

Conformal Field Theory

Lecture notes

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This is the transcript of my notes for the class “[Conformal Field Theory](#)” at the Institute for Theoretical Physics of the University of Bern. These notes will be updated as the class advances, with the latest version always available at:

https://github.com/gillioz/CFT_course

Suggestions for improvements are welcome and can be directly submitted on the GitHub page, or sent to me by email at marc.gillioz@sissa.it.

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1 Introduction

1.1 What is conformal field theory?

Conformal field theory is a quantum field theory with conformal symmetry:

$$\text{conformal field theory} = \text{quantum field theory} + \text{conformal symmetry}, \quad (1.1)$$

where under conformal symmetry we understand

- translations in time and space,
- rotations and Lorentz boosts,
- scale transformations, also known as dilatations,
- special conformal transformations.

The course will begin in section 2 with a precise definition of all these symmetries, even though the first two (the Poincaré symmetry) should be familiar from an elementary quantum field theory class, and you should even have heard of the importance of dilatations in the context of the *renormalization group*.

Most courses on CFT put the emphasis on conformal symmetry as a whole, which makes a lot of sense because it forms a group (more on this latter). However, this approach often requires to work in Euclidean space (we will see why), and the connection with quantum field theory as we know it appears very late in the course, if at all.

Instead, in this lecture we would like to stay as close as possible to quantum field theory; in some sense, we will define conformal field theory as

$$\begin{aligned} \text{conformal field theory} = & \text{relativistic quantum field theory} \\ & + \text{scale symmetry} \\ & + \text{special conformal symmetry}, \end{aligned} \quad (1.2)$$

and go through this definition carefully. The first element is quantum field theory in flat Minkowski space-time. We have a good intuition of this part from particle physics. However, we do not want to restrict our attention to theories that have a nice classical limit and an understanding in terms of perturbation theory! Therefore, we will discuss in section 3 the basics of a non-perturbative quantum field theory. Even though we care about unitary quantum field theory in Minkowski space-time, it will be useful to establish

a connection with the same theory defined in flat Euclidean space. For this reason, we will work with the Minkowski metric convention

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_{\mu\nu}, \quad \mu, \nu = 0, 1, \dots, d-1, \quad (1.3)$$

so that going from Minkowski space-time to Euclidean space will be achieved through a rotation of the time coordinate in the complex plane, $x^0 = t \rightarrow i\tau$. Since for us the space-time will always be flat (no matter whether Minkowski or Euclidean), we will raise and lower the indices at will.

After giving a non-perturbative definition of relativistic quantum field theory, we will examine the role of scale symmetry. The main novelty here is that scale invariance forbids the presence of particles with a definite mass; instead, in a scale-invariant theory there are excitations of *any* energy (unless we are in the very special case of a free theory).

Finally, special conformal symmetry will be discussed last. The point is that it (nearly always) come along with scale symmetry. Nevertheless, it provides extremely powerful tools to understand the theory from a non-perturbative point of view. This will lead us to a comprehensive study of conformally-invariant correlation functions in section ??, and to the formulation of an *operator product expansion* (OPE) that has very nice features in section ??.

1.2 Why is conformal field theory interesting?

Nature is certainly not scale invariant, so why do we care so much about conformal field theory?

First of all, scale transformations play an essential role in quantum field theory, even when this is not a symmetry of the theory. The Callan-Symanzik equation in renormalization is essentially a Ward identity associated with the (broken) scale symmetry. We will see that conformal transformations can be viewed as a *local* generalization of scale transformations, in which the rescaling factor is different at every point in space-time, while still being consistent with Poincaré symmetry.

We also know that all end points of renormalization group trajectories are described by scale-invariant fixed points, and it turns out that *all* known fixed points are in fact conformally invariant. Therefore, conformal field theory is automatically relevant for the description of the low- and high-energy limits of any quantum field theory.

But one of the main reasons why it is important to study conformal field theory now (or at least have a basic understanding of it) is that it has been yielding exciting results in the last 10-15 years. In section ??, we will give a short review of the *conformal bootstrap*, a technique that allows to completely solve conformal field theory in certain cases. Even though the modern conformal bootstrap was invented to solve a particle physics problem, its results so far have had more impact in condensed matter physics. There, the concept of *universality* is playing a central role: it turns out that the critical exponents of very different systems coincide. This has a simple explanation from the fact that there are only “few” conformal field theories that can exist at all (at least once global symmetries are specified together with some information about what are the relevant operators). In other words, the space of all conformal field theories is sparse, unlike in QFT where it is always possible to add some new operator in a Lagrangian to slightly deform the theory. This concept is extremely successful in $d = 2$ and 3 dimensions, but there is also a big hope to classify all possible conformal field theories in $d = 4$ (or higher) dimensions. This goal would have a huge impact also for QFT with scale dependence, because it would strongly constrain the possible renormalization group flow. But this is still a young field, and much more work needs to be done before anything like this can be claimed.

Besides its importance in quantum field theory and condensed matter physics, conformal field theory has also given very nice insights into:

- Quantum gravity, through the AdS/CFT correspondence.
- String theory, as the worldsheet action is a CFT in $d = 2$.
- Foundations of quantum field theory beyond the perturbative approach. CFT admits a mathematically rigorous definition, which does not require to “quantize” a classical Lagrangian theory.¹

1.3 Examples of conformal field theories

Conformal field theories are everywhere in physics, and not necessarily limited to very special situations. Here are some examples, some of which you may already have encountered:

- Free massless theories are conformal. This is the case of the free boson and free fermion in any number of dimensions, but also of the n -form

¹In fact there needs not be a classical limit; for instance, in $d = 6$ dimensions there exists a theory with $(2, 0)$ extended supersymmetry for which no Lagrangian can be written down. The theory is known to exist from string theory arguments, and it has also been “observed” in a conformal bootstrap setup.

gauge theory in $d = 2n + 2$ dimensions (e.g. the free vector theory in $d = 4$). The same is true of theories with any number of free fields.

- Theories in which the β -function admits a perturbative fixed points. Typical examples are deformations of the free scalar theory with a potential of the type ϕ^n . For instance, consider

$$S = \int d^d x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{g}{4!} \phi^4 \right]. \quad (1.4)$$

g is dimensionless in $d = 4$ dimensions. If one considers the theory defined in $d = 4 - \varepsilon$ dimensions, then the beta function for g is

$$\beta_g = \mu \frac{dg}{d\mu} = -\varepsilon + \frac{3g}{(4\pi)^2} + \mathcal{O}(g^2). \quad (1.5)$$

It vanishes when

$$\frac{g_*}{(4\pi)^2} = \frac{\varepsilon}{3}. \quad (1.6)$$

Corrections of higher order in g can be neglected in the limit $\varepsilon \ll 1$. There are similar fixed points for a ϕ^3 interaction in $d = 6 + \varepsilon$ dimensions, or for a ϕ^6 interaction around $d = 3$.

- A similar type of fixed point can be found in the beta function of a $SU(N_c)$ gauge theory with N_f fermions (in the fundamental representation), which is given at leading order in the gauge coupling $\alpha = g^2/(4\pi)^2$ by

$$\beta_\alpha = \mu \frac{d\alpha}{d\mu} = -\frac{2}{3} \alpha^2 (11N_c - 2N_f) + \mathcal{O}(\alpha^3). \quad (1.7)$$

When $N_f = \frac{11}{2} N_c$, the leading order term in this beta function vanishes, so the next-to-leading term becomes important. Around that value, the first two terms are of similar importance, and one finds a perturbative fixed point of when $N_f \lesssim \frac{11}{2} N_c$. This is called the (Caswell-)Banks-Zaks fixed point. A theory in this situation is asymptotically free like QCD, but in the low-energy limit it approaches an interacting conformal field theory. For an $SU(3)$ gauge theory like QCD this critical value is at $N_f = 16.5$. There is strong evidence from lattice simulation that a theory with $N_f = 16$ is conformal. On the other hand a theory with low N_f (QCD has only 3 light quarks) is clearly confining, meaning that its low-energy limit is a theory massless Goldstone bosons (if the quarks are massless), or of massive pions. There is a critical value

$N_f^* \approx 12$ above which we expect an interacting conformal field theory in the low-energy limit. The domain $N_f^* \leq N_f \leq \frac{11}{2}N_c$ is called the *conformal window*. Note that gauge theories with different gauge groups and/or fermions coupling differently to the gauge fields (i.e. in different representations) can also have a conformal window. It is even possible to engineer gauge theories with perturbative UV fixed points using not only fermions but also scalars.

