Adventures in Covariance

EES 5891-03

Bayesian Statistical Methods

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Adventures in Covariance

Varying Intercepts and Slopes

- Coffee Robot Example
  - A robot visits N cafes
  - Estimate the average waiting time W to get a coffee.
  - Model:

$$
\begin{aligned}
W_{\text{cafe}} &\sim \text{Normal}(\mu_{\text{cafe}}, \sigma) \\
\mu_{\text{cafe}} &= \alpha_{\text{cafe}} \\
\alpha_{\text{cafe}} &\sim \text{Normal}(5, 2) \\
\sigma &\sim \text{Expoonential}(1)
\end{aligned}
$$

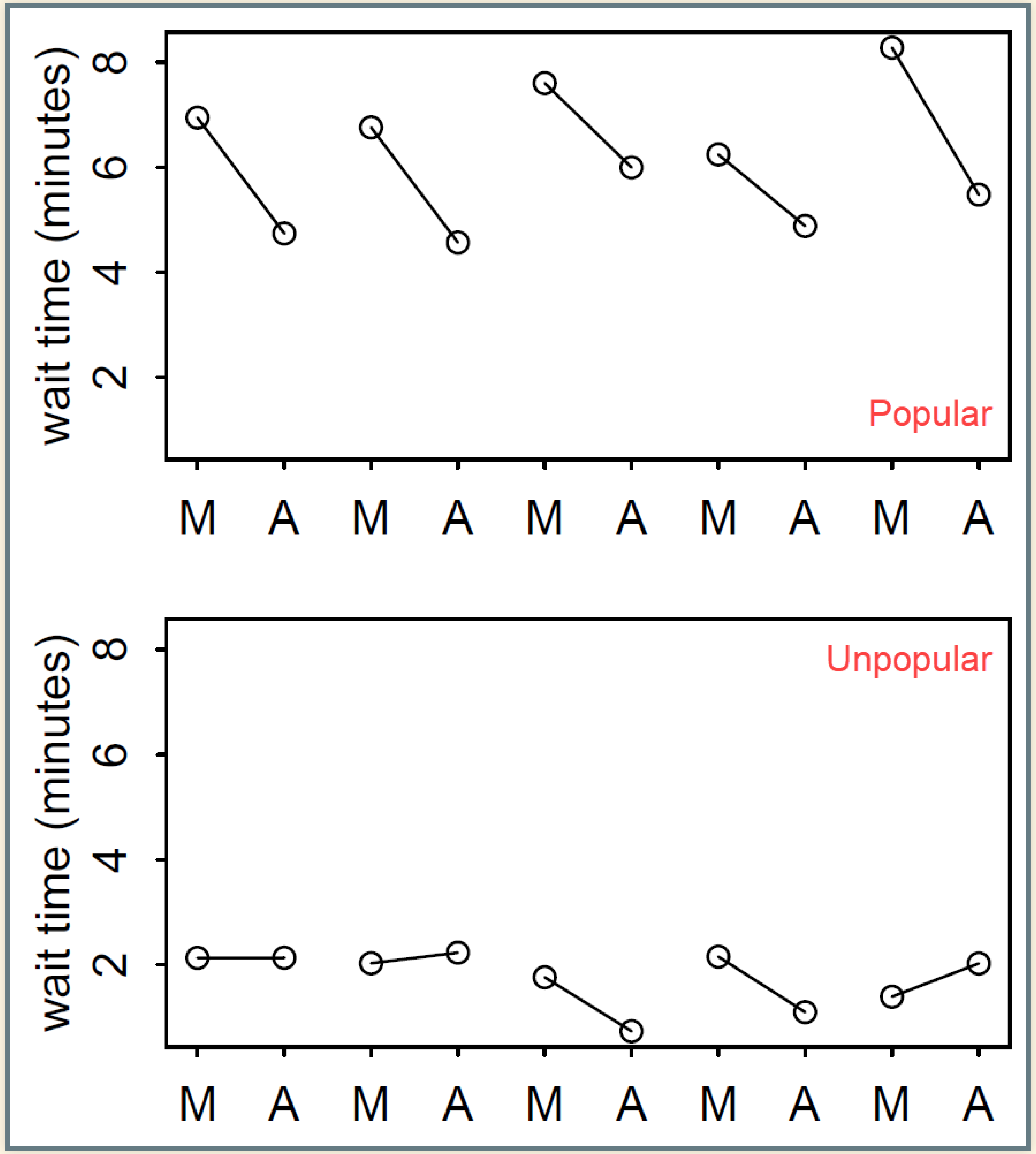
- Different times of day
  - Morning (busier) vs. afternoon (less busy)
  - Model:

$$
\begin{aligned}
W_{\text{cafe}} &\sim \text{Normal}(\mu_{\text{cafe}}, \sigma) \\
\mu_{\text{cafe}} &= \alpha_{\text{cafe}} + \beta_{\text{cafe}} A \\
\alpha_{\text{cafe}} &\sim \text{Normal}(5, 2) \\
\beta_{\text{cafe}} &\sim \text{Normal}(-1, 0.5) \\
\sigma &\sim \text{Expoonential}(1)
\end{aligned}
$$

  - Something's missing ...

Correlations

- Some cafes are very popular, and others are not popular
- Popular cafes are busy
  - Big differences between morning and afternoon
- Unpopular cafes aren't busy
  - Not much difference between morning and afternoon
- **Covariance** between *slopes* (β) and *intercepts* (α).



Modeling covariance

- Covariance matrix

$$\begin{pmatrix} \text{variance of intercepts} & \text{covariance of intercepts and slopes} \\ \text{covariance of intercepts and slopes} & \text{variance of slopes} \end{pmatrix}$$

$$\begin{pmatrix} \sigma_\alpha^2 & \sigma_\alpha \sigma_\beta \rho \\ \sigma_\alpha \sigma_\beta \rho & \sigma_\beta^2 \end{pmatrix}$$

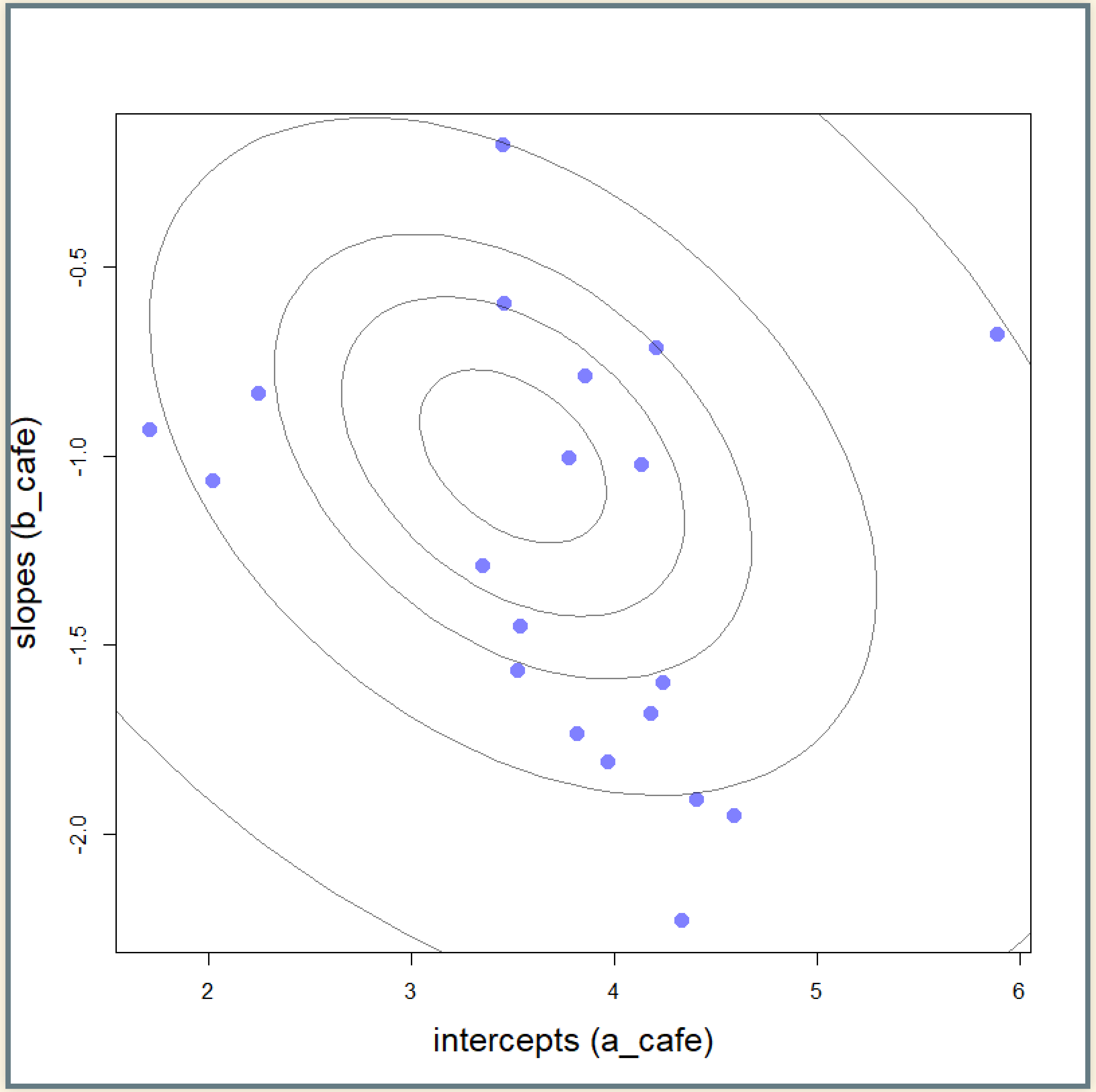
  - ρ is the correlation between slopes and intercepts ($-1 \leq \rho \leq 1$).

