Conformal Field Theory

Lecture notes

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This is an introductory course in which the students will learn what is conformal field theory, why it is special, and get a glimpse of modern developments. The course will begin with a non-perturbative formulation of quantum field theory (Wightman functions, spectral representation), and then gradually focus on understanding the implications of scale and special conformal symmetry. The study of practical tools (embedding space formalism, radial quantization, state-operator correspondence, conformal blocks) will finally lead to the formulation of the modern conformal bootstrap and a review of its most recent results. Some advanced topics will be discussed depending on the students' interests (Virasoro symmetry in 2 dimensions, UV and IR divergences, Mellin representation, superconformal symmetry).

> *[Courses] are fantastically good for learning physics. The lecturer learns a lot of physics. After my first few studies, just about everything I learned about physics came from teaching it. I don't know if the students learned a lot, but I certainly did. So I consider teaching physics very important.* — Leonard Susskind [1]

Contents

1 Introduction

$$\text{conformal field theory} = \text{quantum field theory} + \text{conformal symmetry} \quad (1.1)$$

where Poincaré symmetry means translations and Lorentz transformations (rotations and boosts)

$$\text{conformal field theory} = \text{relativistic quantum field theory} + \text{scale} + \text{special conformal symmetry} \quad (1.2)$$

most reviews focus first on conformal symmetry as a whole "geometric" approach to CFT

for good reasons! but often makes it difficult to connect with standard understanding of relativistic quantum field theory that we have from particle physics

things that are hard to connect to: - spectral representation, particles (is CFT a theory of massless particles? no!) - UV/IR divergences and anomalies these are precisely subjects that will be covered in these lectures

peculiarity of these lecture notes: "algebraic" approach to CFT review non-perturbative approach of QFT, then add scale and special conformal symmetry; "anatomical" approach allows to understand what are the consequences of each of them

possible motivation: study scale transformation to get Wilsonian RG

why not local scale transformations that respect Poincare symmetry?

Jacobian of conformal symmetry is composition of dilatation and rotation (see Rychkov's Lorentzian)

1.1 What is conformal field theory?

Introduction: What is CFT? Why is it useful. Examples of CFT. Where does it fit into modern theoretical physics.

in the last few years, CFT has been dominated by

string theory: 2d CFT

holography: geometric approach

condensed matter physics: Euclidean

here focus on the "old" quantum field theory approach

links with: lattice, perturbation theory, etc.

fun fact: the conformal bootstrap was invented by particle physicists

strongly-coupled QFT

no need for action principle

1.2 Examples of conformal field theories

perturbative examples: ϕ^n theory in non-integer d

Caswell-Banks-Zaks

superconformal: $\mathcal{N} = 4$

any IR fixed point (maybe trivial)

use beyond CFT: correlators that have the isometries of the conformal group gravity in AdS but also late-time correlators in de Sitter

1.3 Outline

originally covered in 14 periods of 45 minutes each

split into 7 chapters?

1.4 Literature

excellent reviews, in order of relevance for the present course:

modern reviews, by conformal bootstrap experts

- Slava Rychkov's EPFL lectures [2] also historical references

- David Simmons-Duffin's TASI lectures [3]

- Shai Chester's Weizmann Lectures [4]

other reviews:

- state-of-the art [5]

- Hugh Osborn's course https://www.damtp.cam.ac.uk/user/ho/CFTNotes.pdf

- Joshua Qualls [6]

most Euclidean CFT, for Lorentzian perspective:

- Slava Rychkov's *Lorentzian methods in conformal field theory* https://courses.ipht.fr/node/226

focus on CFT in $d = 2$:

- Sylvain Ribault [7]
  also available on GitHub https://github.com/ribault/CFT-Review

- Schellekens "Conformal Field Theory" lecture notes [8]
  recent version available at https://www.nikhef.nl/~t58/CFT.pdf

also older literature on the subject in $d = 2$:

- Yellow book [9]

- Polchinski String theory vol. 1 [10]

- older lectures by Paul Ginsparg [11] emphasis on statistical physics and string theory applications

going further:

- superconformal symmetry: Lorenz Eberhardt [12]

