Spring 2015 | 6 May 2015

Synthesis Modulo Bisimulation

Guides: Prof. Madhavan Mukund | Guna Prasaad, Vipul Venkataraman

In this report we describe the results we have obtained during the project. The descriptions are brief, kindly contact the authors for any queries.

1 Introduction

Let us begin with the definition of bisimulation.

Definition 1. *A* bisimulation *between a pair of transition systems $TS_1 = (Q_1, \rightarrow_1, q_1^{in})$ and $TS_2 = (Q_2, \rightarrow_2, q_2^{in})$ is a relation $R \subseteq (Q_1 \times Q_2)$ such that:*

- $(q_1^{in}, q_2^{in}) \in R$
- *If $(q_1, q_2) \in R$ and $q_1 \xrightarrow{a}_1 q_1'$, there exists q_2', $q_2 \xrightarrow{a}_2 q_2'$ and $(q_1', q_2') \in R$*
- *If $(q_1, q_2) \in R$ and $q_2 \xrightarrow{a}_2 q_2'$, there exists q_1', $q_1 \xrightarrow{a}_1 q_1'$ and $(q_1', q_2') \in R$*

Definition 2. *We call a pair of transition systems $TS_1 = (Q_1, \rightarrow_1, q_1^{in})$ and $TS_2 = (Q_2, \rightarrow_2, q_2^{in})$* bisimilar *to each other if there exists a bisimulation relation R between them. We denote this as $TS_1 \cong_B TS_2$.*

1.1 The synthesis problem modulo bisimilarity

The systhesis problem modulo bisimilarity can now be defined as follows. If $TS = (Q, \rightarrow, q^{in})$ is a transition system over Σ, and $\widetilde{\Sigma} = \langle \Sigma_1, ..., \Sigma_k \rangle$ is a distribution of Σ, does there exist a product system $||_{i \in \{1..k\}} TS_i$ over $\widetilde{\Sigma}$ such that $||_{i \in \{1..k\}} TS_i \cong_B TS$?

We worked on the sub-problem that has the following setting:

$$\Sigma = \{a, b, c\}, \Sigma_1 = \{a, c\}, \Sigma_2 = \{b, c\}$$

Theorem 3. *Let $TS_1 = (Q_1, \rightarrow_1, q_1^{in})$ and $TS_2 = (Q_2, \rightarrow_2, q_2^{in})$ be finite state transition systems over Σ_1 such that $TS_1 \cong_B TS_2$ and let $TS_3 = (Q_3, \rightarrow_3, q_3^{in})$ be a finite state transition system over Σ_2, then $TS_1 || TS_3 \cong_B TS_2 || TS_3$.*

Proof: Let $R_1 \subseteq (Q_1 \times Q_2)$ be a bisimulation relation over TS_1 and TS_2. Such an R_1 exists by Definition 2. We consider a relation $R_2 \subseteq ((Q_1 \times Q_3) \times (Q_2 \times Q_3))$ such that $((q_1, q_3), (q_2, q_4)) \in R_2$ if and only if:

(i) $q_3 = q_4$

(ii) $(q_1, q_2) \in R_1$

Constructing such an R_2 is trivial. We wish to show that R_2 is a bisimulation relation between $TS_1||TS_3$ and $TS_2||TS_3$. It is easy to see that $((q_1^{in}, q_3^{in}), (q_2^{in}, q_3^{in})) \in R_2$, where (q_1^{in}, q_3^{in}) is the initial state of $TS_1||TS_3$ and (q_2^{in}, q_3^{in}) is the initial state of $TS_2||TS_3$.

Now let's consider $((q_1, q_3), (q_2, q_3)) \in R_2$, and $(q_1, q_3) \xrightarrow{a} (q_1', q_3')$. If the a-move is disabled in TS_1 (in other words, $(q_1' = q_1)$), $(q_2, q_3) \xrightarrow{a} (q_2, q_3')$, and $((q_1, q_3'), (q_2, q_3')) \in R_2$. If the a-move is enabled in TS_1, we have $q_1 \xrightarrow{a} q_1'$. Using R_1, we know that there exists q_2', such that $q_2 \xrightarrow{a} q_2'$ and $(q_1', q_2') \in R_1$. Therefore there exists q_2', such that $(q_2, q_3) \xrightarrow{a} (q_2', q_3')$ and $((q_1', q_3'), (q_2', q_3')) \in R_2$.