Simulation modeling

Simulation modeling

What Is Covariance?

```
library(ellipse)

plot(a_cafe, b_cafe, col = rangi2, pch=19, cex=1.5,
        cex.lab=1.5,
    xlab="intercepts (a_cafe)", ylab="slopes (b_cafe)")

# overlay population distribution
for ( l in c(0.1,0.3,0.5,0.8,0.99) )
    lines(ellipse(Sigma, centre=Mu, level=l),
          col = col.alpha("black",0.5), lwd=1)
```



Fitting the Model:
Varying Slopes, Varying Intercept

Fitting the Model

- Model:

$$
\begin{aligned}
W &\sim \text{Normal}(\mu, \sigma) \\
\mu &= \alpha_{\text{cafe}} + \beta_{\text{cafe}} A \\
\begin{bmatrix} \alpha_{\text{cafe}} \\ \beta_{\text{cafe}} \end{bmatrix} &\sim \text{MVNormal}\left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}, S \right) \\
S &= \begin{pmatrix} \sigma_\alpha & 0 \\ 0 & \sigma_\beta \end{pmatrix} R \begin{pmatrix} \sigma_\alpha & 0 \\ 0 & \sigma_\beta \end{pmatrix}
\end{aligned}
$$

- Priors

$$
\begin{aligned}
\alpha &\sim \text{Normal}(5, 2) \\
\beta &\sim \text{Normal}(-1, 0.5) \\
\sigma &\sim \text{Exponential}(1) \\
\sigma_\alpha &\sim \text{Exponential}(1) \\
\sigma_\beta &\sim \text{Exponential}(1) \\
R &\sim \text{LKJcorr}(2)
\end{aligned}
$$

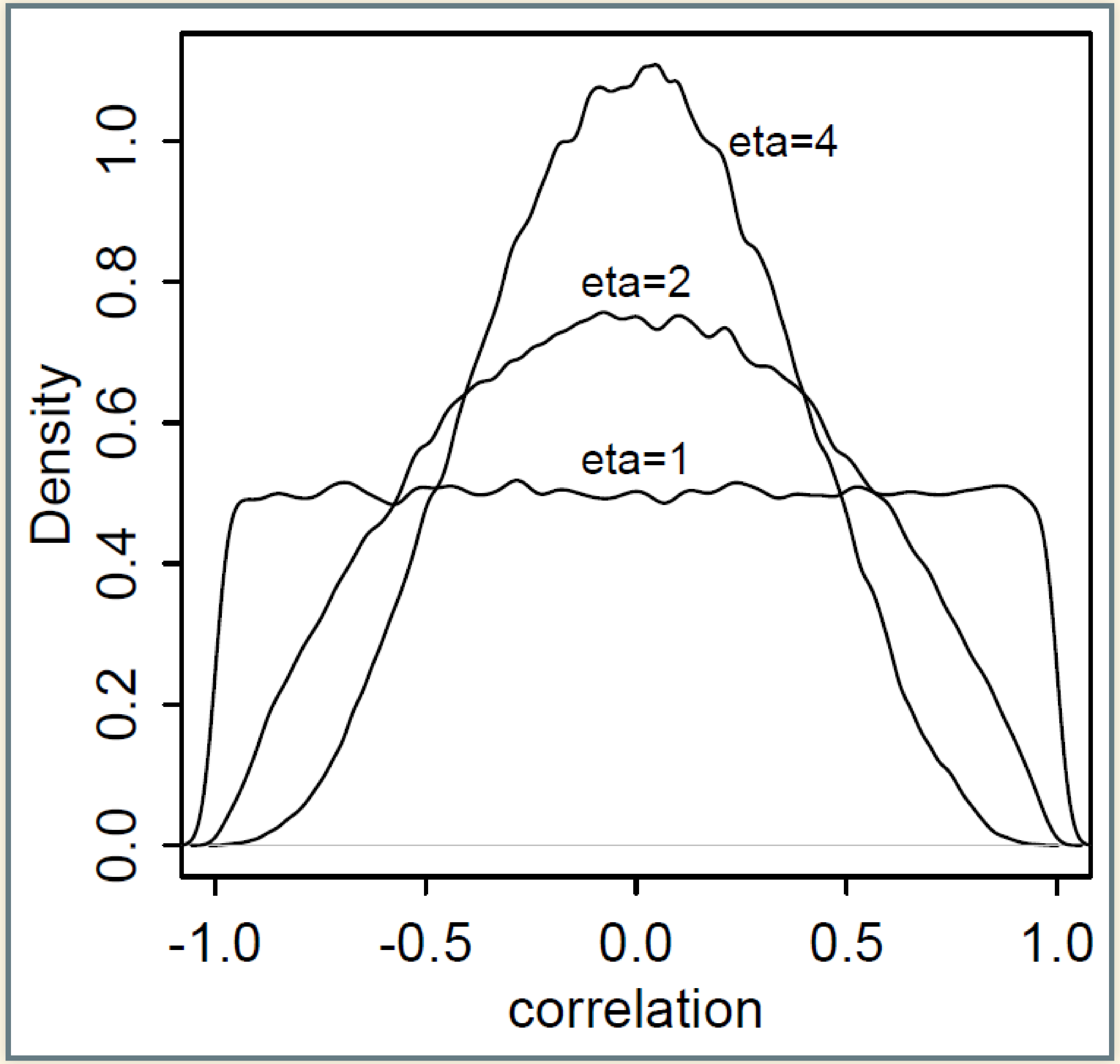
- LKJ prior:
  - Weakly-informative prior for correlation matrices.

