Sampling the Imaginary

EES 4891-06/5891-01

Bayesian Statistical Methods

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Class #3: Monday, January 12 2026

Philosophical Prelude about Sampling

Dichotomies

- This chapter has a lot of dichotomies:
  - Bayesian vs. non-Bayesian (*frequentist*) statistics
  - Formulas (analytical math) vs. Frequencies (samples)
  - Parametric vs. Non-parametric approaches.
- Dichotomies don’t mean one alternative is right and the other is wrong, or one is better than the other.
  - Newtonian vs. Lagrangian mechanics in physics
  - Inclusive fitness vs. multilevel selection in evolutionary biology
  - Bayesian vs. non-Bayesian statistics
- In each case, using both perspectives gives us a richer view of each, helps us pick the right tool for a particular job.

Formulas vs. Frequencies

- Samples and natural frequencies can be more intuitive and easier to understand.

> *Many scientists are uncomfortable with integral calculus, even though they have strong and valid intuitions about how to summarize data. Working with samples transforms a problem in calculus into a problem in data summary, into a frequency format problem.*
>
> *…*
>
> *An integral … can be a challenging calculus problem. But once you have samples from the probability distribution, it's just a matter of counting values.*
>
> — *Statistical Rethinking, p. 51.*

Sampling from the Posterior

Grid Sampling

- There are two levels here:

1. Generate the posterior from grid-samples of the prior and posterior
2. Draw a random sample of points from the grid-sampled posterior.

- Grid-sampled posterior for 9 tosses of the globe, with 6 water:

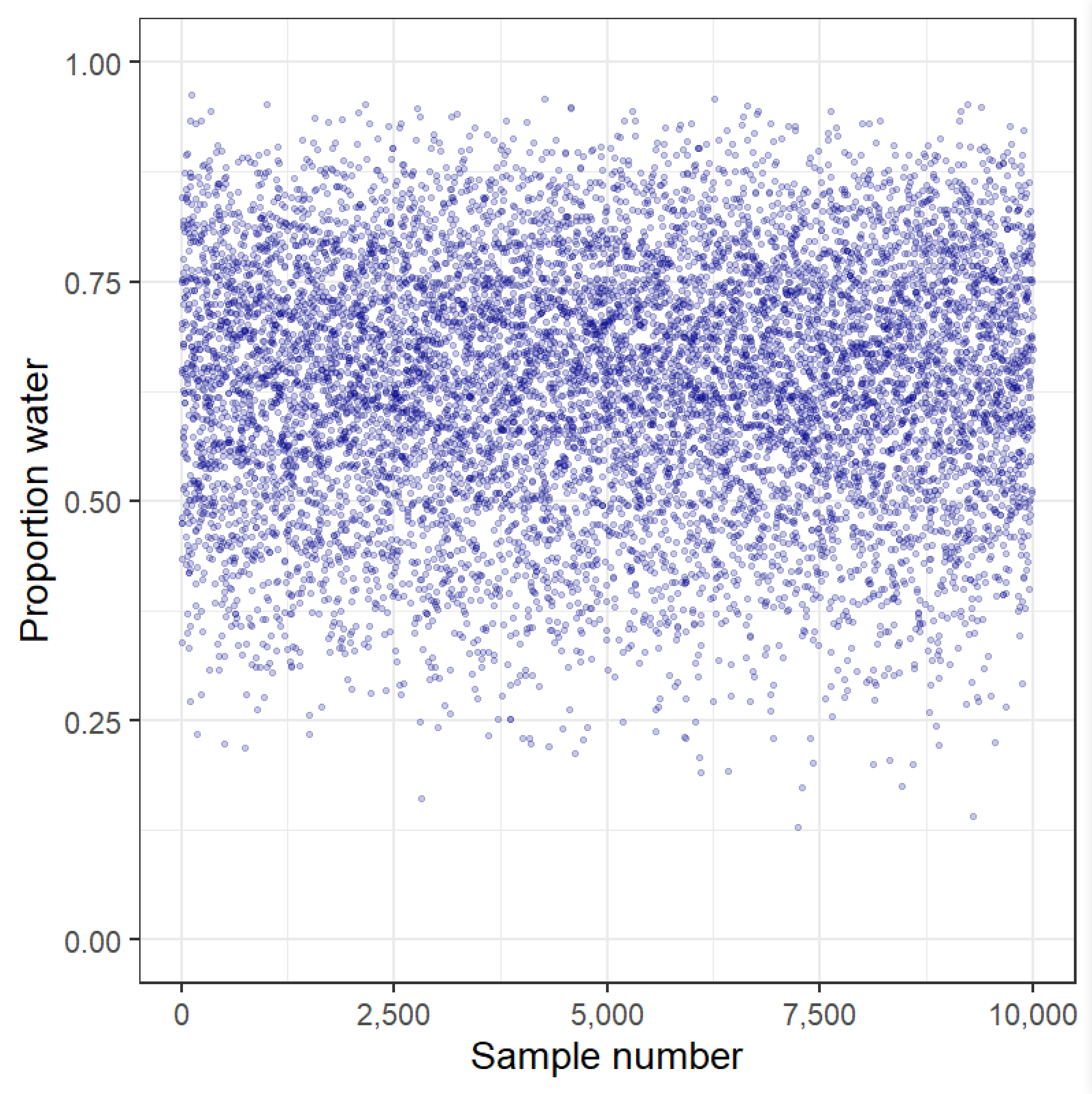
```
p_grid <- seq(from = 0, to = 1, length.out = 1000) # grid from 0 to 1
prior <- rep(1,1000)                               # uniform prior: all values are 1
likelihood <- dbinom(6, size = 9, prob=p_grid)     # binomial distribution
post_unstd <- likelihood * prior                   # unstandardized posterior
posterior <- post_unstd / sum(post_unstd)          # standardized posterior
```

