

Conformal Field Theory

Lecture notes

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This is the transcript of my notes for the class “[Conformal Field Theory](#)” at the Institute for Theoretical Physics of the University of Bern. These notes will be updated as the class advances, with the latest version always available at:

https://github.com/gillioz/CFT_course

Suggestions for improvements are welcome and can be directly submitted on the GitHub page, or sent to me by email at marc.gillioz@sissa.it.

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1 Introduction

1.1 What is conformal field theory?

Conformal field theory (CFT) is a quantum field theory (QFT) with conformal symmetry:

$$\text{CFT} = \text{QFT} + \text{conformal symmetry}, \quad (1.1)$$

where under conformal symmetry we understand

- translations in time and space,
- rotations and Lorentz boosts,
- scale transformations, also known as dilatations,
- special conformal transformations.

The course will begin in section 2 with a precise definition of all these symmetries, even though the first two (forming together the Poincaré symmetry) should be familiar from an elementary quantum field theory class, and you should even have heard of the importance of dilatations in the context of renormalization.

Most introductory courses on CFT put the emphasis on conformal symmetry as a whole, which makes a lot of sense because it forms a group (more on this latter). However, this approach often requires to work in Euclidean space (we will see why), and the connection with quantum field theory as we know it appears very late in the course, if at all.

Instead, in this lecture we would like to stay as close as possible to quantum field theory; in some sense, we will define conformal field theory as

$$\begin{aligned} \text{CFT} = & \text{relativistic QFT} \\ & + \text{scale symmetry} \\ & + \text{special conformal symmetry}, \end{aligned} \quad (1.2)$$

and go through each element on the right-hand side one after the other. The first element is quantum field theory in flat Minkowski space-time. We have a good intuition of this part from particle physics. However, we do not want to restrict our attention to theories that have a nice classical limit and an understanding in terms of perturbation theory. Therefore, we will see in section 3 the basics of a non-perturbative quantum field theory. Even though we care about unitary quantum field theory in Minkowski space-time,

it will be useful to establish a connection with the same theory defined in flat Euclidean space. For this reason, we will work with the Minkowski metric convention

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}_{\mu\nu}, \quad \mu, \nu = 0, 1, \dots, d-1, \quad (1.3)$$

so that going from Minkowski space-time to Euclidean space will be achieved through a rotation of the time coordinate in the complex plane, $x^0 = t \rightarrow i\tau$. Since for us space-time will always be flat (no matter whether Minkowski or Euclidean), we will raise and lower the indices at will.

After giving a non-perturbative definition of relativistic quantum field theory, we will examine the role of scale symmetry. The main novelty here is that scale invariance forbids the presence of particles with a definite mass; instead, in a scale-invariant theory there are excitations of *any* energy (unless we are in the very special case of a free theory).

Finally, special conformal symmetry will be discussed last. The point is that it (nearly always) come along with scale symmetry. Nevertheless, it provides extremely powerful tools to understand the theory from a non-perturbative point of view. This will lead us to a comprehensive study of conformally-invariant correlation functions in section 4, and to the formulation of an *operator product expansion* (OPE) that has very nice features in section 5.

1.2 Why is conformal field theory interesting?

Nature is certainly not scale invariant, so why do we care so much about conformal field theory?

First of all, scale transformations play an essential role in quantum field theory, even when they are not symmetries of the theory. The Callan-Symanzik equation in renormalization is essentially a Ward identity associated with the (broken) scale symmetry. We will see that conformal transformations can be viewed as a *local* generalization of scale transformations, in which the rescaling factor is different at every point in space-time, while still being consistent with Poincaré symmetry.

We also know that all end points of renormalization group trajectories are described by scale-invariant fixed points, and it turns out that *all* known fixed points are in fact conformally invariant.¹ Therefore, conformal field theory is

¹Unless one gives up unitarity.

automatically relevant for the description of the low- and high-energy limits of any quantum field theory.

But one of the main reasons to study conformal field theory right now (or at least have a basic understanding of it) is that it has been producing many exciting results in the last 10-15 years. In section 6, we will give a short review of the *conformal bootstrap*, a technique that allows to completely solve conformal field theory in certain cases. Even though the modern conformal bootstrap was invented to solve a particle physics problem, its results so far have had more impact in condensed matter physics. There, the concept of *universality* is playing a central role: it turns out that the critical exponents of very different systems coincide. This has a simple explanation from the fact that there are only “few” conformal field theories that can exist at all (at least once global symmetries are specified, together with some information about relevant operators). In other words, the space of all conformal field theories is sparse, unlike in QFT where it is always possible to add some new operator in a Lagrangian to slightly deform the theory. This concept is extremely successful in $d = 2$ and 3 dimensions, but there is also a big hope to classify all possible conformal field theories in $d = 4$ (or higher) dimensions. This goal would have a huge impact also in QFT, because it would strongly constrain the possible renormalization group flows. But the conformal bootstrap is still a young field, and much more work needs to be done before anything like this can be claimed.

Besides its importance in quantum field theory and condensed matter physics, conformal field theory has also given very nice insights into:

- Quantum gravity, through the AdS/CFT correspondence.
- String theory, as the worldsheet action is a CFT in $d = 2$.
- Foundations of quantum field theory beyond the perturbative approach. CFT admits a mathematically rigorous definition, which does not require to “quantize” a classical Lagrangian theory.²

1.3 Examples of conformal field theories

Conformal field theories can be found everywhere in physics. Here are some examples, some of which you may already have encountered:

²In fact there needs not be a classical limit; for instance, in $d = 6$ dimensions there exists a theory with $(2, 0)$ extended supersymmetry for which no Lagrangian can be written down. The theory is known to exist from string theory arguments, and it has also been “observed” in a conformal bootstrap setup.

- Free massless theories are conformal. This is the case of the free boson and free fermion in any number of dimensions, but also of the n -form gauge theory in $d = 2n + 2$ dimensions (e.g. the free vector theory in $d = 4$). The same is true of theories with any number of free fields.
- Theories in which the β -function has a perturbative fixed point. Typical examples are deformations of the free scalar theory with a potential of the type ϕ^n . For instance, consider

$$S = \int d^d x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{g}{4!} \phi^4 \right]. \quad (1.4)$$

g is dimensionless in $d = 4$ dimensions. If one considers the same theory in $d = 4 - \varepsilon$ dimensions (this makes sense in perturbation theory), then the β -function for g is

$$\beta_g = \mu \frac{dg}{d\mu} = -\varepsilon + \frac{3g}{(4\pi)^2} + \mathcal{O}(g^2). \quad (1.5)$$

It vanishes when

$$\frac{g_*}{(4\pi)^2} = \frac{\varepsilon}{3}. \quad (1.6)$$

Higher-order corrections are negligible in the limit $\varepsilon \ll 1$. There are similar fixed points with a ϕ^3 interaction in $d = 6 + \varepsilon$ dimensions, or with a ϕ^6 interaction around $d = 3$.

- A similar type of fixed point can be found in the β -function of a $SU(N_c)$ gauge theory with N_f fermions (in the fundamental representation), which is given at leading order in the gauge coupling $\alpha = g^2/(4\pi)^2$ by

$$\beta_\alpha = \mu \frac{d\alpha}{d\mu} = -\frac{2}{3} \alpha^2 (11N_c - 2N_f) + \mathcal{O}(\alpha^3). \quad (1.7)$$

When $N_f = \frac{11}{2} N_c$, the leading order term in this β -function vanishes, so the next-to-leading term becomes important. Around that value, the first two terms are of similar importance, and one finds a perturbative fixed point of when $N_f \lesssim \frac{11}{2} N_c$. This is called the (Caswell-)Banks-Zaks fixed point. A theory in this situation is asymptotically free like QCD, but in the low-energy limit it approaches an interacting conformal field theory. For an $SU(3)$ gauge theory like QCD this critical value is at $N_f = 16.5$. There is strong evidence from lattice simulation that a theory with $N_f = 16$ is conformal. On the other hand a theory with low N_f (QCD has only 3 light quarks) is clearly confining, meaning that its low-energy limit is a theory massless Goldstone bosons (if

the quarks are massless), or of massive pions. There is a critical value $N_f^* \approx 12$ above which we expect an interacting conformal field theory in the low-energy limit. The domain $N_f^* \leq N_f \leq \frac{11}{2}N_c$ is called the *conformal window*. Note that gauge theories with different gauge groups and/or fermions coupling differently to the gauge fields (i.e. in different representations) can also have a conformal window. It is even possible to engineer gauge theories with perturbative UV fixed points using not only fermions but also scalars.

- There are theories with extended supersymmetry in which the β -function is exactly zero at all orders in perturbation theory, for instance $\mathcal{N} = 4$ supersymmetric Yang-Mills. More generically, in theories with sufficiently many supersymmetries, it is often sufficient to engineer the matter content to make the β -function zero at leading order, and then non-renormalization theorems ensure their vanishing at all orders. Theories with conformal symmetry and supersymmetry are called *superconformal*.
- An example of truly non-perturbative fixed point in $d = 3$ dimensions is given by the following action:

$$S = \int d^3x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!} \phi^4 \right], \quad (1.8)$$

in which $g > 0$ for the potential to be bounded below. Note that the free scalar field has mass dimension $[\phi] = \frac{1}{2}$ in $d = 3$, and therefore

$$[m^2] = 2, \quad [g] = 1. \quad (1.9)$$

Since both m^2 and g have positive mass dimensions, they are *relevant* operators: they determine the dynamics in the low-energy limit (IR), but their importance decreases at high energy (UV): at energies $E \gg g, |m|$, this approaches the theory of a massless free scalar. On the other hand, at low energies the physics depends obviously on g and m : when $m^2 \gg g^2$, it is a theory of a massive scalar of mass m , whereas when $m^2 \ll -g^2$, then the potential has two minima at

$$\langle \phi \rangle = \pm \sqrt{-\frac{6m^2}{g}}, \quad (1.10)$$

with excitations of mass $\sqrt{2}|m|$ around it. Clearly this theory has two phases, and there must therefore be an intermediate value of m^2 where

the phase transition happens (or working in units set by g , a specific value of the dimensionless ratio m^2/g^2). Note that this theory has a \mathbb{Z}_2 symmetry in the UV corresponding to $\phi \rightarrow -\phi$, which is spontaneously broken in one phase and not in the other.

It turns out that the theory describing the IR physics exactly at the phase transition is a conformal field theory. Unlike the two phases surrounding it, it admits excitations of arbitrarily small energies (but they are not particles). How do we know that? Feynman diagram computations cannot be trusted in the IR: the approximation given by the (asymptotic) perturbative series is valid in the UV, but it breaks down in the IR, as in QCD. One way of understanding the phase transition is to examine the theory in $d \neq 3$ dimensions: the same theory in $d = 4 - \varepsilon$ has a perturbative fixed point. The value of the fixed point depends on the renormalization scheme, but there are other quantities that are scheme-independent. This is for instance the case of the anomalous dimension of the operator ϕ . The state-of-the-art 6-loop beta-function computation gives

$$\gamma_\phi \approx 0.0182. \quad (1.11)$$

This is more conveniently expressed as the *scaling dimension* of ϕ ,

$$\Delta_\phi = \frac{d-2}{2} + \gamma_\phi \approx 0.5182, \quad (1.12)$$

which in position space describes how the correlation between two points decays with the distance

$$\langle \phi(x)\phi(0) \rangle \propto |x|^{-2\Delta_\phi}. \quad (1.13)$$

A surprising thing about this scaling dimension is that it coincides precisely with that of a statistical physics model. The *Ising model* is a theory of classical spins $\sigma_i = \pm 1$ on a lattice with nearest-neighbor interactions, characterized by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j. \quad (1.14)$$

This model has a critical value of J whose continuum limit is described by a CFT. At this value, Monte-Carlo simulations show that the correlation between two spins decay with the distance with power given by

$$\Delta_\sigma \approx 0.5181. \quad (1.15)$$

The two theories have completely different microscopic descriptions: one is a quantum field theory describing particles in Minkowski space-time, the other a simple theory on a Euclidean lattice. Even more surprisingly, the same critical exponent is found in other systems, such as the critical point of water and other liquids. This is an example of *universality*.

The explanation for this coincidence is that there are not many candidate conformal field theories to describe the phase transition of the ϕ^4 theory and the Ising model. In fact, these two theories have in common:

- A global \mathbb{Z}_2 symmetry ($\sigma_i \rightarrow -\sigma_i$) that is broken in one phase and unbroken in the other;
- Exactly two relevant operators (in addition to J , the Ising model can be characterized by its coupling to an external magnetic field $\delta H = -\mu \sum s_i$).

The conformal bootstrap philosophy is quite different from the two examples above, in the sense that it does not care about the microscopic details: instead, it attempts to solve the conformal field theory that is expected to be there at the fixed point, based on symmetry arguments only. We shall see that the conformal bootstrap shows that there exists a unique conformal field theory with two relevant operators and a \mathbb{Z}_2 symmetry, and that the scaling dimension of its leading operator is equal to

$$\Delta_\sigma \approx 0.5181489. \quad (1.16)$$

In fact, the conformal bootstrap does not only establish that, but it gives access to the whole spectrum of operators in that conformal field theory. This is truly a success story of the bootstrap.

- Finally, another example of CFT is the family of theories obtained by considering a quantum field theory in $d + 1$ -dimensional anti-de-Sitter (AdS) space-time. AdS admits a compactification to a sphere with a boundary, and correlation functions on that boundary are isomorphic with that of a d -dimensional conformal field theory. A generic (e.g. perturbative) theory in AdS gives rise to a CFT on the boundary that does not satisfy all the ordinary assumptions of a quantum field theory (for instance it does not have an energy-momentum tensor), but there is evidence that a theory that includes gravity does.³

³The same type of correspondence applies to late-time correlators in de Sitter space-time.

1.4 Literature

There are several excellent modern reviews on conformal field theory, and large parts of these lecture notes are directly inspired by them. The essentials are covered in the works of conformal bootstrap experts:

- Slava Rychkov’s EPFL lectures [1]; this review also contains interesting historical comments.
- David Simmons-Duffin’s TASI lectures [2].
- Shai Chester’s Weizmann Lectures [3].

Some other useful material can be found in:

- A review article on the state-of-the-art bootstrap methods [4].
- Hugh Osborn’s lecture notes at Cambridge, available at: <https://www.damtp.cam.ac.uk/user/ho/CFTNotes.pdf>
- Joshua Qualls’ lectures [5].

The reviews mentioned above are all mostly focused on the Euclidean approach to CFT. For a Lorentzian perspective, see:

- Slava Rychkov’s *Lorentzian methods in conformal field theory*, available at <https://courses.ipht.fr/node/226> (link to the lecture notes)

There are also modern reviews specifically on the topic of CFT in $d = 2$ dimensions:

- Sylvain Ribault’s notes [6], also accepting suggestions for improvement on GitHub: <https://github.com/ribault/CFT-Review>.
- Schellekens “Conformal Field Theory” lecture notes [7], and a recent version available at: <https://www.nikhef.nl/~t58/CFT.pdf>.

More classic literature on the subject can be found in:

- The “yellow book” by Di Francesco, Mathieu, and Sénéchal [8].
- Polchinski’s *String theory* vol. 1 [9].
- Lectures by Paul Ginsparg [10].
- A review by Matthias Gaberdiel [11].

For theories combining supersymmetry and conformal symmetry:

- Lorenz Eberhardt’s lecture notes [12].

2 Classical conformal transformations

One of the most fundamental principles of physics is independence of the reference frame: observers living at different points might have different perspectives, but the underlying physical laws are the same. This is true in space (invariance under translations and rotations), but also in space-time (e.g. invariance under Lorentz boosts).

2.1 Infinitesimal transformations

In mathematical language, this means that if we have a coordinate system x^μ , the laws of physics do not change under a transformations

$$x^\mu \rightarrow x'^\mu. \quad (2.1)$$

This principle applies to all maps that are invertible (isomorphisms) and differentiable (smooth transformations), hence it is usually called *diffeomorphism* invariance. Being differentiable, the transformation (2.1) can be Taylor-expanded to write

$$x^\mu \rightarrow x'^\mu = x^\mu + \varepsilon^\mu(x), \quad (2.2)$$

in terms of an infinitesimal vector ε^μ (meaning that we will always ignore terms of order ε^2).

In addition to the coordinate system, the description of a physical system requires a way of measuring distances that is provided by a metric $g_{\mu\nu}(x)$. Distances are measured integrating the line element

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu. \quad (2.3)$$

Since all observers should agree on the measure of distances, we must have

$$g'_{\mu\nu}(x')dx'^\mu dx'^\nu = g_{\mu\nu}(x)dx^\mu dx^\nu, \quad (2.4)$$

Here $g_{\mu\nu}$ could be the Euclidean metric $\delta_{\mu\nu}$ or the Minkowski metric $\eta_{\mu\nu}$; for simplicity we only consider the case in which $g_{\mu\nu}$ is flat, i.e. $\partial_\alpha g_{\mu\nu} = 0$. In this case, we can write

$$\begin{aligned} g'_{\mu\nu} &= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \\ &= g_{\alpha\beta} (\delta_\mu^\alpha - \partial_\mu \varepsilon^\alpha) (\delta_\nu^\beta - \partial_\nu \varepsilon^\beta) \\ &= g_{\mu\nu} - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu). \end{aligned} \quad (2.5)$$

If we require the different observers to also agree on the metric, then we must have $g'_{\mu\nu} = g_{\mu\nu}$, which gives a constraint on what kind of coordinate transformations are possible: they must satisfy

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 0. \quad (2.6)$$

This condition admits as most general solution

$$\varepsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu, \quad (2.7)$$

where a^μ is a constant vector and $\omega_{\mu\nu} = g_{\mu\rho}\omega^\rho{}_\nu$ an antisymmetric tensor, i.e. $\omega_{\nu\mu} = -\omega_{\mu\nu}$. The transformation

$$x^\mu \xrightarrow{P} x^\mu + a^\mu \quad (2.8)$$

is obviously a translation and

$$x^\mu \xrightarrow{M} (\delta^\mu{}_\nu + \omega^\mu{}_\nu) x^\nu \quad (2.9)$$

a rotation/Lorentz transformation around the origin $x = 0$: the matrix $R^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ satisfies $R^\mu{}_\alpha g^{\alpha\beta} R^\beta{}_\nu = g^{\mu\nu}$. The composition of these two operations generates the Poincaré group. This is the fundamental symmetry of space-time underlying all relativistic quantum field theory. It is a symmetry of nature to a very good approximation, at least up to energy scales at which quantum gravity becomes important.

However, one can also consider the situation in which the two observers use different systems of units, i.e. they disagree on the overall definition of scale, but agree otherwise on the metric being flat. In this case we must have $g'_{\mu\nu} \propto g_{\mu\nu}$, and therefore the constraint becomes

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 2\lambda g_{\mu\nu}, \quad (2.10)$$

for some positive real number λ , with the most general solution

$$\varepsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu. \quad (2.11)$$

The new infinitesimal transformation is

$$x^\mu \xrightarrow{D} (1 + \lambda)x^\mu. \quad (2.12)$$

It is a scale transformation. Note that scale symmetry is not a good symmetry of nature: there is in fact a fundamental energy scale on which all observer must agree (this can be for instance chosen to be the mass of the electron).

Nevertheless, there are systems in which this is a very good approximate symmetry, making it worth studying.

If one pushes this logic further, in a scale-invariant world in which observers have no physical mean of agreeing on a fundamental scale, they might even decide to change their definition of scale as they walk around, or as time passes. This would correspond to the case in which the metric $g'_{\mu\nu}$ of one observer can differ from the original metric $g_{\mu\nu}$ by a function of space-time:

$$g'_{\mu\nu}(x) = \Omega(x)g_{\mu\nu}. \quad (2.13)$$

Note that we are not saying that $g'_{\mu\nu}$ is completely arbitrary: at every point in space time it is related to the flat metric by a scale transformation. But the scale factor is different at every point. The condition on ε^μ becomes in this case

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 2\sigma g_{\mu\nu}, \quad (2.14)$$

where σ is the infinitesimal version of Ω , i.e. $\Omega(x) = e^{-2\sigma(x)} \approx 1 - 2\sigma(x)$. To find the most general solution to this equation, note that contracting the indices with $g^{\mu\nu}$ gives

$$\partial_\mu \varepsilon^\mu = d\sigma, \quad (2.15)$$

where d is the space(-time) dimension, while acting with ∂^ν gives

$$\partial_\mu \partial_\nu \varepsilon^\nu + \partial^2 \varepsilon_\mu = 2\partial_\mu \sigma, \quad (2.16)$$

so that we get

$$\partial^2 \varepsilon_\mu = (2 - d)\partial_\mu \sigma. \quad (2.17)$$

Acting once again with ∂^μ , we arrive at

$$(d - 1)\partial^2 \sigma = 0, \quad (2.18)$$

while acting with ∂^ν and symmetrizing the indices, we find

$$(2 - d)\partial_\mu \partial_\nu \sigma = g_{\mu\nu} \partial^2 \sigma = 0. \quad (2.19)$$

In $d > 2$, we obtain therefore the condition $\partial_\mu \partial_\nu \sigma = 0$, which is solved by

$$\sigma(x) = \lambda + 2b \cdot x. \quad (2.20)$$

The corresponding value of ε^μ is

$$\varepsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + 2(b \cdot x)x^\mu - x^2 b^\mu. \quad (2.21)$$

Therefore, in addition to the transformations found before, we also have

$$x^\mu \xrightarrow{K} x^\mu + 2(b \cdot x)x^\mu - x^2 b^\mu, \quad (2.22)$$

which is called *special conformal transformation*. If we examine the Jacobian for this transformation, we find

$$\frac{\partial x'^\mu}{\partial x^\nu} = (1 + 2b \cdot x) \delta_\nu^\mu + 2(b_\nu x^\mu - x_\nu b^\mu) \approx (1 + 2b \cdot x) R_\nu^\mu(x). \quad (2.23)$$

We have written this as a position-dependent scale factor $(1 + 2b \cdot x)$, multiplying an orthogonal matrix

$$R_\nu^\mu(x) = \delta_\nu^\mu + 2(b_\nu x^\mu - x_\nu b^\mu). \quad (2.24)$$

This shows that special conformal transformation act locally as the composition of a scale transformation and a rotation (or Lorentz transformation). This also shows that conformal transformations preserve angles, which is the origin of their name. Eq. (2.14) is called the conformal Killing equation and its solutions (2.21) the Killing vectors.