- There are theories with extended supersymmetry in which the beta function is exactly zero at all orders in perturbation theory, for instance $\mathcal{N} = 4$ supersymmetric Yang-Mills. More generically, in theories with sufficiently many supersymmetries, it is often sufficient to engineer the matter content to make the beta function zero at leading order, and then non-renormalization theorems ensure their vanishing at all orders. Theories with conformal symmetry and supersymmetry are called *superconformal*.
- An example of truly non-perturbative fixed point in $d = 3$ dimensions is given by the following action:

$$S = \int d^3x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!} \phi^4 \right], \quad (1.8)$$

in which $g > 0$ for the potential to be bounded below. Note that the free scalar field has mass dimension $[\phi] = \frac{1}{2}$ in $d = 3$, and therefore

$$[m^2] = 2, \quad [g] = 1. \quad (1.9)$$

Since both m^2 and g have positive mass dimensions, they are *relevant* operators: they determine the dynamics in the low-energy limit (IR), but their importance decreases at higher and higher energies (UV): at energies $E \gg g, |m|$, this approaches the theory of a massless free scalar. On the other hand, at low energies the physics depends obviously on g and m : when $m^2 \gg g^2$, it is a theory of a massive scalar of mass m , whereas when $m^2 \ll -g^2$, then the potential has two minima at

$$\langle \phi \rangle = \pm \sqrt{-\frac{6m^2}{g}}, \quad (1.10)$$

with excitations of mass $\sqrt{2}|m|$ around it. Clearly this theory has two phases, and there must therefore be an intermediate value of m^2 where the phase transition happens (or working in units set by g , a specific value of the dimensionless ratio m^2/g^2). Note that this theory has a \mathbb{Z}_2

symmetry in the UV corresponding to $\phi \rightarrow -\phi$, which is spontaneously broken in one phase and not in the other.

It turns out that the theory describing the IR physics exactly at the phase transition is a conformal field theory. Unlike the two phases surrounding it, it admits excitations of arbitrarily small energies (but they are not particles). How do we know that? Feynman diagram computations cannot be trusted in the IR: the approximation given by the (asymptotic) perturbative series is valid in the UV, but it breaks down in the IR, as in QCD. One way of understanding the phase transition is examining the theory in $d \neq 3$ dimensions: the same theory in $d = 4 - \varepsilon$ expansion has a perturbative fixed point, and computations at the fixed point can be performed in an expansion in ε . The value of the fixed point depends on the renormalization scheme, but there are other quantities that are scheme-independent. This is for instance the case of the anomalous dimension of the operator ϕ . Using a 6-loop beta-function computation, one finds

$$\gamma_\phi \approx 0.0182. \quad (1.11)$$

This is more conveniently expressed as a *scaling dimension* of ϕ ,

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi \approx 0.5182, \quad (1.12)$$

which in position space describes how the correlation between two points decays with the distance

$$\langle \phi(x)\phi(0) \rangle \propto |x|^{-2\Delta_\phi}. \quad (1.13)$$

A surprising thing about this scaling dimension is that it coincides precisely with that of a statistical physics model. The *Ising model* is a theory of classical spins $s_i = \pm 1$ on a lattice with nearest-neighbor interactions, characterized by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} s_i s_j. \quad (1.14)$$

This model has a critical value of J whose continuum limit is described by a CFT. At this value, Monte-Carlo simulations show that the correlation between two spins decay with the distance with power given by

$$\Delta_s \approx 0.5181. \quad (1.15)$$

The two theories have completely different microscopic descriptions: one is a quantum field theory describing particles in Minkowski space-time, the other a simple theory on a Euclidean lattice. Even more surprisingly, the same critical exponent is found in other systems, such as the critical point of water and other liquids. This is an example of *universality*.

The explanation for this coincidence is that there are not many candidate conformal field theories to describe the phase transition of the ϕ^4 theory and the Ising model. In fact, these two theories have in common:

- A global \mathbb{Z}_2 symmetry ($s_i \rightarrow -s_i$) that is broken in one phase and unbroken in the other;
- Exactly 2 relevant operators (in addition to J the Ising model can be parametrized in terms of a coupling to an external magnetic field $\delta H = -\mu \sum s_i$).

The conformal bootstrap philosophy is quite different from the two examples above, in the sense that it does not care about the microscopic details of the theory: instead, it attempts to solve the conformal field theory that is expected to be there at the fixed point, based on symmetry arguments only. We shall see that the conformal bootstrap can be used to show that there exists a unique conformal field theory with 2 relevant operators and a \mathbb{Z}_2 symmetry, and that the scaling dimension of its leading operator is equal to

$$\Delta_s \approx 0.5181489. \quad (1.16)$$

In fact, the conformal bootstrap does not only establish that, but it gives access to the whole spectrum of operators in that conformal field theory. This is truly a success story of the bootstrap.

- Finally, another example of CFT is the family of theories obtained by considering a quantum field theory in $d + 1$ -dimensional anti de Sitter (AdS) space-time. AdS admits a compactification to a sphere with a boundary, and correlation functions on that boundary are isomorphic with that of a d -dimensional conformal field theory. A generic (e.g. perturbative) theory in AdS gives rise to a CFT on the boundary that does not satisfy all the ordinary assumptions of a quantum field theory (for instance it does not have an energy-momentum tensor), but there is evidence that a theory that includes gravity does. Note that the same type of correspondence applies to late-time correlators in de Sitter space-time.

1.4 Literature

There are several excellent modern reviews on conformal field theory, and large parts of these lecture notes are directly inspired by them. The essentials are covered in the works of conformal bootstrap experts:

- Slava Rychkov’s EPFL lectures [2]; this review also contains interesting historical comments.
- David Simmons-Duffin’s TASI lectures [3].
- Shai Chester’s Weizmann Lectures [4].

Some other useful material can be found in:

- A review article on the state-of-the-art bootstrap methods [5].
- Hugh Osborn’s lecture notes at Cambridge, available at: <https://www.damtp.cam.ac.uk/user/ho/CFTNotes.pdf>
- Joshua Qualls’ lectures [6].

The reviews mentioned above are all mostly focused on the Euclidean approach to CFT. For a Lorentzian perspective, see:

- Slava Rychkov’s *Lorentzian methods in conformal field theory*, available at <https://courses.ipht.fr/node/226> (link to the lecture notes)

There are also modern reviews specifically on the topic of CFT in $d = 2$ dimensions:

- Sylvain Ribault’s notes [7], also accepting suggestions on GitHub: <https://github.com/ribault/CFT-Review>
- Schellekens “Conformal Field Theory” lecture notes [8], and a recent version available at: <https://www.nikhef.nl/~t58/CFT.pdf>.

More classical literature on the subject can be found in:

- The “yellow book” by Di Francesco, Mathieu, and Sénéchal [9].
- Polchinski’s *String theory* vol. 1 [10].
- Lectures by Paul Ginsparg [11].

For theories combining supersymmetry and conformal symmetry:

- Lorenz Eberhardt’s lecture notes [12].

2 Classical conformal transformations

One of the most fundamental principles of physics is independence of the reference frame: observers living at different points might have different perspectives, but the underlying physical laws are the same. This is true in space (invariance under translations and rotations), but also in space-time (e.g. invariance under Lorentz boosts).

2.1 Infinitesimal transformations

In mathematical language, this means that if we have a coordinate system x^μ , the laws of physics do not change under a transformations

$$x^\mu \rightarrow x'^\mu. \quad (2.1)$$

This principle applies to all maps that are invertible (isomorphisms) and differentiable (smooth transformations), hence it is usually called *diffeomorphism* invariance. Being differentiable, the transformation (2.1) can be Taylor-expanded to write

$$x^\mu \rightarrow x'^\mu = x^\mu + \varepsilon^\mu(x), \quad (2.2)$$

in terms of an infinitesimal vector ε^μ (meaning that we will always ignore terms of order ε^2).

