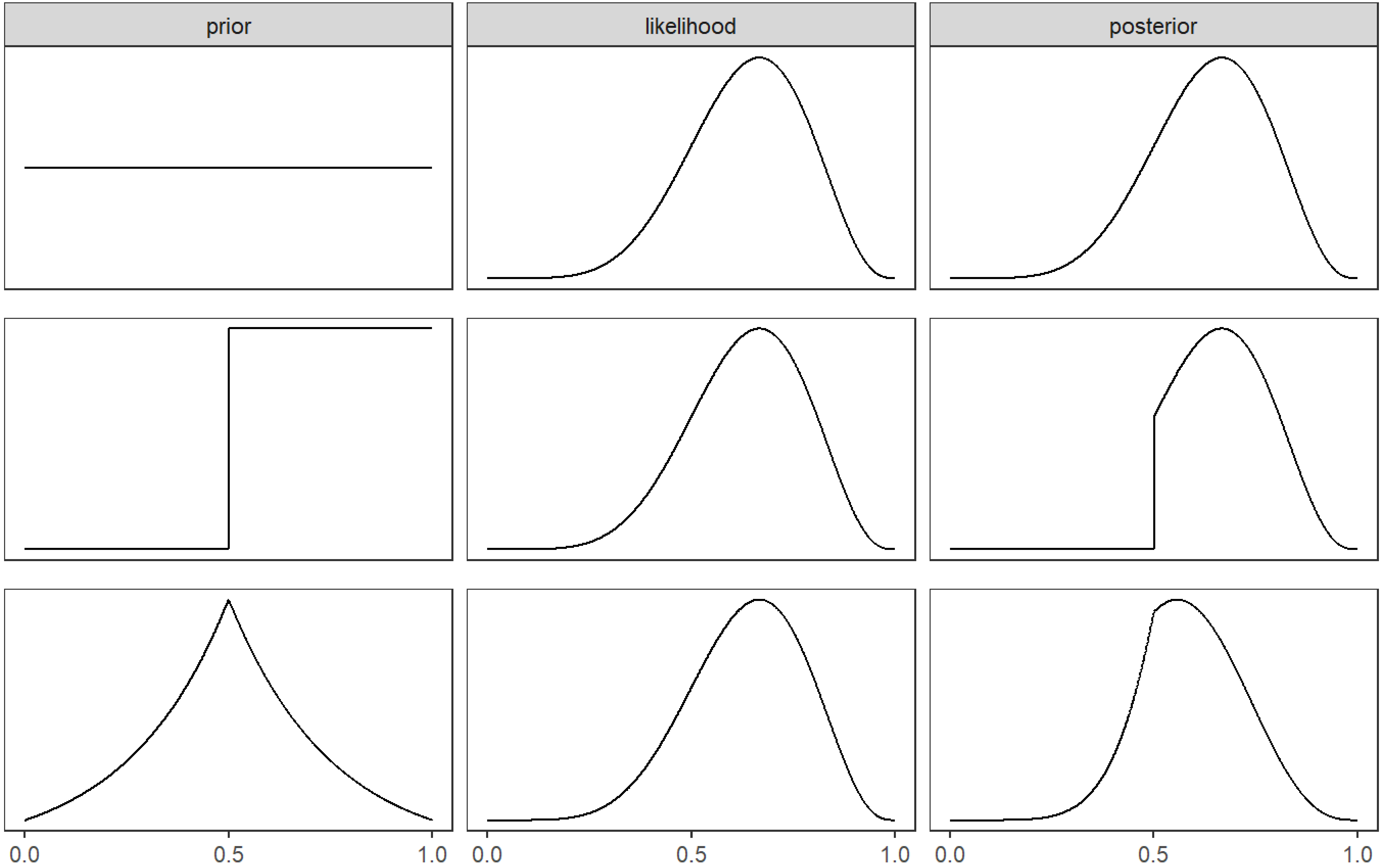
Bayesian analysis incrementally combines our *prior* knowledge with new *data* to produce an updated, improved *posterior*

Developing a Model

Developing a Model

- Observations and parameters are drawn from probability distributions:
  - Likelihood: $W \sim \text{Binomial}(N, p)$, where
    - N is the total number of tosses and
    - p is the fraction of the planet covered by water.
  - Our initial *prior* for p is uniform: $p \sim \text{Uniform}(0, 1)$
  - "$\sim$" means a random variable drawn from a probability distribution.
- We use the likelihood and the prior to calculate the posterior.
- We can't easily do this with analytical math using pencil and paper.
  - Computational methods:
    - Grid approximation
    - Quadratic approximation
    - Monte Carlo sampling

