Rethinking Statistics

EES 5891-03

Bayesian Statistical Methods

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Golems and Rethinking

Statistical Tools as Golems

- Statistical tools will do what you tell them to do …
  - but if you're not careful, what you tell them to do may not be what you want them to do
- The goal of this book is to help you:
  - Learn to use statistical golems wisely
  - Learn to choose the right golem for the job
  - Learn to engineer your own golems if the ready-to-use golems aren't right for your job.

The Golem of Prague (Photo: Prague Post)

Hypothesis Testing

- Karl Popper (1902–1994)
  - Science can never prove that a hypothesis is true
  - But it can prove that an incorrect hypothesis is false
  - The more false hypotheses we rule out, we narrow down the list of potentially true hypotheses.

When you have eliminated the impossible,
whatever remains, however improbable,
must be the truth
— Arthur Conan Doyle/Sherlock Holmes

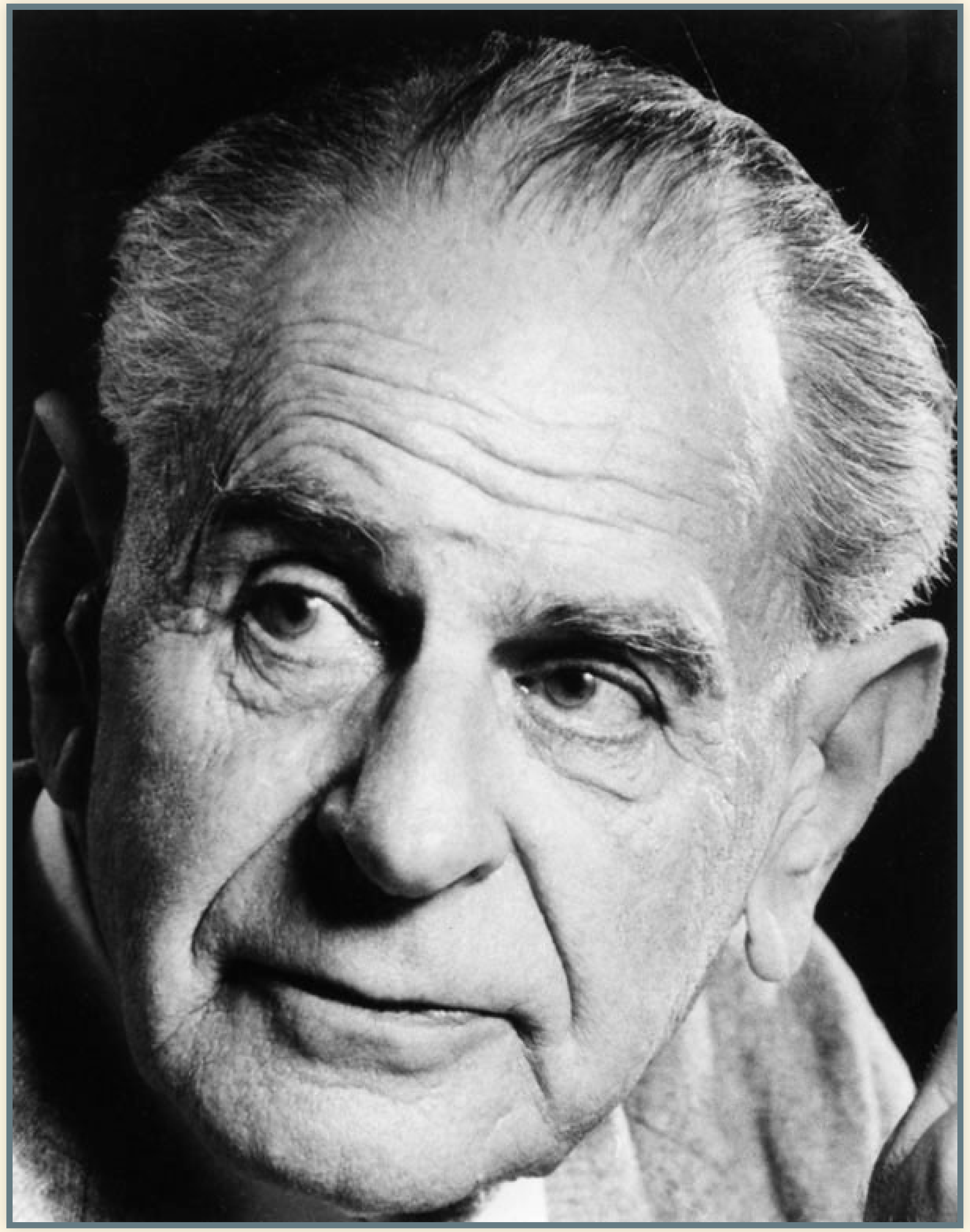
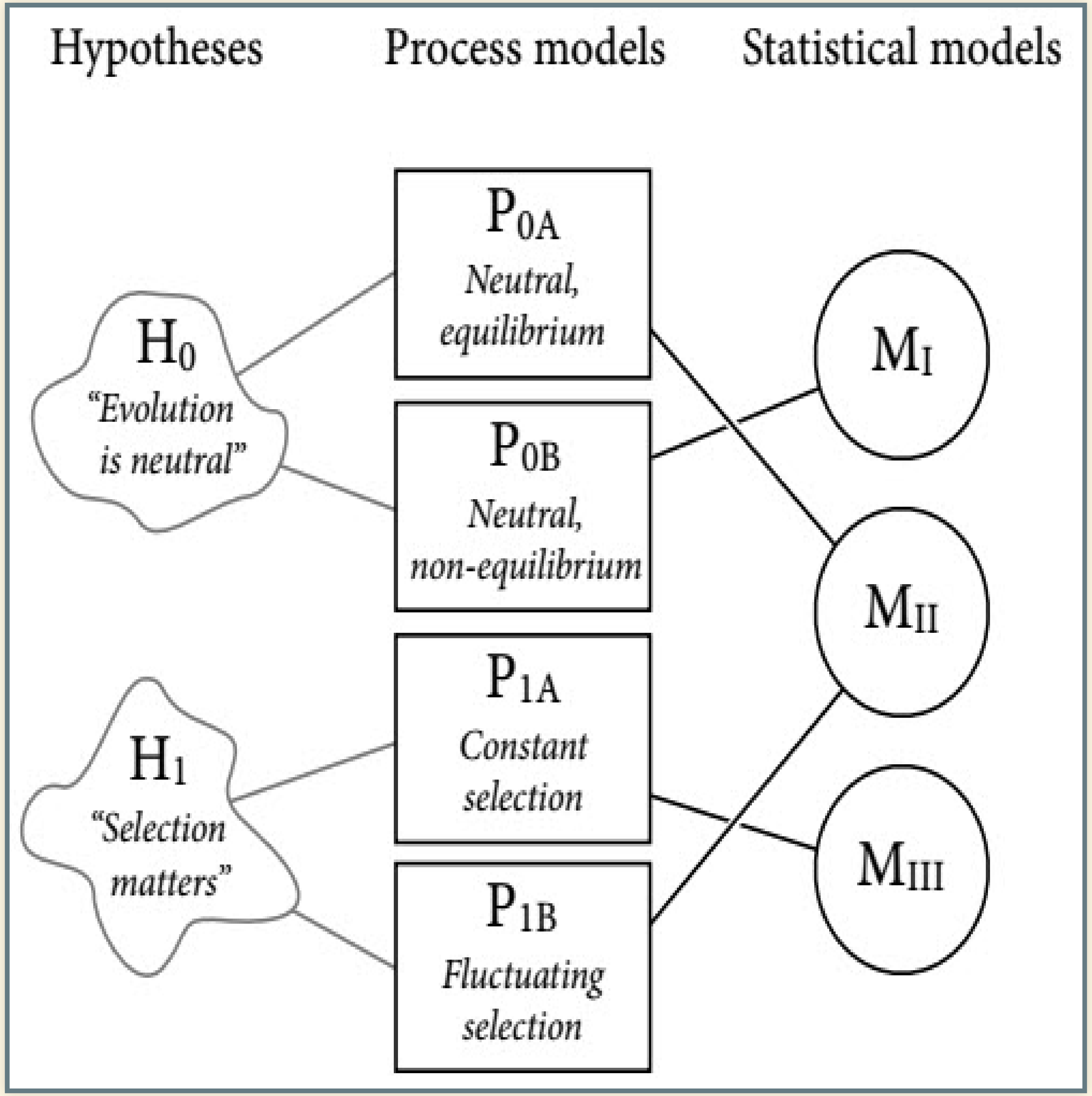


