Probability Distributions

EES 4891/5891

Probability & Statistics for Geosciences

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Class #6: Thursday, January 23 2025

Learning Goals

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- Learn what probability distributions are
  - Continuous and discrete distributions
- Learn how we characterize probability distributions
  - Moments of a distribution
  - Graphical summaries
- Learn the difference between exploratory and confirmatory data analysis
- Learn practices for avoiding mistakes in statistical analysis
- Learn about exploratory data analysis
  - **Robustness** and **Resiliance**
- Learn about exploratory analysis of multivariate data
- Learn about pitfalls of relying too much on numerical summaries of statistical data

Homework

Homework

- I will post a homework assignment tomorrow
- It will cover data wrangling in R, focusing on selected exercises in *R for Data Science*.
- Start working on it before Tuesday, but it will not be due until the following week.
- Come to class Tuesday with questions about the exercises from the reading for Tuesday (these will be clearly indicated in the assignment)
    - We will discuss them and prepare you to compelete them after class.
- You do not need to do all the exercises in *R for Data Science*. Only the ones in the homework.

Probability Distributions

Probability Distributions

Discrete vs. Continuous

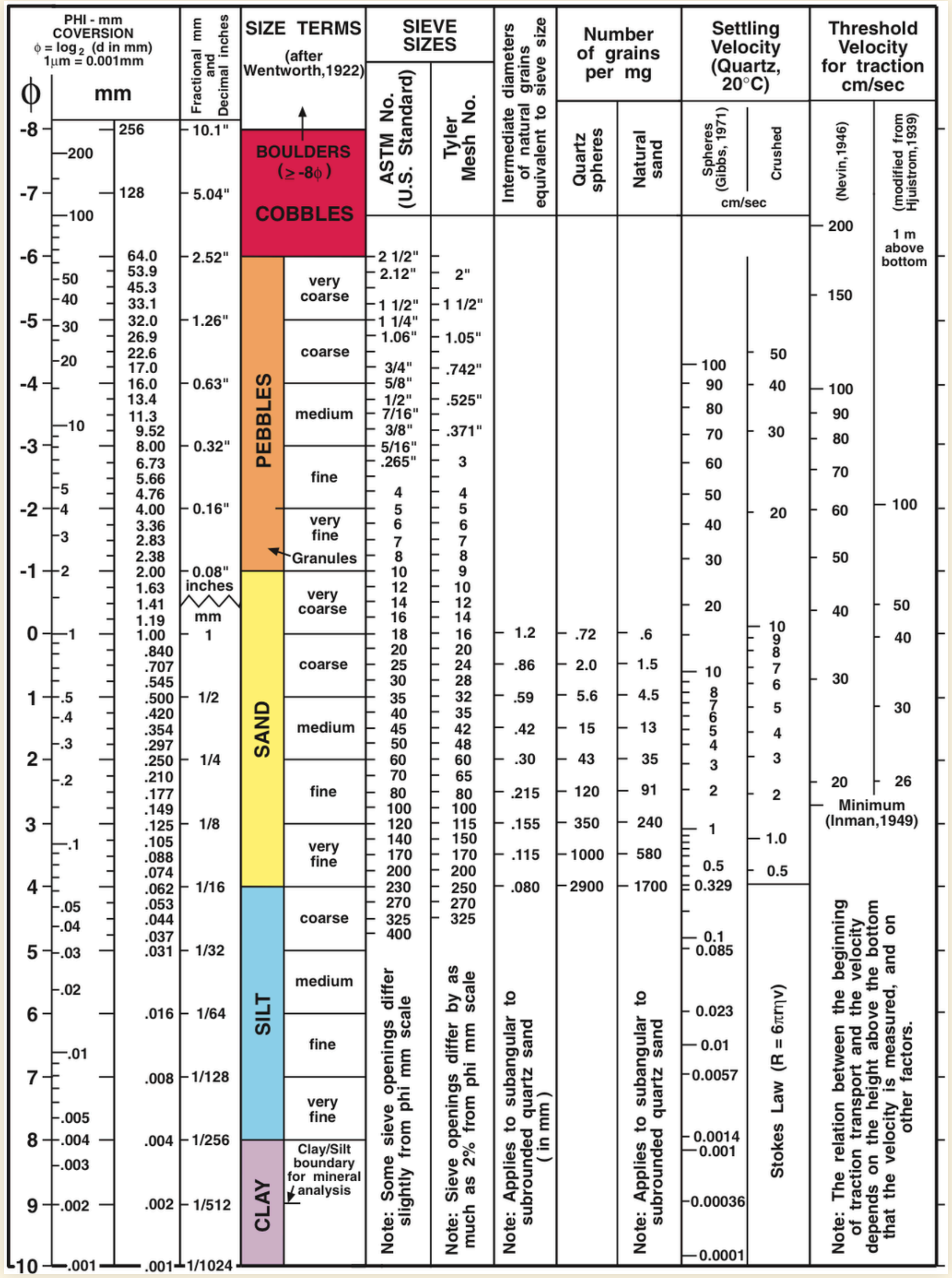
- Discrete:
  - Examples:
    - Counts: how many earthquakes happen in a year?
    - Discrete outcomes: Dice rolls, mineral types
    - Ranges: group grain sizes into ranges (Wentworth classes)
  - Distribution:
    - Probability Mass Function (PMF)

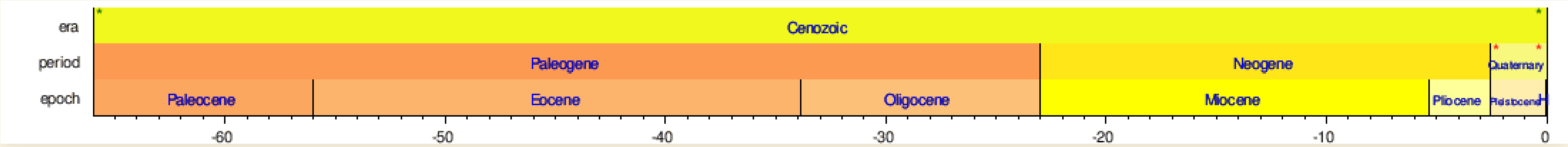
$$P(X = x_i)$$

- Continuous:
  - Examples:
    - Weights
    - Ages
    - Temperatures
  - Distributions:
    - Probability Distribution Function (PDF):

$$P(X = x)$$

Discrete Probability Ranges





Probability Distributions

Discrete vs. Continuous

- Discrete:
  - Probability Mass Function (PMF)

$$P(X = x_i)$$

  - Cumulative Distribution Function

$$F(x) = P(X \leq x) = \sum_{i=1}^{N} P(X = x_i),$$

where $x_i \leq x$ for $i = 1, \ldots, N$

  - Survival Function

$$P(X > x) = 1 - F(x)$$

- Contiunous:
  - Probability Distribution Function (PDF):

