Conformal Field Theory

Lecture notes

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Spring semester 2022

[Courses] are fantastically good for learning physics. The lecturer learns a lot of physics. After my first few studies, just about everything I learned about physics came from teaching it. I don't know if the students learned a lot, but I certainly did. So I consider teaching physics very important. — Leonard Susskind

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1 Introduction

$$\text{conformal field theory} = \text{quantum field theory} + \text{conformal symmetry} \quad (1.1)$$

where Poincaré symmetry means translations and Lorentz transformations (rotations and boosts)

$$\text{conformal field theory} = \text{relativistic quantum field theory} + \text{scale} + \text{special conformal symmetry} \quad (1.2)$$

most reviews focus first on conformal symmetry as a whole “geometric” approach to CFT

for good reasons! but often makes it difficult to connect with standard understanding of relativistic quantum field theory that we have from particle physics

things that are hard to connect to: - spectral representation, particles (is CFT a theory of massless particles? no!) - UV/IR divergences and anomalies these are precisely subjects that will be covered in these lectures

peculiarity of these lecture notes: “algebraic” approach to CFT review non-perturbative approach of QFT, then add scale and special conformal symmetry; “anatomical” approach allows to understand what are the consequences of each of them

1.1 What is conformal field theory?

Introduction: What is CFT? Why is it useful. Examples of CFT. Where does it fit into modern theoretical physics.

in the last few years, CFT has been dominated by

string theory: 2d CFT

holography: geometric approach

condensed matter physics: Euclidean

here focus on the “old” quantum field theory approach

links with: lattice, perturbation theory, etc.

fun fact: the conformal bootstrap was invented by particle physicists

strongly-coupled QFT

no need for action principle

1.2 Examples of conformal field theories

perturbative examples: ϕ^n theory in non-integer d

Caswell-Banks-Zaks

superconformal: $\mathcal{N} = 4$

any IR fixed point (maybe trivial)

use beyond CFT: correlators that have the isometries of the conformal group gravity in AdS but also late-time correlators in de Sitter

1.3 Outline

originally covered in 14 periods of 45 minutes each

split into 7 chapters?

2 Classical conformal symmetry

Poincaré symmetry: physics is the same in every coordinate frame

linear transformation of the coordinates

$$x'^{\mu} = x^{\mu} + \omega^{[\mu\nu]}x_{\nu} + a^{\mu} \tag{2.1}$$

infinitesimal line element $dx^2 = \eta_{\mu\nu}dx^{\mu}dx^{\nu}$ is invariant:

$$dx'^2 = dx^2 \tag{2.2}$$

scale symmetry:

$$x'^{\mu} = \lambda x^{\mu} \tag{2.3}$$

note that this can be seen as a rescaling of the metric, of the form $\eta'^{\mu\nu} = \#\eta^{\mu\nu}$

can we generalize this? general transformations of the metric $\eta'^{\mu\nu} = \Omega(x)\eta^{\mu\nu}$ called a *Weyl transformation*

in general changes geometry of space-time; unless this transformation is equivalent to a coordinate transformation (diffeomorphism)

in infinitesimal form

$$x'^{\mu} = x^{\mu} + v^{\mu}(x) \tag{2.4}$$

requiring that this is equal to the Weyl transformation gives the conformal Killing equation:

$$\partial^{\mu}v^{\nu}(x) + \partial^{\nu}v^{\mu}(x) = 2\sigma(x)\eta^{\mu\nu} \tag{2.5}$$

work out solution in $d = 1$, $d = 2$ and $d > 2$;

conformal Killing vectors (see Osborn's notes): translation Lorentz transformations dilatation special conformal symmetry

in $d = 2$, use z, $\bar{z}$ notation and Laurent series

finite transformations

use inversions: discrete transformations not connected to identity but inversion followed by translation followed by inversion gives special conformal transformations

group of conformal transformation is a subgroup of diffeomorphisms; in fact the largest *finite-dimensional* subgroup

defining property

$$g'_{\mu\nu}(x') = \Omega(x)^2 \eta_{\mu\nu} \tag{2.6}$$

gives Jacobian of conformal transformation

$$\frac{\partial x^\mu}{\partial x^\nu} = \Omega(x) M^\mu{}_\nu(x) \tag{2.7}$$

M: $\mathrm{SO}(d-1,1)$ matrix

conformal transformations look locally like Lorentz and scale transformations

2.1 The conformal algebra

generators corresponding to Killing vectors

compute commutators, find Lorentz and Poincaré subalgebras, and new commutations involving D and K

isomorphic to $\mathrm{SO}(d,2)$

(exercise)

note: everything we said is also valid in Euclidean space: replace $\eta^{\mu\nu} \to \delta^{\mu\nu}$

get the Euclidean conformal group

2.2 Examples

conformal transformations map circles to circles (including circles with infinite radius, aka straight lines): explanation

picture

think of it as renormalization group transformation that depends on space-time

Escher?