Photo: London School of Economics

Problems with falsification

- The predictions of a hypothesis may not be as clear as many people assume.
- Depending on what other assumptions you make, two different hypotheses may predict the same kind of data.
  - If your data looks like M_{I}, it rules out (falsifies) H_1
  - But if your data looks like M_{II}, it doesn't rule out either hypothesis.
- A given hypothesis may predict may different possible kinds of data, depending on what other assumptions you make.
  - If your data doesn't look like M_{I}, that doesn't imply that it's less likely H_1 is true.



Null-Hypothesis Significance Testing

- Most statistical tests aim to rule out a *null hypothesis*, not to falsify the actual research hypothesis.
- Often, there's not one unique alternative hypothesis to the null hypothesis, so even if we reject the null hypothesis, there are many other possibilities.
- Bayesian methods give us better, more powerful golems to answer the questions we're really interested in.
  - But they're still golems and we have to be thoughtful and careful about how to use them.

Discussion

Bayes's Theorem

Bayes's Theorem

- The core of this part of the course is Bayes's theorem.
- Notation:
  - Conditional probability: $P(a|b)$ means *the probability of a, given b*.
- Bayes's theorem:

$$P(H|D) = \frac{P(D|H) \times P(H)}{P(D)},$$

  where
  - $P(H|D)$ is the *posterior*: The probability that H is true, given that you observed D.
  - $P(D|H)$ is the *likelihood*: The probability that you would observe D, if H is true.
  - $P(H)$ is the *prior* probability of H, based on what you knew before observing D
  - $P(D)$ is the *evidence*: The probability that you would observe D, regardless whether H is true.
    - If H is binary (true or false), then

$$P(D) = P(D|H) \times P(H) + P(D|\mathbf{not}H) \times (1 - P(H))$$

Bayes's Theorem (cont)

- We can apply Bayes's theorem to a numbers too. For a variable x that we want to predict:

$$P(x|D) = \frac{P(D|x) \times P(x)}{P(D)}$$

- In this case,

$$P(D) = \int_{-\infty}^{\infty} P(D|x) \times P(x) \mathrm{d}x,$$

where the *prior* $P(x)$ is a probability density function.

Sampling

Sampling

- You have a globe and want to figure out what fraction of the earth’s surface is water.
- Toss the globe in the air, catch it, and note whether your index finger is on water or land: outcomes are W and L.
- At every toss, use Bayes’s theorem to update your estimate of the fraction that is water.

