The free scalar theory in 2 dimensions

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Let us consider the conformal field theory defined by the action

$$S = \int d^2x \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right] = \int dz d\bar{z}\, \partial_z \phi \partial_{\bar{z}} \phi. \tag{1}$$

This theory is invariant under the shift symmetry $\phi \to \phi + c$, and therefore ϕ should be thought of as a gauge field, with the equivalence $\phi \sim \phi + c$.[1] Gauge-invariant operators can be constructed from the elementary building blocks $\partial_z \phi \equiv \partial \phi$ and $\partial_{\bar{z}} \phi \equiv \overline{\partial} \phi$. By the equation of motion $\partial_z \partial_{\bar{z}} \phi = 0$, one can see that $\partial \phi$ is independent of $\bar{z}$ and $\overline{\partial} \phi$ independent of z. In other words, there are (anti-)holomorphic operators. $\partial \phi$ and $\overline{\partial} \phi$ are also primary operators with conformal weights $(h, \bar{h}) = (1, 0)$ and $(0, 1)$ respectively, and their 2-point functions obey therefore

$$\left\langle 0 \middle| \partial \phi(z_1) \partial \phi(z_2) \middle| 0 \right\rangle = \frac{1}{(z_1 - z_2 + i\epsilon)^2}, \qquad \left\langle 0 \middle| \overline{\partial} \phi(\bar{z}_1) \overline{\partial} \phi(\bar{z}_2) \middle| 0 \right\rangle = \frac{1}{(\bar{z}_1 - \bar{z}_2 - i\epsilon)^2}. \tag{2}$$

Other primary operators of the theory can be constructed as products of $\partial \phi$ and $\overline{\partial} \phi$ and their derivatives (modulo descendants). Of particular interest to us will be the (anti-)holomorphic operators

$$T = \tfrac{1}{\sqrt{2}} (\partial \phi)^2, \qquad \overline{T} = \tfrac{1}{\sqrt{2}} (\overline{\partial} \phi)^2, \tag{3}$$

with conformal weights $(2, 0)$ and $(0, 2)$, as well as the two operators

$$\mathcal{L} = \partial \phi \overline{\partial} \phi, \qquad T\overline{T} = \tfrac{1}{2} (\partial \phi)^2 (\overline{\partial} \phi)^2, \tag{4}$$

with conformal weights $(1, 1)$ and $(2, 2)$ respectively.

1 The correlator $\langle \partial \phi \partial \phi \partial \phi \partial \phi \rangle$

The simplest 4-point function that one might consider is

$$\left\langle 0 \middle| \partial \phi(z_4) \partial \phi(z_3) \partial \phi(z_2) \partial \phi(z_1) \middle| 0 \right\rangle = \frac{1}{z_{12}^2 z_{34}^2} + \frac{1}{z_{13}^2 z_{24}^2} + \frac{1}{z_{14}^2 z_{23}^2}. \tag{5}$$

1.1 Conformal block expansion in position space

A correlation function of 4 identical primary operators $\mathcal{O}$ with conformal weights $(h_\mathcal{O}, \bar{h}_\mathcal{O})$ can always be written as

$$\left\langle 0 \middle| \mathcal{O}(z_4, \bar{z}_4) \mathcal{O}(z_3, \bar{z}_3) \mathcal{O}(z_2, \bar{z}_2) \mathcal{O}(z_1, \bar{z}_1) \middle| 0 \right\rangle = \frac{1}{z_{12}^{2h_\mathcal{O}} z_{34}^{2h_\mathcal{O}} \bar{z}_{12}^{2\bar{h}_\mathcal{O}} \bar{z}_{34}^{2\bar{h}_\mathcal{O}}} G\left(\frac{z_{12} z_{34}}{z_{13} z_{24}}, \frac{\bar{z}_{12} \bar{z}_{34}}{\bar{z}_{13} \bar{z}_{24}} \right) \tag{6}$$