2 Classical conformal symmetry

Poincaré symmetry: physics is the same in every coordinate frame
linear transformation of the coordinates

$$x'^{\mu} = x^{\mu} + \omega^{[\mu\nu]}x_{\nu} + a^{\mu} \tag{2.1}$$

infinitesimal line element $dx^2 = \eta_{\mu\nu}dx^{\mu}dx^{\nu}$ is invariant:

$$dx'^2 = dx^2 \tag{2.2}$$

scale symmetry:

$$x'^{\mu} = \lambda x^{\mu} \tag{2.3}$$

note that this can be seen as a rescaling of the metric, of the form $\eta'^{\mu\nu} = \#\eta^{\mu\nu}$

can we generalize this? general transformations of the metric $\eta'^{\mu\nu} = \Omega(x)\eta^{\mu\nu}$ called a *Weyl transformation*

in general changes geometry of space-time; unless this transformation is equivalent to a coordinate transformation (diffeomorphism)

in infinitesimal form

$$x'^{\mu} = x^{\mu} + v^{\mu}(x) \tag{2.4}$$

requiring that this is equal to the Weyl transformation gives the conformal Killing equation:

$$\partial^{\mu}v^{\nu}(x) + \partial^{\nu}v^{\mu}(x) = 2\sigma(x)\eta^{\mu\nu} \tag{2.5}$$

work out solution in $d = 1$, $d = 2$ and $d > 2$;
conformal Killing vectors (see Osborn's notes): translation Lorentz transformations dilatation special conformal symmetry

in $d = 2$, use z, $\bar{z}$ notation and Laurent series
finite transformations

use inversions: discrete transformations not connected to identity

but inversion followed by translation followed by inversion gives special conformal transformations

group of conformal transformation is a subgroup of diffeomorphisms; in fact the largest *finite-dimensional* subgroup

defining property

$$g'_{\mu\nu}(x') = \Omega(x)^2 \eta_{\mu\nu} \tag{2.6}$$

gives Jacobian of conformal transformation

$$\frac{\partial x^\mu}{\partial x^\nu} = \Omega(x) M^\mu{}_\nu(x) \tag{2.7}$$

M: $SO(d-1,1)$ matrix

conformal transformations look locally like Lorentz and scale transformations

translation move 0, leave ∞ fixed

special conformal transformation move ∞, leave 0 fixed

finite transformation derived from infinitesimal: for rotations, translations and scale exactly the same; for sct

$$x'^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2b \cdot x + b^2 x^2} \tag{2.8}$$

or

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu \tag{2.9}$$

2.1 The conformal algebra

follow Shester

generators corresponding to Killing vectors

when applied to function of x, gives derivative action

compute commutators, find Lorentz and Poincaré subalgebras, and new commutations involving D and K

isomorphic to $SO(d,2)$

(exercise)

compact vs. non-compact

note: everything we said is also valid in Euclidean space: replace $\eta^{\mu\nu} \to \delta^{\mu\nu}$

get the Euclidean conformal group

2.2 Examples

conformal transformations map circles to circles (including circles with infinite radius, aka straight lines): explanation

picture

think of it as renormalization group transformation that depends on space-time

Escher?

3 Conformal symmetry in quantum field theory

Wightman axioms

operators are not necessarily invariant under conformal symmetry, but they transform unitarily

unitary representations on Hilbert space

(e.g. pair of point-like particle

no action principle (i.e. no need for "quantization")

only deal with **local** operators

warning about meaning of locality: related to causality?

generator of translations plays a special role: energy and momentum vector

positivity of energy

3.1 Non-perturbative QFT

take P^0 as the Hamiltonian

space-time is foliated by surfaces of equal time

each equal-time surface has a Hilbert space; evolution between different surfaces at times t_1 and t_2 is given by the unitary evolution operator $U = e^{\pm i(t_1 - t_2)P^0}$

because of time-translation symmetry, the Hilbert space is the same on each surface!

states characterized by energy and momentum: $P^\mu |p\rangle = p^\mu |p\rangle$

there is a unique vacuum state: $|0\rangle$

there are local operators: an infinity of them example: free scalar theory

$\phi(x)$ are not operators, but operator-valued distributions

smeared operator $\phi[f]$ have finite norm; $\phi(x)$ don't

but don't worry for the rest of these lectures

positivity of norm of smeared operators implies positivity of thing multiplying delta function!

good observable: Wightman correlation function (note: operator need not be time-ordered; time evolution is unitary, so it can go both ways!)

operators define correlation functions, but the opposite is also true (Wightman reconstruction theorem); so use the latter and forget about Hilbert space

but then lose connection with path integral formulation: this gives time-ordered correlation functions!

representation theory: follow Shester's notes commutation relations for infinitesimal transformations exponentiated form is an abstract unitary operator

forget about unitary rep of SCT: only act on Lorentz cylinder, infinite cover of Minkowski space-time; we know this rep exists but keep infinitesimal form

