Conformal Field Theory

Lecture notes

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Spring semester 2022

[Courses] are fantastically good for learning physics. The lecturer learns a lot of physics. After my first few studies, just about everything I learned about physics came from teaching it. I don't know if the students learned a lot, but I certainly did. So I consider teaching physics very important. — Leonard Susskind [1]

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1 Introduction

$$\text{conformal field theory} = \text{quantum field theory} + \text{conformal symmetry} \quad (1.1)$$

where Poincaré symmetry means translations and Lorentz transformations (rotations and boosts)

$$\text{conformal field theory} = \text{relativistic quantum field theory+scale+special conformal symmetry} \quad (1.2)$$

most reviews focus first on conformal symmetry as a whole "geometric" approach to CFT

for good reasons! but often makes it difficult to connect with standard understanding of relativistic quantum field theory that we have from particle physics

things that are hard to connect to: - spectral representation, particles (is CFT a theory of massless particles? no!) - UV/IR divergences and anomalies these are precisely subjects that will be covered in these lectures

peculiarity of these lecture notes: "algebraic" approach to CFT review non-perturbative approach of QFT, then add scale and special conformal symmetry; "anatomical" approach allows to understand what are the consequences of each of them

possible motivation: study scale transformation to get Wilsonian RG

why not local scale transformations that respect Poincare symmetry?

Jacobian of conformal symmetry is composition of dilatation and rotation (see Rychkov's Lorentzian)

massive ϕ^4 theory in $d = 3$; statistical Ising model; critical point of boiling water (and other liquids);

same critical exponents!

universality of IR behavior is a hint that the space of CFTs is sparse

unlike QFT, where you can always add/remove some operator in your favorite Lagrangian

both Euclidean and Lorentzian!!!

bootstrap philosophy: focus on CFT, not on microscopic details

in the framework of quantum field theory, push mathematical understanding of conformal symmetry to the limit, with amazing results

non-perturbative; in fact, another strong motivation to study CFT is that it gives a mathematically rigorous definition of a QFT (a special one, but anyway); no need for "quantization" of classical Lagrangian theory (in fact no classical limit!)

1.1 What is conformal field theory?

Introduction: What is CFT? Why is it useful. Examples of CFT. Where does it fit into modern theoretical physics.

in the last few years, CFT has been dominated by

string theory: 2d CFT

holography: geometric approach

condensed matter physics: Euclidean

here focus on the "old" quantum field theory approach

links with: lattice, perturbation theory, etc.

fun fact: the conformal bootstrap was invented by particle physicists

strongly-coupled QFT

no need for action principle

1.2 Examples of conformal field theories

- massless free theories
- perturbative fixed points: example ϕ^n theory in non-integer d
- some theories with extended supersymmetry, e.g. $\mathcal{N} = 4$ Yang-Mills; generally engineer matter content to make beta function zero at leading order is sufficient thanks to non-renormalization theorems
- Caswell-Banks-Zaks fixed points perturbative in some domains, not in other
- truly non-perturbative fixed points: example of Ising model universality class, leading to bootstrap
- AdS/CFT: any theory in space with one more dimension and anti de-Sitter space geometry

use beyond CFT: correlators that have the isometries of the conformal group gravity in AdS but also late-time correlators in de Sitter

1.3 Outline

originally covered in 14 periods of 45 minutes each

split into 7 chapters?

1.4 Literature

excellent reviews, in order of relevance for the present course:

modern reviews, by conformal bootstrap experts

- Slava Rychkov's EPFL lectures [2] also historical references
- David Simmons-Duffin's TASI lectures [3]
- Shai Chester's Weizmann Lectures [4]

other modern CFT reviews:

- state-of-the art [5]
- Hugh Osborn's course https://www.damtp.cam.ac.uk/user/ho/CFTNotes.pdf
- Joshua Qualls [6]

most Euclidean CFT, for Lorentzian perspective:

- Slava Rychkov's *Lorentzian methods in conformal field theory* https://courses.ipht.fr/node/226

focus on CFT in $d = 2$:

- Sylvain Ribault [7]
  also available on GitHub https://github.com/ribault/CFT-Review
- Schellekens "Conformal Field Theory" lecture notes [8]
  recent version available at https://www.nikhef.nl/~t58/CFT.pdf

also older literature on the subject in $d = 2$:

- Yellow book [9]
- Polchinski String theory vol. 1 [10]
- older lectures by Paul Ginsparg [11] emphasis on statistical physics and string theory applications

going further:

- superconformal symmetry: Lorenz Eberhardt [12]

2 Classical conformal transformations

One of the most fundamental principles of physics is independence of the reference frame: observers living at different points might have different perspectives, but the underlying physical laws are the same. This is true in space (invariance under translations and rotations), but also in space-time (e.g. invariance under Lorentz boosts).

2.1 Infinitesimal transformations

In mathematical language, this means that if we have a coordinate system x^μ, the laws of physics do not change under a transformations

$$x^\mu \to x'^\mu. \tag{2.1}$$

This principle applies to all maps that are invertible (isomorphisms) and differentiable (smooth transformations), hence it is usually called *diffeomorphism* invariance. Being differentiable, the transformation (2.1) can be Taylor-expanded to write

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x), \tag{2.2}$$

in terms of an infinitesimal vector ε^μ (meaning that we will always ignore terms of order ε^2).

In addition to the coordinate system, the description of a physical system requires a way of measuring distances that is provided by a metric $g_{\mu\nu}(x)$. Distances are measured integrating the line element

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{2.3}$$

Since all observers should agree on the measure of distances, we must have

$$g'_{\mu\nu}(x') dx'^\mu dx'^\nu = g_{\mu\nu}(x) dx^\mu dx^\nu, \tag{2.4}$$

Here $g_{\mu\nu}$ could be the Euclidean metric $\delta_{\mu\nu}$ or the Minkowski metric $\eta_{\mu\nu}$; for simplicity we only consider the case in which $g_{\mu\nu}$ is flat, i.e. $\partial_\alpha g_{\mu\nu} = 0$. In this case, we can write

$$\begin{aligned} g'_{\mu\nu} &= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \\ &= g_{\alpha\beta} \left(\delta^\alpha_\mu - \partial_\mu \varepsilon^\alpha\right) \left(\delta^\beta_\nu - \partial_\nu \varepsilon^\beta\right) \\ &= g_{\mu\nu} - \left(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\right). \end{aligned} \tag{2.5}$$

If we require the different observers to also agree on the metric, then we must have $g'_{\mu\nu} = g_{\mu\nu}$, which gives a constraint on what kind of coordinate transformations are possible: we must have

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 0. \tag{2.6}$$

This condition admits as a most general solution

$$\varepsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu, \tag{2.7}$$

where a^μ is a constant vector and $g_{\mu\rho}\omega^\rho{}_\nu = \omega_{\mu\nu}$ an antisymmetric tensor. The transformation

$$x^\mu \xrightarrow{P} x^\mu + a^\mu \tag{2.8}$$

is obviously a translation and

$$x^\mu \xrightarrow{M} \omega^\mu{}_\nu x^\nu \tag{2.9}$$

a rotation/Lorentz transformation around the origin $x = 0$. The composition of these two operations generates the Poincaré group. This is the fundamental symmetry of space-time underlying all relativistic quantum field theory. It is a symmetry of nature to a very good approximation, at least up to energy scales in which quantum gravity becomes important.

However, one can also consider the situation in which the two observers use different systems of units, i.e. they disagree on the overall definition of scale, but agree otherwise on the metric being flat. In this case we must have $g'_{\mu\nu} \propto g_{\mu\nu}$, and therefore the constraint becomes

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 2\lambda g_{\mu\nu}, \tag{2.10}$$

for some real number λ, with the most general solution

$$\varepsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu. \tag{2.11}$$

The new infinitesimal transformation is

$$x^\mu \xrightarrow{D} (1 + \lambda)x^\nu. \tag{2.12}$$

It is a scale transformation. Note that scale symmetry is not a good symmetry of nature: there is in fact a fundamental energy scale on which all observer must agree (this can be for instance chosen to be the mass of the electron). Nevertheless, there are some systems in which this is a very good approximate symmetry. One can also make very interesting *Gedankenexperimente* that have scale symmetry built in. These are reasons that make it worth studying.