In addition to the coordinate system, the description of a physical system requires a way of measuring distances that is provided by a metric $g_{\mu\nu}(x)$. Distances are measured integrating the line element

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu. \quad (2.3)$$

Since all observers should agree on the measure of distances, we must have

$$g'_{\mu\nu}(x') dx'^\mu dx'^\nu = g_{\mu\nu}(x) dx^\mu dx^\nu, \quad (2.4)$$

Here $g_{\mu\nu}$ could be the Euclidean metric $\delta_{\mu\nu}$ or the Minkowski metric $\eta_{\mu\nu}$; for simplicity we only consider the case in which $g_{\mu\nu}$ is flat, i.e. $\partial_\alpha g_{\mu\nu} = 0$. In this case, we can write

$$\begin{aligned} g'_{\mu\nu} &= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \\ &= g_{\alpha\beta} (\delta_\mu^\alpha - \partial_\mu \varepsilon^\alpha) (\delta_\nu^\beta - \partial_\nu \varepsilon^\beta) \\ &= g_{\mu\nu} - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu). \end{aligned} \quad (2.5)$$

If we require the different observers to also agree on the metric, then we must have $g'_{\mu\nu} = g_{\mu\nu}$, which gives a constraint on what kind of coordinate transformations are possible: we must have

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 0. \quad (2.6)$$

This condition admits as a most general solution

$$\varepsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu, \quad (2.7)$$

where a^μ is a constant vector and $g_{\mu\rho}\omega^\rho{}_\nu = \omega_{\mu\nu}$ an antisymmetric tensor. The transformation

$$x^\mu \xrightarrow{P} x^\mu + a^\mu \quad (2.8)$$

is obviously a translation and

$$x^\mu \xrightarrow{M} (\delta^\mu{}_\nu + \omega^\mu{}_\nu) x^\nu \quad (2.9)$$

a rotation/Lorentz transformation around the origin $x = 0$: the matrix $R^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ satisfies $R^\mu{}_\alpha g^{\alpha\beta} R_\beta{}^\nu = g^{\mu\nu}$. The composition of these two operations generates the Poincaré group. This is the fundamental symmetry of space-time underlying all relativistic quantum field theory. It is a symmetry of nature to a very good approximation, at least up to energy scales in which quantum gravity becomes important.

However, one can also consider the situation in which the two observers use different systems of units, i.e. they disagree on the overall definition of scale, but agree otherwise on the metric being flat. In this case we must have $g'_{\mu\nu} \propto g_{\mu\nu}$, and therefore the constraint becomes

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 2\lambda g_{\mu\nu}, \quad (2.10)$$

for some real number λ , with the most general solution

$$\varepsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu. \quad (2.11)$$

The new infinitesimal transformation is

$$x^\mu \xrightarrow{D} (1 + \lambda)x^\mu. \quad (2.12)$$

It is a scale transformation. Note that scale symmetry is not a good symmetry of nature: there is in fact a fundamental energy scale on which all observer must agree (this can be for instance chosen to be the mass of the electron). Nevertheless, there are some systems in which this is a very good

approximate symmetry. One can also make very interesting *Gedankenexperimente* that have scale symmetry built in. These are reasons that make it worth studying.

If one pushes this logic further, in a scale-invariant world in which observers have no physical mean of agreeing on a fundamental scale, they might even decide to change their definition of scale as they walk around, or as time passes. This would correspond to the case in which the metric $g'_{\mu\nu}$ of one observer can differ from the original metric $g_{\mu\nu}$ by a function of space(-time):

$$g'_{\mu\nu}(x) = \Omega(x)g_{\mu\nu}. \quad (2.13)$$

Note that we are not saying that $g'_{\mu\nu}$ is completely arbitrary: at every point in space time it is related to the flat metric by a scale transformation. But the scale factor is different at every point. The condition on ε^μ becomes in this case

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 2\sigma g_{\mu\nu}, \quad (2.14)$$

where $\Omega(x) = e^{-2\sigma(x)} \cong 1 - 2\sigma(x)$. To find the most general solution to this equation, note that contracting the indices with $g^{\mu\nu}$ gives

$$\partial_\mu \varepsilon^\mu = d\sigma, \quad (2.15)$$

where d is the space(-time) dimension, while acting with ∂^ν gives

$$\partial_\mu \partial_\nu \varepsilon^\nu + \partial^2 \varepsilon_\mu = 2\partial_\mu \sigma, \quad (2.16)$$

so that we get

$$\partial^2 \varepsilon_\mu = (2 - d)\partial_\mu \sigma. \quad (2.17)$$

Acting once again with ∂^μ , we arrive at

$$(d - 1)\partial^2 \sigma = 0, \quad (2.18)$$

while acting with ∂^ν and symmetrizing the indices, we find

$$(2 - d)\partial_\mu \partial_\nu \sigma = g_{\mu\nu} \partial^2 \sigma = 0. \quad (2.19)$$

In $d > 2$, we obtain therefore the condition $\partial_\mu \partial_\nu \sigma = 0$, which is solved by

$$\sigma(x) = \lambda + 2b \cdot x. \quad (2.20)$$

We have therefore

$$\varepsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + 2(b \cdot x)x^\mu - x^2 b^\mu. \quad (2.21)$$

In addition to the transformations found before, we also find

$$x^\mu \xrightarrow{K} x^\mu + 2(b \cdot x)x^\mu - x^2 b^\mu, \quad (2.22)$$

which is a *special conformal transformation*. If we examine the Jacobian for this transformation, we find

$$\frac{\partial x'^\mu}{\partial x^\nu} = (1 + 2b \cdot x) \delta_\nu^\mu + 2(b_\nu x^\mu - x_\nu b^\mu) \approx (1 + 2b \cdot x) R^\mu{}_\nu(x). \quad (2.23)$$

We have written this as a position-dependent scale factor $(1 + 2b \cdot x)$, multiplying an orthogonal matrix

$$R^\mu{}_\nu(x) = \delta_\nu^\mu + 2(b_\nu x^\mu - x_\nu b^\mu). \quad (2.24)$$

This shows that special conformal transformation act locally as the composition of a scale transformation and a rotation (or Lorentz transformation). This also shows that conformal transformations preserve angles, which is the origin of their name. Eq. (2.14) is sometimes called the (conformal) Killing equation and its solutions (2.21) the Killing vectors.

Note that in our derivation the original metric $g_{\mu\nu}$ was flat, but the new metric $g'_{\mu\nu}$ is not. It is however conformally flat: it is always possible to make a change of coordinate after which it is flat. In general, transformations

$$g_{\mu\nu}(x) \rightarrow \Omega(x)g_{\mu\nu}(x) \quad (2.25)$$

are called *Weyl transformations*. They change the geometry of space-time. We found that any Weyl transformation which is at most quadratic in x can be compensated by a change of coordinates to go back to flat space. The corresponding flat-space transformation is called conformal transformation.²

Note that in $d = 2$ the situation is a bit different: the conditions $\partial^2 \sigma = 0$ is sufficient to ensure that the Killing equation has a solution. This is most easily seen in light-cone coordinates,

$$x^+ = \frac{x^0 + x^1}{2}, \quad x^- = \frac{x^0 - x^1}{2}, \quad (2.26)$$

in terms of which

$$\partial^2 \sigma = \partial_+ \partial_- \sigma. \quad (2.27)$$

This is satisfied by taking for σ the sum of an arbitrary function of the left-moving variable x^+ and of another function of the right-moving variable x^- .

²This implies that the group of conformal transformation is a subgroup of diffeomorphisms. It is in fact the largest *finite-dimensional* subgroup.

In fact, if we write $\varepsilon^\pm = \varepsilon^0 \pm \varepsilon^1$, we can take arbitrary functions $\varepsilon^+(x^+)$ and $\varepsilon^-(x^-)$, and verify that eq. (2.14) is satisfied with $\sigma = \frac{1}{2}(\partial_+\varepsilon_+ + \partial_-\varepsilon_-)$. In the Euclidean case we take

$$z = \frac{x^1 + ix^2}{2}, \quad \bar{z} = \frac{x^1 - ix^2}{2}, \quad (2.28)$$

complex-conjugate to each other, and the same logic follows: we can apply arbitrary holomorphic and anti-holomorphic transformations on z and \bar{z} , and the conformal Killing equation is always satisfied. This shows that there are infinitely more conformal transformations in $d = 2$ than in $d > 2$, and also that there is no significant difference between Euclidean and Minkowski conformal transformation in $d = 2$, as the transformation acts essentially on the two light-cone/holomorphic coordinates independently.

2.2 The conformal algebra

The conformal Killing equation (2.21) determines the most general form of *infinitesimal* conformal transformations. *Finite* conformal transformations follow from a sequence of infinitesimal transformations. However, one has to bear in mind that the infinitesimal conformal transformations do not all commute: for instance, a translation followed by a rotation is not the same as the opposite. In fact, the conformal transformations form a *group*: the composition of conformal transformations is again a conformal transformation.