We constructed an R_2 that satisfies all the three properties of a bisimulation relation mentioned in Definition 1. We proved the first two conditions and the third is symmetrical to the second. Since we have found a bisimulation relation R_2 between $TS_1||TS_3$ and $TS_2||TS_3$, both these transition systems are bisimilar. □

Lemma 4. *Let $TS_1 = (Q_1, \rightarrow_1, q_1^{in})$, $TS_2 = (Q_2, \rightarrow_2, q_2^{in})$, $TS_3 = (Q_3, \rightarrow_3, q_3^{in})$ and $TS_4 = (Q_4, \rightarrow_4, q_4^{in})$ be four finite state transition systems such that $TS_1 \cong_B TS_3$ and $TS_2 \cong_B TS_4$. Then $TS_1||TS_2 \cong_B TS_3||TS_4$.*

Proof: The proof of the lemma follows from Theorem 3. Since $TS_1 \cong_B TS_3$, we can say $TS_1||TS_2 \cong_B TS_3||TS_2$. Similarly, $TS_3||TS_2 \cong_B TS_3||TS_4$ follows because $TS_2 \cong_B TS_4$.

Using the above two observations and the transitivity of bisimulation, we can conclude that $TS_1||TS_2 \cong_B TS_3||TS_4$. □

Definition 5. *Let us consider a transition system $TS = (Q, \rightarrow, q^{in})$ over Σ and $a \in \Sigma$. We define another transition system $TS_{[a]} = (Q_{[a]}, \rightarrow_{[a]}, q^{in})$ as follows:*

(i) $q^{in} \in Q_{[a]}$

(ii) $q_1 \rightarrow_{[a]} q_2$ *iff* $q_1, q_2 \in Q, q_1 \xrightarrow{a} q_2, q_1 \in Q_{[a]}$

Notice that $TS_{[a]}$ is essentially the original transition system TS with only the a-moves enabled. It can be constructed through a BFS over TS from the start node and traversing only through the a-moves.

Theorem 6. *Let $TS = (Q, \rightarrow, q^{in})$ be a finite state transition system over $\Sigma = \{a, b\}$. Let $TS_1 = (Q_1, \rightarrow_1, q_1^{in})$ and $TS_2 = (Q_2, \rightarrow_2, q_2^{in})$ be finite state transition systems over $\Sigma_1 = \{a\}$ and $\Sigma_2 = \{b\}$ respectively, such that $TS \cong_B TS_1||TS_2$. Then $TS_1 \cong_B TS_{[a]}$ and $TS_2 \cong_B TS_{[b]}$.*

Proof: Let $R \subseteq (Q \times (Q_1 \times Q_2))$ be a bisimulation relation over TS and $TS_1||TS_2$. Such an R exists by Definition 2. Consider $TS_{[a]} = (Q_{[a]}, \rightarrow_{[a]}, q^{in})$ and $TS_{[b]} = (Q_{[b]}, \rightarrow_{[b]}, q^{in})$. We construct a relation $R_1 \subseteq (Q_1 \times Q_{[a]})$ such that $(q_1, q_2) \in R_1$ if and only if:

(i) $q_1 \in Q_1$ and $q_2 \in Q_{[a]}$

(ii) $(q_2, (q_1, q_2^{in})) \in R$

We notice that constructing such an R_1 is trivial. We wish to show that R_1 is a bisimulation relation between TS_1 and $TS_{[a]}$. It is easy to see that $(q_1^{in}, q^{in}) \in R_1$, where q_1^{in} is the initial state of TS_1 and q^{in} is the initial state of $TS_{[a]}$.

Now let's consider $(q_1, q_2) \in R_1$. This means $(q_2, (q_1, q_2^{in})) \in R$. Let $q_1 \xrightarrow{a} q_1'$. This implies $(q_1, q_2^{in}) \xrightarrow{a} (q_1', q_2^{in})$. Therefore, $\exists q_2' \in Q$, such that $q_2 \xrightarrow{a} q_2'$ and $(q_2', (q_1', q_2^{in})) \in R$. Since $q_2' \in Q_{[a]}$ and $(q_2', (q_1', q_2^{in})) \in R$, we have $(q_2', q_1') \in R_1$.

Let's say $q_2 \xrightarrow{a} q_2'$. Therefore, $\exists q_1' \in Q_1$, such that $(q_1, q_2^{in}) \xrightarrow{a} (q_1', q_2^{in})$ and $(q_2', (q_1', q_2^{in})) \in R$. This is due to the bisimulation relation R. Clearly, $q_2' \in Q_{[a]}$ since $q_2 \in Q_{[a]}$. We also know, $q_1' \in Q_1$ and $(q_2', (q_1', q_2^{in})) \in R$, therefore $(q_2', q_1') \in R$.