$$P(X = x)$$

  - Cumulative Distribution Function

$$F(x) = P(X \leq x) = \int_{z=-\infty}^{x} P(X = z)$$

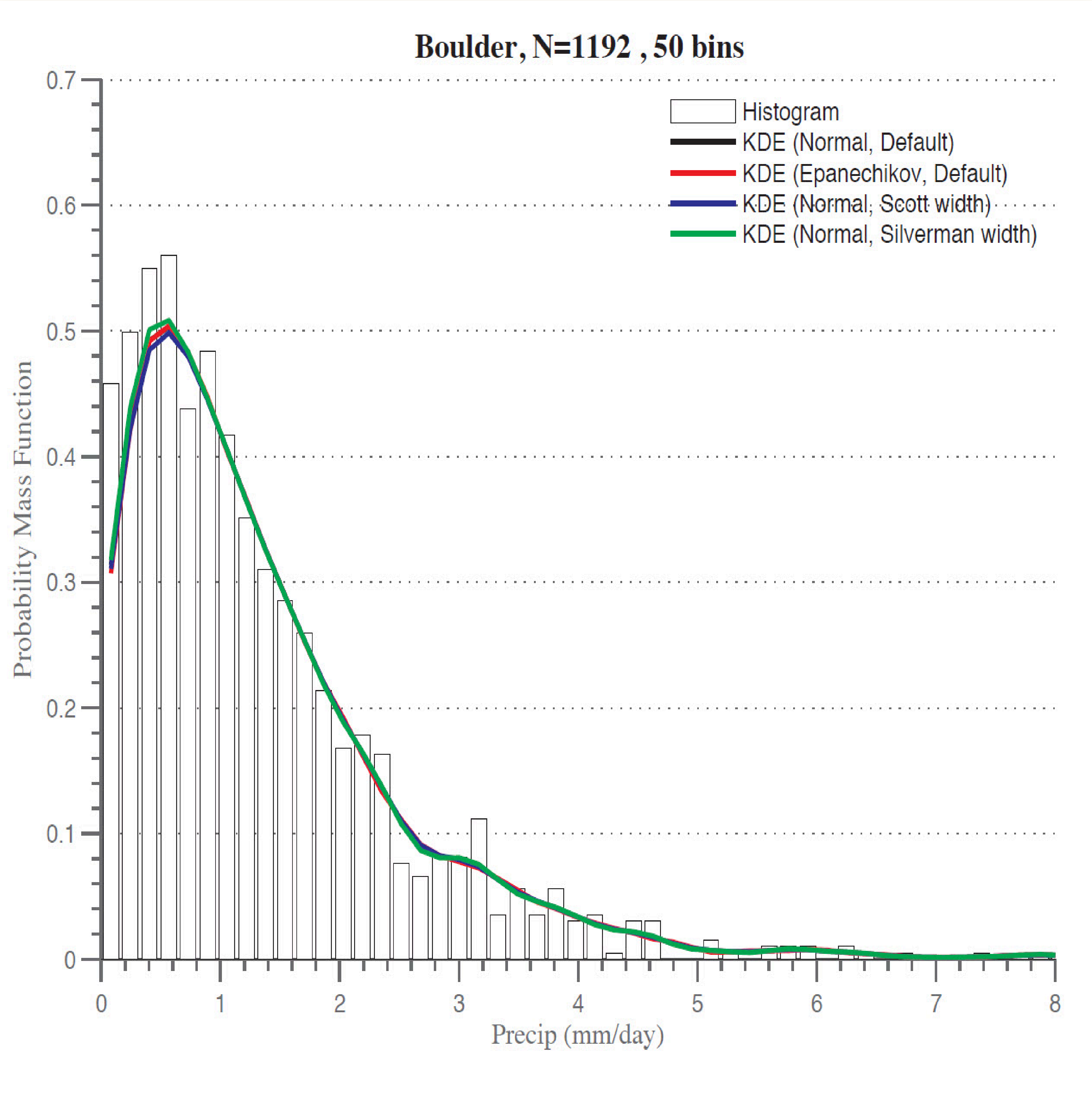
  - Survival Function

$$P(X > x) = 1 - F(x)$$

Empirical Probability Distributions

Empirical Probability Distributions

- Discrete PMF:
  - Histograms
- Continuous PDF:
  - Kernel-Density Estimation (KDE)
    - You choose the *kernel* function and the *bandwidth*.
      - It's usually fine to just use the defaults for the KDE function
        - There's not much difference
      - But you may want to manually adjust the *bandwidth* if your data set is unusual.
  - The figure shows four different kernel density estimates for the data, and they are all almost identical.



Describing an Empirical Distribution

- Quantiles and Percentiles
  - A *quantile* describes a value of x such that a certain fraction of the data x_i are less than x
    - **median:** what value has exactly 50% of the points below it?
    - **quartiles:** what values of x have exactly 25%, 50%, and 75% of the data below them?
    - **deciles:** 10%, 20%, … 90%
    - **percentiles:** 1%, 2%, …, 98%, 99%

Parametric Distributions

- Functions like the normal distribution, which are defined by parameters (mean, standard deviation), which we try to estimate from the data.
  - We’ll look at this in more detail in Chapter 5

Moments of a Distribution

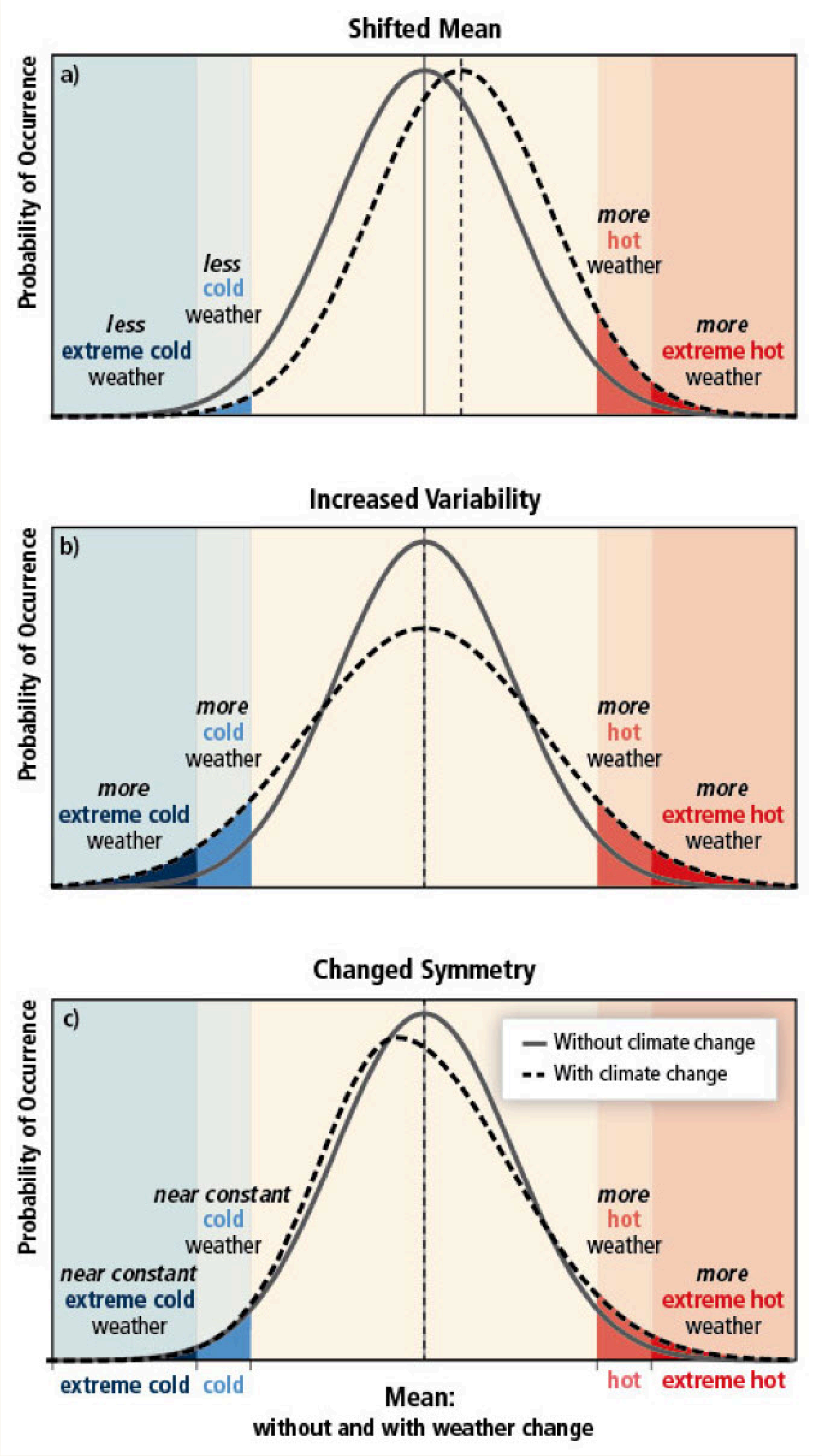
Moments of a Distribution

- **Expected Value:** On average, what value do we expect?