Note that in our derivation the original metric $g_{\mu\nu}$ was flat, but the new metric $g'_{\mu\nu}$ is not. It is however conformally flat: it is always possible to make a change of coordinate after which it is flat again. In general, transformations

$$g_{\mu\nu}(x) \rightarrow \Omega(x) g_{\mu\nu}(x) \quad (2.25)$$

are called *Weyl transformations*. They change the geometry of space-time. We found that any Weyl transformation which is at most quadratic in x can be compensated by a change of coordinates to go back to flat space. The corresponding flat-space transformation is called conformal transformation.⁴

In $d = 2$ the situation is a bit different: the conditions $\partial^2 \sigma = 0$ is sufficient to ensure that the Killing equation has a solution. This is most easily seen in light-cone coordinates,

$$x^+ = \frac{x^0 + x^1}{2}, \quad x^- = \frac{x^0 - x^1}{2}, \quad (2.26)$$

in terms of which

$$\partial^2 \sigma = \partial_+ \partial_- \sigma. \quad (2.27)$$

This is satisfied by taking for σ the sum of an arbitrary function of the left-moving variable x^+ and of another function of the right-moving variable x^- . In fact, if we write $\varepsilon^\pm = \varepsilon^0 \pm \varepsilon^1$, we can take arbitrary functions $\varepsilon^+(x^+)$ and $\varepsilon^-(x^-)$, and verify that eq. (2.14) is satisfied with $\sigma = \frac{1}{2}(\partial_+ \varepsilon_+ + \partial_- \varepsilon_-)$. In the Euclidean case we take

$$z = \frac{x^1 + ix^2}{2}, \quad \bar{z} = \frac{x^1 - ix^2}{2}, \quad (2.28)$$

⁴This implies that the group of conformal transformation is a subgroup of diffeomorphisms. It is in fact the largest finite-dimensional subgroup.

complex-conjugate to each other, and the same logic follows: we can apply arbitrary holomorphic and anti-holomorphic transformations on z and \bar{z} , and the conformal Killing equation is always satisfied. This shows that there are (infinitely) many more conformal transformations in $d = 2$ than in $d > 2$, and also that there is no significant difference between Euclidean and Minkowski conformal transformation in $d = 2$, as the transformation acts essentially on the two light-cone/holomorphic coordinates independently.

2.2 The conformal algebra

The conformal Killing equation (2.21) determines the most general form of *infinitesimal* conformal transformations. *Finite* conformal transformations follow from a sequence of infinitesimal transformations. However, one has to bear in mind that the infinitesimal conformal transformations do not commute: for instance, a translation followed by a rotation is not the same as the opposite. In fact, the conformal transformations form a *group*: the composition of conformal transformations is again a conformal transformation.

As we all know from quantum field theory, a group is characterized by its *generators* and their commutation relations (the *algebra*). A generator G describes an infinitesimal transformation in some direction, and finite transformations are obtained using exponentiation, $e^{i\theta G}$, with parameter θ (the factor of i is the physicist's convention that makes the generators Hermitian). A representation of the conformal group can be obtained from the functions of the coordinates, $f(x)$. For instance, under an infinitesimal translation, we have

$$f(x) \xrightarrow{P} f(x') = f(x + a) \approx f(x) + a^\mu \partial_\mu f(x) \quad (2.29)$$

and we require this to be equal to $e^{-ia_\mu P^\mu} f(x)$, which means

$$P_\mu = i\partial_\mu. \quad (2.30)$$

Performing the same analysis for the other infinitesimal transformations given in eq. (2.21), we obtain for the other generators⁵

$$\text{rotations/Lorentz transformations:} \quad M^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \quad (2.31)$$

$$\text{scale transformations:} \quad D = ix^\mu \partial_\mu, \quad (2.32)$$

$$\text{special conformal transformations:} \quad K^\mu = i(2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu). \quad (2.33)$$

⁵The sign of these generators is an arbitrary convention. It defines once and for all the commutation relations that we will derive next. After that, we will always refer to the commutation relations as defining the generators.

The number of generators matches that of the Killing vectors: there are d translations, d special conformal transformations, $d(d-1)/2$ rotations/Lorentz transformations ($M^{\mu\nu}$ is a $d \times d$ antisymmetric matrix), and one scale transformation. Therefore the total number of generators, i.e. the dimension of this group, is $(d+1)(d+2)/2$. In $d = 4$ space-time dimension, the conformal group has 15 generators.

Using the above definition, one can verify that the following commutation relations are satisfied,

$$\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= -i(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}) \\
[M^{\mu\nu}, P^\rho] &= -i(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu) \\
[M^{\mu\nu}, K^\rho] &= -i(g^{\mu\rho}K^\nu - g^{\nu\rho}K^\mu) \\
[D, P^\mu] &= -iP^\mu \\
[D, K^\mu] &= iK^\mu \\
[P^\mu, K^\nu] &= 2i(g^{\mu\nu}D - M^{\mu\nu})
\end{aligned} \tag{2.34}$$

while all other commutators vanish:

$$[M^{\mu\nu}, D] = [P^\mu, P^\nu] = [K^\mu, K^\nu] = 0. \tag{2.35}$$

The first two relations in eq. (2.34) are the familiar Poincaré algebra. The next one states that K^μ transforms like a vector (as P^μ does), whereas D is obviously a scalar. The next two relations remind us that K^μ and P^μ have respectively the dimension of length and inverse length.

Exercise 2.1 Derive the commutation relations from the action (2.30)–(2.33) of the generators on functions of x .

Even though this is not immediately obvious, this algebra is isomorphic to that of the group $\text{SO}(d+1, 1)$ (if $g^{\mu\nu}$ is the Euclidean metric) or $\text{SO}(d, 2)$ (if it is the Minkowski metric). To see that it is the case, let us introduce a $(d+2)$ -dimensional space with coordinates

$$X^\mu, \quad X^{d+1}, \quad X^{d+2}, \tag{2.36}$$

and a metric defined by the line element

$$ds^2 = g_{\mu\nu}dX^\mu dX^\nu + dX^{d+1}dX^{d+1} - dX^{d+2}dX^{d+2} \equiv \eta_{MN}dX^M dX^N. \tag{2.37}$$

Then we can write all conformal commutation relations as being defined by the Lorentzian algebra

$$[J^{MN}, J^{RS}] = -i(\eta^{MR}J^{NS} - \eta^{MS}J^{NR} - \eta^{NR}J^{MS} + \eta^{NS}J^{MR}), \tag{2.38}$$

provided that we identify the antisymmetric generators J^{MN} with the conformal generators as follows:

$$\begin{aligned} M^{\mu\nu} &= J^{\mu\nu}, \\ P^\mu &= J^{\mu,d+1} + J^{\mu,d+2}, \\ K^\mu &= J^{\mu,d+1} - J^{\mu,d+2}, \\ D &= J^{d+1,d+2}. \end{aligned} \tag{2.39}$$

2.3 Finite transformations

We just saw that the infinitesimal conformal transformations generate a group. But how can we describe finite conformal transformations? Let us see how each generator exponentiates into an element of the group; the most general conformal transformation can then be obtained as a composition of such finite transformations.

In some cases the exponentiation is trivial. For instance, with translations we obtain immediately

$$x^\mu \xrightarrow{P} x^\mu + a^\mu, \tag{2.40}$$

where a is now any d -dimensional vector, not necessarily small. The same is true of scale transformations,

$$x^\mu \xrightarrow{D} \lambda x^\mu \tag{2.41}$$

with finite λ . Rotations or Lorentz transformations exponentiate as

$$x^\mu \xrightarrow{M} \Lambda^\mu{}_\nu x^\nu \tag{2.42}$$

where $\Lambda^\mu{}_\nu$ is a $\text{SO}(d)$ or $\text{SO}(1, d-1)$ matrix, depending whether the metric is Euclidean or Minkowski. All of this is well-known and not surprising.

On the contrary, special conformal transformations do not exponentiate trivially. The easiest way to derive their finite form is to make the following observation: recall that in infinitesimal form we have

$$x'^\mu = x^\mu + 2(b \cdot x)x^\mu - x^2 b^\mu, \tag{2.43}$$

which implies $x'^2 = (1 + 2b \cdot x)x^2$, and therefore (as always neglecting terms of order b^2)

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu. \tag{2.44}$$

The ratio x^μ/x^2 appearing on both side of the equation is the *inverse* of the coordinate x^μ , respectively x'^μ : let us define the inversion as

$$x^\mu \xrightarrow{I} \frac{x^\mu}{x^2}. \tag{2.45}$$

This transformation does not have an infinitesimal form, but otherwise it shares the essential properties of a conformal transformation: its Jacobian is

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \frac{1}{x^2} \left[\delta_{\nu}^{\mu} - 2 \frac{x^{\mu} x_{\nu}}{x^2} \right], \quad (2.46)$$

which is the product of a position-dependent scale factor (x^{-2}) with an orthogonal matrix. To understand what this transformation does globally, let us consider a point $\vec{x} = (a, 0, \dots, 0) \in \mathbb{R}^d$. Then the matrix is square bracket is diagonal, and equates $\text{diag}(-1, 1, \dots, 1)$. This is an orthogonal matrix with determinant -1 , which is part of $O(d)$ but not $SO(d)$. This shows that the inversion is a discrete transformations not connected to identity. A conformally invariant theory might be invariant under inversions, but it needs not be.

Eq. (2.44) shows that infinitesimal special conformal transformations are obtained taking an inversion followed by a translation, followed by an inversion again. Since this process involves the inversion twice, and since inversion is its own inverse, it does not matter whether inversion is a true symmetry of the system or not. The advantage of this representation is that it can easily be exponentiated: the composition of (infinitely) many infinitesimal special conformal transformation can be written as an inversion followed by a finite translation, followed by an inversion again. In other words, eq. (2.44) holds for finite b^{μ} . This can be used to show that

$$x^{\mu} \xrightarrow{K} x'^{\mu} = \frac{x^{\mu} - x^2 b^{\mu}}{1 - 2b \cdot x + b^2 x^2}. \quad (2.47)$$

Exercise 2.2 Use eq. (2.44) to show (2.47).

What do special transformation do globally? Let us look specifically in Euclidean space. There are some special points:

- The origin of the coordinate system $x = 0$ is mapped onto itself.
- The point b^{μ}/b^2 is mapped to ∞ .
- Conversely, the “point” $x \rightarrow \infty$ is mapped to the finite vector $-b^{\mu}/b^2$.

These properties can be understood from the fact that special conformal transformations and translations are related by inversion: special conformal transformations keep the origin fixed but move every other points, including

∞ ; translations move every point except ∞ . The other two transformations, rotations and scale transformations, keep both 0 and ∞ fixed.

An essential property of conformal transformations is that they allow to map any 3 points (x_1, x_2, x_3) onto another set (x'_1, x'_2, x'_3) . This can be seen as follows: first, apply a translation to place x_1 at the origin, followed by a special conformal transformation that takes x_3 to ∞ , after which the image of the original triplet is $(0, x''_2, \infty)$; then use rotations and scale transformations to move x''_2 to another point x'''_2 , while keeping 0 and ∞ fixed; finally apply again a special conformal transformation that takes ∞ to $x'_3 - x'_1$, and a translation by x'^1 to reach the configuration (x'_1, x'_2, x'_3) . This property has an immediate physical consequence: in correlation functions involving 2 or 3 local operators (see next sections for a definition), all kinematics is fixed by conformal symmetry. The only freedom is about the operators themselves, not about their position in space.

Another interesting property of conformal transformations is that they map spheres to spheres: this is an obvious property of translations, rotations and scale transformations, but it is also true of special conformal transformations.

Exercise 2.3 Show that under the special conformal transformation (2.47), a sphere centered at the point a^μ and with radius R gets mapped to a sphere centered at the point

$$a'^\mu = \frac{a^\mu - (a^2 - R^2)b^\mu}{1 - 2a \cdot b + (a^2 - R^2)b^2}$$

and with radius

$$R' = \frac{R}{|1 - 2a \cdot b + (a^2 - R^2)b^2|}.$$

In the special case in which b^μ/b^2 is on the original sphere, show that the sphere gets mapped to a plane orthogonal to the vector $a^\mu + (R^2 - a^2)b^\mu$.

Note that in $d = 2$, the additional conformal transformations (infinitely many of them!) mean that (nearly) any shape can be mapped onto another.⁶

⁶This is known as the Riemann mapping theorem. A neat example of how a disk can be mapped onto a polygon is available at <https://herbert-mueller.info/uploads/3/5/2/3/35235984/circletopolygon.pdf>.

2.4 Compactifications

It is often said that conformal symmetry is a symmetry of flat space(-time). This is true, as we have just seen, provided that we treat the point ∞ as being part of the space. This is quite straightforward in Euclidean space, but much more subtle in Minkowski space-time, as there are different, inequivalent ways of reaching ∞ there.

To gain a better understanding of this, it is useful to map the flat Euclidean space \mathbb{R}^d or the Minkowski space-time $\mathbb{R}^{1,d-1}$ onto a curved manifold. In Euclidean space, this is for instance achieved by the (inverse) stereographic projection that maps $\mathbb{R}^d \cup \{\infty\}$ to the unit sphere S^d . Geometrically, the stereographic projection is constructed as follows: embed \mathbb{R}^d as a plane in \mathbb{R}^{d+1} , together with a sphere of unit radius centered at the origin. Every point on the plane has an image on the sphere obtained drawing a segment between the original point and the north pole of the sphere, and noting where it intersects the sphere. The origin is mapped to the south pole, ∞ to the north pole, and the sphere S^{d-1} of unit radius to the equator. Algebraically, this is achieved as follows: first write the Euclidean metric in spherical coordinates,

$$ds^2 = dr^2 + r^2 d\Omega_{d-1}^2, \quad (2.48)$$

where the solid angle is given in $d = 2$ by $d\Omega_1^2 = d\phi^2$, in $d = 3$ by $d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2$, and more generically by the recursion relation $d\Omega_n^2 = d\theta^2 + \sin^2 \theta d\Omega_{n-1}^2$. Let us perform the change of variable

$$r = \frac{\sin \varphi}{1 - \cos \varphi}, \quad (2.49)$$

and interpret $\varphi \in [0, \pi]$ as the zenith angle on the sphere: $\varphi = 0$ is the north pole, corresponding to $r \rightarrow \infty$, and $\varphi = \pi$ the south pole, corresponding to $r = 0$. In these coordinates we have

$$ds^2 = \frac{1}{(1 - \cos \varphi)^2} (d\varphi^2 + \sin^2 \varphi d\Omega_{d-1}^2) = \frac{1}{(1 - \cos \varphi)^2} d\Omega_d^2. \quad (2.50)$$

This shows that the metric is flat up to an overall Weyl factor that depends on φ . In these new coordinates, conformal transformations are always non-singular. For this reason, it is often very convenient to study *classical* conformal transformations on the sphere S^d instead of Euclidean space.

However, this compactification is not as nice in a *quantum* theory in which one wants to foliate the space along some direction: if one chooses φ as the “Euclidean time”, then the “space” direction is a sphere S^{d-1} whose volume depends on φ . In other words, the generator of “time” translations

is not a symmetry of the system. This is also true of any other choice of time direction on the sphere.

Instead, another compactification is often preferred to the sphere: from the Euclidean metric in spherical coordinates, one can make the change of variable

$$\tau = \log(r) \quad \Leftrightarrow \quad r = e^\tau, \quad (2.51)$$

after which

$$ds^2 = r^2 (d\tau^2 + d\Omega_{d-1}^2). \quad (2.52)$$

This is again a flat metric, up to a Weyl factor $r^2 = e^{2\tau}$. In this case however the metric is completely independent of τ . This space has the geometry of the *cylinder*, $\mathbb{R} \times S^{d-1}$. It is not fully compact: τ goes from $-\infty$ to $+\infty$. But it has an important advantage: translations in τ are generated by dilatations D , which will be taken to be a symmetry of the quantum theory. Foliating the space into surfaces of constant τ will later lead us to radial quantization in conformal field theory.

Note that in $\text{SO}(d+1, 1)$ language, the generator $D = J^{d+1, d+2}$ is completely equivalent to any other generator $J^{\mu, d+2} = \frac{1}{2}(P^\mu - K^\mu)$, since they are related by $\text{SO}(d+1)$ rotations. So we might as well look for a cylinder compactification in which the non-compact direction corresponds to transformation generated by $\frac{1}{2}(P^0 - K^0)$. This combination of generators obeys

$$\frac{1}{2}(P^0 - K^0) = i \left(\frac{1 - (x^0)^2 + \vec{x}^2}{2} \partial_0 - x^0 \vec{x} \cdot \vec{\partial} \right), \quad (2.53)$$

with two fixed points at $x^0 = \pm 1$ with $\vec{x} = 0$. The foliation of space generated by this linear combination of generators is what is used in N-S quantization,⁷ discussed later in section 5. Figure 1 illustrates the two foliations of Euclidean space by D and $\frac{1}{2}(P^0 - K^0)$, and the corresponding cylinder interpretations are shown in figure 2.

Exercise 2.4 Find the change of coordinate that makes the Euclidean metric Weyl-equivalent to a cylinder in which translations in the non-compact direction are generated by $\frac{1}{2}(P^0 - K^0)$.

Hint: Find a special conformal transformation followed by a translation that take $(0, \infty)$ to $(-1, 1)$, and apply it to the cylinder coordinates.

⁷The lines drawn when evolving a point with this Hamiltonian look like magnetic field lines connecting the north (N) and south (S) poles of a magnet, hence the name N-S quantization.

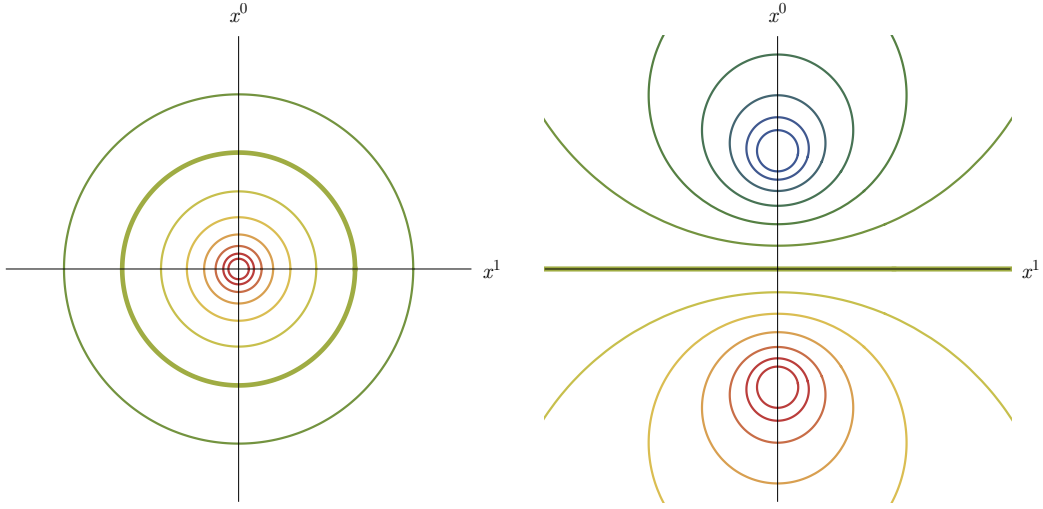


Figure 1: Foliation of Euclidean space (here in $d = 2$) in radial (left) and N-S quantization (right). Circles of identical colors are mapped onto each other by a conformal transformation. In particular, the plane $x^0 = 0$ in N-S quantization is mapped to the unit sphere in radial quantization.

The cylinder compactifications of Euclidean space are interesting by themselves, but they are also extremely convenient to understand the connection between Euclidean and Minkowski space-times: in this last form, performing a Wick rotation $\tau \rightarrow -it$ defines a cylinder on which the Lorentzian conformal group $SO(d, 2)$ acts naturally. But before going there, let us go back to flat Minkowski space-time and make some general remarks.

2.5 Minkowski space-time

Translations and Lorentz transformations in Minkowski space-time are familiar, and even dilatations are a standard tool in the renormalization group analysis. But what do conformal transformations do?

To understand this, let us place an observer at the origin of Minkowski space-time. The presence of this observer breaks translations, but not Lorentz transformations (the observer is point-like), nor dilatations or special conformal transformations. For the observer, space-time is split into three regions: a future light cone, a past light cone, and a space-like region from which they know nothing. Lorentz and scale transformation preserve this causal structure: the future and past light-cones are mapped onto themselves. In other words, if a point x is space-like separated from the observed, it will remain space-like separated no matter the choice of scale and Lorentz frame. With-

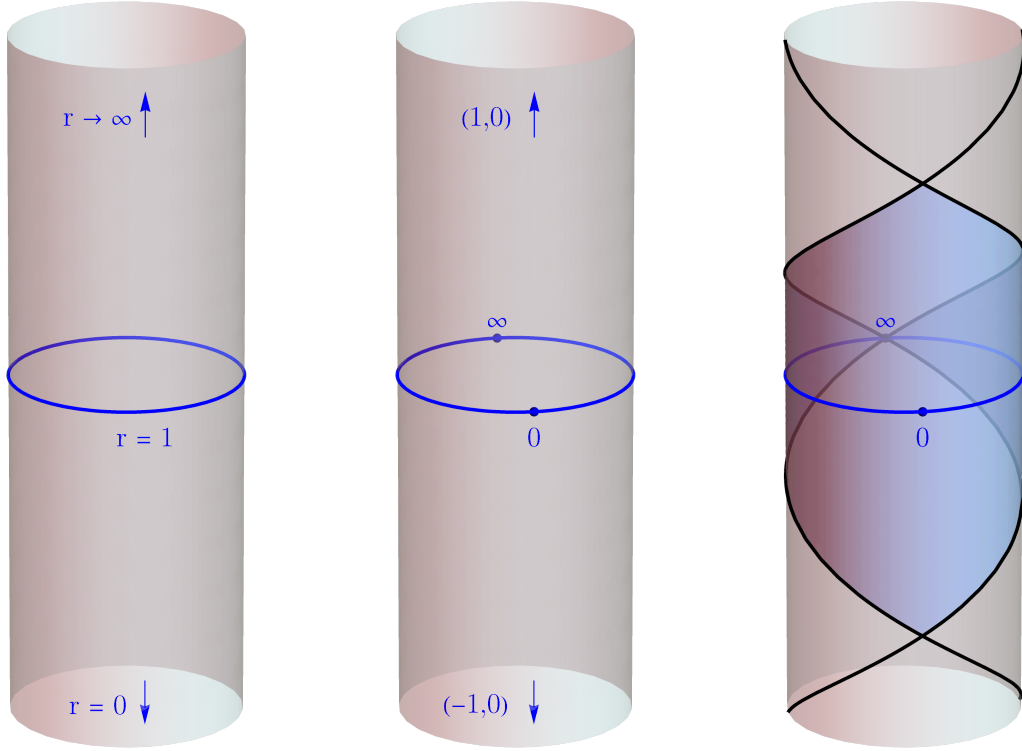


Figure 2: Left: the Euclidean cylinder corresponding the radial coordinates (2.52), in which the evolution in the non-compact direction is given by the generator D of scale transformations; the blue circle corresponds to the unit sphere. Middle: same Euclidean cylinder after a conformal transformation, so that the evolution is given by $\frac{1}{2}(P^0 - K^0)$; the blue circle is now the plane $x^0 = 0$, which includes the origin and the point at infinity. Right: the Lorentzian cylinder, obtained after a Wick rotation in x^0 , so that the surface $x^0 = 0$ is unchanged; the image of this surface under time translations generated by P^0 is the Poincaré patch, shown as the blue diamond on the cylinder's surface; only special conformal transformations can move a point from inside the Poincaré patch to the outside.

out loss of generality, let us choose this point to be at position $x = (0, \vec{n})$, where \vec{n} is a unit vector (units can be chosen so that this is the case). Now apply a special conformal transformation with parameter $b^\mu = (-\alpha, \alpha\vec{n})$, with α varying between 0 and 1. This draws a curve y^μ in space-time, parameterized by α , with

$$y^0(\alpha) = \frac{\alpha}{1-2\alpha}, \quad \vec{y}(\alpha) = \frac{1-\alpha}{1-2\alpha}\vec{n}. \quad (2.54)$$

This curve begins at the space-like point $y = (0, \vec{n})$, and ends in the past light cone at $y = (-1, \vec{0})$. Note that y never crosses a light cone: the image of x is never null if x is not itself null, since $x'^2 = x^2/(1-2x \cdot b + x^2b^2)$. Instead, we have

$$y^2(\alpha) = \frac{1}{1-2\alpha} \neq 0. \quad (2.55)$$

What happens is that the point travels all the way to space-like infinity at $\alpha = \frac{1}{2}$, and comes back from past infinity. Clearly, special conformal transformation break causality!

