The Normal Distribution

EES 4891/5891

Probability & Statistics for Geosciences

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Learning Goals

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- Understand the basic properties of the normal distribution
- Understand the central limit theorem
  - Why do measurements of variables with non-normal distributions lead us to the normal distribution?
  - How can we use R to explore the central limit theorem?
  - What does this mean for estimating the true value of a variable from uncertain measurements?
- Know two other limit theorems:
  - Binomial to Poisson
  - Poisson to Normal

Historical Background

Carl Friedrich Gauss (1777–1855)

- Mathematician, astronomer, geographer, physicist
- 1823: Theory of errors
  - If you have several approximate measurements m_1, m_2, ..., m_n of a quantity v,
    - What is the best estimate of the true value of v?
    - The *arithmetic mean*

$$v_{\text{est}} = \frac{1}{n} \sum_{i=1}^{n} m_i$$

  - But this only works if the errors in m_i are *normally distributed*.

$$\mathcal{P}(m_i = x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-v)^2}{2\sigma^2}}$$

Portrait by Christian Albrecht Jensen, 1840. Public domain.

Pierre-Simon Laplace (1749–1827)

- Discovered the Central Limit Theorem (1810)
- Method of Least Squares
  - Find the best estimate of a quantity v based on a number of measurements m_i with errors
    - The best estimate minimizes the sum of the squares of the differences between the estimate and the measurements.

$$v_{\text{est}} \textbf{ minimizes } \sum_i (v_{\text{est}} - m_i)^2$$

  - History
    - Originated with Legendre (1805)
    - Developed by Gauss (1809)
    - Fully developed by Laplace (1810–11) using the Central Limit Theorem