Examples



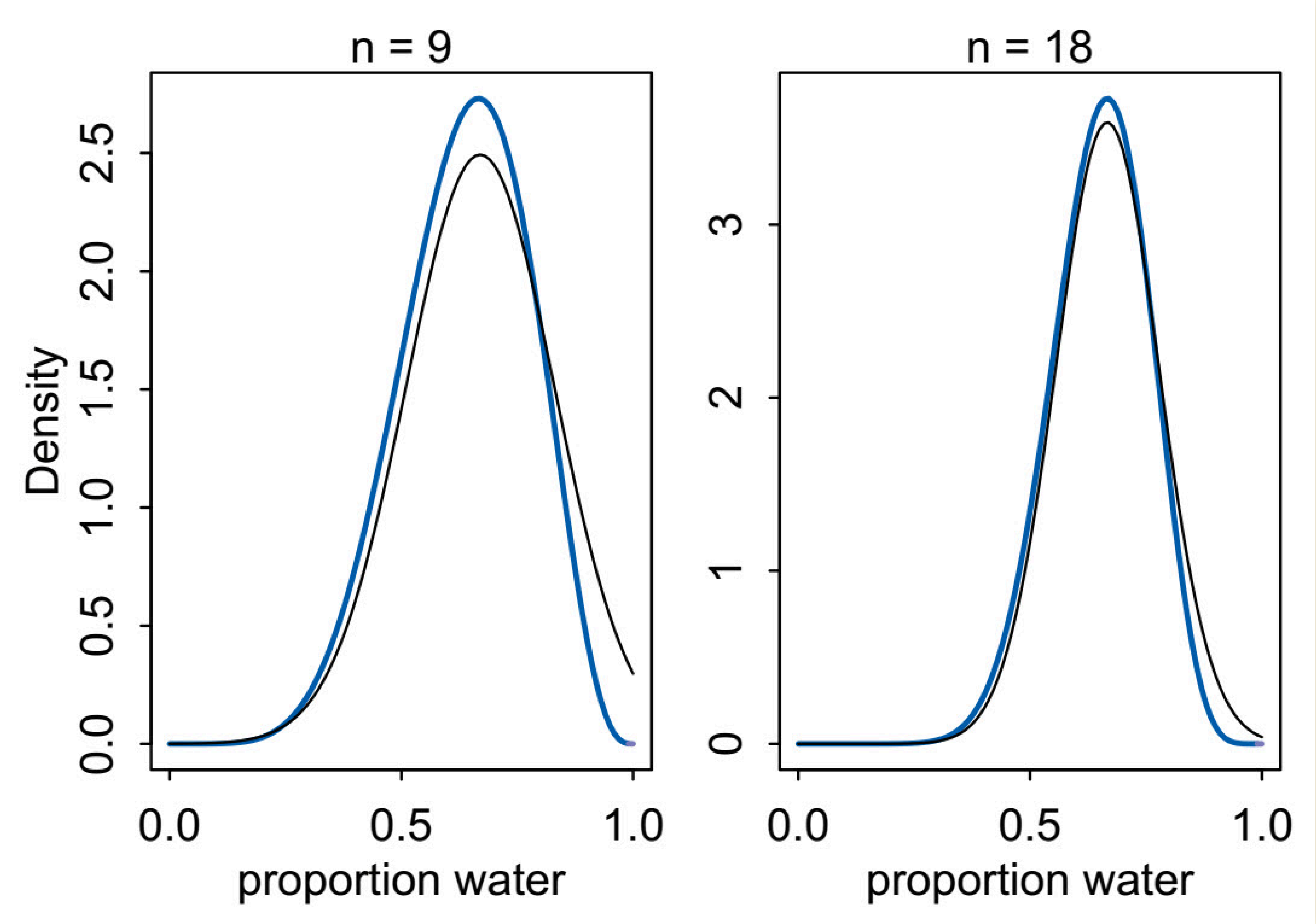
Grid Approximation

1. Define a grid:
   - specify a number of points to sample your function at.
   - Take evenly spaced values for each parameter (e.g., the proportion of water).
     - This example uses one parameter and a one-dimensional grid for simplicity.
     - For models with more than one parameter, the grid has 2, 3, or more dimensions—one per parameter.
2. Calculate the value of the *prior* at each grid point
3. Calculate the *likelihood* at each grid point
4. Compute an *unstandardized posterior* by multiplying the *prior* and *posterior* at each grid point.
5. Finally, standardize the *posterior* by dividing each value by the sum of all values in the *unstandardized posterior*.

- The more grid points you use, the more accurate your estimate will be, but the more computer power you'll need.
  - For one parameter, a 1000 point grid is simple.
  - For 2 parameters, a 1000 point grid for each of them means 1 million points.
  - For 3 parameters, it means 1 billion points.
  - For 30 parameters, your grid would have more points than there are atoms in the universe.

Quadratic Approximation

- Focus our attention near the part of the distribution that has the highest probability density.
- This region looks a lot like a Gaussian (normal) distribution.
  1. Find the posterior mode (the highest value).
     - Hill-climbing algorithms
  2. Estimate the curvature of the posterior near the mode
     - Approximate probability density as a Gaussian
     - Approximate the logarithm of the probability density as a quadratic function.
  3. We can calculate the integral of a Gaussian easily.