As we all know from quantum field theory, a group is characterized by its *generators* and their commutation relations (the *algebra*). A generator G describes an infinitesimal transformation in some direction, and finite transformations are obtained using exponentiation, $e^{i\theta G}$, with parameter θ (the factor of i is the physicist's convention that makes the generators Hermitian). A representation of the conformal group is given on the functions of the coordinates, $f(x)$. For instance, under an infinitesimal translation, we have

$$f(x) \xrightarrow{P} f(x') = f(x + a) \approx f(x) + a^\mu \partial_\mu f(x) \quad (2.29)$$

and we require this to be equal to $e^{-ia_\mu P^\mu} f(x)$, which means

$$P_\mu = i\partial_\mu. \quad (2.30)$$

Performing the same analysis for the other infinitesimal transformations given

in eq. (2.21), we obtain for the other generators³

$$\text{rotations/Lorentz transformations:} \quad M^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \quad (2.31)$$

$$\text{dilations:} \quad D = ix^\mu \partial_\mu, \quad (2.32)$$

$$\text{special conformal transformations:} \quad K^\mu = i(2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu). \quad (2.33)$$

The number of generators matches that of the Killing vectors: there are d translations, d special conformal transformations, $d(d-1)/2$ rotations/Lorentz transformations ($M^{\mu\nu}$ is a $d \times d$ antisymmetric matrix), and one scale transformation. Therefore the total number of generators, i.e. the dimension of this group, is $(d+1)(d+2)/2$. In $d = 4$ space-time dimension, the conformal group has 15 generators.

Using the above definition, one can verify that the following commutation relations are satisfied,

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= -i(g^{\mu\rho} M^{\nu\sigma} - g^{\mu\sigma} M^{\nu\rho} - g^{\nu\rho} M^{\mu\sigma} + g^{\nu\sigma} M^{\mu\rho}) \\ [M^{\mu\nu}, P^\rho] &= -i(g^{\mu\rho} P^\nu - g^{\nu\rho} P^\mu) \\ [M^{\mu\nu}, K^\rho] &= -i(g^{\mu\rho} K^\nu - g^{\nu\rho} K^\mu) \\ [D, P^\mu] &= -iP^\mu \\ [D, K^\mu] &= iK^\mu \\ [P^\mu, K^\nu] &= 2i(g^{\mu\nu} D - M^{\mu\nu}) \end{aligned} \quad (2.34)$$

while all other commutators vanish:

$$[M^{\mu\nu}, D] = [P^\mu, P^\nu] = [K^\mu, K^\nu] = 0. \quad (2.35)$$

The first two relations in eq. (2.34) are the familiar Poincaré algebra. The next one states that K^μ transforms like a vector (as P^μ does), whereas D is obviously a scalar. The next two relations remind us that K^μ and P^μ have respectively the dimension of length and inverse length.

Exercise 2.1 Derive the commutation relations from the action (2.30)–(2.33) of the generators on functions of x .

Even though this is not immediately obvious, this algebra is isomorphic to that of the group $\text{SO}(d+1, 1)$ (if $g^{\mu\nu}$ is the Euclidean metric) or $\text{SO}(d, 2)$

³The sign of these generators is an arbitrary convention. It defines once and for all the commutations relations that we will derive next. After that, we will always refer to the commutation relations as the definition of the generators.

(if it is the Minkowski metric). To see that it is the case, let us introduce a $(d+2)$ -dimensional space with coordinates

$$X^\mu, \quad X^{d+1}, \quad X^{d+2}, \quad (2.36)$$

and a metric defined by the line element

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu + dX^{d+1} dX^{d+1} - dX^{d+2} dX^{d+2} \equiv \eta_{MN} dX^M dX^N. \quad (2.37)$$

Then we write all conformal commutation relations as being defined by the Lorentzian algebra

$$[J^{MN}, J^{RS}] = -i (\eta^{MR} J^{NS} - \eta^{MS} J^{NR} - \eta^{NR} J^{MS} + \eta^{NS} J^{MR}), \quad (2.38)$$

provided that we identify the antisymmetric generators J^{MN} with the conformal generators as follows:

$$\begin{aligned} M^{\mu\nu} &= J^{\mu\nu}, \\ P^\mu &= J^{\mu,d+1} + J^{\mu,d+2}, \\ K^\mu &= J^{\mu,d+1} - J^{\mu,d+2}, \\ D &= J^{d+1,d+2}. \end{aligned} \quad (2.39)$$

2.3 Finite transformations

We just saw that the infinitesimal conformal transformations generate a group. But how can we describe finite conformal transformations? Let us see how each generator exponentiates into an element of the group; the most general conformal transformation can then be obtained as a composition of such finite transformations.

In some cases the exponentiation is trivial. For instance, with translations we obtain immediately

$$x^\mu \xrightarrow{P} x^\mu + a^\mu, \quad (2.40)$$

where a is now any d -dimensional vector, not necessarily small. The same is true of scale transformations,

$$x^\mu \xrightarrow{D} \lambda x^\mu \quad (2.41)$$

with finite λ . Rotations or Lorentz transformations exponentiate as

$$x^\mu \xrightarrow{M} \Lambda^\mu{}_\nu x^\nu \quad (2.42)$$

where $\Lambda^\mu{}_\nu$ is a $\text{SO}(d)$ or $\text{SO}(1, d-1)$ matrix, depending whether the metric is Euclidean or Minkowski. All of this is well-known and not surprising.

On the contrary, special conformal transformations do not exponentiate trivially. The easiest way to derive their finite form is to make the following observation: recall that in infinitesimal form we have

$$x'^\mu = x^\mu + 2(b \cdot x)x^\mu - x^2 b^\mu, \quad (2.43)$$

which implies $x'^2 = (1 + 2b \cdot x)x^2$, and therefore (as always neglecting terms of order b^2)

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu. \quad (2.44)$$

The ratio x^μ/x^2 appearing on both side of the equation is the *inverse* of the coordinate x^μ , respectively x'^μ : let us define the inversion as

$$x^\mu \xrightarrow{I} \frac{x^\mu}{x^2}. \quad (2.45)$$

This transformation does not have an infinitesimal form, but otherwise it shares the essential properties of a conformal transformation: its Jacobian is

$$\frac{\partial x'^\mu}{\partial x^\nu} = \frac{1}{x^2} \left[\delta_\nu^\mu - 2 \frac{x^\mu x_\nu}{x^2} \right], \quad (2.46)$$

which is the product of a position-dependent scale factor x^{-2} with an orthogonal matrix. To understand what this transformation does globally, let us consider a point $\vec{x} = (x, 0, \dots, 0) \in \mathbb{R}^d$. Then the matrix in square bracket is diagonal, and equates $\text{diag}(-1, 1, \dots, 1)$. This is an orthogonal matrix with determinant -1 , which is part of $O(d)$ but not $SO(d)$. This shows that the inversion is a discrete transformations not connected to identity. A conformally invariant theory might be invariant under inversions, but it needs not be.

Eq. (2.44) shows that infinitesimal special conformal transformations are obtained taking an inversion followed by a translation, followed by an inversion again. Since this process involves the inversion twice, and since inversion is its own inverse, it does not matter whether inversion is a true symmetry of the system or not. The advantage of this representation is that it can easily be exponentiated: the composition of (infinitely) many infinitesimal special conformal transformation can be written as an inversion followed by a finite translation, followed by an inversion again. In other words, eq. (2.44) holds for finite b^μ . This can be used to show that

$$x^\mu \xrightarrow{K} x'^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2b \cdot x + b^2 x^2}. \quad (2.47)$$

Exercise 2.2 Use eq. (2.44) to show (2.47).

What do special transformation do globally? Let us look specifically in Euclidean space. There are some special points:

- The origin of the coordinate system $x = 0$ is mapped onto itself.
- The point b^μ/b^2 is mapped to ∞ .
- Conversely, the “point” $x \rightarrow \infty$ is mapped to the finite vector $-b^\mu/b^2$.

These properties can be understood from the fact that special conformal transformations and translations are related by inversion: special conformal transformations keep the origin fixed but move every other points, including ∞ ; translations move every point *except* ∞ . The other two transformations, rotations and scale transformations, keep both 0 and ∞ fixed.

An essential property of conformal transformations is that they allow to map any 3 points (x_1, x_2, x_3) onto another set (x'_1, x'_2, x'_3) . This can be seen as follows: first, apply a translation to place x_1 at the origin, followed by a special conformal transformation that takes x_3 to ∞ , after which the image of the original triplet is $(0, x''_2, \infty)$; then use rotations and scale transformations to move x''_2 to another point x'''_2 , while keeping 0 and ∞ fixed; finally apply again a special conformal transformation that takes ∞ to $x'_3 - x'_1$, and a translation by x'^1 to reach the configuration (x'_1, x'_2, x'_3) . This property has an immediate physical consequence: in correlation functions of 2 or 3-points (see next sections for a definition), all kinematics is fixed by conformal symmetry. The only freedom encodes information about the operators themselves, not about their position in space.

Another interesting property of conformal transformations is that they map spheres to spheres: this is an obvious property of translations, rotations and scale transformations, but it is also true of special conformal transformations.