If one pushes this logic further, in a scale-invariant world in which observers have no physical mean of agreeing on a fundamental scale, they might even decide to change their definition of scale as they walk around. This would correspond to the case in which the metric $g'_{\mu\nu}$ of one observer can differ from the original metric $g_{\mu\nu}$ by a function of space(-time):

$$g'_{\mu\nu}(x) = \Omega(x) g_{\mu\nu}. \tag{2.13}$$

Note that we are not saying that $g'_{\mu\nu}$ is completely arbitrary: at every point in space time it is related to the flat metric by a scale transformation. But the scale factor is different at every point. The condition on ε^μ becomes in this case

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = 2\sigma g_{\mu\nu}, \tag{2.14}$$

where $\Omega(x) = 1 + 2\sigma(x)$. To find the most general solution to this equation, note that contracting the indices with $g^{\mu\nu}$ gives

$$\partial_\mu \varepsilon^\mu = d\sigma, \tag{2.15}$$

where d is the space(-time) dimension, while acting with ∂^ν gives

$$\partial_\mu \partial_\nu \varepsilon^\nu + \partial^2 \varepsilon_\mu = 2\partial_\mu \sigma, \tag{2.16}$$

so that we get

$$\partial^2 \varepsilon_\mu = (2-d)\partial_\mu \sigma. \tag{2.17}$$

Acting once again with ∂^μ, we arrive at

$$(d-1)\partial^2 \sigma = 0, \tag{2.18}$$

while acting with ∂^ν and symmetrizing the indices, we find

$$(2-d)\partial_\mu \partial_\nu \sigma = g_{\mu\nu} \partial^2 \sigma = 0. \tag{2.19}$$

In $d > 2$, we obtain therefore the condition $\partial_\mu \partial_\nu \sigma = 0$, which is solved by

$$\sigma(x) = \lambda + 2b \cdot x. \tag{2.20}$$

We have therefore

$$\varepsilon^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + 2(b \cdot x) x^\mu - x^2 b^\mu. \tag{2.21}$$

In addition to the transformations found before, we also find

$$x^\mu \xrightarrow{K} 2(b \cdot x) x^\mu - x^2 b^\mu, \tag{2.22}$$

which is a *special conformal transformation*. If we examine the Jacobian for this transformation, we find

$$\frac{\partial x'^\mu}{\partial x^\nu} = (1 + 2b\cdot x)\,\delta^\mu_\nu + 2\,(b_\nu x^\mu - x_\nu b^\mu) \approx (1 + 2b\cdot x)\,R^\mu{}_\nu(x). \tag{2.23}$$

We have written this as a position-dependent scale factor $(1 + 2b\cdot x)$, multiplying an orthogonal matrix

$$R^\mu{}_\nu(x) = \delta^\mu_\nu + 2\,(b_\nu x^\mu - x_\nu b^\mu)\,. \tag{2.24}$$

This shows that special conformal transformation act locally as the composition of a scale transformation and a rotation (or Lorentz transformation). This also shows that conformal transformations preserve angles, which is the origin of their name. Eq. (2.14) is sometimes called the (conformal) Killing equation and its solutions (2.21) the Killing vectors.

Note that in our derivation the original metric $g_{\mu\nu}$ was flat, but the new metric $g'_{\mu\nu}$ is not. It is however conformally flat: it is always possible to make a change of coordinate after which it is flat. In general, transformations

$$g_{\mu\nu}(x) \to \Omega(x)^2 g_{\mu\nu}(x) \tag{2.25}$$

are called *Weyl transformations*. They change the geometry of space-time. We found that any Weyl transformation which is at most quadratic in x can be compensated by a change of coordinates to go back to flat space. The corresponding flat-space symmetry is called conformal symmetry.[1]