The resolution of this paradox is that conformal transformation do not act directly on Minkowski space, but rather on its universal cover that is isomorphic to the Lorentzian cylinder described above, and shown in figure 2. Evolution on that cylinder is given by the Hamiltonian $H = \frac{1}{2}(P^0 - K^0)$, but this differs from the Minkowski time evolution by P^0 . On any given slice of the Lorentzian cylinder, space is compactified in such a way that the notion of infinite distance is unequivocal: space-like infinity corresponds to a point on the sphere, antipodal to the origin. If one takes any other point of that sphere and apply finite translations using the generator P^μ , then this defines a compact *Poincaré patch*. The full Lorentzian cylinder is a patchwork of Poincaré patches, but every observer only has access to one.

The lesson that we must learn from this is that only the infinitesimal form of special conformal transformation can be used in Minkowski space-time: *any* finite special conformal transformation would bring part of space-time into another patch on the cylinder. This is sometimes called *weak conformal invariance*.

A quantum field theory defined on Minkowski space-time with weak conformal invariance uniquely defines a quantum field theory on the Lorentzian cylinder with global conformal invariance. This non-trivial fact was proven by Martin Lüscher and Gerhard Mack in 1974 [13] (both were employed at the University of Bern at the time). In their own words:

In picturesque language, [the superworld] consists of Minkowski space, infinitely many “spheres of heaven” stacked above it and infinitely many “circles of hell” below it.

2.6 Conformal symmetry in classical field theory

So far we have been discussing conformal transformations of the coordinates. The next step is to consider a field theory (for the moment a classical one) that has conformal symmetry built in. The simplest example is the free scalar field theory, defined by the action

$$S = \int d^d x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right]. \quad (2.56)$$

We shall see in the next section that the action principle can in fact be dropped in CFT, but for now it is a convenient starting point.

In this context, a conformal transformation is a transformation *of the fields*. There are two different and complementary perspectives one can adopt. It is often convenient to think of the metric tensor as a field in its own right, and to define a conformal transformation as a (position-dependent) scale transformation of the field

$$\phi(x) \rightarrow e^{\Delta \sigma(x)} \phi(x), \quad (2.57)$$

combined with a Weyl transformation of the metric

$$g_{\mu\nu}(x) \rightarrow e^{2\sigma(x)} g_{\mu\nu}(x). \quad (2.58)$$

Δ is the *scaling dimension* of the field ϕ . In a free theory it coincides with the engineering dimension of the field in units of energy (inverse units of length), namely

$$\Delta = \frac{d-2}{2}. \quad (2.59)$$

$\sigma(x)$ is an infinitesimal scale factor that satisfies $\partial_\mu \partial_\nu \sigma = 0$. The advantage of this perspective is that the conformal transformations are simple, multiplicative transformations of the fields. The disadvantage is that it requires thinking of theory in curved space-time. This means that the metric that is implicit in the action (2.56) must be made explicit, but also that the action could be augmented with a term depending on the scalar curvature tensor R as

$$S = \int d^d x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \alpha R \phi^2 \right]. \quad (2.60)$$

Since R vanishes in flat space, it looks like this additional term could appear with an arbitrary coefficient α without modifying the original flat-space action, but this is not the case.

Exercise 2.5 Verify that there is a specific value of α for which this action is invariant under the infinitesimal conformal transformations (2.57) and (2.58), after the explicit dependence on the curved-space metric has been taken into account in the contraction of Lorentz indices and in the measure of integration. What is this value of α ?

For this reason, it is also convenient to consider the opposite perspective in which conformal transformations are transformations of the dynamical fields and of the coordinates, but not of the metric. In this case the conformal transformation can be defined as

$$\phi(x) \rightarrow e^{\Delta\sigma(x)}\phi(x + \varepsilon), \quad (2.61)$$

where the parameters σ and ε are related by the conformal Killing equation (2.14), i.e. $\sigma = \frac{1}{d}\partial_\mu\varepsilon^\mu$. In infinitesimal form, this transformation becomes

$$\phi(x) \rightarrow \left[1 + \frac{d-2}{2d}(\partial_\nu\varepsilon^\nu) + \varepsilon^\nu\partial_\nu\right]\phi(x). \quad (2.62)$$

Exercise 2.6 Show that under the transformation (2.62), the free scalar field Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi,$$

is shifted by a total derivative term

$$\delta\mathcal{L} = \partial_\mu\left(-\frac{1}{2}\varepsilon^\mu\partial_\nu\phi\partial^\nu\phi - \frac{d-2}{2d}\partial_\nu\varepsilon^\nu\phi\partial^\mu\phi\right),$$

hence proving that this is a symmetry of the action. You will need to use the fact that ε is at most quadratic in x .

Note that Poincaré symmetry is a special case of this transformation, corresponding to constant ε (and thus $\sigma = 0$).

By Noether's theorem, whenever an action is invariant under some transformation of the field

$$\phi \rightarrow \phi + \delta_\varepsilon\phi, \quad (2.63)$$

i.e. whenever the Lagrangian varies by a total derivative term,

$$\mathcal{L} \rightarrow \mathcal{L} + \partial_\mu\Lambda_\varepsilon^\mu, \quad (2.64)$$

then there exists a conserved current

$$J_\varepsilon^\mu = \Lambda_\varepsilon^\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\varepsilon \phi. \quad (2.65)$$

In our example, this conserved current is therefore

$$J_\varepsilon^\mu = \varepsilon_\nu \left(\partial^\mu \phi \partial^\nu \phi - \frac{1}{2} g^{\mu\nu} \partial_\rho \phi \partial^\rho \phi \right) \equiv \varepsilon_\nu T_c^{\mu\nu}. \quad (2.66)$$

The 2-index tensor on the right-hand side is called the *canonical energy-momentum tensor*. Its divergence satisfies

$$\partial_\nu T_c^{\mu\nu} = \partial^\mu \phi \partial^2 \phi, \quad (2.67)$$

therefore vanishing by the equation of motion for the free field, $\partial^2 \phi = 0$. This implies in turn that the Noether current (2.66) is conserved *for constant* ε . If on the contrary ε depends on space(-time), then we have

$$\partial_\mu J_\varepsilon^\mu = (\partial_\nu \varepsilon_\mu) T_c^{\mu\nu}. \quad (2.68)$$

In our example the canonical energy-momentum tensor is symmetric in its indices μ and ν , and therefore we can write

$$\partial_\mu J_\varepsilon^\mu = \frac{1}{2} (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) T_c^{\mu\nu} = \sigma g_{\mu\nu} T_c^{\mu\nu} = -\frac{d-2}{2} \sigma \partial_\mu \phi \partial^\mu \phi. \quad (2.69)$$

In dimensions $d > 2$, this current is only conserved when $\sigma = 0$. This is surprising, because we just showed that scale and special conformal transformations are also symmetries of the action, so why is the Noether current not conserved?

The reason is that the version of Noether's theorem given above does not straightforwardly apply to the case of a space-time dependent parameter ε . In fact, the energy-momentum tensor that we computed in this example is not unique: one can always add to it a piece proportional to

$$(\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi^2, \quad (2.70)$$

without affecting the conservation equation (2.67), but changing the value of its trace. The combination

$$T^{\mu\nu} = T_c^{\mu\nu} + \frac{d-2}{2(d-1)} (\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi^2 \quad (2.71)$$

is for instance traceless in any d . It turns out that it is always possible in a field theory with conformal symmetry to construct an energy-momentum tensor that is:

- symmetric ($T^{\mu\nu} = T^{\nu\mu}$),
- traceless ($g_{\mu\nu}T^{\mu\nu} = 0$), and
- conserved ($\partial_\nu T^{\mu\nu} = 0$) once the equation of motions are imposed.

This is a non-trivial fact, but we will skip its proof (as already mentioned, we are interested in theories that are not necessarily defined through an action).

Strictly speaking, Noether's theorem only applies to theories that have a Lagrangian description, but we will assume the existence of a traceless energy-momentum tensor in all cases (in some sense this is going to be one of the “axioms” of conformal field theory). From this assumption, we can deduce that the theory is invariant under conformal transformations: a conserved current can always be built from the energy-momentum tensor and a conformal Killing vector ε as

$$J^\mu = \varepsilon_\nu T^{\mu\nu} \quad \Rightarrow \quad \partial_\mu J^\mu = \varepsilon_\nu \partial_\mu T^{\mu\nu}. \quad (2.72)$$

As always, conserved charges can be constructed as the integral of the time component of a conserved current over space. The simplest example is

$$P^\mu = -i \int d^{d-1}\vec{x} T^{0\mu}(x). \quad (2.73)$$

This is a priori a function of x^0 , but it is in fact constant in time, since

$$\partial_0 P^\mu = -i \int d^{d-1}\vec{x} \partial_0 T^{0\mu}(x) = i \int d^{d-1}\vec{x} \partial_i T^{i\mu}(x) = 0. \quad (2.74)$$

This conserved charge is the *momentum*, associated with translation symmetry. Similarly, there are conserved charges associated with Lorentz transformations,

$$M^{\mu\nu} = -i \int d^{d-1}\vec{x} [x^\mu T^{0\nu}(x) - x^\nu T^{0\mu}(x)] \quad (2.75)$$

with scale transformations,

$$D = -i \int d^{d-1}\vec{x} x_\mu T^{0\mu}(x), \quad (2.76)$$

and with special conformal transformations

$$K^\mu = -i \int d^{d-1}\vec{x} [2x^\mu x_\nu T^{0\nu}(x) - x^2 T^{0\mu}(x)]. \quad (2.77)$$

Exercise 2.7 Show that the charges (2.75), (2.76) and (2.77) are conserved in time.

Note that the conservation of all these charges relies on the vanishing divergence of the energy-momentum tensor, $\partial_\nu T^{\mu\nu}$, which itself relies on the equation of motion being satisfied. This is certainly true in the absence of sources. But when source terms are added to the action, the equation of motion is modified. In our example of the free scalar field theory, adding to the action (2.56) a source term of the form

$$S_{\text{source}} = \int d^d x J(x) \phi(x), \quad (2.78)$$

modifies the equation of motion to

$$\partial^2 \phi + J = 0. \quad (2.79)$$

Instead of the conservation equation (2.67) for the canonical energy-momentum tensor, we must replace it with

$$\partial_\nu T^{\mu\nu}(x) = -J \partial^\mu \phi. \quad (2.80)$$

In the presence of such a source, the charges (2.73)–(2.77) are not conserved anymore. However, if the source is local, say $J(x) = \delta^d(x - x_\odot)$, so that

$$\partial_\nu T^{\mu\nu}(x) = -\delta^d(x - x_\odot) \partial^\mu \phi(x_\odot), \quad (2.81)$$

then we can determine the change in one of the charge — say P^μ — between a time $x^0 < x_\odot^0$ anterior to the local source, and a time $x^0 > x_\odot^0$ posterior to it, and call this difference the momentum of the source, $P_\mu(\phi_\odot)$. By our definition, this is equal to

$$P_\mu(\phi_\odot) = -i \int_{\partial\Sigma} d^{d-1} n_\nu T^{\mu\nu}, \quad (2.82)$$

where $\partial\Sigma$ is a codimension-one surface in $\mathbb{R}^{1,d-1}$ that consists in two planes of constant x^0 , joining at spatial infinity, and surrounding the point x_\odot . By the divergence theorem, this is equal to

$$P_\mu(\phi_\odot) = -i \int_\Sigma d^d x \partial_\nu T^{\mu\nu}(x) = i \int_\Sigma d^d x \delta^d(x - x_\odot) \partial^\mu \phi(x) = i \partial^\mu \phi(x_\odot). \quad (2.83)$$

This equation says that the charge P_μ associated with a local source for the field ϕ is equal to $i\partial^\mu\phi$. This is strikingly similar to the action of the generator (2.30) on functions. In fact, it is easy to verify that the other charges (2.75), (2.76), and (2.77) also act on the classical field $\phi(x)$ exactly like the generators (2.31), (2.32), and (2.33) respectively, and therefore that the charges obey the conformal algebra.

We worked with the free scalar field theory as example, but our discussion can be generalized to arbitrary classical field theories. This shows that a traceless energy-momentum tensor can be used to give a field-theoretical realization of the conformal generators discussed before.

3 Conformal quantum field theory

Let us now turn to quantum field theory and study the implications of conformal symmetry in that context. The most standard approach to quantum field theory is to think of a classical field theory, in which we now have a basic understanding of what conformal symmetry does, and then quantize it by promoting the fields to operators acting on some Hilbert space. But this is not the approach that we will take here. There will still be states and operators, but the latter will not necessarily be associated with fields appearing in a Lagrangian.

3.1 Non-perturbative quantum field theory

To define a quantum field theory non-perturbatively, we need the following ingredients:

1. The Minkowski space-time is foliated into surfaces of equal-time, and to each time slice we associate a Hilbert space of quantum states. This Hilbert space includes a vacuum state $|0\rangle$.
2. There are a number of (in fact infinitely many) *local* operators that act on this Hilbert space. For instance, let us take $\phi(x^0, \vec{x})$ to be an operator acting on the Hilbert space at time $t = x^0$. We call this operator *local* in \vec{x} and we require that it commutes with any other local operators on the same time slice inserted at a point $\vec{x}' \neq \vec{x}$:

$$[\phi(x^0, \vec{x}), \phi(x^0, \vec{x}')] = 0, \quad \vec{x} \neq \vec{x}'. \quad (3.1)$$

3. One of the local operators of the theory is the energy-momentum tensor, and from it we can define conserved charges, including P^μ and $M^{\mu\nu}$ as in

eqs. (2.73) and (2.75). In a generic QFT the energy-momentum tensor needs not be traceless, so the charges D and K^μ cannot be considered. P^μ and $M^{\mu\nu}$ are conserved in time, so they are good operator valid on all Hilbert spaces at any time t . In particular, we shall assume that the vacuum state has zero charge,

$$P^\mu |0\rangle = M^{\mu\nu} |0\rangle = 0. \quad (3.2)$$

4. The value of P^μ and $M^{\mu\nu}$ changes however every time an operator is inserted at some point (x^0, \vec{x}) . In analogy with eq. (2.83), we require that this change is encoded in the commutator

$$[P^\mu, \phi(x)] = i\partial^\mu \phi(x). \quad (3.3)$$

Note that while P^μ is not a local operator, its commutator with any local operator is again local. Since this equation is solved by

$$\phi(x) = e^{-ix \cdot P} \phi(0) e^{ix \cdot P}, \quad (3.4)$$

we see that translations are realized as *unitarity* transformations of the operators (note that P^μ is Hermitian). This is true of time translations as well, and therefore the Hamiltonian that defines the time-evolution of states can be identified with P^0 : a state $|\Psi\rangle$ defined at time $t = 0$ is translated to another state at time t by

$$|\Psi\rangle_t = e^{itP^0} |\Psi\rangle. \quad (3.5)$$

Since this evolution is unitary, the Hilbert space will be the same on all surfaces.

5. Lorentz transformations are similarly realized as unitary transformations, generated by the charge $M^{\mu\nu}$. We choose to decompose local operators inserted at the origin of space-time into irreducible representations of the Lorentz group and denote these with $\phi^a(0)$, with a standing for a collection of Lorentz indices, so that

$$[M^{\mu\nu}, \phi_a(0)] = i(\mathcal{S}^{\mu\nu})_{ab} \phi^b(0), \quad (3.6)$$

where $(\mathcal{S}^{\mu\nu})_{ab}$ is a matrix that satisfies the Lorentz algebra. It vanishes for a scalar operator. For a vector operator with one Lorentz index, it is given by

$$(\mathcal{S}^{\mu\nu})_{\alpha\beta} = \delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu. \quad (3.7)$$

When combined with eq. (3.4), this implies

$$[M^{\mu\nu}, \phi^a(x)] = -i(x^\mu \partial^\nu - x^\nu \partial^\mu) \phi^a(x) + i(\mathcal{S}^{\mu\nu})^a_b \phi^b(0). \quad (3.8)$$

The fact that the quantum theory is Poincaré invariant implies that the locality condition (3.1) can be rewritten as

$$[\phi(x), \phi(y)] = 0 \quad \text{if } (x - y)^2 > 0. \quad (3.9)$$

In other words, local operators commute as long as they are space-like separated.⁸

6. Finally, the last condition has to do with the positivity of energy. Working in a basis of eigenstates of energy and momentum,

$$P^\mu |p\rangle = p^\mu |p\rangle, \quad (3.10)$$

we require $p^0 \geq 0$ for *all* states in the Hilbert space. Since this must be true in any Lorentz frame, p must be contained inside the forward light-cone, i.e.

$$p^0 \geq |\vec{p}| \quad \Leftrightarrow \quad p^2 < 0 \text{ and } p^0 > 0. \quad (3.11)$$

Note that eigenstates of P^μ can be constructed taking the Fourier transform of a local operator acting on the vacuum,

$$\tilde{\phi}(p) |0\rangle = \int d^d x e^{ip \cdot x} \phi(x) |0\rangle. \quad (3.12)$$

Exercise 3.1 Using eq. (3.4), show that

$$P^\mu \tilde{\phi}(p) |0\rangle = p^\mu \tilde{\phi}(p) |0\rangle$$

Note that the operators P^μ and $M^{\mu\nu}$ are in fact topological charges (they can be defined on any codimension-1 surfaces, not only time slices), so the commutation relations (3.3) and (3.8) can be motivated in *any* quantization of the theory. However, it is obvious from point 6 that the generator P^0 plays a special role in quantum field theory in Minkowski space-time: not all of the conformal generators are treated on the same footing.

⁸For operators of half-integer spins, the commutator must be replaced by an anti-commutator.

The six items essentially give the most general non-perturbative definition of quantum field theory. They are nearly in one-to-one correspondence with the so-called *Wightman axioms*. The only difference (besides the level of mathematical rigor) is that the Wightman axioms do not rely on the existence of an energy-momentum tensor, but assume directly that the Poincaré transformations are realized as unitary transformations on the Hilbert space.

3.2 Wightman functions

With these axioms in mind, the next thing we can do is compute the vacuum expectation value of products of local operators,

$$\langle 0 | \phi_1(x_1) \cdots \phi_n(x_n) | 0 \rangle. \quad (3.13)$$

This can be viewed as an overlap of the vacuum state, $\langle 0 |$, with a state created acting on the vacuum with a sequence of local operators. Note that these operators need not be time-ordered: the translation operator is unitary, so it can go both ways. This is very different from time-ordered correlation functions obtained from the path integral.

Correlators of this type are called *Wightman functions*. They are really the fundamental observables in non-perturbative quantum field theory. In fact, it is even possible to completely define a quantum field theory just by its Wightman functions: the Wightman reconstruction theorem states that the Hilbert space of a quantum field theory can be constructed from all its Wightman function. A convenient perspective is therefore to forget about the Hilbert space and focus on correlation functions.

Symmetry is encoded in this correlation functions through Ward identities: given a conserved charge G that annihilates the vacuum, $G|0\rangle = 0$ (this could be P^μ or $M^{\mu\nu}$), the following equation must be satisfied

$$\begin{aligned} & \langle 0 | [G, \phi_1(x_1)] \phi_2(x_2) \cdots \phi_n(x_n) | 0 \rangle \\ & + \langle 0 | \phi_1(x_1) [G, \phi_2(x_2)] \cdots \phi_n(x_n) | 0 \rangle \\ & + \dots \\ & + \langle 0 | \phi_1(x_1) \phi_2(x_2) \cdots [G, \phi_n(x_n)] | 0 \rangle = 0. \end{aligned} \quad (3.14)$$

This is an equation that is obvious in the Hilbert space picture, but it is also valid as a differential equation for the Wightman function, since each commutator is again related to the local operator. Let us see some examples.

The simplest Wightman function is that involving a single scalar local operator,

$$\langle 0 | \phi(x) | 0 \rangle. \quad (3.15)$$

In this case the Ward identity associated with translations implies

$$\frac{\partial}{\partial x^\mu} \langle 0 | \phi(x) | 0 \rangle = 0, \quad (3.16)$$

or in other words that the vacuum expectation value of the operator is a constant over all of space-time. Lorentz symmetry does not give more information about that constant, but it forbids a vacuum expectation value for any operator transforming non-trivially under the Lorentz group.

Let us consider next a Wightman 2-point function of identical scalar operators,

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle. \quad (3.17)$$

In this case, translation symmetry tells us that

$$\left(\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial y^\mu} \right) \langle 0 | \phi(x) \phi(y) | 0 \rangle. \quad (3.18)$$

If we think of the correlator as being a function of $x + y$ and $x - y$, then this Ward identities establishes that there is no dependence on the former, i.e.

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = W(y - x), \quad (3.19)$$

where W denotes so far an arbitrary function.

In general, the consequences of translation symmetry are easier to see in momentum space, using the Fourier transform of the local operators, analogous to eq. (3.12):

$$\tilde{\phi}(p) = \int d^d x e^{ip \cdot x} \phi(x). \quad (3.20)$$

By this definition, we can establish that

$$[P^\mu, \tilde{\phi}(p)] = p^\mu \tilde{\phi}(p), \quad (3.21)$$

and therefore the Fourier transform of the Wightman 2-point function obeys

$$(p^\mu + q^\mu) \langle 0 | \tilde{\phi}(p) \tilde{\phi}(q) | 0 \rangle. \quad (3.22)$$

From this, we conclude that

$$\langle 0 | \tilde{\phi}(p) \tilde{\phi}(q) | 0 \rangle = (2\pi)^d \delta^d(p + q) \widetilde{W}(q). \quad (3.23)$$

\widetilde{W} is another unknown function. As the notation suggests, it is actually the Fourier transform of W ,

$$\widetilde{W}(q) = \int d^d x e^{ip \cdot x} W(x). \quad (3.24)$$

Exercise 3.2 Verify that the Fourier transform of eq. (3.19) with respect to both x and y reproduces eq. (3.23).

Taking into account Lorentz symmetry, one can also establish that the Wightman function $W(x)$ can only depend on the Lorentz-invariant distance x^2 , although there is a subtlety: this can be a different function depending whether x is space-like or time-like (future- or past-directed), as Lorentz transformations act separately on each of these regions. In momentum space, the same arguments says that $\widetilde{W}(q)$ must be a function of q^2 . In this case, the condition that only states of positive energy can exist requires that $\widetilde{W}(q)$ vanishes unless $q^2 < 0$ with $q^0 > 0$, and therefore we can unambiguously write

$$\widetilde{W}(q) = 2\pi \theta(q^0 - |\vec{q}|) \rho(-q^2), \quad (3.25)$$

where ρ is a function of the positive quantity $-q^2$, and θ is the Heaviside step function ($\theta(a) = 0$ for $a < 0$, $\theta(a) = 1$ for $a > 0$).