LKJ Priors

- Correlation matrix:

$$R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

- LKJcorr(η)
  - $\eta > 1$: the greater η is, the more unlikely extreme correlations are



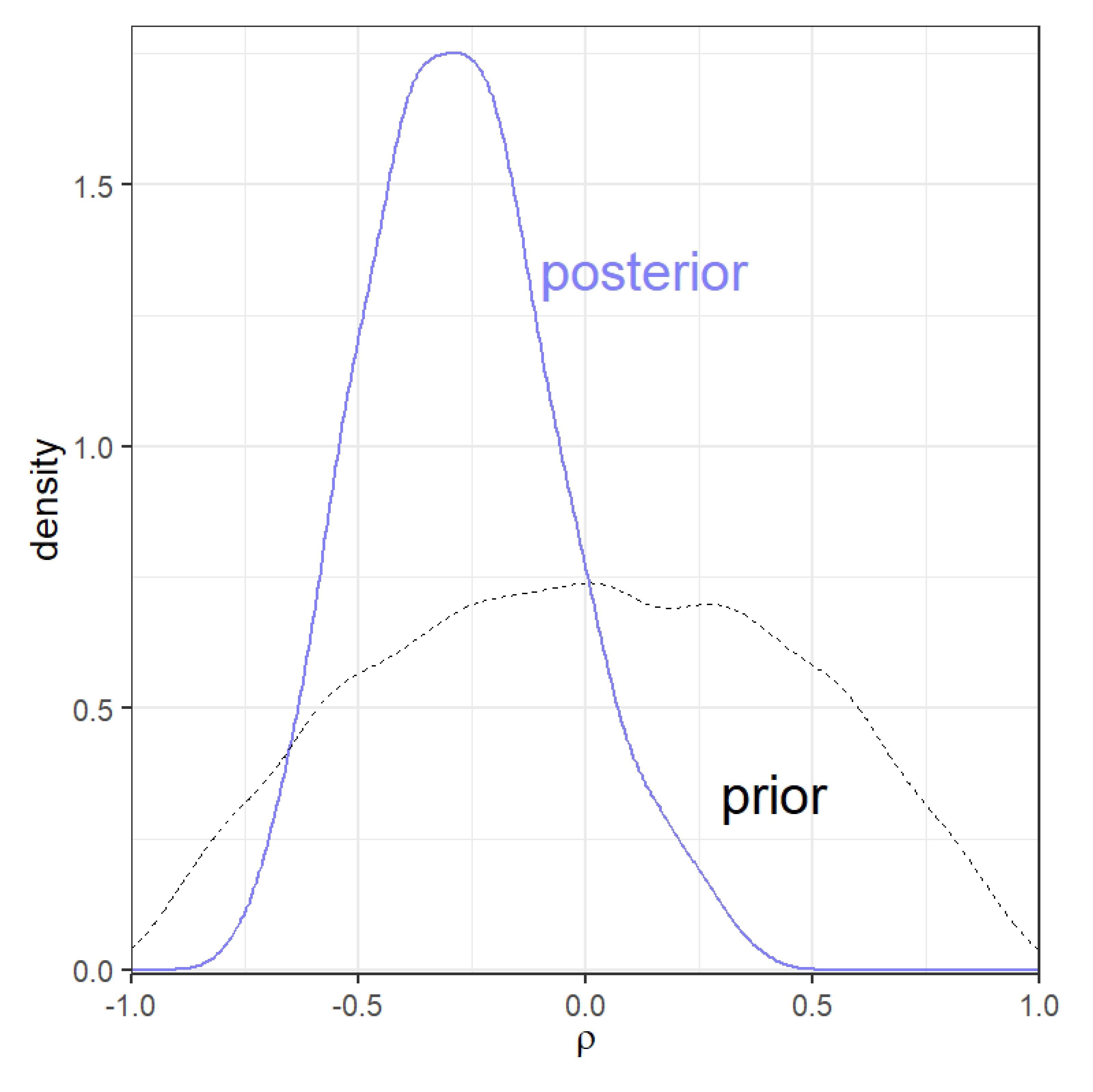
Fitting the model

- Model

```
set.seed(867530)
mdl_cafe <- ulam(
    alist(
        wait ~ normal(mu, sigma),
        mu <- a_cafe[cafe] + b_cafe[cafe] * afternoon,
        c(a_cafe,b_cafe)[cafe] ~ multi_normal(c(a, b), Rho,
         sigma_cafe),
        a ~ normal(5, 2),
        b ~ normal(-1, 0.5),
        sigma_cafe ~ exponential(1),
        sigma ~ exponential(1),
        Rho ~ lkj_corr(2)
    ), data=d, chains = 4, cores = 4)
```

- Extract samples from posterior

```
post <- tidy_draws(mdl_cafe) %>% mutate(label = "posterior")
        %>%
  rename(rho = `Rho[1,2]`)
prior <- tibble(rho = rlkjcorr(1e4, K=2, eta=2)[,1,2], label =
        "prior")
labels <- tibble(rho = c(-0.1, 0.3), y = c(1.3, 0.3),
                 label = c("posterior", "prior"))

ggplot(post, aes(x = rho, color = label)) +
geom_density(bw = 0.05, linewidth=1) +
geom_density(data = prior, linetype="dashed") +
geom_text(data = labels, aes(y = y, label = label), size = 10,
          vjust = 0, hjust = 0) +
scale_x_continuous(limits = c(-1,1), expand = c(0,0)) +
scale_y_continuous(expand = expansion(c(0.005, 0), c(0, 0.05)))
```

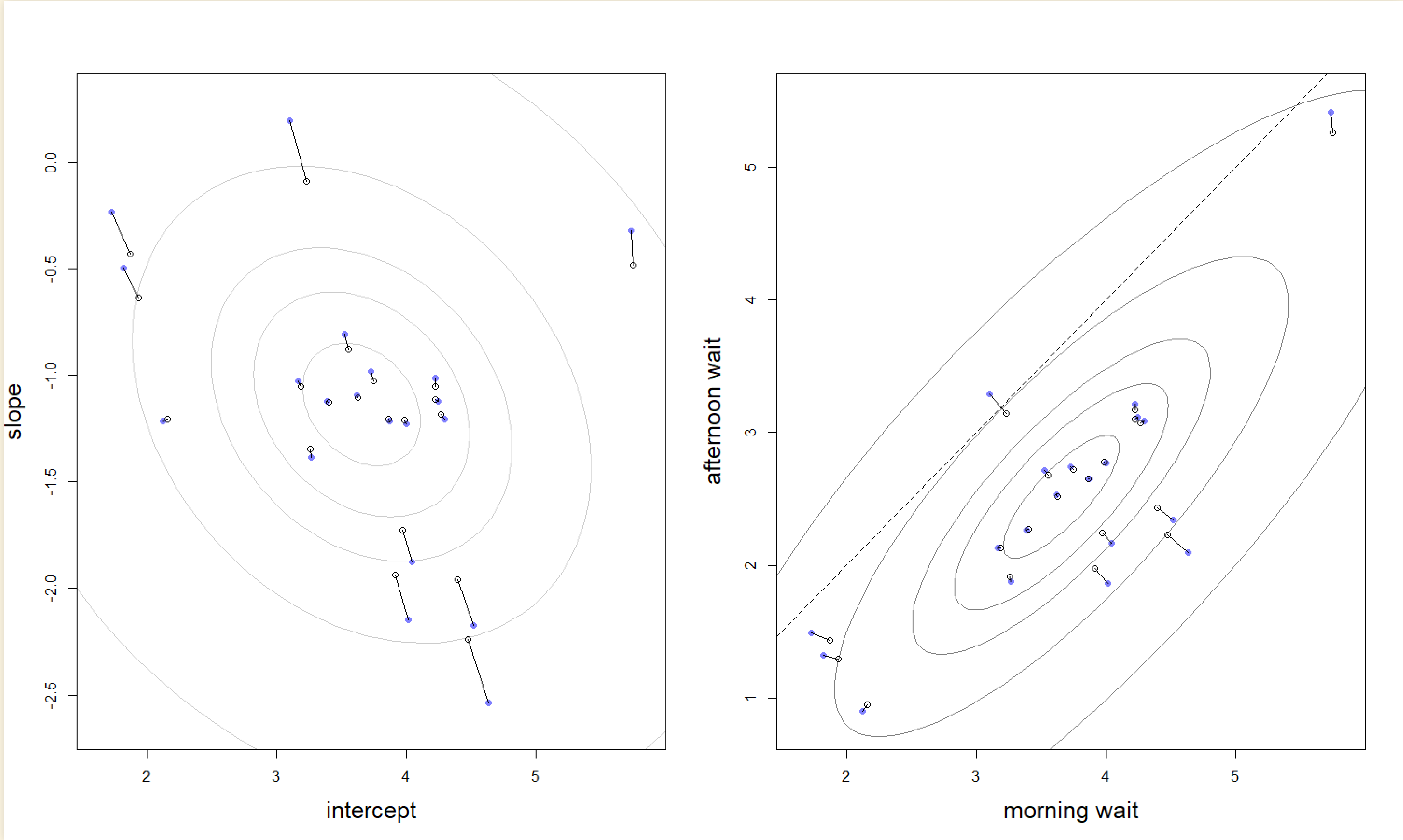


- Posterior is well within the prior:
  - The data are determining the posterior

```
        +
scale_color_manual(values = c(prior = "black", posterior =
        rangi2),
                   guide = "none") +
labs(x = expr(rho), y = "density") +
theme(plot.margin = unit(c(1,2,1,1), "lines"))
```

