Conformal Field Theory

Lecture notes

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[Courses] are fantastically good for learning physics. The lecturer learns a lot of physics. After my first few studies, just about everything I learned about physics came from teaching it. I don't know if the students learned a lot, but I certainly did. So I consider teaching physics very important. — Leonard Susskind [**?**]

Contents

1 Introduction

$$\text{conformal field theory} = \text{quantum field theory} + \text{conformal symmetry} \quad (1.1)$$

where Poincaré symmetry means translations and Lorentz transformations (rotations and boosts)

$$\text{conformal field theory} = \text{relativistic quantum field theory+scale+special conformal symmetry} \quad (1.2)$$

most reviews focus first on conformal symmetry as a whole "geometric" approach to CFT

for good reasons! but often makes it difficult to connect with standard understanding of relativistic quantum field theory that we have from particle physics

things that are hard to connect to: - spectral representation, particles (is CFT a theory of massless particles? no!) - UV/IR divergences and anomalies these are precisely subjects that will be covered in these lectures

peculiarity of these lecture notes: "algebraic" approach to CFT review non-perturbative approach of QFT, then add scale and special conformal symmetry; "anatomical" approach allows to understand what are the consequences of each of them

possible motivation: study scale transformation to get Wilsonian RG

why not local scale transformations that respect Poincare symmetry?

Jacobian of conformal symmetry is composition of dilatation and rotation (see Rychkov's Lorentzian)

massive ϕ^4 theory in $d = 3$; statistical Ising model; critical point of boiling water (and other liquids);

same critical exponents!

universality of IR behavior is a hint that the space of CFTs is sparse

unlike QFT, where you can always add/remove some operator in your favorite Lagrangian

both Euclidean and Lorentzian!!!

bootstrap philosophy: focus on CFT, not on microscopic details

in the framework of quantum field theory, push mathematical understanding of conformal symmetry to the limit, with amazing results

non-perturbative; in fact, another strong motivation to study CFT is that it gives a mathematically rigorous definition of a QFT (a special one, but anyway); no need for "quantization" of classical Lagrangian theory (in fact no classical limit!)

1.1 What is conformal field theory?

Introduction: What is CFT? Why is it useful. Examples of CFT. Where does it fit into modern theoretical physics.

in the last few years, CFT has been dominated by

string theory: 2d CFT

holography: geometric approach

condensed matter physics: Euclidean

here focus on the "old" quantum field theory approach

links with: lattice, perturbation theory, etc.

fun fact: the conformal bootstrap was invented by particle physicists

strongly-coupled QFT

no need for action principle

1.2 Examples of conformal field theories

perturbative examples: ϕ^n theory in non-integer d

Caswell-Banks-Zaks

superconformal: $\mathcal{N} = 4$

any IR fixed point (maybe trivial)

use beyond CFT: correlators that have the isometries of the conformal group gravity in AdS but also late-time correlators in de Sitter

1.3 Outline

originally covered in 14 periods of 45 minutes each

split into 7 chapters?

1.4 Literature

excellent reviews, in order of relevance for the present course:

modern reviews, by conformal bootstrap experts

- Slava Rychkov's EPFL lectures [?] also historical references
- David Simmons-Duffin's TASI lectures [?]
- Shai Chester's Weizmann Lectures [?]

other reviews:

- state-of-the art [?]

- Hugh Osborn's course https://www.damtp.cam.ac.uk/user/ho/CFTNotes.pdf

- Joshua Qualls [?]

most Euclidean CFT, for Lorentzian perspective:

- Slava Rychkov's *Lorentzian methods in conformal field theory* https://courses.ipht.fr/node/226

focus on CFT in $d = 2$:

- Sylvain Ribault [?]
  also available on GitHub https://github.com/ribault/CFT-Review

- Schellekens "Conformal Field Theory" lecture notes [?]
  recent version available at https://www.nikhef.nl/~t58/CFT.pdf

also older literature on the subject in $d = 2$:

- Yellow book [?]