We constructed an R_1 that satisfies all the three properties of a bisimulation relation mentioned in Definition 1. We proved all the three conditions. This bisimulation relation R_1 between TS_1 and $TS_{[a]}$ implies $TS_1 \cong_B TS_{[a]}$.

The proof of $TS_2 \cong_B TS_{[b]}$ is very similar to that of $TS_1 \cong_B TS_{[a]}$ and is hence omitted. $\square$

Lemma 7. *Let $TS = (Q, \rightarrow, q^{in})$ be a finite state transition system over $\Sigma = \{a, b\}$. Let $TS_1 = (Q_1, \rightarrow_1, q_1^{in})$ and $TS_2 = (Q_2, \rightarrow_2, q_2^{in})$ be finite state transition systems over $\Sigma_1 = \{a\}$ and $\Sigma_2 = \{b\}$ respectively, such that $TS \cong_B TS_1||TS_2$. Then $TS \cong_B TS_{[a]}||TS_{[b]}$.*

Proof: The proof of the lemma follows directly from Lemma 4 and Theorem 6. Since $TS_1 \cong_B TS_{[a]}$ and $TS_2 \cong_B TS_{[b]}$, we have $TS_1||TS_2 \cong_B TS_{[a]}||TS_{[b]}$. We know that $TS \cong_B TS_1||TS_2$.

Using the above observations and the transitivity of bisimulation, we can conclude that $TS \cong_B TS_{[a]}||TS_{[b]}$. $\square$

Definition 8. *Let us consider an alphabet set Σ. We call $\widetilde{\Sigma} = \langle \Sigma_1, ..., \Sigma_k \rangle$ a* partition *of Σ if:*

(i) $\bigcup_{i=1}^{i=k} \Sigma_i = \Sigma$

(ii) $\Sigma_i \bigcap \Sigma_j = \emptyset$*, for all $i, j \in [1..k]$ and $i \neq j$*

Corollary 9. *Notice that if $TS = (Q, \rightarrow, q^{in})$ is a finite state transition system over Σ, and if $\widetilde{\Sigma} = \langle \Sigma_1, ..., \Sigma_k \rangle$ is a partition of Σ, we can effectively determine whether there exists a product system $||_{i \in \{1..k\}} TS_i$ over $\widetilde{\Sigma}$ such that $||_{i \in \{1..k\}} TS_i \cong_B TS$ using Lemma 7.*

Figure 1: $TS1$

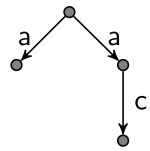
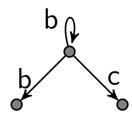


Figure 2: $TS2$



1.2 Quotient transition system

Let $TS = (Q, \rightarrow, q^{in})$ be a finite state transition sytem and let R be a bisimulation relation between TS and itself. The relation R defines an equivalence relation over Q. For $q \in Q$, let $[q]$ denote the R-equivalence class containing q. The *quotient* of TS is the transition system $TS/_R = (\hat{Q}, \rightsquigarrow, [q^{in}])$ where

(i) $\hat{Q} = \{[q] \mid q \in Q\}$

(ii) $[q] \rightsquigarrow [q']$ if there exists $q_1 \in [q]$ and $q_2 \in [q']$ such that $q_1 \rightsquigarrow q_2$

Definition 10. *Let $TS = (Q, \rightarrow, q^{in})$ be a finite state transition sytem and let $\sim_{TS}$ be the largest bisimulation relation between TS and itself. Then the transition system $TS/_{\sim_{TS}} = (\hat{Q}, \rightsquigarrow, [q^{in}])$ is called the* bisimulation quotient *of TS.*

The bisimulation quotient becomes very useful while expressing a class of transition systems bisimilar to each other, owing to the absence of any redundant states. Henceforth, we denote the bisimulation quotient of a transition system TS by $[TS]_{BQ}$.

A trivial but important property is to note that the bisimulation quotient need not necessary be a product all the time. This property is captured in figures 1, 2, 3 and 4.

The following lemma shows us that there can't be redundant states in a bisimulation quotient.