Note that in $d = 2$ the situation is a bit different: the conditions $\partial^2\sigma = 0$ is sufficient to ensure that the Killing equation has a solution. This is most easily seen in light-cone coordinates,

$$x^+ = \frac{x^0 + x^1}{2}, \qquad x^- = \frac{x^0 - x^1}{2}, \tag{2.26}$$

in terms of which

$$\partial^2\sigma = \partial_+\partial_-\sigma. \tag{2.27}$$

This is satisfied by taking for σ the sum of an arbitrary function of the left-moving variable x^+ and of another function of the right-moving variable x^-. In fact, if we write $\varepsilon^\pm = \varepsilon^0 \pm \varepsilon^1$, we can take arbitrary functions $\varepsilon^+(x^+)$ and $\varepsilon^-(x^-)$, and verify that eq. (2.14) is satisfied with $\sigma = \frac{1}{2}\,(\partial_+\varepsilon_+ + \partial_-\varepsilon_-)$. In the Euclidean case we take

$$z = \frac{x^1 + ix^2}{2}, \qquad \bar{z} = \frac{x^1 - ix^2}{2}, \tag{2.28}$$

[1] This implies that the group of conformal transformation is a subgroup of diffeomorphisms. It is in fact the largest *finite-dimensional* subgroup.

complex-conjugate to each other, and the same logic follows: we can apply arbitrary holomorphic and anti-holomorphic transformations on z and $\bar{z}$, and the conformal Killing equation is always satisfied. This shows that there are infinitely more conformal transformations in $d = 2$ than in $d > 2$, and also that there is no significant difference between two-dimensional Euclidean and Minkowski conformal symmetry, as the symmetry acts essentially on the two light-cone/holomorphic coordinates independently.

2.2 The conformal algebra

The conformal Killing equation (2.21) determines the most general form of *infinitesimal* conformal transformations. *Finite* conformal transformations follow from a sequence of infinitesimal transformations. However, one has to bear in mind that the infinitesimal conformal transformations do not all commute: for instance, a translation followed by a rotation is not the same as the opposite. In fact, the conformal transformations form a *group*: the composition of conformal transformations is again a conformal transformations.

As we all know from quantum field theory, a group is characterized by its *generators* and their commutation relations (the *algebra*). A generator G describes an infinitesimal transformation in some direction, and finite transformation are obtained using exponentiation, $e^{i\theta G}$, with parameter θ (the factor of i is the physicist's convention that make the generators Hermitian). A representation of the conformal group is given on the functions of the coordinates, $f(x)$. For instance, under an infinitesimal translation, we have

$$f(x) \xrightarrow{P} f(x') = f(x+a) \approx f(x) + a^\mu \partial_\mu f(x) \tag{2.29}$$

and we require this to be equal to $e^{-ia_\mu P^\mu} f(x)$, which means

$$P_\mu = i\partial_\mu. \tag{2.30}$$

Performing the same analysis for the other infinitesimal transformations given in eq. (2.21), we obtain for the other generators[2]

rotations/Lorentz transformations:	$M^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)$	(2.31)
dilatations:	$D = ix^\mu \partial_\mu,$	(2.32)
special conformal transformations:	$K^\mu = i\left(2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu\right).$	(2.33)

[2]The sign of these generators is an arbitrary convention. It defines once and for all the commutations relations that we will derive next. After that, we will always refer to the commutation relations as the definition of the generators.

The number of generators matches that of the Killing vectors: there are d translations, d special conformal transformations, $d(d-1)/2$ rotations/Lorentz transformations ($M^{\mu\nu}$ is a $d\times d$ antisymmetric matrix), and one scale transformation. Therefore the total number of generators, i.e. the dimension of this group, is $(d+1)(d+2)/2$. In $d=4$ space-time dimension, the conformal group has 15 generators.

Using the above definition, one can verify that the following commutation relations are satisfied,

$$\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= -i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}\right] \\
[M^{\mu\nu}, P^{\rho}] &= -i\left(g^{\mu\rho}P^{\nu} - g^{\nu\rho}P^{\mu}\right) \\
[M^{\mu\nu}, K^{\rho}] &= -i\left(g^{\mu\rho}K^{\nu} - g^{\nu\rho}K^{\mu}\right) \\
[D, P^{\mu}] &= -iP^{\mu} \\
[D, K^{\mu}] &= iK^{\mu} \\
[P^{\mu}, K^{\nu}] &= 2i\left(g^{\mu\nu}D - M^{\mu\nu}\right)
\end{aligned} \tag{2.34}$$

while all other commutators vanish:

$$[M^{\mu\nu}, D] = [P^{\mu}, P^{\nu}] = [K^{\mu}, K^{\nu}] = 0. \tag{2.35}$$

The first two relations in eq. (2.34) are the familiar Poincaré algebra. The next one states that K^{μ} transforms like a vector (as P^{μ} does), whereas D is obviously a scalar. The next two relations remind us that K^{μ} and P^{μ} have respectively the dimension of length and inverse length.