$$E(X) = \mu = \begin{cases} \sum_{i=1}^{N} x_i P(X = x_i) & \text{Discrete} \\ \int_{-\infty}^{\infty} x P(x) dx & \text{Continuous} \end{cases}$$

  - This is the *first moment* of P, and it's equivalent to the *mean* of P.
- **Higher moments:**
  - The second moment is the *variance* of P
  - The third moment is related to the *skewness*, or asymmetry, of P
  - The fourth moment is related to the *kurtosis*, or how sharply peaked, versus spread out, P is.
  - The textbook gives the mathematical formulas for a generic *n*-th order moment
  - If you know all the moments (up to $n = \infty$), you know everything about P.

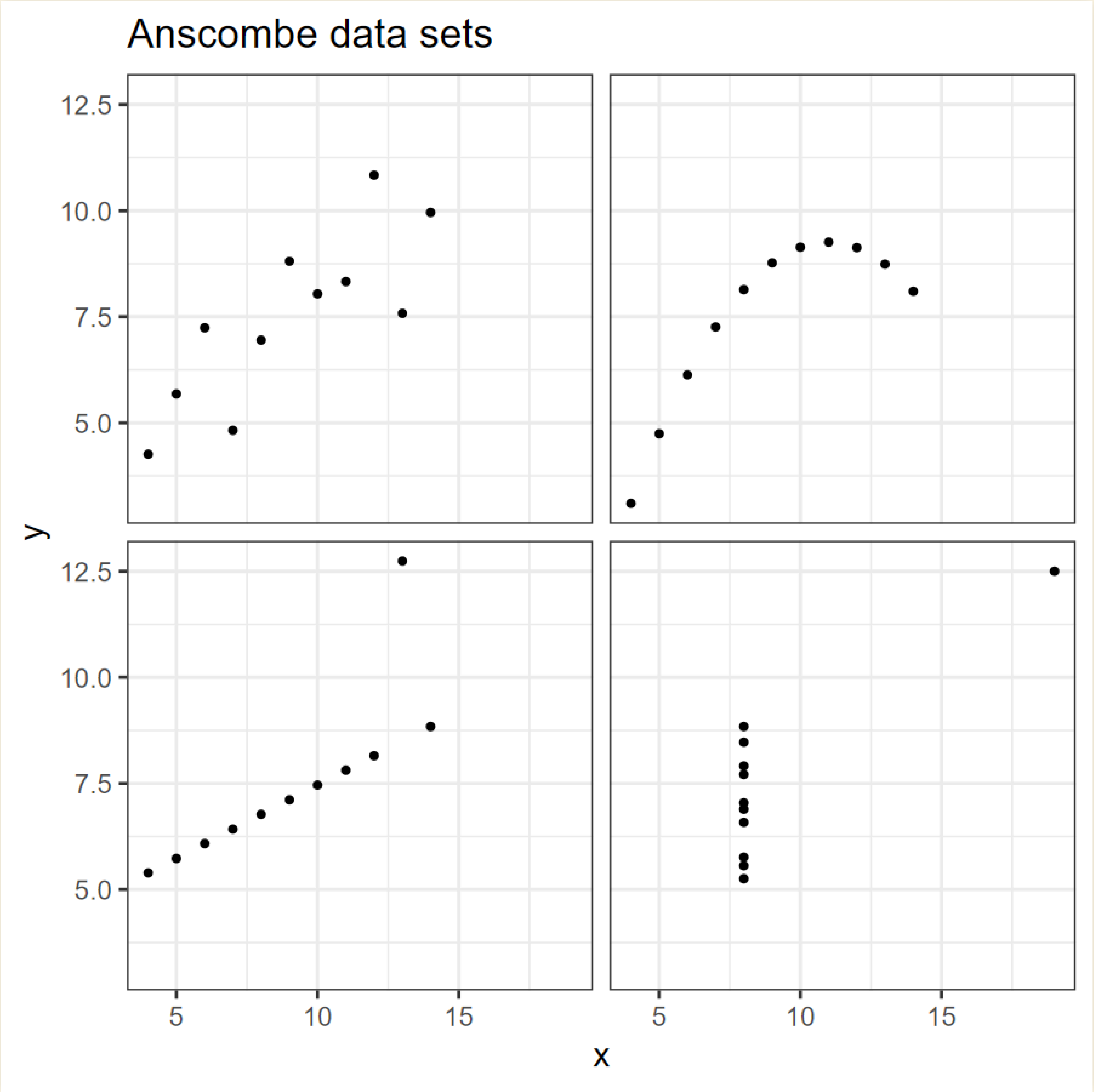
Practical Application: Climate Change



Caution

- Francis J. Anscombe data sets:
  - 4 sets of 11 data points in *x* and *y*
  - Identical means and standard deviations

set	mean_x	mean_y	sd_x	sd_y
1	9	7.501	3.317	2.032
2	9	7.501	3.317	2.032
3	9	7.500	3.317	2.030
4	9	7.501	3.317	2.031



Exploratory Data Analysis

Data Analysis

- **Exploratory data analysis:** examines data to try to find patterns and develop hypotheses about it.
- **Confirmatory data analysis:** examines data to test previously developed hypotheses.
- Scientific Process:
  - If you use the same data for *exploratory* and *confirmatory* analysis, you are likely to fool yourself and make errors.
    - **Overfitting data:** Your data may not be perfectly representative of all possible measurements. the hypothesis that best describes your data may not describe newer data very well.
  - It’s important to *test* hypotheses with different, independent data to the data you used to *generate* the hypothesis.

Avoiding Mistakes in Exploratory Analysis

- For *exploratory analysis*, we want to use methods that are *robust* and *resistant*
  - **robust** methods are not sensitive to assumptions about the data (e.g., assuming the data are normally distributed)
  - **resistant** methods are not sensitive to small numbers of unsual observations (outliers).
    - If there are 100 ordinary people in a coffee shop, and Elon Musk walks in, suddenly the *mean* wealth of people in the coffee shop is several billion dollars.
      - But the *median* wealth doesn’t change very much
    - The *median* is much more *resistant* than the *mean* for describing average properties of observations

Numerical Summaries

- **Range:** difference between largest and smallest
- **Location:**
  - *mean*
  - *median*: If the *mean* and *median* are different, the distribution must have *skewness*
  - Other measures *trimean*, *trimmed mean* are rarely used

Numerical Summaries

- **Spread:**
  - *standard deviation*

$$s = \sqrt{E((X - E(X))^2)} = \sqrt{E((X - \text{mean})^2)}$$

    - Not very *reistant*
    - The *variance* $v = s^2$
  - *interquartile range* $\mathrm{IQR} = q_{0.75} - q_{0.25}$ is much more *resistant* than standard deviation
    - Difference between 1st and 3rd quartiles (25th and 75th percentiles)
  - *mean absolute deviation* MAD is also *resisant*.

$$\mathrm{MAD} = E(|X - q_{0.50}|) = E(|X - \text{median}|)$$

Examples:

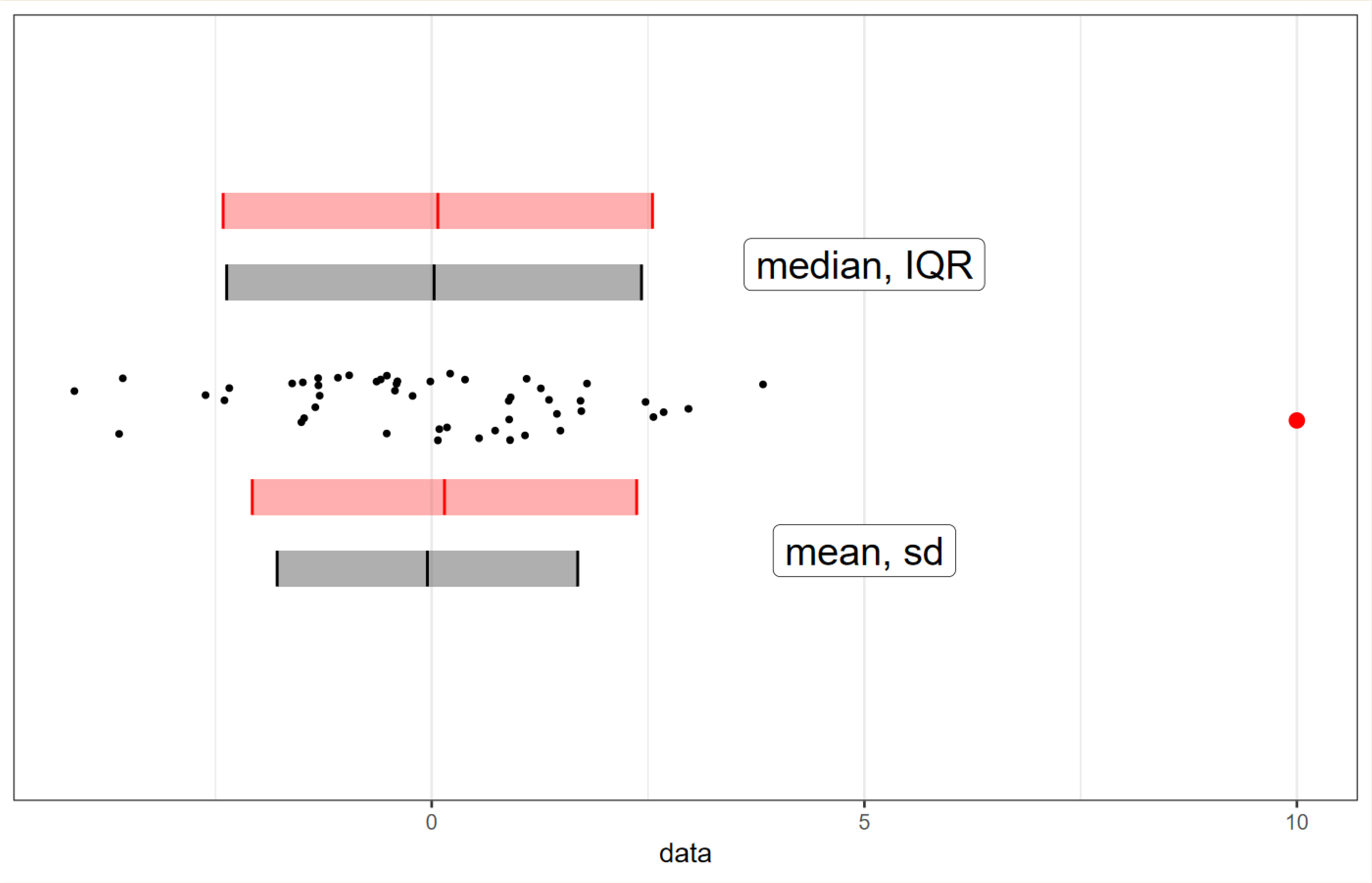
```
data = rnorm(50, 0, 2)
```

- Mean: `mean(data)` = -0.049
- Median: `median(data)` = 0.028
- Standard deviation: `sd(data)` = 1.737
- Interquartile range: `IQR(data)` = 2.398
- Mean absolute range: `mad(data)` = 1.896

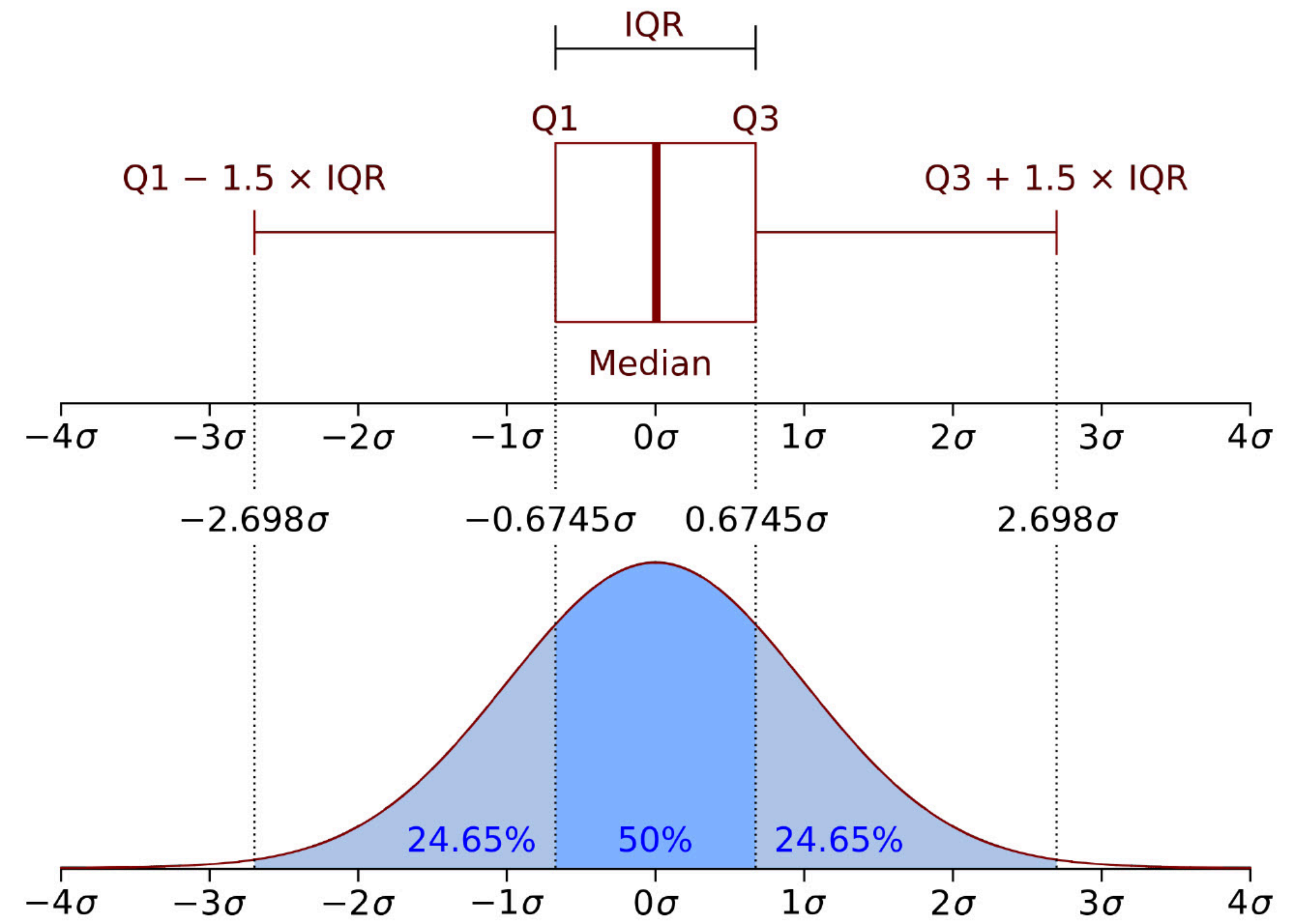
```
data_2 = c(data, 10)
```

- Mean: `mean(data_2)` = 0.148
- Median: `median(data_2)` = 0.071
- Standard deviation: `sd(data_2)` = 2.222
- Interquartile range: `IQR(data_2)` = 2.482
- Mean absolute range: `mad(data_2)` = 1.906

Example (cont.)