Exercise 2.3 Show that under the special conformal transformation (2.47), a sphere centered at the point a^μ and with radius R gets mapped to a sphere centered at the point

$$a'^\mu = \frac{a^\mu - (a^2 - R^2)b^\mu}{1 - 2a \cdot b + (a^2 - R^2)b^2}$$

and with radius

$$R' = \frac{R}{|1 - 2a \cdot b + (a^2 - R^2)b^2|}.$$

In the special case in which b^μ/b^2 is on the original sphere, show that the sphere gets mapped to a plane orthogonal to the vector $a^\mu + (R^2 - a^2)b^\mu$.

Note that in $d = 2$, the additional conformal transformations (infinitely many of them!) mean that (nearly) any shape can be mapped onto another. This is known as the Riemann mapping theorem.⁴

2.4 Compactifications

It is often said that conformal symmetry is a symmetry of flat space(-time). This is true, as we have just seen, provided that we treat the point ∞ as being part of the space. This is quite straightforward in Euclidean space, but much more subtle in Minkowski space-time, as there are different ways of reaching ∞ there.

To gain a better understanding of this, it can be useful to map the flat Euclidean space \mathbb{R}^d or the Minkowski space-time $\mathbb{R}^{1,d-1}$ onto a curved (but hopefully compact) manifold. In Euclidean space this is for instance achieved by the (inverse) stereographic projection that maps $\mathbb{R}^d \cup \{\infty\}$ to the unit sphere S^d . Geometrically, the stereographic projection is constructed as follows: embed \mathbb{R}^d as a (hyper-)plane in \mathbb{R}^{d+1} , together with a sphere of unit radius centered at the origin. Every point on the plane has an image on the sphere obtained drawing a segment between the original point and the north pole of the sphere, and noting where it intersects the sphere. The origin is mapped to the south pole, ∞ to the north pole, and the sphere S^{d-1} of unit radius to the equator. Algebraically, this is achieved as follows: first write the Euclidean metric in spherical coordinates,

$$ds^2 = dr^2 + r^2 d\Omega_{d-1}^2, \quad (2.48)$$

where the solid angle is given in $d = 2$ by $d\Omega_1^2 = d\phi^2$, in $d = 3$ by $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$, and more generically by the recursion relation $d\Omega_n^2 = d\theta^2 + \sin^2\theta d\Omega_{n-1}^2$. Let us perform the change of variable

$$r = \frac{\sin \varphi}{1 - \cos \varphi}, \quad (2.49)$$

⁴A neat example of how a disk is mapped onto a polygon is available at <https://herbert-mueller.info/uploads/3/5/2/3/35235984/circletopolygon.pdf>.

and interpret $\varphi \in [0, \pi]$ as the zenith angle on the sphere: $\varphi = 0$ is the north pole, corresponding to $r \rightarrow \infty$, and $\varphi = \pi$ the south pole, corresponding to $r = 0$. In these coordinates we have

$$ds^2 = \frac{1}{(1 - \cos \varphi)^2} (d\varphi^2 + \sin^2 \varphi d\Omega_{d-1}^2) = \frac{1}{(1 - \cos \varphi)^2} d\Omega_d^2. \quad (2.50)$$

This shows that the metric is flat up to an overall Weyl factor that depends on φ . In these new coordinates, conformal transformations are always non-singular. For this reason, it is often very convenient to study *classical* conformal transformations on the sphere S^d instead of Euclidean space.

However, this compactification is not as nice in a *quantum* theory in which one wants to foliate the space along some direction: if one chooses φ as the “Euclidean time”, then the “space” direction is a sphere S^{d-1} whose volume depends on φ . In other words, the generator of “time” translations is not a symmetry of the system. Being on a perfectly symmetric sphere, this is also true of any other choice of time direction. This is not forbidden, but it complicates vastly the analysis.

Instead, another compactification is often preferred to the sphere: from the Euclidean metric in spherical coordinates, one can make the change of variable

$$\tau = \log(r) \quad \Rightarrow \quad r = e^\tau, \quad (2.51)$$

after which

$$ds^2 = r^2 (d\tau^2 + d\Omega_{d-1}^2). \quad (2.52)$$

This is again a flat metric, up to a Weyl factor $r^2 = e^{2\tau}$. In this case however the metric is completely independent of τ . This space has the geometry of the *cylinder*, $\mathbb{R} \times S^{d-1}$. It is not fully compact: τ goes from $-\infty$ to $+\infty$. But it has a important advantage: translations in τ are generated by dilatations D , which will be taken to be a symmetry of quantum field theory. Foliating the space into surfaces of constant τ will later lead us to the famous radial quantization in conformal field theory.

Note that in $\text{SO}(d+1, 1)$ language, the generator $D = J^{d+1, d+2}$ is completely equivalent to any other generator $J^{\mu, d+2} = \frac{1}{2} (P^\mu - K^\mu)$. So for instance we might as well look for a cylinder compactification in which the non-compact direction corresponds to translations in the direction of $\frac{1}{2} (P^0 - K^0)$. This generator obeys

$$\frac{1}{2} (P^0 - K^0) = i \left(\frac{1 - (x^0)^2 + \vec{x}^2}{2} \partial_0 - x^0 \vec{x} \cdot \vec{\partial} \right), \quad (2.53)$$

with two fixed points at $x^0 = \pm 1$ with $\vec{x} = 0$.

Exercise 2.4 Find the change of coordinate that make the Euclidean metric Weyl-equivalent to a cylinder in which translations in the non-compact direction are generated by $\frac{1}{2}(P^0 - K^0)$.

Hint: Find a special conformal transformation followed by a translation that take $(0, \infty)$ to $(-1, 1)$, and apply it to the cylinder coordinates.

The cylinder compactifications of Euclidean space are interesting by themselves, but they are also extremely convenient to understand the connection between Euclidean and Minkowski space-times: in this last form, performing a Wick rotation $\tau \rightarrow \pm it$ defines a Lorentzian cylinder on which the conformal group with a simple action of the Lorentzian conformal group $SO(d, 2)$. But before going there, let us go back to flat Minkowski space-time and make some general remarks.

2.5 Minkowski space-time

Everybody is familiar with translations and Lorentz transformations in Minkowski space-time, and even with dilatations as this is a standard tool in the renormalization group analysis. But what do conformal transformations do?

To understand this, let us place an observer at the origin of Minkowski space-time. The presence of this observer breaks translations, but not Lorentz transformations (the observer is point-like), nor dilatations and special conformal transformations. For the observer, space-time is split into three regions: a future light cone, a past light cone, and a space-like region from which they know nothing. Lorentz and scale transformation preserve this causal structure: the future and past light-cones are mapped onto themselves. In other words, if a point x is space-like separated from the observed, it will remain space-like separated no matter the choice of scale and Lorentz frame. Without loss of generality, let us choose this point to be at position $x = (0, \vec{n})$, where \vec{n} is a unit vector (units can be chosen so that this is the case). Now apply a special conformal transformation with parameter $b^\mu = (-\alpha, \alpha\vec{n})$, with α varying between 0 and 1. This draws a curve y^μ in space-time, parameterized by α , with

$$y^0(\alpha) = \frac{\alpha}{1 - 2\alpha}, \quad \vec{y}(\alpha) = \frac{1 - \alpha}{1 - 2\alpha} \vec{n}. \quad (2.54)$$

This curve begins at the space-like point $y = (0, \vec{n})$, and ends in the past light cone at $y = (-1, \vec{0})$! Note that y never crosses a light cone: the image of x is never null if x is not itself null, since $x'^2 = x^2/(1 - 2x \cdot b + x^2 b^2)$.

Instead, we have

$$y^2(\alpha) = \frac{1}{1-2\alpha} \neq 0. \quad (2.55)$$

The point goes all the way to space-like infinity when $\alpha = \frac{1}{2}$, and comes back from past infinity. Clearly, special conformal transformation break causality.

The resolution of this paradox is that conformal transformation do not act directly on Minkowski space, but rather on its universal cover that is isomorphic to the Lorentzian cylinder described in the previous section. Time evolution on that cylinder is given by the Hamiltonian $H = \frac{1}{2}(P^0 - K^0)$. At any given time t_0 , space is compactified in such a way that the notion of infinite distance is unequivocal: space-like infinity corresponds to a point on the sphere. If one takes any other point of that sphere and apply finite translations using the generator P^μ , then this defines a compact *Poincaré patch*. The full cylinder is a patchwork of Minkowski space-times, but every observer only has access to one.

The lesson that we need to learn from this is that only the infinitesimal form of special conformal transformation can be used in Minkowski space-time: *any* finite special conformal transformation would bring part of space-time into another patch on the cylinder. This is sometimes called *weak conformal invariance*.