Lemma 11. *Let $TS = (Q, \rightarrow, q^{in})$ be a finite state transition system over $\Sigma = \{a, b, c\}$. Let $TS_1 = (Q_1, \rightarrow_1, q_1^{in})$ and $TS_2 = (Q_2, \rightarrow_2, q_2^{in})$ be finite state transition systems over $\Sigma_1 = \{a, c\}$ and $\Sigma_2 = \{b, c\}$ respectively, such that $TS \cong_B TS_1 || TS_2$. Consider the bisimulation quotient $[TS]_{BQ} = (\hat{Q}, \rightsquigarrow, [q^{in}])$. Let $R \subseteq (\hat{Q} \times (Q_1 \times Q_2))$*

Figure 3: $TS1||TS2$

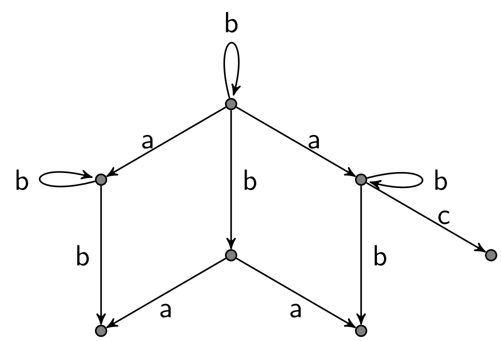
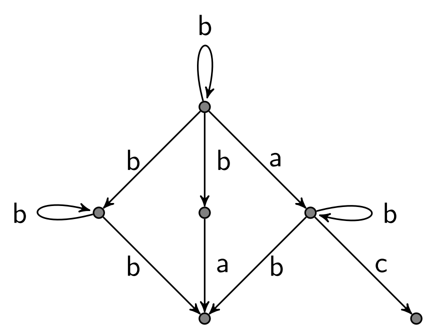


Figure 4: $[TS1||TS2]_{BQ}$



be a bisimulation relation between $[TS]_{BQ}$ and $TS_1||TS_2$. Now $(q_0, (q_1, q_2)) \in R$ and $(q_0', (q_1, q_2)) \in R$ iff $q_0 = q_0'$.

Proof: Clearly, if $TS \cong_B TS_1||TS_2$, then $[TS]_{BQ} \cong_B TS_1||TS_2$. Therefore, such an R exists through Definition 2. Consider a relation $C \subseteq (\hat{Q} \times \hat{Q})$ such that:

(i) $(q^{in}, q^{in}) \in C$

(ii) $(q_i, q_j) \in C$ iff $\exists\ q_k \in (Q_1 \times Q_2)$ such that $(q_i, q_k) \in R$ and $(q_j, q_k) \in R$

Let $(q_i, q_j) \in C$. This means $\exists\ q_k \in (Q_1 \times Q_2)$ such that $(q_i, q_k) \in R$ and $(q_j, q_k) \in R$. Let $q_i \xrightarrow{a} q_i'$. We can necessitate that $\exists\ q_k' \in (Q_1 \times Q_2)$ such that $(q_i', q_k') \in R$ and $q_k \xrightarrow{a} q_k'$. Now since $(q_j, q_k) \in R$, there exists $q_j' \in \hat{Q}$ such that $(q_j', q_k') \in R$ and $q_j \xrightarrow{a} q_j'$. This implies $(q_i', q_j') \in C$. Therefore, it is easy to see that C is a bisimulation relation between $[TS]_{BQ}$ and itself.

Notice that if q_i and q_j are related in the bisimulation relation C between $[TS]_{BQ}$ and itself, then $q_i = q_j$ because we quotient all states that are related into a single state in the bisimulation quotient (see Definition 13). In other words, $q_i = q_j$ since there are no redundant states in the bisimulation quotient. $\square$

Theorem 12. *Let $TS = (Q, \rightarrow, q^{in})$ be a finite state transition system over $\Sigma = \{a, b, c\}$. Let $TS_1 = (Q_1, \rightarrow_1, q_1^{in})$ and $TS_2 = (Q_2, \rightarrow_2, q_2^{in})$ be finite state transition systems over $\Sigma_1 = \{a, c\}$ and $\Sigma_2 = \{b, c\}$ respectively, such that $TS \cong_B TS_1||TS_2$. Consider the bisimulation quotient $[TS]_{BQ} = (\hat{Q}, \rightsquigarrow, [q^{in}])$. Let $R \subseteq (\hat{Q} \times (Q_1 \times Q_2))$ be a bisimulation relation between $[TS]_{BQ}$ and $TS_1||TS_2$. Consider $(q_0, (q_1, q_2)) \in R$, $q_0 \xrightarrow{a} q_3$, $q_0 \xrightarrow{b} q_4$. Then there exists $q_5 \in \hat{Q}$, such that $q_3 \xrightarrow{b} q_5$, $q_4 \xrightarrow{a} q_5$.*