Exercise 2.1 Derive the commutation relations from the action (2.30)–(2.33) of the generators on functions of x.

Even though this is not immediately obvious, this algebra is isomorphic to that of the group SO$(d+1,1)$ (if $g^{\mu\nu}$ is the Euclidean metric) or SO$(d,2)$ (if it is the Minkowski metric). To see that it is the case, let us introduce a $(d+2)$-dimensional space with coordinates

$$X^{\mu}, \quad X^{d+1}, \quad X^{d+2}, \tag{2.36}$$

and a metric defined by the line element

$$ds^2 = g_{\mu\nu}dX^{\mu}dX^{\nu} + dX^{d+1}dX^{d+1} - dX^{d+2}dX^{d+2} \equiv \eta_{MN}dX^{M}dX^{N}. \tag{2.37}$$

Then we write all conformal commutation relations as being defined by the Lorentzian algebra

$$\left[J^{MN}, J^{RS}\right] = -i\left(\eta^{MR}J^{NS} - \eta^{MS}J^{NR} - \eta^{NR}J^{MS} + \eta^{NS}J^{MR}\right], \tag{2.38}$$

provided that we identify the antisymmetric generators J^{MN} with the conformal generators as follows:

$$\begin{aligned} M^{\mu\nu} &= J^{\mu\nu}, \\ P^{\mu} &= J^{\mu,d+1} + J^{\mu,d+2}, \\ K^{\mu} &= J^{\mu,d+1} - J^{\mu,d+2}, \\ D &= J^{d+1,d+2}. \end{aligned} \tag{2.39}$$

2.3 Finite transformations

We just saw that the infinitesimal conformal transformations generate a group. But how can we describe finite conformal transformations? Let us see how each generator exponentiates into an element of the group; the most general conformal transformation can then be obtained as a composition of such finite transformations.

In some cases the exponentiation is trivial. For instance, with translations we obtain immediately

$$x^{\mu} \xrightarrow{P} x^{\mu} + a^{\mu}, \tag{2.40}$$

where a is now any d-dimensional vector, not necessarily small. The same is true of scale transformations,

$$x^{\mu} \xrightarrow{D} \lambda x^{\mu} \tag{2.41}$$

with finite λ. Rotations or Lorentz transformations exponentiate as

$$x^{\mu} \xrightarrow{M} \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{2.42}$$

where $\Lambda^{\mu}{}_{\nu}$ is a SO(d) or SO$(1, d-1)$ matrix, depending whether the metric is Euclidean or Minkowski. All of this is well-known and not surprising.

On the contrary, special conformal transformations do not exponentiate trivially. The easiest way to derive their finite form is to make the following observation: recall that in infinitesimal form we have

$$x'^{\mu} = x^{\mu} + 2(b \cdot x)x^{\mu} - x^2 b^{\mu}, \tag{2.43}$$

which implies $x'^2 = (1 + 2b \cdot x)\, x^2$, and therefore (as always neglecting terms of order b^2)

$$\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - b^{\mu}. \tag{2.44}$$

The ratio x^{μ}/x^2 appearing on both side of the equation is the *inverse* of the coordinate x^{μ}, respectively x'^{μ}: let us define the inversion as

$$x^{\mu} \xrightarrow{I} \frac{x^{\mu}}{x^2}. \tag{2.45}$$

This transformation does not have an infinitesimal form, but otherwise it shares the essential properties of a conformal transformation: its Jacobian is

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \frac{1}{x^2}\left[\delta^{\mu}_{\nu} - 2\frac{x^{\mu}x_{\nu}}{x^2}\right], \tag{2.46}$$

which is the product of a position-dependent scale factor x^{-2} with an orthogonal matrix. To understand what this transformation does globally, let us consider a point $\vec{x} = (x, 0, \dots 0) \in \mathbb{R}^d$. Then the matrix is square bracket is diagonal, and equates $\mathrm{diag}(-1, 1, \dots, 1)$. This is an orthogonal matrix with determinant -1, which is part of $\mathrm{O}(d)$ but not $\mathrm{SO}(d)$. This shows that the inversion is a discrete transformations not connected to identity. A conformally invariant theory might be invariant under inversions, but it needs not be.