A quantum field theory defined on Minkowski space-time with weak conformal invariance uniquely defines a quantum field theory on the Lorentzian cylinder with global conformal invariance. This non-trivial fact was proven by Martin Lüscher and Gerhard Mack in 1974 [13], both employed at the University of Bern at the time. In their own words:

In picturesque language, [the superworld] consists of Minkowski space, infinitely many “spheres of heaven” stacked above it and infinitely many “circles of hell” below it.

2.6 Conformal symmetry in classical field theory

So far we have been discussing conformal transformations of the coordinates. The next step is to consider a field theory (for the moment a classical one) that has conformal symmetry built in. For this, consider a field theory defined by an action. The simplest example is the free scalar field theory

$$S = \int d^d x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right]. \quad (2.56)$$

We shall see in the next section that the action principle can in fact be dropped in the conformal field theory, but for now it is a convenient starting point.

In this context, a conformal transformation is a transformation *of the fields*. There are two different and complementary perspectives one can adopt. It is often convenient to think of the metric tensor as a field in its own right, and to define a conformal transformation as a (position-dependent) scale transformation of the field

$$\phi(x) \rightarrow e^{\Delta\sigma(x)}\phi(x), \quad (2.57)$$

combined with a Weyl transformation of the metric

$$g_{\mu\nu}(x) \rightarrow e^{2\sigma(x)}g_{\mu\nu}(x). \quad (2.58)$$

Δ is the *scaling dimension* of the field ϕ . In a free theory it coincides with the engineering dimension of the field in units of energy (inverse units of length), namely

$$\Delta = \frac{d-2}{2}. \quad (2.59)$$

$\sigma(x)$ is an infinitesimal scale factor that satisfies $\partial^2\sigma = 0$. The advantage of this perspective is that the conformal transformations are simple, multiplicative transformations of the fields. The disadvantage is that it requires thinking of theory in curved space-time. This means that the metric that is implicit in the action (2.56) must be made explicit.

Exercise 2.5 Verify that the action (2.56) is invariant under the infinitesimal conformal transformations (2.57) and (2.58), after the explicit dependence on the curved-space metric has been taken into account in the contraction of Lorentz indices and in the measure of integration.

Moreover, the action could be augmented with a term depending on the scalar curvature tensor R as

$$S = \int d^d x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \alpha R \phi^2 \right]. \quad (2.60)$$

Since R vanishes in flat space, it looks like this additional term could appear with an arbitrary coefficient α without modifying the original flat-space action.

For this reason, it is also convenient to consider the opposite perspective in which conformal transformations are transformations of the (dynamical) fields and of the coordinates, but not of the metric. In this case the conformal transformation can be defined as

$$\phi(x) \rightarrow e^{\Delta\sigma(x)}\phi(x + \varepsilon), \quad (2.61)$$

where the parameters σ and ϵ are related by the conformal Killing equation (2.14), e.g. $d\sigma = \partial_\mu \varepsilon^\mu$. In infinitesimal form, this transformation becomes

$$\phi(x) \rightarrow \left[1 + \frac{d-2}{2d} (\partial_\nu \varepsilon^\nu) + \varepsilon^\nu \partial_\nu \right] \phi(x). \quad (2.62)$$

Exercise 2.6 Show that under the transformation (2.62), the free scalar field Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi,$$

is shifted by a total derivative term

$$\delta \mathcal{L} = \partial_\mu \left(-\frac{1}{2} \varepsilon^\mu \partial_\nu \phi \partial^\nu \phi - \frac{d-2}{2d} \partial_\nu \varepsilon^\nu \phi \partial^\mu \phi \right),$$

hence proving that this is a symmetry of the action. You will need the fact that ε is at most quadratic in x .

Note that Poincaré symmetry is a special case of this transformation, corresponding to constant ε (and thus $\sigma = 0$).

By Noether's theorem, whenever an action is invariant under some transformation of the field

$$\phi \rightarrow \phi + \delta_\varepsilon \phi, \quad (2.63)$$

i.e. whenever the Lagrangian varies by a total derivative term,

$$\mathcal{L} \rightarrow \mathcal{L} + \partial_\mu \Lambda_\varepsilon^\mu, \quad (2.64)$$

then there exists a conserved current

$$J_\varepsilon^\mu = \Lambda_\varepsilon^\mu - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta_\varepsilon \phi. \quad (2.65)$$

In our example, this conserved current is therefore

$$J_\varepsilon^\mu = \varepsilon_\nu \left(\partial^\mu \phi \partial^\nu \phi - \frac{1}{2} g^{\mu\nu} \partial_\rho \phi \partial^\rho \phi \right) \equiv \varepsilon_\nu T_c^{\mu\nu}. \quad (2.66)$$

The 2-index tensor on the right-hand side is called the *canonical energy-momentum tensor*. Its divergence satisfies

$$\partial_\nu T_c^{\mu\nu} = \partial^\mu \phi \partial^2 \phi, \quad (2.67)$$

therefore vanishing by the equation of motion for the free field, $\partial^2\phi = 0$. This implies in turn that the Noether current (2.66) is conserved *for constant* ε . If on the contrary ε depends on space(-time), then we have

$$\partial_\mu J_\varepsilon^\mu = (\partial_\nu \varepsilon_\mu) T_c^{\mu\nu}. \quad (2.68)$$

In our example the canonical energy-momentum tensor is symmetric in its indices μ and ν , and therefore we can write

$$\partial_\mu J_\varepsilon^\mu = \frac{1}{2}(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) T_c^{\mu\nu} = \sigma g_{\mu\nu} T_c^{\mu\nu} = -\frac{d-2}{2} \sigma \partial_\mu \phi \partial^\mu \phi. \quad (2.69)$$

In dimensions $d > 2$, this current is only conserved when $\sigma = 0$. This is surprising, because we just showed that scale and special conformal transformations are also symmetries of the action, so why is the Noether current not conserved?

The reason is that the version of Noether's theorem given above does not straightforwardly apply to the case of a space-time dependent parameter ε . In fact, the energy-momentum tensor that we computed in this example is not unique: one can always add to it a piece proportional to

$$(\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi^2, \quad (2.70)$$

without affecting the conservation equation (2.67), but changing the value of its trace. The combination

$$T^{\mu\nu} = T_c^{\mu\nu} + \frac{d-2}{2(d-1)} (\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi^2 \quad (2.71)$$

is for instance traceless in any d . It turns out that it is always possible in a field theory with conformal symmetry to construct an energy-momentum tensor that is:

- symmetric ($T^{\mu\nu} = T^{\nu\mu}$),
- traceless ($g_{\mu\nu} T^{\mu\nu} = 0$), and
- conserved ($\partial_\nu T^{\mu\nu} = 0$) once the equation of motions are imposed.

This is a non-trivial fact, but we will skip its proof. As already mentioned, we are interested in theories that are not necessarily defined through an action.

Strictly speaking, Noether's theorem only applies to theories that have a Lagrangian description, but we will assume the existence of a traceless energy-momentum tensor in all cases (in some sense this is going to be one of the “axioms” of conformal field theory). From this assumption, we can

deduce that the theory is invariant under conformal transformations. A conserved current can always be built from the energy-momentum tensor and a conformal Killing vector ε as

$$J^\mu = \varepsilon_\nu T^{\mu\nu} \quad \Rightarrow \quad \partial_\mu J^\mu = \varepsilon_\nu \partial_\mu T^{\mu\nu}. \quad (2.72)$$

As always, conserved charges can be constructed as the integral of the time component of a conserved current over space. The simplest example is

$$P^\mu = -i \int d^{d-1} \vec{x} T^{0\mu}(x). \quad (2.73)$$

This is a priori a function of x^0 , but it is in fact constant in time, since

$$\partial_0 P^\mu = -i \int d^{d-1} \vec{x} \partial_0 T^{0\mu}(x) = i \int d^{d-1} \vec{x} \partial_i T^{i\mu}(x) = 0. \quad (2.74)$$

This conserved charge is the *momentum*, associated with translation symmetry. Similarly, conserved charges associated with Lorentz transformations,

$$M^{\mu\nu} = -i \int d^{d-1} \vec{x} [x^\mu T^{0\nu}(x) - x^\nu T^{0\mu}(x)] \quad (2.75)$$

with scale transformations,

$$D = -i \int d^{d-1} \vec{x} x_\mu T^{0\mu}(x), \quad (2.76)$$

and with special conformal transformations

$$K^\mu = -i \int d^{d-1} \vec{x} [2x^\mu x_\nu T^{0\nu}(x) - x^2 T^{0\mu}(x)]. \quad (2.77)$$

Exercise 2.7 Show that the charges (2.75), (2.76) and (2.77) are conserved in time.

