Confirmatory Data Analysis

EES 4891/5891

Probability & Statistics for Geosciences

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Learning Goals

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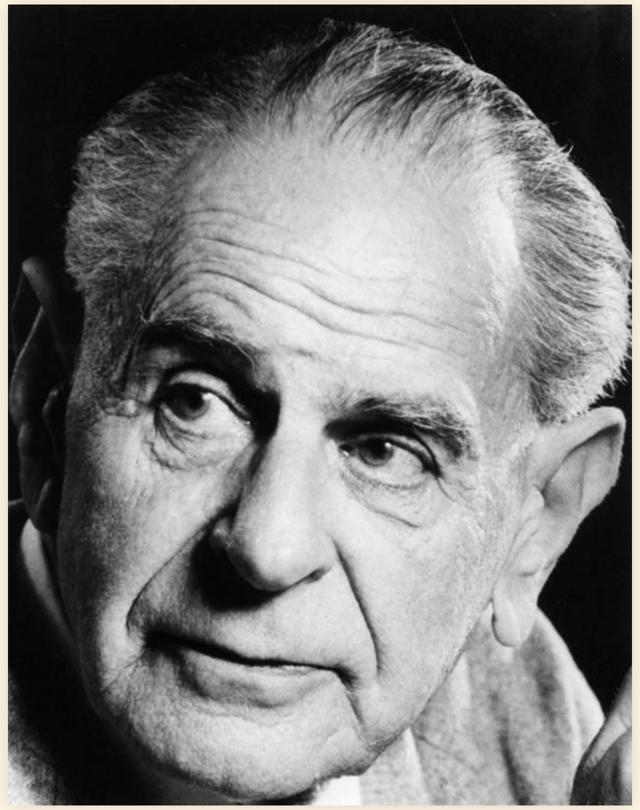
- Become familiar with *falsification* as a scientific method
- Understand what a *statistical null hypothesis* is
- Learn how to use a z-test to test hypotheses when the data is normal and you know the variance
- Learn how to use a t-test to test hypotheses when the data is normal and you don't know the variance
  - One-sample t-test to test hypotheses about one sample of data
  - Two-sample t-test to test whether two sets of data were sampled from the same distribution
- Learn the general structure and logic of testing a statistical hypothesis

Testing Hypotheses

Philosophical Background

- What make science scientific?
  - Demarcation Problem:
    - How can we distinguish science from non-science or pseudo-science?
  - How does a scientific method produce truth?
- David Hume (1711–1776)
  - You can’t rationally prove scientific principles from observations
- Karl Popper (1902–1994)
  - *The Logic of Scientific Discovery* (1934)
    - You can’t prove that a scientific principle is true, but you can prove that a false principle is false

David Hume



Karl Popper

Falsification and Scientific Method

- Falsification and scientific method:
  - You can't prove that a scientific principle is true, but you can prove that a false principle is false
  - A hypothesis is scientific if it allows us to make *risky predictions*
  - A prediction is *risky* if a small number of observations can prove it false
    - "All swans are white"
    - "This swan is black"
  - Every time you make a risky proposition and it comes true, you gain confidence in the hypothesis
  - We can never be certain that a hypothesis is true.
    - But we can have a lot of confidence if it survives a great deal of testing

Statistics and Hypothesis Testing

- Statistical tests focus on a *null hypothesis* H_0.
  - If data make us doubt the *null hypothesis*, then we become more confident that another hypothesis may be true.
- *Null hypothesis* should be the most conventional or boring idea.
  - "These two distributions are the same."
  - "Newton's laws of motion are correct."
- Statistical tests can't prove the null hypothesis is false
  - Why not?
  - But they can cast doubt on the null hypothesis

Which Tribe Arrived First?

Two Tribes

- Two tribes lived in an area for more than 1000 years
- There are disputes about which tribe arrived first
  - Hypothesis 1: Tribe A arrived in 622 CE
  - Hypothesis 2: Tribe B arrived in 615 CE
- Archaeologists ask you to use ^{14}C dating to estimate ages of wood artifacts from early settlements of both tribes

$$\begin{cases} \overline{t_A} = 650\text{ CE} \pm 50\text{y} \\ \overline{t_B} = 750\text{ CE} \pm 50\text{y} \end{cases} \quad (1\sigma)$$

- Three questions:
  1. How confident are you about each date (confidence or credible intervals)
  2. Are the observations compatible with the hypotheses? (hypothesis tests)
  3. How confident are you about which tribe got there first? (*p*-values)