The fact that the Wightman 2-point function (3.23) is proportional to a delta function raises an important concern: in spite of their names, Wightman functions are *not* functions but rather distributions (this is also part of the Wightman axioms: they are in fact *tempered distributions*). Note also that the same function computes the overlap between the two states

$$\widetilde{\phi}(q) |0\rangle \quad \text{and} \quad \widetilde{\phi}(-p) |0\rangle \quad (3.26)$$

(Hermitian conjugation flips the sign of momenta). Therefore, the limit $p \rightarrow -q$ corresponds to the norm of either of these states. But this limit is clearly discontinuous, or the norm of the state infinite. The resolution of this issue is that the objects $\phi(x)$ and its Fourier transform $\widetilde{\phi}(p)$ are not operators, but rather *operator-valued distributions*. In other words, $\phi(x) |0\rangle$ and $\widetilde{\phi}(p) |0\rangle$ are *not* states of the theory, as they have in fact infinite norm. Formally, these operator-valued distributions only make sense when they are integrated against test functions, defining

$$\phi[f] = \int d^d x f(x) \phi(x), \quad (3.27)$$

or

$$\widetilde{\phi}[\tilde{f}] = \int d^d p \tilde{f}(p) \widetilde{\phi}(p), \quad (3.28)$$

where f and \tilde{f} are Schwartz-class test functions (smooth functions decaying faster than any power at infinity). When acting on the vacuum, these

smearred operator give well-defined states, with finite norms. For instance, we have

$$\begin{aligned}\left\|\widetilde{\phi}[f] |0\rangle\right\|^2 &= \int d^d p d^d q \tilde{f}^*(p) \tilde{f}(q) \langle 0 | \widetilde{\phi}(-p) \widetilde{\phi}(q) | 0 \rangle \\ &= 2\pi \int_{q^0 > |\vec{q}|} d^d q \left| \tilde{f}(q) \right|^2 \rho(-q^2).\end{aligned}\quad (3.29)$$

Test functions will not appear further in these lectures. For physicists, they are mostly an annoyance that we prefer to avoid. However, it is important to know that there exists a mathematically rigorous way of dealing with Wightman functions. For one thing, this gives a proper justification of why it is always fine to take the Fourier transform way and back between the position- and momentum-space representation, as tempered distribution always admit a Fourier transform. This is however only true of Wightman functions, not of time-ordered correlators (see below).

3.3 Spectral representation

The norm (3.29) is also giving away important information: it can only be positive if the function ρ is positive,

$$\rho(\mu^2) \geq 0, \quad \forall \mu^2 > 0. \quad (3.30)$$

The function ρ is in fact the *spectral density* encountered in standard quantum field theory textbooks, where we can often find it in the form

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = 2\pi \int \frac{d^d k}{(2\pi)^d} \int_0^\infty d\mu^2 e^{ik \cdot (x-y)} \theta(k^0) \delta(k^2 + \mu^2) \rho(\mu^2). \quad (3.31)$$

The spectral density is an essential tool in non-perturbative quantum field theory. It can for instance be used to construct the time-ordered correlation function

$$\langle \phi(x) \phi(y) \rangle_T \equiv \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle. \quad (3.32)$$

Unlike the Wightman function, this is not a tempered distribution, because the θ functions are not differentiable at the origin. Nevertheless, the time-ordered product admits the simple representation

$$\langle \phi(x) \phi(y) \rangle_T = \int \frac{d^d k}{(2\pi)^d} \int_0^\infty d\mu^2 e^{ik \cdot (x-y)} \frac{i}{-k^2 - \mu^2 + i\varepsilon} \rho(\mu^2) \quad (3.33)$$

where the limit $\varepsilon \rightarrow 0_+$ is understood. This is the *Källén-Lehmann representation* for the time-ordered 2-point function.

Exercise 3.3 Derive the Källén-Lehmann representation. An elegant derivation is to first show that the time-ordered 2-point function can be written as the difference between the Wightman function and the vacuum expectation value of a retarded commutator,

$$\langle \phi(x)\phi(y) \rangle_T = W(y-x) - \theta(y^0 - x^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle.$$

The next step is to Fourier transform both terms in y after setting $x = 0$. We know that the Wightman function (3.23) has a nice Fourier transform

$$\widetilde{W}(q) = 2\pi \int_0^\infty d\mu^2 \theta(q^0) \delta(q^2 + \mu^2) \rho(\mu^2),$$

which can be equivalently written

$$\widetilde{W}(q) = \int_0^\infty d\mu^2 \theta(q^0) \left[\frac{i}{q^2 + \mu^2 + i\varepsilon} - \frac{i}{q^2 + \mu^2 - i\varepsilon} \right] \rho(\mu^2).$$

The retarded commutator is only non-zero in the forward light cone in y , and therefore it also admits a Fourier transform that converges provided that we give a small imaginary part to q . Compute this Fourier transform, and show that it is equal to

$$\int_0^\infty d\mu^2 \left[\theta(q^0) \frac{i}{q^2 + \mu^2 + i\varepsilon} + \theta(-q^0) \frac{i}{q^2 + \mu^2 - i\varepsilon} \right] \rho(\mu^2).$$

The difference of these last two integrals can then easily be turned into the Källén-Lehmann representation (3.33).

The spectral representation for the 2-point function gives familiar results in non-interacting theories. A massive scalar field has for instance the spectral density

$$\rho(\mu^2) = \delta(\mu^2 - m^2), \quad (3.34)$$

and from this we recover the known massive propagator

$$\langle \phi(x)\phi(y) \rangle_T = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot (x-y)} \frac{i}{-k^2 - m^2 + i\varepsilon}. \quad (3.35)$$

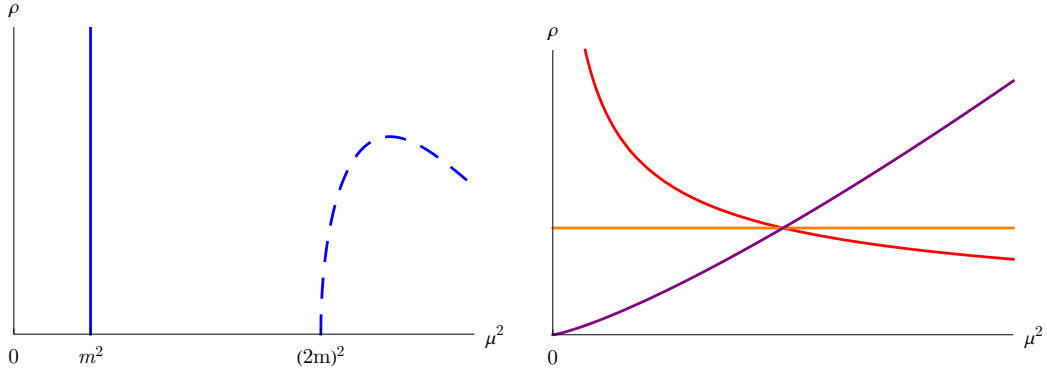


Figure 3: Left: spectral density in the theory of a free massive field, and typical contributions above the threshold for particle production (dashed line). Right: possible spectral densities in a scale-invariant theory.

In an interacting theory, the spectral density will generically have a non-zero contribution above a certain threshold corresponding to particle production (see figure 3).

The discussion applied so far to a generic quantum field theory without conformal symmetry. Let us now examine the role of scale and special conformal invariance, starting with the former.

3.4 Scale symmetry

The assumption of scale symmetry coincides with the existence of a third conserved charge besides P^μ and $M^{\mu\nu}$, namely the operator D . We obtained before the commutator of a generic local operator with $M^{\mu\nu}$ assuming that it transforms in some irreducible representation of the Lorentz group at the origin $x = 0$ (an operator inserted at some other point is not in an irreducible representation because P^μ does not commute with $M^{\mu\nu}$). Since $M^{\mu\nu}$ commutes with D , the same assumption can be made about scale: any local operator can be decomposed into irreducible representations of the group of scaling transformations, meaning that we can write

$$[D, \phi(0)] = -i\Delta\phi(0). \quad (3.36)$$

Δ is called the *scaling dimension* of the operator ϕ (each local operator of the theory has its own scaling dimension). The factor of i ensures that Δ is a real number when ϕ is a real operator. As before, we can use eq. (3.4) to obtain the commutator at any other point x :

$$[D, \phi(x)] = -i(x^\mu \partial_\mu + \Delta)\phi(x). \quad (3.37)$$

Note that this is precisely equivalent to the transformation rule (2.62) for a classical field: the scaling dimension Δ coincides with the mass dimension of the operator ϕ in a free theory.

Using this new commutator and assuming that the vacuum state is invariant under scale transformations, a new Ward identity can be obtained for the Wightman 2-point function,

$$\left(x^\mu \frac{\partial}{\partial x^\mu} + y^\mu \frac{\partial}{\partial y^\mu} + 2\Delta\right) \langle 0 | \phi(x) \phi(y) | 0 \rangle = 0, \quad (3.38)$$

or equivalently

$$\left(x^\mu \frac{\partial}{\partial x^\mu} + 2\Delta\right) W(x) = 0. \quad (3.39)$$

The corresponding condition on the momentum-space 2-point function is obtained from the Fourier transform, using integration by parts, and we find

$$\left(-q^\mu \frac{\partial}{\partial q^\mu} + 2\Delta - d\right) \widetilde{W}(q) = 0. \quad (3.40)$$

The solution to this equation consistent with the form (3.25) is unique, up to a constant multiplicative coefficient C ,

$$\widetilde{W}(q) = 2\pi C \theta(q^0 - |\vec{q}|) (-q^2)^{\Delta-d/2}, \quad (3.41)$$

implying that the spectral density is a power law,

$$\rho(\mu^2) = C (\mu^2)^{\Delta-d/2}. \quad (3.42)$$

This kind of spectral density shown in figure 3 is very different from that of a massive interacting theory: the operator ϕ describes excitations of all energies. At the same time, its simplicity is striking: it is characterized by a single number Δ .

It is instructive to compute the time-ordered 2-point function using the Källén-Lehmann representation: performing the integral over μ^2 , we find

$$\langle \phi(x) \phi(y) \rangle_T = \frac{i\pi C}{\sin\left[\pi\left(\Delta - \frac{d}{2}\right)\right]} \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot (x-y)} (k^2 - i\varepsilon)^{\Delta-d/2}. \quad (3.43)$$

The term $(k^2 - i\varepsilon)^{\Delta-d/2}$ in the integral looks like the propagator of a massless scalar field raised to an arbitrary power. This is in fact what we expect in perturbation theory when the β -function has a non-trivial fixed point: the

renormalized 2-point function has logarithms that can be resummed into a power controlled by the anomalous dimension γ of the field ϕ ,

$$\frac{i}{-q^2} [1 + \gamma \log(-q^2) + \dots] \approx i(-q^2)^{-1+\gamma}. \quad (3.44)$$

In this case, the scaling dimension of the scalar operator corresponding to the renormalized field is

$$\Delta = \frac{d-2}{2} + \gamma. \quad (3.45)$$

In the limit $\gamma \rightarrow 0$, we recover the free propagator mentioned above. Note however that this requires the coefficient C to obey

$$C \approx \gamma = \Delta - \frac{d-2}{2} \quad (3.46)$$

in order to recover the ordinary, finite normalization of the propagator. In this case, the spectral density obeys

$$\rho(\mu^2) \approx \gamma (\mu^2)^{-1+\gamma} \xrightarrow{\gamma \rightarrow 0} \delta(\mu^2). \quad (3.47)$$

(This identity can be verified integrating both sides in μ^2 and taking the limit $\gamma \rightarrow 0$ afterwards). This is precisely the spectral density that is expected in the free scalar field theory. In this case (and this case only!), the operator ϕ describes a massless scalar particle.

It turns out that the case $\Delta = \frac{d-2}{2}$ is the lowest possible value for Δ : for any Δ below that value, the spectral density is not integrable in the limit $\mu^2 \rightarrow 0$. One might also worry about the opposite limit $\mu^2 \rightarrow \infty$ in the integral: for any $\Delta > \frac{d}{2}$, the spectral density increases with μ^2 . However, remember that this spectral density is in fact a Wightman function, i.e. a tempered distribution that should be understood as integrated against test functions that decay faster than any power at large q^2 . Therefore, arbitrarily large values of Δ are possible, but there is a lower bound on Δ below which the states smeared with test functions have infinite norm. The inequality

$$\Delta \geq \frac{d-2}{2}. \quad (3.48)$$

is known in the literature as the *unitarity bound* for scalar operators. Note that any scalar operator that saturates this unitarity bound has

$$\langle 0 | \tilde{\phi}(p) \tilde{\phi}(q) | 0 \rangle \propto \delta(q^2), \quad (3.49)$$

which implies

$$q^2 \langle 0 | \tilde{\phi}(p) \tilde{\phi}(q) | 0 \rangle = 0, \quad (3.50)$$

or in position space

$$\langle 0 | \phi(x) \partial^2 \phi(y) | 0 \rangle. \quad (3.51)$$

Since this is true for any x and y , it implies that

$$\partial^2 \phi(x) = 0 \quad (3.52)$$

is true as an operator equation. Since this is the equation of motion for a free field, a theory in which $\Delta = \frac{d-2}{2}$ is a free field theory.

The simplicity of the momentum-space 2-point function in a scale-invariant theory also means that it can easily be Fourier transformed back to position space using

$$W(x) = \int \frac{d^d q}{(2\pi)^d} e^{-iq \cdot x} \widetilde{W}(q). \quad (3.53)$$

Exercise 3.4 Perform the Fourier transform explicitly. You can use the fact that the integral is Lorentz invariant to determine that $W(x)$ is in fact a function of x^2 . Moreover, since the integrand only has support for q in the forward light-cone, this defines a function of x that is analytic in x as long as $\text{im } x$ is contained in the future light cone: the integrand is damped by the exponential $e^{q \cdot \text{im } x}$ with $q \cdot \text{im } x < 0$ in that case (this domain of analyticity is known as the “future tube”). This means that we are free to evaluate the integral at a point $x = (i\tau, 0)$, and then use $\tau^2 = x^2$ to recover the general solution. The integral at that point is convergent for all Δ satisfying the unitarity bound, and you should find

$$W(\tau) = C \frac{2^{2\Delta} \Gamma(\Delta) \Gamma(\Delta - \frac{d-2}{2})}{(4\pi)^{d/2}} \tau^{-2\Delta}$$

This can be used to obtain the result below.

The result of this integral can be written as

$$W(x) = \frac{C'}{[-(x^0 + i\varepsilon)^2 + \vec{x}^2]^\Delta} \quad (3.54)$$

where the limit $\varepsilon \rightarrow 0_+$ is understood to make sense of the case in which $x^2 \leq 0$, and the coefficient C' is related to C by

$$C' = \frac{2^{2\Delta} \Gamma(\Delta) \Gamma(\Delta - \frac{d-2}{2})}{(4\pi)^{d/2}} C. \quad (3.55)$$

Note that the proportionality factor is positive for all Δ satisfying the unitary bound (3.48). This implies that the 2-point correlation function is always decreasing with the distance, and not the other way around.

It is in fact customary in conformal field theory to normalize the scalar operator ϕ so that $C' = 1$. In this case, we have

$$C = \frac{(4\pi)^{d/2}}{2^{2\Delta}\Gamma(\Delta)\Gamma\left(\Delta - \frac{d-2}{2}\right)}. \quad (3.56)$$

Note that C vanishes as in eq. (3.46) in the limit $\Delta \rightarrow \frac{d-2}{2}$.

Finally, let us conclude the analysis of scale symmetry with a comment about one-point functions. We saw in the previous section that a constant vacuum expectation value for a scalar operator was compatible with Poincaré symmetry. However, the commutator (3.37) requires then that $\Delta = 0$, which violates the unitarity bound. We conclude that all one-point functions must vanish in a scale-invariant theory.

3.5 Special conformal symmetry

As with scale symmetry, the presence of special conformal symmetry is associated with the existence of the conserved charges K^μ , organized in a d -dimensional vector. Unlike D , however, K^μ does not commute with $M^{\mu\nu}$, so it cannot be diagonalized at the point $x = 0$. Nevertheless, we can use the conformal algebra to establish that, if ϕ is a local operator with scaling dimension Δ , then $[K^\mu, \phi]$ has scaling dimension $\Delta - 1$:

$$\begin{aligned} [D, [K^\mu, \phi(0)]] &= [K^\mu, [D, \phi(0)]] + [[D, K^\mu], \phi(0)] \\ &= [K^\mu, -i\Delta\phi(0)] + [iK^\mu, \phi(0)] = -i(\Delta - 1)[K^\mu, \phi(0)]. \end{aligned} \quad (3.57)$$

This is similar to the observation that the commutator $[P^\mu, \phi(x)]$ has scaling dimension $\Delta + 1$,

$$[D, [P^\mu, \phi(0)]] = -i(\Delta + 1)[P^\mu, \phi(0)], \quad (3.58)$$

consistent with the fact that the derivative ∂^μ has mass dimension $+1$. The fact that K^μ lowers the scaling dimensions seems to be in contradiction with our findings of the last section that Δ is bounded below: given any local operator, one can always construct other local operators with arbitrarily smaller scaling dimension.

The only way out of this paradox is to assume that at some point the action of K^μ annihilates the operator. In other words, there exists some local operator such that

$$[K^\mu, \phi(0)] = 0. \quad (3.59)$$

We call this local operator a *primary*. Any other local operator obtained acting on a primary with P^μ is called a *descendant*. Since the action of P^μ coincides with taking derivatives, a primary operator is simply an operator that cannot be written as the derivative of some other operator. Unless specified otherwise, we shall from now on only consider Wightman correlation functions of primary operators. Descendants will be explicitly denoted with a derivative.

The transformation of a primary operator away from the origin can once again be obtained from eq. (3.4). Note that since the commutator of P^μ and K^μ involves both D and $M^{\mu\nu}$, this transformation depends on the scaling dimension and on the Lorentz representation of the operators, i.e. on the eigenvalues Δ and $S^{\mu\nu}$. We find

$$[K^\mu, \phi(x)] = -i (2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu + 2\Delta x^\mu - 2S^{\mu\nu} x_\nu) \phi(x). \quad (3.60)$$

This equation also defines the commutator of a momentum-space operator: using integration by parts in the definition (3.20), one can show that this amounts to replacing $\partial_\mu \rightarrow -iq_\mu$ and $x^\mu \rightarrow -i\partial/\partial q_\mu$, so that

$$[K^\mu, \tilde{\phi}(q)] = \left[2 \frac{\partial^2}{\partial q_\mu \partial q_\nu} q_\nu - \frac{\partial^2}{\partial q_\nu \partial q^\nu} q^\mu - 2\Delta \frac{\partial}{\partial q_\mu} + 2S^{\mu\nu} \frac{\partial}{\partial q^\nu} \right] \tilde{\phi}(q), \quad (3.61)$$

or after permuting the derivatives with q ,

$$[K^\mu, \tilde{\phi}(q)] = \left[2q^\nu \frac{\partial^2}{\partial q_\mu \partial q^\nu} - q^\mu \frac{\partial^2}{\partial q_\nu \partial q^\nu} + 2(d - \Delta) \frac{\partial}{\partial q_\mu} + 2S^{\mu\nu} \frac{\partial}{\partial q^\nu} \right] \tilde{\phi}(q). \quad (3.62)$$

This is now a second-order differential acting on the operator expressed in momentum space. This is the price to pay for using a representation in which the commutator (3.21) takes a simple diagonal form.

The first thing we can do with this commutator is to examine the related Ward identity for the Wightman 2-point function. Remember that this function can be written as

$$W(x) = \langle 0 | \phi(0) \phi(x) | 0 \rangle. \quad (3.63)$$

The commutator acts trivially on the operator inserted at the origin, so that we must have (note that $S^{\mu\nu} = 0$ for scalar operators)

$$(2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu + 2\Delta x^\mu) W(x) = 0. \quad (3.64)$$

Using $W(x) = 1/(x^2)^\Delta$ at space-like x , we have $\partial_\mu W(x) = -2\Delta W(x) x^\mu / x^2$, and therefore the differential equation is readily satisfied. The same can be verified in momentum space: by definition, we have

$$\widetilde{W}(q) = \langle 0 | \phi(0) \tilde{\phi}(q) | 0 \rangle, \quad (3.65)$$

where only the operator on the right is Fourier transformed while the one on the left is kept fixed at the origin in position space, and so the commutator above implies that

$$\left[2q^\nu \frac{\partial^2}{\partial q_\mu \partial q^\nu} - q^\mu \frac{\partial^2}{\partial q_\nu \partial q^\mu} + 2(d - \Delta) \frac{\partial}{\partial q_\mu} \right] \widetilde{W}(q) = 0. \quad (3.66)$$

With $\widetilde{W}(q) = (-q^2)^{\Delta-d/2}$, this equation is again satisfied.

Distinct operators The fact that $W(x)$ and $\widetilde{W}(q)$ readily satisfy the constraint imposed by special conformal symmetry is very specific to identical scalar operators. In every other case, special conformal symmetry adds more constraints than Poincaré and scale symmetry alone. The simplest example is that of a 2-point function of distinct scalar operators,

$$\langle 0 | \phi_1(x) \phi_2(y) | 0 \rangle. \quad (3.67)$$

This is still a function of $(x - y)^2$ by Poincaré symmetry. But there are now two distinct scaling dimensions Δ_1 and Δ_2 corresponding to the operators ϕ_1 and ϕ_2 , and the Ward identity for scale symmetry becomes (for simplicity setting ϕ_1 at the origin)

$$\left(x^\mu \frac{\partial}{\partial x^\mu} + \Delta_1 + \Delta_2 \right) \langle 0 | \phi_1(0) \phi_2(x) | 0 \rangle = 0. \quad (3.68)$$

The solution is fixed up to a multiplicative constant to be (assuming x space-like for simplicity)

$$\langle 0 | \phi_1(0) \phi_2(x) | 0 \rangle = \frac{C_{12}}{(x^2)^{(\Delta_1 + \Delta_2)/2}}. \quad (3.69)$$

The Ward identity for special conformal transformation is obtained from the commutator (3.60), giving

$$(2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu + 2\Delta_2 x^\mu) \langle 0 | \phi_1(0) \phi_2(x) | 0 \rangle = 0. \quad (3.70)$$

Using $\partial_\mu [(x^2)^{-(\Delta_1 + \Delta_2)/2}] = -(\Delta_1 + \Delta_2) x^\mu / x^2$, this implies

$$(\Delta_2 - \Delta_1) C_{12} \frac{x^\mu}{(x^2)^{(\Delta_1 + \Delta_2)/2}} = 0. \quad (3.71)$$

If the scaling dimensions are different ($\Delta_1 \neq \Delta_2$), then C_{12} must vanish. In conformal field theory, only primary operators of identical scaling dimensions can have non-zero 2-point functions.

In fact, if there are several scalar operators with the same scaling dimension ϕ_i with $i = 1, \dots, N$, then

$$\langle 0 | \phi_i(0) \phi_j(x) | 0 \rangle = \frac{C_{ij}}{(x^2)^\Delta}, \quad (3.72)$$

where C_{ij} is a symmetric $N \times N$ matrix. By unitarity, this matrix must be positive-definite: if this were not the case, then one could define a negative-norm state by taking an appropriate linear combination of the ϕ_i and smearing. Therefore, it is always possible to choose a basis of operators in which C_{ij} is diagonal. Moreover, the operators can be normalized so that $C_{ij} = \delta_{ij}$. From now on, we will therefore always assume that the only non-zero 2-point functions are those involving identical operators.