Note that the conservation of all these charges relies on vanishing divergence of the energy-momentum tensor, $\partial_\nu T^{\mu\nu}$, which itself relies on the equation of motion being satisfied. This is certainly true in the absence of sources. But when source terms are added to the action, the equation of motion is modified. In our example of the free scalar field theory, this would correspond to adding to the action (2.56) a source term of the form

$$S_{\text{source}} = \int d^d x J(x) \phi(x), \quad (2.78)$$

which modifies the equation of motion to

$$\partial^2 \phi + J = 0. \quad (2.79)$$

Instead of the conservation equation (2.67) for the canonical energy-momentum tensor, we must replace it with

$$\partial_\nu T^{\mu\nu}(x) = -J \partial^\mu \phi. \quad (2.80)$$

In the presence of such a source, the charges (2.73)–(2.77) are not conserved anymore. However, let us assume that the source is local, say $J(x) = \delta^d(x - x_\odot)$, so that

$$\partial_\nu T^{\mu\nu}(x) = -\delta^d(x - x_\odot) \partial^\mu \phi(x_\odot). \quad (2.81)$$

Then we determine the change in one of the charge — say P^μ — between a time $x^0 < x_\odot^0$ anterior to the local source, and a time $x^0 > x_\odot^0$ posterior to it, and call this difference the momentum of the source, $P_\mu(\phi_\odot)$. By our definition, this is equal to

$$P_\mu(\phi_\odot) = -i \int_{\partial\Sigma} d^{d-1} n_\nu T^{\mu\nu}, \quad (2.82)$$

where $\partial\Sigma$ is a codimension-one surface in $\mathbb{R}^{1,d-1}$ that consists in two planes of constant x^0 , joining at spatial infinity, and surrounding the point x_\odot . By the divergence theorem, this is equal to

$$P_\mu(\phi_\odot) = -i \int_\Sigma d^d x \partial_\nu T^{\mu\nu} = i \int_\Sigma d^d x J \partial^\mu \phi = i \partial^\mu \phi(x_\odot). \quad (2.83)$$

This equation says that the charge P_μ associated with a local source for the field ϕ is equal to $\partial^\mu \phi$. This is strikingly similar to the action of the generator (2.30) on functions. In fact, it is easy to verify that the other charges (2.75), (2.76), and (2.77) also act on the classical field $\phi(x)$ exactly like the generators (2.31), (2.32), and (2.33) respectively.

We worked with the free scalar field theory example, but it is straightforward to generalize this to arbitrary classical field theories. This shows that a traceless energy-momentum tensor can be used to give a field-theoretical realization of the conformal generators discussed before.

3 Conformal quantum field theory

Let us now turn to quantum field theory and study the implications of conformal symmetry in that context. The most standard approach to quantum

field theory is to think of a classical field theory, for which we now have a basic understanding of what conformal symmetry does, and then quantize it by promoting the fields to operators acting on some Hilbert space. But this is not the approach we will take here. There will still be states and operators, but the latter will not *necessarily* be associated with fields appearing in a Lagrangian.

3.1 Non-perturbative quantum field theory

To define a quantum field theory non-perturbatively, we need the following ingredients:

1. The Minkowski space-time is foliated into surfaces of equal-time, and to each time slice we associate a Hilbert space of quantum states. This Hilbert space includes a vacuum state $|0\rangle$.
2. There are a number of (in fact infinitely many) *local* operators that act on this Hilbert space. For instance, let us take $\phi(x^0, \vec{x})$ to be an operator acting on the Hilbert space at time $t = x^0$. We call this operator *local* in \vec{x} because it commutes with any other local operators on the same time slice inserted at a point $\vec{x}' \neq \vec{x}$:

$$[\phi(x^0, \vec{x}), \phi(x^0, \vec{x}')] = 0 \quad \vec{x} \neq \vec{x}'. \quad (3.1)$$

3. One of the local operators of the theory is the energy-momentum tensor, and from it we can define conserved charges, including P^μ as in eq. (2.73). P^μ is conserved in time, so it is a good operator valid on all Hilbert spaces at any time t . In particular, we shall assume that the vacuum state has zero charge,

$$P^\mu |0\rangle = 0. \quad (3.2)$$

The value of P^μ changes however every time an operator is inserted at some point (x^0, \vec{x}) . In analogy with eq. (2.83), this change is encoded in the commutator

$$[P^\mu, \phi(x)] = i\partial^\mu \phi(x). \quad (3.3)$$

This equation is solved by

$$\phi(x) = e^{-ix \cdot P} \phi(0) e^{ix \cdot P}. \quad (3.4)$$

Note that while P^μ is not a local operator, its commutator with any local operator is again local.

Since P^μ is Hermitian (this can be seen from (3.3) in the case of a Hermitian local operator ϕ), this equation indicates that translations are represented by a unitary operator on the Hilbert space. This is true both of space- and time-translations, and therefore the Hamiltonian that defines the time-evolution of states is P^0 : a state $|\Psi\rangle$ defined at time $t = 0$ is translated to another state at time t by

$$|\Psi\rangle_t = e^{itP^0} |\Psi\rangle. \quad (3.5)$$

Since time evolution is unitary, the Hilbert space will be the same on all surfaces.

4. Lorentz transformations are similarly realized as unitary transformations of the Hilbert space. The charge $M^{\mu\nu}$ associated with Lorentz symmetry is again a conserved operator whose commutator with any local operator can be expressed as another local operator. If we choose to decompose a local operator inserted at the origin of space-time into irreducible representations of the Lorentz group and denote these with $\phi^a(0)$, a collectively denoting some Lorentz indices, then we must have

$$[M^{\mu\nu}, \phi_a(0)] = i (\mathcal{S}^{\mu\nu})_{ab} \phi^b(0), \quad (3.6)$$

where $(\mathcal{S}^{\mu\nu})_{ab}$ is a matrix that satisfies the $SO(1, d-1)$ algebra. It vanishes for a scalar operator. For a vector operator with one Lorentz index, it is given by

$$(\mathcal{S}^{\mu\nu})_{\alpha\beta} = \delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu. \quad (3.7)$$

When combined with eq. (3.4), this implies

$$[M^{\mu\nu}, \phi^a(x)] = i (x^\mu \partial^\nu - x^\nu \partial^\mu) \phi^a(x) + i (\mathcal{S}^{\mu\nu})^a_b \phi^b(0). \quad (3.8)$$

As with translations, we shall also assume that the vacuum is invariant under rotations, namely

$$M^{\mu\nu} |0\rangle = 0. \quad (3.9)$$

The fact that the quantum theory is Poincaré invariant implies that the locality condition (3.1) can be rewritten as

$$[\phi(x), \phi(y)] = 0 \quad \text{if } (x - y)^2 > 0. \quad (3.10)$$

In other words, local operators commute as long as they are space-like separated.

5. Finally, the last condition has to do with the positivity of energy. Working in a basis of eigenstates of energy and momentum,

$$P^\mu |p\rangle = p^\mu |p\rangle, \quad (3.11)$$

then we require $p^0 \geq 0$. Since this must be true in any Lorentz frame, p must be contained inside the forward light-cone, i.e.

$$p^0 \geq |\vec{p}| \quad \Leftrightarrow \quad p^2 < 0 \text{ and } p^0 > 0. \quad (3.12)$$

This gives a strong constraint on local operators: eigenstates of P^μ can be constructed taking the Fourier transform of a local operator acting on the vacuum,

$$\tilde{\phi}(p) |0\rangle = \int d^d x e^{ip \cdot x} \phi(x) |0\rangle. \quad (3.13)$$

Exercise 3.1 Using eq. (3.4), show that

$$P^\mu \tilde{\phi}(p) |0\rangle = p^\mu \tilde{\phi}(p) |0\rangle$$

Note that the operators P^μ and $M^{\mu\nu}$ are in fact topological charges (they can be defined on any codimension-1 surfaces, not only time slices), so the commutation relations (3.3) and (3.8) can be motivated in *any* quantization of the theory, including on the Lorentzian cylinder with Hamiltonian $\frac{1}{2}(P^0 - K^0)$. The points 1 to 4 in the list above can actually directly be used to define a Euclidean quantum field theory. However, it is obvious from point 5 that the generator P^0 plays a special role in quantum field theory in Minkowski space-time: not all of the conformal generators are treated on the same footing.⁵

These five points are essentially the most general definition a non-perturbative quantum field theory. They are nearly in one-to-one correspondence with the so-called *Wightman axioms*. The only difference (besides the level of mathematical rigor) is that the Wightman axioms do not rely on the existence of an energy-momentum tensor, but assume directly that the Poincaré transformations are realized as unitary transformations on the Hilbert space.