- Polchinski String theory vol. 1 [?]

- older lectures by Paul Ginsparg [?] emphasis on statistical physics and string theory applications

going further:

- superconformal symmetry: Lorenz Eberhardt [?]

2 Classical conformal transformations

One of the most fundamental principles of physics is independence of the reference frame: observers living at different points might have different perspectives, but the underlying physical laws are the same. This is true in space (invariance under translations and rotations), but also in space-time (e.g. invariance under Lorentz boosts).

2.1 Infinitesimal transformations

In mathematical language, this means that if we have a coordinate system x^μ, the laws of physics do not change under a transformations

$$x^\mu \to x'^\mu. \tag{2.1}$$

This principle applies to all maps that are invertible (isomorphisms) and differentiable (smooth transformations), hence it is usually called *diffeomorphism* invariance. Being differentiable, the transformation (2.1) can be Taylor-expanded to write

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x), \tag{2.2}$$

in terms of an infinitesimal vector ε^μ (meaning that we will always ignore terms of order ε^2).

In addition to the coordinate system, the description of a physical system requires a way of measuring distances that is provided by a metric $g_{\mu\nu}(x)$. Distances are measured integrating the line element

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu. \tag{2.3}$$

Since all observers should agree on the measure of distances, we must have

$$g'_{\mu\nu}(x')dx'^\mu dx'^\nu = g_{\mu\nu}(x)dx^\mu dx^\nu, \tag{2.4}$$

Here $g_{\mu\nu}$ could be the Euclidean metric $\delta_{\mu\nu}$ or the Minkowski metric $\eta_{\mu\nu}$; for simplicity we only consider the case in which $g_{\mu\nu}$ is flat, i.e. $\partial_\alpha g_{\mu\nu} = 0$. In this case, we can write

$$\begin{aligned} g'_{\mu\nu} &= g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} \\ &= g_{\alpha\beta}\left(\delta^\alpha_\mu - \partial_\mu\varepsilon^\alpha\right)\left(\delta^\beta_\nu - \partial_\nu\varepsilon^\beta\right) \\ &= g_{\mu\nu} - \left(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu\right). \end{aligned} \tag{2.5}$$

If we require the different observers to also agree on the metric, then we must have $g'_{\mu\nu} = g_{\mu\nu}$, which gives a constraint on what kind of coordinate transformations are possible: we must have

$$\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu = 0. \tag{2.6}$$

This condition admits as a most general solution

$$\varepsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu, \tag{2.7}$$

where a^μ is a constant vector and $g_{\mu\rho}\omega^\rho{}_\nu = \omega_{\mu\nu}$ an antisymmetric tensor. The transformation

$$x^\mu \xrightarrow{P} x^\mu + a^\mu \tag{2.8}$$

is obviously a translation and

$$x^\mu \xrightarrow{M} \omega^\mu{}_\nu x^\nu \tag{2.9}$$

a rotation/Lorentz transformation around the origin $x = 0$. The composition of these two operations generates the Poincaré group. This is the fundamental symmetry of space-time underlying all relativistic quantum field theory. It is a symmetry of nature to a very good approximation, at least up to energy scales in which quantum gravity becomes important.

However, one can also consider the situation in which the two observers use different systems of units, i.e. they disagree on the overall definition of scale, but agree otherwise on the metric being flat. In this case we must have $g'_{\mu\nu} \propto g_{\mu\nu}$, and therefore the constraint becomes

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 2\lambda g_{\mu\nu}, \tag{2.10}$$

for some real number λ, with the most general solution

$$\varepsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu. \tag{2.11}$$

The new infinitesimal transformation is

$$x^\mu \xrightarrow{D} (1 + \lambda) x^\nu. \tag{2.12}$$

It is a scale transformation. Note that scale symmetry is not a good symmetry of nature: there is in fact a fundamental energy scale on which all observer must agree (this can be for instance chosen to be the mass of the electron). Nevertheless, there are some systems in which this is a very good approximate symmetry. One can also make very interesting *Gedankenexperimente* that have scale symmetry built in. These are reasons that make it worth studying.

