# Conformal Field Theory

## Lecture notes

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Spring semester 2022

*[Lectures] are fantastically good for learning physics. The lecturer learns a lot of physics. After my first few studies, just about everything I learned about physics came from teaching it. I don't know if the students learned a lot, but I certainly did. So I consider teaching physics very important.* — Leonard Susskind

# Contents

# 1 Introduction

## 1.1 What is conformal field theory?

Introduction: What is CFT? Why is it useful. Examples of CFT. Where does it fit into modern theoretical physics.

in the last few years, CFT has been dominated by

string theory: 2d CFT

holography: geometric approach

condensed matter physics: Euclidean

here focus on the "old" quantum field theory approach

links with: lattice, perturbation theory, etc.

fun fact: the conformal bootstrap was invented by particle physicists

**strongly-coupled QFT**

no need for action principle

## 1.2 Examples of conformal field theories

perturbative examples: $\phi^n$ theory in non-integer $d$

Caswell-Banks-Zaks

superconformal: $\mathcal{N} = 4$

any IR fixed point (maybe trivial)

use beyond CFT: correlators that have the isometries of the conformal group gravity in AdS but also late-time correlators in de Sitter

## 1.3 Outline

originally covered in 14 periods of 45 minutes each

split into 7 chapters?

# 2 Classical conformal symmetry

Poincaré symmetry: physics is the same in every coordinate frame

linear transformation of the coordinates

$$x'^{\mu} = x^{\mu} + \omega^{[\mu\nu]}x_{\nu} + a^{\mu} \tag{2.1}$$

infinitesimal line element $dx^2 = \eta_{\mu\nu}dx^{\mu}dx^{\nu}$ is invariant:

$$dx'^2 = dx^2 \tag{2.2}$$

scale symmetry:

$$x'^{\mu} = \lambda x^{\mu} \tag{2.3}$$

note that this can be seen as a rescaling of the metric, of the form $\eta'^{\mu\nu} = \#\eta^{\mu\nu}$

can we generalize this? general transformations of the metric $\eta'^{\mu\nu} = \Omega(x)\eta^{\mu\nu}$ called a *Weyl transformation*

in general changes geometry of space-time; unless this transformation is equivalent to a coordinate transformation (diffeomorphism)

in infinitesimal form

$$x'^{\mu} = x^{\mu} + v^{\mu}(x) \tag{2.4}$$

requiring that this is equal to the Weyl transformation gives the conformal Killing equation:

$$\partial^{\mu} v^{\nu}(x) + \partial^{\nu} v^{\mu}(x) = 2\sigma(x)\eta^{\mu\nu} \tag{2.5}$$

work out solution in $d = 1$, $d = 2$ and $d > 2$;

conformal Killing vectors (see Osborn's notes): translation Lorentz transformations dilatation special conformal symmetry

in $d = 2$, use $z$, $\bar{z}$ notation and Laurent series

finite transformations

use inversions: discrete transformations not connected to identity

but inversion followed by translation followed by inversion gives special conformal transformations

group of conformal transformation is a subgroup of diffeomorphisms; in fact the largest *finite-dimensional* subgroup

defining property

$$g'_{\mu\nu}(x') = \Omega(x)^2 \eta_{\mu\nu} \tag{2.6}$$

gives Jacobian of conformal transformation

$$\frac{\partial x^{\mu}}{\partial x^{\nu}} = \Omega(x) M^{\mu}{}_{\nu}(x) \tag{2.7}$$

$M$: $\mathrm{SO}(d-1,1)$ matrix

conformal transformations look locally like Lorentz and scale transformations

## 2.1 The conformal algebra

generators corresponding to Killing vectors

compute commutators, find Lorentz and Poincaré subalgebras, and new commutations involving $D$ and $K$

isomorphic to $\mathrm{SO}(d,2)$

(exercise)

note: everything we said is also valid in Euclidean space: replace $\eta^{\mu\nu} \to \delta^{\mu\nu}$

get the Euclidean conformal group

## 2.2 Examples

conformal transformations map circles to circles (including circles with infinite radius, aka straight lines): explanation

picture

think of it as renormalization group transformation that depends on space-time

Escher?