Confidence Intervals

- Can we assume that our estimate of $\overline{t_A}$ is drawn from a normal distribution?
- Assume $\overline{t_A} \sim \mathcal{N}(\hat{\mu}_A, \sigma_A)$
  - Find t_{min} and t_{max} such that $\mathbb{P}(t_{\text{min}} \leq t_A \leq t_{\text{max}}) = 95\%$
- Transform to a z-score (standardize):

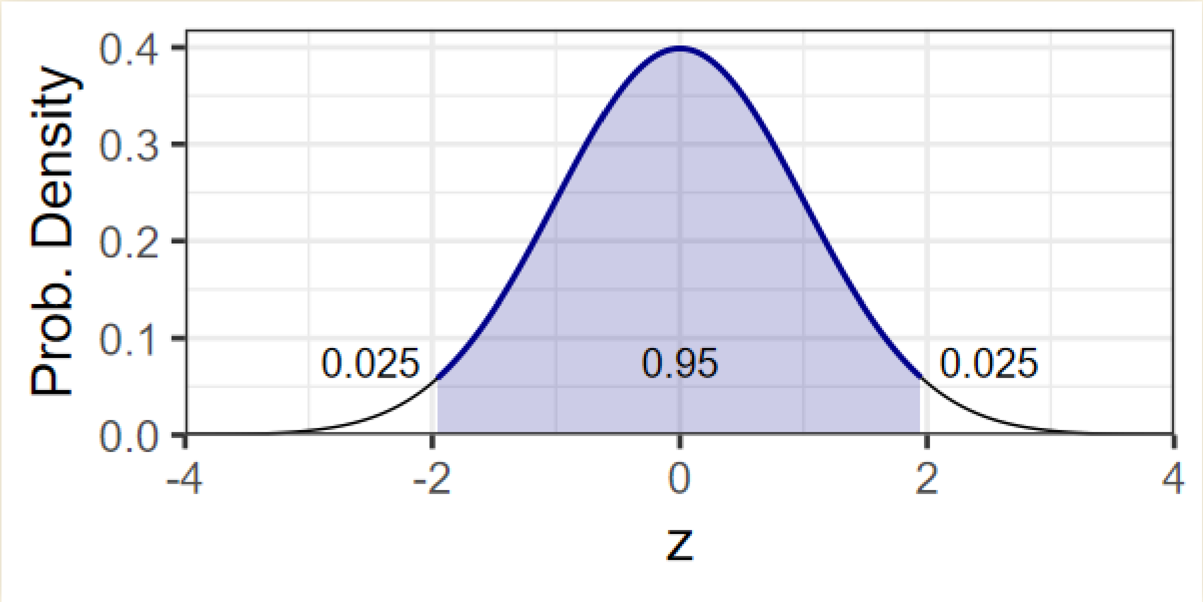
$$z_A = \frac{t_A - \hat{\mu}_A}{\sigma_A} \sim \mathcal{N}(0, 1)$$

- The normal distribution is symmetrical, so we can write our condition:

$$\mathbb{P}(z_{\text{min}} \leq z_A \leq z_{\text{max}}) = \mathbb{P}(|z_A| \leq z_\alpha)$$

- Find z_α such that $\mathbb{P}(|z_A| \leq z_\alpha) = (100 - \alpha)\%$

- For 95% confidence interval, $\alpha = 0.05$



$$\begin{aligned} z_\alpha &= \texttt{qnorm(0.975, 0, 1)} \\ &= \texttt{qnorm(0.025, 0, 1, FALSE)} \\ &= 1.96 \\ t_{\text{min}} &= \hat{\mu}_A - 1.96\sigma_A = 552 \\ t_{\text{max}} &= \hat{\mu}_A + 1.96\sigma_A = 748 \end{aligned}$$

- $\mathbb{P}(552 \leq t_A \leq 748) = 95\%$
- $\mathbb{P}(652 \leq t_B \leq 848) = 95\%$

Interpreting Confidence Intervals

- What does it mean to say:
  - $\mathbb{P}(552 \leq t_A \leq 748) = 95\%$
  - $\mathbb{P}(652 \leq t_B \leq 848) = 95\%$
- If you do the experiment many times, and calculate a 95% confidence interval for each,
  - then for 95% of the experiments, the true value of t_A will lie within the confidence interval for that experiment.
- It does not mean that *for your experiment*, there is a 95% probability that the true value of t_A lies within the interval.