If one pushes this logic further, in a scale-invariant world in which observers have no physical mean of agreeing on a fundamental scale, they might even decide to change their definition of scale as they walk around. This would correspond to the case in which the metric $g'_{\mu\nu}$ of one observer can differ from the original metric $g_{\mu\nu}$ by a function of space(-time):

$$g'_{\mu\nu}(x) = \Omega(x) g_{\mu\nu}. \tag{2.13}$$

Note that we are not saying that $g'_{\mu\nu}$ is completely arbitrary: at every point in space time it is related to the flat metric by a scale transformation. But the scale factor is different at every point. The condition on ε^μ becomes in this case

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 2\sigma g_{\mu\nu}, \tag{2.14}$$

where $\Omega(x) = 1 + 2\sigma(x)$. To find the most general solution to this equation, note that contracting the indices with $g^{\mu\nu}$ gives

$$\partial_\mu \varepsilon^\mu = d\sigma, \tag{2.15}$$

where d is the space(-time) dimension, while acting with ∂^ν gives

$$\partial_\mu \partial_\nu \varepsilon^\nu + \partial^2 \varepsilon_\mu = 2\partial_\mu \sigma, \tag{2.16}$$

so that we get

$$\partial^2 \varepsilon_\mu = (2 - d)\partial_\mu \sigma. \tag{2.17}$$

Acting once again with ∂^μ, we arrive at

$$(d - 1)\partial^2 \sigma = 0, \tag{2.18}$$

while acting with ∂^ν and symmetrizing the indices, we find

$$(2 - d)\partial_\mu \partial_\nu \sigma = g_{\mu\nu} \partial^2 \sigma = 0. \tag{2.19}$$

In $d > 2$, we obtain therefore the condition $\partial_\mu \partial_\nu \sigma = 0$, which is solved by

$$\sigma(x) = \lambda + 2b \cdot x. \tag{2.20}$$

We have therefore

$$\varepsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + 2(b \cdot x)x^\mu - x^2 b^\mu. \tag{2.21}$$

In addition to the transformations found before, we also find

$$x^\mu \xrightarrow{K} 2(b \cdot x)x^\mu - x^2 b^\mu, \tag{2.22}$$

which is a *special conformal transformation.* If we examine the Jacobian for this transformation, we find

$$\frac{\partial x'^\mu}{\partial x^\nu} = (1 + 2b \cdot x)\, \delta^\mu_\nu + 2\left(b_\nu x^\mu - x_\nu b^\mu\right) \approx (1 + 2b \cdot x)\, R^\mu{}_\nu(x). \tag{2.23}$$

We have written this as a position-dependent scale factor $(1 + 2b \cdot x)$, multiplying an orthogonal matrix

$$R^\mu{}_\nu(x) = \delta^\mu_\nu + 2\left(b_\nu x^\mu - x_\nu b^\mu\right). \tag{2.24}$$

This shows that special conformal transformation act locally as the composition of a scale transformation and a rotation (or Lorentz transformation). Eq. (2.14) is sometimes called the (conformal) Killing equation and its solutions (2.21) the Killing vectors.

Note that in our derivation the original metric $g_{\mu\nu}$ was flat, but the new metric $g'_{\mu\nu}$ is not. It is however conformally flat: it is always possible to make a change of coordinate after which it is flat. In general, transformations

$$g_{\mu\nu}(x) \to \Omega(x)^2 g_{\mu\nu}(x) \tag{2.25}$$

are called *Weyl transformations*. They change the geometry of space-time. We found that any Weyl transformation which is at most quadratic in x can be compensated by a change of coordinates to go back to flat space. The corresponding flat-space symmetry is called conformal symmetry.[1]