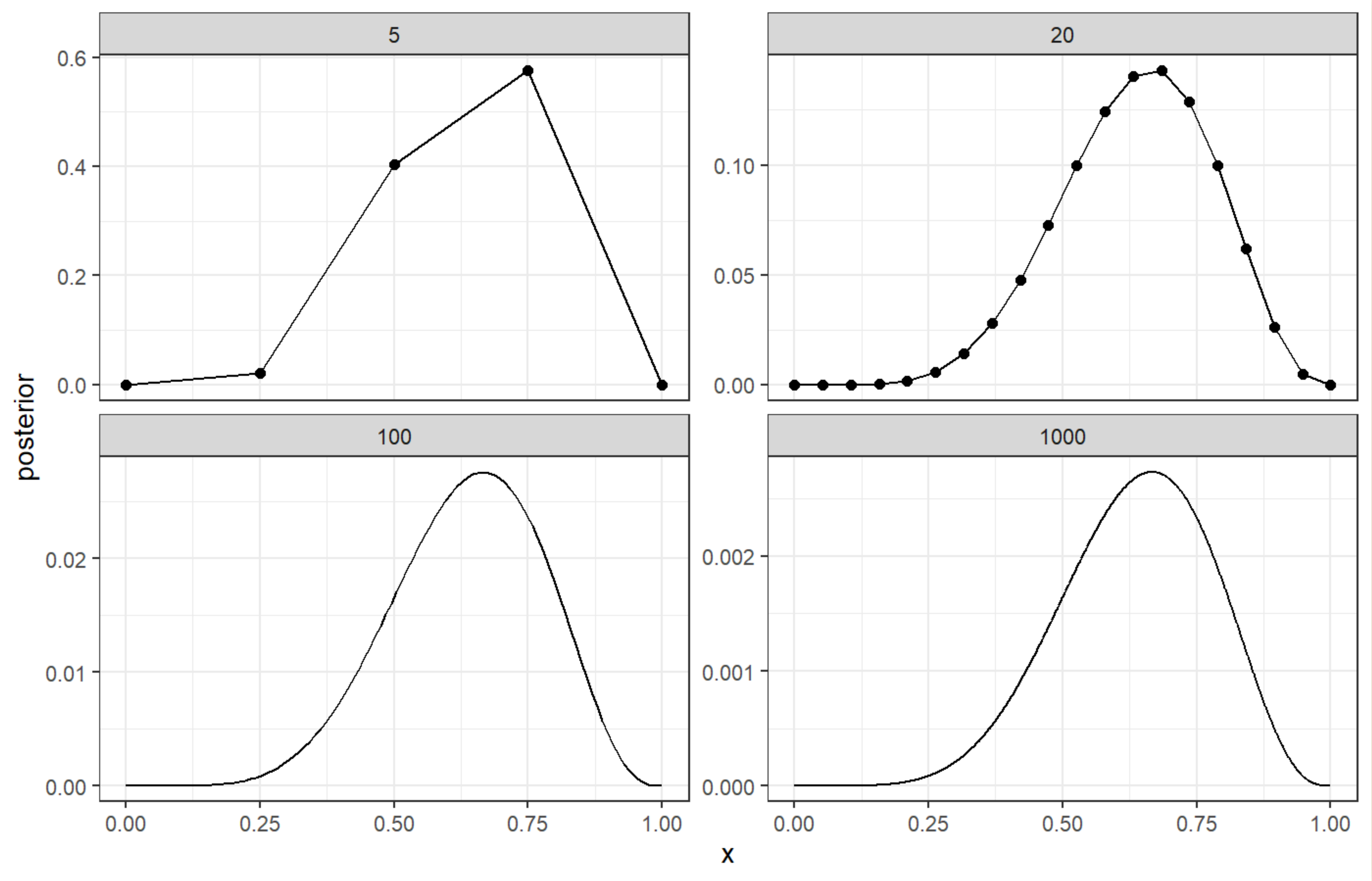
- Blue = exact, Black = quadratic approximation
- Advanced topic: Integrated Nested Laplace Approximation (INLA)
  - Powerful generalization of the quadratic approximation

Programming Models in R

Grid approximation

- Sample functions on a regular grid and approximate integrals by the sum of the samples.

```
# define number of points in the grid

grid_points = 200

# define grid
p_grid <- seq(from = 0, to = 1,
              length.out = grid_points)

# define prior
prior <- rep(1, grid_points)

# compute likelihood at each value in grid
likelihood <- dbinom(6, size = 9, prob = p_grid)

# compute product of likelihood and prior
unstd.posterior <- likelihood * prior

# standardize the posterior, so it sums to 1
posterior <- unstd.posterior / sum(unstd.posterior)
```



Quadratic Approximation

```
library(rethinking)

globe_qa <- quap(
  alist(
    W ~ dbinom( W + L, p),  # binomial likelihood
    p ~ dunif(0, 1)         # uniform prior
  ),
  data = list(W = 6, L = 3)
)
```

Now display a summary:

```
precis(globe_qa)
```

```
##        mean        sd      5.5%     94.5%
## p 0.6666664 0.1571338 0.4155362 0.9177967
```