Portrait by James Posselwhite. Public domain

The Normal Distribution

The Normal Distribution

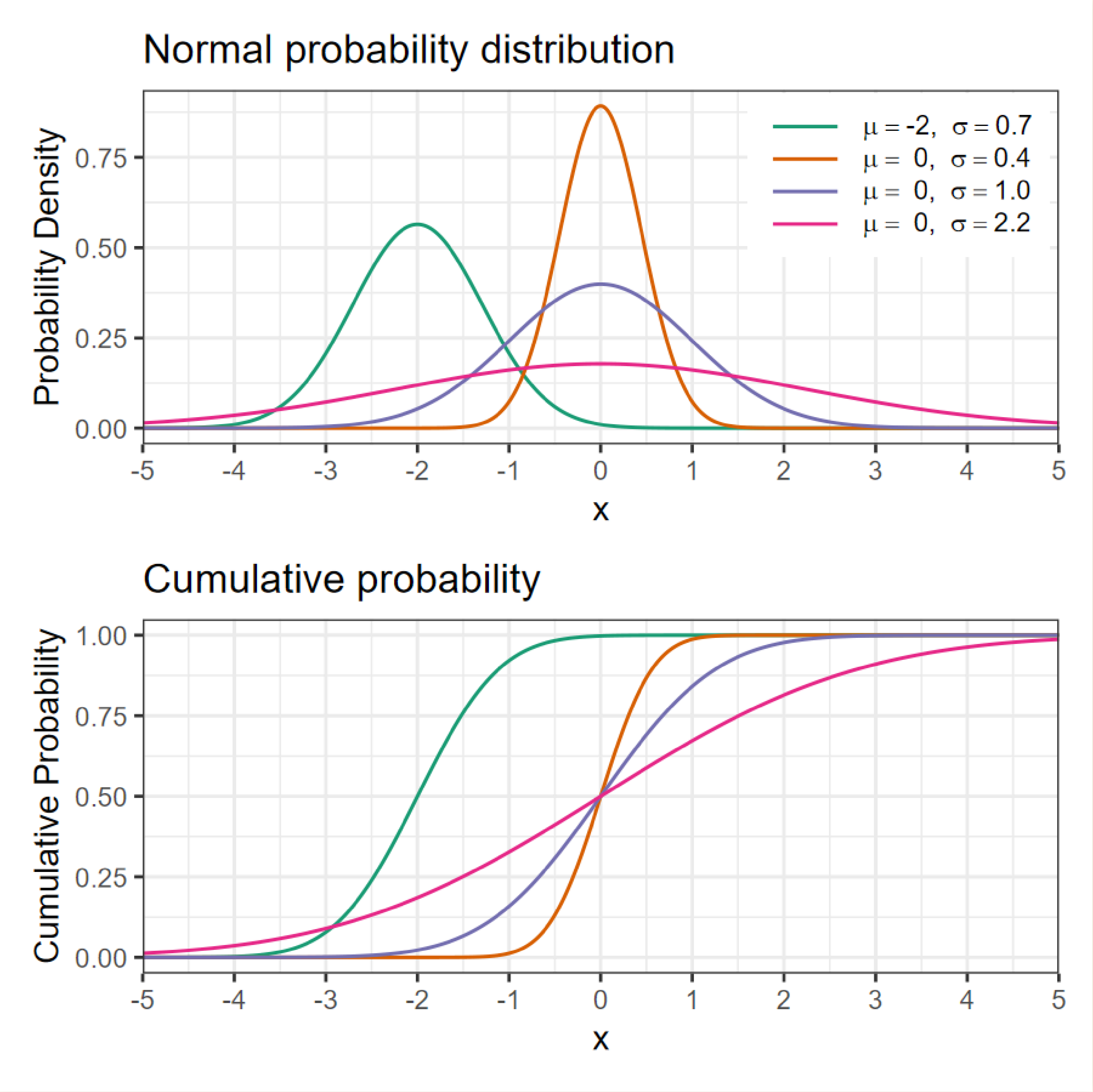
- Normal or Gaussian distribution

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

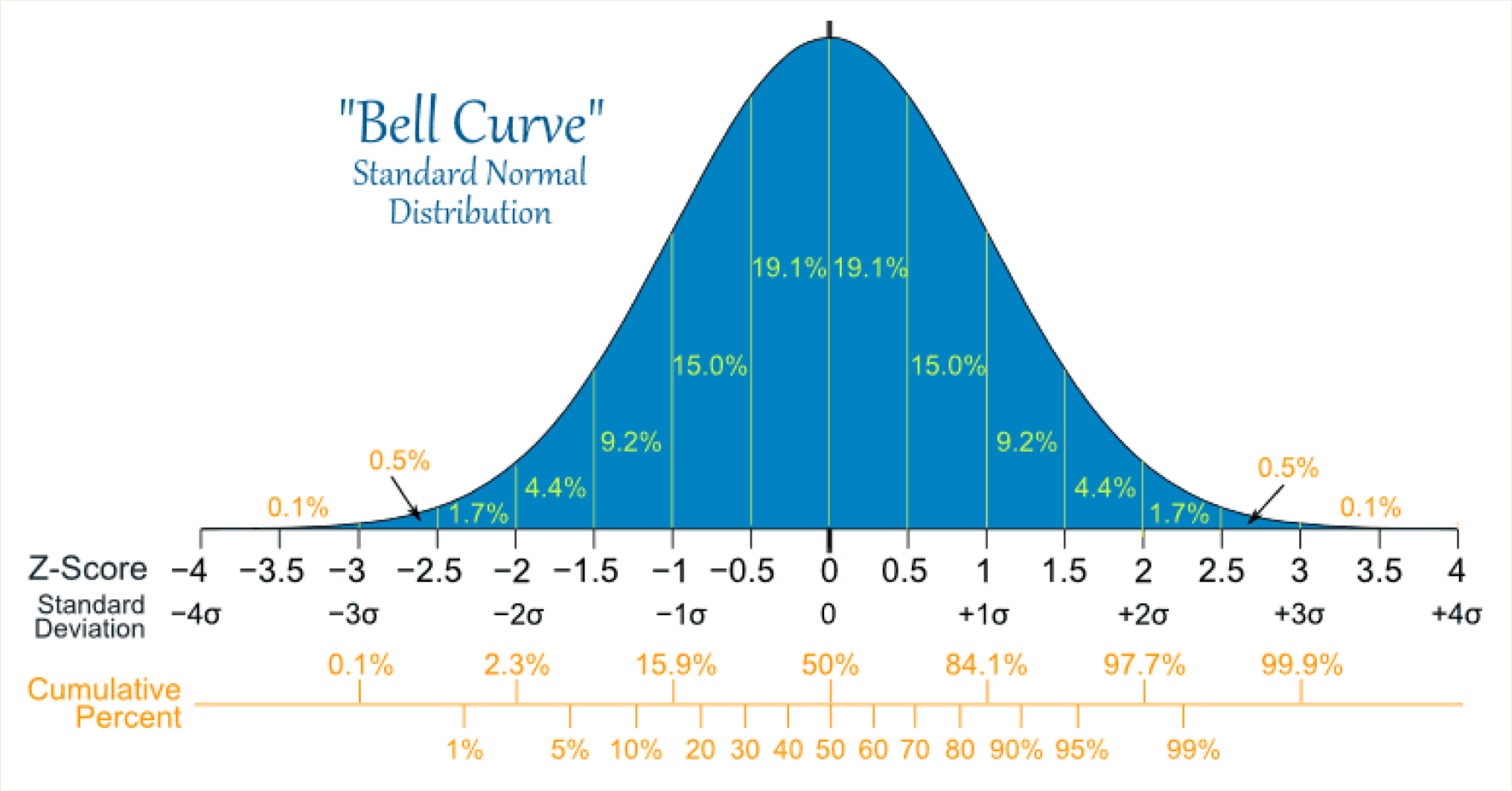
- Cumulative probability distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- R functions:
  - Probability distribution: `dnorm(x, mu, sigma)`
  - Cumulative probability: `pnorm(x, mu, sigma)`