Graphical Summaries



Multivariate Data

Multivariate Data

- So far we've looked at observations of one variable at a time.
- What do we do if we measure several variables for each observation?
- Covariance:

$$\begin{aligned}\mathrm{Cov}(X, Y) &= E(\,(X - E(X))\,(Y - E(Y))\,) \\ &= E(XY) - E(X)E(Y)\end{aligned}$$

  - If there is no relationship between X and Y, $\mathrm{Cov}(X, Y) = 0$
  - Standardized data: mean = 0, sd = 1

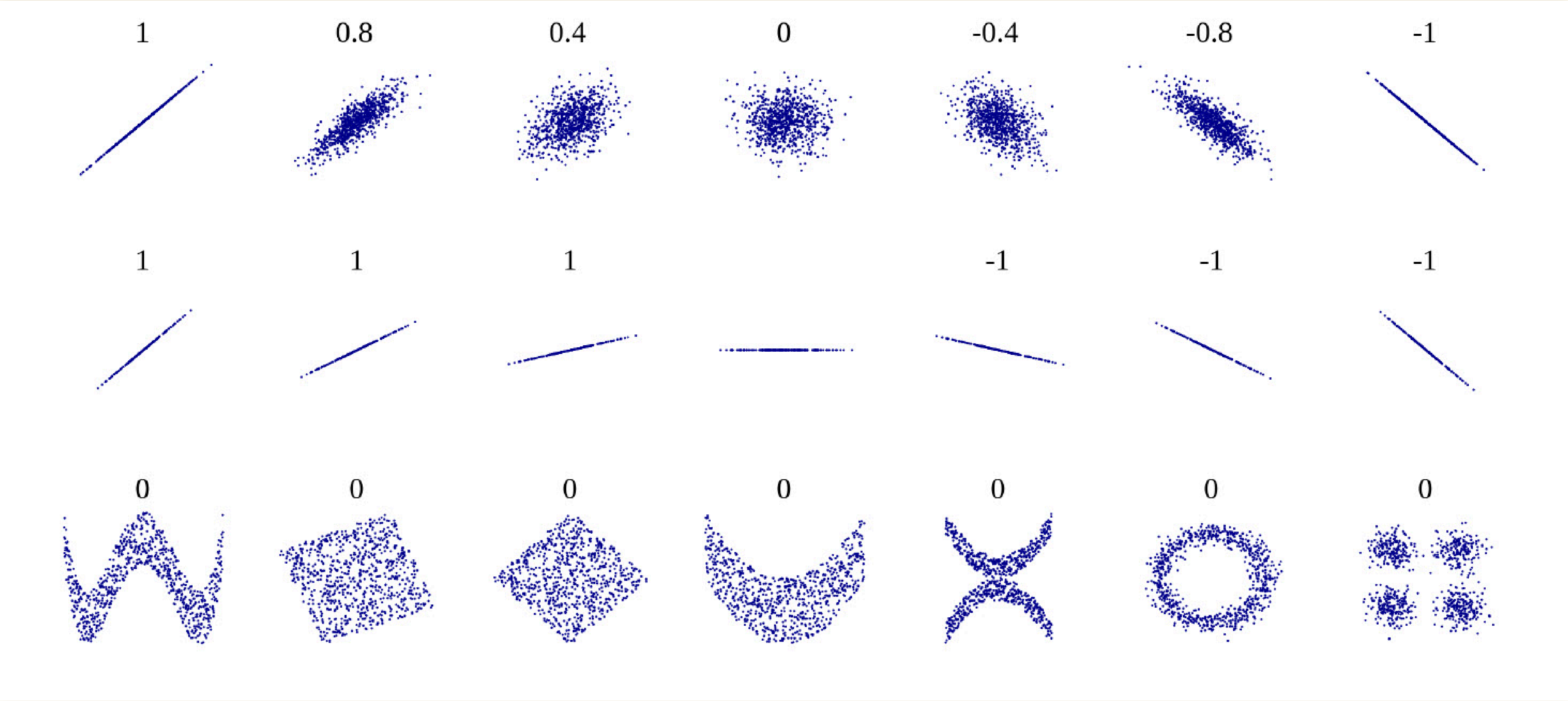
$$X_{\text{std}} = \frac{X - E(X)}{\sigma_X}$$

  - Correlation coefficient ρ is the covariance of standardized variables

$$\rho_{XY} = \mathrm{Cov}(X_{\text{std}}, Y_{\text{std}})$$

Correlation

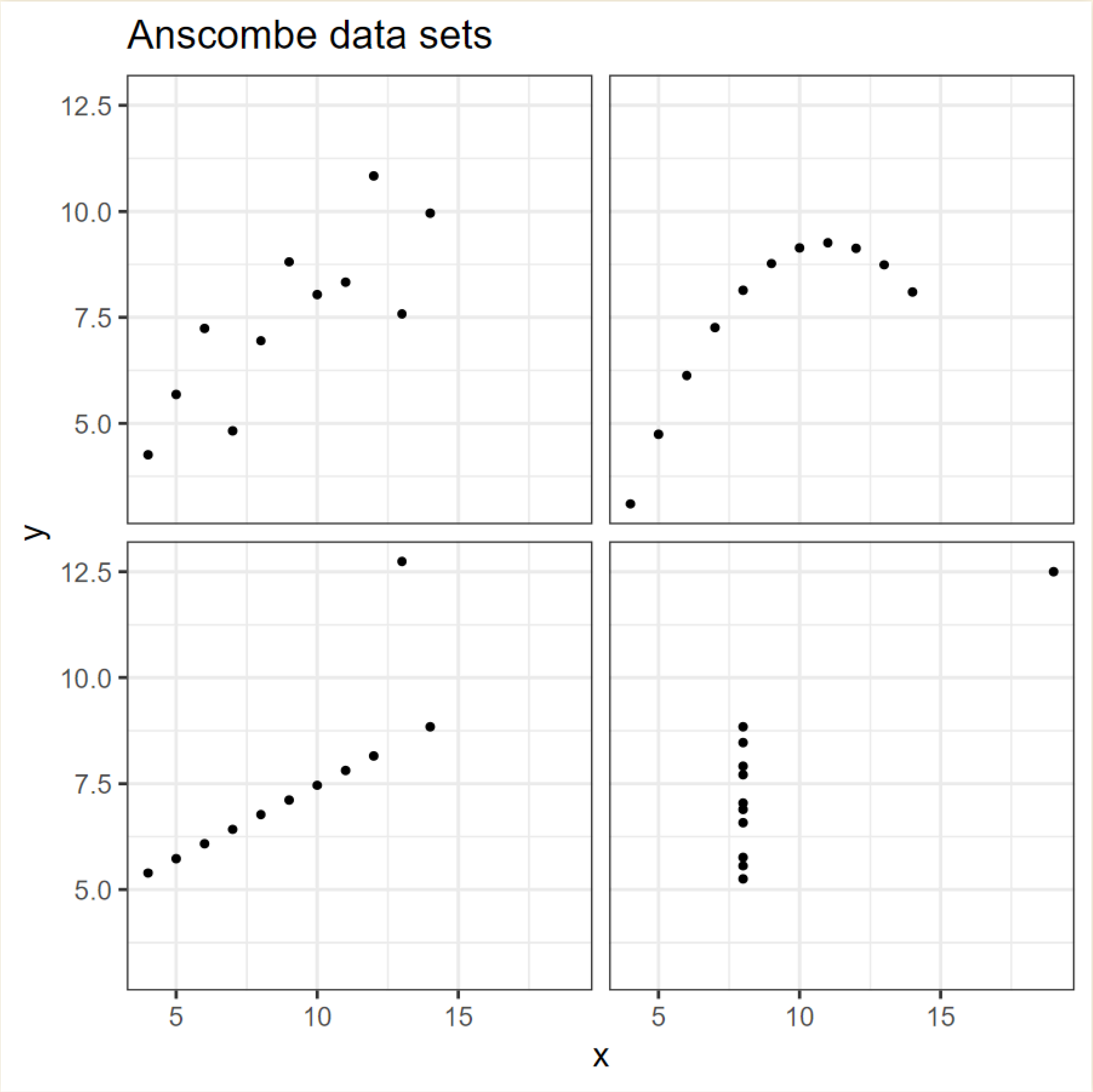
- $\rho_{XY} = 0$: There is no relationshp between X and Y
- $|\rho_{XY}| > 0$: The value of X tells us something about Y
- $|\rho_{XY}| = 1$: The values of X and Y lie exactly on a line