Shrinkage and Regularization

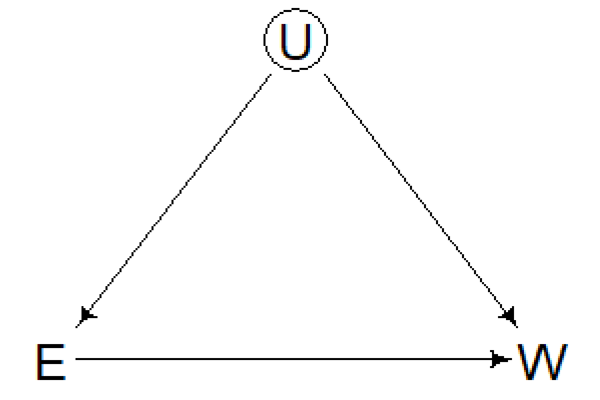
- **Shrinkage**: Estimates for each cafe move toward the average



Instrumental Variables

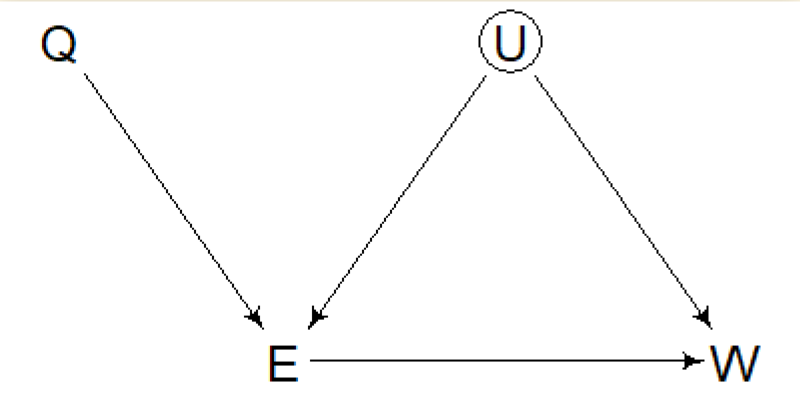
Instrumental Variables

- Effect of education on wages
  - Does more school improve future wages?
  - Complication: People who are motivated and disciplined are likely to stay in school longer, and also earn more regardless of schooling:



  - U is an unobserved variable that describes the person's work ethic and self-discipline.
  - The backdoor path $E \leftarrow U \rightarrow W$ prevents us from drawing causal inferences from correlations between E and W.

- **Instrumental variables:**

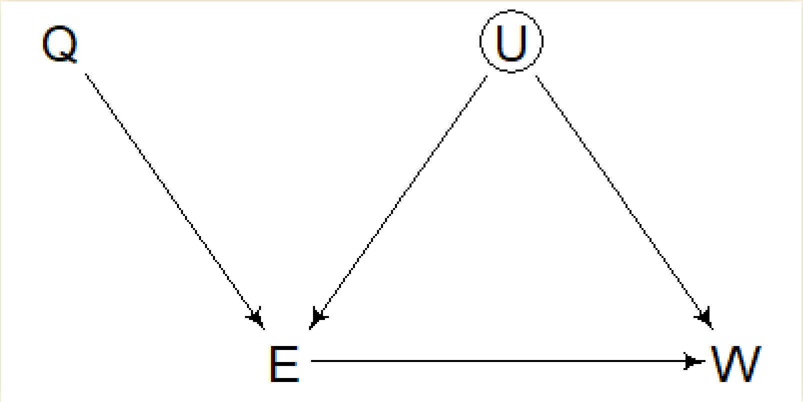


  - Q can be an **instrumental variable** if:
    1. It's independent of U.
    2. It's not independent of E.
    3. Q cannot influence W except through E.

Models with Instrumental Variables

- What are instrumental variables?
- Natural experiments
- We don't have enough control of the system to do *controlled experiments*
- SO we use natural variations in *instrumental variables* as *natural experiments*

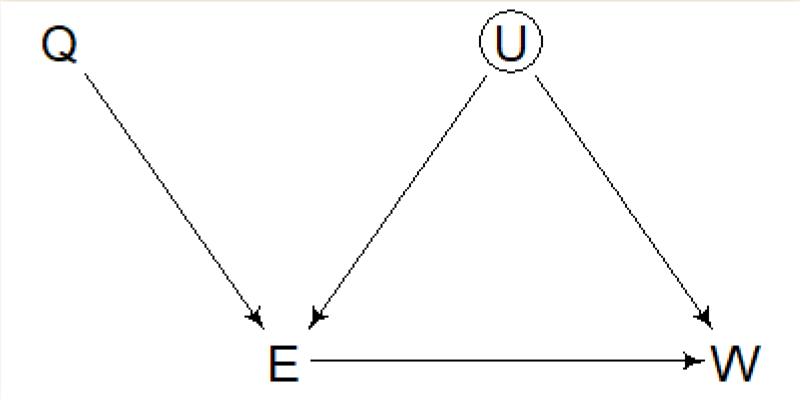
- DAG:



- We can't just put Q in a regression model
  - E is a *collider* of Q and U.
  - The collider opens up a *non-causal* path from Q to W, which creates a false correlation between Q and W, even if E doesn't influence W.
    - Q is a **bias amplifier**.

Example of Instrumental Variable

- Example:



- Q is the season of the year
  - Students born earlier complete fewer grades by the time they reach the age where they can leave school (drop out).
  - Q is not influenced by any of the other variables

- Generate simulated data:

```
set.seed(73)
N <- 500

U_sim <- rnorm(N)
Q_sim <- sample(1:4, size = N, replace = TRUE)
E_sim <- rnorm(N, U_sim + Q_sim)
W_sim <- rnorm(N, U_sim + 0 * E_sim)

dat_sim <- list(
  W=standardize(W_sim),
  E=standardize(E_sim),
  Q=standardize(Q_sim)
)
```

  - Notice that in the simulated data, E has no effect on W.

Writing a model

```
mdl_wages <- ulam(
  alist(
    W ~ dnorm(mu, sigma),
    mu <- aW + bEW * E,
    aW ~ dnorm(0, 0.2),
    bEW ~ dnorm(0, 0.5),
    sigma ~ dexp(1)
  ), data = dat_sim, chains = 4, cores = 4)
```

```
precis_show(precis(mdl_wages, digits = 2))
```

```
##        mean   sd  5.5% 94.5% n_eff Rhat4
## aW     0.00 0.04 -0.06  0.07  1737     1
## bEW    0.40 0.04  0.33  0.46  1648     1
## sigma  0.92 0.03  0.88  0.96  1924     1
```

- The influence of U causes the model to incorrectly think E has a significant effect on W.

```
mdl_wages_q <- ulam(
  alist(
    W ~ dnorm(mu, sigma),
    mu <- aW + bEW * E + bQW * Q,
    aW ~ dnorm(0, 0.2),
    bEW ~ dnorm(0, 0.5),
    bQW ~ dnorm(0, 0.5),
    sigma ~ dexp(1)
  ), data = dat_sim, chains = 4, cores = 4)
```

```
precis_show(precis(mdl_wages_q, digits = 2))
```

```
##         mean   sd  5.5% 94.5% n_eff Rhat4
## aW      0.00 0.04 -0.06  0.06  1876     1
## bEW     0.64 0.05  0.57  0.71  1189     1
## bQW    -0.41 0.05 -0.48 -0.33  1533     1
## sigma   0.86 0.03  0.82  0.90  1875     1
```

- This is even worse:
  - The estimate of the effect of E is even larger
  - The model also thinks Q has a direct effect on W.