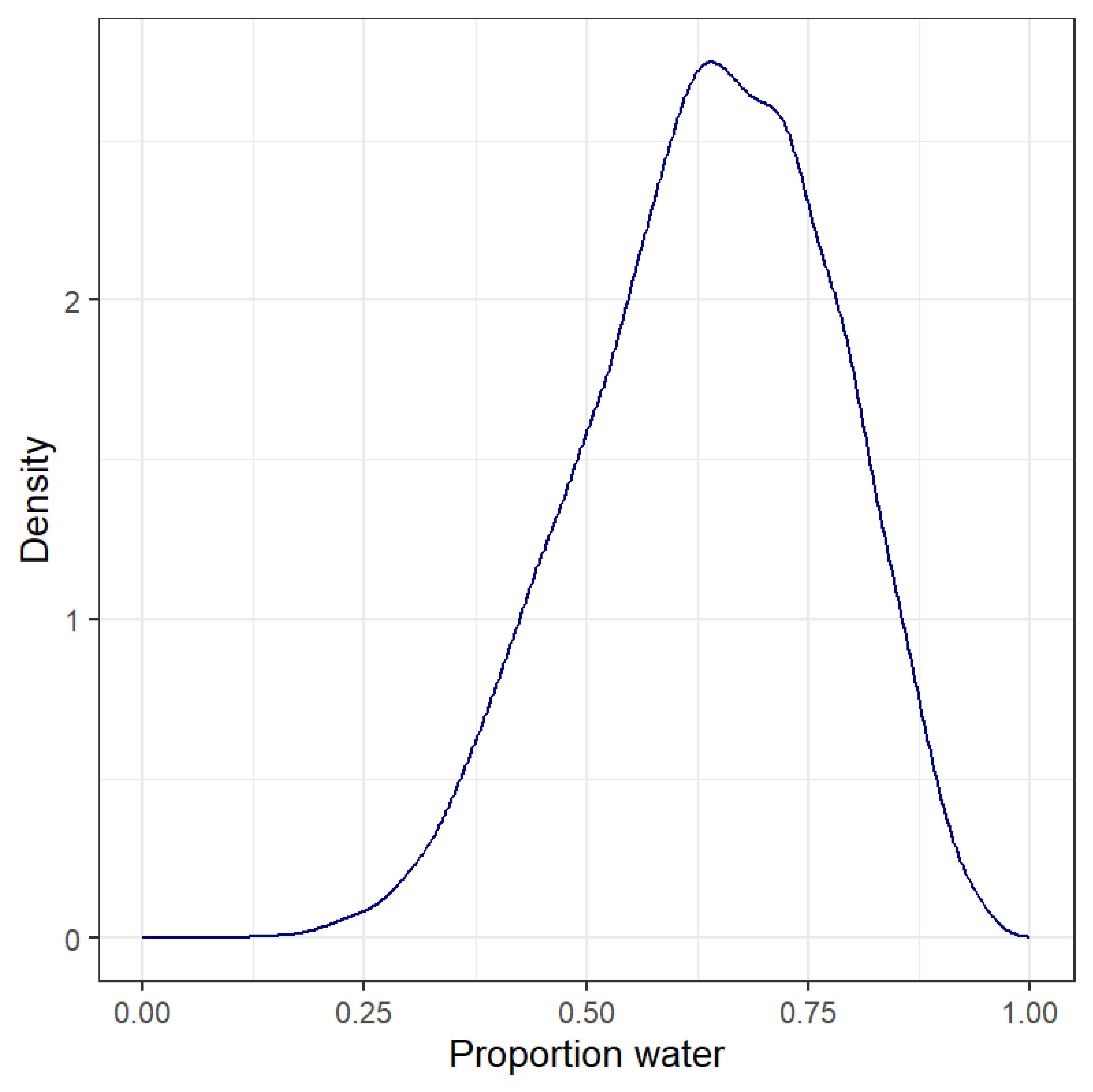
  - The *likelihood* is the probability of seeing Water 6 times in 9 tosses, *if* the probability of water in each toss is ***p***.