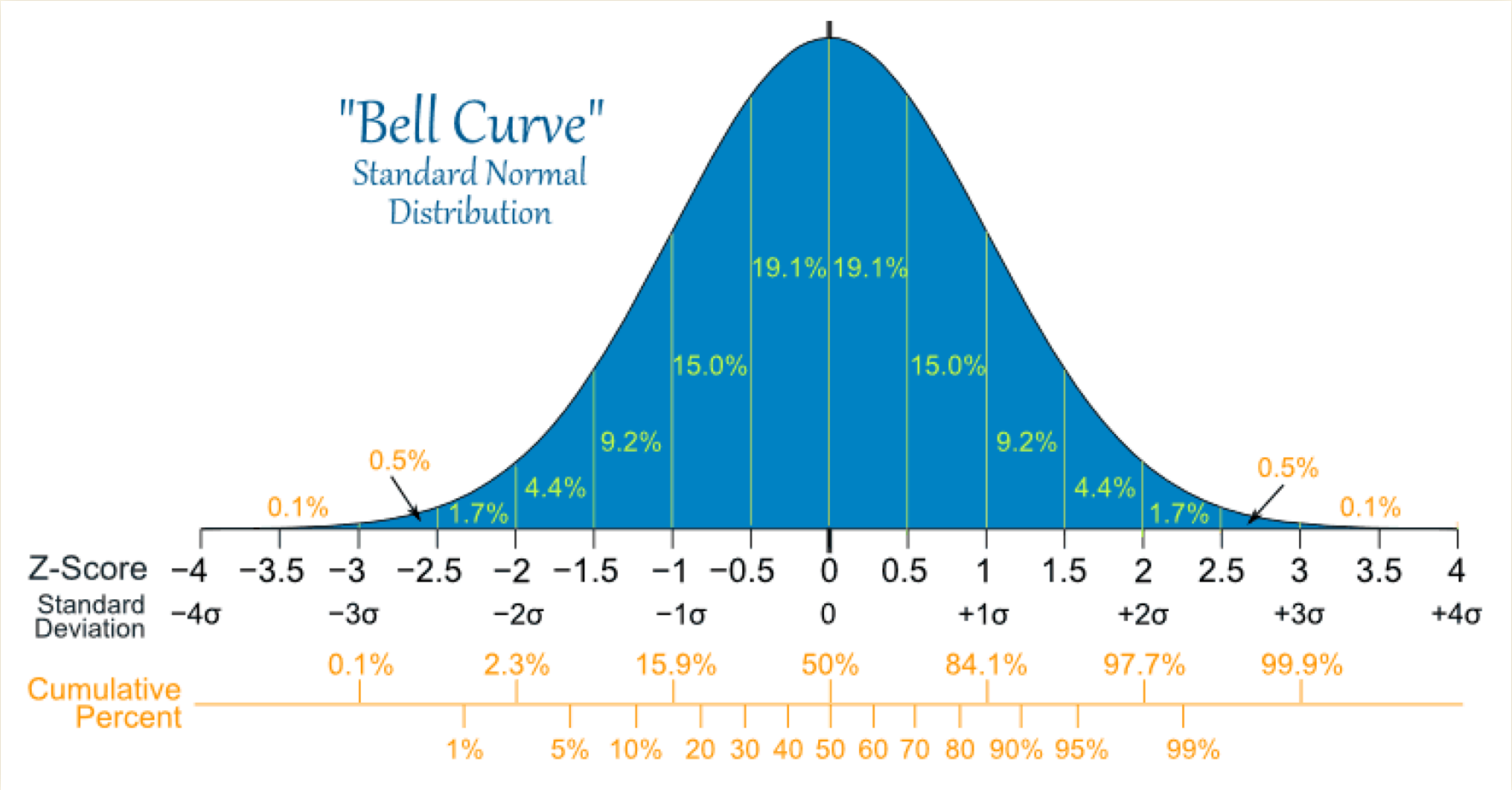


The Normal Distribution



Cumulative Probabilities

- Between $-\sigma$ and σ: 68.3%
  - Roughly $1/3$ of measurements will be outside $\pm 1\sigma$
- Between -2σ and 2σ: 95.4%
  - Roughly 5% of measurements will be outside $\pm 2\sigma$
- Between -3σ and 3σ: 99.7%
  - Roughly 0.3% of measurements will be outside $\pm 3\sigma$



Standardizing Data

Standardizing Data

- Consider $X = x_1, x_2, \ldots, x_N$
  - Standardized data: $X_{\text{std}} = \frac{X - \text{mean}(X)}{\text{sd}(X)}$
    - If X is described by a normal distribution ($X \sim \mathcal{N}(\mu, \sigma)$),
      - $X_{\text{std}}\ \mathcal{N}(0, 1)$
- The *standard normal distribution* is a normal distribution with $\mu = 0$ and $\sigma = 1$.