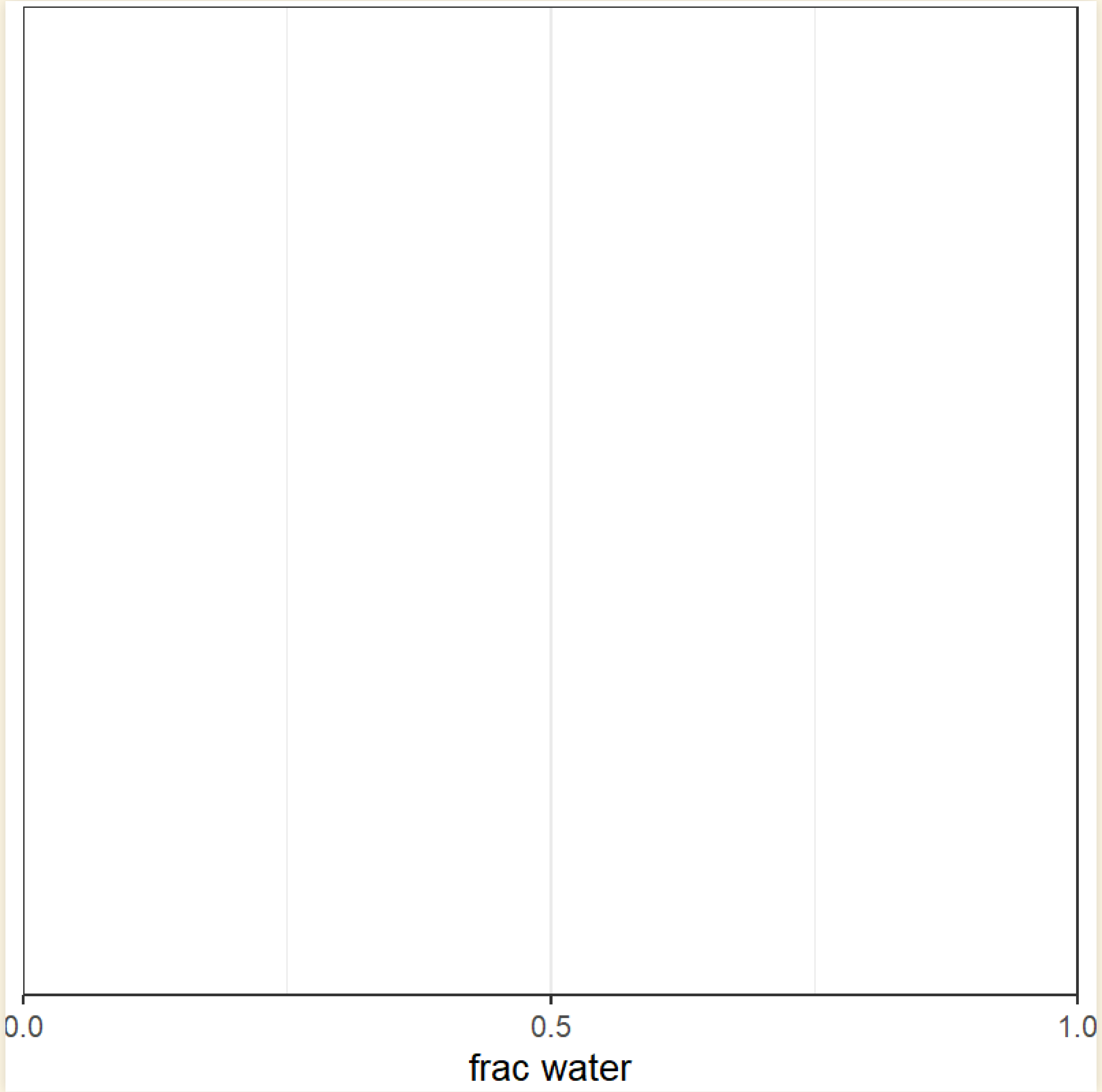
First toss

- Before you toss the globe, pick a prior probability distribution for the fraction that's water.
- Suppose we don't know anything.
  - Pick $p \sim \mathbf{Uniform}(0, 1)$, a uniform prior:
- Toss the globe and your finger lands on water.
- Update the probability:

$$P(p|W) = \frac{P(W|p)P(p)}{p(W)},$$

where p is the probability of water, and W is measuring water.

Prior:



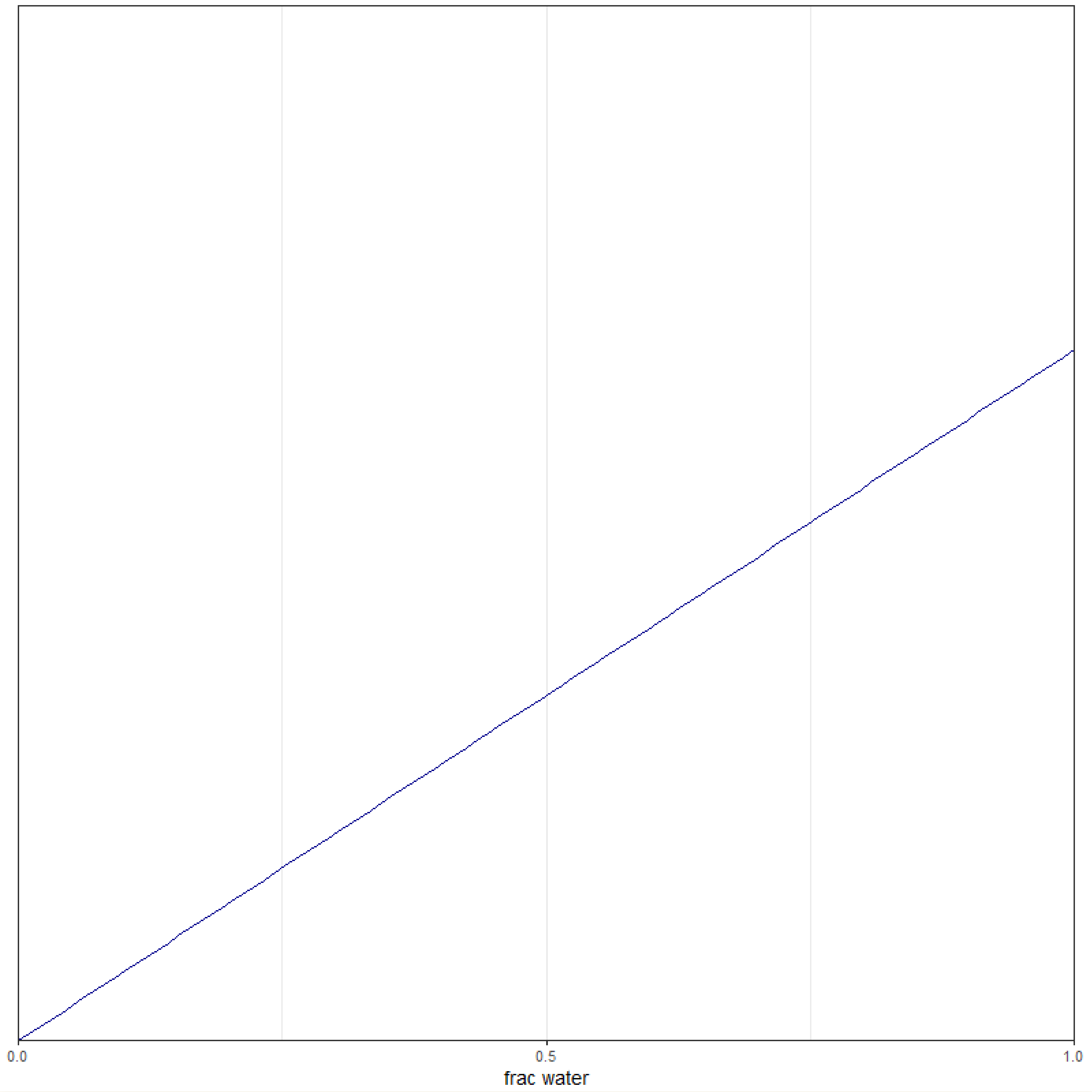
The calculation:

$$P(p|W) = \frac{P(W|p)P(p)}{p(W)},$$

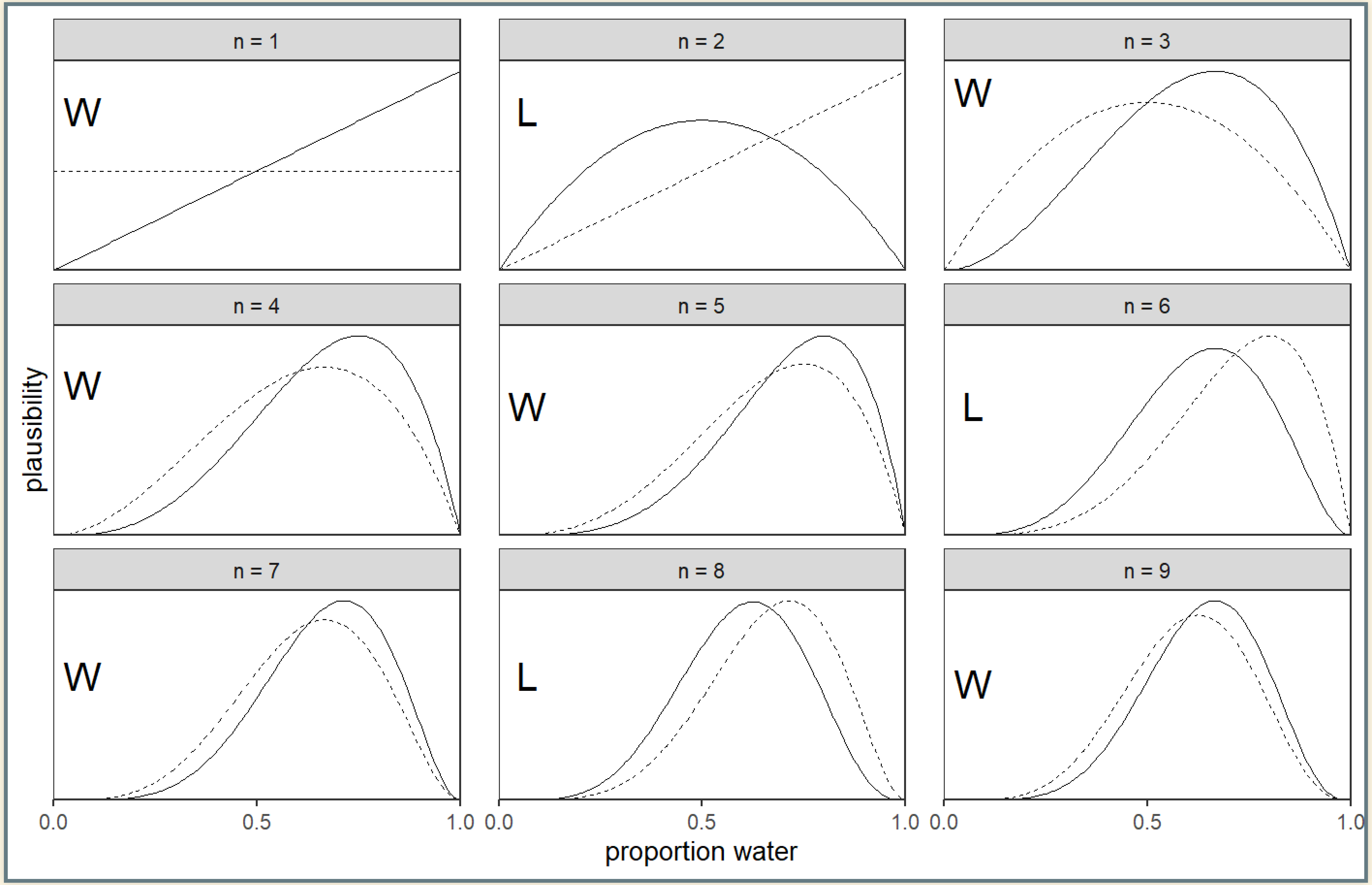
- $P(W|p) = p$
- $P(p) = 1$ (it's a uniform distribution

$$P(W) = \int_0^1 P(W|p)P(p)\mathrm{d}p = \int_0^1 p \times 1\mathrm{d}p$$
$$= (p^2/2)\Big|_0^1 = 1/2$$

- so the posterior $P(p|W) = 2p$
- Use this posterior as the prior for the next toss...