Testing Hypotheses

Testing Hypotheses

- Assume t_A is normally distributed
- Null hypothesis: H_0: $\mu_A = 622$ CE

Known Variance: Z-test

- Assume we know the precision of the measurements

$$\sigma_A = \sigma_B = \sigma$$
$$z_A = \frac{t_A - \mu_A}{\sigma} = \frac{t_A - 622}{50}$$
$$z_A \sim \mathcal{N}(0, 1)$$

- Use the Z test, based on the cumulative probability of the normal distribution.

Unknown Variance: t-test

- We don't know the precision of the measurements
- Take n_A measurements and estimate precision from sample variance S^2
- Calculate the T statistic

$$\hat{T}_A = \frac{\overline{t_A} - \mu_A}{S_A / \sqrt{n_A}}$$

  - use Student's t-test, which is based on the T-distribution.

Known Variance: Z-test

- We know the precision of the measurements: $\sigma_A = \sigma_B = \sigma$

- Null hypothesis:
  - $H_{0,A}$: $\mu_A = 622$

$$z_A = \frac{t_A - \mu_A}{\sigma} = \frac{t_A - 622}{50}$$
$$z_A \sim \mathcal{N}(0, 1)$$
$$\begin{aligned}\mathbb{P}(t_A \geq 650) &= \mathbb{P}\left(z_A \geq \frac{650 - 622}{50}\right) \\ &= \mathbb{P}\left(z_A \geq 0.56\right) \\ &= 1 - \Phi(0.56) \approx 29\%\end{aligned}$$

  - There is a very good chance that we could measure a date of 650 CE or later if $\mu_A = 622$.

- Null Hypothesis:
  - $H_{0,B}$: $\mu_B = 615$

$$z_B = \frac{t_B - \mu_B}{\sigma} = \frac{t_B - 615}{50}$$
$$z_B \sim \mathcal{N}(0, 1)$$
$$\begin{aligned}\mathbb{P}(t_B \geq 750) &= \mathbb{P}\left(z_B \geq \frac{750 - 615}{50}\right) \\ &= \mathbb{P}\left(z_B \geq 2.7\right) \\ &= 1 - \Phi(2.7) \approx 0.35\%\end{aligned}$$

  - It is very unlikely that we'd measure a date of 750 CE or later if $\mu_B = 615$.

Unknown Variance: t-test

- Take n_A measurements of artifacts from tribe A, which give dates of $\{t_{A,1}, t_{A,2}, \ldots, t_{A,n_A}\}$
- Assume t_A are normally distributed: $t_A \sim \mathcal{N}(\mu_A, \sigma_A)$.
- The Central Limit Theorem tells us that

$$E(\overline{t_A}) = \mu_A$$
$$V(\overline{t_A}) = \frac{\sigma_A^2}{n_A},$$

but we don't know σ_A, so we estimate it from the sample variance:

$$S_A^2 = \frac{1}{n_A - 1} \sum_{i=1}^{n_A} \left(t_{A,i} - \overline{t_A}\right)^2$$

- $t_{A,i} - \overline{t_A}$ have independent values drawn from a normal distribution $\mathcal{N}(0, \sigma_A)$, so we can scale them:

$$\frac{1}{\sigma_A}(t_{A,i} - \overline{t_A})$$

will have unit variance.
- The quantity

$$(n-1)\frac{S^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{n} \left(t_i - \bar{t}\right)^2$$

is distributed according to the χ^2_ν, or *chi-squared* distribution for $\nu = n - 1$ degrees of freedom.

Chi-Squared Distribution

- If t_i are n independent normally distributed measurements with variance σ^2, then

$$(n-1)\frac{S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(t_i - \bar{t}\right)^2$$

  follows the χ^2_ν, or *chi-squared* distribution for $\nu = n - 1$ degrees of freedom.
- What are "degrees of freedom"?
  - The number of measurements that could change independently.
  - We lose one degree of freedom for each constraint on the data.
    - If the data have to average to $\bar{t}$, this removes one degree of freedom
    - If $n = 1$, then $\bar{t} = t_1$, so you can't change the measurement without changing $\bar{t}$ and you have zero degrees of freedom ($1 - 1 = 0$)
    - If $n = 2$, then you can change one variable, $t_1 \to t_1 + \delta$, but t_2 would have to make the opposite change $t_2 \to t_2 - \delta$, so there is 1 degree of freedom $(2 - 1 = 1)$.
    - For $n = 3$, if $t_1 \to t_1 + \delta_1$ and $t_2 \to t_2 + \delta_2$, then you have to have $t_3 \to t_3 - (\delta_1 + \delta_2)$. t_3 isn't independent from t_1 and t_2, so there are 2 degrees of freedom (3 - 1 = 2).