A Better Model of Instrumental Variables

- Formula:

$$
\begin{aligned}
W &\sim \text{Normal}(\mu_W, \sigma_W) \\
\mu_W &= \alpha_W + \beta_{EW} E + U \\
E &\sim \text{Normal}(\mu_E, \sigma_E) \\
\mu_E &= \alpha_E + \beta_{QE} Q + U \\
Q &\sim \text{Categorical}([0.25, 0.25, 0.25, 0.25]) \\
U &\sim \text{Normal}(0, 1) \\
\begin{pmatrix} W \\ E \end{pmatrix} &\sim \text{MVNormal}\left(\begin{pmatrix} \mu_W \\ \mu_E \end{pmatrix}, S\right) \\
\mu_W &= \alpha_W + \beta_{EW} E \\
\mu_E &= \alpha_E + \beta_{QE} Q
\end{aligned}
$$

- Model code

```
mdl_wages_inst <- ulam(
  alist(
    c(W,E) ~ multi_normal(c(muW, muE), Rho, Sigma),
    muW <- aW + bEW * E,
    muE <- aE + bQE * Q,
    c(aW,aE) ~ normal(0, 0.2),
    c(bEW,bQE) ~ normal(0, 0.5),
    Rho ~ lkj_corr(2),
    Sigma ~ exponential(1)
  ), data = dat_sim, chains = 4, cores = 4)
```

```
precis_show(precis(mdl_wages_inst, digits = 2, depth=3))
```

```
##            mean   sd  5.5% 94.5% n_eff Rhat4
## aE         0.00 0.03 -0.06  0.05  1782     1
## aW         0.00 0.05 -0.07  0.07  1643     1
## bQE        0.59 0.04  0.53  0.64  1482     1
## bEW       -0.05 0.07 -0.17  0.07   916     1
## Rho[1,1]   1.00 0.00  1.00  1.00   NaN   NaN
## Rho[1,2]   0.54 0.05  0.46  0.62   972     1
## Rho[2,1]   0.54 0.05  0.46  0.62   972     1
## Rho[2,2]   1.00 0.00  1.00  1.00   NaN   NaN
## Sigma[1]   1.03 0.05  0.96  1.10  1145     1
## Sigma[2]   0.81 0.02  0.77  0.85  1736     1
```

- μ_{EW} is consistent with zero, as it ought to be.

New Data

- Generate new data, where E has a positive effect on wages.

```
set.seed(73)
N <- 500

U_sim <- rnorm(N)
Q_sim <- sample(1:4, size = N, replace = TRUE)
E_sim <- rnorm(N, U_sim + Q_sim)
W_sim <- rnorm(N, U_sim + 0.3 * E_sim)

dat_sim_2 <- list(
  W=standardize(W_sim),
  E=standardize(E_sim),
  Q=standardize(Q_sim)
)
```

- Rerun the model with new data:

```
mdl_wages_inst_2 <- ulam(mdl_wages_inst, data = dat_sim_2,
                         chains = 4, cores = 4)
```

```
precis_show(precis(mdl_wages_inst_2, digits = 2, depth=3))
```

```
##           mean   sd  5.5% 94.5% n_eff Rhat4
## aE        0.00 0.04 -0.05  0.06  1578     1
## aW        0.00 0.04 -0.06  0.06  1555     1
## bQE       0.59 0.04  0.53  0.64  1167     1
## bEW       0.23 0.07  0.12  0.33   869     1
## Rho[1,1]  1.00 0.00  1.00  1.00   NaN   NaN
## Rho[1,2]  0.54 0.05  0.46  0.63   870     1
## Rho[2,1]  0.54 0.05  0.46  0.63   870     1
## Rho[2,2]  1.00 0.00  1.00  1.00   NaN   NaN
## Sigma[1]  0.88 0.04  0.81  0.94   910     1
## Sigma[2]  0.81 0.03  0.77  0.85  1827     1
```

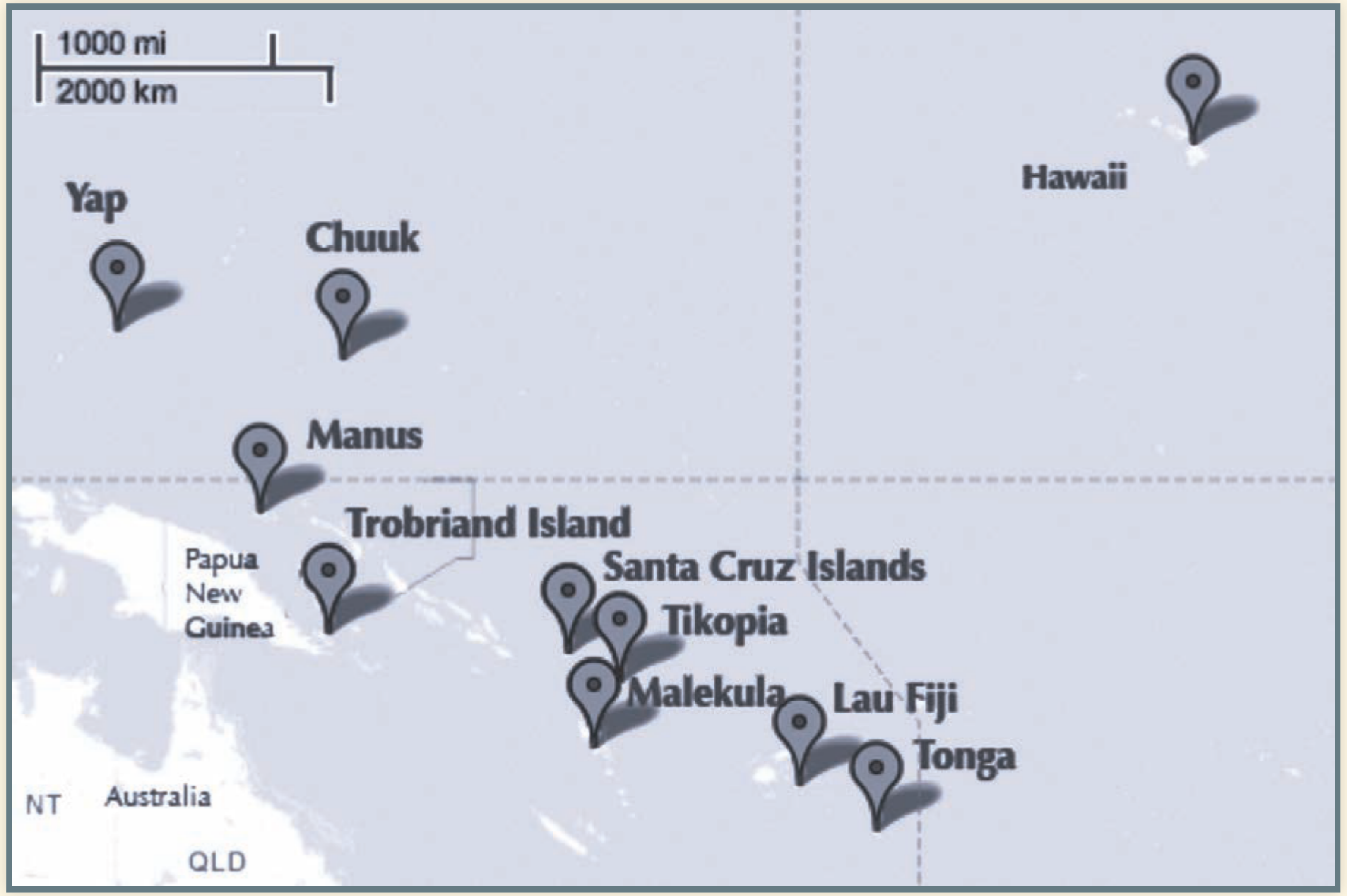
  - β_{EW} is consistent with the correct value 0.3.

Gaussian Processes

Continuous Categories

- Everything we've looked at with multilevel models so far has studied the varying effects of discrete categories (cafes, individual chimpanzees, academic departments, etc.).
  - Data is organized in discrete clusters.
  - There's no special order to the clusters, no significance of being cluster #1.
  - What about clustering on continuously distributed variables?
    - Height
    - Age
    - Income
    - …
  - **Gaussian Process** models allow us to study variation in slope or intercept for continuously distributed predictor variables.