Subsequent tosses

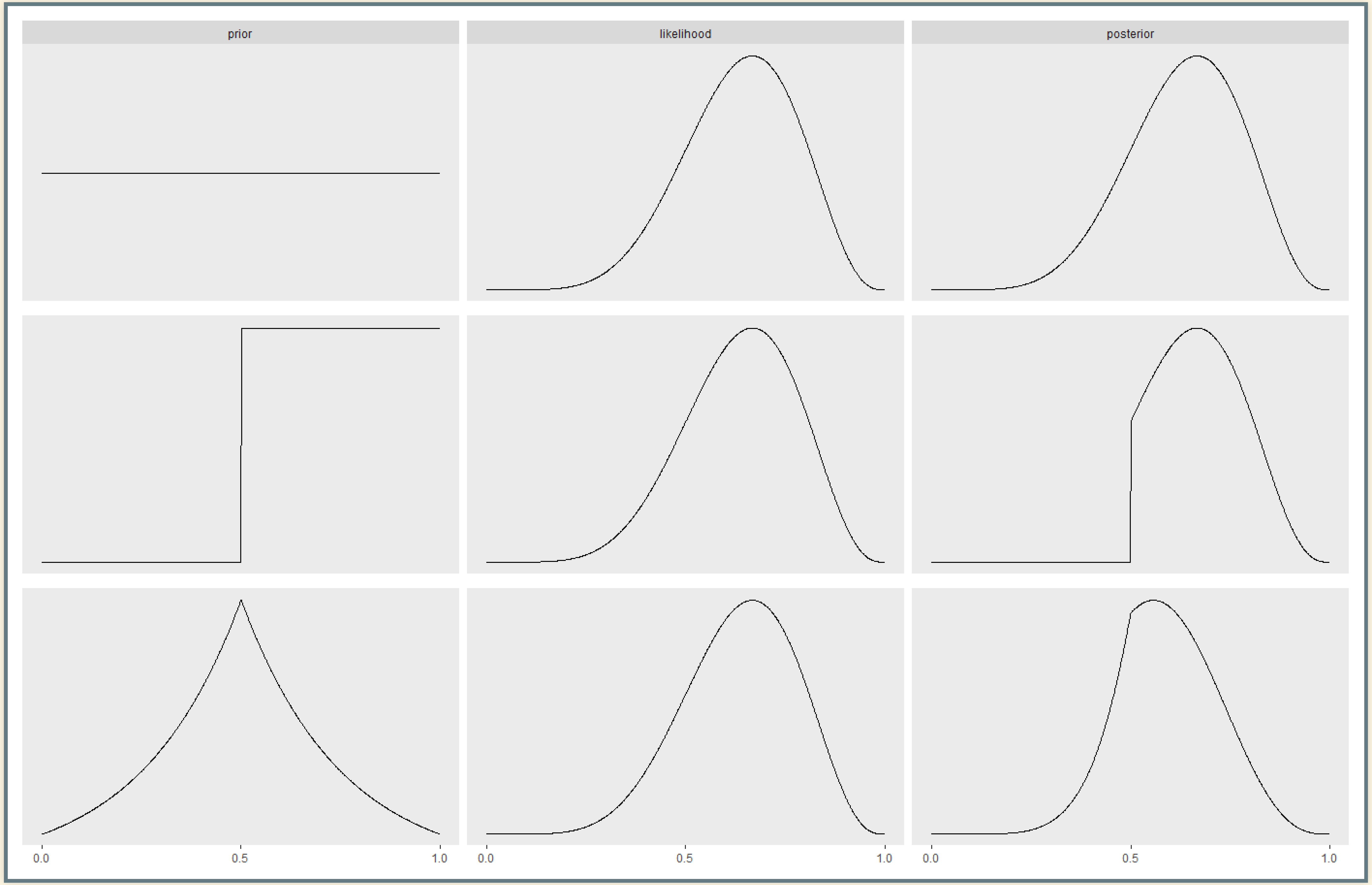


Developing a Model

Developing a Model

- Observations and parameters are drawn from probability distributions:
  - Likelihood: $W \sim \text{Binomial}(N, p)$, where N is the total number of tosses.
  - Prior $p \sim \text{Uniform}(0, 1)$
  - "$\sim$" means a random variable drawn from a probability distribution.
- We use the likelihood and the prior to calculate the posterior.
- We can't easily do this with analytical math using pencil and paper.
  - Computational methods:
    - Grid approximation
    - Quadratic approximation
    - Monte Carlo sampling