- Now randomly sample 10,000 points from the grid-sampled posterior

```
n_samples = 10000
samples <- sample(p_grid, prob = posterior, size = n_samples,
                  replace = TRUE)
```

Sampled Posterior



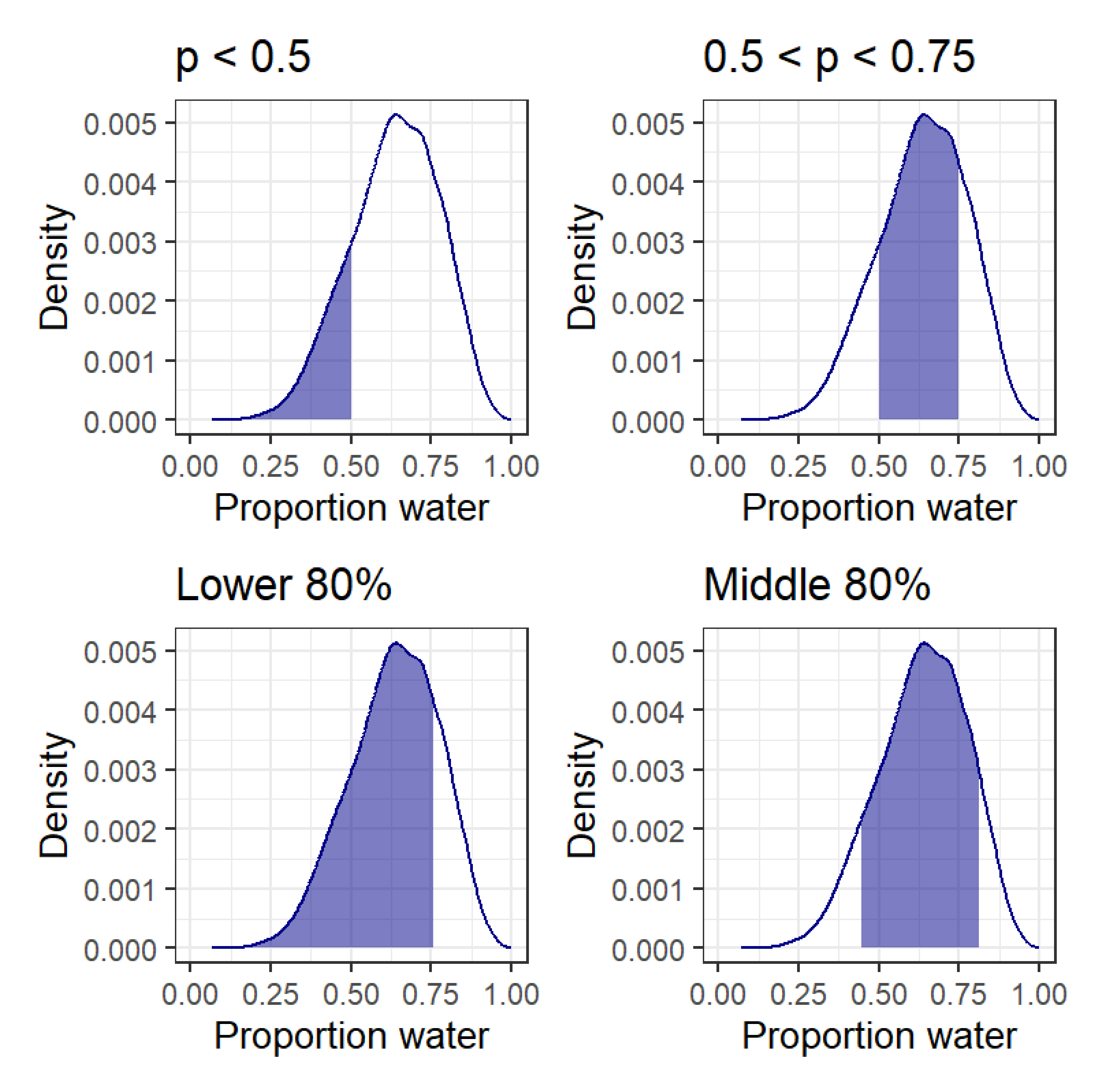


Analyzing Sampled Posterior

- What can we do with samples drawn from the posterior?
- Summarize the posterior distribution
  - Percentile intervals (PI)
  - Highest probability-density intervals (HPDI)
  - Point estimates
- Simulate prediction
  - Designing models
    - Will your model be able to infer parameters from data?
  - Checking correctness of models and code
    - Does your model accurately describe the data?
  - Designing research
    - Will data from a planned project be sufficient to answer relevant questions?
  - Forecasting
    - What does your model imply about the future?