- Distribution of Tools across the islands of Oceania



The Data

```
library(rethinking)
data(Kline)
data("islandsDistMatrix")

d <- Kline
dist <- islandsDistMatrix

kable(d)
```

culture	population	contact	total_tools	mean_TU
Malekula	1100	low	13	3.2
Tikopia	1500	low	22	4.7
Santa Cruz	3600	low	24	4.0
Yap	4791	high	43	5.0
Lau Fiji	7400	high	33	5.0
Trobriand	8000	high	19	4.0
Chuuk	9200	high	40	3.8
Manus	13000	low	28	6.6
Tonga	17500	high	55	5.4
Hawaii	275000	low	71	6.6

- **culture:** Name of island culture
- **population:** Historical population size
- **contact:** low or high contact rate with other islands
- **total tools**: number of tools in historical tool kit
- **mean TU:** a measure of tool complexity
- Distance matrix (thousands of km):

	Malekula	Tikopia	Santa Cruz	Yap	Lau Fiji	Trobriand
Malekula	0.000	0.475	0.631	4.363	1.234	2.036
Tikopia	0.475	0.000	0.315	4.173	1.236	2.007
Santa Cruz	0.631	0.315	0.000	3.859	1.550	1.708
Yap	4.363	4.173	3.859	0.000	5.391	2.462
Lau Fiji	1.234	1.236	1.550	5.391	0.000	3.219
Trobriand	2.036	2.007	1.708	2.462	3.219	0.000

Modeling Tool Acquisition

- Single-level Model:

$$T \sim \text{Poisson}(\lambda)$$
$$\lambda = \alpha P^{\beta}/\gamma$$

- Multilevel model:

$$T \sim \text{Poisson}(\lambda)$$
$$\lambda = \exp(k_{\text{society}})\alpha P^{\beta}/\gamma$$

  - k is the varying intercept.
  - Instead of assigning it randomly to each society category, we're going to try to predict it from the distance of other islands.

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ \dots \\ k_{10} \end{pmatrix} \sim \text{MVNormal}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}, K \right)$$

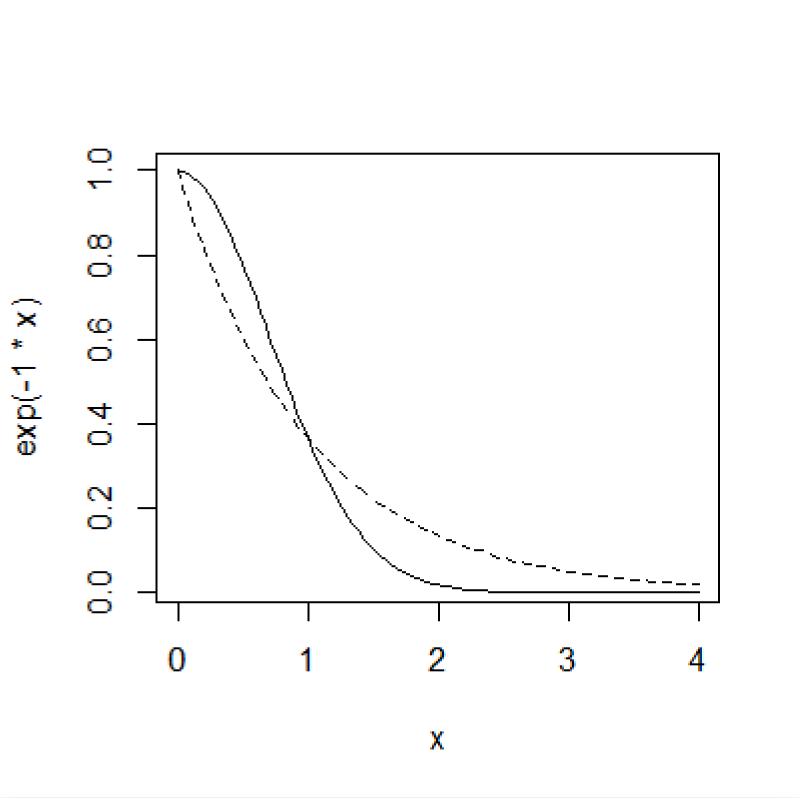
$$K_{i,j} = \eta^2 \exp(-\rho^2 D_{i,j}^2) + \delta_{i,j}\sigma^2$$

- $D_{i,j}$ is the distance between islands i and j
- $\delta_{i,j}$ is 1 if $i == j$ and 0 otherwise.
- 10-dimensional prior
- Covariance matrix K: $K_{i,j}$ is the covariance between societies i and j.
  - $\exp(-\rho^2 D_{i,j}^2)$ gives the covariance its shape.

Covariance Function

- Shape of the covariance function:
  - Set $\rho = 1$

```
curve( exp(-1*x) , from=0 , to=4 , lty=2 )
curve( exp(-1*x^2) , add=TRUE )
```



  - The dashed line is a linear distance, the solid has squared distance.

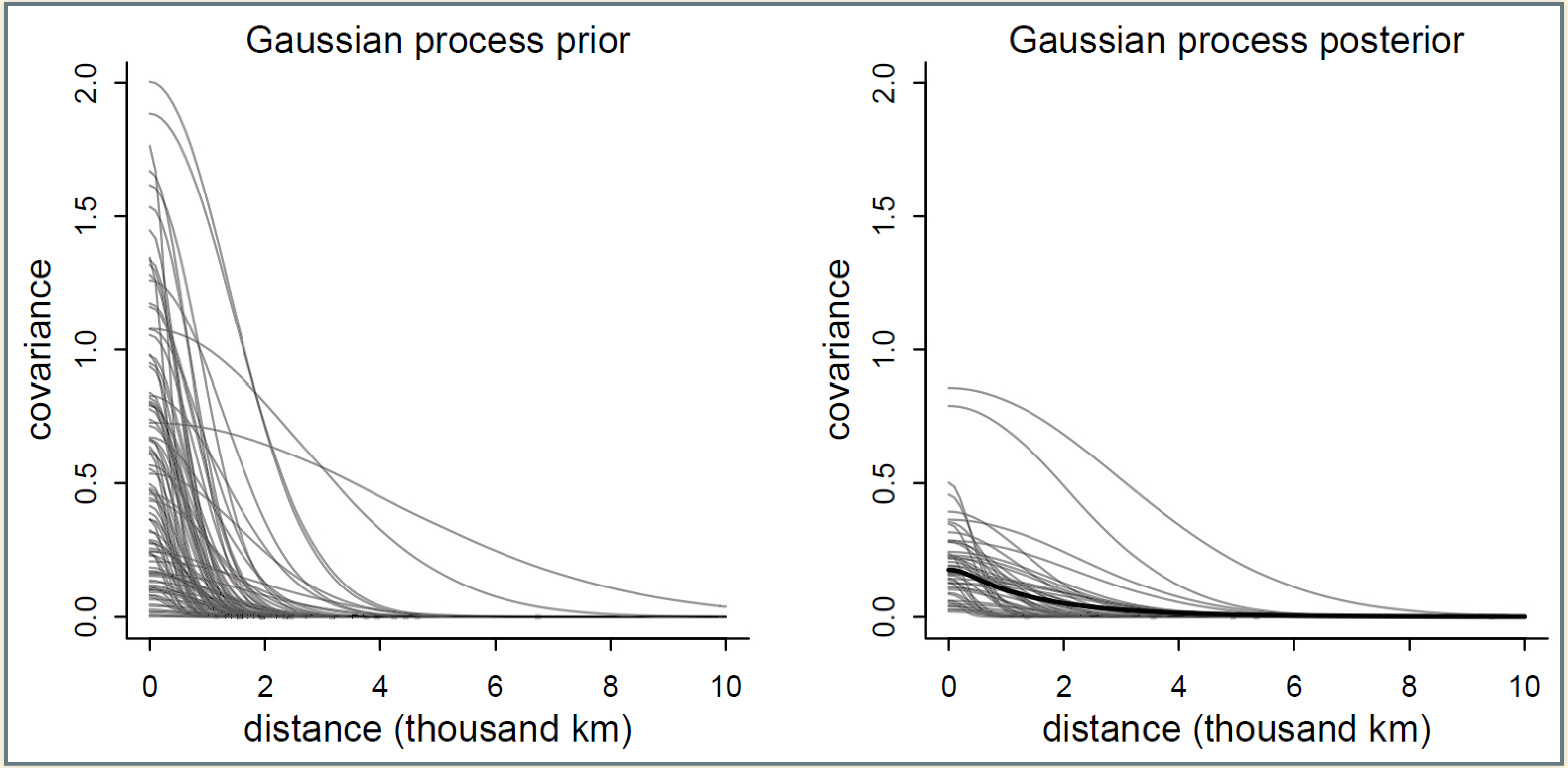
Gaussian Process Model

```
data(Kline2) # load the ordinary data, now with coordinates
d <- Kline2
d$society <- 1:10 # index observations

dat_list <- list(
    T = d$total_tools,
    P = d$population,
    society = d$society,
    Dmat=islandsDistMatrix )

mdl_gp <- ulam(
    alist(
        T ~ dpois(lambda),
        lambda <- (a * P^b / g) * exp(k[society]),
        vector[10]:k ~ multi_normal(0, SIGMA),
        matrix[10,10]:SIGMA <- cov_GPL2(Dmat, etasq, rhosq,
         0.01),
        c(a,b,g) ~ dexp(1),
        etasq ~ dexp(2),
        rhosq ~ dexp(0.5)
    ), data=dat_list, chains=4, cores=4, iter=2000 )
```

```
precis_show(precis(mdl_gp, digits=2, depth=3))
```

```
##          mean   sd  5.5% 94.5% n_eff Rhat4
## k[1]    -0.17 0.29 -0.66  0.29   918     1
## k[2]    -0.03 0.29 -0.48  0.43   871     1
## k[3]    -0.07 0.27 -0.50  0.35   854     1
## k[4]     0.34 0.26 -0.04  0.76   880     1
## k[5]     0.07 0.25 -0.31  0.45   867     1
## k[6]    -0.38 0.27 -0.83  0.01   943     1
## k[7]     0.14 0.25 -0.24  0.52   860     1
## k[8]    -0.21 0.25 -0.61  0.17   898     1
## k[9]     0.26 0.24 -0.10  0.64   865     1
## k[10]   -0.16 0.34 -0.69  0.37  1213     1
## g        0.58 0.56  0.07  1.62  1983     1
## b        0.27 0.08  0.14  0.41  1309     1
## a        1.40 1.08  0.25  3.43  2208     1
## etasq    0.18 0.19  0.03  0.52   763     1
## rhosq    1.36 1.70  0.08  4.58  1861     1
```