Student's t-test

One-Sample t-test

- Null hypothesis H_0: $\mu_A = 622$
- Alternate hypothesis H_a: $\mu_A > 622$
- One-sided, one-sample t-test:
  - T-statistic

$$\hat{T} = \frac{\overline{t_A} - \mu_A}{S_A/\sqrt{n_A}}$$

  - Compute $\mathbb{P}(t > \hat{T}) = 1 - F_{t_\nu}(\hat{T})$, where F_{t_ν} is the cumulative distribution function of the t-distribution for ν degrees of freedom.

- Suppose $\hat{T} = 1.94$.
  - If $n_A = 4$, $1 - F_{t_4}(1.94) = 13\%$, so we can't reject H_0
  - If $n_A = 12$, $1 - F_{t_{12}}(1.94) = 4\%$, so we reject H_0 at the 5% level.
- 4 measurements aren't enough to tell the difference between tribe A arriving at 622 versus 650 CE.
  - 12 measurements are sufficient to tell the difference, and confidently say that the tribe probably arrived after 622.

One-Sample *t*-Test in R

- Sample some data:

```
set.seed(2357)
t_A <- rnorm(4, 650, 50)
```

  - Run a t-test

```
t.test(t_A, mu = 622, alternative = "greater")
```

```
##
##  One Sample t-test
##
## data:  t_A
## t = -0.48173, df = 3, p-value = 0.6685
## alternative hypothesis: true mean is greater
than 622
## 95 percent confidence interval:
##  581.4812      Inf
## sample estimates:
## mean of x
##  615.1151
```

- Now try with 12 samples

```
t_A <- rnorm(12, 650, 50)
t.test(t_A, mu = 622, alternative = "greater")
```

```
##
##  One Sample t-test
##
## data:  t_A
## t = 2.7604, df = 11, p-value = 0.009271
## alternative hypothesis: true mean is greater
than 622
## 95 percent confidence interval:
##  636.7262      Inf
## sample estimates:
## mean of x
##  664.1454
```

Two-Sample t-Test

- Null hypothesis H_0: $\mu_A = \mu_B$
- Alternate hypothesis H_a: $\mu_B > \mu_A$
- One-sided two-sample t-test:
  - Compute the two-sample T-statistic

$$\hat{T} = \frac{\overline{t_B} - \overline{t_A}}{\sqrt{\frac{S_B^2}{n_B} + \frac{S_A^2}{n_A}}} \sim t_{\nu'}$$

    where ν' depends on what we know about whether t_A and t_B have the same variance.

  - R will calculate ν' so we don't have to worry about the formulas in the textbook

- Try it in R

```
t_B <- rnorm(9, 750, 50)
t.test(t_B, t_A, alternative = "less", var.equal =
         TRUE)
```

```
##
##  Two Sample t-test
##
## data:  t_B and t_A
## t = 3.074, df = 19, p-value = 0.9969
## alternative hypothesis: true difference in
means is less than 0
## 95 percent confidence interval:
##      -Inf 102.8415
## sample estimates:
## mean of x mean of y
##  729.9636  664.1454
```

The Logic of Statistical Tests

The Logic of Statistical Tests

- Five Steps:
  1. Identify the appropriate test and test statistic
     - e.g., t-test and T statistic
  2. Define the null hypothesis
     - e.g., H_0: $\mu_1 = \mu_2$
  3. Define an alternate hypothesis:
     - e.g., H_a: $\mu_1 > \mu_2$ (one-sided)
     - H_a: $\mu_1 \neq \mu_2$ (two-sided)
  4. Obtain the *null distribution*
     - Distribution of the test statistic if H_0 is true
  5. Compute p-value
     - Probability that you'd see values as extreme as the observed test statistic if H_0 is true
     - Compare to test level α
       - e.g., $\alpha = 0.05$
     - $p < \alpha$: Reject H_0 (guilty)
     - $p \geq \alpha$: Insufficient evidence to reject H_0 (not guilty $\neq$ innocent)