[1]Note that this shift symmetry of the free scalar theory is only present in 2 dimensions. In higher dimensions, the energy-momentum tensor must be improved with a descendant of the operator ϕ^2 for the theory to be conformal, or equivalently one must add the term $R\phi^2$ in the action. In both case the shift symmetry is broken.

where we have denoted $z_{ij} = z_i - z_j - i\epsilon$ and $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j + i\epsilon$. The function G admits an expansion

$$G(z,\bar{z}) = \sum_{h,\bar{h}} \lambda^{\mathcal{O}}_{h,\bar{h}}\, g_h(z) g_{\bar{h}}(\bar{z}) \tag{7}$$

in terms of the (holomorphic) conformal blocks

$$g_h(z) = z^h\, {}_2F_1(h,h;2h;z). \tag{8}$$

In the case of the correlator (5), the function G is

$$G(z,\bar{z}) = 1 + z^2 + \frac{z^2}{(1-z)^2}. \tag{9}$$

It is holomorphic, which means that the sum over $\bar{h}$ in (7) is trivial: only terms with $\bar{h} = 0$ will contribute (note that $g_0(\bar{z}) = 1$). Matching the expansion of G in powers of z with the conformal block expansion, one finds the OPE coefficients

$$\lambda^{\partial\phi}_{0,0} = 1, \qquad \lambda^{\partial\phi}_{h,0} = \frac{(h-1)!h!}{(2h-3)!} \quad (h = 2,4,6,\ldots) \tag{10}$$

Beside the identity operator with $(h,\bar{h}) = (0,0)$ and unit OPE coefficient, there is an infinite family of operators of conformal weights $(h,0)$ with even h that enter the OPE $\partial\phi \times \partial\phi$.

1.2 Fourier transform

The Fourier transform of the 4-point function (5) can be easily computed using the Schwinger integral

$$\frac{1}{(z-i\epsilon)^\alpha} = \frac{e^{i\pi\alpha/2}}{\Gamma(\alpha)} \int_0^\infty dk\, k^{\alpha-1} e^{-ikz}. \tag{11}$$

Defining $\widetilde{\partial\phi}(p) = \int dz\, e^{ipz} \partial\phi(z)$, we find

$$\begin{aligned}\big\langle 0 \big| \widetilde{\partial\phi}(p_4)\widetilde{\partial\phi}(p_3)\widetilde{\partial\phi}(p_2)\widetilde{\partial\phi}(p_1) \big| 0 \big\rangle = (2\pi)^4 \Big[&\Theta(p_1)\Theta(p_2) p_1 p_2 \delta(p_1+p_4)\delta(p_2+p_3) \\ &+ \Theta(p_1)\Theta(p_2) p_1 p_2 \delta(p_1+p_3)\delta(p_2+p_4) \\ &+ \Theta(p_1)\Theta(p_3) p_1 p_3 \delta(p_1+p_2)\delta(p_3+p_4) \Big].\end{aligned} \tag{12}$$

If will often be useful to use a different parametrization of the momenta, defining

$$p_i = p_1, \qquad p_f = -p_4, \qquad k = p_1 + p_2 = -p_3 - p_4, \tag{13}$$

and using the notation

$$\big\langle 0 \big| \widetilde{\mathcal{O}}(-p_f) \widetilde{\mathcal{O}}(p_f - k') \widetilde{\mathcal{O}}(k - p_i) \widetilde{\mathcal{O}}(p_i) \big| 0 \big\rangle = \delta(k-k') W^{\mathcal{O}}(p_i, p_f, k). \tag{14}$$

We will always assume in what follows that all three momenta p_i, p_f and k are physical, i.e. strictly inside the future light cone, which means $p_i, p_f, k > 0$. Under these assumptions, we have

$$W^{\partial\phi}(p_i,p_f,k) = (2\pi)^4 \Big[p_i p_f \delta(p_i + p_f - k) + p_i (k-p_i) \delta(p_i - p_f) \Theta(k - p_i) \Big]. \tag{15}$$