Moments of the Normal Distribution

- If $X \sim \mathcal{N}(\mu, \sigma)$
- First moment: mean $E(X) = \mu$
- Second moment: variance $E((X-\mu)^2) = \sigma^2$
- Third moment:
  - $E((X-\mu)^3) = 0$
  - $\mathcal{N}$ is unimodal (one peak) and symmetric, so the mean and median are the same and it has no skewness.
- Fourth moment: $E((X-\mu)^4) = 3\sigma^4$
  - *kurtosis*, measures how sharply peaked a distribution is:

$$\textbf{kurtosis} = \frac{E[(X-\mu)^4]}{(E[(X-\mu)^2])^2} = \frac{E[(X-\mu)^4]}{\sigma^4}$$

    - The normal distribution has *kurtosis* of 3 and we define *excess kurtosis* as

$$\textbf{kurtosis} - 3 = \frac{E[(X-\mu)^4]}{\sigma^4} - 3$$

    - Positive excess kurtosis: *leptokurtic*, more sharply peaked than a normal distribution
    - Negative excess kurtosis: *platykurtic*, flatter peak than normal
  - You don't need to memorize these

Stability of the Normal Distribution

- Scaling, adding, and subtracting normal distributions produces more normal distributions.
  - If $X \sim \mathcal{N}(\mu_x, \sigma_x)$ and $Y \sim \mathcal{N}(\mu_y, \sigma_y)$, then
    - $aX + b \sim \mathcal{N}(a\mu_x + b, a\sigma_x)$
    - $X + Y \sim \mathcal{N}(\mu_x + \mu_y, \sqrt{\sigma_x^2 + \sigma_y^2})$
    - $X - Y \sim \mathcal{N}(\mu_x - \mu_y, \sqrt{\sigma_x^2 + \sigma_y^2})$

Central Limit Theorem

Central Limit Theorem

- Consider a set of N independent and identically distributed random variables $X_1, X_2, \ldots, X_N$, with identical mean μ and variance σ^2.
  - The Xs are not necessarily normally distributed.
- Central Limit Theorem:

$$\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$$

As N gets larger, the distribution of $\overline{X}$ converges to a normal distribution

$$\overline{X} \sim \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{N}}\right)$$

- This applies to any distribution of X_i that has a finite mean μ and variance σ^2
- The standard deviation of $\overline{X}$ is $\sigma/\sqrt{N}$, so more measurements means less uncertainty.
- Usually 30 measurements is more than sufficient to guarantee that the average is normally distributed.

Explore Normal Distributions in R

Set Things Up

Working with the Samples

- In RStudio, type the following in the console:

```
library(tidyverse)
mu <- 5
x <- map(1:n_rep, \(x) rnorm(N, mu, sigma))
```

- Pick up where we left off…
- We want to figure out how the average of N variables is distributed.
  - To figure out a distribution, we need to repeat taking the average of N variables many times.
  - We're going to repeat the samples 1000 times, and refer to these repetitions as "replicates".

```
n_rep <- 1000
```

- Create 1000 replicates, each of which contains N samples from a distribution.

```
x <- map(1:n_rep, \(x) rnorm(N, mu, sigma))
```

- `map` a function for each value in a list or sequence, and returns a list of results.
  - `\(x)` tells R that what follows is an *anonymous function*, where `x` is the name of the argument.

```
map(1:4, \(x) c(x, x^2, x^3))
```

```
## [[1]]
##
## [[2]]
## [1] 2 4 8
##
## [[3]]
## [1] 3 9 27
##
## [[4]]
## [1] 4 16 64
```

- `x` is a list of 1000 replicates, each a vector of 30 samples from the normal distribution.
- Average across each of the 1000 replicates:

```
x_bar <- map(x, \(x) mean(x))
```

- `x_bar` is a list of 1000 numbers, each of which is the mean of the 30 samples in that replicate.
- We could also say `x_bar <- map(x, mean)`.
  - `map` can take an *anonymous function*, or the name of a function with a single argument.
  - We used an *anonymous function* for `rnorm` to provide arguments `N`, `mu`, and `sigma`.
- We want to calculate the mean and standard deviation of the 1000 replicates.
  - We can't run `mean()` or `sd()` on a list, so we want to convert the list of 1000 numbers and turn it into a *vector*. There are two ways to do this:

```
x_bar <- unlist(x_bar)
```

or

```
x_bar <- map_dbl(x, mean)
```

  - `map_dbl()` is like `map()`, but it assumes that the formula returns a single floating point number (the `dbl` stands for "double-precision", which is the way the computer represents floating point numbers)

Properties of the averages

- Now, we're ready to take the mean and standard deviation:

```
print(str_c("Mean = ", mean(x_bar), " and std. dev. = ", sd(x_bar)))
```

```
## [1] "Mean = 5.00654847532592 and std. dev. = 0.368562825499823"
```

```
print(str_c("mu = ", mu, ", sigma = ", sigma,
            ", sigma / sqrt(N) = ", sigma / sqrt(N)))
```

```
## [1] "mu = 5, sigma = 2, sigma / sqrt(N) = 0.365148371670111"
```