⁵On the other hand the existence of the conserved charge $M^{\mu\nu}$ is not needed: one can for instance consider quantum field theories in which the boost symmetry is broken

3.2 Wightman functions

With these axioms in mind, the next thing we can do is compute the vacuum expectation value of products of local operators,

$$\langle 0 | \phi_1(x_1) \cdots \phi_n(x_n) | 0 \rangle. \quad (3.14)$$

This can be viewed as an overlap of the vacuum state ($\langle 0 |$) with a state created acting on the vacuum with a sequence of local operators. Note that these operators need not be time-ordered: the time evolution operator is unitary, so it can go both ways. This is in stark contrast with time-ordered correlation functions obtained from the path integral.

Correlators of this type are called *Wightman functions*. They are really the fundamental observables in non-perturbative quantum field theory. In fact, it is even possible to completely define a quantum field theory just by its Wightman functions: the *Wightman reconstruction theorem* states that the Hilbert space of a quantum field theory can be constructed from all its Wightman function. A convenient perspective is therefore to forget about the Hilbert space and focus on correlation functions.

Symmetry is encoded in this correlation functions in the form of Ward identities. Given a conserved charge G that annihilates the vacuum, $G | 0 \rangle = 0$ (this could be P^μ or $M^{\mu\nu}$), the following equation must be satisfied

$$\begin{aligned} & \langle 0 | [G, \phi_1(x_1)] \phi_2(x_2) \cdots \phi_n(x_n) | 0 \rangle \\ & + \langle 0 | \phi_1(x_1) [G, \phi_2(x_2)] \cdots \phi_n(x_n) | 0 \rangle \\ & + \dots \\ & + \langle 0 | \phi_1(x_1) \phi_2(x_2) \cdots [G, \phi_n(x_n)] | 0 \rangle = 0. \end{aligned} \quad (3.15)$$

This is an equation that is obvious in the Hilbert space picture, but it is also valid as a differential equation for the Wightman function, since each commutator is again related to the local operator. Let us consider some examples

The simplest Wightman function is that involving a single scalar local operator,

$$\langle 0 | \phi(x) | 0 \rangle. \quad (3.16)$$

In this case the Ward identity associated with translations implies

$$\frac{\partial}{\partial x^\mu} \langle 0 | \phi(x) | 0 \rangle = 0, \quad (3.17)$$

or in other words that the vacuum expectation value of the operator is a constant over all of space-time. Lorentz symmetry does not give more information about that constant, but it forbids a vacuum expectation value for any operator transforming non-trivially under the Lorentz group.

Let us consider next a Wightman 2-point function of identical scalar operators,

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle. \quad (3.18)$$

In this case, translation symmetry tells us that

$$\left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial y^\mu} \right) \langle 0 | \phi(x) \phi(y) | 0 \rangle. \quad (3.19)$$

If we think of the correlator as being a function of $x + y$ and $x - y$, then this Ward identities establishes that there is no dependence on the former, i.e.

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = f(x - y). \quad (3.20)$$

In general, the consequences of translation symmetry are easier to see in momentum space, using the Fourier transform (3.13) of the local operators. By this very definition, we can establish that

$$[P^\mu, \tilde{\phi}(p)] = p^\mu \tilde{\phi}(p), \quad (3.21)$$

and therefore the Fourier transform of the Wightman 2-point function obeys

$$(p^\mu + q^\mu) \langle 0 | \tilde{\phi}(p) \tilde{\phi}(q) | 0 \rangle. \quad (3.22)$$

From this, we conclude that

$$\langle 0 | \tilde{\phi}(p) \tilde{\phi}(q) | 0 \rangle \propto \delta^d(p + q). \quad (3.23)$$

Taking into account Lorentz symmetry, one can also establish that the 2-point function in position space must only depend on the invariant distance $(x - y)^2$ between the two points, although there are subtleties associated to possible light-cone crossings (see below). In momentum space, the delta function in $p + q$ also multiplies a function ρ of the only Lorentz invariant q^2 (or equivalently p^2). When combined with the condition that only states of positive energy can exist, we deduce (up to factors of 2π that are purely conventional)

$$\langle 0 | \tilde{\phi}(p) \tilde{\phi}(q) | 0 \rangle = (2\pi)^{d+1} \delta^d(p + q) \theta(q^0 - |\vec{q}|) \rho(-q^2). \quad (3.24)$$

This result is in fact raising an important point (also part of the Wightman axioms): in spite of their names, Wightman functions are *not* functions but rather distributions. The above 2-point function computes in fact the overlap between the two states (note that the Hermitian conjugation flips the sign of momenta)

$$\tilde{\phi}(q) | 0 \rangle \quad \text{and} \quad \tilde{\phi}(-p) | 0 \rangle, \quad (3.25)$$

and therefore one would expect that it computes the norm of this state in the limit $p \rightarrow q$. However, this limit is clearly discontinuous, or the norm of the state infinite.

The resolution of this issue is that the objects $\phi(x)$ and its Fourier transform $\tilde{\phi}(p)$ are not operators, but rather *operator-valued distributions*. In other words, $\phi(x) |0\rangle$ is *not* a state of the theory, as it has in fact infinite norm (see example below). Formally, these operator-valued distributions only make sense when they are integrated against test functions, defining

$$\phi[f] = \int d^d x f(x) \phi(x), \quad (3.26)$$

or

$$\tilde{\phi}[\tilde{f}] = \int d^d p \tilde{f}(p) \tilde{\phi}(p), \quad (3.27)$$

where f and \tilde{f} are Schwartz-class test functions (i.e. functions decaying faster than any power at infinity). When acting on the vacuum, these smeared states give well-defined states, with finite norms. For instance, we have

$$\begin{aligned} \left\| \tilde{\phi}[\tilde{f}] |0\rangle \right\|^2 &= \int d^d p d^d q \tilde{f}^*(p) \tilde{f}(q) \langle 0 | \tilde{\phi}(p) \tilde{\phi}(q) | 0 \rangle \\ &= 2\pi \int_{q^0 > |\vec{q}|} d^d q \left| \tilde{f}(q) \right|^2 \rho(-q^2). \end{aligned} \quad (3.28)$$

Test functions won't appear in these lectures anymore. For physicists, this is mostly an annoyance that we prefer to avoid. However, it is important to know that there exists a mathematically rigorous way of dealing with Wightman functions. For one thing, this gives a proper justification of why it is always fine to take the Fourier transform way and back between the position- and momentum-space representation, at least as long as we are dealing with Wightman functions.

3.3 Spectral representation

The norm (3.28) is also giving away important information: it is positive if and only if the function ρ is positive for any $-q^2 > 0$. This is not an accident: the function ρ is in fact the *spectral density* encountered in standard quantum field theory textbooks, where we can often find it in the form

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = 2\pi \int \frac{d^d k}{(2\pi)^d} \int_0^\infty d\mu^2 e^{ik \cdot (x-y)} \theta(k^0) \delta(k^2 + \mu^2) \rho(\mu^2). \quad (3.29)$$

Exercise 3.2 Verify that this expression matches eq. (3.24) upon Fourier transforming in x and y .

The spectral representation is also used to construct the time-ordered correlation function

$$\langle \phi(x)\phi(y) \rangle_T \equiv \theta(x^0 - y^0) \langle 0 | \phi(x)\phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y)\phi(x) | 0 \rangle. \quad (3.30)$$

Unlike the Wightman function, this is not a tempered distribution, because the θ functions are not differentiable at the origin. Nevertheless, the time-ordered product admits the simple representation

$$\langle \phi(x)\phi(y) \rangle_T = \int \frac{d^d k}{(2\pi)^d} \int_0^\infty d\mu^2 e^{ik \cdot (x-y)} \frac{-i\rho(\mu^2)}{k^2 + \mu^2 - i\varepsilon} \quad (3.31)$$

to be understood in the limit $\varepsilon \rightarrow 0_+$. This is the *Källén-Lehmann* representation for the time-ordered 2-point function.

Exercise 3.3 Derive the Källén-Lehmann representation starting from eq. (3.29).

In the theory of a free, massive scalar field, the spectral density obeys

$$\rho(\mu^2) = \delta(\mu^2 - m^2), \quad (3.32)$$

and therefore

$$\langle 0 | \phi(x)\phi(y) | 0 \rangle = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot (x-y)} \frac{-i}{k^2 + m^2 - i\varepsilon} \quad (3.33)$$

is the ordinary propagator for a massive field.

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