Operators with spin The other situation in which special conformal symmetry plays an essential role is when the operators carry spin. Let us take the simplest example of a vector operator $A^\mu(x)$, and denote its 2-point function by

$$W^{\mu\nu}(x) = \langle 0 | A^\mu(0) A^\nu(x) | 0 \rangle. \quad (3.73)$$

As with scalars, one can also take the Fourier transform of this tempered distribution, defining

$$\widetilde{W}^{\mu\nu}(p) = \int d^d x e^{ip \cdot x} \langle 0 | A^\mu(0) A^\nu(x) | 0 \rangle = \langle 0 | A^\mu(0) \widetilde{A}^\nu(p) | 0 \rangle. \quad (3.74)$$

Again, this function corresponds to the momentum-space correlation function without the delta-function imposing momentum conservation, namely

$$\langle 0 | \widetilde{A}^\mu(p) \widetilde{A}^\nu(q) | 0 \rangle = (2\pi)^d \delta^d(p + q) \widetilde{W}^{\mu\nu}(q). \quad (3.75)$$

By Lorentz symmetry, this function of a single momentum can be decomposed into two different tensor structures multiplying scalar functions,

$$\widetilde{W}^{\mu\nu}(p) = (p^\mu p^\nu - p^2 g^{\mu\nu}) \widetilde{W}_1(p) + p^\mu p^\nu \widetilde{W}_0(p). \quad (3.76)$$

Moreover, using scale symmetry and the positive-energy condition, we can infer that the functions $\widetilde{W}_{1,0}$ are just powers of p^2 over the forward light cone,

$$\widetilde{W}_{1,0}(p) = \theta(p^0 - |\vec{p}|) (-p^2)^{\Delta-d/2-1} C_{1,0}, \quad (3.77)$$

where Δ is the scaling dimensions of the operator A^μ and C_1, C_0 two constants that cannot be related by scale and Poincaré symmetry only.

There is a good reason for using precisely the two tensor structures in eq. (3.76) and not, say, $g^{\mu\nu}$. Thanks to the positive-energy condition, it is always possible to choose a Lorentz frame in which $\vec{p} = 0$.⁹ In this frame, the momentum is invariant under the group $\text{SO}(d-1)$ of spatial rotations, and therefore the 2-point function can be decomposed into irreducible representations of that group: The part proportional to W_0 only appears in the component \widetilde{W}^{00} , and it transforms like a scalar under rotations. Conversely, the part proportional to W_1 only has non-zero entries for spatial Lorentz indices; it is in fact proportional to the identity in the $d-1$ subspace, i.e. it is the invariant tensor for the vector representation of $\text{SO}(d-1)$. In particle physics language, we would call these two parts respectively *longitudinal* and *transverse*.

Being able to use irreducible representations of $\text{SO}(d-1)$ is an advantage of working in momentum space: there is no obvious Lorentz frame in which such a decomposition can be made in position space, since the 2-point function has support over all of Minkowski space-time. The disadvantage of working in momentum space is that the Ward identity for special conformal transformation is a second-order differential equation in p , while it is a first-order differential in position space. This Ward identity can nevertheless be straightforwardly applied to eq. (3.76), and it yields a relation between the longitudinal and transverse parts, i.e. between the coefficients C_0 and C_1 , given by (see exercise)

$$C_0 = \frac{\Delta - d + 1}{\Delta - 1} C_1. \quad (3.78)$$

This is a very important consequences of special conformal symmetry: while in a scale-invariant theory the longitudinal and transverse polarizations are independent, in a conformal theory they are related.

Exercise 3.5 Using the definition (3.74) for the function $\widetilde{W}^{\mu\nu}(p)$, show that it satisfies a special conformal Ward identity

$$\left(2p^\beta \frac{\partial^2}{\partial p_\alpha \partial p^\beta} - p^\alpha \frac{\partial^2}{\partial p_\beta \partial p^\beta} + 2(d - \Delta) \frac{\partial}{\partial p_\alpha} + 2S^{\alpha\beta} \frac{\partial}{\partial p_\beta} \right) \widetilde{W}^{\mu\nu}(p) = 0,$$

where $S^{\alpha\beta}$ is given by eq. (3.7), and use this to prove the relation (3.78).

This has consequences on the possible values that Δ can take. As before, this 2-point function computes the norm of a state, and its positivity requires:

⁹This is similar to going to a massive particle's rest frame in Wigner's group-theoretical approach to quantum field theory.

- $\Delta > \frac{d}{2}$ so that the 2-point function is integrable at $p^2 \rightarrow 0$;
- C_0 and C_1 are both positive so that the norm is positive for any choice of external polarization vector (i.e. the tensor $\widetilde{W}^{\mu\nu}$ is positive-definite). This requires that $\Delta - 1$ and $\Delta - d + 1$ have the same sign.

The combination of these two conditions in $d > 2$ dimensions (spin is treated differently in $d = 2$) implies

$$\Delta \geq d - 1. \quad (3.79)$$

This is known as the unitarity bound for a vector operator.

As in the scalar case, something special happens when the unitarity bound is saturated ($\Delta = d - 1$). In this case the 2-point function has no longitudinal component, $C_0 = 0$, and $\widetilde{W}^{\mu\nu}$ vanishes when contracted with p_μ or p_ν . This implies that the longitudinal part of the state is null,

$$p_\mu \widetilde{A}^\mu(p) |0\rangle = 0, \quad (3.80)$$

or equivalently that $\widetilde{A}^\mu(p) |0\rangle$ is a state that only describes transverse excitations. The equivalent statement in position-space is

$$\partial_\mu A^\mu(x) |0\rangle = 0, \quad (3.81)$$

in other words that A^μ is a conserved current. The equivalence goes both way: any vector operator with $\Delta = d - 1$ is a conserved current, and any conserved current must have scaling dimension $\Delta = d - 1$. This also shows that conserved currents are primary operators: they cannot be written as ∂^2 acting on another current (that current would have $\Delta = d - 3$, below the unitarity bound), nor as $\partial^\mu \phi$ where ϕ is a scalar operator, because the conservation requirement implies $\partial^2 \phi = 0$, which is only possible if ϕ has scaling dimensions $(d - 2)/2$, and thus the current $\Delta = d/2$ (again below the unitarity bound).

The fact that 2-point functions of primary operators are completely fixed by conformal symmetry up to a coefficient that corresponds to a choice of normalization is not specific to scalar and vector operators. In fact, any local operator specified by a representation under the Lorentz group and a scaling dimension defines an irreducible representation of the conformal group $SO(d, 2)$, and as such its 2-point function is fixed by group theory (up to a conventional normalization). This also explains on more general grounds why 2-point functions of distinct operators vanish. The construction of all unitary representations of the conformal group in $d = 4$ dimensions was performed by Mack in 1975 [14], and similar constructions can be done in other dimensions. Some Lorentz representations are specific to a given dimension d , and others

exist in any d , like the scalar and all symmetric traceless representations with spin ℓ , of which the vector discussed above is a special case corresponding to $\ell = 1$. All such symmetric tensors satisfy the unitarity bound

$$\Delta \geq d - 2 + \ell, \quad (3.82)$$

and they are in general described by $\ell + 1$ distinct polarizations (irreducible representations of rotation group), except when the bound is saturated, where there is just a single, transverse polarization: in this case the operators are higher-spin conserved currents. In terms of representations of the conformal group, generic operators are said to belong to *long multiplets*, whereas special cases such as scalars with $\Delta = (d - 2)/2$ or symmetric tensors with $\Delta = d - 2 + \ell$ are said to be in *short multiplets* (they contain fewer descendants).

Exercise 3.6 Construct explicitly the 2-point function of a 2-index symmetric traceless operator $B^{\mu\nu}(x)$. As a starting point, let us decompose the momentum-space correlation functions into tensors that transform covariantly under rotations in the rest frame. Using the transverse projector

$$g_{\perp}^{\mu\nu} = g^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2}$$

satisfying $p_{\mu}g_{\perp}^{\mu\nu} = 0$, this can be done as

$$\begin{aligned} \langle 0 | B^{\mu\nu}(0) \tilde{B}^{\rho\sigma}(p) | 0 \rangle = & \left[\frac{1}{2} (g_{\perp}^{\mu\rho} g_{\perp}^{\nu\sigma} + g_{\perp}^{\mu\sigma} g_{\perp}^{\nu\rho} - \text{traces}) C_2 \right. \\ & + \frac{1}{4} \left(g_{\perp}^{\mu\rho} \frac{p^{\nu}p^{\sigma}}{p^2} + \text{permutations} \right) C_1 \\ & \left. + \frac{p^{\mu}p^{\nu}p^{\rho}p^{\sigma}}{(p^2)^2} C_0 \right] \theta(p^0 - |\vec{p}|) (-p^2)^{\Delta-d/2+1}. \end{aligned}$$

Then write down the Ward identity for special conformations (including the spin operator for a 2-index tensor, which you have to determine), and show that it leads to the conditions

$$\begin{aligned} C_1 &= 2 \frac{\Delta - d}{\Delta} C_0, \\ C_2 &= \frac{d}{d-1} \frac{(\Delta - d)(\Delta - d + 1)}{\Delta(\Delta - 1)}. \end{aligned}$$

Argue that this gives rise to the unitarity bound $\Delta \geq d$, in agreement

with eq. (3.82). Conclude that the energy-momentum tensor $T^{\mu\nu}$ is a primary operator with $\Delta = d$.

3.6 UV/IR divergences and anomalies

The discussion has been focused so far on Wightman functions. Beside having a Hilbert-space interpretation and satisfying conformal Ward identities that give strong constraints on their possible form, Wightman functions are also free of divergences. In momentum space, regularity at small momenta (IR) is enforced by the unitarity bound, whereas the power-law growth at large momenta (UV) is compatible with the damping provided by Schwartz-class test functions. In position space, the apparent singularity at short distance (UV) is resolved by the $i\varepsilon$ that provides a unequivocal prescription for deforming a contour of integration.

These properties are not true of time-ordered products. Let us consider once again the scalar 2-point function. Using the standard CFT normalization (3.56) and the Källén-Lehmann representation, we find

$$\langle \phi(x)\phi(y) \rangle_T = -i \frac{(4\pi)^{d/2} \Gamma(\frac{d}{2} - \Delta)}{2^{2\Delta} \Gamma(\Delta)} \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot (x-y)} (k^2 - i\varepsilon)^{\Delta-d/2}. \quad (3.83)$$

This expression diverges whenever $\Delta = \frac{d}{2} + n$ with integer n due to the Γ -function multiplying the integral. This corresponds precisely to a case in which the 2-point function has scaling dimension $d + 2n$, and is therefore compatible with contact terms of the form

$$(\partial^2)^n \delta^d(x - y). \quad (3.84)$$

In path integral language, this is the situation in which the source field J for the operator ϕ has scaling dimension $d - \Delta = \frac{d}{2} - n$, and therefore contact terms of the form $J(\partial^2)^n J$ can (and must) be added to the action. These contact terms are obviously covariant under Poincaré and scale transformations, so they can be added to the correlation function without affecting the Ward identities at separated points. (3.84) is the only possible contact term appearing in a scalar 2-point function, but more terms can appear in higher-point correlation functions, also involving operators that carry spin.

When Fourier transformed to momentum space, all such contact terms become polynomials in the momenta. In the scalar 2-point function case, these are $(p^2)^n$. Polynomial terms are incompatible with the positive-energy condition of Wightman functions, as they have support over all causal regions. But they are allowed (and in fact required) in time-ordered products.

Time-ordered products are defined in position space as the product of Wightman functions, which are tempered distributions, with step functions, which are not: therefore they do not necessarily have a Fourier transform. Contact terms can be seen as a way of “fixing” the time-ordered product so that they can be Fourier transformed.

This can be understood by analytic continuation in scaling dimension Δ . Let us assume that our scalar operator has scaling dimension $\Delta = \frac{d}{2} + n - \epsilon$, and take the limit $\epsilon \rightarrow 0$ (note that this ϵ is different from the one in the $i\varepsilon$ limit). Then the time-ordered 2-point function in momentum space takes the form

$$\Gamma(-n + \epsilon)(p^2)^{n-\epsilon} + Z(p^2)^n, \quad (3.85)$$

where the first term with a pole in ϵ comes from the expression (3.83) valid at $x \neq y$, and the second term is the counterterm added to the action. Choosing $Z \propto \epsilon^{-1}$ allows to cancel the divergence as $\epsilon \rightarrow 0$, but it also gives rise to a logarithm,

$$(p^2)^n \log \left(\frac{p^2}{\mu^2} \right). \quad (3.86)$$

The fact that we need to introduce a dimensionful quantity is the sign of a *conformal anomaly*: a 2-point function of this form does not satisfy the Ward identity for scale transformations (3.40),

$$\left[-p^\mu \frac{\partial}{\partial p^\mu} + 2n \right] (p^2)^n \log \left(\frac{p^2}{\mu^2} \right) = -2(p^2)^n \neq 0. \quad (3.87)$$

However, the anomalous term on the right-hand side is a polynomial in the momenta, corresponding to a contact term, indicating that the anomaly is local. In QFT language, this is typical of a (renormalized) UV divergence.

Note that the presence of contact term is associated with a very special type of operators of (half-)integer scaling dimension. Such operators are generically absent in an interacting conformal field theory, where the scaling dimensions take arbitrary real values. However, an exception to this rule concerns conserved currents in even space-time dimension d . For instance, a conserved current J^μ in $d = 4$ has scaling dimensions $\Delta = 3$; it is therefore associated in the path integral language with a source a_μ that carries dimension 1. This source is also subject to a gauge symmetry, $a_\mu \sim a_\mu + \partial_\mu \alpha$, and therefore a possible contact term is generated by $f_{\mu\nu} f^{\mu\nu}$, where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. As in the scalar case, this term must be used as a counterterm to cancel a divergence arising in the time-ordered correlation function, leading generically to logarithms in correlation functions involving the transverse polarization

of the current,

$$\langle J^\mu(p)J^\nu(q) \rangle_T \propto (2\pi)^4 \delta^4(p+q)(q^\mu q^\nu - q^2 g^{\mu\nu}) \log\left(\frac{q^2}{\mu^2}\right). \quad (3.88)$$

As in the scalar case, this is the manifestation of a UV divergence. However, the logarithm also implies that the limit $q^2 \rightarrow 0$ diverges: this is what we would call an IR divergence in quantum field theory. In the context of CFT, there is no clear distinction between UV and IR divergences, as the two are closely related. They both arise from an ambiguity in taking the Fourier transform. UV divergences can be cured with local counterterms, at the price of introducing a reference scale, but IR divergences are physical.

Note finally that time-ordered correlation functions involving a conserved current do not only have transverse polarization: while the conservation condition implies the vanishing of the state

$$\partial_\mu J^\mu(x) |0\rangle = 0, \quad (3.89)$$

and thus of Wightman correlation functions constructed from that state, this is not true of the divergence of J^μ appearing in a time-ordered product: the conservation equation $\partial_\mu J^\mu(x) = 0$ is only true as an *operator equation*, namely away from coincident points. On general grounds, one expects

$$\begin{aligned} \langle \phi_1(x_1) \cdots \phi_n(x_n) \partial_\mu J^\mu(y) \rangle_T &= -\delta^d(x_1 - y) \langle \delta\phi_1(x_1) \cdots \phi_n(x_n) \rangle_T \\ &\quad - \cdots \\ &\quad - \delta^d(x_n - y) \langle \phi_1(x_1) \cdots \delta\phi_n(x_n) \rangle_T, \end{aligned} \quad (3.90)$$

where $\delta\phi_i$ indicates the charge of the field ϕ under J^μ .¹⁰

4 Conformal correlation functions

The method presented in the last section using conformal Ward identities and the Wightman axioms could in principle be used to determine 3- and higher-point correlation functions.¹¹ However, it is quite inconvenient, and

¹⁰Note that it is conventional to normalize the conserved current J so that this equation is true, which means that the 2-point function of J cannot be arbitrarily normalized like a scalar operator in eq. (3.56). The same is true of the energy-momentum tensor, and of higher-spin conserved currents. For all of these operator the normalization of the 2-point function involves a coefficient of physical relevance.

¹¹The conformal Ward identities give differential equations and the Wightman axioms boundary conditions. Together, these are sufficient to constrain the conformal correlation functions. Note however that the positive-energy condition alone is not sufficient: the micro-causality axiom is required as well (see e.g. ref. [15] for a discussion at the level of 3-point functions).

it hides the simplicity of the result. To put things in perspective, note that the Wightman 3-point function of scalar operators in momentum space was only constructed in 2019 [16], whereas the position-space correlator has been known since the work of Polyakov in 1970 [17].

4.1 From Minkowski space-time to Euclidean space

For a start, let us go back to the Wightman 2-point function of scalar operators, given in position space by

$$W(x) = \frac{1}{[-(x^0 + i\varepsilon)^2 + \vec{x}^2]^\Delta}. \quad (4.1)$$

The very definition of this 2-point function includes an $i\varepsilon$ prescription with its time component and suggests therefore that x^0 should be thought of as a complex variable: as a function of a complex x^0 , W is analytic in the upper-half complex plane. Taking x^0 to be purely imaginary, i.e.

$$x^0 = i\tau, \quad \tau > 0, \quad (4.2)$$

we obtain the *Schwinger function*

$$\langle \phi(0)\phi(x) \rangle = \frac{1}{(\tau^2 + \vec{x}^2)^\Delta} \equiv \frac{1}{(x_E^2)^\Delta}. \quad (4.3)$$

We denote with $x_E = (\tau, \vec{x})$ the Euclidean vector that is contracted with the d -dimensional Euclidean metric. Schwinger functions will always be written using the “average” notation $\langle \cdots \rangle$, as opposed to the Wightman functions $\langle 0 | \cdots | 0 \rangle$ that can be interpreted as vacuum expectation values; since these are Euclidean functions by definition, we denote the position of the operator by x instead of x_E , even though the latter is implied.

Note that the Schwinger function transforms covariantly under the Euclidean conformal group $SO(d+1, 1)$ which is obtained by replacing the Minkowski metric by the Euclidean one. This includes both $SO(d)$ rotations and translations, and as a consequence we have

$$\langle \phi(0)\phi(x) \rangle \stackrel{\text{rotation}}{=} \langle \phi(0)\phi(-x) \rangle \stackrel{\text{translation}}{=} \langle \phi(x)\phi(0) \rangle, \quad (4.4)$$

showing that it is symmetric under the exchange of the order of the two operators. Also note that the Schwinger 2-point function is positive, and that it is not defined at the point $x_E = 0$, unlike the Wightman function whose $i\varepsilon$ prescription indicates how to approach any null point.

Exercise 4.1 There is another way to get to the same result, starting from the momentum-space representation of the 2-point function. The Wightman function is not well-suited to do so (at least not without exploring its analyticity properties following from the micro-causality axiom), but the time-ordered function is: starting from the Källén-Lehmann representation (3.83), which in momentum space becomes (using a notation in which the momentum-conservation delta function is implicit)

$$\langle \phi(-p)\phi(p) \rangle_T = -i \frac{(4\pi)^{d/2} \Gamma(\frac{d}{2} - \Delta)}{2^{2\Delta} \Gamma(\Delta)} (p^2 - i\varepsilon)^{\Delta - d/2},$$

one can perform a Wick rotation in which x^0 and p^0 are simultaneously rotated in opposite directions in the complex plane, to arrive at the Euclidean result

$$\langle \phi(-p)\phi(p) \rangle = \frac{(4\pi)^{d/2} \Gamma(\frac{d}{2} - \Delta)}{2^{2\Delta} \Gamma(\Delta)} (p^2)^{\Delta - d/2}.$$

Perform the Fourier transform in p to recover the Schwinger function (4.3). Note that you will need to make assumptions about Δ for the Fourier integral to converge. The result is however analytic in Δ , so you can argue a posteriori that it must be valid for all scaling dimensions.

This construction of a Euclidean function by analytic continuation from the Wightman function can in fact be generalized to any number of operator insertions. Consider the n -point Wightman function parameterized as

$$\langle 0 | \phi_n(x_n - x_{n-1}) \cdots \phi_2(x_2 - x_1) \phi_1(x_1) | 0 \rangle, \quad (4.5)$$

and complexify all time components x_i^0 . This function of several complex variables (and many more real ones) is in fact analytic in every upper-half complex plane in x_1^0 to x_n^0 , because it can be written as a Fourier transform of a function that only has support when the dual variable p_1^0 to p_n^0 are all positive. Therefore, going to purely imaginary times $x_i^0 = i\tau_i$, one obtains the Schwinger n -point function

$$\langle \phi_n(x_n - x_{n-1}) \cdots \phi_2(x_2 - x_1) \phi_1(x_1) \rangle, \quad (4.6)$$

in which the Euclidean times all satisfy $\tau_i > 0$, i.e. the operators are ordered along the Euclidean time direction. This latter observation is however irrelevant, because the ordering of operators does not matter in a Schwinger

function: the analytic continuation can be performed starting with a configuration in which all real Minkowski times are equal, in which case the operators commute by micro-causality. The observation made on the 2-point function is therefore valid more generally:

- Schwinger functions are symmetric under the exchange of operators.

The other observations made before are also true in general:

- Schwinger functions transform covariantly under the Euclidean conformal group $SO(d+1, 1)$. This property will be very useful because it means that we can use finite conformal transformations that act nicely in Euclidean space (including ∞ as a point) to simplify our computations.
- Schwinger functions are not defined at coincident points. This is not a bug but a feature: using functions that are only defined at separated points means that we do not need to worry about contact terms and UV divergences.¹² It means however that we cannot simply take the Fourier transform of these functions to obtain momentum-space Schwinger functions.
- Schwinger functions enjoy a property called *reflection positivity*: if the operators are organized in a configuration that is invariant under reflection across a plane (e.g. four points on the edges of a square; or trivially any two points), then the correlator is positive.¹³

We have seen that the Wightman functions let us define Schwinger functions by analytic continuation. But it turns out that the opposite is also true: the Osterwalder-Schrader theorem states that the properties of Schwinger functions listed above are sufficient to reconstruct Wightman functions.¹⁴

¹²It is in fact possible to describe contact terms (local and semi-local ones) using a generalization of the embedding formalism described in the next section [18]. But these contact terms are ambiguous: they carry more information than the Schwinger or Wightman functions themselves, and as such are unphysical.

¹³This property would not be needed if we had been studying Euclidean conformal field theory from the start. Indeed, there are interesting critical fixed points in condensed matter or statistical physics that are described by CFTs that are not reflection-positive (often called non-unitary). However, the conformal bootstrap described in section 6 relies on this property in an essential way.

¹⁴There is in fact another property that is needed in the proof, called *linear growth condition*. This property is very difficult to establish in quantum field theory. But within conformal field theory the linear growth condition is not necessary if one works instead with a set of “Euclidean CFT axioms”, which are otherwise equivalent to the Osterwalder-Schrader axioms, and from which the Wightman axioms can be recovered [19, 20].

This means that we can in fact focus on the Euclidean Schwinger functions for all our purposes, as any other physical observable can be reconstructed from them (we do not claim that this reconstruction is easy, though).

The only piece of information that we shall take for granted from the analysis of section 3 is the unitarity bound on the scaling dimension of operators. Working out these bounds purely from the Schwinger functions is possible, but the procedure is not particularly enlightening.

4.2 From Euclidean space to embedding space

Once we are dealing with correlation functions that transform under the Euclidean conformal group $SO(d+1, 1)$, it makes a lot of sense to try and make use of our understanding of the Lorentz group in $d+2$ dimensions to gain mileage. We already introduced in section 2 a set of coordinates in this $(d+2)$ -dimensional, *embedding* space, and a metric obeying

$$\eta_{MN}dX^M dX^N = (dX^\mu)^2 + (dX^{d+1})^2 - (dX^{d+2})^2. \quad (4.7)$$

Note that the directions labeled with indices μ are all space-like: the time-like direction is X^{d+2} .