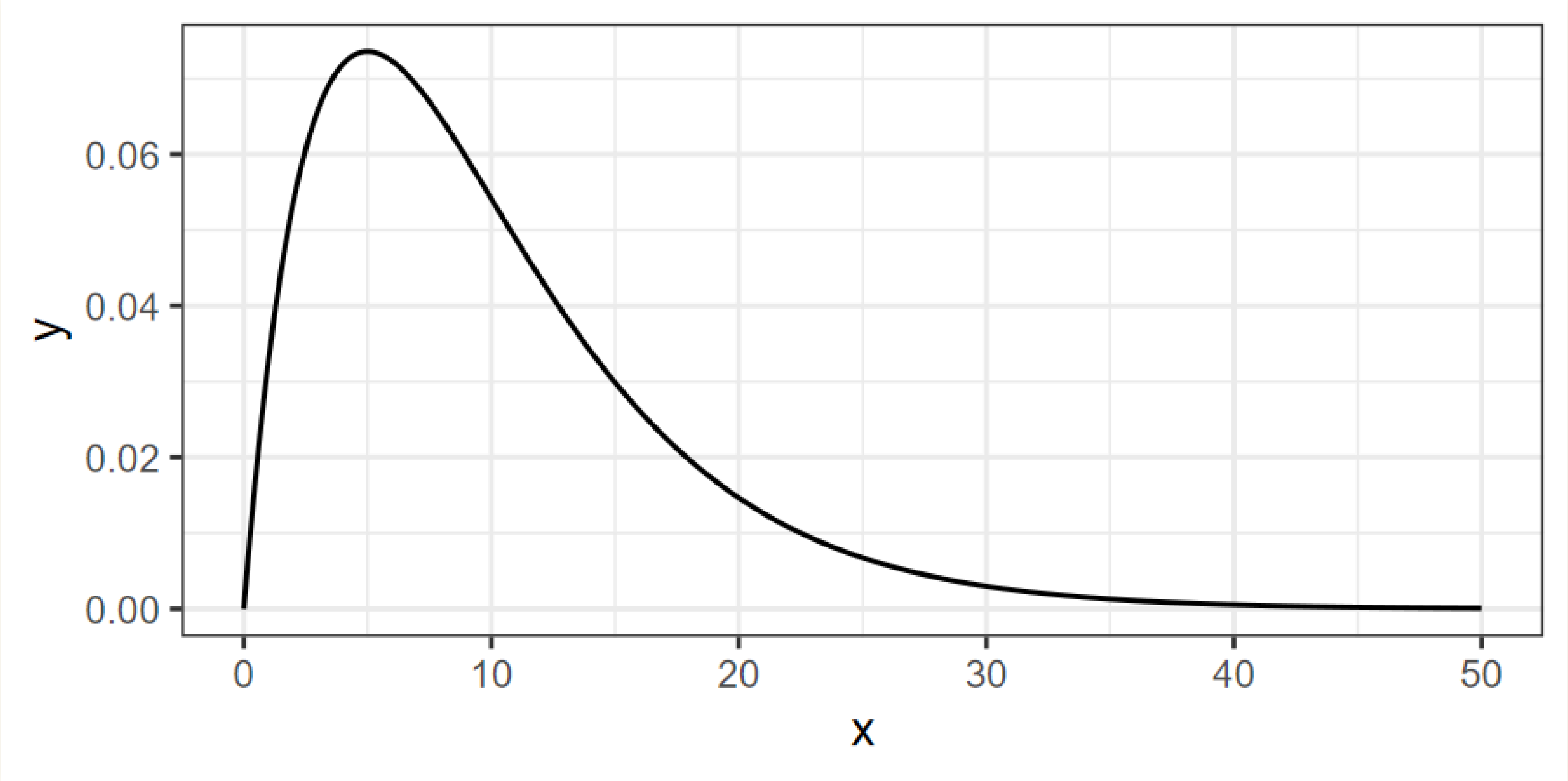
What about Gamma distribution?

- Set up:

```
k <- 2
theta <- 5
```

- Plot the PDF:

```
df <- tibble(x = seq(0, 50, 0.1),
             y = dgamma(x, shape = k, scale = theta))
ggplot(df, aes(x = x, y = y)) + geom_line(linewidth =
        1)
```



- Generate the replicates

```
x <- map(1:n_rep, \(x) rgamma(N, shape = k, scale =
        theta))
```

- Averate the samples in each replicate

```
x_bar <- map_dbl(x, mean)
```

- How are the replicates distributed?

```
print(str_c("Mean = ", mean(x_bar), " and std. dev. =
        ", sd(x_bar)))
```

```
## [1] "Mean = 10.0211472739298 and std. dev. =
1.28821892268581"
```