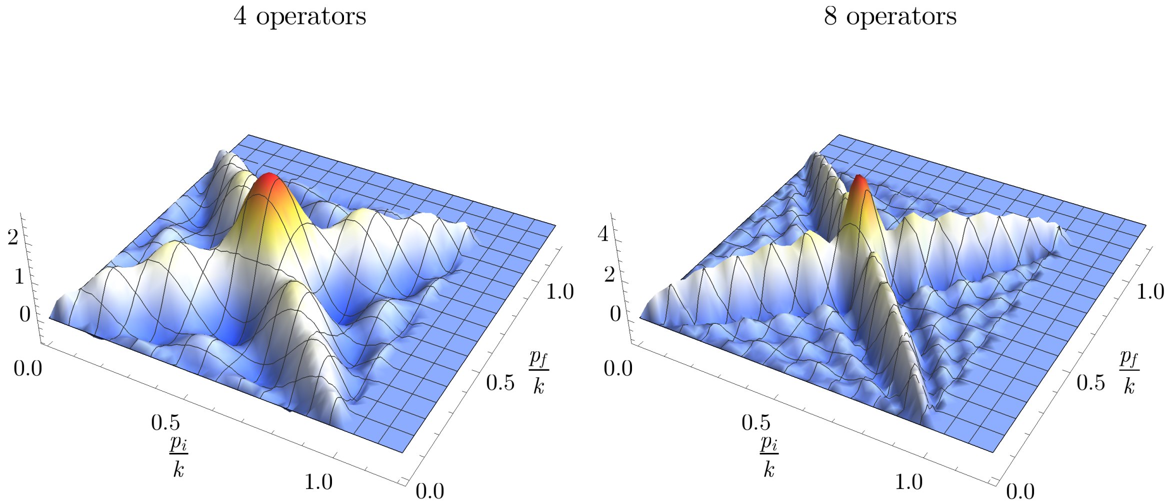

Figure 1: Result of the truncated conformal block expansion for the 4-point function $\langle\partial\phi\partial\phi\partial\phi\partial\phi\rangle$ in momentum space. We can see by eye that the expansion taken as a distribution converges to the expression (15) in terms of δ-functions.

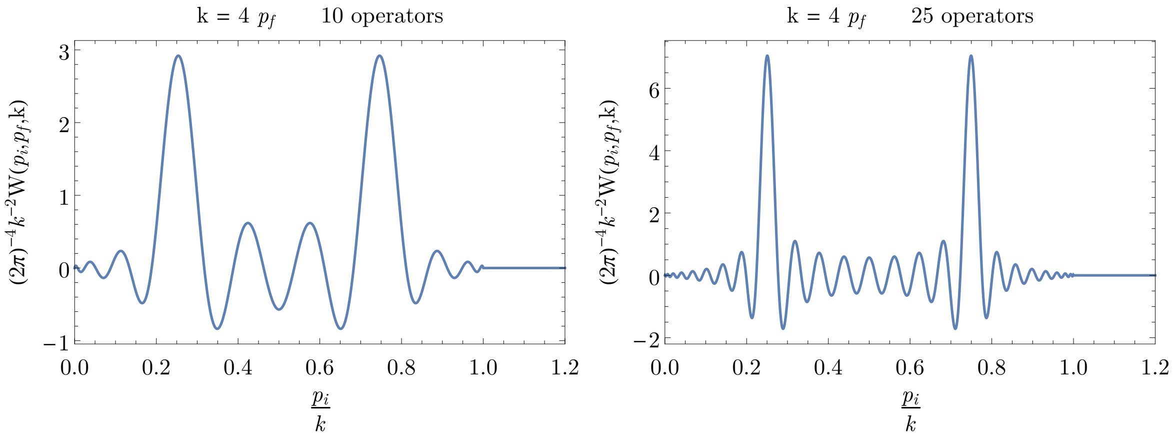

Figure 2: Cross-section of Fig. 1 along the line $p_f/k = 0.25$ for different number of operators in the conformal block expansion.