The question is how do we go from $X^M \in \mathbb{R}^{d+1,1}$ to $x^\mu \in \mathbb{R}^d$ without explicitly breaking the Lorentz symmetry. This can be done in two steps:

1. Restrict our attention to the future light-cone $X^2 = 0$ with $X^{d+2} > 0$, which is an invariant subspace.
2. Identify any two points related by a scale transformation on this light cone, i.e. $X^M \sim \lambda X^M$ with $\lambda > 0$.

This means that we are essentially considering a map between a point x^μ in d -dimensional Euclidean space and a light-ray in a $(d+2)$ -dimensional Minkowski space-time. To make the map explicit, we chose a section of the cone, which we will take to be $X^{d+1} + X^{d+2} = 0$, and identify

$$X^\mu = x^\mu, \quad X^{d+1} = \frac{1-x^2}{2}, \quad X^{d+2} = \frac{1+x^2}{2}. \quad (4.8)$$

Conformal transformations now act linearly in embedding space,

$$X^M \rightarrow X'^M = \Lambda^M_N X^N \quad (4.9)$$

To obtain the action on x^μ , we first map it to our preferred section of the cone using eq. (4.8), apply the Lorentz transformation on X^M , then perform

a (space-time dependent) rescaling $X'^M \rightarrow \lambda(X')X'^M$ to get back on the section, and read off $x'^\mu = \lambda(X')X'^\mu$.

Local operators in Euclidean space must also be lifted to the embedding space, or at least to the null cone. On our preferred section, we declare

$$\phi(X) \equiv \phi(x), \quad (X^2 = 0, X^{d+1} + X^{d+2} = 1). \quad (4.10)$$

Then primary operators are defined on the rest of the cone by the scaling rule

$$\phi(\lambda X) = \lambda^{-\Delta} \phi(X). \quad (4.11)$$

Note that this rule can only apply to *primary* operators: descendants obtained acting with derivatives do not satisfy the same scaling property. This choice gives the right transformation rules for primary operators under all infinitesimal conformal transformations. This can be verified explicitly (see the exercise below), or argued as follows: a conformal transformation is a composition of a $(d+2)$ -dimensional Lorentz transformation, which locally acts on the operator like a rotation or a boost, followed by a position-dependent scale transformation to get back to the preferred section, which by the rule (4.11) amounts to a local scale transformation with weight given by the scaling dimension Δ of the operator. The combination of these two local transformations is precisely what we expect from the conformal transformation of an operator.

Exercise 4.2 Verify that a Lorentz boost in the direction of X^{d+1} corresponds to a scale transformation in Euclidean space, in agreement with eq. (2.39), and that operators transform accordingly.

A bit more care is required to define operators that carry spin on the projective null cone, as the additional Lorentz indices in the directions of X^{d+1} and X^{d+2} imply that there are more degrees of freedom that need to be addressed. This can be done imposing transversality in embedding space, together with a gauge symmetry condition, but we do want to dive into this here. We will therefore restrict our attention to 3- and higher-point functions of scalar primary operators only.

Once the transformation properties of local operators are clear, the construction of correlation functions of n points X_1 to X_n follows two simple rules:

- The correlators must depend on Lorentz-invariant quantities in embedding space, i.e. scalar products of the form $X_i \cdot X_j$. Since $X_i^2 = 0$ on the null cone, only scalar products with $i \neq j$ can appear.

- Applying a local scale transformation $X^M \rightarrow \lambda(X)X^M$ under which all primary operators ϕ_i with scaling dimension Δ_i satisfy

$$\phi_i(X_i) \rightarrow \lambda(X_i)^{-\Delta_i} \phi_i(X_i), \quad (4.12)$$

the correlator must transform homogeneously as

$$\langle \phi_1(X_1) \cdots \phi_n(X_n) \rangle \rightarrow \lambda(X_1)^{-\Delta_1} \cdots \lambda(X_n)^{-\Delta_n} \langle \phi_1(X_1) \cdots \phi_n(X_n) \rangle. \quad (4.13)$$

These rules imply immediately that there cannot be one-point functions, as there simply is no corresponding Lorentz-invariant quantity on the projective null cone. The simplest non-trivial case is that of a 2-point function, which must obey

$$\langle \phi(X_1) \phi(X_2) \rangle \propto (X_1 \cdot X_2)^{-\Delta}. \quad (4.14)$$

Note that this is only consistent with the homogeneous scaling rule if both operators have the same scaling dimension, another property that was derived in the hard way in section 3. To recover the dependence on the d -dimensional Euclidean coordinates, one simply needs to use the identification (4.8), yielding

$$X_1 \cdot X_2 = -\frac{1}{2}(x_1 - x_2)^2. \quad (4.15)$$

Upon fixing the proportionality factor in the above result, one recovers the expected result

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{1}{[(x_1 - x_2)^2]^\Delta}. \quad (4.16)$$

4.3 3-point functions

The embedding space formalism becomes very interesting when examining the 3-point function

$$\langle \phi_1(X_1) \phi_2(X_2) \phi_3(X_3) \rangle, \quad (4.17)$$

where the 3 scalar operators have now a priori distinct scaling dimensions Δ_1 , Δ_2 and Δ_3 . Solving the conformal Ward identities for this 3-point function would be an annoying task. Instead, by the above rules we immediately know that this is a function of the 3 invariant quantities

$$X_1 \cdot X_2, \quad X_1 \cdot X_3, \quad X_2 \cdot X_3. \quad (4.18)$$

Moreover, by the homogeneous scaling rule, the only possible form of the 3-point function is

$$\langle \phi_1(X_1) \phi_2(X_2) \phi_3(X_3) \rangle \propto (X_1 \cdot X_2)^{\alpha_{12}} (X_1 \cdot X_3)^{\alpha_{13}} (X_2 \cdot X_3)^{\alpha_{23}} \quad (4.19)$$

with the exponents satisfying

$$\begin{aligned}\alpha_{12} + \alpha_{13} &= -\Delta_1, \\ \alpha_{12} + \alpha_{23} &= -\Delta_2, \\ \alpha_{13} + \alpha_{23} &= -\Delta_3.\end{aligned}\tag{4.20}$$

This system of equations admits as unique solution

$$\alpha_{12} = -\frac{\Delta_1 + \Delta_2 - \Delta_3}{2},\tag{4.21}$$

$$\alpha_{13} = -\frac{\Delta_1 + \Delta_3 - \Delta_2}{2},\tag{4.22}$$

In terms of Euclidean coordinates, this can be written

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{\lambda_{123}}{(x_{12}^2)^{\Delta_{12,3}}(x_{13}^2)^{\Delta_{13,2}}(x_{23}^2)^{\Delta_{23,1}}},\tag{4.23}$$

where we have introduced the compact notation

$$x_{ij}^2 = (x_i - x_j)^2,\tag{4.24}$$

and

$$\Delta_{ij,k} = \frac{\Delta_i + \Delta_j - \Delta_k}{2}.\tag{4.25}$$

Eq. (4.23) is truly special. It should be compared with the most general 3-point function invariant under Poincaré and scale symmetry only: in this case, any term of the form

$$(x_{12}^2)^\alpha (x_{13}^2)^\beta (x_{23}^2)^\gamma\tag{4.26}$$

with

$$\alpha + \beta + \gamma = \frac{\Delta_1 + \Delta_2 + \Delta_3}{2}\tag{4.27}$$

satisfies the symmetry requirement, so that the 3-point function could take the form

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \sum_i \frac{c_i}{(x_{12}^2)^{\alpha_i}(x_{13}^2)^{\beta_i}(x_{23}^2)^{\gamma_i}},\tag{4.28}$$

with infinitely many free coefficients c_i . Instead, the conformal 3-point function (4.23) is fixed up to a *unique* multiplicative coefficient λ_{123} .

To give another point of comparison, let us examine a 3-point function that involves a descendant operator. Since the focus is on scalar operators, let us act on the first operator in (4.23) with $\partial_\mu \partial^\mu$, after which we obtain

$$\begin{aligned} \langle \partial^2 \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \rangle = 4\lambda_{123} & \left[\frac{(\Delta_1 - \frac{d-2}{2}) \Delta_{12,3}}{(x_{12}^2)^{\Delta_{12,3}+1} (x_{13}^2)^{\Delta_{13,2}} (x_{23}^2)^{\Delta_{23,1}}} \right. \\ & + \frac{(\Delta_1 - \frac{d-2}{2}) \Delta_{13,2}}{(x_{12}^2)^{\Delta_{12,3}} (x_{13}^2)^{\Delta_{13,2}+1} (x_{23}^2)^{\Delta_{23,1}}} \\ & \left. + \frac{\Delta_{12,3} \Delta_{13,2}}{(x_{12}^2)^{\Delta_{12,3}+1} (x_{13}^2)^{\Delta_{13,2}+1} (x_{23}^2)^{\Delta_{23,1}-1}} \right]. \end{aligned} \quad (4.29)$$

Unlike the correlation function of primary operators, this is now the sum of three terms with distinct powers of the distances x_{ij}^2 , all of which are individually consistent with Poincaré and scale symmetry.

Thanks to the peculiar form of the coefficients multiplying all three terms, there are special cases in which this descendant 3-point function takes the general primary form of eq. (4.23). These are not accidental, but correspond to physically interesting cases:

- If $\Delta_1 = |\Delta_2 - \Delta_3|$, then either $\Delta_{12,3} = 0$ or $\Delta_{13,2} = 0$, and in both cases two terms on the right-hand of eq. (4.29) vanish. This is a very special situation in which the primary 3-point function factorizes into a product of 2-point function, e.g. when $\Delta_3 = \Delta_1 + \Delta_2$,

$$\langle \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) \rangle \propto \langle \phi_1(x_1) \phi_1(x_1) \rangle \langle \phi_2(x_2) \phi_2(x_2) \rangle. \quad (4.30)$$

This situation is realized in generalized free field theory (definition in section 6 below) when ϕ_3 is a composite operator $\phi_3 \approx \phi_1 \phi_2$.

- If $\Delta_1 = \frac{d-2}{2}$, then the first two terms in eq. (4.29) vanish. As we saw in section 3, only a free scalar field satisfying the equation of motion $\partial^2 \phi_1(x) = 0$ can have such a scaling dimension. Since then obviously $[K^\mu, \partial^2 \phi_1(x)] = 0$, it is natural that the correlator involving the equation of motion takes the form of a primary 3-point function. However, we also know that it must vanish identically, which implies that either the 3-point coefficient λ_{123} vanishes, or that the additional condition $\Delta_1 = |\Delta_2 - \Delta_3|$ is satisfied: this is for instance the case of a 3-point function involving the primary composite operator ϕ^2 ,

$$\langle \phi(x_1) \phi(x_2) \phi^2(x_3) \rangle, \quad (4.31)$$

which is non-zero but vanishes under the action of $\partial^2/(\partial x_1)^2$.

Finally, let us go back to Minkowski space-time through an analytic continuation in the opposite direction of what we did before. It is not hard to see that the Wightman function of scalar primary operators satisfies

$$\langle 0 | \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) | 0 \rangle = \frac{\lambda_{123}}{(x_{12}^2)^{\Delta_{12,3}} (x_{13}^2)^{\Delta_{13,2}} (x_{23}^2)^{\Delta_{23,1}}}, \quad (4.32)$$

where now the Minkowski distances between two points are defined by

$$x_{ij}^2 = -(x_i^0 - x_j^0 - i\varepsilon) + (\vec{x}_i - \vec{x}_j)^2. \quad (4.33)$$

Note that $x_{ij}^2 \neq x_{ji}^2$, and as a consequence the Wightman 3-point function is not symmetric under the exchange of operators. Nevertheless, for real operators satisfying $\phi_i(x)^\dagger = \phi_i(x)$, we must have

$$\langle 0 | \phi_1(x_1) \phi_2(x_2) \phi_3(x_3) | 0 \rangle = \langle 0 | \phi_3(x_3) \phi_2(x_2) \phi_1(x_1) | 0 \rangle^*, \quad (4.34)$$

which is only compatible with the $i\varepsilon$ prescriptions appearing above if the coefficient λ_{123} is real. This property will in fact be essential in the conformal bootstrap discussed in section 6.

4.4 4-point functions

The embedding space technique can be used for higher-point functions as well, but we shall see that things get more complicated. To avoid dealing with four distinct operators and as many scaling dimensions, let us focus our attention on the case of 4 identical scalar operators,

$$\langle \phi(X_1) \phi(X_2) \phi(X_3) \phi(X_4) \rangle. \quad (4.35)$$

In these case there are 6 Lorentz invariants $X_i \cdot X_j$ with $i \neq j$. It is easy to see that a term of the form

$$\frac{1}{(X_1 \cdot X_2)^\Delta (X_3 \cdot X_4)^\Delta} \quad (4.36)$$

satisfies all the constraints of conformal symmetry. It is however definitely not unique: so does

$$\frac{1}{(X_1 \cdot X_3)^\Delta (X_2 \cdot X_4)^\Delta}. \quad (4.37)$$

This shows right away that there is no hope of constraining the 4-point function as much as we did with the 3-point function. In fact, one can construct two *invariant* quantities out of the X_i :

$$u = \frac{(X_1 \cdot X_2)(X_3 \cdot X_4)}{(X_1 \cdot X_3)(X_2 \cdot X_4)}, \quad v = \frac{(X_1 \cdot X_4)(X_2 \cdot X_3)}{(X_1 \cdot X_3)(X_2 \cdot X_4)}. \quad (4.38)$$

These are called *conformal cross-ratios*. Any function of u and v is conformally invariant, and the most general 4-point function must take the form

$$\langle \phi(X_1)\phi(X_2)\phi(X_3)\phi(X_4) \rangle \propto \frac{g(u, v)}{(X_1 \cdot X_2)^\Delta (X_3 \cdot X_4)^\Delta}. \quad (4.39)$$

In terms of Euclidean coordinates, this can be written

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{g(u, v)}{(x_{12}^2 x_{34}^2)^\Delta} \quad (4.40)$$

with

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \quad (4.41)$$

To see that this is the most general result, let us consider the following argument based on a sequence of finite conformal transformations:

1. Using translations, it is always possible to choose a reference frame in which $x_1 = 0$.
2. Using a special conformal transformation, one can then send $x_4 \rightarrow \infty$ without moving x_1 .
3. A rotation can then be used to place x_3 along some chosen direction, followed by a scale transformation to get $x_3 = (0, \dots, 0, 1)$, without touching the origin nor the point at infinity.
4. Finally, there is still a subset of rotations that do not affect x_3 , those around the last axis. They can be used to move the point x_2 to the position $x_2 = (b, 0, \dots, 0, a)$, shown later in figure 4.

Note that the first 3 steps can be used to fix entirely the kinematics of a 3-point function, which explains why its only freedom is in the form of a multiplicative coefficient. Instead, with a 4-point function we are left with two quantities, a and b , which correspond to the freedom parameterized by the two conformal cross ratios. It is in fact convenient to replace these two real numbers with a complex $z = a + ib$ and its conjugate $\bar{z} = a - ib$. The cross-ratios are then related to z and \bar{z} by

$$u = z\bar{z}, \quad v = (1 - z)(1 - \bar{z}). \quad (4.42)$$

Since Schwinger functions are analytic at all non-coincident configurations of points, the function $g(z)$ is a single-valued function over the complex

plane minus the points $\{0, 1, \infty\}$. Note that g is also subject to the *crossing symmetry* of the 4-point function, which requires¹⁵

$$g(u, v) = g\left(\frac{u}{v}, \frac{1}{v}\right) \quad (4.43)$$

upon exchanging the operators $\phi(x_1)$ and $\phi(x_2)$, as well as

$$g(u, v) = \left(\frac{u}{v}\right)^\Delta g(v, u), \quad (4.44)$$

upon $\phi(x_1) \leftrightarrow \phi(x_3)$.

The situation is however much more complicated in Minkowski space-time. One can still define cross ratios by eq. (4.41), with Minkowski distances defined through $i\varepsilon$ prescriptions as in (4.33). But the Wightman functions are not analytic (they lie at the boundary of the domain of analyticity in complexified time), and therefore g is a multi-valued function. Its values can be obtained by analytic continuation from a configuration in which all 4 operators live on a time slice, in which case it coincides with the Schwinger function. For instance, keeping x_1 , x_3 and x_4 on that time slice but letting the time component b of x_2 become imaginary, one ends up in a configuration in which z and \bar{z} are both real but distinct. This procedure is in general tedious, and it is fair to say that the current understanding of conformal Wightman 4-point functions is incomplete.¹⁶

5 State-operator correspondence and OPE

The construction of correlation functions could be continued beyond 4 points, but there is a good reason to stop there (at least in this course): we shall now see that any n -point function can be reduced to a $(n - 1)$ -point function using the operator product expansion (OPE), and that this procedure can be iterated until everything is expressed in terms of 2- and 3-point functions. The 4-point function will be used as a typical situation in which this OPE can be applied, and higher-point functions will not be considered further.

¹⁵It is standard in the CFT literature to call “crossing symmetry” the property of Schwinger functions to be symmetric under the exchange of operators, in analogy with scattering amplitudes. The two types of crossing are related, but they are not quite the same.

¹⁶Working in momentum space is not helping, even though the positive-energy condition reduces the number of non-trivial configurations: since the Ward identity for special conformal transformation is a second-order differential equation, the solution cannot be formulated in terms of invariants. The fact that there exists conformal invariants in position space is intimately connected with Ward identities being first-order differential equations.

5.1 The OPE in quantum field theory

The operator product expansion in quantum field theory is the statement that when two local operators are “sufficiently close” to each other, they can be replaced by another local operator, or rather a sum of them:

$$\phi_1(x)\phi_2(y) \xrightarrow{x \rightarrow y} \sum_i C_i(x-y)\phi_i(y), \quad (5.1)$$

It does not matter whether the operator ϕ_i on the right-hand side is inserted at x or at y , or at the middle point $(x+y)/2$, since this is understood in the limit in which x and y coincide. In fact, the proportionality factor C_i might diverge in the limit $x \rightarrow y$, so this is in general to be understood in the sense of an asymptotic limit, whose radius of convergence is strictly-speaking zero.

In non-perturbative quantum field theory, the OPE can be formulated in terms of the Hilbert space: when the product of operators in eq. (5.1) is acting on the vacuum, then the completeness of the Hilbert space implies that we can write

$$\phi_1(x)\phi_2(y)|0\rangle = \sum_{|\Psi\rangle} |\Psi\rangle \langle\Psi| \phi_1(x)\phi_2(y)|0\rangle, \quad (5.2)$$

where the sum is over states $|\Psi\rangle$ forming an orthonormal basis in the Hilbert space. Among these are states obtained acting with a local operator $\phi_i(y)$ on the vacuum, so that we have

$$\phi_1(x)\phi_2(y)|0\rangle = \sum_i C_i(x-y)\phi_i(y)|0\rangle + \dots, \quad (5.3)$$

where the constant C_i is related to a ratio of Wightman functions

$$C_i(x-y) \approx \lim_{z \rightarrow y} \frac{\langle 0 | \phi_i(z) \phi_1(x) \phi_2(y) | 0 \rangle}{\langle 0 | \phi_i(z) \phi_i(y) | 0 \rangle}. \quad (5.4)$$

This formulation remains however quite heuristic, and it is not obvious how to make it rigorous using only general principles.

In conformal field theory, however, the operator product expansion goes to a completely new level, thanks to the following observations:

- 2-point functions of primary operators are *diagonal* (i.e. only identical primaries have a 2-point function), which implies that states created by different primaries are readily orthogonal. The norm of primary states is also known in terms of the operators’ normalization.

- 3-point functions are known (see section 4), which means that the proportionality coefficient C_i are in fact fixed up to an overall multiplicative factor.¹⁷
- There are no other contributions to the OPE beyond those of local operators acting on the vacuum. This property is due to the state/operator correspondence that is discussed next.

5.2 The state/operator correspondence

In conformal field theory there exists a one-to-one correspondence between the states on a given time slice and local operators defined by their scaling dimension and representation under the Lorentz group.

The fact that local operators define states is obvious in Minkowski space-time. But it is also true in the analytic continuation to Euclidean space. To see this, remember that a local operator inserted at a Minkowski coordinate x can be expressed using eq. (3.4) as an operator living on the surface $x^0 = 0$ and evolved unitarily,

$$\phi(x) |0\rangle = e^{ix^0 P^0} e^{-i\vec{x}\cdot\vec{P}} \phi(0) |0\rangle. \quad (5.5)$$

Since the spectrum of P^0 is non-negative, this can be analytically continued to any value of x^0 in the upper-half complex plane; on the contrary, if x^0 had a negative imaginary part, then there are states of arbitrarily high energy that would become singular. By going to purely imaginary values, $x^0 = i\tau$, we can therefore define a state of the theory through an operator insertion in Euclidean space, provided that this operator is inserted at Euclidean time $\tau > 0$.

This has an immediate interpretation in terms of Euclidean path integral: a state on the Euclidean surface $\tau = 0$ (which is *identical* to the original Minkowski time slice $x^0 = 0$) is defined by a path integral over all field configurations restricted to the region $\tau < 0$. This is also valid for any number of operators inserted at separated points of Euclidean time $\tau < 0$, including the case of no operators, corresponding to the vacuum state. But the identification also works the other way: a given state on the surface $\tau = 0$ defines a boundary condition for the path integral at $\tau < 0$, which might correspond to some number of local operators inserted in the bulk (or a superposition of such configurations). In this way, any Euclidean correlation function in which the operators are ordered in Euclidean time τ can be given a Hilbert space interpretation.

¹⁷There is more than one factor if the operators ϕ_1 and ϕ_2 carry themselves spin.

Once we adopt the Euclidean path integral perspective, then it does not matter whether we use P^0 as the Hamiltonian, or an other conformal generator, as long as the surface at $\tau = 0$ is part of the foliation of space that it defines. For instance, one can use the so-called conformal Hamiltonian $\frac{1}{2}(P^0 - K^0)$, which foliates Euclidean space as shown in figure 1. We mentioned already in section 2 that this combination of generators is in fact equivalent to the dilatation generator D , in the sense that they are related by a rotation in the compact subgroup $\text{SO}(d+1)$ of the conformal group $\text{SO}(d+1, 1)$. This means that there exists a conformal transformation that maps the surface $\tau = 0$ to the unit sphere, and the other surfaces related by Hamiltonian evolution to spheres of different radii (see figure 1). Foliating Euclidean space with spheres centered at the origin and using the dilatation generator as a Hamiltonian is called *radial quantization*. In comparison, the physically-equivalent choice of Hamiltonian $\frac{1}{2}(P^0 - K^0)$ is often called *N-S quantization*.

The importance of these two quantizations in conformal field theory is due to the fact that they have fixed points, unlike P^0 : evolving back in “time”, the path integral shrinks to a ball of arbitrarily small radius surrounding one of these fixed points (the origin or the south pole). This means that any state on the unit sphere, or on the plane $\tau = 0$, can be related by Hamiltonian evolution to a state *localized* at the fixed point, and it is therefore equivalent to a state created by a local operator inserted there. This is in essence the argument showing that the state/operator correspondence goes both way: a local operator defines a state, but any state also defines a local operator (or rather a superposition of local operators).

This correspondence is in fact justifying a posteriori our choice made in section 3 to organize the operators into irreducible representations of the Lorentz group with a definite scaling dimension: the diagonalization can be performed at the level of the Hilbert space, and then radial quantization used to argue that each state corresponds to a local operator. Note that there are both primary and descendant states, in the sense that the local operator needs not be a primary.