Proof: Since $q_0 \xrightarrow{a} q_3$ and $q_0 \xrightarrow{b} q_4$, there exists $q_6 \in Q_1$, $q_1 \xrightarrow{a} q_6$ and $q_7 \in Q_2$, $q_2 \xrightarrow{a} q_7$ since $[TS]_{BQ}$ and $TS_1||TS_2$ are bisimilar. We now know the following:

(i) $(q_0, (q_1, q_2)) \in R$

(ii) $(q_3, (q_6, q_2)) \in R$

(iii) $(q_4, (q_1, q_7)) \in R$

Since $(q_6, q_2) \xrightarrow{b} (q_6, q_7)$ and $(q_3, (q_6, q_2)) \in R$, there exists $q_5 \in \hat{Q}$, $q_3 \xrightarrow{b} q_5$ and $(q_5, (q_6, q_7)) \in R$. Similarly since $(q_1, q_7) \xrightarrow{a} (q_6, q_7)$ and $(q_4, (q_1, q_7)) \in R$, there exists $q_8 \in \hat{Q}$, $q_4 \xrightarrow{a} q_8$ and $(q_8, (q_6, q_7)) \in R$. We now have two states q_5 and q_8 that is related to the same state $(q_6, q_7) \in (Q_1 \times Q_2)$. Therefore, using Lemma 11 we have $q_5 = q_8$. $\square$

We call theorem 12 as the *diamond closure* property and this is one of the fundamental results of this report. This does not hold when TS is not the bisimulation quotient. Consider $TS1$ from figure 5 and $TS2$ from figure 6. $TS3$ from figure 7 doesn't satisfy the diamond closure property.

Figure 5: $TS1$

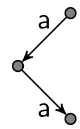


Figure 6: $TS2$

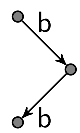


Figure 7: $TS3, TS3 \cong_B [TS1||TS2]_{BQ}$

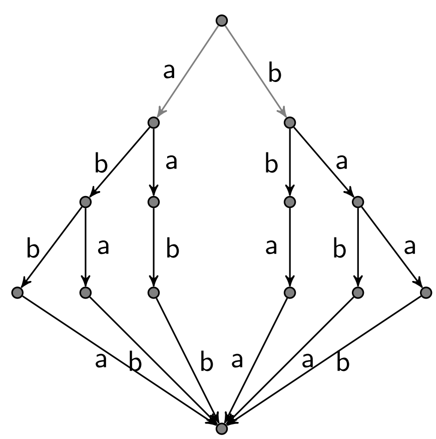


Figure 8: $TS1$

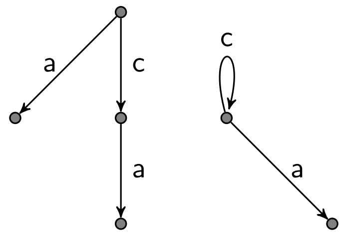
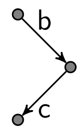


Figure 9: $TS2$



Definition 13. *Let $TS_1 = (Q_1, \rightarrow_1, q_1^{in})$ and $TS_2 = (Q_2, \rightarrow_2, q_2^{in})$ be two finite state transition systems. Let $TS = (Q, \rightarrow, q^{in})$ represent the product system $TS_1 || TS_2$. Let R represent the largest bisimulation relation between TS and itself. Then we say that given $q_1, q_1' \in Q_1$, $q_1 \cong_P q_1'$ iff $\forall q_2 \in Q_2$, $((q_1, q_2), (q_1', q_2)) \in R$.*

Lemma 14. *It is easy to show that $\cong_P$ is an equivalence relation on Q_1.*

Consider $TS_1' = (Q_1', \rightarrow_1', q_1^{in'})$, where $Q_1' = Q_1 / \cong_P$. An interesting problem that naturally arises from lemma 14 is if to check if $\rightarrow_1'$ is *well defined* i.e. $([q_1], a, [q_1']) \in \rightarrow_1'$ if $(q_1, a, q_1') \in \rightarrow_1$.

Theorem 15. *$\rightarrow_1'$ as defined above is not well defined.*

Proof: We prove the theorem through a counter example. Figures 8 and 9 completes the proof. Upon inspection, you will find that the two nodes in TS_1 having c transitions are $\cong_P$ equivalent. But the c transitions do not lead to nodes of the same class. $\square$

References

[1] I Castellani, M Mukund and P S Thiagarajan: Synthesizing distributed transition systems from global specifications, Proc. FSTTCS 19, Springer LNCS 1739 (1999) 219-231.

[2] M Mukund: From global specifications to distributed implementations, Synthesis and Control of Discrete Event Systems, B Caillaud, P Darondeau, L Lavagno (eds), Kluwer (2002) 19-34.