1.3 Conformal block expansion in momentum space

The function W defined in Eq. (14) admits an expansion similar to Eq. (7),

$$W^{\partial\phi}(p_i, p_f, k) = \sum_h \lambda^{\partial\phi}_{h,0} W^{\partial\phi}_h(p_i, p_f, k). \tag{16}$$

In our case the conformal blocks are

$$W^{\partial\phi}_h(p_i, p_f, k) = \Gamma(2h) k^{1-2h} V^{\partial\phi}_h(p_i, k-p_i) V^{\partial\phi}_h(p_f, k-p_f) \tag{17}$$

where V are homogeneous polynomials in the momenta

$$V^{\partial\phi}_0(p_1, p_2) = 0, \tag{18}$$

$$V^{\partial\phi}_h(p_1, p_2) = (2\pi)^2 \Theta(p_2) \sum_{n=1}^{h-1} \frac{(-1)^{n+1}(h-2)!}{n!(n-1)!(h-n)!(h-n-1)!} p_1^n p_2^{h-n} \quad (h = 2, 4, 6, \ldots). \tag{19}$$

The result of this expansion is shown in Figures 1 and 2. It can be verified that the series converge to Eq. (5) as a distribution, i.e. when integrated against well-behaved test functions. However, the conformal block expansion does not actually converge at any single point in the region $p_i, p_f < k$. In other words, the amplitude of the oscillations visible in Fig. 2 does not decay when the number of operators increases.

2 The correlator $\langle TTTT\rangle$

The second correlator that we consider is

$$\begin{aligned}\big\langle 0\big|T(z_4)T(z_3)T(z_2)T(z_1)\big|0\big\rangle &= \frac{1}{z_{12}^4 z_{34}^4} + \frac{1}{z_{13}^4 z_{24}^4} + \frac{1}{z_{14}^4 z_{23}^4} \\ &+ 4\left[\frac{1}{z_{12}^2 z_{13}^2 z_{24}^2 z_{34}^2} + \frac{1}{z_{12}^2 z_{14}^2 z_{23}^2 z_{34}^2} + \frac{1}{z_{13}^2 z_{14}^2 z_{23}^2 z_{24}^2}\right].\end{aligned} \tag{20}$$

2.1 Conformal block expansion in position space

We will focus on the connected part (the second line in the equation above), for which

$$G_{\text{conn.}}(z) = \frac{8z^2(1-z+z^2)}{(1-z)^2}. \tag{21}$$

The conformal block expansion gives a spectrum that consists in operator of conformal weights $(h,0)$ with $h = 2,4,6,\ldots$, and OPE coefficients

$$\lambda^T_{2,0} = 8, \qquad \lambda^T_{4,0} = \frac{44}{5}, \qquad \lambda^T_{6,0} = \frac{116}{63}, \qquad \lambda^T_{8,0} = \frac{10}{39}, \qquad \lambda^T_{10,0} = \frac{356}{12155}, \qquad \ldots \tag{22}$$

2.2 Fourier transform

The Fourier transform of Eq. (20) can be again computed using Schwinger integral (11). Writing

$$\big\langle 0\big|\widetilde{T}(-p_f)\widetilde{T}(p_f-k')\widetilde{T}(k-p_i)\widetilde{T}(p_i)\big|0\big\rangle_{\text{conn.}} = \delta(k-k')W^T(p_i,p_f,k), \tag{23}$$

with $p_i, p_f, k > 0$, we find that W^T is a piecewise polynomial function. Since by symmetry under Hermitian conjugation we have $W^T(p_i,p_f,k) = W^T(p_f,p_i,k)$, we can assume without loss of generality that $p_i \le p_f$, and we obtain then

$$W^T(p_i,p_f,k) = (2\pi)^4\frac{2}{3}\times\begin{cases} k^3(2p_i-k)(2p_f-k) & \text{if } 0\le k\le p_i, \\ p_i^3(2p_f-k)(2k-p_i) & \text{if } p_i\le k\le p_f, \\ p_i^3(2p_f-p_i)(2k-p_i-p_f) \\ \quad -(p_f-p_i-k)(p_f-p_i+k)(p_i+p_f-k)^3 & \text{if } p_f\le k\le p_i+p_f, \\ p_i^3(2p_f-p_i)(2k-p_i-p_f) & \text{if } k\ge p_i+p_f. \end{cases} \tag{24}$$