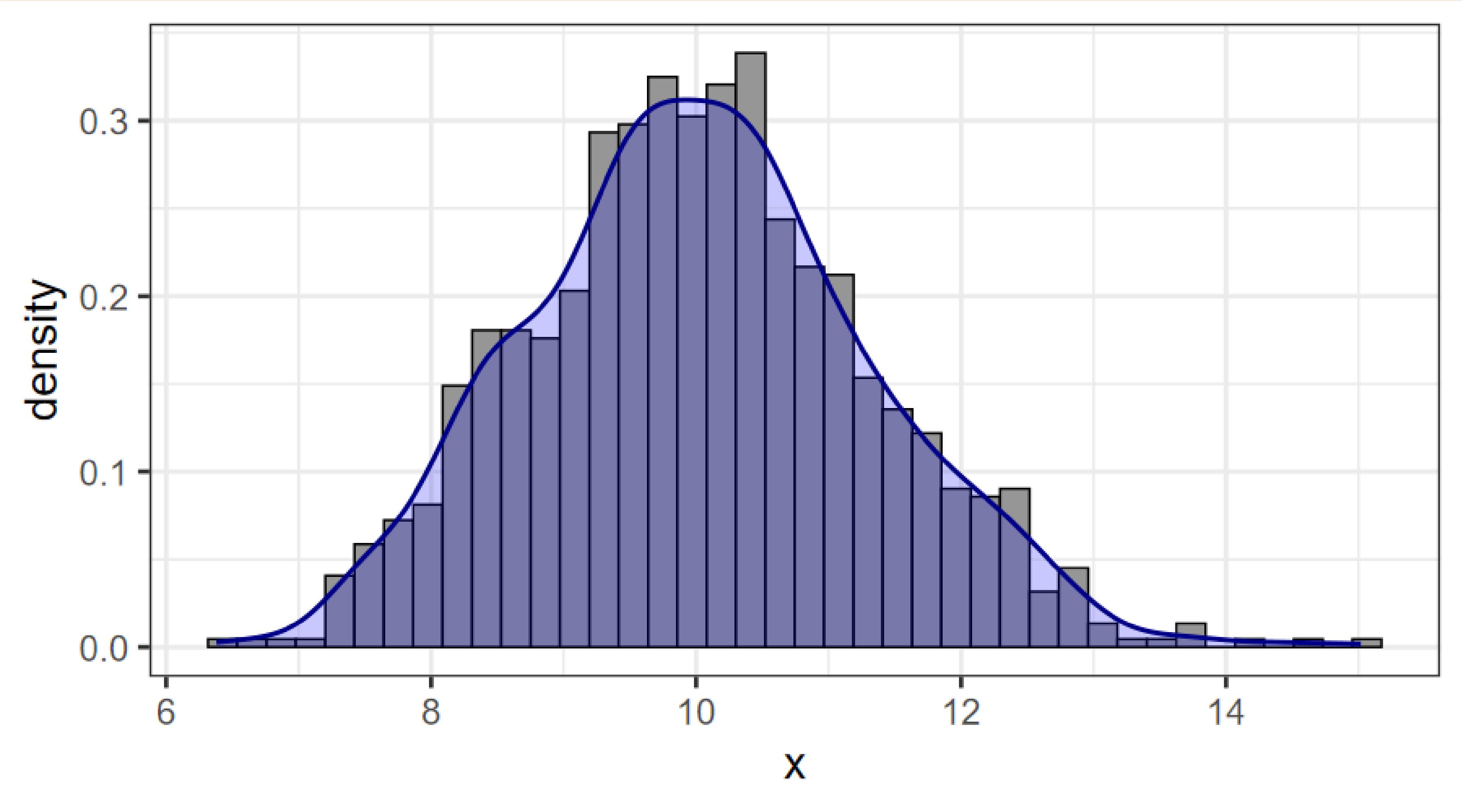
```
print(str_c("mu  = ", k * theta, ", sigma = ", sqrt(k)
        * theta,
           ", sigma / sqrt(N) = ", sqrt(k) * theta /
        sqrt(N)))
```

```
## [1] "mu  = 10, sigma = 7.07106781186548, sigma /
sqrt(N) = 1.29099444873581"
```