Eq. (2.44) shows that infinitesimal special conformal transformations are obtained taking an inversion followed by a translation, followed by an inversion again. Since this process involves the inversion twice, and since inversion is its own inverse, it does not matter whether inversion is a true symmetry of the system or not. The advantage of this representation is that it can easily be exponentiated: the composition of (infinitely) many infinitesimal special conformal transformation can be written as an inversion followed by a finite translation, followed by an inversion again. In other words, eq. (2.44) holds for finite b^{μ}. This can be used to show that

$$x^{\mu} \xrightarrow{K} x'^{\mu} = \frac{x^{\mu} - x^2 b^{\mu}}{1 - 2b\cdot x + b^2 x^2}. \tag{2.47}$$

Exercise 2.2 Use eq. (2.44) to show (2.47).
Hint: Contract both sides of eq. (2.44) with x_{μ}, x'_{μ} and b_{μ}, and use the three equations that you get to solve for x'^2.

What do special transformation do globally? Let us look specifically in Euclidean space. There are some special points:

- The origin of the coordinate system $x = 0$ is mapped onto itself.
- The point b^{μ}/b^2 is mapped to ∞.
- Conversely, the "point" $x \to \infty$ is mapped to the finite vector $-b^{\mu}/b^2$.

These properties can be understood from the fact that special conformal transformations and translations are related by inversion: special conformal transformations keep the origin fixed but move every other points, including ∞; translations move every point *except* ∞. The other two transformations, rotations and scale transformations, keep both 0 and ∞ fixed.

An essential property of conformal transformations is that they allow to map any 3 points (x_1, x_2, x_3) onto another set (x'_1, x'_2, x'_3). This can be seen as follows: first, apply a translation to place x_1 at the origin, followed by a special conformal transformation that takes x_3 to ∞, after which the image of the original triplet is $(0, x''_2, \infty)$; then use rotations and scale transformations to move x''_2 to another point x'''_2, while keeping 0 and ∞ fixed; finally apply again a special conformal transformation that takes ∞ to $x'_3 - x'_1$, and a translation by x'^1 to reach the configuration (x'_1, x'_2, x'_3). This property has an immediate physical consequence: in correlation functions of 2 or 3-points (see next sections for a definition), all kinematics is fixed by conformal symmetry. The only freedom encodes information about the operators themselves, not about their position in space.

Another interesting property of conformal transformations is that they map spheres to spheres: this is is an obvious property of translations, rotations and scale transformations, but it is also true of special conformal transformations.

Exercise 2.3 Show that under the special conformal transformation (2.47), a sphere centered at the point a^μ and with radius R gets mapped to a sphere centered at the point

$$a'^\mu = \frac{a^\mu - (a^2 - R^2)b^\mu}{1 - 2a \cdot b + (a^2 - R^2)b^2}$$

and with radius

$$R' = \frac{R}{|1 - 2a \cdot b + (a^2 - R^2)b^2|}.$$

In the special case in which b^μ/b^2 is on the original sphere, show that the sphere gets mapped to a plane orthogonal to the vector $a^\mu + (R^2 - a^2)b^\mu$.

Note that in $d = 2$, the additional conformal transformations (infinitely many of them!) mean that (nearly) any shape can be mapped onto another. This is known as the Riemann mapping theorem.[3]

[3] A neat example of how a disk is be mapped onto a polygon is available at https://herbert-mueller.info/uploads/3/5/2/3/35235984/circletopolygon.pdf.

2.4 Compactifications

It is often said that conformal symmetry is a symmetry of flat space(-time). This is true, as we have just seen, provided that we treat the point ∞ as being part of the space. This is quite straightforward in Euclidean space, but much more subtle in Minkowski space-time, as there are different ways of reaching ∞ there.