Caution

- Anscombe data

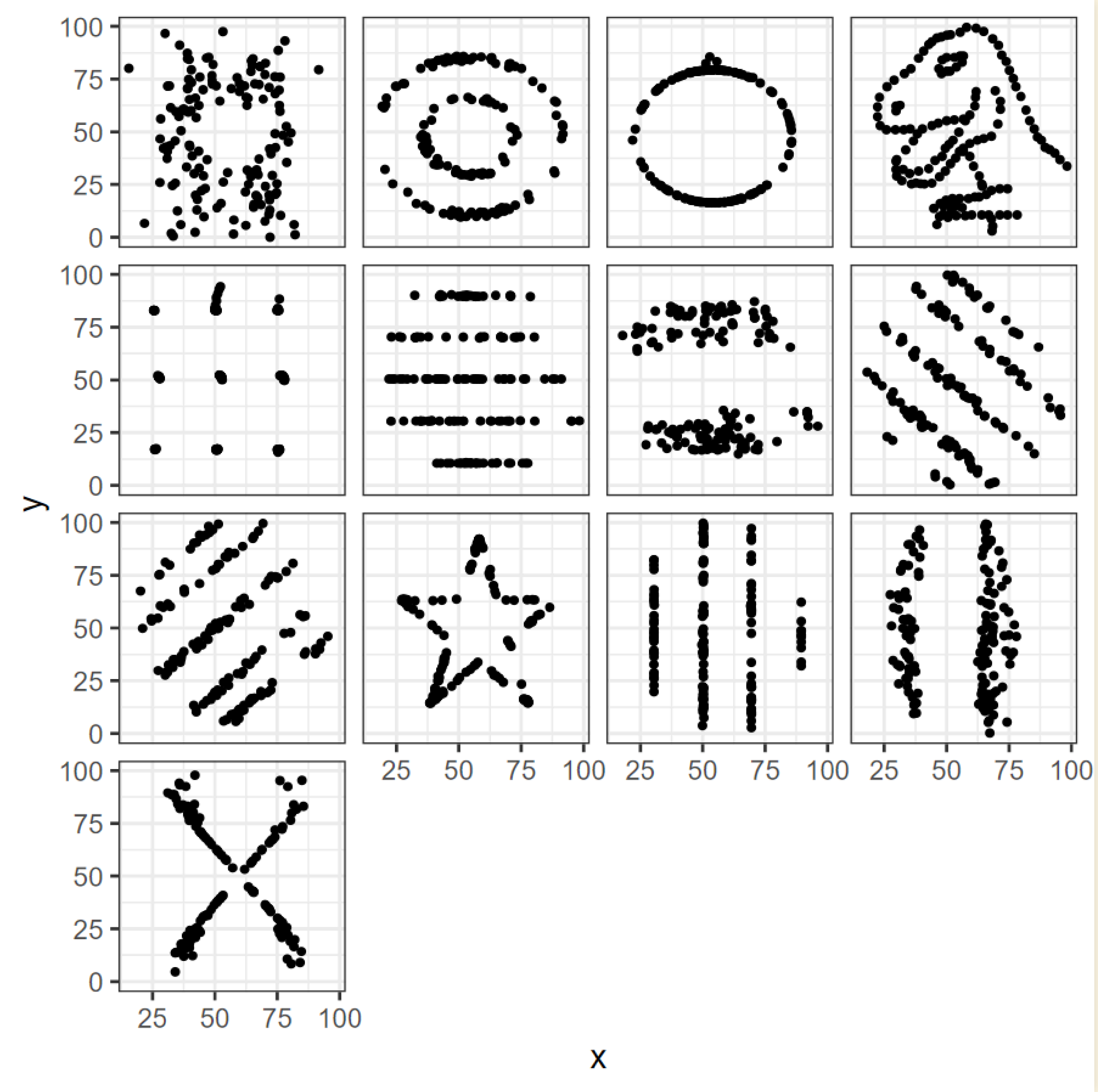
set	mean_x	mean_y	sd_x	sd_y	cor
1	9	7.501	3.317	2.032	0.816
2	9	7.501	3.317	2.032	0.816
3	9	7.500	3.317	2.030	0.816
4	9	7.501	3.317	2.031	0.817



More Fun with Correlations

- "Datasaurus" data set

dataset	mean_x	mean_y	sd_x	sd_y	cor
dino	54.263	47.832	16.765	26.935	-0.064
away	54.266	47.835	16.770	26.940	-0.064
h_lines	54.261	47.830	16.766	26.940	-0.062
v_lines	54.270	47.837	16.770	26.938	-0.069
x_shape	54.260	47.840	16.770	26.930	-0.066
star	54.267	47.840	16.769	26.930	-0.063
high_lines	54.269	47.835	16.767	26.940	-0.069
dots	54.260	47.840	16.768	26.930	-0.060
circle	54.267	47.838	16.760	26.930	-0.068
bullseye	54.269	47.831	16.769	26.936	-0.069
slant_up	54.266	47.831	16.769	26.939	-0.069
slant_down	54.268	47.836	16.767	26.936	-0.069
wide_lines	54.267	47.832	16.770	26.938	-0.067