Many introductory courses in conformal field theory begin in fact with radial quantization, as it provides a compelling picture in Euclidean space. The connection with CFT being a unitary quantum field theory is not easy to establish, though. Hermitian conjugation in the unitary theory corresponds to taking $x^0 = i\tau$ to $(x^0)^* = -i\tau$, i.e. it corresponds to a reflection across the surface of $\tau = 0$ in N-S quantization. In radial quantization, this becomes inversion $r \rightarrow r^{-1}$: correlation functions compute the overlap between states corresponding to a path integral inside the unit sphere with states corresponding to a path integral outside the sphere. The smearing of states

required in Minkowski space-time is replaced by the limit $r \rightarrow \infty$ needed to define conjugate states in terms of local operators. Once this limit is taken into account, the unitarity bounds can be derived using the interplay of generators P_μ and K_μ , which are in fact conjugate to each other under inversion. This procedure reproduces the results of our section 3, but it more obscure than examining Wightman functions projected onto eigenstates of spin.

5.3 The conformal OPE

The lesson we learn from radial quantization is that every state in the Hilbert space can be written as a linear combination of states created by the action of a single local operator on the vacuum, including primaries as well as descendants. Applying this lesson to Wightman functions in Minkowski space-time, we can be tempted to write the complete Hilbert space as

$$\mathcal{H} \stackrel{?}{=} \text{span}\{ |0\rangle, \phi(0) |0\rangle, \partial_\mu \phi(0) |0\rangle, \dots \}. \quad (5.6)$$

The problem is that these are not normalizable states. Instead, we could consider states created by primary operators inserted at any point x in Minkowski space-time,

$$\mathcal{H} \sim \text{span}\{ |0\rangle, \phi(x) |0\rangle, \dots \}, \quad (5.7)$$

where we implicitly understand that the local operators should be smeared against test functions. This gives a good description of the Hilbert space, but it is not very practical because primary operators inserted at different points (or smeared against distinct test functions) are not orthogonal to each other. This leads us to consider yet another representation of the Hilbert space, using local operators in momentum space,

$$\mathcal{H} \sim \text{span}\{ |0\rangle, \tilde{\phi}(p) |0\rangle, \dots \}. \quad (5.8)$$

In these case states carrying different momenta are orthogonal to each other, and moreover it is sufficient to consider momenta p inside the forward light cone. This means that we can express the completeness of the Hilbert space through the resolution of the identity

$$\mathbb{1} = |0\rangle \langle 0| + \sum_i \int_{p^0 > |\vec{p}|} \frac{d^d p}{(2\pi)^d} \frac{\tilde{\phi}_i(p) |0\rangle \langle 0| \tilde{\phi}_i(-p)}{(-p^2)^{\Delta-d/2}} + \dots, \quad (5.9)$$

where the sum is over all primary operators of the theory. The denominator is simply the scalar 2-point function, and states that carry spin have been

omitted for simplicity (this is the meaning of the dots): they could be included in the same way after inverting the Lorentz tensor appearing in the 2-point function, which is positive definite by unitarity.

This identity can be applied whenever we encounter a product of two local operators acting on the vacuum, to rewrite the product as a sum of local operators. This takes its simplest form when the operators are all expressed in momentum space:

$$\tilde{\phi}_1(p_1)\tilde{\phi}_2(p_2)|0\rangle = \sum_i \tilde{C}_i \tilde{\phi}_i(p_1 + p_2)|0\rangle. \quad (5.10)$$

This is the conformal OPE in Minkowski space-time, which reduces higher-point functions to lower-point ones. It is trivial when applied to 2-point functions, as only the vacuum state appears in the sum (the identity operator can be interpreted as a primary operator in its own right). When applied to a 3-point function, it allows us to determine the proportionality factor \tilde{C}_i in all generality in terms of the scaling dimension, Lorentz representation, and 3-point coefficients. Using this OPE further, any 4-point function (or higher) can be decomposed into a sum (or multiple sums) of terms whose kinematics is completely determined by conformal symmetry.

This form of the OPE is however rarely used, for two reasons: our understanding of Wightman 3-point functions in momentum space is incomplete, and the convergence of this OPE is not particularly nice. By this we mean that the OPE converges in the distributional sense, but this will never be as nice as the convergence properties of the Euclidean OPE discussed next.

Since we saw before that even Schwinger functions can be given a Hilbert space interpretation, we might as well formulate the OPE directly in Euclidean space. We write

$$\phi_1(x_1)\phi_2(x_2) = \sum_i C_i(x_1 - x_2, \partial_{x_2}) \phi_i(x_2). \quad (5.11)$$

Unlike the Minkowskian OPE (5.10), the convergence of this OPE is conditional: it is only true inside a correlation function in which the points x_1 and x_2 can be separated from any other insertions of local operators by a quantization surface, say a sphere in radial quantization. However, since the radius and center of the sphere can be chosen at will thanks to translation and scale symmetry, this includes many (in fact nearly all) configurations of points, as we shall see below.

In the notation of eq. (5.11), the coefficient C_i does not only depend on the distance between the two operators, but also on derivatives acting on the primary operator ϕ_i . This is a compact way of including all descendants in the

sum. Keep in mind however that there are infinitely many such descendants, so C_i is itself an infinite series. The only exception is the identity operator, which enters the OPE if and only if the two primary operators on the left-hand side of (5.11) are identical: in a 2-point function, the OPE takes the form

$$\langle \phi(x)\phi(0) \rangle = \frac{1}{(x^2)^{\Delta_\phi}} \langle \mathbb{1} \rangle, \quad (5.12)$$

with $\langle \mathbb{1} \rangle = 1$, from which we deduce

$$C_{\mathbb{1}}(x, \partial) = \frac{1}{(x^2)^{\Delta_\phi}}. \quad (5.13)$$

When viewed as a primary operator, the identity does not have any descendants. In all other cases, the value of $C_i(x, \partial)$ can be determined from a 3-point function: using the OPE in the scalar 3-point function (4.23), one can for instance deduce that

$$C_i(x, \partial) = \frac{\lambda_{12i}}{(x^2)^{\Delta_{12,i}}} [1 + ax^\mu \partial_\mu + b_1 x^\mu x^\nu \partial_\mu \partial_\nu + b_2 x^2 \partial^2 + \dots] \quad (5.14)$$

where

$$\begin{aligned} a &= \frac{\Delta_{i1,2}}{\Delta_i}, \\ b_1 &= \frac{\Delta_{i1,2}(\Delta_{i1,2} + 1)}{2\Delta_i(\Delta_i + 1)}, \\ b_2 &= -\frac{\Delta_{i1,2}\Delta_{i2,1}}{4\Delta_i(\Delta_i + 1)(\Delta_i - \frac{d-2}{2})}, \end{aligned} \quad (5.15)$$

and so on. This expression is valid when ϕ_i is a scalar operator. But in general there are also operators with spin entering the OPE. This is the case even when ϕ_1 and ϕ_2 are scalars: the operators being inserted at separated points in space, they carry angular momentum; for a point-like operator on the other side of the OPE, this angular momentum is expressed as internal spin. Note however that not all Lorentz representations appear in the OPE of two scalars, but only symmetric tensors. Moreover, when the operators are identical, then only symmetric tensors with an even number of Lorentz indices appear. The list of representations that appear can be determined from the group theory associated with Lorentz symmetry. At the same time, any scaling dimension Δ can in general be present.

There are therefore still many unknown in the OPE (we will see in the next section that there are in fact always infinitely many primary operators in the sum). But when compared with the OPE in a generic quantum field theory,

even in the presence of scale symmetry, the conformal OPE is extremely rigid: in a scale-invariant QFT the coefficients a, b_1, b_2, \dots , entering the definition of the coefficient $C_i(x, \partial)$ are all theory-dependent factors. In CFT, on the contrary, they are completely fixed by the kinematics. The only dynamical information is contained in the OPE coefficients λ_{12i} multiplying the contribution of a primary and all of its descendants.

6 The conformal bootstrap

Our approach to conformal field theory so far has been an *algebraic* one: in section 3 we have defined primary operators as irreducible representations of the conformal group, characterized by their transformations under Lorentz symmetry and by a scaling dimension. In section 5 we have seen that these operators can be combined using the operator product expansion. The OPE coefficients play the role of the structure constants of this algebra. The set of all scaling dimensions and representations under the Lorentz group of a theory, together with the OPE coefficients,

$$\{(\Delta_i, R_i), \lambda_{ijk}\} \tag{6.1}$$

is called the *CFT data*. A conformal field theory is completely defined by its CFT data: any correlation function can be computed from it, by repeated use of the OPE.

But does any CFT data define a good theory? The answer to this question is no: the algebra of local operators must close, giving very constraining consistency conditions.¹⁸ The precise formulation of this closure condition depends whether we are talking about the Euclidean or Lorentzian conformal groups. For a conformal field theory in Minkowski space-time, the crucial point is that the OPE must be consistent with the micro-causality condition (3.9). But this logic is quite difficult to implement in practice.

Instead, there is a much simpler consistency condition for Schwinger functions in Euclidean space: since these functions are symmetric under the exchange of operators (a property that is in fact related to micro-causality), then the Euclidean OPE must be associative. To illustrate this, let us focus on 4-point functions (imposing associativity of the OPE on *all* 4-point functions is in fact sufficient to guarantee associativity of higher-point functions), and for simplicity consider as before the case of 4 identical scalar primary

¹⁸This is very similar in spirit to the classification of simple Lie group, leading to the well-known families $SU(N)$, $SO(N)$, $Sp(2N)$ and a few exceptional groups.

operators with scaling dimension Δ_ϕ :

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} g(u, v). \quad (6.2)$$

6.1 Conformal blocks

The simplest way to understand how this 4-point function can be expressed in terms of the CFT data is to use the OPE (5.11) on it twice:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \sum_i C_i(x_1 - x_2, \partial_{x_2}) C_i(x_3 - x_4, \partial_{x_4}) \langle \phi_i(x_2)\phi_i(x_4) \rangle \quad (6.3)$$

There is a single sum on the right-hand side because the 2-point function is diagonal: it vanishes unless the two primary operators are identical. To put this in a convenient form, let us extract the OPE coefficient λ_i corresponding to the 3-point function $\langle \phi\phi\phi_i \rangle$, and a power of the distance, defining

$$C_i(x, \partial) = \frac{\lambda_i}{(x^2)^{\Delta_\phi}} C'_i(x, \partial), \quad (6.4)$$

The new function C'_i is then of the form

$$C'_i(x, \partial) = (x^2)^{\Delta_i/2} [1 + \dots], \quad (6.5)$$

where Δ_i is the scaling dimension of the operator entering the OPE (the *internal* operator), not to be confused with the scaling dimension Δ_ϕ of the operators in the original correlation function (the *external* operators). We have now

$$\begin{aligned} \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \\ = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} \sum_i \lambda_i^2 g_i(u, v), \end{aligned} \quad (6.6)$$

where we have defined

$$g_i(u, v) = C'_i(x_1 - x_2, \partial_{x_2}) C'_i(x_3 - x_4, \partial_{x_4}) \frac{1}{(x_{24})^{\Delta_i}}. \quad (6.7)$$

This should be compared with the representation (6.2) of the 4-point function, in terms of which

$$g(u, v) = \sum_i \lambda_i^2 g_i(u, v). \quad (6.8)$$

The g_i are called *conformal blocks*: they represent the contribution of a single primary and of all its descendants to the 4-point function stripped of

an overall covariant factor. They are therefore conformally invariant, and must be functions of the cross-ratios u and v , even though this is not at all obvious from their definition above.

This definition is in fact not very convenient to compute the conformal blocks: working out all the terms in the series of C'_i is difficult, and there are also Lorentz indices that need to be contracted when the intermediate operator carries spin. But it is convenient to examine the limit $x_1 \rightarrow x_2$, $x_3 \rightarrow x_4$, or equivalently $u \rightarrow 0$: in this case, the leading term in the OPE indicates that

$$g_i(u, v) = u^{\Delta_i/2} [1 + \dots]. \quad (6.9)$$

Operators with the lowest scaling dimensions give the leading contribution to the 4-point function in this limit. The operator with the absolute lowest scaling dimension is the identity, for which there are also no descendant so that¹⁹

$$g_{\mathbf{1}}(u, v) = 1. \quad (6.10)$$

Note that this statement is related to the general *cluster decomposition principle* in quantum field theory: since there is no absolute scale in CFT, the limit $x_1 \rightarrow x_2$ is equivalent to a limit in which the other points x_3 and x_4 are sent very far away, all the way to infinity, and in this case one expects on general grounds that the correlation function factorizes as

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle \xrightarrow{x_3, x_4 \rightarrow \infty} \langle \phi(x_1) \phi(x_2) \rangle \langle \phi(x_3) \phi(x_4) \rangle. \quad (6.11)$$

This factorization corresponds to the OPE in the limit $u \rightarrow 0$.

There exists a much simpler way of computing the conformal block, based on the Casimir invariant of the conformal group. From the group algebra, one can verify that the combination of generators (2.39)

$$\mathcal{C}_2 = -\frac{1}{2} J_{MN} J^{MN} \quad (6.12)$$

commutes with all individual generator, e.g.

$$[\mathcal{C}_2, P^\mu] = 0. \quad (6.13)$$

In group theory language, \mathcal{C}_2 is called the quadratic Casimir invariant. Since it commutes with translations, its action on a state created by a local operator is the same no matter where the operator is inserted, or whether it is a primary or a descendant: in all cases,

$$\mathcal{C}_2 \phi(x) |0\rangle = [\Delta(\Delta - d) + \ell(\ell + d - 2)] \phi(x) |0\rangle. \quad (6.14)$$

¹⁹In this special case it does not make much sense to speak of OPE coefficient, but it is conventional to also take $\lambda_{\mathbf{1}} = 1$.

The second term in the square bracket is the quadratic Casimir of the Lorentz/rotation group for a symmetric tensor with ℓ indices.

Exercise 6.1 Compute the eigenvalue of the quadratic Casimir operator from the commutators of the conformal generators with $\phi(x)$. For simplicity, work at $x = 0$.

The idea is to evaluate the value of this Casimir operator between states that are each created by two local operators, i.e. compute the correlation function

$$\langle \phi(x_1)\phi(x_2)\mathcal{C}_2\phi(x_3)\phi(x_4) \rangle \quad (6.15)$$

Applying the OPE once, this can be written

$$\langle \phi(x_1)\phi(x_2)\mathcal{C}_2\phi(x_3)\phi(x_4) \rangle = \sum_i C_i(x_3 - x_4, \partial_{x_4}) \langle \phi(x_1)\phi(x_2)\mathcal{C}_2\phi_i(x_4) \rangle, \quad (6.16)$$

where there are now two possible ways of computing each term on the right-hand side: either \mathcal{C}_2 acts to the right, and then the eigenvalue equation (6.14) can be used, or it acts to the left. In the second case, the individual generators forming \mathcal{C}_2 must be commuted with $\phi(x_1)$ and $\phi(x_2)$ successively. This is most easily done in a convenient reference frame, for instance in the z -frame described in section 4 and shown in figure 4. In this case, one finds that the action of \mathcal{C}_2 on the 4-point function or on any individual conformal block is in both case given by a second-order differential operator in z and \bar{z} , which we denote by $D_{z,\bar{z}}$. This means that the conformal blocks satisfy an equation of the form

$$D_{z,\bar{z}} g_i(z, \bar{z}) = [\Delta_i(\Delta_i - d) + \ell_i(\ell_i + d - 2)] g_i(z, \bar{z}). \quad (6.17)$$

Together with a boundary condition provided by eq. (6.9), this is sufficient to determine completely the conformal blocks. In even space-time dimensions, the solution is in the form of products of hypergeometric functions. In odd dimension there is no known closed-form solution, but the Casimir equation can be conveniently solved term-by-term in a series expansion.

Note that in this specific case with identical external operators, the conformal block do not depend on the scaling dimension Δ_ϕ , but only on the scaling dimension Δ_i and spin ℓ_i of the internal operator.

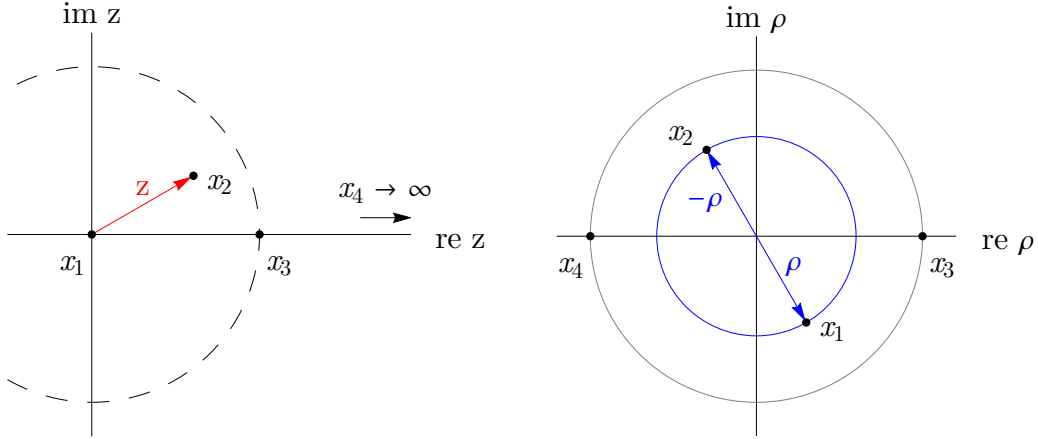


Figure 4: Two convenient conformal frames that describe the 4-point function. On the left, 3 points are mapped to 0, 1 and ∞ using a conformal transformation, and a complex coordinate z describes the position of the fourth point. Applying a conformal transformation on the plane, one can reach the configuration on the right, with operators placed at antipodal points on a pair of concentric circles, with the larger circle having unit radius; this configuration is parameterized by the coordinate ρ , satisfying $|\rho| \leq 1$.

6.2 OPE convergence

So far we have been using the OPE in the 4-point function without worrying about its convergence. This is fine as long as the points x_1 and x_2 can be separated from x_3 and x_4 by a sphere. In terms of the coordinate z , this is obviously true for all $|z| < 1$ (the dashed circle in figure 4): then the product of operators $\phi(x_1)\phi(x_2)$ can be replaced by a sum of local operators $\phi_i(x_1)$ by the state-operator correspondence, and therefore the OPE converges since it is a Hilbert-space sum. But in fact the domain of convergence is much larger: one can for instance use radial quantization around any point in space, not just x_1 , and it is not hard to see that in most configurations one can draw a circle surrounding x_1 and x_2 , but excluding x_3 and x_4 . This is in fact possible in all cases except when $z > 1$.²⁰

This is even more easily seen in a different reference frame: using a conformal transformation on the plane of z , one can map the 4 points to a configuration depicted on the right-hand side of figure 4, with the pairs of points (x_1, x_2) and (x_3, x_4) placed at antipodal points on two circles centered

²⁰In a generic frame, the only case in which the OPE $\phi(x_1) \times \phi(x_2)$ does not converge is when all four points are on a circle (this includes a line), but the points x_1 and x_2 are not consecutive, i.e. they are separated by x_3 and x_4 .

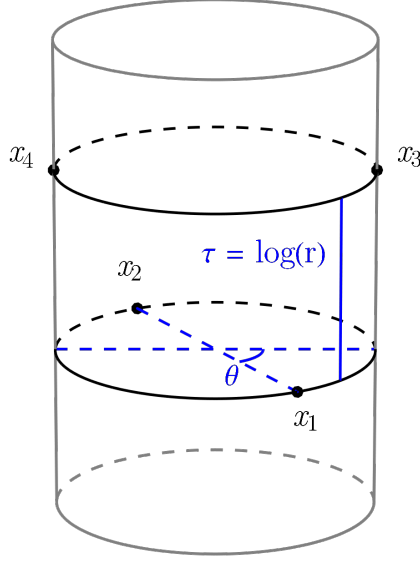


Figure 5: The ρ -coordinate configuration of figure 4 as seen on the radial quantization cylinder. The two segment $x_1 - x_2$ and $x_3 - x_4$ form an angle θ , and they are separated by in cylinder time $\tau = \log(r)$, related to the flat-space configuration by $\rho = r e^{i\theta}$.

at the origin. The map is given by

$$\rho = \frac{z}{(1 + \sqrt{1 - z})^2} \quad \Leftrightarrow \quad z = \frac{4\rho}{(1 + \rho)^2}, \quad (6.18)$$

and it takes all of the complex z plane to the unit disk in ρ , with the half-line $z > 1$ being mapped to the unit circle. In this frame, it is now obvious that the OPE converges in radial quantization unless $|\rho| = 1$.

This conformal frame is also very useful for understanding the convergence properties of the OPE. It can be seen as a configuration on the radial quantization cylinder, as in figure 5, with coordinates

$$x_1 = (\tau, \vec{n}), \quad x_2 = (\tau, -\vec{n}), \quad x_3 = (0, \vec{n}'), \quad x_4 = (0, -\vec{n}'), \quad (6.19)$$

where $\tau = \log(r)$ is the “time” component, and the “space” components are unit vectors \vec{n} and \vec{n}' that parameterize a direction on the sphere S^{d-1} , and form an angle θ . The connection with the flat-space configuration is through

$$\rho = r e^{i\theta}. \quad (6.20)$$

A conformal block corresponds to the projection onto intermediate primary and descendant states with scaling dimensions $\Delta_i + n$ (n being an integer)

and spin j (whose range is determined by the spin ℓ of the primary). On general grounds, one can therefore expect the conformal block to take the form

$$g_i(r, \theta) = \sum_{n,j} B_{n,j} r^{\Delta_i+n} \mathcal{C}_j^{(d-2)/2}(\cos \theta), \quad (6.21)$$

where $\mathcal{C}_j^{(d-2)/2}(\cos \theta)$ is a Gegenbauer polynomial obtained from the contraction of a symmetric and traceless tensor $\vec{n}^{\mu_1} \dots \vec{n}^{\mu_j}$ with $\vec{n}'^{\mu_1} \dots \vec{n}'^{\mu_j}$. The coefficients $B_{n,j}$ are positive by unitarity, because they are the norm of a descendant projected onto eigenstates of definite spin (labeled by n and j). They are in fact rational functions of Δ_i . This representation is very useful: on the one hand, it gives an efficient way of evaluating the conformal block at any given level of precision by truncating the series in n , since by assumption $r < 1$; on the other hand, it shows that the g_i are nice (i.e. analytic) functions of scaling dimension Δ_i for all values of Δ_i above the unitarity bound.

Beyond individual conformal blocks, the ρ coordinates also show how the OPE converges as a function of r : in a configuration with $r \ll 1$, the series is dominated by operators of low Δ (both primary and descendants), and a good approximation of the 4-point function can be obtained from a truncated series. For instance, a configuration in which the operators are placed on the corners of a square corresponds to $z = \frac{1}{2}$, or $|\rho| = (1 + \sqrt{2})^{-2} \approx 0.17$. Even though the point x_1 and x_2 are not obviously close to each other in this configuration, $r = |\rho|$ is a small number and the convergence of the OPE is very good!

This is a fantastic property of the *Euclidean* OPE. The situation is not as nice in Minkowski space-time. The power-law dependence in r in eq. (6.21) is the consequence of exponential damping in Euclidean “time” τ ($r^{\Delta_i} = e^{\tau \Delta_i}$ with negative τ), through Hamiltonian evolution in radial quantization. In Minkowski space-time, independently of the choice of quantization, the evolution is always *unitary*, e.g. e^{iEt} : there is in principle no reason why operators of higher scaling dimension contribute less than those of lower dimension in a generic configuration. This does not mean that the Minkowskian OPE is uninteresting, but it is more difficult to harness than the Euclidean one.