Wigner construction: use reference momentum

generator of Lorentz transformation can be diagonalize at a point: the origin of space-time

get the action of $M^{\mu\nu}$ at arbitrary point

conformal Ward identities

operator vs. field

3.2   Scale symmetry

action of generator of translations

no mass (often said)

or better said: all masses!

reference vector, then construct spectral density

remark: only operator that is a scalar with scaling dimension zero can have a one-point function

3.3   Special conformal symmetry

special conformal generator implies two things:

primary and descendant operators

special condition on 2-point function of primary operators: that they have identical scaling dimensions

note that scale invariance and unitarity generally imply that the trace of the energy-momentum tensor is zero: then Hamiltonian is invariant under change of the metric $\delta g_{\mu\nu} = \sigma(x)\eta_{\mu\nu}$,

$$\delta H = \int d^d x\, T_{\mu\nu}\delta g^{\mu\nu} = \int d^x\, \sigma(x)T_\mu^\mu = 0 \tag{3.1}$$

invariance under infinitesimal Weyl transformations

note that $T_{\mu\nu}$ is only defined in flat space, or rather that it acquires a vacuum expectation value in curved space: the Weyl anomaly so let us only consider Weyl transformations that do not change the geometry of space-time: those that are equivalent to a coordinate transformation (a diffeomorphism)

exercise: compute conformal generators in momentum-space representation

Mack's classification in 4d

representation: long and short multiplets!

compare this with Slava's approach:

organize operators into representation of conformal group:

call some operators *primary*

operators are local, which mean that they do not feel the effect of the transformation of the metric, or only through coordinate dependence: $\phi(x) \rightarrow \phi(x') = \Omega(x)^{-\Delta}\phi(x)$

note: if ϕ transforms like this, then $\partial_\mu\phi(x)$ does not! call it a *descendant*

comment on scale vs. conformal invariance

correlation functions:

no need for action principle

3.4   The energy-momentum tensor

locality

allows to construct conformal charges

3.5   Exercises

show that the spectral density in the case $\Delta = \frac{d}{2} - 1$ is that of a free scalar field;

show also that in the case $\Delta = \frac{d}{2}$ it is given by the phase space of two massless particle states (see Rychkov eq. 1.13)

4   Conformal correlators

2-pt function: go from Wightman 2-pt function in momentum space to Euclidean 2-pt function in position space in two ways:

- Fourier transform first, then analytic continuation of Wightman function - first construct T product (Källen-Lehmann representation), then Wick rotation to Euclidean momentum space, then Fourier transform

T-products vs. Wightman functions

use retarded products and micro-causality

Ward identities

more identities for conserved currents and the energy-momentum tensor (there cannot be higher-spin conserved currents otherwise the theory must be free; reference?)

generalities:

one-point functions vanish (in flat space; not so on conformally flat manifolds such as S^d, where 1-point function might depend on radius)

4.1 Spectral representation for spinning operators

unitary bounds

4.2 From momentum to position

taking fourier transforms

4.3 From Minkowski to Eulidean space

analytic continuation of Wightman functions vs. Wick rotation of T-products

reflection positivity

going in the other direction: Osterwalder-Schrader

do we need to use the cylinder interpretation? I prefer not: only CFT in flat space here

4.4 Embedding-space formalism

conformal group acts linearly in embedding space with $d + 2$ dimensions, among which two times

$X'^M = \Lambda^M_N X^N$

get rid of two dimensions by restricting to light-cone $X^2 = 0$, and identifying $X^+ \sim \lambda X^+$

can take $X^+ = 1$, so that

$$(X^+, X^-, X^\mu) = (1, x^2, x^\mu) \tag{4.1}$$

fields defined on the cone, depending homogeneously on X

rewrite 2-point function

see section 2.1 of Rychkov

4.5   3-point functions

scalar 3-point function

    remarkable fact! fixed up to a single coefficient

    if only scale invariance, this could be arbitrarily complicated

    historical note (cite Rychkov, around eq. 2.43): birth of conformal field theory

    check: 2-point functions involving conserved currents and energy-momentum tensors are automatically conserved; for 3-point functions this gives additional constraint: scaling dimensions of operators must be equal