To gain a better understanding of this, it can be useful to map the flat Euclidean space $\mathbb{R}^d$ or the Minkowski space-time $\mathbb{R}^{1,d-1}$ onto a curved (but hopefully compact) manifold. In Euclidean space this is for instance achieved by the (inverse) stereographic projection that maps $\mathbb{R}^d \cup \{\infty\}$ to the unit sphere S^d. Geometrically, the stereographic projection is constructed as follows: embed $\mathbb{R}^d$ as a (hyper-)plane in $\mathbb{R}^{d+1}$, together with a sphere of unit radius centered at the origin. Every point on the plane has an image on the sphere obtained drawing a segment between the original point and the north pole of the sphere, and noting where it intersects the sphere. The origin is mapped to the south pole, ∞ to the north pole, and the sphere S^{d-1} of unit radius to the equator. Algebraically, this is achieved as follows: first write the Euclidean metric in spherical coordinates,

$$ds^2 = dr^2 + r^2 d\Omega_{d-1}^2, \tag{2.48}$$

where the solid angle is given in $d = 2$ by $d\Omega_1^2 = d\phi^2$, in $d = 3$ by $d\Omega_2^2 = d\theta^2 + \sin\theta^2 d\phi^2$, and more generically by the recursion relation $d\Omega_n^2 = d\theta^2 + \sin\theta^2 d\Omega_{n-1}^2$. Let us perform the change of variable

$$r = \frac{\sin\varphi}{1-\cos\varphi}, \tag{2.49}$$

and interpret $\varphi \in [0, \pi]$ as the zenith angle on the sphere: $\varphi = 0$ is the north pole, corresponding to $r \to \infty$, and $\varphi = \pi$ the south pole, corresponding to $r = 0$. In these coordinates we have

$$ds^2 = \frac{1}{(1-\cos\varphi)^2}\left(d\varphi^2 + \sin\varphi^2 d\Omega_{d-1}^2\right) = \frac{1}{(1-\cos\varphi)^2} d\Omega_d^2. \tag{2.50}$$

This shows that the metric is flat up to an overall Weyl factor that depends on φ. In these new coordinates, conformal transformations are always non-singular. For this reason, it is often very convenient to study *classical* conformal symmetry on the sphere S^d instead of Euclidean space.

However, this compactification is not as nice in a *quantum* theory in which one wants to foliate the space along some direction: if one chooses φ as the "Euclidean time", then the "space" direction is a sphere S^{d-1} whose volume depends on φ. In other words, the generator of "time" translations is not a symmetry of the system. Being on a perfectly symmetric sphere, this is also true of any other choice of time direction. This is not forbidden, but it complicates vastly the analysis.

Instead, another compactification is often preferred to the sphere: from the Euclidean metric in spherical coordinates, one can make the change of variable

$$\tau = \log(r) \qquad \Rightarrow \qquad r = e^{\tau}, \tag{2.51}$$

after which

$$ds^2 = r^2 \left(d\tau^2 + d\Omega_{d-1}^2\right). \tag{2.52}$$

This is again a flat metric, up to a Weyl factor $r^2 = e^{2\tau}$. In this case however the metric is completely independent of τ. This space has the geometry of the *cylinder*, $\mathbb{R} \times S^{d-1}$. It is not fully compact: τ goes from $-\infty$ to $+\infty$. But it has a important advantage: translations in τ are generated by dilatations D, which will be taken to be a symmetry of quantum field theory. Foliating the space into surfaces of constant τ will later lead us to the famous radial quantization in conformal field theory.

Note that in $\mathrm{SO}(d+1,1)$ language, the generator $D = J^{d+1,d+2}$ is completely equivalent to any other generator $J^{\mu,d+2} = \frac{1}{2}\left(P^\mu - K^\mu\right)$. So for instance we might as well look for a cylinder compactification in which the non-compact direction corresponds to translations in the direction of $\frac{1}{2}\left(P^0 - K^0\right)$. This generator obeys

$$\frac{1}{2}\left(P^0 - K^0\right) = i\left(\frac{1-(x^0)^2+\vec{x}^2}{2}\partial_0 - x^0\,\vec{x}\cdot\vec{\partial}\right), \tag{2.53}$$

with two fixed points at $x^0 = \pm 1$ with $\vec{x} = 0$.