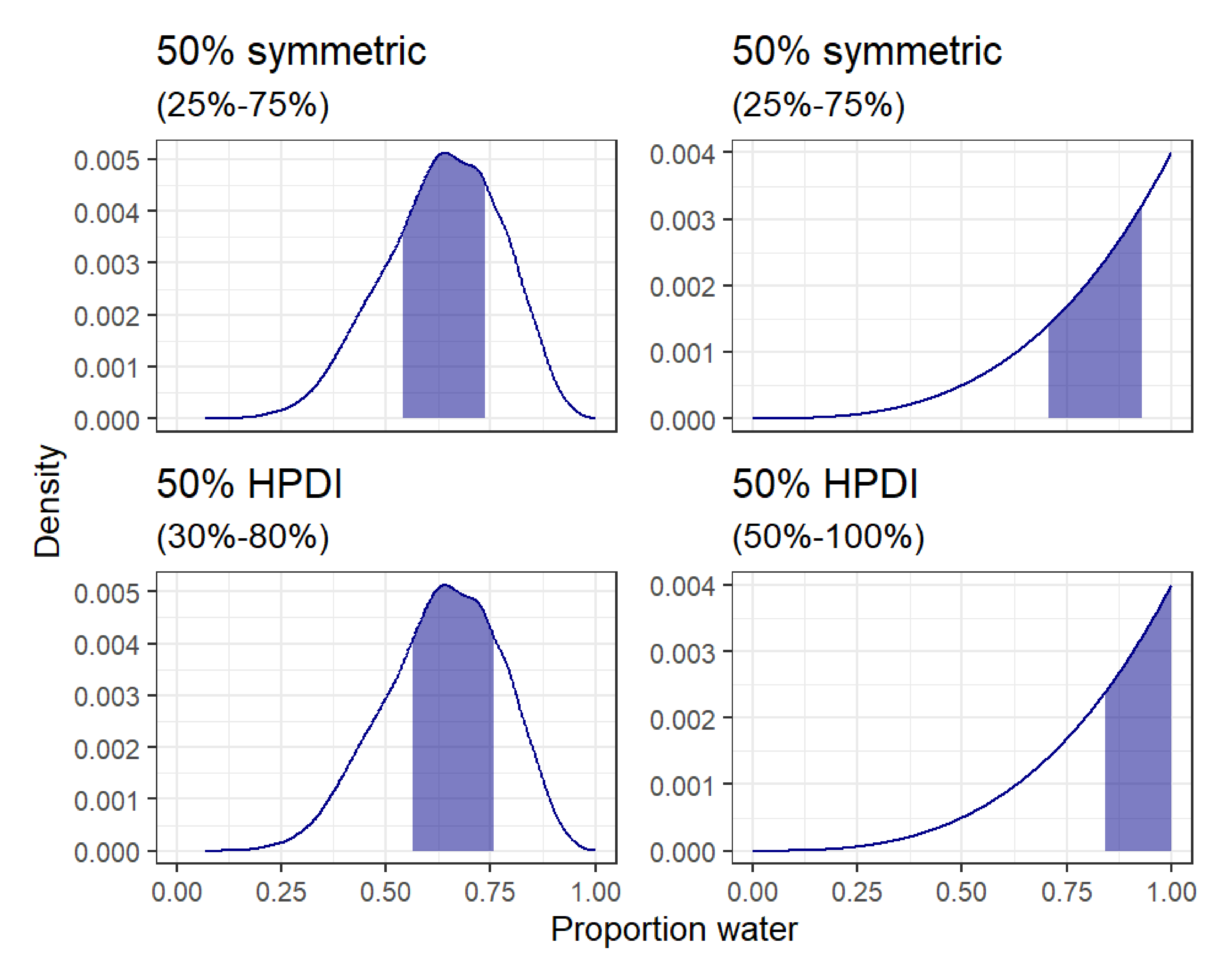
Probability Intervals

- **Vocabulary:** for posterior probability density intervals
  - **Compatibility Intervals**: Less likely to be misunderstood than *"confidence intervals"* or *"credibility intervals"*.
  - **Probability Mass:** The area under an interval of the posterior density curve.
    - Related to *cumulative probability*
- **Kinds of intervals:**
  - **Percentile Intervals** contain a certain probability mass, with equal tails on either side
    - e.g., middle 80% of probability mass
  - **Highest Probability-Density Intervals (HPDI)** contain a certain probability mass located at the region of highest probability density.



Percentile vs. Highest Probability Density Intervals

- **Symmetric Percentile Intervals** are symmetric, but may omit the highest probability density
- **Highest Probability-Density Intervals (HPDI)** Make sure to contain the highest density region, but may be unstable from one sample to another.
  - The lowest probability density in HDPI > highest probability density outside.
- If the posterior is reasonably smooth and symmetrical, these two intervals should be very similar.