Posterior vs. Prior Covariance



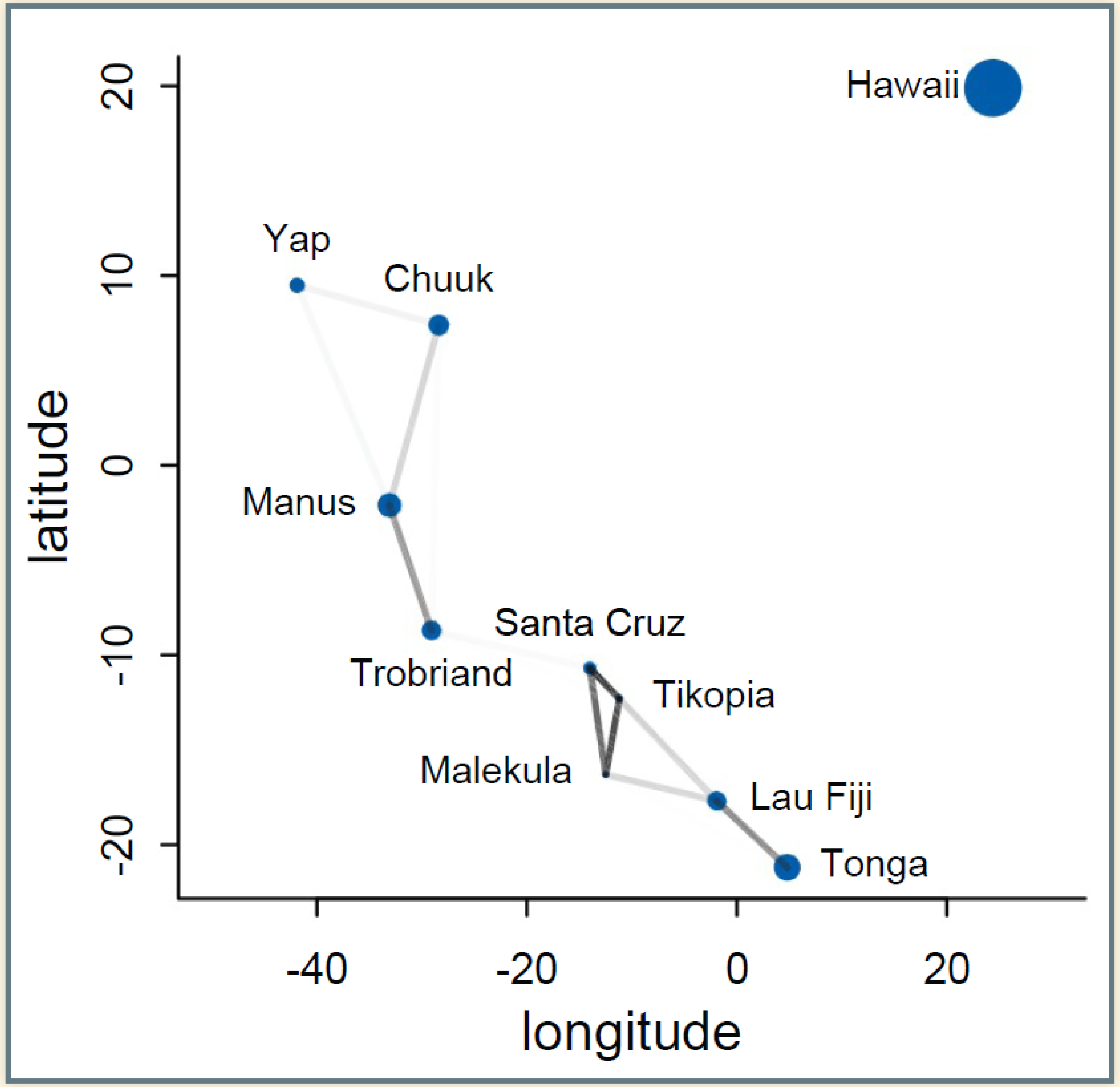
Posterior on the distance matrix

- Computer posterior median covariance:

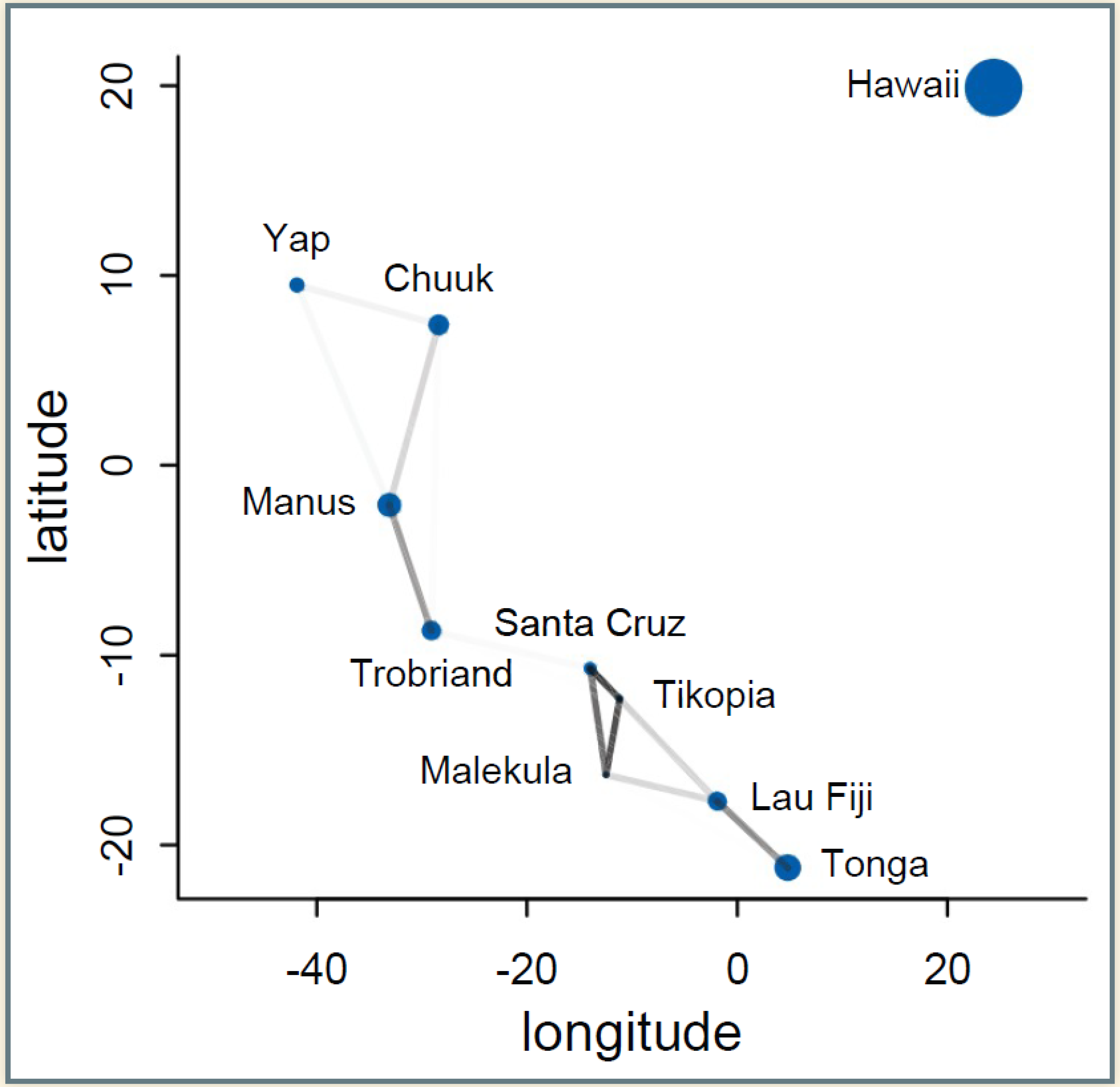
```
# compute posterior median covariance among societies
K <- matrix(0,nrow=10,ncol=10)
for ( i in 1:10 )
  for ( j in 1:10 )
    K[i,j] <- median(post$etasq) *
  exp( -median(post$rhosq) * islandsDistMatrix[i,j]^2 )
diag(K) <- median(post$etasq) + 0.01

## R code 14.43
# convert to correlation matrix
Rho <- round( cov2cor(K) , 2 )
# add row/col names for convenience
colnames(Rho) <-
        c("Ml","Ti","SC","Ya","Fi","Tr","Ch","Mn","To","Ha")
rownames(Rho) <- colnames(Rho)
Rho
```

```
##       Ml   Ti   SC   Ya   Fi   Tr   Ch   Mn   To Ha
## Ml 1.00 0.79 0.69 0.00 0.31 0.05 0.00 0.00 0.08  0
## Ti 0.79 1.00 0.86 0.00 0.31 0.05 0.00 0.01 0.06  0
## SC 0.69 0.86 1.00 0.00 0.16 0.11 0.01 0.02 0.02  0
## Ya 0.00 0.00 0.00 1.00 0.00 0.01 0.16 0.14 0.00  0
## Fi 0.31 0.31 0.16 0.00 1.00 0.00 0.00 0.00 0.61  0
## Tr 0.05 0.05 0.11 0.01 0.00 1.00 0.09 0.55 0.00  0
## Ch 0.00 0.00 0.01 0.16 0.00 0.09 1.00 0.32 0.00  0
## Mn 0.00 0.01 0.02 0.14 0.00 0.55 0.32 1.00 0.00  0
## To 0.08 0.06 0.02 0.00 0.61 0.00 0.00 0.00 1.00  0
## Ha 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00 0.00  1
```

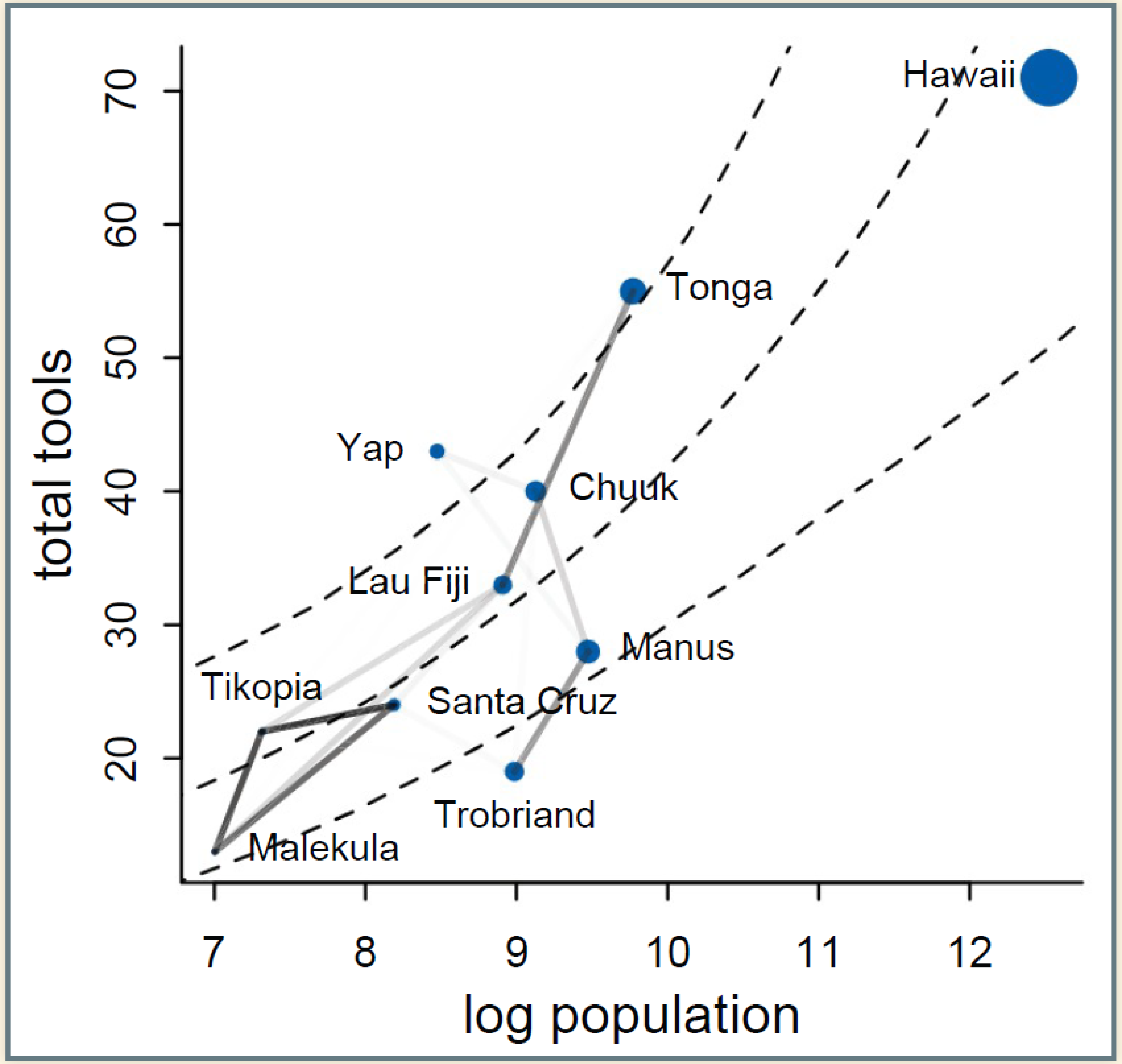
- Notice the cluster at the top left
- Map



Maps



Hawaii
Yap
Chuuk
Manus
Santa Cruz
Trobriand
Tikopia
Malekula
Lau Fiji
Tonga
latitude
20
10
0
-10
-20
-40
-20
0
20
longitude



Hawaii
Tonga
Yap
Chuuk
Lau Fiji
Manus
Tikopia
Santa Cruz
Trobriand
Malekula
total tools
70
60
50
40
30
20
7
8
9
10
11
12
log population