Note that in $d = 2$ the situation is a bit different: the conditions $\partial^2\sigma = 0$ is sufficient to ensure that the Killing equation has a solution. This is most easily seen in light-cone coordinates,

$$x^+ = \frac{x^0 + x^1}{2}, \qquad x^- = \frac{x^0 - x^1}{2}, \tag{2.26}$$

in terms of which

$$\partial^2\sigma = \partial_+\partial_-\sigma. \tag{2.27}$$

This is satisfied by taking for σ the sum of an arbitrary function of the left-moving variable x^+ and of another function of the right-moving variable x^-. In fact, if we write $\varepsilon^\pm = \varepsilon^0 \pm \varepsilon^1$, we can take arbitrary functions $\varepsilon^+(x^+)$ and $\varepsilon^-(x^-)$, and verify that eq. (2.14) is satisfied with $\sigma = \frac{1}{2}\left(\partial_+\varepsilon_+ + \partial_-\varepsilon_-\right)$. In the Euclidean case we take

$$z = \frac{x^1 + ix^2}{2}, \qquad \bar{z} = \frac{x^1 - ix^2}{2}, \tag{2.28}$$

complex-conjugate to each other, and the same logic follows: we can apply arbitrary holomorphic and anti-holomorphic transformations on z and $\bar{z}$, and the conformal Killing equation is always satisfied. This shows that there are infinitely more conformal transformations in $d = 2$ than in $d > 2$, and also that there is no significant difference between two-dimensional Euclidean and Minkowski conformal symmetry, as the symmetry acts essentially on the two light-cone/holomorphic coordinates independently.

[1] This implies that the group of conformal transformation is a subgroup of diffeomorphisms. It is in fact the largest *finite-dimensional* subgroup.

2.2 The conformal algebra

The conformal Killing equation (2.21) determines the most general form of *infinitesimal* conformal transformations. *Finite* conformal transformations follow from a sequence of infinitesimal transformations. However, one has to bear in mind that the infinitesimal conformal transformations do not all commute: for instance, a translation followed by a rotation is not the same as the opposite. In fact, the conformal transformations form a *group*: the composition of conformal transformations is again a conformal transformations.

As we all know from quantum field theory, a group is characterized by its *generators* and their commutation relations (the *algebra*). A generator G describes an infinitesimal transformation in some direction, and finite transformation are obtained using exponentiation, $e^{i\theta G}$, with parameter θ (the factor of i is the physicist's convention that make the generators Hermitian). A representation of the conformal group is given on the functions of the coordinates, $f(x)$. For instance, under an infinitesimal translation, we have

$$f(x) \xrightarrow{P} f(x') = f(x+a) \approx f(x) + a^\mu \partial_\mu f(x) \tag{2.29}$$

and we require this to be equal to $e^{-ia_\mu P^\mu} f(x)$, which means

$$P_\mu = i\partial_\mu . \tag{2.30}$$

Performing the same analysis for the other infinitesimal transformations given in eq. (2.21), we obtain for the other generators[2]

rotations/Lorentz transformations: $$M^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right) \tag{2.31}$$

dilatations: $$D = i x^\mu \partial_\mu , \tag{2.32}$$

special conformal transformations: $$K^\mu = i\left(2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu\right) . \tag{2.33}$$

The number of generators matches that of the Killing vectors: there are d translations, d special conformal transformations, $d(d-1)/2$ rotations/Lorentz transformations ($M^{\mu\nu}$ is a $d \times d$ antisymmetric matrix), and one scale transformation. Therefore the total number of generators, i.e. the dimension of this group, is $(d+1)(d+2)/2$. In $d = 4$ space-time dimension, the conformal group has 15 generators.

[2]The sign of these generators is an arbitrary convention. It defines once and for all the commutations relations that we will derive next. After that, we will always refer to the commutation relations as the definition of the generators.