Exercise 2.4 Find the change of coordinate that make the Euclidean metric Weyl-equivalent to a cylinder in which translations in the non-compact direction are generated by $\frac{1}{2}\left(P^0 - K^0\right)$.
Hint: Find a special conformal transformation followed by a translation that take $(0,\infty)$ *to* $(-1,1)$*, and apply it to the cylinder coordinates.*

The cylinder compactifications of Euclidean space are interesting by themselves, but they are also extremely convenient to understand the connection between Euclidean and Minkowski space-times: in this last form, performing a Wick rotation $\tau \to \pm it$ defines a Lorentzian cylinder on which the conformal group with a simple action of the Lorentzian conformal group $\mathrm{SO}(d,2)$. But before going there, let us go back to flat Minkowski space-time and make some general remarks.

2.5 Minkowski space-time

Everybody is familiar with translations and Lorentz transformations in Minkowski space-time, and even with dilatations as this is a standard tool in the renormalization group analysis. But what do conformal transformations do?

To understand this, let us place an observer at the origin of Minkowski space-time. The presence of this observer breaks translations, but not Lorentz transformations (the observer is point-like), nor dilatations and special conformal transformations. For the observer, space-time is split into three regions: a future light cone, a past light cone, and a space-like region from which they know nothing. Lorentz and scale transformation preserve this causal structure: the future and past light-cones are mapped onto themselves. In other words, if a point x is space-like separated from the observed, it will remain space-like separated no matter the choice of scale and Lorentz frame. Without loss of generality, let us choose this point to be at position $x = (0, \vec{n})$, where $\vec{n}$ is a unit vector (units can be chosen so that this is the case). Now apply a special conformal transformation with parameter $b^\mu = (-\alpha, \alpha\vec{n})$, with α varying between 0 and 1. This draws a curve y^μ in space-time, parameterized by α, with

$$y^0(\alpha) = \frac{\alpha}{1-2\alpha}, \qquad \vec{y}(\alpha) = \frac{1-\alpha}{1-2\alpha}\vec{n}. \tag{2.54}$$

This curves begins at the space-like point $y = (0, \vec{n})$, and ends in the past light cone at $y = (-1, \vec{0})$! Note that y never crosses a light cone: the image of x is never null if x is not itself null, since $x'^2 = x^2/(1 - 2x \cdot b + x^2 b^2)$. Instead, we have

$$y^2(\alpha) = \frac{1}{1-2\alpha} \neq 0. \tag{2.55}$$

The point goes all the way to space-like infinity when $\alpha = \frac{1}{2}$, and comes back from past infinity. Clearly, special conformal transformation break causality.

The resolution of this paradox is that conformal transformation do not act directly on Minkowski space, but rather on its universal cover that is isomorphic to the Lorentzian cylinder described in the previous section. Time evolution on that cylinder is given by the Hamiltonian $H = \frac{1}{2}(P^0 - K^0)$. At any given time t_0, space is compactified in such a way that the notion of infinite distance is unequivocal: space-like infinity corresponds to a point on the sphere. If one takes any other point of that sphere and apply finite translations using the generator P^μ, then this defines a compact *Poincaré patch*. The full cylinder is a patchwork of Minkowski space-times, but every observer only has access to one.

The lesson that we need to learn from this is that only the infinitesimal form of special conformal transformation can be used in Minkowski space-time: *any* finite special conformal transformation would bring part of space-time into another patch on the cylinder. This is sometimes called *weak conformal invariance*.

The fact that a quantum field theory defined on Minkowski space-time with *weak conformal invariance* uniquely defines a quantum field theory on the Lorentzian cylinder with global conformal invariance is quite non-trivial. The proof was given by Martin Lüscher and Gerhard Mack in 1974 [13], both employed at the University of Bern at the time. In their own words:

> *In picturesque language, [the superworld] consists of Minkowski space, infinitely many "spheres of heaven" stacked above it and infinitely many "circles of hell" below it.*

2.6 The energy-momentum tensor

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