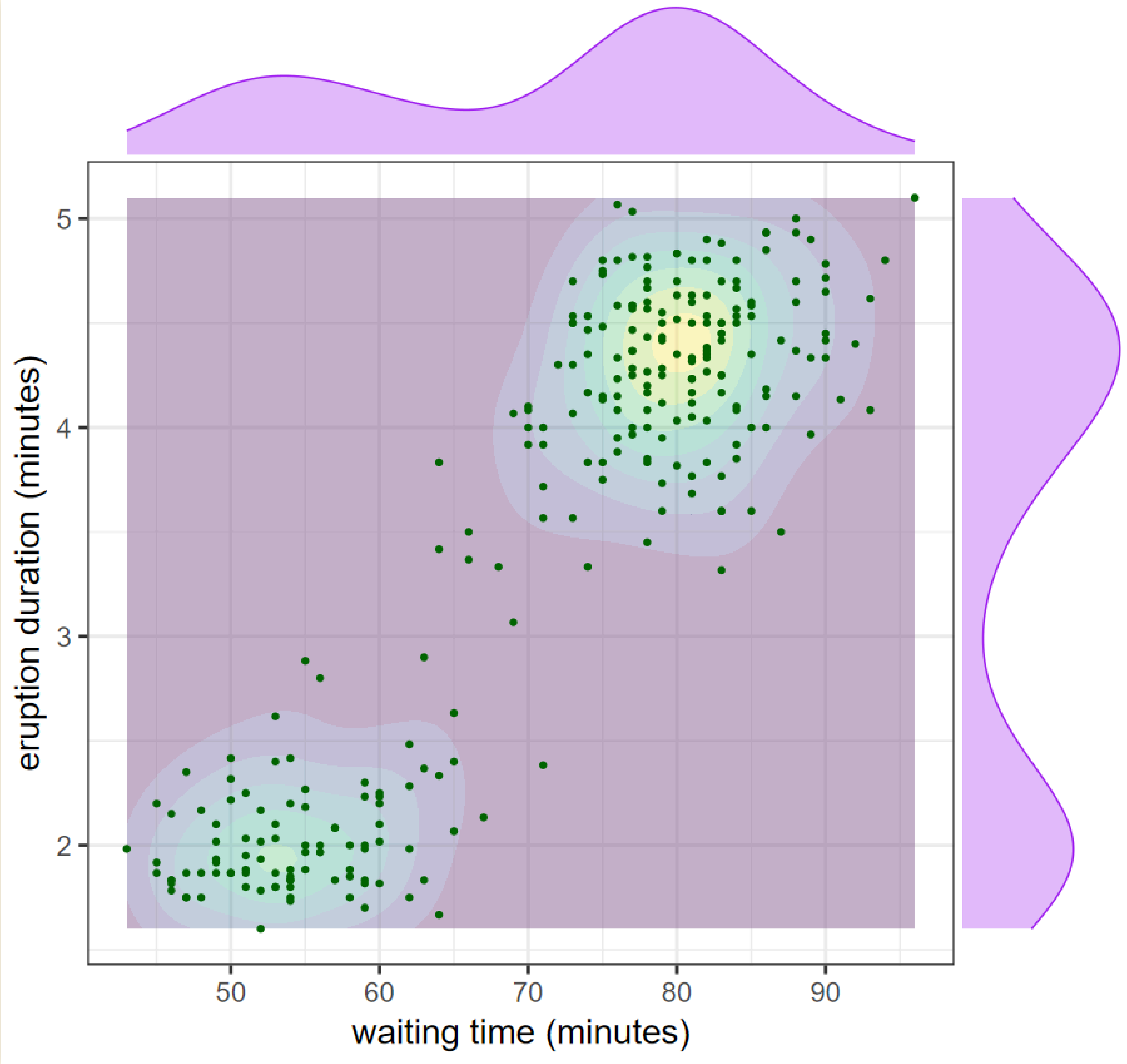
Lesson

- Numerical summaries are useful, but limited
- Summary graphs are better, but still limited
- **Always plot your data**