Using the above definition, one can verify that the following commutation relations are satisfied,

$$
\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= -i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}\right] \\
[M^{\mu\nu}, P^{\rho}] &= -i\left(g^{\mu\rho}P^{\nu} - g^{\nu\rho}P^{\mu}\right) \\
[M^{\mu\nu}, K^{\rho}] &= -i\left(g^{\mu\rho}K^{\nu} - g^{\nu\rho}K^{\mu}\right) \\
[D, P^{\mu}] &= -iP^{\mu} \\
[D, K^{\mu}] &= iK^{\mu} \\
[P^{\mu}, K^{\nu}] &= 2i\left(g^{\mu\nu}D - M^{\mu\nu}\right)
\end{aligned}
\tag{2.34}
$$

while all other commutators vanish:

$$
[M^{\mu\nu}, D] = [P^{\mu}, P^{\nu}] = [K^{\mu}, K^{\nu}] = 0. \tag{2.35}
$$

The first two relations in eq. (2.34) are the familiar Poincaré algebra. The next one states that K^{μ} transforms like a vector (as P^{μ} does), whereas D is obviously a scalar. The next two relations remind us that K^{μ} and P^{μ} have respectively the dimension of length and inverse length.

Exercise 2.1 Derive the commutation relations from the action (2.30)–(2.33) of the generators on functions of x.

Even though this is not immediately obvious, this algebra is isomorphic to that of the group SO$(d+1,1)$ (if $g^{\mu\nu}$ is the Euclidean metric) or SO$(d,2)$ (if it is the Minkowski metric). To see that it is the case, let us introduce a $(d+2)$-dimensional space with coordinates

$$
X^{\mu}, \quad X^{d+1}, \quad X^{d+2}, \tag{2.36}
$$

and a metric defined by the line element

$$
ds^2 = g_{\mu\nu}dX^{\mu}dX^{\nu} + dX^{d+1}dX^{d+1} - dX^{d+2}dX^{d+2} \equiv \eta_{MN}dX^{M}dX^{N}. \tag{2.37}
$$

Then we write all conformal commutation relations as being defined by the Lorentzian algebra

$$
\left[J^{MN}, J^{RS}\right] = -i\left(\eta^{MR}J^{NS} - \eta^{MS}J^{NR} - \eta^{NR}J^{MS} + \eta^{NS}J^{MR}\right], \tag{2.38}
$$

provided that we identify the antisymmetric generators J^{MN} with the conformal generators as follows:

$$
\begin{aligned}
M^{\mu\nu} &= J^{\mu\nu}, \\
P^{\mu} &= J^{\mu,d+1} + J^{\mu,d+2}, \\
K^{\mu} &= J^{\mu,d+1} - J^{\mu,d+2}, \\
D &= J^{d+1,d+2}.
\end{aligned}
\tag{2.39}
$$

2.3 Finite transformations

most general conformal transformation can be written as combination of exponentiated form of infinitesimal special conformal transformation

translation: easy to exponentiate

$$x^\mu \xrightarrow{P} x^\mu + a^\mu \tag{2.40}$$

a not necessarily infinitesimal

same is true of special conformal transformations:

$$x^\mu \to \lambda x^\mu \tag{2.41}$$

with finite λ

rotations:

$$x^\mu \xrightarrow{M} \Lambda^\mu{}_\nu x^\nu \tag{2.42}$$

where $\Lambda^\mu{}_\nu$ is a $\mathrm{SO}(d)$ or $\mathrm{SO}(1, d-1)$ matrix

special conformal transformation: in infinitesimal form:

$$x'^\mu = x^\mu + 2(b \cdot x)x^\mu - x^2 b^\mu \tag{2.43}$$

this implies

$$x'^2 = x^2 + 2(b \cdot x)x^2 \tag{2.44}$$

neglecting terms of order b^2

hence

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu \tag{2.45}$$

use inversions: discrete transformations not connected to identity (no infinitesimal form!)