Examples



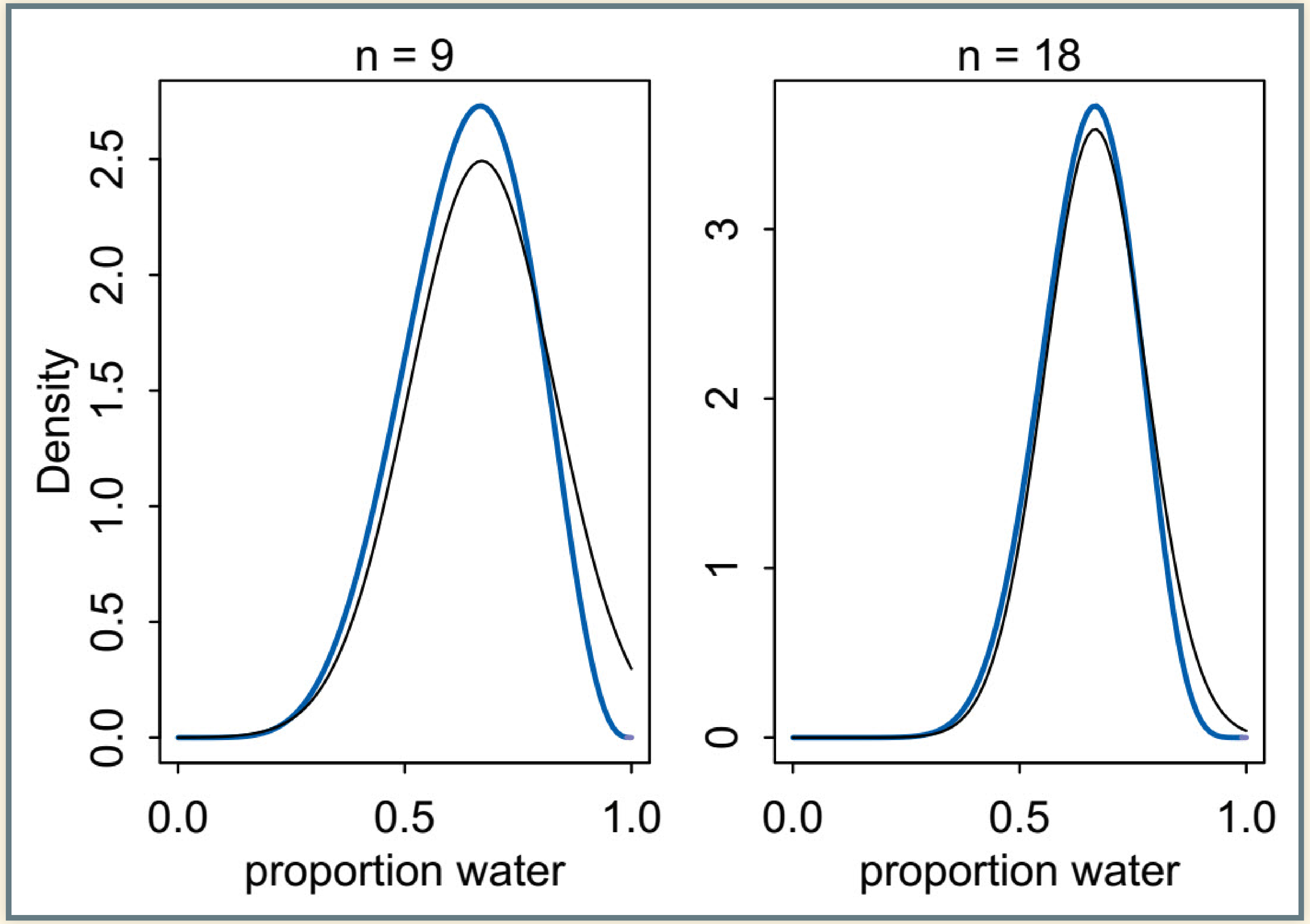
Grid Approximation

1. Define a grid:
   - specify a number of points to sample your function at.
   - Take evenly spaced values for each parameter (e.g., the proportion of water).
     - This example uses one parameter and a one-dimensional grid for simplicity.
     - For models with more than one parameter, the grid has 2, 3, or more dimensions—one per parameter.
2. Calculate the value of the *prior* at each grid point
3. Calculate the *likelihood* at each grid point
4. Compute an *unstandardized posterior* by multiplying the *prior* and *posterior* at each grid point.
5. Finally, standardize the *posterior* by dividing each value by the sum of all values in the *unstandardized posterior*.

- The more grid points you use, the more accurate your estimate will be, but the more computer power you'll need.
  - For one parameter, a 1000 point grid is simple.
  - For 2 parameters, a 1000 point grid for each of them means 1 million points.
  - For 3 parameters, it means 1 billion points.
  - For 30 parameters, your grid would have more points than there are atoms in the universe.

Quadratic Approximation

- Focus our attention near the part of the distribution that has the highest probability density.
- This region looks a lot like a Gaussian (normal) distribution.
  1. Find the posterior mode (the highest value).
     - Hill-climbing algorithms
  2. Estimate the curvature of the posterior near the mode
     - Approximate probability density as a Gaussian
     - Approximate the logarithm of the probability density as a quadratic function.
  3. We can calculate the integral of a Gaussian easily.