Point estimates

- Extract a single point and ignore the rest of the distribution.
  - **MAP** (maximum *a-posteriori*), the mode of the posterior
  - **median** of the posterior
  - **mean** of the posterior
- How would you choose what point estimate to use?
- Loss function: What are the consequences of being wrong?
  - Hurricane forecasting: Consequences are much worse if the wind is stronger than forecast, than if the wind is weaker.
- There’s no universal best method for point estimation. You need to choose a method to fit the specific problem you’re trying to solve.

Sampling to Simulate Prediction

Sampling to Simulate Prediction

1. **Model design:** Will your model be able to produce good analysis from the kind of data you expect to get?
   - Working with *dummy data* can be very helpful.
2. **Model checking:** After you fit a model to real data, do you trust the fit?
3. **Software validation:** Does your computer code work to fit your model to data?
   - *Dummy data* can be helpful here.
4. **Research design:** Will your research design produce the *quality* and *quantity* of data necessary for your model to answer your research questions?
   - Generate *dummy data* representative of what you expect your experiment to produce.
5. **Forecasting:** Use your model to make predictions about the world.

Dummy Data

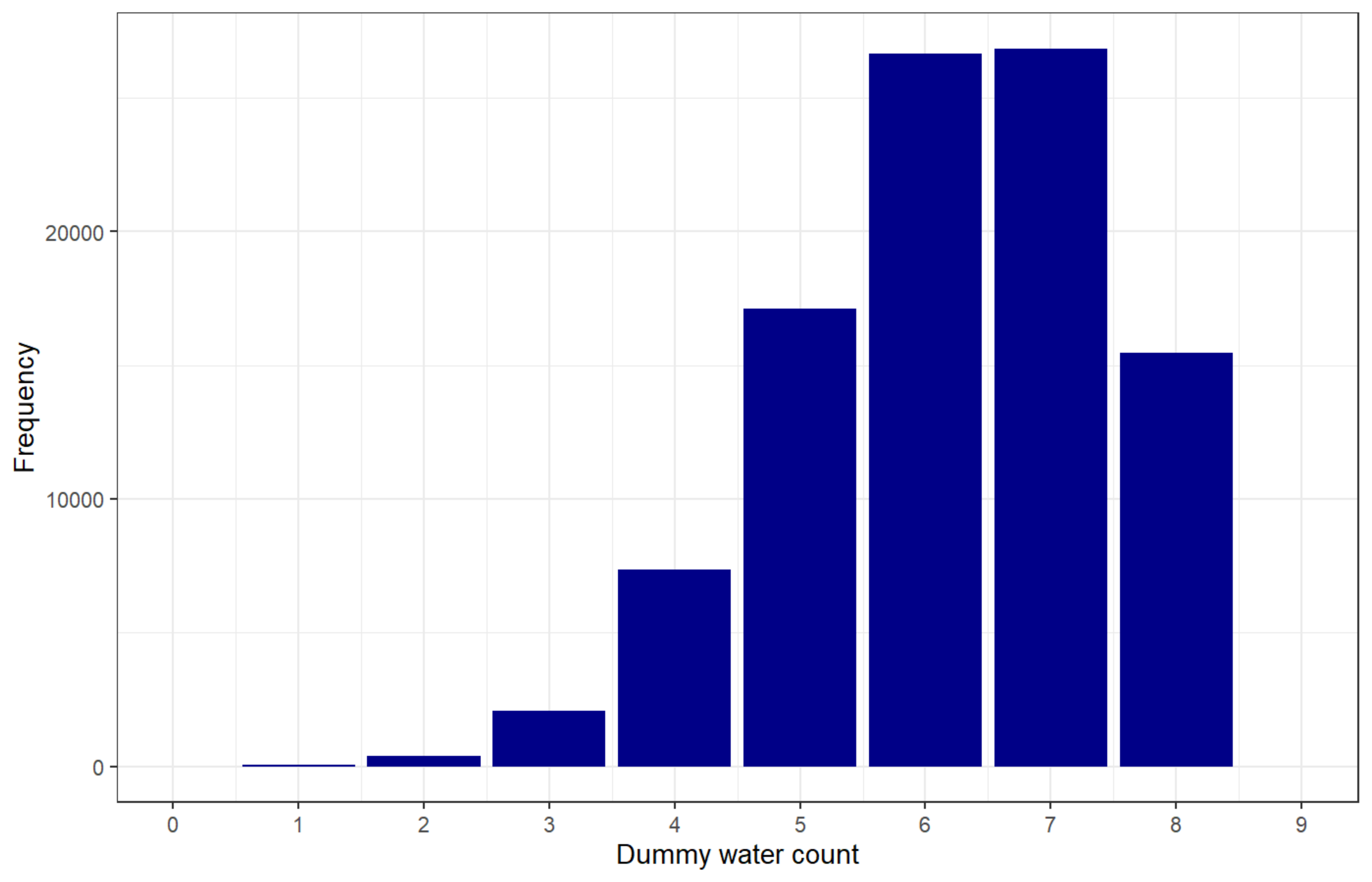
- Before using your model on real data, always test it on dummy data.