# 3 Non-perturbative quantum field theory

Wightman axioms

operators are not necessarily invariant under conformal symmetry, but they transform unitarily

unitary representations on Hilbert space

(e.g. pair of point-like particle

no action principle (i.e. no need for “quantization”)

only deal with **local** operators

warning about meaning of locality: related to causality?

generator of translations plays a special role: energy and momentum vector

positivity of energy

Wigner construction: use reference momentum

generator of Lorentz transformation can be diagonalize at a point: the origin of space-time

get the action of $M^{\mu\nu}$ at arbitrary point

## 3.1 Spectral representation

operator vs. field

## 3.2 Scale symmetry

action of generator of translations

no mass (often said)

or better said: all masses!
reference vector, then construct spectral density

## 3.3 Special conformal symmetry

special conformal generator implies two things:

primary and descendant operators

special condition on 2-point function of primary operators: that they have identical scaling dimensions

note that scale invariance and unitarity generally imply that the trace of the energy-momentum tensor is zero: then Hamiltonian is invariant under change of the metric $\delta g_{\mu\nu} = \sigma(x)\eta_{\mu\nu}$,

$$\delta H = \int d^d x\, T_{\mu\nu}\delta g^{\mu\nu} = \int d^x\, \sigma(x) T^\mu_\mu = 0 \tag{3.1}$$

invariance under infinitesimal Weyl transformations

note that $T_{\mu\nu}$ is only defined in flat space, or rather that it acquires a vacuum expectation value in curved space: the Weyl anomaly so let us only consider Weyl transformations that do not change the geometry of space-time: those that are equivalent to a coordinate transformation (a diffeomorphism)

exercise: compute conformal generators in momentum-space representation

Mack's classification in 4d

representation: long and short multiplets!

compare this with Slava's approach:

organize operators into representation of conformal group:

call some operators *primary*

operators are local, which mean that they do not feel the effect of the transformation of the metric, or only through coordinate dependence: $\phi(x) \to \phi(x') = \Omega(x)^{-\Delta}\phi(x)$

note: if $\phi$ transforms like this, then $\partial_\mu\phi(x)$ does not! call it a *descendant*

comment on scale vs. conformal invariance

correlation functions:

no need for action principle

## 3.4 Exercises

show that the spectral density in the case $\Delta = \frac{d}{2} - 1$ is that of a free scalar field;

show also that in the case $\Delta = \frac{d}{2}$ it is given by the phase space of two massless particle states (see Rychkov eq. 1.13)

### 3.5 Bibliography

itzikson zuber

## 4 Conformal correlation functions

2-pt function: go from Wightman 2-pt function in momentum space to Euclidean 2-pt function in position space in two ways:

- Fourier transform first, then analytic continuation of Wightman function - first construct T product (Källen-Lehmann representation), then Wick rotation to Euclidean momentum space, then Fourier transform

T-products vs. Wightman functions

use retarded products and micro-causality

Ward identities

more identities for conserved currents and the energy-momentum tensor (there cannot be higher-spin conserved currents otherwise the theory must be free; reference?)

### 4.1 Spectral representation for spinning operators

unitary bounds

### 4.2 From momentum to position

taking fourier transforms

### 4.3 From Minkowski to Eulidean space

Osterwalder-Schrader

analytic continuation of Wightman functions vs. Wick rotation of T-products

reflection positivity

note that this is usually done in the other direction!

### 4.4 Embedding-space formalism

conformal group acts linearly in embedding space with $d + 2$ dimensions, among which two times

$X'^M = \Lambda^M_{\ N} X^N$

get rid of two dimensions by restricting to light-cone $X^2 = 0$, and identifying $X^+ \sim \lambda X^+$

can take $X^+ = 1$, so that

$$(X^+, X^-, X^\mu) = (1, x^2, x^\mu) \tag{4.1}$$

fields defined on the cone, depending homogeneously on $X$
rewrite 2-point function
see section 2.1 of Rychkov

### 4.5 3-point functions

scalar 3-point function
remarkable fact! fixed up to a single coefficient
if only scale invariance, this could be arbitrarily complicated
historical note (cite Rychkov, around eq. 2.43): birth of conformal field theory

check: 2-point functions involving conserved currents and energy-momentum tensors are automatically conserved; for 3-point functions this gives additional constraint: scaling dimensions of operators must be equal

### 4.6 4-point functions

invariant cross-ratios for 4-point functions

## 5 State-operator correspondence and OPE

important remark on OPE convergence in Euclidean space!

## 6 The conformal bootstrap

### 6.1 Conformal blocks

### 6.2 The numerical boostrap

generalized free fields/Gaussian

### 6.3 Results

discussion about universality: see Poland, Rychkov, Vichi

# 7 Selected advanced topics

light-cone limit and ?

Virasoro symmetry in 2d:

conformal anomalies: anomalies are contact term in the action for the source

superconformal bootstrap? note classification of all superconformal algebras

## 7.1 How to continue from here

- holography: see Joao's TASI lectures
- superconformal: Tajikawa pedestrian lectures
- bootstrap: Shai Chester's lectures
- condensed matter: note that unitarity is unnecessary: many non-unitary fixed points, yet they have conformal invariance (why?)