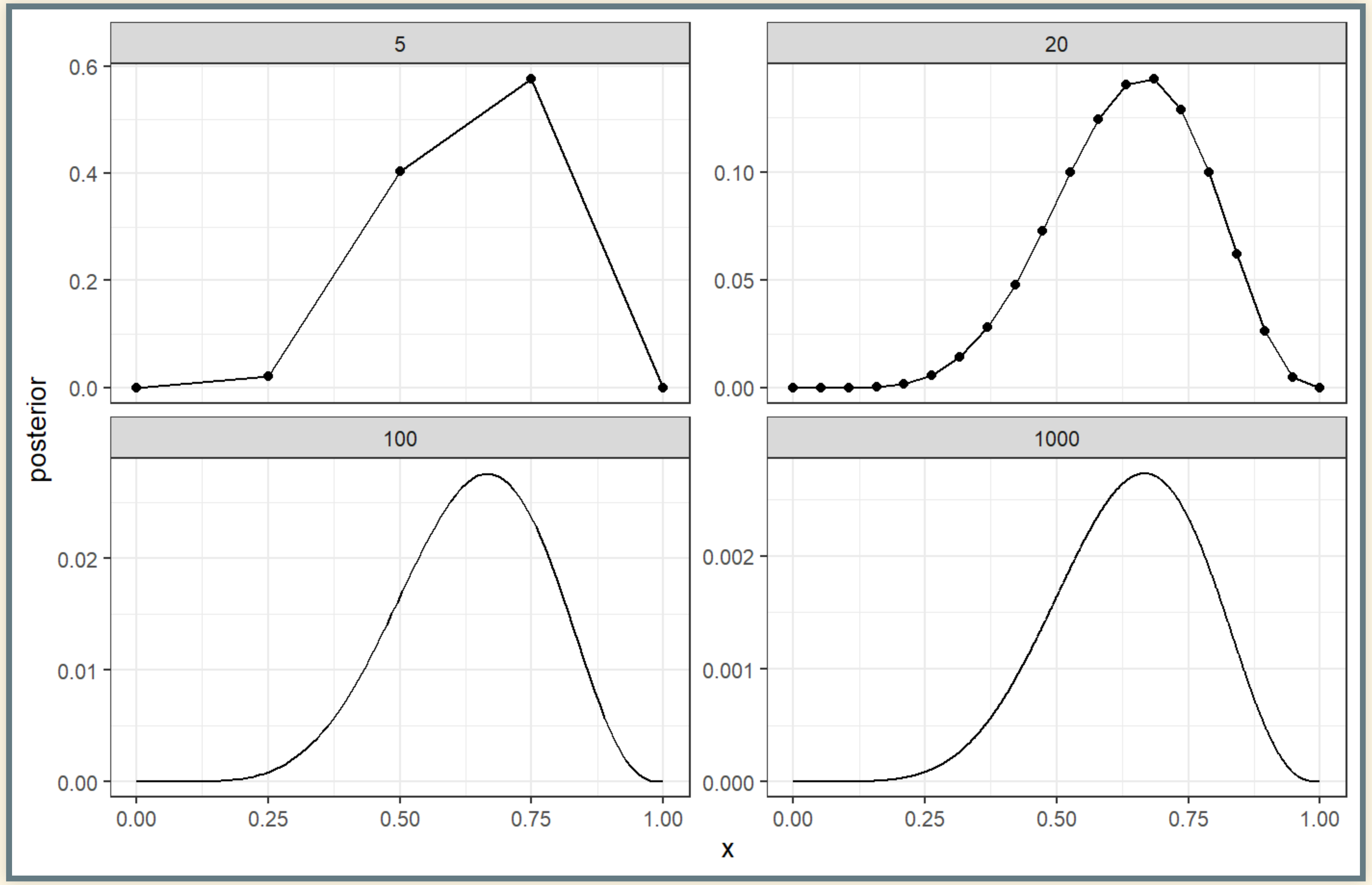


Programming Models in R

Grid approximation

- Sample functions on a regular grid and approximate integrals by the sum of the samples.

```
# define number of points in the grid

grid_points = 200

# define grid
p_grid <- seq(from = 0, to = 1,
              length.out = grid_points)

# define prior
prior <- rep(1, grid_points)

# compute likelihood at each value in grid
likelihood <- dbinom(6, size = 9, prob = p_grid)

# compute product of likelihood and prior
unstd.posterior <- likelihood * prior

# standardize the posterior, so it sums to 1
posterior <- unstd.posterior / sum(unstd.posterior)
```



Quadratic Approximation

```
library(rethinking)

globe_qa <- quap(
  alist(
    W ~ dbinom( W + L, p),  # binomial likelihood
    p ~ dunif(0, 1)        # uniform prior
  ),
  data = list(W = 6, L = 3)
)
```

Now display a summary:

```
precis(globe_qa)
```

```
##          mean        sd      5.5%     94.5%
## p 0.6666761 0.1571315 0.4155496 0.9178027
```