2.3 Conformal block expansion in momentum space

For the momentum-space conformal block, we have

$$W_2^T(p_i,p_f,k) = (2\pi)^4\frac{1}{24}\times\begin{cases} k^3(2p_i-k)(2p_f-k) & \text{if } 0\le k\le p_i, \\ p_i^3(2p_f-k)(2k-p_i) & \text{if } p_i\le k\le p_f, \\ \dfrac{p_i^3p_f^3(2k-p_i)(2k-p_f)}{k^3} & \text{if } k\ge p_f, \end{cases} \tag{25}$$

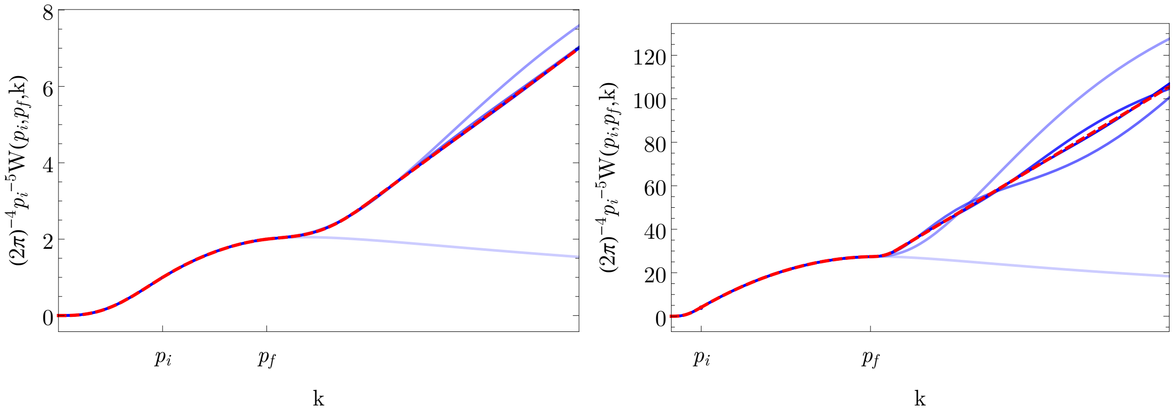

Figure 3: The Fourier transform of the connected correlation function $\langle TTTT \rangle$ taken from Eq. (24) (red dashed line), compared with the truncated conformal block expansion, including, 1, 2 3, 4 or 5 operators (from lighter to darker blue). The two panels correspond to different kinematics, with a large hierarchy between p_i and p_f on the right-hand side.

$$W_4^T(p_i, p_f, k) = (2\pi)^4 \frac{35}{9} \times \begin{cases} 0 & \text{if } k \leq p_f, \\ \dfrac{p_i^3 p_f^3 (k-p_i)^3 (k-p_f)^3}{k^7} & \text{if } k \geq p_f, \end{cases} \tag{26}$$

and blocks with $h > 4$ are similar to W_4^T in the sense that they vanish when $k \leq p_f$ and are polynomials in p_i/k and p_f/k otherwise. In the regime where $p_f \geq k$ or $p_i \geq k$, we see that the connected 4-point function is saturated by the first operator in the OPE, namely T itself. Beyond that regime, the conformal block expansion must necessarily contain an infinite number of contribution, because each conformal block is an analytic function in k over the interval $k \in (p_f, \infty)$ while the connected 4-point function has a non-analyticity at $k = p_i + p_f$. Nevertheless, as illustrated in Figure 3, the domain of convergence of the conformal block expansion extends all the way to $k \to \infty$.