6.3 The crossing equation and simple solutions

We have so far only discussed one particular OPE, in which the points x_1 and x_2 were supposed to be closed to each other (in a conformal sense). But since Schwinger functions are symmetric under the exchange of operators, there is no reason not to also consider different OPEs, e.g. between the operators

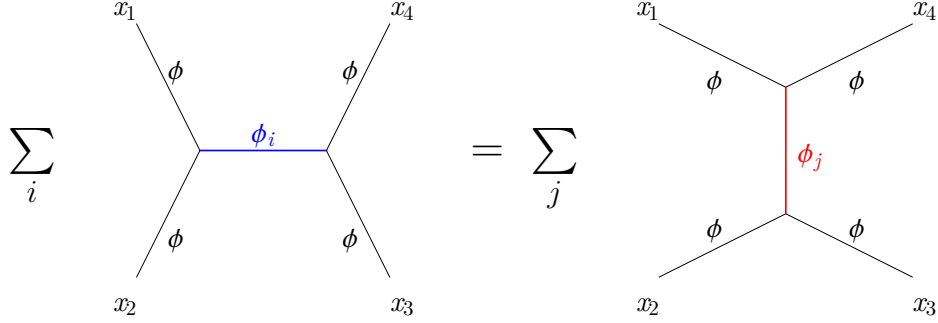


Figure 6: Diagrammatic representation of the crossing equation at the core of the conformal bootstrap.

$\phi(x_1)$ and $\phi(x_3)$, or $\phi(x_1)$ and $\phi(x_4)$. In the vast majority of configurations of 4 points, the space can be split in half by a sphere in three inequivalent ways. Even in the special case in which the 4 points are on a circle, there are still two inequivalent ways of surrounding pairs of points with a sphere. This means that we expect different OPEs to converge in all cases.²¹

This implies that besides the expansion

$$g(u, v) = \sum_i \lambda_i^2 g_i(u, v), \quad (6.22)$$

one can write a similar expansion for the 4-point function in a configuration related by symmetry, e.g.

$$\left(\frac{u}{v}\right)^{\Delta_\phi} g(v, u) = \left(\frac{u}{v}\right)^{\Delta_\phi} \sum_i \lambda_i^2 g_i(v, u), \quad (6.23)$$

and since the two are related, that we must have the equality

$$\sum_i \lambda_i^2 g_i(u, v) = \left(\frac{u}{v}\right)^{\Delta_\phi} \sum_i \lambda_i^2 g_i(v, u). \quad (6.24)$$

This equation is illustrated in figure 6. Note that in the case of identical external operators the two sums are over the same set of primary operators, but in general these could be different sums.

This is a relatively simple equation, but it is quite difficult to solve. We saw above that the OPE is dominated by intermediate operators of low Δ_i in the limit $u \rightarrow 0$, but this only applies to the left-hand side: taking the

²¹In term of Hilbert space, this is a consequence of the fact that the same Schwinger function can be obtained by analytic continuation from distinct Wightman functions with different orderings of operators.

same limit on the right-hand side, one reaches the boundary of the domain of convergence of the OPE. Using the known expression for the conformal blocks in terms of the hypergeometric functions, one can see that the limit $z \rightarrow 1$ approaches a branch cut around which

$$g_i(z, \bar{z}) \propto \log(1 - z). \quad (6.25)$$

This means that the right-hand side of the crossing equation is dominated by terms of the form

$$\left(\frac{u}{v}\right)^{\Delta_\phi} g_i(v, u) \propto z^{2\Delta_\phi} \log(z), \quad (6.26)$$

in the limit $z \rightarrow 0$. There is no way a finite sum of terms can reproduce the leading constant contribution from the identity operator on the left-hand side. This is the first lesson we learn from crossing symmetry: the crossing equation can never be satisfied block-by-block, but only by an infinite sum of conformal blocks, and there must therefore be an infinite number of primary operators in the OPE!

Before looking into the clever way in which the conformal bootstrap deals with this problem below, let us examine the simplest solution to crossing symmetry that we know. A possible function that transforms covariantly under the conformal group and is crossing symmetric is the following linear combination of 2-point functions:

$$\begin{aligned} &\langle \phi(x_1) \phi(x_2) \rangle \langle \phi(x_3) \phi(x_4) \rangle + \langle \phi(x_1) \phi(x_3) \rangle \langle \phi(x_2) \phi(x_4) \rangle \\ &\quad + \langle \phi(x_1) \phi(x_4) \rangle \langle \phi(x_2) \phi(x_3) \rangle, \end{aligned} \quad (6.27)$$

corresponding to

$$g(u, v) = 1 + u^{\Delta_\phi} + \left(\frac{u}{v}\right)^{\Delta_\phi}. \quad (6.28)$$

Performing an expansion at small u and matching each term with a hypothetical primary operator, one finds an infinite spectrum of operators characterized by their spin ℓ and scaling dimension

$$\Delta_i = 2\Delta_\phi + 2n + \ell \quad (6.29)$$

where $n = 0, 1, 2, \dots$ takes non-negative integer values. This defines therefore a valid 4-point correlation function in a theory called *generalized free field theory* (or sometimes mean free theory, or Gaussian theory). It is similar to a free theory in the sense that the OPE $\phi \times \phi$ consists of operators that have the same scaling dimension and spin as *composites* of the field ϕ , of the form

$$[\phi(\partial^2)^n \partial^{\mu_1} \dots \partial^{\mu_\ell} \phi]. \quad (6.30)$$

The free scalar theory is a special case in which only operators with $n = 0$ are present in the OPE (the other vanish by the equation of motion $\partial^2\phi = 0$), and these operators are higher-spin conserved current. Generalized free field theory is usually considered non-local: its definition through the above 4-point function does not include an energy-momentum tensor.²² It is nevertheless an physically interesting theory, because the large- N limit of gauge theory is of this form: if one considers the 4-point function of a gauge-invariant composite operator such as the fermion bilinear $\bar{\psi}_a\psi_a$, then its decomposition into conformal blocks is given at leading order in $1/N$ by a sum of “double-trace” operators with scaling dimensions $2\Delta_\phi + 2n + \ell$, whereas “single-trace” operators (among which the energy-momentum tensor) only enter at sub-leading order in $1/N$.

6.4 The numerical bootstrap

The revolutionary approach to the crossing equation that gave birth to the modern conformal bootstrap appeared in 2008 [21]. The starting point is to rewrite the crossing equation as

$$\sum_i \lambda_i^2 [v^{\Delta_\phi} g_i(u, v) - u^{\Delta_\phi} g_i(v, u)] = 0, \quad (6.31)$$

and to realize that this can be viewed as an equation in an infinite vector space

$$\sum_i \lambda_i^2 \vec{F}_i(\Delta_\phi, \Delta_i, \ell_i) = 0. \quad (6.32)$$

The components of the vector \vec{F}_i correspond to $v^{\Delta_\phi} g_i(u, v) - u^{\Delta_\phi} g_i(v, u)$ evaluated at an infinite set of points (u, v) , or equivalently as the coefficient of a Taylor expansion around some preferred point.²³ \vec{F}_i has infinitely many components, but the equation is also true if we truncate it to any finite subset. Some interesting results can already be obtained from considerations about a 2-dimensional subset, but typically the more stringent bounds that are shown below require scanning over spaces with a large number of dimensions.

The norm of the vector \vec{F}_i is irrelevant in this equation, as it gets multiplied by an unknown positive factor λ_i^2 . But the direction in which the vector \vec{F}_i points is crucial: if all vector \vec{F}_i point in the same general direction, for all

²²Also it can be defined through the non-local Lagrangian $\mathcal{L} = \frac{1}{2}\phi(\partial^2)^{d/2-\Delta_\phi}\phi$.

²³Traditionally the numerical bootstrap uses a Taylor expansion around the point $u = v = \frac{1}{4}$, which corresponds to $z = \frac{1}{2}$, i.e. the symmetric point that has the fastest convergence rate in both channels.

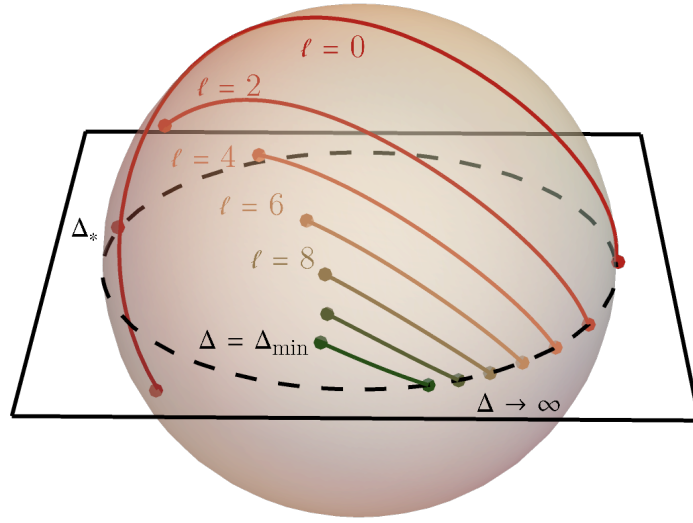


Figure 7: A toy example of the behavior of vectors \vec{F}_i in a 3-dimensional space. For each spin $\ell_i = 0, 2, 4, 6, \dots$, the vector \vec{F}_i draws a smooth curve on the unit sphere as Δ_i varies between the unitarity bound (Δ_{\min}) and infinity. All curves are contained in the upper hemisphere, except for the scalar curve $\ell = 0$ which is in the lower hemisphere when $\Delta < \Delta_*$. This means that a putative theory with no scalar primary operator with $\Delta < \Delta_*$ is excluded, as the crossing equation (6.32) cannot be satisfied: all vectors \vec{F}_i are on the same side of the horizontal *separating plane*. Note that the curves shown here are made up and do not correspond to actual functionals acting on conformal blocks.

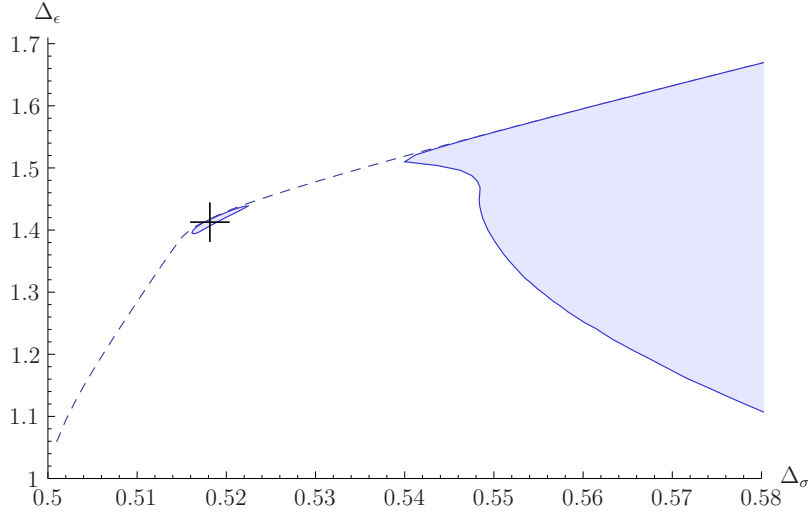


Figure 8: Figure taken from ref. [22], showing in blue the allowed scaling dimensions of two scalar operators σ and ϵ , assuming an OPE of the form of eq. (6.34). For comparison, the dashed line shows an upper bound on the scaling dimension of the first operator in the OPE of two identical scalar operators, without making any theory-specific assumptions, already displaying a “kink” close to the Ising CFT. By now the size of the “island” surrounding the cross has shrunk to something smaller than the best Monte-Carlo simulations of the 3-dimensional Ising model by several orders of magnitude.

spins ℓ_i and all Δ_i above the corresponding unitarity bound, then the equation does not have a solution, because no non-trivial linear combinations of \vec{F}_i can ever sum to zero. This observation is at the core of all numerical bootstrap algorithms, which go along the following lines: we begin with making a hypothesis about the scaling dimension Δ_ϕ , and say the operator with the lowest scaling dimension appearing in the OPE $\phi \times \phi$,²⁴ then we study the behavior of the vector \vec{F}_i under this assumption, and try to find a separating plane so that all vectors point on the same side of the plane (this part is hard!); if such a plane is found, the equation cannot be satisfied, which implies that the hypothesis is wrong. Iterating over this strategy allows to exclude entire regions in the space of possible parameters. Figure 8 shows the result of this procedure for scalar operators in $d = 3$ dimensions.

Very often, a lot of mileage is gained when one can input assumptions about some target theory. For instance, let us go back to the model discussed

²⁴We know that there exist a crossing symmetric solution for all values of Δ_ϕ , namely the generalized free field theory one. Therefore it is not sufficient to make a hypothesis on Δ_ϕ alone.

in the introduction:

$$S = \int d^3x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{g}{4!} \phi^4 \right]. \quad (6.33)$$

This theory is free at high energy, and it is supposed to flow to a strongly-coupled interacting fixed-point at low energy. We can gain some intuition about the theory by examining the list of its relevant primary operators, given in the first column of table 1. Two of these primaries can reasonably be expected to still be primary operators at the interacting fixed point: ϕ and ϕ^2 have sufficiently low scaling dimensions to start with. On the contrary, ϕ^4 is the relevant operator that triggers the renormalization group flow in the UV, so we expect it to become irrelevant in the IR (otherwise the flow would continue). ϕ^3 is a special case: as soon as the coupling g is turned on, the equation of motion $(-\partial^2 + m^2)\phi^2 = \frac{1}{3!}\phi^3$ implies that it is not an independent primary operator anymore, but rather a descendant of ϕ . Among the operators that carry spin, the only relevant one is the energy-momentum tensor $T^{\mu\nu}$: this operator exists in any QFT, and therefore it can be expected to be there in the IR as well, with unchanged scaling dimension. In addition to this, the Lagrangian (6.33) is invariant under a global \mathbb{Z}_2 symmetry that takes $\phi \rightarrow -\phi$. Based on these assumptions, we can make the following claims:

- The IR fixed point has only two relevant primary operators, both of them scalars, which we will denote σ (\mathbb{Z}_2 odd) and ϵ (\mathbb{Z}_2 even).
- The OPEs of all combinations of σ and ϵ have the following form,

$$\begin{aligned} \sigma \times \sigma &= \mathbb{1} + \epsilon + T^{\mu\nu} + \dots \\ \sigma \times \epsilon &= \sigma + \dots \\ \epsilon \times \epsilon &= \mathbb{1} + \epsilon + T^{\mu\nu} + \dots \end{aligned} \quad (6.34)$$

where the dots indicate contribution from all (infinitely many) irrelevant primary operators.

Using these properties as an input, and studying all 4-point functions involving σ and ϵ , one is able to make the allowed region of parameters $(\Delta_\sigma, \Delta_\epsilon)$ shrink to a island surrounding the value known experimentally, see figure 8. With the more recent numerical analysis, the size of this allowed island has shrunk to become much smaller than the best Monte-Carlo simulations, so that the current best theoretical prediction for the critical exponents of the Ising model come from the conformal bootstrap.

UV primary	Δ	IR primary	Δ	\mathbb{Z}_2
ϕ	0.5	σ	0.51815	odd
ϕ^2	1	ϵ	1.14126	even
ϕ^3	1.5	—	—	—
ϕ^4	2	ϵ'	> 3	even
$T^{\mu\nu} \sim \partial^\mu \phi \partial^\nu \phi$	3	$T^{\mu\nu}$	3	even

Table 1: List of primary operators in the free UV limit of the theory described by the Lagrangian (6.33), and corresponding primaries in the IR CFT.

This is very impressive, and it illustrates admirably how the concept of *universality* in statistical physics works: given some very general assumptions (conformal symmetry, only two relevant operators, and a \mathbb{Z}_2 symmetry), the conformal bootstrap essentially establishes that there is a unique theory and provides excellent numerical results with rigorous error bars. Moreover, at the intersection between the allowed and disallowed regions (i.e. on the boundary of the island), the crossing equation (6.32) must be satisfied in a peculiar way: many vectors must lie precisely on the separating plane, and only these vectors can have non-zero OPE coefficients for the equation to be satisfied. Reading out the scaling dimensions Δ_i and spins ℓ_i associated with these vectors, one gets an numerical estimate of the spectrum of primary operators entering the OPE. Figure 9 shows such an estimate. The spectrum of operators is shown there as a function of the spin ℓ_i and of the difference $\Delta_i - \ell_i$, so that an underlying geometric structure appears clearly: the operators are organized into *conformal Regge trajectories*.

This observation matches the state-of-the-art analytical understanding of the crossing equation: every operators with a low scaling dimension Δ_i (and also low spin by the unitarity bound) is related by crossing symmetry to families of operators with scaling dimensions $2\Delta_i + 2n + \ell$. This is what we found in generalized free field theory, but it works reasonably well in the Ising model, as seen in the figure: for $\Delta - \ell \lesssim 4$, there are 3 families of operators that are aligned with the values $\Delta_i - \ell_i = 2\Delta_\sigma$, $2\Delta_\epsilon$, and $2\Delta_\sigma + 2$. The equivalence between these families and operators of low scaling dimension has been established rigorously in the limit of infinite spin. We do not only know what is the asymptotic value as $\ell \rightarrow \infty$, but we also know the leading correction that is proportional to a power of the spin ℓ .

The reach of the conformal bootstrap is not limited to the Ising model. Our understanding of many conformal field theories has been significantly improved by the conformal bootstrap. This is particularly true of theories in 2 and 3 dimensions. In 3 dimensions, there is a generalization of the

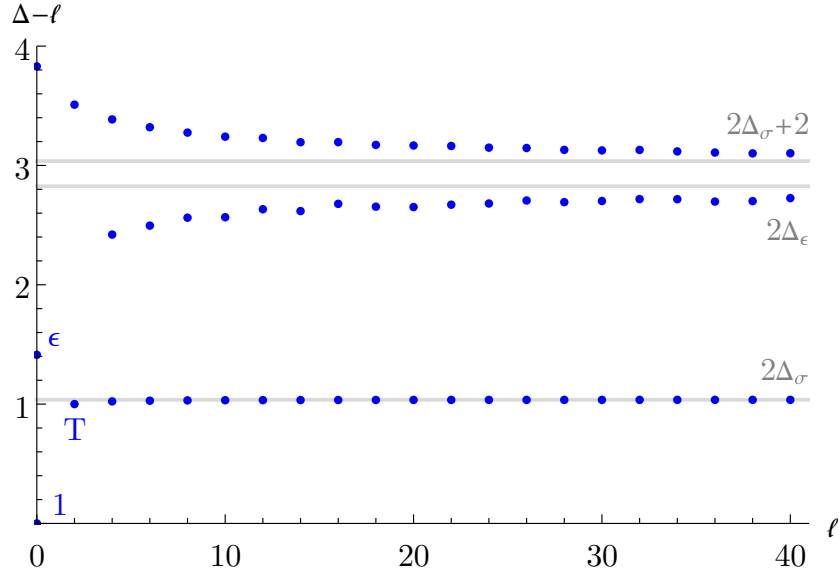


Figure 9: Spectrum of operators entering the OPE $\sigma \times \sigma$ in the 3-dimensional Ising model, for spins up to 40 and scaling dimensions of the order of the spin, based on data taken from ref. [23].

Lagrangian (6.33) to N scalar fields ϕ_a , with an interaction term of the form $(\phi_a \phi_a)^2$, which are called $O(N)$ models, which has led to interesting physics predictions from the bootstrap:

- The $O(2)$ model describes for instance the superfluid transition of liquid helium. There is an large unresolved discrepancy between the best measurement of this transition (an experiment performed in the space shuttle) and the best Monte-Carlo simulations. In this case, the prediction of the conformal bootstrap is consistent with the Monte-Carlo result [24].
- The $O(3)$ model is perhaps even more exciting, because it describes the critical behavior of Heisenberg magnets. These are statistical physics models in which the magnetization is isotropic: it can point in any direction in space, without preferred direction. However, in practice it seems very difficult to achieve the required level of isotropy on a lattice: the $O(3)$ symmetry tends to be spontaneously broken, for instance to its cubic subgroup (the finite group of symmetries of the cube). The interesting question here is whether the 4-index operator \mathcal{O}_{ijkl} that could trigger an RG flow from the $O(3)$ model to a CFT with cubic symmetry is relevant or not. Since its scaling dimension is accidentally

very close to 3, this is very hard to determine from lattice simulations. The conformal bootstrap has now established on rigorous grounds that the scaling dimension of this operator is below 3, and hence that the $O(3)$ naturally tends to be broken [25].

There are many more interesting results and exciting open questions related to the conformal bootstrap, and this is not the place to list them exhaustively (the list would not stay up-to-date for long). For a recent summary, see for instance the *Snowmass White Papers* on the numerical [26] and analytic bootstrap [27]. So far, for physics in $d = 3 + 1$ Minkowski space, generic bounds have remained relatively weak and featureless. The CFTs that are known to exist seem to all be gauge theories (e.g. various conformal windows for different gauge groups and matter content), and this is an additional difficulty since the information on the gauge group is encoded in a complicated way in the CFT data and cannot be directly used as for instance the global symmetry in the $O(N)$ models in 3 dimensions. But this should not be taken as a failure of the conformal bootstrap, but rather as a challenge for the next generation of physicists!

7 Going beyond

There are many topics in conformal field theory and beyond that have not been addressed in these lectures. To list a few:

- Conformal symmetry in two dimensions. We mentioned in section 2 that there are many more Killing vectors, and hence conformal generators when $d = 2$. In theories that have an energy-momentum tensor, this gives rise to the infinite-dimensional Virasoro algebra, which has been used to solve completely a class of theories called *minimal models*. One can also place a 2-dimensional CFT on a torus and study the modular properties of its partition function in a bootstrap setup. This is a vast topic that has its own literature.
- Superconformal symmetry [28]. When combined with supersymmetry, the conformal algebra also gets extended to a bigger and more rigid structure. All superconformal algebras have been classified and are restricted to dimensions $d \leq 6$.
- AdS/CFT and the holographic principle. Correlation functions of operators living at the boundary of AdS have the isometries of a conformal field theory, which means that most CFT techniques can be applied there. There are two quite distinct situations: On the one

hand, one can consider theories of quantum gravity that only approach AdS space-time asymptotically; these would be dual to a true CFT with an energy-momentum tensor, but the question is which CFT. On the other hand, one can place a quantum field theory in a fixed AdS background, and study the CFT correlators on its boundary; this CFT does not necessarily have an energy-momentum tensor, and it is in fact very similar to the generalized free field theory (or a large-N theory). It should also be noted that one can study the limit of infinite AdS radius in this latter approach to recover flat-space scattering amplitude. This observation is at the heart of the recently-revived S-matrix bootstrap techniques [29].

- Hamiltonian truncation [30]. This is a numerical technique that studies renormalization group flows using the CFT data as an input: the Hilbert space of a UV CFT is truncated with some cutoff, before a relevant deformation is introduced; the new Hamiltonian can then be diagonalized numerically in the truncated basis, and IR observables measured; iterating the process with a varying cutoff gives a sense of how the observables depend on the cutoff scale.

[Courses] are fantastically good for learning physics. The lecturer learns a lot of physics. After my first few studies, just about everything I learned about physics came from teaching it. I don't know if the students learned a lot, but I certainly did. So I consider teaching physics very important.

— Leonard Susskind [31]

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