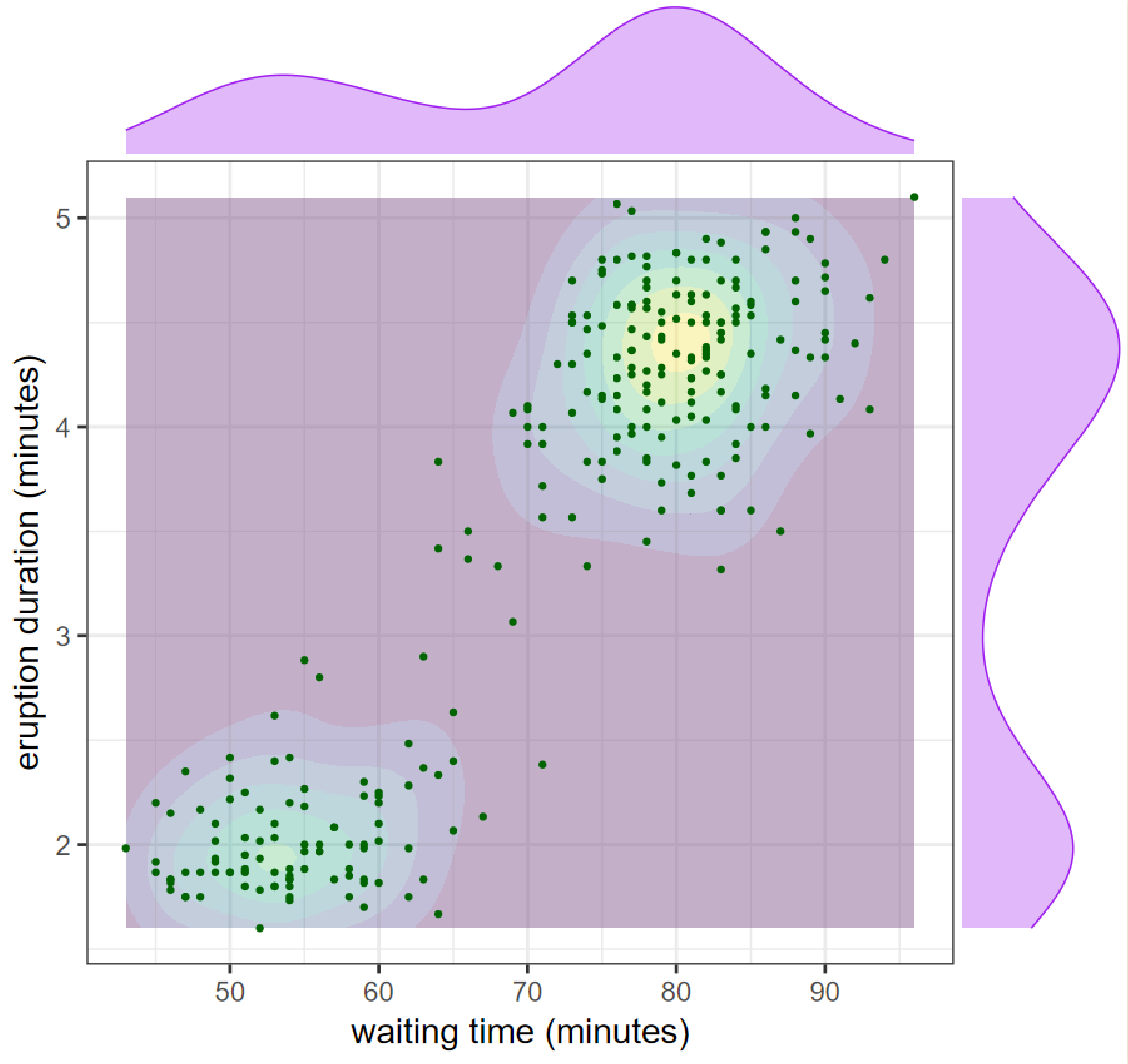
Bivariate Plots

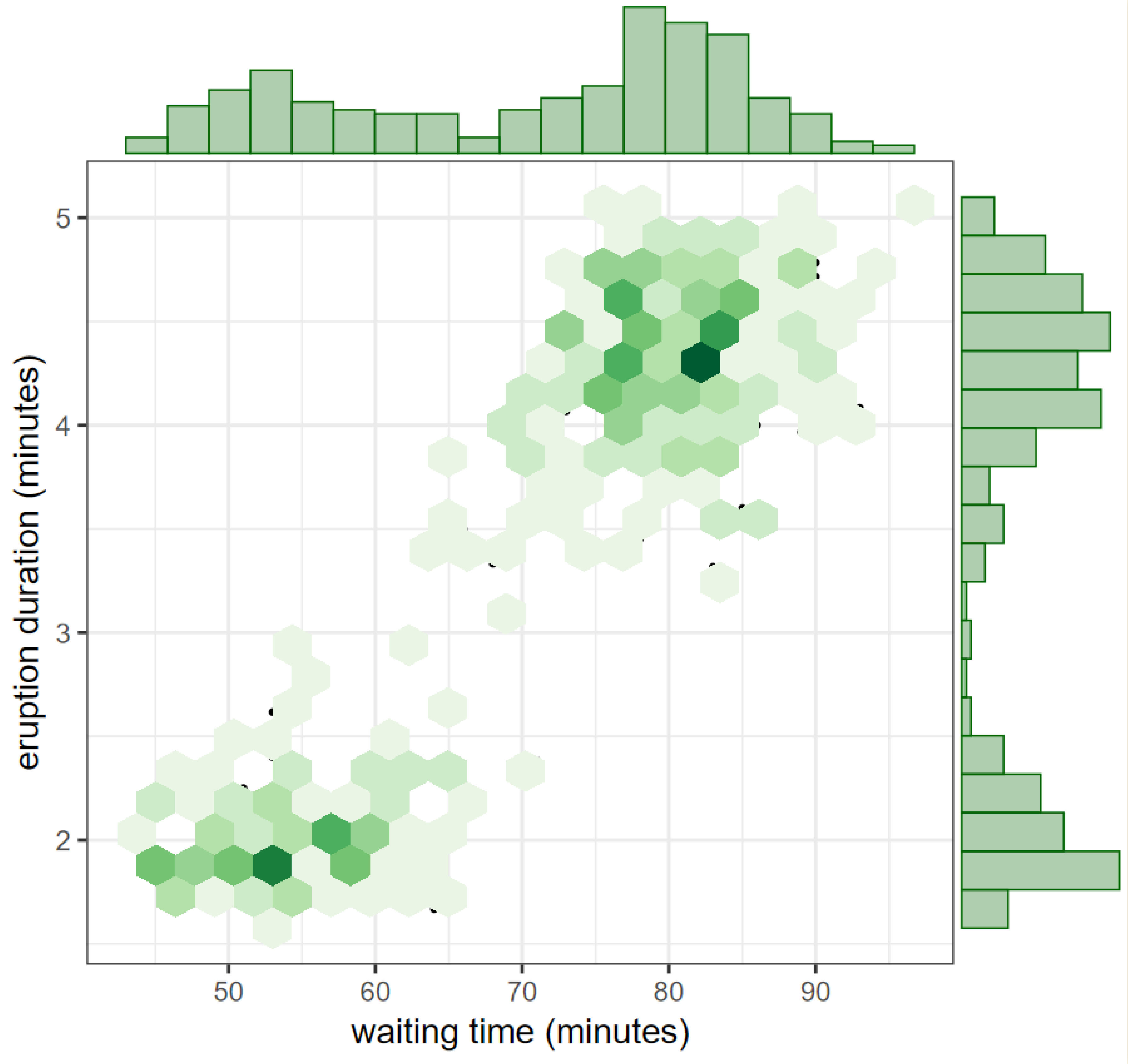
Scatterplots

- Old Faithful geyser in Yellowstone National Park is known for regular eruptions.
  - Plot the eruption duration versus the waiting time between eruptions



Bivariate Histograms





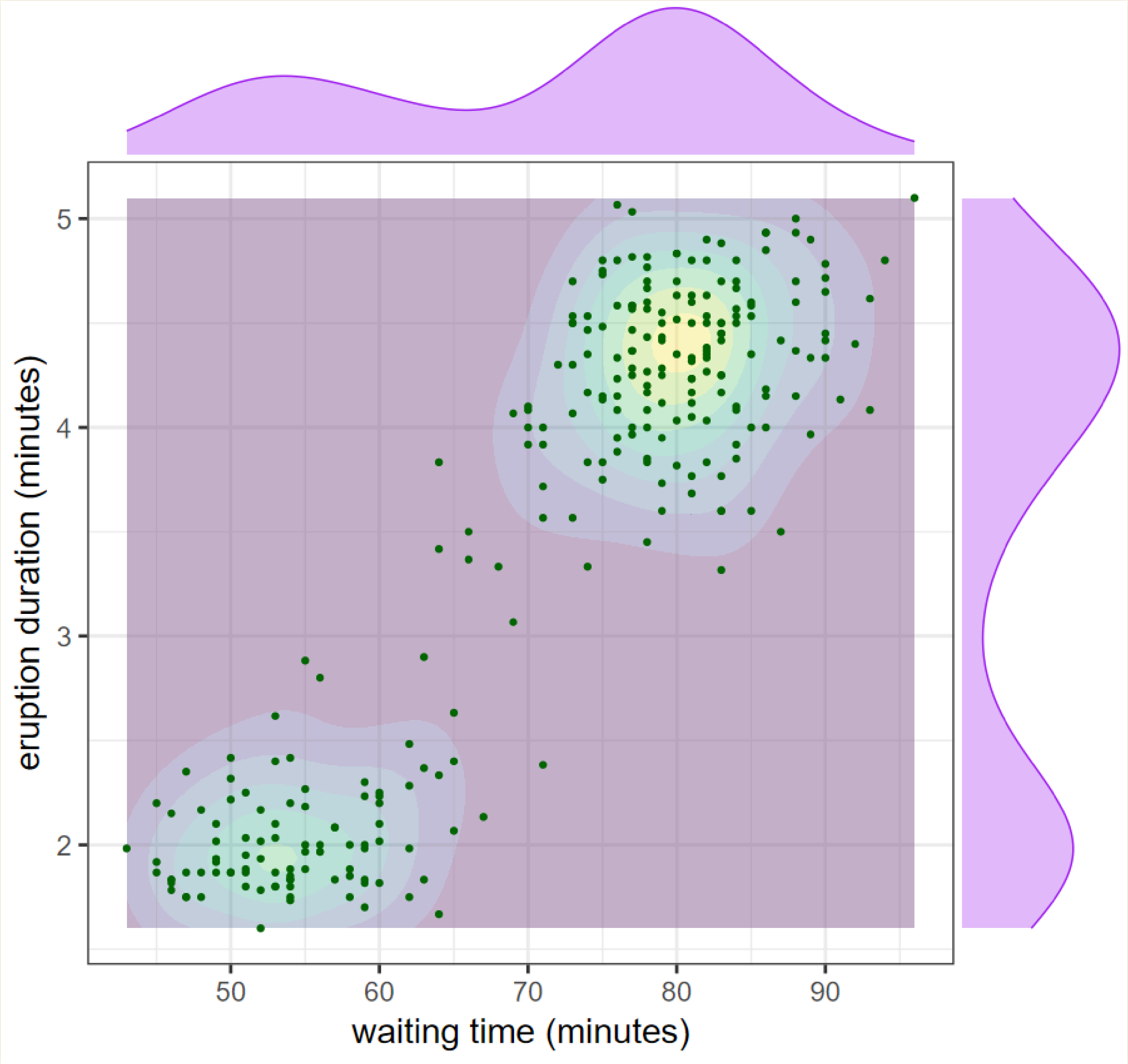
Bonus: How to Make the Plots

Scatterplot

```
p = ggplot(faithful,
           aes(x = waiting,
               y = eruptions)) +
  geom_contour_filled(data = faithfuld,
                      aes(z = density),
                      alpha = 0.3) +
  geom_point(color = "darkgreen") +
  guides(fill = "none") +
  labs(x = "waiting time (minutes)",
       y = "eruption duration (minutes)") +
  theme_bw(base_size = 20)

pm = ggMarginal(p, type = "density",
                color = "purple",
                fill = "purple",
                alpha = 0.3)

pm
```



Hexagon Plot

```
p2 = ggplot(faithful,
            aes(x = waiting,
                y = eruptions)) +
  geom_point() +
  geom_hex(bins = 20) +
  scale_fill_distiller(palette = "Greens",
                       direction = 1) +
  guides(fill = "none") +
  labs(x = "waiting time (minutes)",
       y = "eruption duration (minutes)") +
  theme_bw(base_size = 20)

p2m = ggMarginal(p2,
                 type = "histogram",
                 bins = 20,
                 color = "darkgreen",
                 fill = "darkgreen",
                 alpha = 0.3)

p2m
```