3 Non-perturbative quantum field theory

Wightman axioms

operators are not necessarily invariant under conformal symmetry, but they transform unitarily

unitary representations on Hilbert space (e.g. pair of point-like particle

no action principle (i.e. no need for "quantization")

only deal with **local** operators

warning about meaning of locality: related to causality?

generator of translations plays a special role: energy and momentum vector

positivity of energy

take P^0 as the Hamiltonian

space-time is foliated by surfaces of equal time

each equal-time surface has a Hilbert space; evolution between different surfaces at times t_1 and t_2 is given by the unitary evolution operator $U = e^{\pm i(t_1-t_2)P^0}$

because of time-translation symmetry, the Hilbert space is the same on each surface!

states characterized by energy and momentum: $P^\mu |p\rangle = p^\mu |p\rangle$

there is a unique vacuum state: $|0\rangle$

Wigner construction: use reference momentum

generator of Lorentz transformation can be diagonalize at a point: the origin of space-time

get the action of $M^{\mu\nu}$ at arbitrary point

3.1 Spectral representation

operator vs. field

3.2 Scale symmetry

action of generator of translations

no mass (often said)

or better said: all masses!

reference vector, then construct spectral density

3.3 Special conformal symmetry

special conformal generator implies two things:

primary and descendant operators

special condition on 2-point function of primary operators: that they have identical scaling dimensions

note that scale invariance and unitarity generally imply that the trace of the energy-momentum tensor is zero: then Hamiltonian is invariant under change of the metric $\delta g_{\mu\nu} = \sigma(x)\eta_{\mu\nu}$,

$$\delta H = \int d^d x\, T_{\mu\nu}\delta g^{\mu\nu} = \int d^x\, \sigma(x) T^\mu_\mu = 0 \tag{3.1}$$

invariance under infinitesimal Weyl transformations

note that $T_{\mu\nu}$ is only defined in flat space, or rather that it acquires a vacuum expectation value in curved space: the Weyl anomaly so let us only consider Weyl transformations that do not change the geometry of space-time: those that are equivalent to a coordinate transformation (a diffeomorphism)

exercise: compute conformal generators in momentum-space representation

Mack's classification in 4d

representation: long and short multiplets!

compare this with Slava's approach:

organize operators into representation of conformal group:

call some operators *primary*

operators are local, which mean that they do not feel the effect of the transformation of the metric, or only through coordinate dependence: $\phi(x) \to \phi(x') = \Omega(x)^{-\Delta}\phi(x)$

note: if ϕ transforms like this, then $\partial_\mu\phi(x)$ does not! call it a *descendant*

comment on scale vs. conformal invariance

correlation functions:

no need for action principle

3.4 Exercises

show that the spectral density in the case $\Delta = \frac{d}{2} - 1$ is that of a free scalar field;

show also that in the case $\Delta = \frac{d}{2}$ it is given by the phase space of two massless particle states (see Rychkov eq. 1.13)

3.5 Bibliography

itzikson zuber

4 Conformal correlation functions

2-pt function: go from Wightman 2-pt function in momentum space to Euclidean 2-pt function in position space in two ways:

- Fourier transform first, then analytic continuation of Wightman function - first construct T product (Källen-Lehmann representation), then Wick rotation to Euclidean momentum space, then Fourier transform

T-products vs. Wightman functions

use retarded products and micro-causality

Ward identities

more identities for conserved currents and the energy-momentum tensor (there cannot be higher-spin conserved currents otherwise the theory must be free; reference?)

4.1 Spectral representation for spinning operators

unitary bounds

4.2 From momentum to position

taking fourier transforms

4.3 From Minkowski to Eulidean space

Osterwalder-Schrader

analytic continuation of Wightman functions vs. Wick rotation of T-products

reflection positivity

note that this is usually done in the other direction!

4.4 Embedding-space formalism

conformal group acts linearly in embedding space with $d+2$ dimensions, among which two times

$X'^M = \Lambda^M_N X^N$

get rid of two dimensions by restricting to light-cone $X^2 = 0$, and identifying $X^+ \sim \lambda X^+$

can take $X^+ = 1$, so that

$$(X^+, X^-, X^\mu) = (1, x^2, x^\mu) \tag{4.1}$$

fields defined on the cone, depending homogeneously on X

rewrite 2-point function

see section 2.1 of Rychkov

4.5 3-point functions

scalar 3-point function

remarkable fact! fixed up to a single coefficient

if only scale invariance, this could be arbitrarily complicated

historical note (cite Rychkov, around eq. 2.43): birth of conformal field theory

check: 2-point functions involving conserved currents and energy-momentum tensors are automatically conserved; for 3-point functions this gives additional constraint: scaling dimensions of operators must be equal

4.6 4-point functions

invariant cross-ratios for 4-point functions

5 State-operator correspondence and OPE

important remark on OPE convergence in Euclidean space!

6 The conformal bootstrap

6.1 Conformal blocks

6.2 The numerical boostrap

generalized free fields/Gaussian

6.3 Results

discussion about universality: see Poland, Rychkov, Vichi

7 Selected advanced topics

light-cone limit and ?

Virasoro symmetry in 2d:

conformal anomalies: anomalies are contact term in the action for the source

superconformal bootstrap? note classification of all superconformal algebras

7.1 How to continue from here

- holography: see Joao's TASI lectures
- superconformal: Tajikawa pedestrian lectures
- bootstrap: Shai Chester's lectures
- condensed matter: note that unitarity is unnecessary: many non-unitary fixed points, yet they have conformal invariance (why?)