Plot the distribution

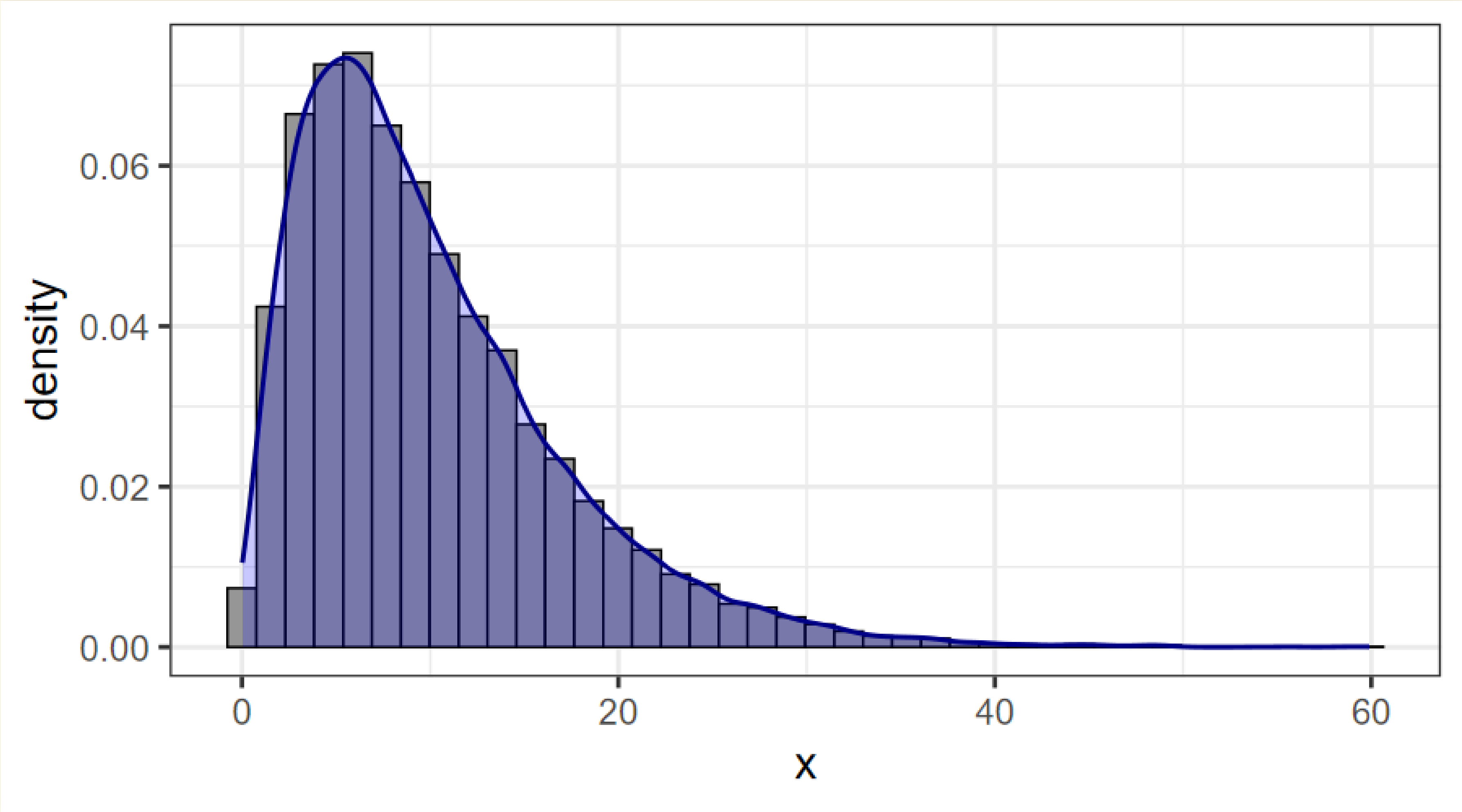
- Distribution of `x_bar`

```
tibble(x = x_bar) |>
ggplot(aes(x = x)) +
  geom_histogram(aes(y = after_stat(density)), bins = 40,
                 color = "black", fill = "gray60") +
  geom_density(color = "darkblue", fill = alpha("blue",
        0.2),
               linewidth = 1)
```



- Distribution of `x`

```
tibble(x = unlist(x)) |>
ggplot(aes(x = x)) +
  geom_histogram(aes(y = after_stat(density)), bins = 40,
                 color = "black", fill = "gray60") +
  geom_density(color = "darkblue", fill = alpha("blue",
        0.2),
               linewidth = 1)
```



Compare `x_bar` to True Normal

What Does This Mean?