  1. Generate simulated data by sampling from a distribution with known parameters.

     Example: Simulate 100,000 repetitions of an experiment with 9 tosses of a globe that's 70% water.

     ```
     dummy_w <- rbinom(1E5, size = 9, prob = 0.7)
     ```

  2. Analyze the simulated data the same way you'd analyze your real data.
  3. Compare the posterior estimates of the parameters to the actual values you used to simulate the data.
     - Fit the 100,000 repetitions. How close are the estimates of *p* to the true value (0.7)?



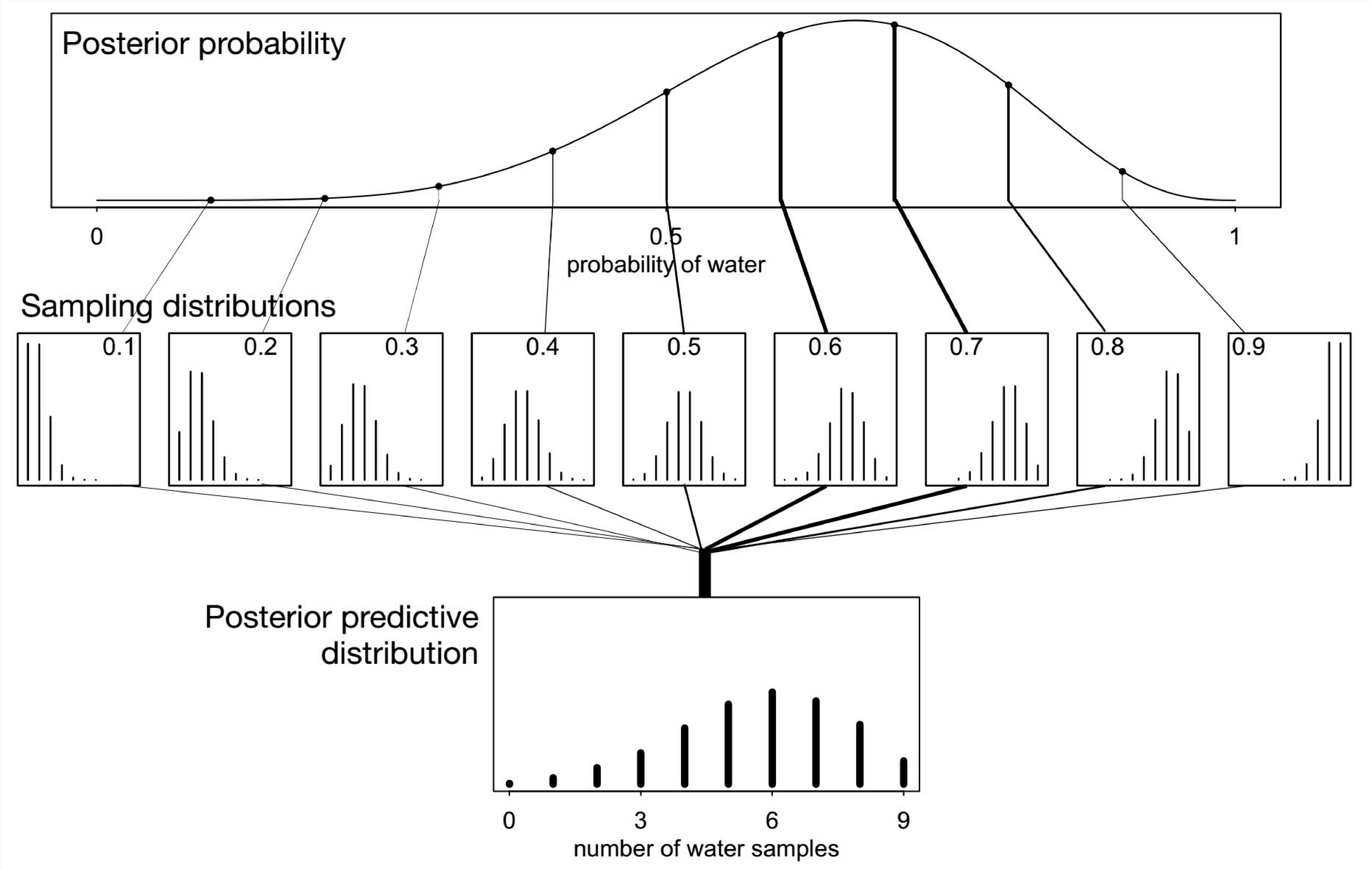
Using Samples to Diagnose Model Fits

- You have fit a model to real data
- You want to understand how well the model fits the data.
- **Posterior Predictive Distributions**