4.6   4-point functions

invariant cross-ratios for 4-point functions

5   State-operator correspondence and OPE

goal of this section: prove state/operator correspondence heuristically, justifying a posteriori some facts assumed before

    not only a curiosity: also useful computationally

5.1   N-S quantization

different foliation of space-time than the one discussed above

    same Hilbert space on a slice at $t = 0$; use different evolution operator

    conjugation in Euclidean space?

    takes $x_d \to -x_d$

    use Hamiltonian

$$H = P_d + K_d \tag{5.1}$$

instead of P_d

    fixed points

    operators can be evolved to fixed points by Hamiltonian

5.2   Radial quantization

in Euclidean, take the generator of dilatations as Hamiltonian

    simple rotation in $SO(d + 1, 1)$

    apply conformal transformation mapping the two fixed points to 0 and ∞

    conjugation is inversion $r \to 1/r$

generators P_μ and K_μ are conjugate under inversion!

raising and lowering operators

equivalence between state living on a sphere (arbitary radius) and operators inserted at the origin

organize states as eigenstates of scale and Lorentz (commuting): matches the construction we did above!

this shows that every state can be created by local primary operator (and possibly action of translation generators); conversely, every operator defines a state (this part was already clear)

not necessarily so in QFT! (e.g. gauge theories? do we also need non-local operators? Wilson lines?)

how to define an operator from a state? see Rychkov (3.26) operator is defined by all of its correlation functions

note: state/operator correspondence validates our interpretation of Wightman 2-point function

note 2: operator inside the unit sphere need not be smeared anymore; they have a well-defined norm!

exercise: unitarity bound in radial quantization (Rychkov section 3.2)

path integral interpretation: state is created by performing path integral inside unit sphere

5.3 Conformal OPE

derive OPE from radial quantization

relate this to Lorentzian OPE on vacuum

use in 3-point function: relates OPE coefficient with 3-pt fct coefficient (structure constants of the operator algebra)

exercise: compute function entering the OPE

5.4 OPE convergence

Hilbert-space argument

radius?

absolute convergence as long as one can find a sphere separating the points:

note that a plane is also a sphere with radius infinity

concluding remarks:

CFT data: spectrum of operators and OPE coefficients completely determines the theory!

OPE reduces any higher-point function to 2-point

6 The conformal bootstrap

any CFT data defines a good theory? no!

different quantization surfaces gives different OPEs in the same theory

OPE associativity

often called "crossing symmetry" (note: quite different from crossing symmetry of scattering amplitudes)

(in fact this argument shows that Schwinger functions are symmetric!)

focus on 4-point functions: all 4-point functions of all operators contain all the associativity constraints

simplifications: only 4 identical operators, all scalars

6.1 Conformal blocks

rather conformal partial waves

diagrammatics (not to be confused with Feynman diags)

square configuration and expansion in rho coordinates

6.2 The numerical boostrap

discussion about universality: see Poland, Rychkov, Vichi also Shai Chester's notes

bound on scaling dimensions

also spectrum at the boundary

6.3 Generalized free field theory

aka mean free theory, Gaussian theory

double-twist operator

corresponds to large-N limit of gauge theory

6.4 Analytic bootstrap

Euclidean $z^* = \bar{z}$ $z = \bar{z} = \frac{1}{2}$ gives bootstrap equations that constrain operators of low scaling dimension Δ

Minkowski: $z \neq \bar{z}$, both real light-cone limit $z \to 0$, $\bar{z}$ fixed is dominated by operators of low twist

rigorous bound in the limit $\ell \to \infty$

Regge trajectories

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7 Selected advanced topics

light-cone limit and ?

Virasoro symmetry in 2d:

conformal anomalies: anomalies are contact term in the action for the source

superconformal bootstrap? note classification of all superconformal algebras

7.1 The 2d conformal bootstrap

historical presentation

see Rychkov section 4.3.1

advantage 1: full Virasoro symmetry

advantage 2: easier to deal with spin: left- and right-moving, i.e. holomorphic and anti-holomorphic

by looking at just the energy-momentum tensor, already get strong constraints on its 2-point function (the central charge)

for $c < 1$, only minimal models possible: match Ising, . . .

finitely many primaries in 2d (still infinitely many quasi-primaries)

note: infinitely many primaries for theories with $c > 1$

also can put the theory on torus and study modular properties of partition function

7.2 How to continue from here

- holography: see Joao's TASI lectures

- superconformal: Tajikawa pedestrian lectures

- bootstrap: Shai Chester's lectures

- condensed matter: note that unitarity is unnecessary: many non-unitary fixed points, yet they have conformal invariance (why?)

References

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