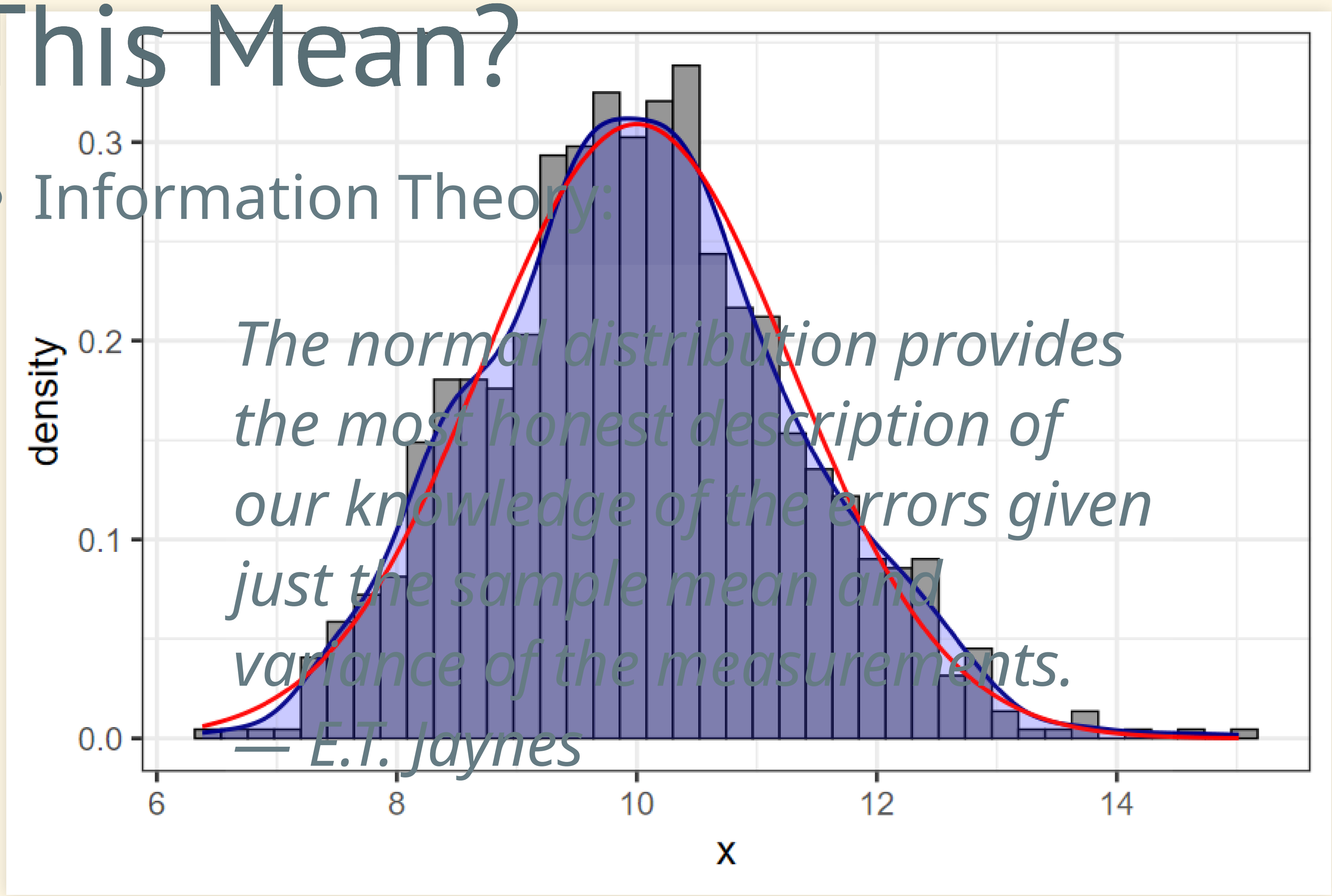
```r
df_norm <- tibble(x = seq(min(x_bar), max(x_bar),
                          length.out = 200),
                  y = dnorm(x, k * theta,
                            sqrt(k) * theta /
                              sqrt(N)))
df_x_bar <- tibble(x = x_bar)

ggplot(df_x_bar, aes(x = x)) +
  geom_histogram(aes(y = after_stat(density)), bins
  = 40,
                 color = "black", fill = "gray60") +
  geom_density(color = "darkblue",
               fill = alpha("blue", 0.2),
               linewidth = 1) +
  geom_line(data = df_norm, mapping = aes(x = x, y =
  y),
            color = "red", linewidth = 1)
```

- Central Limit Theorem:
  - If we have many measurements with errors:
    - Measurements: $m_1, m_2, m_3, \ldots, m_N$
    - True value: v
  - The central limit theorem tells us that the larger N is, the closer

$$\text{mean}(m_i) = \frac{1}{N}\sum_{i=1}^{N} m_i$$

    will be to v
  - The uncertainty of this estimate will shrink as $1/\sqrt{N}$, so taking 4 times more measurements cuts the uncertainty in half.



- Information Theory:

  The normal distribution provides the most honest description of our knowledge of the errors given just the sample mean and variance of the measurements.

  — *E.T. Jaynes*

- Notes:
  - `after_stat(density)` makes the histogram bar height correspond to density rather than count, so they line up with a density plot.
  - `alpha("blue", 0.2)` makes a partially transparent blue (1 = opaque, 0 = completely transparent, 0.2 = 20% opaque)

Other Limit Theorems

Other Limit Theorems

- Binomial $\longrightarrow$ Poisson:

$$X \sim \mathcal{B}(n, p)$$

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

  - $\binom{n}{k}$ becomes hard to calculate when n is large.
  - For large n and small p, the binomial distribution approaches a Poisson distribution with $\lambda = np$

$$\mathbb{P}(X = k) \to e^{-\lambda} \frac{\lambda^k}{k!}$$

- Poisson $\longrightarrow$ Normal
  - This is slightly different to what was presented in the book.
  - As λ gets large, the Poisson distribution approaches a normal distribution with $\mu = \lambda$ and $\sigma = \sqrt{\lambda}$