1. Sample from your posterior for the model parameters
   - *p*, proportion of water
2. For each sample, use your model to simulate the data.
   - `rbinom(n, size = 9, prob = p)` to simulate n repetitions of 9 tosses with proportion `p`.
3. Combine the simulated experiments for different samples of the posterior to produce a *posterior predictive distribution*.
   - This combines:
     - Uncertainty about ***p*** (from the *posterior* distribution)
     - Randomness of the number of *W* results in 9 tosses for any value of ***p*** (`rbinom()`)

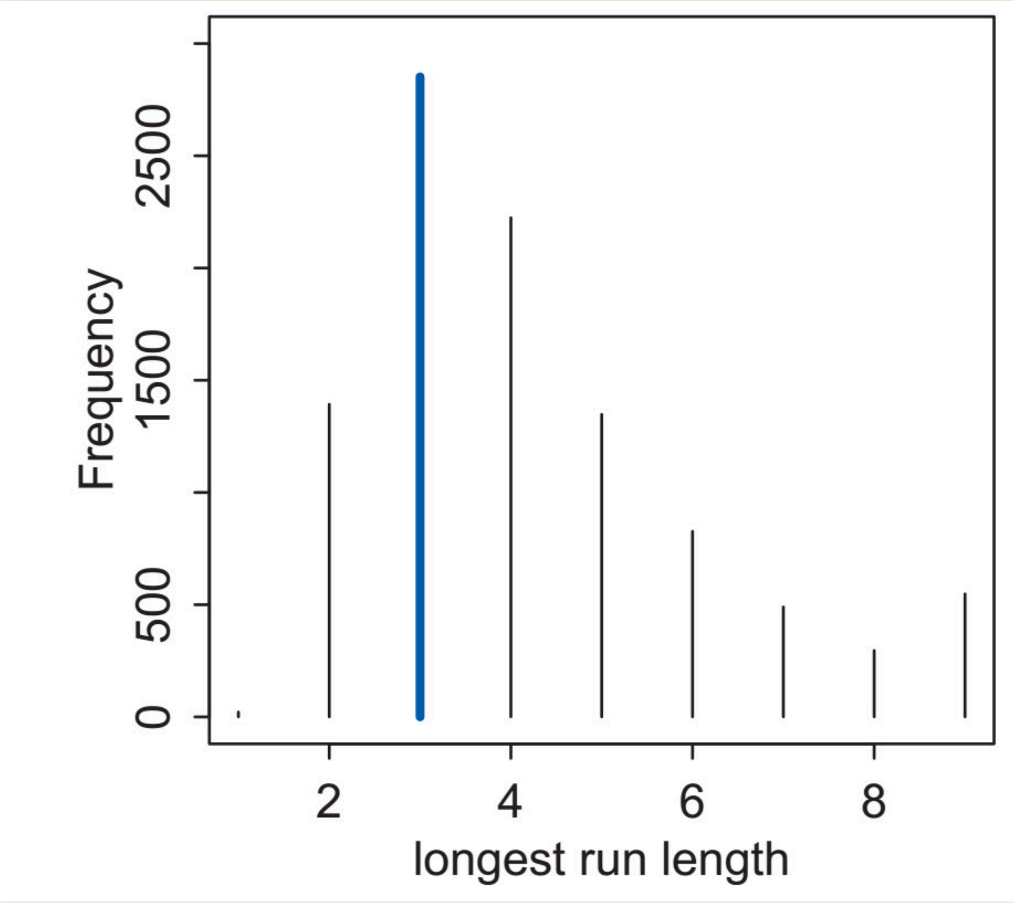
Posterior Predictive Distribution

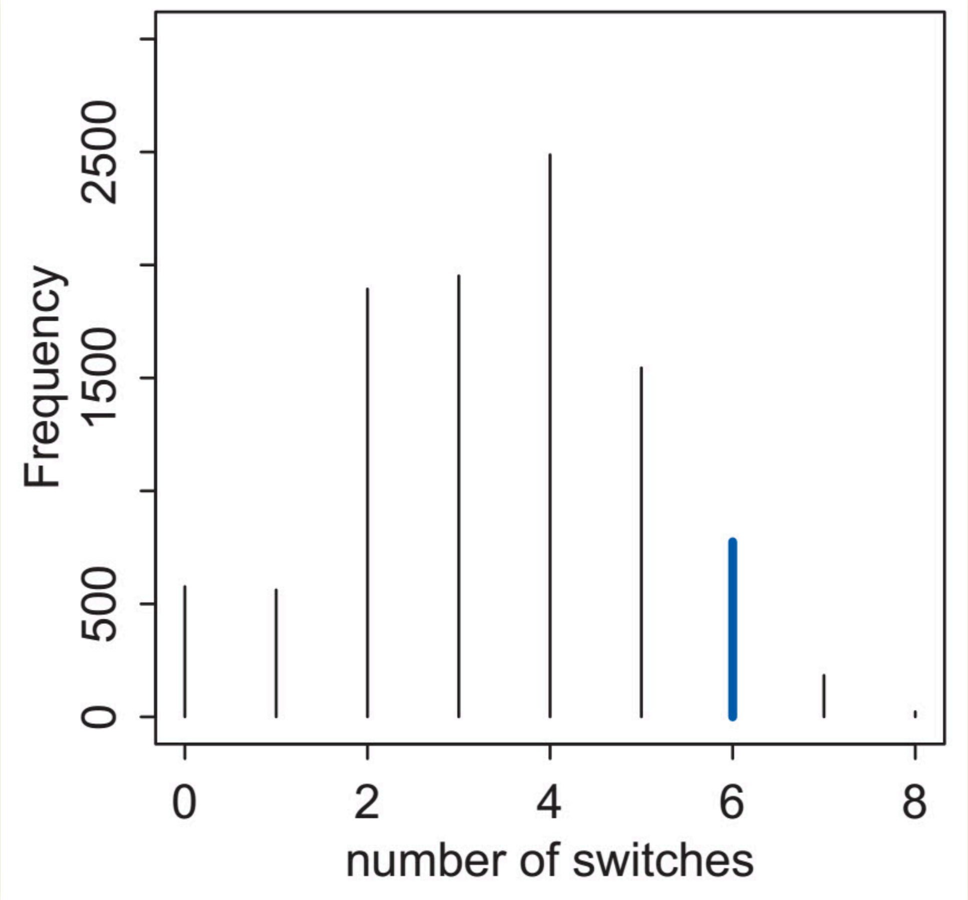
Consistency check: How likely is it that you'd observe the data you did, given the posterior you calculated?



Additional Diagnostics

- Maybe the problem wasn't with your model but with your experiment.
  - If you don't spin the globe enough when you toss it, there will be a correlation between the position when you tossed it and when you catch it.
  - If your finger is in the middle of the Pacific Ocean when you toss it, it's more likely you'll catch it on water.
- Check longest consecutive run of **W** or **L**
  - Experiment saw 3
- Check number of switches between **W** and **L** in 9 tosses.
  - Experiment saw 6





The Hot Hand Phenomenon

Hot Hands: Do Basketball Players Have "Hot" Streaks?

- Many sports fans think basketball players have "hot" streaks, where they are either more likely to sink many shots in a row.
- Statisticians wondered whether this was real, or just a misunderstanding of randomness
- 1985 analysis: Gilovich, Vallone, & Tversky analyze NBA games and controlled free-throw experiments with NCAA collegiate players:
  - **Null hypothesis:** Probability of making any shot is the player's season average.
  - **Alternative hypothesis:** When a player has sunk several shots in a row, he is more likely to sink the next.