$$x^\mu \xrightarrow{I} \frac{x^\mu}{x^2} \tag{2.46}$$

Jacobian of inversion is given by

$$\frac{\partial x'^\mu}{\partial x^\nu} = \frac{1}{x^2}\left[\delta^\mu_\nu - 2\frac{x^\mu x_\nu}{x^2}\right] \tag{2.47}$$

in Euclidean space, choose a frame in which $\vec{x} = (x, 0, \ldots 0)$, then the object in square bracket is the diagonal matrix $\mathrm{diag}(-1, 1, \ldots, 1)$, which is part of $\mathrm{O}(d)$ but not $\mathrm{SO}(d)$

inversion followed by a translation, followed by an inversion again gives SCT

$$x^\mu \xrightarrow{K} x'^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2b \cdot x + b^2 x^2} \tag{2.48}$$

Exercise 2.2 Use eq. (2.45) to show (2.48). *Hint: Contract both sides of eq. (2.45) with x_μ, x'_μ and b_μ, and use these 3 equations to solve for x'^2.*

What do special transformation do globally? Let us look specifically in Euclidean space. There are some special points:

- leaves the origin invariant: the point $x = 0$ is mapped onto itself
- but singular: in Euclidean space, the point b^μ/b^2 is mapped to ∞
- and conversely: ∞ to point $-b^\mu$

Exercise 2.3 Show that under the special conformal transformation (2.48), a sphere centered at the point a^μ and with radius R gets mapped to a sphere centered at the point

$$a'^\mu = \frac{a^\mu - (a^2 - R^2)b^\mu}{1 - 2a\cdot b + (a^2 - R^2)b^2}$$

and with radius

$$R' = \frac{R}{|1 - 2a\cdot b + (a^2 - R^2)b^2|}.$$

In the special case in which b^μ/b^2 is on the original sphere, show that the sphere gets mapped to a plane orthogonal to the vector $a^\mu + (R^2 - a^2)b^\mu$.

translations keep ∞ fixed, and sct keep origin fixed; because related by inversion!

given a set of 3 points $\{x_1, x_2, x_3\}$ that are not on a line, and another set $\{x'_1, x'_2, x'_3\}$ with the same property, there exists a unique conformal transformation that maps the first set onto the second (and conversely)

If the 3 points are on a line it is obvious that a rotation in the space orthogonal to the line won't change the position of the points, so the

physical consequence: in correlation functions of 2 or 3-points (see next sections for a definition), all kinematics is fixed by conformal symmetry; the only freedom encodes information about the operators themselves, not about their position in space

[3]

[3] In $d = 2$, the infinite conformal group means that (nearly) any shape can be mapped onto an other. This is known as the Riemann mapping theorem. For an example of how a circle can be mapped to a polygon, see https://herbert-mueller.info/uploads/3/5/2/3/35235984/circletopolygon.pdf.

2.4 Compactifications

conformal symmetry is a symmetry of flat space(-time)

but sometimes it is also convenient to make a change of variables that maps the flat Euclidean space $\mathbb{R}^d$ or the Minkowski space-time $\mathbb{R}^{1,d-1}$ to a compact manifold

have a gives a more geometric picture of what conformal transformation do

cylinder

2.5 Minkowski space-time

place an observer at the origin: this breaks translations, so I'm not going to use translations anymore, but the rest of the conformal group

Lorentz and scale transformations preserve causality: if a point x is space-like separated from the origin, it will remain space-like separated no matter the choice of scale and Lorentz frame

without loss of generality, can choose it to be at $x^0 = 0$, $x^1 = 1$ (I choose my units in this way), and all other $x^i = 0$

however, I can use a SCT to bring it to the past light cone!

take $b^\mu =$

in fact, we can even follow the path that the point takes when we vary b from 0 to its final value: does it cross light-cone? or does it goes to infinity?

weak vs. strong conformal invariance Martin Lüscher and Gerhard Mack, 1974 [**?**] (also earlier paper [**?**]

Lorentzian cylinder

2.6 The energy-momentum tensor

classical field theory picture

Noether current associated with conformal symmetry

use the metric as source?

condition for Weyl invariance is tracelessness of energy-momentum tensor