  - **Result:** All data is consistent with the null hypothesis
- **Conclusion:** Fans are wrong: Hot streaks are an illusion.

- T. Gilovich, R. Vallone, & A. Tversky. 1985. "The hot hand in basketball: On the misperception of random sequences." *Cognitive Psychology*, **17**, 295. doi: 10.1016/0010-0285(85)90010-6

Hot Hands Analysis, Revisited

- 2015: Miller & Sanjurjo re-analyze Gilovich *et al.*
- Gilovich *et al.* start counting consecutive hits *after* the player makes a hit.
  - This conditional counting introduces bias
  - Gilovich's statistical model is faulty:
    - Their model underestimates the prevalence of streaks
- Miller & Sanjurjo use simulation to measure the bias
- **Result:** After correcting for bias, 5 out of 25 players show significant violations of the null hypothesis.
- **Conclusion:** Hot streaks are real, but uncommon.
  - Gamblers can use "hot hands" to improve predictions and win money.
- Many of the world's best statisticians were consistently wrong about this for 30 years.
  - Simulation and sampling were crucial to revealing the error

- J.B. Miller & A. Sanjurjo. 2018. "Surprised by the hot hand fallacy? A truth in the law of small numbers." *Econometrica*, **86**, 2019. doi: 10.3982/ECTA14943

Example of Bias

- 8 (2^3) possible sequences of heads and tails in three coin tosses.
- If $p(H)$ is always 50%, all 8 sequences are equally likely.
- But if you start counting after *H*, then you only examine 6 of the 8 sequences.
  - Averaging the 6 sequences, the proportion of heads following *H* is 42%, even when each toss has a 50% chance of landing heads.
- Bias gets larger when studying longer streaks (proportion of heads following 2, 3, 4, or more consecutive heads).

	Three-flip sequence	**Proportion of H's on recorded flips**
1	TTT	-
2	TTH	-
3	T*H***T**	0
4	*H***T**T	0
5	T*H***H**	1
6	*H***T**H	0
7	*H***HT**	0.5
8	*H***HH**	1
	Average	$\frac{2.5}{6} = \frac{5}{12} = 42\%$